Pillowcases and quasimodular forms

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To Vladimir Drinfeld on his 50*th birthday.*

Summary. We prove that natural generating functions for enumeration of branched coverings of the pillowcase orbifold are level 2 quasimodular forms. This gives a way to compute the volumes of the strata of the moduli space of quadratic differentials.

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1 Introduction

1.1 Pillowcase covers and quadratic differentials

Consider a complex torus $\mathbb{T}^2 = \mathbb{C}/L$, where $L \subset \mathbb{C}$ is a lattice. Its quotient

$$\mathfrak{P} = \mathbb{T}^2/\pm$$

by the automorphism $z \mapsto -z$ is a sphere with four $(\mathbb{Z}/2)$-orbifold points which is sometimes called the *pillowcase* orbifold. The map $\mathbb{T}^2 \to \mathfrak{P}$ is essentially the Weierstraß $\wp$-function. The quadratic differential $(dz)^2$ on $\mathbb{T}^2$ descends to a quadratic differential on $\mathfrak{P}$. Viewed as a quadratic differential on the Riemann sphere, $(dz)^2$ has simple poles at corner points.

Let μ be a partition and ν a partition of an even number into *odd* parts. We are interested in enumeration of degree $2d$ maps

$$\pi : \mathcal{C} \to \mathfrak{P} \tag{1}$$

with the following ramification data. Viewed as a map to the sphere, π has profile $(\nu, 2^{d-|\nu|/2})$ over $0 \in \mathfrak{P}$ and profile (2^d) over the other three corners of $\mathfrak{P}$. Additionally, π has profile $(\mu_i, 1^{2d-\mu_i})$ over some $\ell(\mu)$ given points of $\mathfrak{P}$ and is unramified elsewhere. Here $\ell(\mu)$ is the number of parts in μ. This ramification data determines the genus of $\mathcal{C}$ by

$$\chi(\mathcal{C}) = \ell(\mu) + \ell(\nu) - |\mu| - |\nu|/2.$$

In principle, one could allow more general ramifications over 0 and the nonorbifold points, but this more general problem is readily reduced to the one above.[1]

Pulling back $(dz)^2$ via π gives a quadratic differential on $\mathcal{C}$ with zeros of multiplicities $\{\nu_i - 2\}$ and $\{2\mu_i - 2\}$. The periods of this differential, by construction, lie in a translate of a certain lattice. The enumeration of covers π is thus related to lattice point enumeration in the natural strata of the *moduli space of quadratic differentials*. In particular, the $d \to \infty$ asymptotics gives the volumes of these strata. These volumes are of considerable interest in ergodic theory, in particular in connection with billiards in rational polygons; see [6, 18]. Their computation was the main motivation for the present work.

A different way to compute the volume of the principal stratum was found by M. Mirzakhani [19].

1.2 Generating functions

1.2.1

Two covers $\pi_i : \mathcal{C}_i \to \mathfrak{P}$, $i = 1, 2$, are identified if there is an isomorphism $f : C_1 \to C_2$ such that $\pi_1 = f \circ \pi_2$. In particular, associated to every cover π is a finite group $\mathrm{Aut}(\pi)$. This group is trivial for most connected covers; see, e.g., [7, Section 3.1]. We form the generating function

$$Z(\mu, \nu; q) = \sum_{\pi} \frac{q^{\deg \pi}}{|\mathrm{Aut}(\pi)|}, \tag{2}$$

where π ranges over all inequivalent covers (1) with ramification data μ and ν as above. Note that the degree of any such π is even.

In particular, for $\mu = \nu = \emptyset$ any connected cover has the form

$$\pi : \mathbb{T}^2 \xrightarrow{\pi'} \mathbb{T}^2 \to \mathbb{T}^2/\pm$$

with π' unramified. We have $|\mathrm{Aut}(\pi)| = 2|\mathrm{Aut}(\pi')|$ corresponding to the lift of $\pm$. From the enumeration of possible π' we obtain,

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[1] From first principles, the count of the branched coverings does not change if one replaces two ramification conditions by the product of the corresponding conjugacy classes in the class algebra of the symmetric group. In this way, one can generate complicated ramifications from simpler ones.

$$Z(\emptyset, \emptyset; q) = \prod_n (1 - q^{2n})^{-1/2}.$$

By definition, we set

$$Z'(\mu, \nu; q) = \frac{Z(\mu, \nu; q)}{Z(\emptyset, \emptyset; q)}. \tag{3}$$

This enumerates covers without unramified connected components. By the usual inclusion-exclusion, one can extract from (3) a generating function for connected covers. This generating function for connected covers will be denoted by $Z^\circ(\mu, \nu; q)$.

1.2.2

Recall the classical level 1 Eisenstein series

$$E_{2k}(q) = \frac{\zeta(1-2k)}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n, \quad k = 1, 2, \ldots.$$

The algebra they generate is called the algebra $\mathcal{QM}(\Gamma(1))$ of *quasimodular forms* for $\Gamma(1) = SL_2(\mathbb{Z})$; see [16] and also below in Section 3.3.7. It is known that E_2, E_4, and E_6 are free commutative generators of $\mathcal{QM}(\Gamma(1))$. The algebra $\mathcal{QM}(\Gamma(1))$ is naturally graded by weight, where $\operatorname{wt} E_{2k} = 2k$. Clearly, for any integer N, $E_{2k}(q^N)$ is a quasimodular form of weight $2k$ for the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c \equiv 0 \bmod N \right\} \subset SL_2(\mathbb{Z}).$$

The quasimodular forms that will appear in this paper will typically be inhomogeneous, so instead of weight grading we will only keep track of the corresponding filtration. We define the weight of a partition by

$$\operatorname{wt} \mu = |\mu| + \ell(\mu).$$

The main result of this paper is the following.

Theorem 1. *The series $Z'(\mu, \nu; q)$ is a polynomial in $E_2(q^2)$, $E_2(q^4)$, and $E_4(q^4)$ of weight* $\operatorname{wt} \mu + |\nu|/2$.

Several explicit examples of the forms $Z'(\mu, \nu; q)$ are given in the appendix.

1.2.3

Quasimodular forms occur in nature, for example, as coefficients of the expansion of the odd genus 1 theta-function

$$\vartheta(x) = (x^{1/2} - x^{-1/2}) \prod_{i=1}^{\infty} \frac{(1 - q^i x)(1 - q^i/x)}{(1 - q^i)^2}$$

at the origin $x = 1$. The techniques developed below give a certain formula for (3) in terms of derivatives of $\vartheta(x)$ at $x = \pm 1$, from which the quasimodularity follows.

1.2.4

The following discussion closely parallels the corresponding discussion for the case of holomorphic differentials in [7, Section 1.2].

Let $\mathsf{Q}(\mu, \nu)$ denote the moduli space of pairs (Σ, ϕ), where ϕ is a quadratic differential on a curve Σ with zeroes of multiplicities $\{\nu_i - 2, 2\mu_i - 2\}$. Note that we allow $\nu_i = 1$; hence our quadratic differentials can have simple poles. For $(\Sigma, \phi) \in \mathsf{Q}(\mu, \nu)$, let $\widetilde{\Sigma}$ denote the double cover of Σ on which the differential

$$\omega = \sqrt{\phi}$$

is well defined. The pair $(\widetilde{\Sigma}, \omega)$ belongs to the corresponding space of holomorphic differentials with zeroes of multiplicity

$$\{\nu_i - 1, \mu_i - 1, \mu_i - 1\}.$$

By construction, Σ is the quotient of $\widetilde{\Sigma}$ by an involution σ. Let P denote the set of zeroes of ω; it is clearly stable under σ. Then σ acts as an involution on the relative homology group $H_1(\widetilde{\Sigma}, P, \mathbb{Z})$. Let H^- denote the subspace of $H_1(\widetilde{\Sigma}, P, \mathbb{Z})$ on which σ acts as multiplication by -1. Choose a basis $\{\gamma_1, \dots, \gamma_n\}$ for H^-, and consider the period map $\Phi : \mathsf{Q}(\mu, \nu) \to \mathbb{C}^n$ defined by

$$\Phi(\Sigma, \phi) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_n} \omega \right).$$

It is known [18] that $\Phi(\Sigma, \phi)$ is a local coordinate system on $\mathsf{Q}(\mu, \nu)$ and, in particular, $n = \dim_{\mathbb{C}} H^- = \dim_{\mathbb{C}} \mathsf{Q}(\mu, \nu)$.

Pulling back the Lebesgue measure from $\mathbb{C}^n$ yields a well-defined measure on $\mathsf{Q}(\mu, \nu)$. However, this measure is infinite since ϕ can be multiplied by any complex number. Thus we define $\mathsf{Q}_1(\mu, \nu)$ to be the subset satisfying

$$\operatorname{Area}(\widetilde{\Sigma}) \equiv \frac{\sqrt{-1}}{2} \int_{\widetilde{\Sigma}} \omega \wedge \overline{\omega} = 2.$$

As in the case of holomorphic differentials, the area function is a quadratic form in the local coordinates on $\mathsf{Q}(\mu, \nu)$, and thus the image under Φ of $\mathsf{Q}_1(\mu, \nu)$ can be identified with an open subset of a hyperboloid in $\mathbb{C}^n$.

Now let $E \subset \mathsf{Q}_1(\mu, \nu)$ be a set lying in the domain of a coordinate chart, and let $C\Phi(E) \subset \mathbb{C}^n$ denote the cone over $\Phi(E)$ with vertex 0. Then we can define a measure ρ on $\mathsf{Q}_1(\mu, \nu)$ via

$$\rho(E) = \operatorname{vol}(C\Phi(E)),$$

where vol is the Lebesgue measure. The proof of [7, Proposition 1.6] shows the analogue

$$\rho(\mathsf{Q}_1(\mu, \nu)) = \lim_{D \to \infty} D^{-\dim_{\mathbb{C}} \mathcal{Q}(\mu, \nu)} \sum_{d=1}^{2D} \operatorname{Cov}^0_d(\mu, \nu),$$

where $\mathrm{Cov}^0_d(\mu, \nu)$ is the number of inequivalent degree d connected covers $\mathcal{C} \to \mathfrak{P}$. Thus, the volume $\rho(\mathsf{Q}_1(\mu, \nu))$ can be read off from the $q \to 1$ asymptotics of the connected generating function $Z^\circ(\mu, \nu; q)$.

Note that the moduli spaces $\mathsf{Q}(\mu, \nu)$ may be disconnected. Ergodic theory applications require the knowledge of volumes of each connected component. Fortunately, connected components of $\mathsf{Q}(\mu, \nu)$ have been classified by E. Lanneau [17] and these spaces turn out to be connected except for hyperelliptic components (whose volume can be computed separately) and finitely many sporadic cases.

1.2.5

The modular transformation

$$q = e^{2\pi i \tau} \mapsto e^{-2\pi i/\tau}$$

relates $q = 0$ and $q = 1$ and thus gives an easy handle on the $q \to 1$ asymptotics of (3). This gives an asymptotic enumeration of pillowcase covers and hence computes the volume of the moduli spaces of quadratic differentials.

1.2.6

In spirit, Theorem 1 is parallel to the results of [1, 8, 13]; see also [2, 3, 5] for earlier results in the physics literature. The main novelty is the occurrence of quasimodular forms of higher level. One might speculate whether similar lattice point enumeration in the space of Nth order differentials leads to level N quasimodular forms. Those spaces, however, do not admit an $SL_2(\mathbb{R})$-action and a natural interpretation of their volumes is not known.

1.2.7

The following enumerative problem is naturally a building block of the enumerative problem that we consider. Consider branched covers of the sphere ramified over 3 points $0, 1, \infty$ with profile $(\nu, 2^{d-|\nu|/2})$, (2^d), and μ, respectively, where μ is an arbitrary partition of $2d$.

The preimage of the segment $[0, 1]$ on the sphere is a graph $\mathcal{G}$ on a Riemann surface (also known as a *ribbon graph*) with many 2-valent vertices (that can be ignored) and a few odd valent vertices (namely, with valencies ν_i). The complement of $\mathcal{G}$ is a union of $\ell(\mu)$ disks (known as *cells*) with perimeters $2\mu_i$ in the natural metric on $\mathcal{G}$. The asymptotic enumeration of such combinatorial objects is, almost by definition, given by integrals of ψ-classes against Kontsevitch's combinatorial cycles in $\overline{\mathcal{M}}_{g,\ell(\mu)}$; see [15]. There is a useful expression for these integrals in terms of Schur Q-functions obtained in [4, 11]. In fact, our original approach to the results presented in this paper was based on these ideas.

While the proof that we give here is more direct, it is still interesting to investigate the connection with combinatorial classes further, especially since a natural geometric interpretation of combinatorial classes is still missing. Perhaps the Gromov–Witten theory of the orbifold $\mathfrak{P}$ is the natural place to look for it. This will be further discussed in [22].

2 Character sums

2.1 Characters of near-involutions

2.1.1

There is a classical way to enumerate branched coverings in terms of irreducible characters, which is reviewed, for example, in [10] or in [21]. Specialized to our case, it gives

$$Z(\mu, \nu; q) = \sum_{\lambda} q^{|\lambda|/2} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \mathbf{f}_{\nu,2,2,\dots}(\lambda) \mathbf{f}_{2,2,\dots}(\lambda)^3 \prod_i \mathbf{f}_{\mu_i}(\lambda) \tag{4}$$

where summation is over all partitions, $\dim \lambda$ is the dimension of the corresponding representation of the symmetric group, and $\mathbf{f}_\eta(\lambda)$ is the *central character* of an element with cycle type η in the representation λ. Recall that the sum of all permutations with cycle type η acts as a scalar operator in any representation λ and, by definition, this number is $\mathbf{f}_\eta(\lambda)$. In (4), as usual, we abbreviate $\mathbf{f}_{k,1,1,\dots}$ to $\mathbf{f}_k$.

2.1.2

A lot is known about the characters of the symmetric group $S(2d)$ in the situation when the representation is arbitrary but the support of the permutation is bounded by some number independent of d. In particular, explicit formulas exist for the functions $\mathbf{f}_k$.

Understanding the function $\mathbf{f}_{\nu,2,2,\dots}$ is the key to evaluation of (4). That is, we must study characters of permutations that are a product of a permutation with finite support and a fixed-point-free involution. We call such permutations *near-involutions*.

2.1.3

By a result of Kerov and Olshanski [14], the functions $\mathbf{f}_k$ belong to the algebra Λ^* generated by

$$\mathbf{p}_k(\lambda) = (1 - 2^{-k})\zeta(-k) + \sum_i \left[(\lambda_i - i + \frac{1}{2})^k - \left(-i + \frac{1}{2} \right)^k \right]; \tag{5}$$

moreover, $\mathbf{f}_k$ has weight $k+1$ in the weight filtration on Λ^* defined by setting

$$\operatorname{wt} \mathbf{p}_k = k + 1.$$

The functions $\mathbf{p}_k$ are central characters of certain distinguished elements in the group algebra of symmetric group known as *completed cycles*. See [21] for the discussion of the relation between $\mathbf{p}_k$ and $\mathbf{f}_k$ from the viewpoint of Gromov–Witten theory.

2.1.4

Our next goal is to generalize the results of [14] to characters of near-involutions. This will require enlarging the algebra of functions. In addition to the polynomials $\mathbf{p}_k$, we will need quasi-polynomial functions $\bar{\mathbf{p}}_k$ defined in (6) below.

It is convenient to work with the generating function

$$\mathbf{e}(\lambda, z) \stackrel{\text{def}}{=} \sum_i e^{z(\lambda_i - i + \frac{1}{2})} = \frac{1}{z} + \sum_k \mathbf{p}_k(\lambda) \frac{z^k}{k!}.$$

By definition, set

$$\begin{aligned} \bar{\mathbf{p}}_k(\lambda) &= i k! [z^k] \mathbf{e}(\lambda, z + \pi i) \\ &= \sum_i \left[(-1)^{\lambda_i - i + 1} \left(\lambda_i - i + \frac{1}{2} \right)^k - (-1)^{-i+1} \left(-i + \frac{1}{2} \right)^k \right] + \text{const}, \end{aligned} \tag{6}$$

where the constant terms are determined by the expansion

$$\sum_k \frac{z^k}{k!} \bar{\mathbf{p}}_k(\emptyset) = \frac{1}{e^{z/2} + e^{-z/2}}.$$

Up to powers of 2, they are Euler numbers.

2.1.5

Define

$$\bar{\Lambda} = \mathbb{Q}[\mathbf{p}_k, \bar{\mathbf{p}}_k]_{k \geq 1}.$$

Setting

$$\text{wt}\, \bar{\mathbf{p}}_k = k$$

gives the algebra $\bar{\Lambda}$ the weight grading. Note that if f is homogeneous, then

$$f(\lambda') = (-1)^{\text{wt}\, f} f(\lambda), \tag{7}$$

where λ' denotes the conjugate partition.

2.1.6

In the definition of $\bar{\Lambda}$, we excluded the function

$$\bar{\mathbf{p}}_0(\lambda) = \frac{1}{2} + \sum_i [(-1)^{\lambda_i - i + 1} - (-1)^{-i+1}],$$

which measures the difference between the number of even and odd numbers among $\{\lambda_i - i + 1\}$, also known as the *2-charge* of a partition λ.

Every partition λ uniquely defines two partitions α and β, known as its *2-quotients*, such that

$$\left\{\lambda_i - i + \frac{1}{2}\right\} = \left\{2\left(\alpha_i - i + \frac{1}{2}\right) + \bar{\mathbf{p}}_0(\lambda)\right\} \sqcup \left\{2\left(\beta_i - i + \frac{1}{2}\right) - \bar{\mathbf{p}}_0(\lambda)\right\}.$$

A partition λ will be called *balanced* if $\bar{\mathbf{p}}_0(\lambda) = \frac{1}{2}$.

Several constructions related to 2-quotients will play an important role in this paper. A modern review of these ideas can be found, for example, in [9]. In particular, it is known that the character $\chi^\lambda_{2,2,\dots}$ of a fixed-point free involution in the representation λ vanishes unless λ is balanced, in which case

$$\left|\chi^\lambda_{2,2,\dots}\right| = \binom{|\lambda|/2}{|\alpha|,\,|\beta|} \dim\alpha \dim\beta. \tag{8}$$

It follows that only balanced partitions contribute to the sum (4).

2.1.7

For a balanced partition λ, define

$$\mathbf{g}_\nu(\lambda) = \frac{\mathbf{f}_{(\nu,2,2,\dots)}(\lambda)}{\mathbf{f}_{(2,2,\dots)}(\lambda)}. \tag{9}$$

We will prove that this function lies in $\bar{\Lambda}$ in the following sense.

Theorem 2. *The ratio* (9) *is the restriction of a unique function* $\mathbf{g}_\nu \in \bar{\Lambda}$ *of weight* $|\nu|/2$ *to the set of balanced partitions.*

Several examples of the polynomials $\mathbf{g}_\nu$ can be found in the appendix.

2.1.8

In view of Theorem 2, it is natural to introduce the *pillowcase weight*

$$\mathsf{w}(\lambda) = \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \mathbf{f}_{2,2,\dots}(\lambda)^4.$$

Theorem 1 follows from (4), Theorem 2, and the following result.

Theorem 3. *For any* $F \in \bar{\Lambda}$, *the average*

$$\langle F\rangle_{\mathsf{w}} = \frac{1}{Z(\emptyset,\emptyset;q)} \sum_\lambda q^{|\lambda|}\mathsf{w}(\lambda) F(\lambda) \tag{10}$$

is a polynomial in $E_2(q^2)$, $E_2(q^4)$, *and* $E_4(q^4)$ *of weight* wt F.

Note that if F is homogeneous of odd weight, then $\langle F\rangle_{\mathsf{w}} = 0$. This can be seen directly from (7). Also note that (10) will *not* in general be of pure weight even if F is a monomial in the generators $\mathbf{p}_k$ and $\bar{\mathbf{p}}_k$. This contrast with [1, 8] hints to the existence of a better set of generators of the algebra $\bar{\Lambda}$. Probably such generators are related to descendents of orbifold points in the Gromov–Witten theory of $\mathfrak{P}$.

2.1.9

It will be convenient to work with the following generating functions for the sums (10):

$$F(x_1, \dots, x_n) = \left\langle \prod \mathbf{e}(\lambda, \ln x_i) \right\rangle_{\mathsf{w}} . \tag{11}$$

The function (11) will be called the *n-point function*.

2.2 Proof of Theorem 2

2.2.1

In the proof of theorems 2 and 3 it will be very convenient to use the fermionic Fock space formalism. This formalism is standard and [12, 20] can be recommended as a reference. A quick review of these techniques can be found, for example, in [21, Section 2]. We follow the notation of [21].

2.2.2

By definition, the space $\Lambda^{\frac{\infty}{2}}{}_0 V$ is spanned by the infinite wedge products

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \cdots , \tag{12}$$

where $\underline{k}$, $k \in \mathbb{Z} + \frac{1}{2}$, is a basis of the underlying space V and λ is a partition. The subscript 0 in $\Lambda^{\frac{\infty}{2}}{}_0 V$ refers to the charge zero condition: the ith factor in (12) is $\underline{-i + \frac{1}{2}}$ for all sufficiently large i.

There is a natural projective representation of the Lie algebra $\mathfrak{gl}(V)$ on $\Lambda^{\frac{\infty}{2}}{}_0 V$. For us, the following elements of $\mathfrak{gl}(V)$ will be especially important:

$$\mathcal{E}_k[f(x)]\underline{i} = f\left(i - \frac{k}{2}\right) \underline{i - k}, \tag{13}$$

where f is a function on the real line. To define the action of $\mathcal{E}_0[f(x)]$ on $\Lambda^{\frac{\infty}{2}}{}_0 V$ one needs to regularize the infinite sum $\sum_{i<0} f(\frac{1}{2} - i)$. This regularization is the source of the central extension in the $\mathfrak{gl}(V)$ action. When f is an exponential as in

$$\mathcal{E}_k(z) = \mathcal{E}_k[e^{zx}],$$

this infinite sum is a geometric series and thus has a natural regularization. By differentiation, this leads to the ζ-regularization for operators $\mathcal{E}_k[f]$ with a polynomial function f.

2.2.3

Other very useful operators are

$$\alpha_k = \mathcal{E}_k[1], \quad k \neq 0.$$

The operator H defined by

$$H v_\lambda = |\lambda| v_\lambda$$

is known as the energy operator. It differs only by a constant from the operator $\mathcal{E}_0[x]$. The operator H defines a natural grading on $\Lambda^{\frac{\infty}{2}}{}_0 V$ and $\mathfrak{gl}(V)$.

2.2.4

A function $F(\lambda)$ on partitions of n can be viewed as a vector

$$\sum_{|\lambda|=n} F(\lambda) v_\lambda \in \Lambda^{\frac{\infty}{2}}{}_0 V$$

of energy n. For example, the vectors

$$|\mu\rangle \stackrel{\text{def}}{=} \frac{1}{\mathfrak{z}(\mu)} \prod \alpha_{-\mu_i} v_\emptyset = \frac{1}{\mathfrak{z}(\mu)} \sum_\lambda \chi^\lambda_\mu v_\lambda, \tag{14}$$

where

$$\mathfrak{z}(\mu) = |\mathrm{Aut}\, \mu| \prod \mu_i,$$

correspond to irreducible characters normalized by the order of the centralizer.

2.2.5

The operator $\mathcal{E}_0(z)$ is the generating function

$$\mathcal{E}_0(z) = \mathcal{E}_0[e^{zx}] = \frac{1}{z} + \sum_k \frac{z^k}{k!} \mathcal{P}_k,$$

for the operators $\mathcal{P}_k$ acting by

$$\mathcal{P}_k v_\lambda = \mathbf{p}_k(\lambda) v_\lambda.$$

In parallel to (6), we define operators $\bar{\mathcal{P}}_k$ by

$$i \mathcal{E}_0(z + \pi i) = \sum_k \frac{z^k}{k!} \bar{\mathcal{P}}_k.$$

Translated into the operator language, the statement of Theorem 2 is the following: the orthogonal projection of $|\nu, 2^{d-|\nu|/2}\rangle$ onto the subspace spanned by the v_λ with λ balanced is a linear combination of vectors

$$\prod \mathcal{P}_{\mu_i} \prod \bar{\mathcal{P}}_{\bar{\mu}_i} |2^d\rangle \tag{15}$$

with

$$\mathrm{wt}\, \mu + |\bar{\mu}| \leq |\nu|/2$$

and coefficients independent of d.

2.2.6

Let us call the span of v_λ with λ balanced the balanced subspace of $\Lambda^{\frac{\infty}{2}}{}_0 V$. A convenient orthogonal basis of it is provided by the vectors

$$|\rho;\bar\rho\rangle \overset{\text{def}}{=} \frac{1}{\mathfrak{z}(\rho)\mathfrak{z}(\bar\rho)} \prod \alpha_{-\rho_i} \prod \bar\alpha_{-\bar\rho_i} v_\emptyset, \qquad \rho_i, \bar\rho_i \in 2\mathbb{Z}, \tag{16}$$

where the operators $\bar\alpha_k$ are defined by

$$\bar\alpha_k = i^{k+1}\mathcal{E}_k(\pi i) = \sum_n (-1)^{n+\frac12} E_{n-k,n} + \frac{\delta_k}{2}, \tag{17}$$

the operators $E_{i,j}$ being the matrix units of $\mathfrak{gl}(V)$. From the commutation relations for the operators $\mathcal{E}_k(z)$, we compute

$$[\bar\alpha_k, \bar\alpha_m] = [(-1)^k - (-1)^m]\alpha_{k+m} + k(-1)^k \delta_{k+m}, \tag{18}$$

$$[\alpha_k, \bar\alpha_m] = [1-(-1)^k]\left(\bar\alpha_{k+m} + \frac{\delta_{k+m}}{2}\right). \tag{19}$$

In particular, when both k and m are even, all these operators commute apart from the central term in $[\bar\alpha_k, \bar\alpha_{-k}]$.

The adjoint of $\bar\alpha_k$ is

$$\bar\alpha_k^* = (-1)^k \bar\alpha_{-k},$$

which gives the inner products

$$\langle \rho;\bar\rho | \rho';\bar\rho'\rangle = \frac{\delta_{\rho,\rho'}\delta_{\bar\rho,\bar\rho'}}{\mathfrak{z}(\rho)\mathfrak{z}(\bar\rho)}, \tag{20}$$

provided all parts of all partitions in (20) are even. In particular, the vectors (16) are orthogonal. It is clear that they lie in the balanced subspace and their number equals the dimension of the space. Therefore, they form a basis.

2.2.7

The projection of $|\nu, 2^{d-|\nu|/2}\rangle$ onto the balanced subspace is given in term of inner products of the form

$$\langle \nu, 2^{d-|\nu|/2} \big| (\rho, 2^{d-|\rho|/2-|\bar\rho|/2}); \bar\rho\rangle$$

where all parts of ν are odd, all parts of ρ and $\bar\rho$ are even, and ρ has no parts equal to 2. From the commutation relations (18) and (19) we conclude that this inner product vanishes unless

$$\rho = \emptyset.$$

The nonvanishing inner products are

$$\langle \nu, 2^k \big| 2^k; \bar\rho \rangle = \frac{2^{\ell(\nu)-\ell(\bar\rho)}}{2^k k! \mathfrak{z}(\nu)\mathfrak{z}(\bar\rho)} \mathsf{C}(\nu, \bar\rho), \tag{21}$$

where the combinatorial coefficient $\mathsf{C}(\nu, \bar\rho)$ equals the number of ways to represent the parts of $\bar\rho$ as sums of parts of ν. For example,

$$\mathsf{C}((3, 1, 1, 1), (4, 2)) = 3, \qquad \mathsf{C}((3, 1, 1, 1), (6)) = 1.$$

2.2.8

The matrix elements

$$\left\langle 2^d \left| \prod \mathcal{P}_{\mu_i} \prod \bar{\mathcal{P}}_{\bar\mu_i} \right| (\rho, 2^{d-|\rho|/2-|\bar\rho|/2}); \bar\rho \right\rangle, \quad \rho_i \neq 2, \tag{22}$$

describe the decomposition of the vectors (15) in the basis (16). Since

$$\mathcal{P}_1 |2^d\rangle = \left(d - \frac{1}{24}\right) |2^d\rangle, \tag{23}$$

we can also assume that $\mu_i \neq 1$.

We claim that (22) vanishes unless

$$\operatorname{wt} \mu + |\bar\mu| \geq \operatorname{wt} \rho/2 + |\bar\rho|/2, \tag{24}$$

where $\rho/2$ is the partition with parts $\rho_i/2$ (recall that all parts of ρ are even).

2.2.9

The usual way to evaluate a matrix element like (22) is to use commutation relations to commute all lowering operators to the right until they reach the vacuum (which they annihilate) and, similarly, commute the raising operators to the left.

We will exploit the following property of the operators $\mathcal{P}_k$ and $\bar{\mathcal{P}}_k$: their commutator with enough operators of the form $\alpha_{-2\rho_i}$ and $\bar\alpha_{-2\bar\rho_i}$ vanishes. All such commutators have the form $\mathcal{E}_k[f]$ with $f(x) = (\pm 1)^x p(x)$, where $p(x)$ is a polynomial. Commutation with $\alpha_{-2\rho_i}$ takes a finite difference of $p(x)$; commutation with $\bar\alpha_{-2\bar\rho_i}$ additionally flips the sign of ± 1.

Since a $(k+1)$-fold finite difference of a degree k polynomial vanishes, the commutator of $\mathcal{P}_k$ with more than $k+1$ operators of the form $\alpha_{-2\rho_i}$ or $\bar\alpha_{-2\bar\rho_i}$ vanishes. In fact, a $(k+1)$-fold commutator may be nonvanishing only because of the central extension term. To pick up this central term, the total energy of all operators involved should be zero and the number of $\bar\alpha$s should be even. The same reasoning applies to $\bar{\mathcal{P}}_k$, but now the number of $\bar\alpha$s should be odd to produce a nontrivial $(k+1)$-fold commutator.

2.2.10

Now look at one of the raising operators involved in (22), say $\bar{\alpha}_{-\bar{\rho}_i}$. This operator commutes with α_2 and its adjoint annihilates the vacuum, so only the terms involving the commutator of $\bar{\alpha}_{-\bar{\rho}_i}$ with one of the $\mathcal{P}_{\mu_i}$ or $\bar{\mathcal{P}}_{\bar{\mu}_i}$ give a nonzero contribution to (22). The commutator $[\mathcal{P}_{\mu_i}, \bar{\alpha}_{-\bar{\rho}_i}]$ has energy $(-\rho_i)$ and so its adjoint again annihilates the vacuum. The same is true for the commutation with $\bar{\mathcal{P}}_{\bar{\mu}_i}$. To bring these commutators back to zero energy, one needs to commute it $\bar{\rho}_i/2$ times with α_2. Given the above bounds on how many commutators we can afford, this implies (24).

2.2.11

When the bound (24) is saturated, then a further condition

$$\ell(\rho) + \ell(\bar{\rho}) \geq \ell(\mu) + \ell(\bar{\mu})$$

is clearly necessary for nonvanishing of (22). The unique nonzero coefficient saturating both bounds corresponds to

$$\rho = 2\mu, \quad \bar{\rho} = 2\bar{\mu}.$$

Moreover, when divided by the norm squared of the vector $|(\rho, 2^{d-|\rho|/2-|\bar{\rho}|/2}); \bar{\rho}\rangle$, this coefficient is independent of d.

2.2.12

For general ρ and $\bar{\rho}$, the similarly normalized coefficient will be a polynomial in d of degree

$$\frac{1}{2}(\operatorname{wt}\mu + |\bar{\mu}| - \operatorname{wt}\rho/2 - |\bar{\rho}|/2) \tag{25}$$

because so many operators α_{-2} can commute with $\mathcal{P}_{\mu_i}$s or $\bar{\mathcal{P}}_{\bar{\mu}_i}$s instead of commuting directly with α_2s.

By induction on weight and length, we can express the basis vectors (16) in terms of (15) with $\mu_i \neq 1$ and coefficients being polynomial in d of degree at most minus the difference (25). By (23), to have d-dependent coefficients and $\mu_i \neq 1$ is the same as to allow $\mu_i = 1$ and make the coefficients independent of d. The bound of degree in d ensures that this transition preserves weight. This concludes the proof of Theorem 2.

3 Proof of Theorem 3

3.1 The pillowcase operator

3.1.1

Consider the operator

$$\mathfrak{W} = \exp\left(\sum_{n>0} \frac{\alpha_{-2n-1}}{2n+1}\right) \exp\left(-\sum_{n>0} \frac{\alpha_{2n+1}}{2n+1}\right). \tag{26}$$

Because this operator is normally ordered, its matrix elements $(\mathfrak{W}v, w)$ are well defined for any vectors v and w of finite energy. The relevance of this operator for our purposes lies in the following.

Theorem 4. *The diagonal matrix elements of* $\mathfrak{W}$ *are*

$$(\mathfrak{W}v_\lambda, v_\lambda) = \begin{cases} \mathsf{w}(\lambda), & \lambda \textit{ is balanced}, \\ 0 & \textit{otherwise}. \end{cases}$$

The proof of this theorem will occupy the rest of Section 3.1.

3.1.2

Let N be chosen so large that $\lambda_{2N+1} = 0$. Then because the operator (26) is a product of an upper unitriangular and lower unitriangular operator, the vectors $\underline{\lambda_i - i + \frac{1}{2}}$ with $i > 2N$ in

$$v_\lambda = \underline{\lambda_1 - \tfrac{1}{2}} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \cdots$$

are inert bystanders for the evaluation of $(\mathfrak{W}v_\lambda, v_\lambda)$. The whole computation is therefore a computation of a matrix element of an operator in a finite exterior power of a finite dimensional vector space $V^{[N]}$ with basis

$$e_k = \underline{-2N + k + \tfrac{1}{2}}, \quad k = 0, \ldots, \lambda_1 + 2N - 1.$$

By definition, matrix elements of $\mathfrak{W}$ in exterior powers of $V^{[N]}$ are determinants of the matrix elements of $\mathfrak{W}$ acting on the space $V^{[N]}$ itself. The latter matrix elements are determined in the following.

Proposition 1. *We have*

$$\frac{(\mathfrak{W}e_k, e_l)}{\mathsf{b}(k)\mathsf{b}(l)} = \begin{cases} 1, & k \equiv l \equiv 0 \mod 2, \\ 0, & k \equiv l \equiv 1 \mod 2, \\ 2/(k-l), & \textit{otherwise}, \end{cases} \tag{27}$$

where

$$\mathsf{b}(k) = \frac{k!}{2^k \lfloor k/2 \rfloor!^2}.$$

3.1.3

For the proof of Proposition 1, form the generating function

$$f(x, y) = \sum_{k,l} x^k y^l (\mathfrak{W} e_k, e_l).$$

From the equality

$$\exp\left(\sum_{n>0} \frac{x^{2n+1}}{2n+1}\right) = \sqrt{\frac{1+x}{1-x}}$$

and definitions, we compute

$$f(x, y) = \frac{1}{1-xy}\sqrt{\frac{1+x}{1-x}}\sqrt{\frac{1-y}{1+y}}.$$

The factorization

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) f(x, y) = \frac{(x+y)(1+x)(1-y)}{(1-x^2)^{3/2}(1-y^2)^{3/2}}$$

by elementary binomial coefficient manipulations proves (27) for $k \neq l$. To compute the diagonal matrix elements observe that the above differential equation uniquely determines $f(x, y)$ from its values on the diagonal $x = y$. On the diagonal, the skew-symmetric terms in (27) cancel out and evaluation is immediate.

3.1.4

We now proceed to the computation of the matrix element $(\mathfrak{W} v_\lambda, v_\lambda)$. We have the following.

Proposition 2. *We have*

$$(\mathfrak{W} v_\lambda, v_\lambda) = \left(2^N \prod_{i=1}^{2N} \mathsf{b}(\lambda_i - i + 2N) \prod_{i<j\leq 2N} (\lambda_i - \lambda_j + j - i)^{(-1)^{\lambda_i-\lambda_j+j-i}}\right)^2,$$

provided λ *is balanced and* $(\mathfrak{W} v_\lambda, v_\lambda) = 0$ *otherwise.*

The proof of this proposition is the following. Observe that by Proposition 1 the matrix element $(\mathfrak{W} v_\lambda, v_\lambda)$ is a determinant of a $2N \times 2N$ block matrix in which the odd-odd block is identically zero, the even-even block is a rank 1 matrix with all elements equal to 1 and the off-diagonal blocks have the form $(\frac{2}{x_i - y_j})$, where $\{x_i\}$ and $\{y_i\}$ are the odd and even subsets of $\{\lambda_i - i + 2N\}$. Since the odd-odd block is identically zero, its size has to be $\leq N$ for the determinant to be nonvanishing. Similarly, if the size of the even-even block is larger than N, then the determinant is easily seen to vanish. It follows that both blocks have size N, which precisely means that the partition λ is balanced. It remains to use the Cauchy determinant

$$\det\left(\frac{1}{x_i+y_j}\right)=\frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)}{\prod(x_i+y_j)}$$

to finish the proof.

3.1.5

Note that decomposition of $\{\lambda_i - i + 2N\}$ into the even and odd subsets is the same as the 2-quotient construction from Section 2.1.6. Theorem 4 follows from formula (8) and the classical formula

$$\frac{\dim\lambda}{|\lambda|!}=\frac{\prod_{i<j\le N}(\lambda_i-\lambda_j+j-i)}{\prod(\lambda_i+N-i)!},$$

where N is any number such that $\lambda_{N+1}=0$.

3.1.6

It would be interesting to find an interpretation of the operator $\mathfrak{W}$ in conformal field theory. Note that

$$\exp\left(\sum_{n>0}\frac{z^{-2n-1}}{2n+1}\right)\exp\left(-\sum_{n>0}\frac{z^{2n+1}}{2n+1}\right)=\sqrt{\frac{1+z^{-1}}{1-z^{-1}}}\sqrt{\frac{1-z}{1+z}}$$

is the Wiener–Hopf factorization of the function taking the value $\mp i$ on the upper/lower half-plane.

3.2 Formula for the n-point function

3.2.1

Theorem 4 yields the following operator formula for the n-point function (11):

$$F(x_1,\dots,x_n)=\frac{1}{Z(\emptyset,\emptyset;q)}\operatorname{tr}q^H\prod\mathcal{E}_0(\ln x_i)\mathfrak{W}, \qquad (28)$$

where the trace is taken in the charge zero subspace of the infinite wedge and H is the energy operator

$$Hv_\lambda=|\lambda|v_\lambda.$$

We have the following expression for the operator $\mathcal{E}_0$ in terms of the fermionic currents:

$$\mathcal{E}_0(\ln x)=[y^0]\psi(xy)\psi^*(y),$$

where $[y^0]$ denotes the constant coefficient in the Laurent series expansion in the variable y. Therefore,

$$F(x_1,\dots,x_n) = \frac{1}{Z(\emptyset,\emptyset;q)} \times [y_1^0\cdots y_n^0]\,\mathrm{tr}\, q^H \psi(x_1y_1)\psi^*(y_1)\cdots\psi(x_ny_n)\psi^*(y_n)\mathfrak{W}. \tag{29}$$

3.2.2

By the main result of [9], we have

$$\mathsf{w}(\lambda) \le 1 \tag{30}$$

for any partition λ. In other words, all diagonal matrix elements of $\mathfrak{W}$ are bounded by 1. For the off-diagonal elements, we prove the following cruder bound.

Proposition 3. *Let* $M = \max\{|\lambda|, |\mu|\}$. *Then*

$$(\mathfrak{W}v_\lambda, v_\mu) \le \exp\left(\frac{1}{2}\sum_{n=0}^{\lfloor\frac{M-1}{2}\rfloor}\frac{1}{2n+1}\right) \sim \mathrm{const}\cdot M^{1/4}. \tag{31}$$

To see this note that

$$(\mathfrak{W}v_\lambda, v_\mu) = (\mathfrak{W}^{[M]}v_\lambda, v_\mu),$$

where $\mathfrak{W}^{[M]}$ is the truncated operator

$$\exp\left(\sum_{2n+1\le M}\frac{\alpha_{-2n-1}}{2n+1}\right)\exp\left(-\sum_{2n+1\le M}\frac{\alpha_{2n+1}}{2n+1}\right).$$

We claim that the operator $\mathfrak{W}^{[M]}$ is a multiple of a unitary operator. Indeed,

$$(\mathfrak{W}^{[M]*})^{-1} = \exp\left(-\sum_{n=0}^{\lfloor\frac{M-1}{2}\rfloor}\frac{1}{2n+1}\right)\mathfrak{W}^{[M]},$$

whence the result.

In fact, we will only use that (31) is bounded by a polynomial in the sizes of the partitions.

3.2.3

By normally ordering all fermionic operators in (29) and using the estimate (31) one sees that the trace converges if

$$|y_n/q| > |x_1y_1| > |y_1| > \cdots > |x_ny_n| > |y_n| > 1. \tag{32}$$

3.2.4

The proof of the following identity is given in [12, Theorem 14.10]:

$$\psi(xy)\psi^*(y) = \frac{1}{x^{1/2} - x^{-1/2}} \times \exp\left(\sum_n \frac{(xy)^n - y^n}{n}\alpha_{-n}\right)\exp\left(\sum_n \frac{y^{-n} - (xy)^{-n}}{n}\alpha_n\right). \tag{33}$$

It allows to express the operator in (29) in terms of bosonic operators α_n.

With respect to the action of the operators α_n, the charge zero subspace of the infinite wedge space decomposes as the infinite tensor product

$$\Lambda^{\frac{\infty}{2}}{}_0 V \cong \bigotimes_{n=1}^{\infty}\bigoplus_{k=0}^{\infty} \alpha_{-n}^k v_\emptyset,$$

the distinguished vector in each factor being $v_\emptyset$. This gives a factorization of the trace in (29). The trace in each tensor factor is computed as follows:

$$\operatorname{tr} e^{A\alpha_{-n}} e^{B\alpha_n}\Big|_{\bigoplus_{k=0}^{\infty}\alpha_{-n}^k v_\emptyset} = \frac{1}{1-q^n}\exp\left(\frac{nABq^n}{1-q^n}\right).$$

For example, this shows that

$$\operatorname{tr} q^H \mathfrak{W} = (q^2)_\infty^{-1/2} = Z(\emptyset,\emptyset;q),$$

where

$$(a)_\infty = \prod_{n\geq 0}(1-aq^n),$$

and so the 0-point function is $F(\,) = 1$, as expected. For the n-point function this gives the following.

Theorem 5. *We have*

$$F(x_1,\dots,x_n) = \prod \frac{1}{\vartheta(x_i)} \times [y_1^0\cdots y_n^0]\prod_{i<j}\frac{\vartheta(y_i/y_j)\vartheta(x_iy_i/x_jy_j)}{\vartheta(x_iy_i/y_j)\vartheta(y_i/x_jy_j)}\prod_i\sqrt{\frac{\vartheta(-y_i)\vartheta(x_iy_i)}{\vartheta(y_i)\vartheta(-x_iy_i)}}, \tag{34}$$

where the series expansion is performed in the domain (32).

3.3 Quasimodular forms

3.3.1

In the computation of (34), we can assume that $1 < |x_i| \ll |q^{-1}|$ for all i and hence

$$|y_i| > |y_j| \prod |x_k|^{\pm 1} > |q y_i|, \quad i < j.$$

The series expansion in (34) can then be performed using the following elementary lemma.

Lemma 4. *We have*

$$\frac{1}{2\pi i} \oint_{|y|=c} \frac{dy}{y} \prod_{i=1}^{n} \frac{\vartheta(y/a_i)}{\vartheta(y/b_i)} = \left(1 - \prod \frac{a_i}{b_i}\right)^{-1} \sum_{i=1}^{n} \frac{\prod_j \vartheta(b_i/a_j)}{\prod_{j \neq i} \vartheta(b_i/b_j)}, \tag{35}$$

provided $c > |b_i| > |q| c$ for $i = 1, \dots, n$.

This is obtained by computing the difference of $\oint_{|y|=c}$ and $\oint_{|y|=|q|c}$ as a sum of residues using

$$\vartheta'(1) = 1.$$

3.3.2

There are two obstacles to literally applying this lemma to the evaluation of (34). The first is the square roots in (34). However, we are ultimately interested in the expansion of (34) about $x_i = \pm 1$. The expansion of the integrand about $x_i = \pm 1$ contains no square roots, only the theta function and its derivatives. Formulas for integrating derivatives can be obtained from (35) by differentiating with respect to parameters.

3.3.3

The other issue is that at $x_i = 1$ the integrand is an elliptic function of the corresponding y_i, and so the left hand side of (35) gives infinity times zero. This can be circumvented, for example, by replacing each factor of x_i in the argument of each theta function an independent variable and specializing them all back to x_i only after integration. By l'Hôpital's rule, this will produce an additional differentiation any time we expand around $x_i = 1$ for some i.

3.3.4

In the end, we will get some rather complicated polynomial in theta functions and their derivatives evaluated at ± 1 divided by a power of $\vartheta(-1)$. This means that we will get a combination of Eisenstein series arising from

$$\ln \frac{z}{\vartheta(e^z)} = 2 \sum_{k \geq 1} \frac{z^{2k}}{(2k)!} E_{2k}(q), \tag{36}$$

and

$$\ln \frac{\vartheta(-e^z)}{\vartheta(-1)} = 2 \sum_{k \geq 1} \frac{z^{2k}}{(2k)!} [E_{2k}(q) - 2^{2k} E_{2k}(q^2)], \tag{37}$$

together with the product

$$\vartheta(-1) = 2i \left(\prod_n \frac{1+q^n}{1-q^n} \right)^2 = \frac{\eta(q^2)^2}{\eta(q)^4}. \tag{38}$$

Note that (38) has weight -1.

3.3.5

Without knowing the precise form of the answer, one can still make some qualitative observations about it.

Suppose we are interested in the coefficient of $z_1^{k_1} \cdots z_n^{k_n}$ in the expansion of

$$F(e^{z_1}, \dots, e^{z_r}, -e^{z_{r+1}}, \dots, -e^{z_n})$$

in powers of z_i. We claim that the weight of this coefficient is at most $\sum k_i + r$. Indeed, we from (36) and (37) we have

$$\operatorname{wt} \left(x \frac{d}{dx} \right)^k \vartheta(x) \Big|_{x=\pm 1} = k - 1.$$

This gives the following count for the weight:

$$n - n + \sum k_i + r,$$

where the first n is added because of the prefactor in (34), the second n is subtracted due to integration in y_i (which, by Lemma 4 changes the balance of θ-factors by 1), $\sum k_i$ is the number of times we need to differentiate the integrand, and, finally, r additional differentiations are needed for reasons explained in Section 3.3.3.

3.3.6

We further claim that (34) is, in fact, a polynomial in the coefficients of (36), (37), and

$$\frac{1}{\vartheta(-1)^2} = -\frac{1}{4} \frac{\eta(q)^8}{\eta(q^2)^4} = 2E_2(q) - 12E_2(q^2) + 16E_2(q^4). \tag{39}$$

First, observe only even powers of (38) appear in the answer. This is because the formula (34) has a balance of minus signs in the arguments of theta functions in the numerator and denominator. Every time we specialize y_i to one of the poles in (35), the balance of minus signs changes by an even number.

Inverse powers of (39) cannot appear in the answer because they grow exponentially as $q \to 1$ and there are no other exponentially large terms to cancel this growth out. The averages (10) may grow only polynomially as $q \to 1$ because of the bound (30).

3.3.7

Recall from [16] that a *quasimodular form* for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is, by definition, the holomorphic part of an almost holomorphic modular form for Γ. A function of $|q| < 1$ is called almost holomorphic if it is a polynomial in $(\ln |q|)^{-1}$ with coefficients in holomorphic functions of q. Quasimodular forms for Γ form a graded algebra denoted by $\mathcal{QM}(\Gamma)$. By a theorem of Kaneko and Zagier [13],

$$\mathcal{QM}(\Gamma) = \mathbb{Q}[E_2] \otimes \mathcal{M}(\Gamma).$$

In particular,

$$E_2(q),\, E_2(q^2),\, E_2(q^4) \in \mathcal{QM}(\Gamma_0(4)) \tag{40}$$

where

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c \equiv 0 \bmod 4 \right\} \subset SL_2(\mathbb{Z}).$$

Hence all averages (10) lie in $\mathcal{QM}(\Gamma_0(4))$.

In fact, the series (40) generate the subalgebra $\mathcal{QM}_{2*}(\Gamma_0(4))$ of even weight quasimodular forms. This is because $\mathcal{M}_{2*}(\Gamma_0(4))$ is freely generated by two generators of weight two, for example, by $E_2^{\text{odd}}(q)$ and $E_2^{\text{odd}}(q^2)$, where

$$E_2^{\text{odd}}(q) = E_2(q) - 2E_2(q^2) = \frac{1}{24} + \sum_{n=1}^{\infty} \left(\sum_{d|n, d \text{ odd}} d \right) q^n.$$

3.3.8

Note that because $\mathsf{w}(\lambda) = 0$ for any partition λ of odd size, the series (10) is in fact a series in q^2. It follows that it is quasimodular with respect to a bigger group, namely

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \Gamma_0(2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \supset \Gamma_0(4).$$

In other words, (10) is, in fact, obtained by substituting $q \mapsto q^2$ into an element of $\mathcal{QM}(\Gamma_0(2))$. We have

$$\mathcal{M}(\Gamma_0(2)) = \mathbb{Q}[E_2^{\text{odd}}(q), E_4(q^2)]$$

and hence

$$\mathcal{QM}(\Gamma_0(2)) = \mathbb{Q}[E_2(q), E_2(q^2), E_4(q^2)].$$

This concludes the proof of Theorem 3.

Appendix A: Examples

In this appendix, we list some simple examples of the quasimodular forms $Z'(\mu, \nu; q)$ appearing in Theorem 1 and polynomials $\mathbf{g}_\nu$ from Theorem 2.

A.1 Quasimodular forms $Z'(\mu, \nu; q)$

$$Z'((1,1),(2)) = 20E_2(q^4)^2 - 20E_2(q^4)E_2(q^2) + 4E_2(q^2)^2 - \frac{5}{3}E_4(q^4).$$

$$\begin{aligned}Z'((3,1),(3)) = &-\frac{2112}{5}E_2(q^4)^3 + \frac{3888}{5}E_2(q^4)^2E_2(q^2)\\ &-\frac{2304}{5}E_2(q^4)E_2(q^2)^2 + \frac{384}{5}E_2(q^2)^3\\ &+48E_4(q^4)E_2(q^4) - 36E_4(q^4)E_2(q^2).\end{aligned}$$

$$\begin{aligned}Z'((3,3),(2)) = &\frac{1056}{5}E_2(q^4)^3 - \frac{1044}{5}E_2(q^4)^2E_2(q^2)\\ &+\frac{252}{5}E_2(q^4)E_2(q^2)^2 - \frac{12}{5}E_2(q^2)^3 - 24E_4(q^4)E_2(q^4)\\ &+3E_4(q^4)E_2(q^2) + \frac{15}{2}E_2(q^4)^2 - \frac{15}{2}E_2(q^4)E_2(q^2)\\ &+\frac{3}{2}E_2(q^2)^2 - \frac{5}{8}E_4(q^4).\end{aligned}$$

$$\begin{aligned}Z'((5,1),(2)) = &\frac{3520}{3}E_2(q^4)^3 - 1160E_2(q^4)^2E_2(q^2) + 280E_2(q^4)E_2(q^2)^2\\ &-\frac{40}{3}E_2(q^2)^3 - \frac{400}{3}E_4(q^4)E_2(q^4) + \frac{50}{3}E_4(q^4)E_2(q^2)\\ &+\frac{125}{3}E_2(q^4)^2 - \frac{125}{3}E_2(q^4)E_2(q^2) + \frac{25}{3}E_2(q^2)^2\\ &-\frac{125}{36}E_4(q^4).\end{aligned}$$

$$Z'((1,1,1,1),\emptyset) = \frac{1}{4}E_2(q^4) + \frac{1}{96}.$$

$$\begin{aligned}Z'((3,3,1,1),\emptyset) = &\frac{9}{256} - 12E_2(q^4)^2 + \frac{27}{2}E_2(q^4)E_2(q^2) - \frac{9}{4}E_2(q^2)^2\\ &+\frac{5}{4}E_4(q^4) + \frac{9}{16}E_2(q^4) + \frac{3}{8}E_2(q^2).\end{aligned}$$

$$\begin{aligned}Z'((5,1,1,1),\emptyset) = &\frac{125}{1152} - 10E_2(q^4)^2 + 15E_2(q^4)E_2(q^2) - \frac{5}{2}E_2(q^2)^2\\ &+\frac{55}{24}E_2(q^4) + \frac{5}{12}E_2(q^2).\end{aligned}$$

$$\begin{aligned}Z'((3,3,3,3),\emptyset) = &-\frac{24}{5}E_2(q^4)^3 - \frac{84}{5}E_2(q^4)^2E_2(q^2) + \frac{423}{20}E_2(q^4)E_2(q^2)^2\\ &-\frac{39}{10}E_2(q^2)^3 + E_4(q^4)E_2(q^4) + \frac{7}{4}E_4(q^4)E_2(q^2)\end{aligned}$$

$$- \frac{33}{4} E_2(q^4)^2 + \frac{141}{16} E_2(q^4) E_2(q^2) - \frac{21}{32} E_2(q^2)^2 + \frac{25}{32} E_4(q^4) + \frac{27}{256} E_2(q^4) + \frac{9}{32} E_2(q^2) + \frac{27}{2048}.$$

$$\begin{aligned} Z'((5,3,3,1),\emptyset) = {} & 132 E_2(q^4)^3 - 708 E_2(q^4)^2 E_2(q^2) + 639 E_2(q^4) E_2(q^2)^2 \\ & - 114 E_2(q^2)^3 - 15 E_4(q^4) E_2(q^4) + 55 E_4(q^4) E_2(q^2) \\ & - 310 E_2(q^4)^2 + \frac{1365}{4} E_2(q^4) E_2(q^2) - \frac{285}{8} E_2(q^2)^2 \\ & + \frac{175}{6} E_4(q^4) + \frac{615}{64} E_2(q^4) + \frac{85}{8} E_2(q^2) + \frac{375}{512}. \end{aligned}$$

A.2 Polynomials $\mathbf{g}_\nu$

$$\mathbf{g}_{1,1} = \frac{1}{2}\bar{\mathbf{p}}_1.$$

$$\mathbf{g}_{3,1} = \frac{1}{6}\bar{\mathbf{p}}_1^2 + \frac{1}{6}\bar{\mathbf{p}}_2 - \frac{1}{2}\mathbf{p}_1.$$

$$\mathbf{g}_{3,3} = -\frac{1}{54}\bar{\mathbf{p}}_1^3 + \frac{1}{18}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 + \frac{1}{54}\bar{\mathbf{p}}_3 - \frac{1}{4}\mathbf{p}_2 + \frac{3}{16}\bar{\mathbf{p}}_1.$$

$$\mathbf{g}_{5,1} = \frac{1}{30}\bar{\mathbf{p}}_1^3 + \frac{1}{10}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 - \frac{1}{2}\bar{\mathbf{p}}_1\mathbf{p}_1 + \frac{1}{15}\bar{\mathbf{p}}_3 - \frac{1}{2}\mathbf{p}_2 + \frac{25}{24}\bar{\mathbf{p}}_1.$$

$$\begin{aligned} \mathbf{g}_{5,3} = {} & -\frac{1}{360}\bar{\mathbf{p}}_1^4 - \frac{1}{60}\bar{\mathbf{p}}_1^2\bar{\mathbf{p}}_2 - \frac{1}{12}\bar{\mathbf{p}}_1^2\mathbf{p}_1 + \frac{2}{45}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1 + \frac{25}{36}\bar{\mathbf{p}}_1^2 + \frac{1}{40}\bar{\mathbf{p}}_2^2 \\ & - \frac{1}{12}\bar{\mathbf{p}}_2\mathbf{p}_1 + \frac{5}{8}\mathbf{p}_1{}^2 + \frac{1}{60}\bar{\mathbf{p}}_4 - \frac{1}{2}\mathbf{p}_3 + \frac{25}{36}\bar{\mathbf{p}}_2 - \frac{25}{12}\mathbf{p}_1. \end{aligned}$$

$$\mathbf{g}_{1,1,1,1} = -\frac{1}{24}\bar{\mathbf{p}}_1^2 + \frac{1}{12}\bar{\mathbf{p}}_2 + \frac{1}{96}.$$

$$\mathbf{g}_{3,1,1,1} = \frac{1}{108}\bar{\mathbf{p}}_1^3 - \frac{1}{36}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 - \frac{1}{4}\bar{\mathbf{p}}_1\mathbf{p}_1 + \frac{2}{27}\bar{\mathbf{p}}_3 + \frac{3}{8}\bar{\mathbf{p}}_1.$$

$$\begin{aligned} \mathbf{g}_{3,3,1,1} = {} & \frac{1}{216}\bar{\mathbf{p}}_1^4 - \frac{1}{12}\bar{\mathbf{p}}_1^2\mathbf{p}_1 + \frac{1}{108}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1 - \frac{1}{8}\mathbf{p}_2\bar{\mathbf{p}}_1 + \frac{9}{32}\bar{\mathbf{p}}_1^2 - \frac{1}{72}\bar{\mathbf{p}}_2^2 \\ & - \frac{1}{12}\bar{\mathbf{p}}_2\mathbf{p}_1 + \frac{1}{8}\mathbf{p}_1{}^2 + \frac{1}{36}\bar{\mathbf{p}}_4 + \frac{9}{16}\bar{\mathbf{p}}_2 - \frac{3}{4}\mathbf{p}_1 + \frac{9}{256}. \end{aligned}$$

$$\begin{aligned} \mathbf{g}_{3,3,3,1} = {} & \frac{1}{4860}\bar{\mathbf{p}}_1^5 + \frac{1}{486}\bar{\mathbf{p}}_1^3\bar{\mathbf{p}}_2 + \frac{1}{108}\bar{\mathbf{p}}_1^3\mathbf{p}_1 - \frac{5}{972}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1^2 - \frac{1}{24}\mathbf{p}_2\bar{\mathbf{p}}_1^2 - \frac{1}{96}\bar{\mathbf{p}}_1^3 \\ & + \frac{1}{324}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2^2 - \frac{1}{36}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2\mathbf{p}_1 + \frac{1}{162}\bar{\mathbf{p}}_4\bar{\mathbf{p}}_1 - \frac{5}{972}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_2 - \frac{1}{108}\bar{\mathbf{p}}_3\mathbf{p}_1 - \frac{1}{24}\mathbf{p}_2\bar{\mathbf{p}}_2 \\ & + \frac{1}{8}\mathbf{p}_2\mathbf{p}_1 + \frac{31}{96}\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 - \frac{19}{32}\bar{\mathbf{p}}_1\mathbf{p}_1 + \frac{1}{4}\bar{\mathbf{p}}_3 - \mathbf{p}_2 + \frac{2}{405}\bar{\mathbf{p}}_5 + \frac{153}{128}\bar{\mathbf{p}}_1. \end{aligned}$$

$$\begin{aligned} \mathbf{g}_{3,3,3,3} = {} & \frac{1}{29160}\bar{\mathbf{p}}_1^6 - \frac{1}{2916}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1^3 + \frac{1}{216}\mathbf{p}_2\bar{\mathbf{p}}_1^3 - \frac{1}{432}\bar{\mathbf{p}}_1^4 + \frac{1}{1944}\bar{\mathbf{p}}_1^2\bar{\mathbf{p}}_2^2 - \frac{1}{972}\bar{\mathbf{p}}_4\bar{\mathbf{p}}_1^2 \\ & + \frac{1}{972}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 - \frac{1}{72}\mathbf{p}_2\bar{\mathbf{p}}_1\bar{\mathbf{p}}_2 - \frac{7}{288}\bar{\mathbf{p}}_1^2\bar{\mathbf{p}}_2 - \frac{1}{12}\bar{\mathbf{p}}_1^2\mathbf{p}_1 - \frac{1}{2916}\bar{\mathbf{p}}_2^3 \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{1944}\bar{\mathbf{p}}_3^2-\frac{1}{216}\bar{\mathbf{p}}_3\mathbf{p}_2+\frac{59}{864}\bar{\mathbf{p}}_3\bar{\mathbf{p}}_1+\frac{1}{32}\mathbf{p}_2{}^2-\frac{3}{64}\mathbf{p}_2\bar{\mathbf{p}}_1+\frac{1}{1215}\bar{\mathbf{p}}_5\bar{\mathbf{p}}_1\\
&+\frac{231}{512}\bar{\mathbf{p}}_1^2+\frac{1}{32}\bar{\mathbf{p}}_2^2-\frac{1}{12}\bar{\mathbf{p}}_2\mathbf{p}_1+\frac{3}{8}\mathbf{p}_1{}^2+\frac{5}{144}\bar{\mathbf{p}}_4-\frac{5}{12}\mathbf{p}_3+\frac{1}{2916}\bar{\mathbf{p}}_6\\
&+\frac{129}{256}\bar{\mathbf{p}}_2-\frac{9}{8}\mathbf{p}_1+\frac{27}{2048}.
\end{aligned}
$$

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