7

Optimization of Signal-to-Noise Ratio in Linear Systems

Useful, deterministic signals passing through various transmission devices often acquire extraneous random components due to, say, thermal noise in conducting materials, radio clutter or aurora borealis magnetic field fluctuations in the atmosphere, or deliberate jamming in warfare. If there exists some prior information about the nature of the original useful signal and the contaminating random noise it is possible to devise algorithms to improve the relative power of the useful compenent of the signal or, in other words, to increase the *signal-to-noise ratio* of the signal, by passing it through a filter designed for the purpose. In this short chapter, we give a few examples of such designs just to show how the previously introduced techniques of analysis of random signals can be applied in this context.

7.1 Fixed filter structure, known input signal

The general problem of optimization (maximization) of the signal-to-noise ratio in a linear system schematically pictured here,

$$x(t) + N(t) \longrightarrow \boxed{\text{h(t)}} \longrightarrow y(t) + M(t),$$

can be formulated as follows: Consider a linear filter (system) characterized by its impulse response function $h(t)$ with the input signal $X(t)$ of the form

$$X(t) = x(t) + N(t), \tag{7.1.1}$$

where $x(t)$ is a deterministic "useful" signal and $N(t)$ is a random stationary "noise" signal with zero mean and autocorrelation function $\gamma_N(t)$. Given the linearity of the system, the output signal $Y(t)$ is of the form

$$Y(t) = y(t) + M(t), \tag{7.1.2}$$

where the deterministic "useful" output component is

$$y(t) = \int_{-\infty}^{\infty} x(s)h(t-s)ds, \tag{7.1.3}$$

and the "noise" output is a stationary zero-mean signal with the autocorrelation function

$$\gamma_M(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_N(\tau - s + u)h(s)h(u)dsdu.$$

The task is as follows: Given the shape of the input signal, design the structure of the filter which would maximize the signal-to-noise power ratio on the output. More precisely, we need to find an impulse response function $h(t)$ such that, for a given detection time t, the signal-to-noise ratio

$$\frac{S}{\mathcal{N}} = \frac{\mathrm{PW}_y(t)}{\mathbf{E}(\mathrm{PW}_M)} \tag{7.1.4}$$

is maximized over all possible impulse response functions; in brief, we want to find $h(t)$ for which

$$\frac{S}{\mathcal{N}} = \max .$$

Here, $\mathrm{PW}_y(t) = y^2(t)$ is the instantaneous power of the output signal, and $\mathbf{E}(\mathrm{PW}_M) = \gamma_M(0) = \sigma_M^2$ is the mean power of the output noise. Hence the optimization problem is to find $h(t)$, and also the detection time t_0, such that

$$\frac{S}{\mathcal{N}} = \frac{y^2(t_0)}{\gamma_M(0)} = \frac{y^2(t_0)}{\sigma_M^2} = \max . \tag{7.1.5}$$

In the present section we will take a look at a relatively simple situation when the general structure of the filter is essentially fixed and only certain parameters, including the detection time t_0, need to be optimized.

To show the essence of our approach, we will just consider the RC filter with the impulse response function

$$h(t) = be^{-bt} \cdot u(t), \tag{7.1.6}$$

with a single parameter $b = \frac{1}{RC}$ to be determined in addition to the optimal detection time t_0.

Suppose that the "useful" input signal we are trying to detect on the output is a rectangular impulse

$$x(t) = \begin{cases} A & \text{for } 0 \le t \le T; \\ 0 & \text{elsewhere} \end{cases} \tag{7.1.7}$$

and that the input noise is a white noise with the autocovariance $\gamma_N(t) = N_0\delta(t)$.

The deterministic "useful" output signal is

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(s)h(-(s-t))ds \\
&= \begin{cases} \int_0^t Abe^{-b(t-s)}ds & \text{for } 0 < t < T; \\ \int_0^T Abe^{-b(t-s)}ds & \text{for } t \geq T, \end{cases} \\
&= \begin{cases} A(1-e^{-bt}) & \text{for } 0 < t \leq T; \\ A(1-e^{-bT})e^{-b(t-T)} & \text{for } t \geq T. \end{cases}
\end{aligned}
\tag{7.1.8}
$$

It is pictured in Figure 7.1.1.

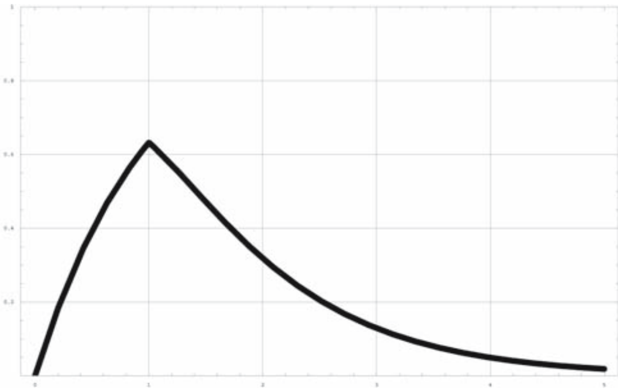

Fig. 7.1.1. Response $y(t)$ (7.1.8) of the RC filter (7.1.6) to the rectangular input signal $x(t)$ (7.1.7). The parameter values are $T = 1$, $A = 1$, and $b = \frac{1}{RC} = 1$. The maximum is clearly attained for $t_0 = T$.

Clearly, the maximum of the output signal is attained at $t_0 = T$. On the other hand, as calculated in Chapter 6, the autocorrelation function of the output noise is

$$\gamma_M(\tau) = N_0\frac{b}{2}e^{-b\tau},$$

so that, at the already optimized detection time $t_0 = T$,

$$\frac{S}{\mathcal{N}} = \frac{y^2(T)}{\gamma_M(0)} = \frac{A^2[1-e^{-bT}]^2}{\frac{bN_0}{2}}.$$

To simplify our calculations we will substitute $z = bT$. Now our final task is to find the maximum of the function

$$\frac{S}{\mathcal{N}}(z) = \frac{2A^2T}{N_0} \cdot \frac{(1-e^{-z})^2}{z} \tag{7.1.9}$$

of one variable z. Function $\frac{S}{\mathcal{N}}(z)$, although simple-looking, is a little tricky and we will start the exploration of its maximum by graphing it; see Figure 7.1.2.

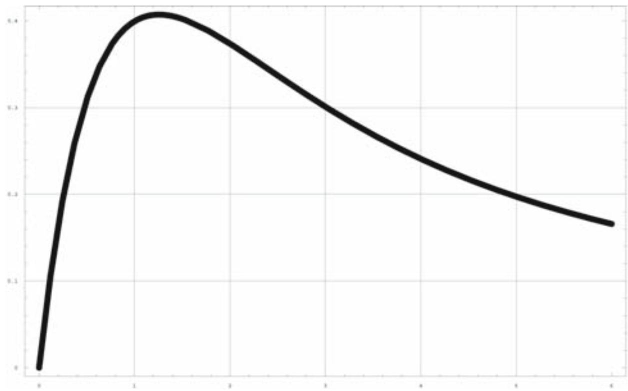

Fig. 7.1.2. Graph of the factor $\frac{(1-e^{-z})^2}{z}$ in formula (7.1.9) for the signal-to-noise ratio $\frac{S}{\mathcal{N}}(z)$.

To find the location of the maximum we calculate the derivative and try to solve the equation

$$\frac{d}{dz}\frac{(1-e^{-z})^2}{z} = \frac{2(1-e^{-z})e^{-z}z-(1-e^{-z})^2}{z^2} = 0.$$

Although the above equation can be easily simplified to the equation

$$e^z - 1 - 2z = 0,$$

the latter cannot be solved explicitly. So, as usual, as the first step we explore the solution graphically; see Figure 7.1.3.

The nontrivial zero is approximately at $z_{\max} = 1.25$, which gives $b_{\max} = \frac{1.25}{T}$ so that the optimal RC constant is

$$RC_{\max} \approx \frac{1}{b_{\max}} = \frac{T}{1.25} = 0.8T. \tag{7.1.10}$$

Evaluated at the optimal values of parameters t_0 and b, the maximum available signal-to-noise ratio is

$$\frac{S}{\mathcal{N}}_{\max} \approx \frac{y^2(T)}{\frac{b_{\max}N_0}{2}} = \frac{2A^2[1-e^{-b_{\max}T}]^2}{b_{\max}N_0} = 0.81\cdot\frac{A^2T}{N_0}. \tag{7.1.11}$$

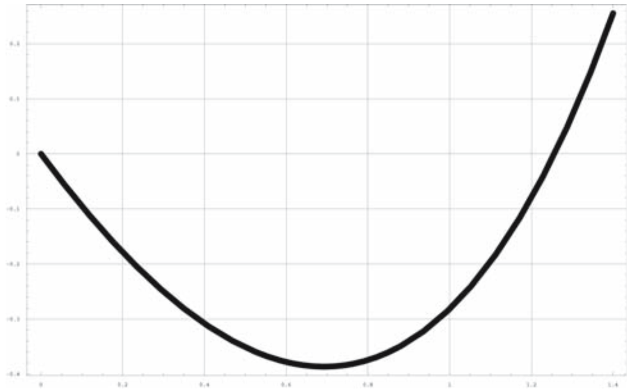

Fig. 7.1.3. A plot of function $e^z - 1 - 2z = 0$. The nontrivial zero is approximately at $z_{\max} = 1.25$.

7.2 Filter structure matched to signal

In this section we will solve a more ambitious problem of designing the structure of the filter to maximize the signal-to-noise ratio on the output rather than just optimizing filter parameters. To be more precise, the task at hand is to find an impulse response function $h(t)$ and the detection time t_0 such that

$$\frac{S}{\mathcal{N}} = \frac{y^2(t_0)}{\sigma_M^2} = \max \tag{7.2.1}$$

for a given deterministic (nonrandom) input signal $x(t)$ transmitted in the presence of the white noise input $N(t)$ with autocorrelation function $\gamma_N(t) = N_0\delta(t)$, where, as before, $x(t) = 0$ for $t \le 0$ and

$$y(t) = \int_0^\infty x(t-s)h(s)ds. \tag{7.2.2}$$

For the output noise,

$$\sigma_M^2 = \gamma_M(0) = \int_0^\infty \Big(\int_0^\infty \delta(u-s)h(u)du\Big)h(s)ds = N_0\int_0^\infty h^2(s)ds. \tag{7.2.3}$$

In this situation,

$$\frac{S}{\mathcal{N}} = \frac{y^2(t_0)}{\sigma_M^2} = \frac{(\int_0^\infty x(t_0-s)h(s)ds)^2}{N_0\int_0^\infty h^2(s)ds}. \tag{7.2.4}$$

In view of the Cauchy–Schwartz inequality,

$$\frac{S}{\mathcal{N}} \le \frac{\int_0^\infty x^2(t_0-s)ds \cdot \int_0^\infty h^2(s)ds}{N_0\int_0^\infty h^2(s)ds} = \frac{1}{N_0}\int_0^\infty x^2(t_0-s)ds, \tag{7.2.5}$$

with the equality, that is, the maximum for $\frac{S}{\mathcal{N}}$, achieved when the two factors, $h(s)$ and $x(t_0 - s)$, in the scalar product in the numerator of (7.2.4) are linearly dependent. In other words, for any constant c, the impulse response function

$$h(s) = cx(t_0 - s)u(s) = cx(-(s - t_0))u(s) \tag{7.2.6}$$

gives the optimal structure of the filter and maximizes the $\frac{S}{\mathcal{N}}$ ratio. This so-called *matching filter* has the impulse response function equal to the input signal $x(t)$ run backwards in time, then shifted to the right by t_0, and, finally, cut off at 0.

With the selection of the matching filter, in view of (7.2.4), the maximal value of the $\frac{S}{\mathcal{N}}$ ratio is

$$\frac{S}{\mathcal{N}}_{\max} = \frac{(\int_0^\infty x(t_0 - s)cx(t_0 - s)u(s)ds)^2}{N_0 \int_0^\infty (cx(t_0 - s)u(s))^2 ds} = \frac{\int_0^\infty x^2(t_0 - s)ds}{N_0}. \tag{7.2.7}$$

Example 7.2.1 (matching filter for a rectangular input signal). Consider a rectangular input signal of the form

$$x(t) = \begin{cases} A & \text{for } 0 < t < T; \\ 0 & \text{elsewhere,} \end{cases}$$

transmitted in the presence of an additive white noise with autocorrelation function $\gamma_N(t) = N_0\delta(t)$. According to formula (7.2.6), its matching filter at detection time t_0 is

$$h(t) = \begin{cases} A & \text{for } 0 < t < t_0; \\ 0 & \text{elsewhere} \end{cases}$$

if $0 \le t_0 \le T$ and

$$h(t) = \begin{cases} A & \text{for } t_0 - T < t < t_0; \\ 0 & \text{elsewhere} \end{cases}$$

if $t_0 > T$. So the $\frac{S}{\mathcal{N}}_{\max}$, as a function of the detection time t_0, is

$$\frac{S}{\mathcal{N}}_{\max}(t_0) = \begin{cases} \frac{A^2 t_0}{N_0} & \text{for } 0 < t_0 < T; \\ \frac{A^2 T}{N_0} & \text{for } t_0 > T. \end{cases}$$

Clearly, the earliest detection time t_0 to maximize $\frac{S}{\mathcal{N}}_{\max}(t_0)$ is $t_0 = T$ (see Figure 7.2.1).

At the optimal detection time $t_0 = T$, or any later detection time,

$$\frac{S}{\mathcal{N}}_{\max} = \frac{A^2 T}{N_0}. \tag{7.2.8}$$

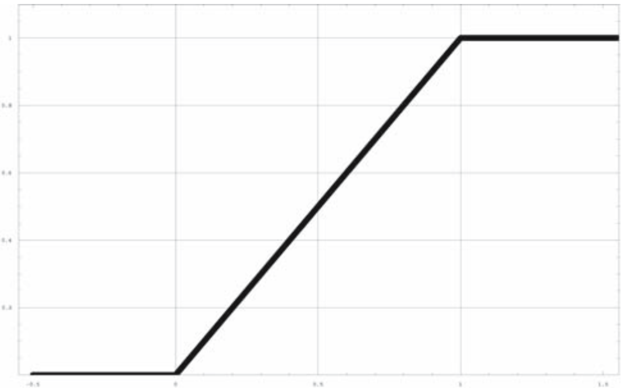

Fig. 7.2.1. The dependence of the optimal signal-to-noise ratio on the detection time t_0 for the matching filter from Example 7.2.1. The input signal is the sum of a rectangular signal of amplitude $A = 1$, duration $T = 1$, and the white noise with autocorrelation function $\gamma_N(t) = \delta(t)$.

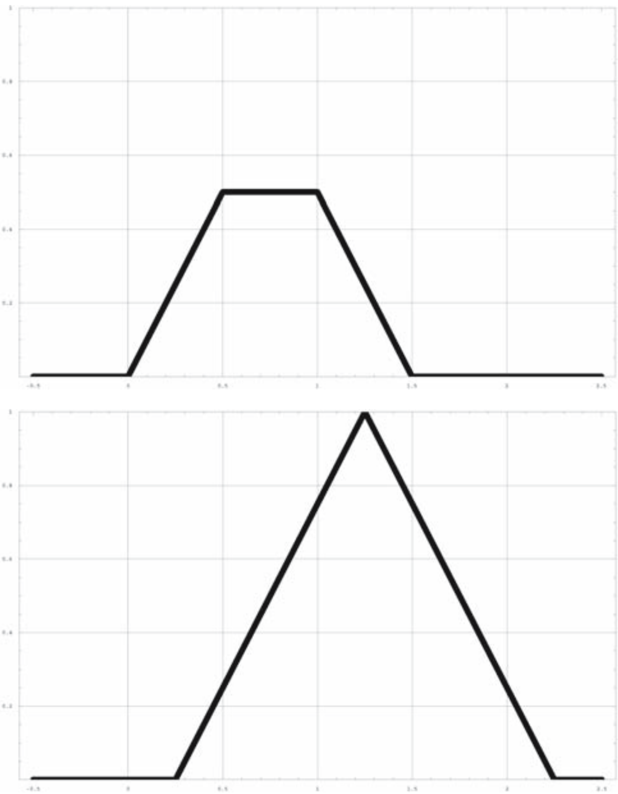

Fig. 7.2.2. The response $y(t)$ of the matching filter for the rectangular input signal with amplitude $A = 1$ and duration $T = 1$ (see Example 7.2.1). *Top*: For detection time $t_0 = 0.25 < T = 1$. *Bottom*: For detection time $t_0 = 1.25 > T = 1$.

This result should be compared with the maximum signal-to-noise ratio $0.81\frac{A^2T}{N_0}$ (see (7.1.11)) obtained in Section 7.1 by optimally tuning the RC filter: the best-matching filter gives about a 25% gain in the signal-to-noise ratio over the best RC filter.

It is also instructive to trace the behavior of the deterministic part $y(t)$ of the output signal for the matching filter as a function of detection time t_0. The formula (7.2.2) applied to the matching filter immediately gives that, for $0 < t_0 < T$,

$$y(t) = \begin{cases} A^2 t & \text{for } 0 < t < t_0; \\ A^2 t_0 & \text{for } t_0 < t < T; \\ -A^2(t-(t_0+T)) & \text{for } T < t < t_0+T; \\ 0 & \text{elsewhere} \end{cases} \tag{7.2.9}$$

and, for $t_0 \geq T$,

$$y(t) = \begin{cases} A^2(t-(t_0-T)) & \text{for } t_0 - T < t < t_0; \\ -A^2(t-(t_0+T)) & \text{for } t_0 < t < t_0+T; \\ 0 & \text{elsewhere.} \end{cases} \tag{7.2.10}$$

These two output signals are pictured in Figure 7.2.2.

7.3 The Wiener filter

Acausal filter. Given stationary random signals $X(t)$ and $Y(t)$, the problem is to find a (not necessarily causal) impulse response function $h(t)$ such that the mean-square distance between $Y(t)$ and the output signal,

$$Y_h(t) = \int_{-\infty}^{\infty} X(t-s)h(s)ds,$$

is the smallest possible. In other words, we need $h(t)$ to minimize the error quantity

$$\mathbf{E}(Y(t) - Y_h(t))^2.$$

In the space of all finite variance (always zero-mean) random quantities equipped with the covariance as the scalar product, the best approximation $Y_h(t)$ of a random quantity $Y(t)$ by elements of the linear subspace $\mathcal{X}$ spanned by linear combinations of values of $X(t-s)$, $-\infty < s < \infty$, is given by the orthogonal projection of $X(t)$ on $\mathcal{X}$.[31] That means that the difference $Y(t) - Y_h(t)$ must be orthogonal to all $X(t-s)$, $-\infty < s < \infty$, or, more formally,

[31] This argument is analagous to the one encountered in Chapter 2, when we discussed the best approximation in power (for a definition, see Section 2.2) of deterministic periodic signals by their Fourier series.

$$\begin{aligned}
&\mathbf{E}((Y(t) - Y_h(t)) \cdot X(t-s)) \\
&\quad = \mathbf{E}(Y(t) \cdot X(t-s)) - \mathbf{E}\left(\int_{-\infty}^{\infty} X(t-u)h(u)du \cdot X(t-s)\right) \\
&\quad = \gamma_{YX}(s) - \int_{-\infty}^{\infty} \gamma_X(s-u)h(u)du = 0,
\end{aligned}$$

for all s, $-\infty < s < \infty$. Hence the optimal $h(t)$ can be found by solving, for each s, the integral equation

$$\gamma_{YX}(s) = \int_{-\infty}^{\infty} \gamma_X(s-u)h(u)du, \tag{7.3.1}$$

which involves only the autocorrelation function $\gamma_X(s)$ and the cross-correlation function $\gamma_{YX}(s)$. The solution is found readily in the frequency domain. Remembering that the Fourier transform of a convolution is the product of Fourier transforms, and denoting by $H(f)$ the transfer function (the Fourier transform of the impulse response function) of the optimal $h(t)$, (7.3.1) can be rewritten in the form

$$S_{YX}(f) = S_X(f) \cdot H(f),$$

which immediately gives the explicit formula for the transfer function of the optimal filter:

$$H(f) = \frac{S_{YX}(f)}{S_X(f)}. \tag{7.3.2}$$

The minimal error can then also be calculated explicitly:

$$\mathbf{E}(Y(t) - Y_h(t))^2 = \gamma_Y(0) - \int_{-\infty}^{\infty} \gamma_{YX}(s)h(s)ds, \tag{7.3.3}$$

or, in terms of the optimal transfer function, using the Parseval formula for the last integral, we have

$$\mathbf{E}(Y(t) - Y_h(t))^2 = \int_{-\infty}^{\infty} (S_Y(f) - S_{YX}^*(f)H(f))df. \tag{7.3.4}$$

Example 7.3.1. Assume that signal $X(t)$ is the sum of a "useful" signal $Y(t)$ and noise $N(t)$, that is, $X(t) = Y(t) + N(t)$, where $Y(t)$ has the power spectrum

$$S_Y(f) = \frac{1}{1+f^2},$$

and is uncorrelated with the white noise $N(t)$, which is assumed to have the power spectrum $S_N(f) \equiv 1$. Then

$$S_{YX}(f) = S_Y(f) = \frac{1}{1+f^2} \quad \text{and} \quad S_X(f) = S_Y(f) + S_N(f) = \frac{2+f^2}{1+f^2}.$$

The transfer function of the optimal filter is then

$$H(f) = \frac{S_{YX}(f)}{S_X(f)} = \frac{1}{2+f^2},$$

with the corresponding impulse response function

$$h(t) = \frac{1}{2\sqrt{2}} e^{-\sqrt{2}|t|},$$

and the error

$$\begin{aligned}\mathbf{E}(Y(t) - Y_h(t))^2 &= \int_{-\infty}^{\infty} \left(\frac{1}{1+f^2} - \frac{1}{1+f^2} \cdot \frac{1}{2+f^2} \right) df \\ &= \int_{-\infty}^{\infty} \frac{1}{2+f^2} df = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Causal filter. For given stationary random signals $X(t)$ and $Y(t)$, the construction of the optimal *causal filter* requires finding a causal impulse response function $h(t) = 0$, for $t \le 0$, such that the error

$$\mathbf{E}\left(Y(t) - \int_0^{\infty} X(t-s) h(s) ds \right)^2$$

is minimal. In other words, we are trying to find the best mean-square approximation to $Y(t)$ by (continuous) linear combinations of the *past* values of $X(t)$. Using the same orthogonality argument we applied for the acausal optimal filter, we obtain another integral equation for the optimal $h(t)$:

$$\gamma_{YX}(s) = \int_0^{\infty} \gamma_X(s-u) h(u) du,$$

this time valid only for all $s > 0$. This equation is traditionally called the *Wiener–Hopf equation.* It is clear that to solve the above equation via an integral transform method we have to replace the Fourier transform used in the acausal case by the Laplace transform. However, the details here are more involved, and for the solution, we refer the reader to the literature of the subject.[32]

7.4 Problems and exercises

7.4.1. The triangular signal $x(t) = 0.01t$ for $0 < t < 0.01$ and 0 elsewhere is combined with white noise having a flat power spectrum of $2\frac{V^2}{\mathrm{Hz}}$. Find the value of the RC-constant such that the signal-to-noise ratio at the output of the RC filter is maximal at $t = 0.01$ second.

[32] N. Wiener's original *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, MIT Press and Wiley, New York, 1950, is still very readable, but also see Chapter 10 of A. Papoulis, *Signal Analysis*, McGraw–Hill, New York, 1977.

7.4.2. A signal of the form $x(t) = 5e^{-(t+2)}u(t)$ is to be detected in the presence of white noise with a flat power spectrum of $0.25\frac{V^2}{\text{Hz}}$ using a matched filter.

(a) For $t_0 = 2$ find the value of the impulse response of the matched filter at $t = 0, 2, 4$.

(b) Find the maximum output signal-to-noise ratio that can be achieved if $t_0 = \infty$.

(c) Find the detection time t_0 that should be used to achieve an output signal-to-noise ratio that is equal to 95% of the maximum signal-to-noise ratio discovered in part (b).

(d) The signal $x(t) = 5e^{-(t+2)}u(t)$ is combined with white noise having a power spectrum of $2\frac{V^2}{\text{Hz}}$. Find the value of RC such that the signal/noise at the output of the RC filter is maximal at $t = 0.01$ second.

7.4.3. Repeat construction of the optimal filter from Example 7.3.1 in the case when the useful signal $Y(t)$ has a more general power spectrum

$$S_Y(f) = \frac{a}{b^2 + f^2},$$

and the uncorrelated white noise $N(t)$ has arbitrary power spectrum $S_N(f) \equiv \mathcal{N}$. Discuss the properties of this filter when the noise power is much bigger than the power of the useful signal, that is, when $\mathcal{N} \gg S_Y(f)$. Construct the optimal acausal filters for other selected spectra of $Y(t)$ and $N(t)$.