6

Transmission of Stationary Signals through Linear Systems

Signals produced in nature are almost never experienced in their original form. Usually, we have access to them after they pass through various sensing and/or transmission devices such as a voltmeter, for electric signals, the ear, for acoustic signals, the eye, for visual signals, a fiber optic cable, for wide-band Internet signals, etc. All of them impose restrictions on the signal being transmitted by attenuating different frequency components of the signal to a different degree. This process is generally called *filtering* and the devices that change the signal's spectrum are traditionally called *filters*.

A typical example here is the so-called *band-pass filter*, which permits transmission of the components of the signal only in a certain frequency band, attenuating the frequencies in that band in a uniform fashion, but totally "killing" the frequencies outside this band. Figure 6.0.1 shows results of filtering a portion of the EEG signal from Figure 4.1.1 through four band-pass filters with frequency bands (top to bottom) 0.5–3.5 Hz, 4–7.5 Hz, 8–12.5 Hz, and 13–17 Hz. In neurological literature the contents of the EEG signal within these frequency bands are traditionally called delta, theta, alpha, and beta waves, respectively.

In this chapter we study how statistical characteristics of random stationary signals are affected by transmission through linear filters. The linearity assumption means that we suppose that there is a linear relationship between the signals on the input and on the output of the filter. In real life this is not always the case, but the study of nonlinear filters is much more difficult than the linear theory presented below, and beyond the scope of this book.

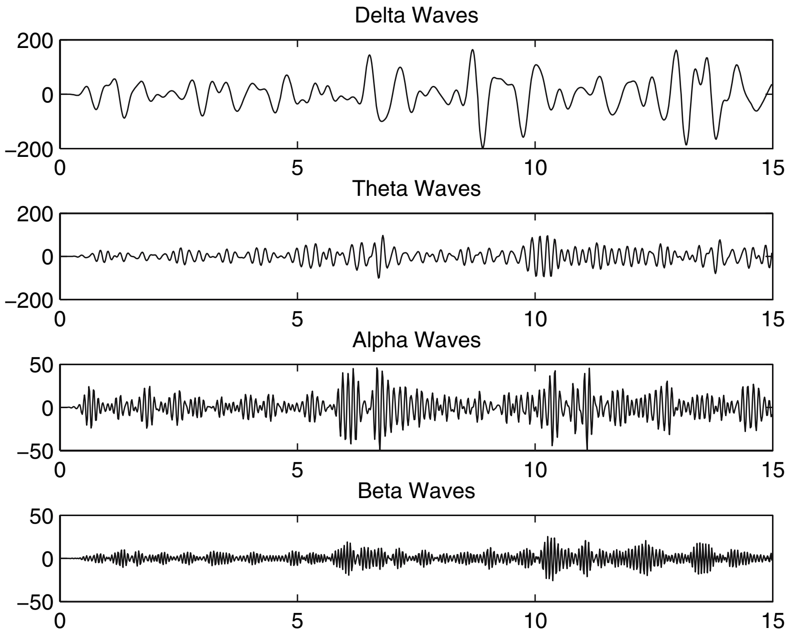

Fig. 6.0.1. A portion of the EEG signal from Figure 4.1.1 filtered through four band-pass filters with frequency bands (top to bottom) 0.5–3.5 Hz, 4–7.5 Hz, 8–12.5 Hz, and 13 – −17 Hz, respectively.

6.1 The time domain analysis

In this section we conduct the time domain analysis of transmission of random signals through a linear system shown schematically below:

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t).$$

The input signal $X(t)$ is assumed to be random and stationary with mean $m_X = \mathbf{E}X(t)$ and autocorrelation function $\gamma_X(\tau) = \mathbf{E}X(t)X(t+\tau)$. The system is identified by a function $h(t)$, and the output signal $Y(t)$ is defined as the continuous-time moving average (convolution):

$$Y(t) = \int_{-\infty}^{\infty} X(s)h(t-s)\,ds = \int_{-\infty}^{\infty} X(t-s)h(s)\,ds. \tag{6.1.1}$$

Note that in the case of a nonrandom Dirac delta impulse input $\delta(t)$ the nonrandom output signal is

$$y(t) = \int_{-\infty}^{\infty} \delta(s)h(t-s)\,ds = h(t-0) = h(t).$$

For this reason the system-identifying time domain function $h(t)$ is usually called the *impulse response function*.

The mean value of the output signal is easily calculated in terms of the input signal and of the impulse response function:

$$\mathbf{E}Y(t) = \int_{-\infty}^{\infty} \mathbf{E}[X(t-s)]h(s)ds = m_X \int_{-\infty}^{\infty} h(s)ds. \tag{6.1.2}$$

The above formula makes sense only if the last integral is well defined. For this reason, we will always assume that the system is *realizable*, that is,

$$\int_{-\infty}^{\infty} |h(s)|ds < \infty. \tag{6.1.3}$$

In view of (6.1.2), for realizable systems, if the input signal has zero mean then the output signal has also zero mean:

$$m_X = 0 \Longrightarrow m_Y = 0.$$

In this situation, henceforth we will restrict our attention only to zero-mean signals.

The calculation of the autocorrelation function of the output signal $Y(t)$ is a little bit more involved. Replacing the product of the integrals by the double integral, we obtain that

$$\begin{aligned}\gamma_Y(\tau) &= \mathbf{E}(Y(t)Y(t+\tau)) \\ &= \mathbf{E}\left[\int_{-\infty}^{\infty} X(t-s)h(s)ds \int_{-\infty}^{\infty} X(t+\tau-u)h(u)du\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}[X(t-s)X(t+\tau-u)]h(s)h(u)dsdu.\end{aligned}$$

Then in view of the stationarity assumption,

$$\mathbf{E}[X(t-s)X(t+\tau-u)] = \mathbf{E}[X(-s)X(\tau-u)] = \gamma_X(\tau-u+s),$$

so that, finally,

$$\gamma_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_X(\tau-u+s)h(s)h(u)dsdu. \tag{6.1.4}$$

A system is said to be *causal* if the current values of the output depend only on the past and present values of the input. This property can be equivalently stated as the requirement that the impulse response function satisfy

$$h(t) = 0 \quad \text{for } t \le 0. \tag{6.1.5}$$

In other words, the moving average is performed only over the past. This condition, in particular, implies that the second output integral in (6.1.1) is restricted to the positive half-line

$$Y(t) = \int_0^\infty X(t-s)h(s)\,ds, \tag{6.1.6}$$

and the autocorrelation function formula (6.1.4) becomes

$$\gamma_Y(\tau) = \int_0^\infty \int_0^\infty \gamma_X(\tau - u + s)h(s)h(u)\,ds\,du. \tag{6.1.7}$$

In what follows, we will just consider causal systems.

Example 6.1.1 (an integrating circuit). A standard integrating circuit with a single capacitor is shown in Figure 6.1.1.

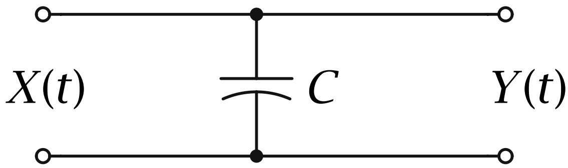

Fig. 6.1.1. A standard integrating circuit. The voltage $Y(t)$ on the output is the integral of the current $X(t)$ on the input.

The impulse response function for this system is the unit step function $u(t)$ multiplied by $\frac{1}{C}$, where the constant C represents the capacitance of the capacitor:

$$h(s) = \frac{1}{C}u(s) = \begin{cases} 0 & \text{for } s < 0; \\ \frac{1}{C} & \text{for } s \geq 0. \end{cases}$$

The output is

$$Y(t) = \frac{1}{C}\int_{-\infty}^\infty X(s)U(t-s)\,ds = \frac{1}{C}\int_{-\infty}^t X(s)\,ds.$$

Obviously, this system, although causal, is not realizable since

$$\int_{-\infty}^\infty |h(t)|\,dt = \int_0^\infty \frac{1}{C}\,dt = \infty.$$

To avoid this difficulty, we need to restrict the integrating circuit to a finite time interval and assume that the adjusted impulse response function is of the form

$$h(s) = \begin{cases} 0 & \text{for } s < 0; \\ \frac{1}{C} & \text{for } 0 \le s \le T; \\ 0 & \text{for } s > T. \end{cases} \tag{6.1.8}$$

In this situation, the system is realizable and the output is

$$Y(t) = \int_{-\infty}^{\infty} X(s)h(t-s)ds = \frac{1}{C}\int_{t-T}^{t} X(s)ds.$$

The autocorrelation is

$$\begin{aligned} \gamma_Y(\tau) &= \int_0^T \int_0^T \gamma_X(\tau - u + s)h(s)h(u)dsdu \\ &= \frac{1}{C^2}\int_0^T \int_0^T \gamma_X(u - (\tau + s))dsdu \end{aligned} \tag{6.1.9}$$

because of the symmetry of the autocorrelation function.

Therefore, if the input signal is the standard white noise $X(t) = W(t)$ with the autocorrelation $\gamma_W(t) = \delta(t)$, then for $\tau \ge 0$, the output autocorrelation function is

$$\gamma_Y(\tau) = \int_0^T \int_0^T \delta(u - (\tau + s))duds = \int_0^T \zeta(s)ds,$$

where

$$\zeta(s) = \begin{cases} 0 & \text{for } \tau + s < 0; \\ \frac{1}{2} & \text{for } \tau + s = 0; \\ 1 & \text{for } 0 < \tau + s < T; \\ \frac{1}{2} & \text{for } \tau + s = T; \\ 0 & \text{for } \tau + s > T. \end{cases}$$

Hence

$$\gamma_Y(s) = \begin{cases} 0 & \text{for } \tau < -T; \\ T - |\tau| & \text{for } -T \le \tau \le T; \\ 0 & \text{for } \tau > T. \end{cases} \tag{6.1.10}$$

If the input signal $X(t)$ is a simple random harmonic oscillation with the autocorrelation function $\gamma_X(\tau) = \cos\tau$, then the output autocorrelation is

$$\gamma_Y(\tau) = \int_0^\infty \int_0^\infty \cos(\tau - u + s)dsdu = -\cos(\tau + T) + 2\cos\tau - \cos(\tau - T). \tag{6.1.11}$$

As simple as the formula (6.1.9) for the output autocorrelation function seems to be, the analytic evaluation of the double convolution may get tedious very quickly. Consider, for example, an input signal $X(t)$ with the autocorrelation function

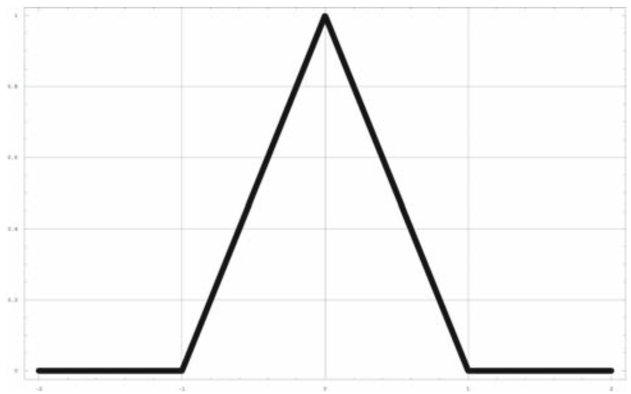

Fig. 6.1.2. The output autocorrelation function $\gamma_Y(\tau)$ (6.1.10) of the integrating system (6.1.8) with $T = 1$, in the case of the standard white noise input $X(t) = W(t)$.

$$\gamma_X(\tau) = \frac{1}{1+\tau^2}, \tag{6.1.12}$$

which corresponds to the exponentially decaying power spectrum (see Section 6.4).

In this case,

$$\begin{aligned}\gamma_Y(\tau) &= \int_0^T \int_0^T \frac{1}{1+(\tau-u+s)^2} ds du \\ &= \frac{1}{2}(2(T-\tau)\arctan(T-\tau) - 2\tau\arctan\tau - \log(1+(T-\tau)^2) \\ &\qquad + \log(1+\tau^2)) \\ &\quad + \frac{1}{2}(-2\tau\arctan(\tau) + 2(\tau+T)\arctan(\tau+T) + \log(1+\tau^2) \\ &\qquad + \log(1+T^2+2T\tau+\tau^2)).\end{aligned} \tag{6.1.13}$$

So even for a relatively simple input autocorrelation function the output autocorrelation is quite complex and unreadable. Yes, you guessed right—we have obtained this formula using *Mathematica*. Figure 6.1.4 traces graphically the dependence of $\gamma_Y(\tau)$ on T.

Example 6.1.2 (an RC filter). A standard RC filter is shown in Figure 6.1.5.

The impulse response function of this circuit is of the form

$$h(t) = \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\cdot u(t), \tag{6.1.14}$$

where $u(t)$ is the unit step function, R is the resistivity, and C is the capacitance. The product RC represent the so-called time constant of the circuit.

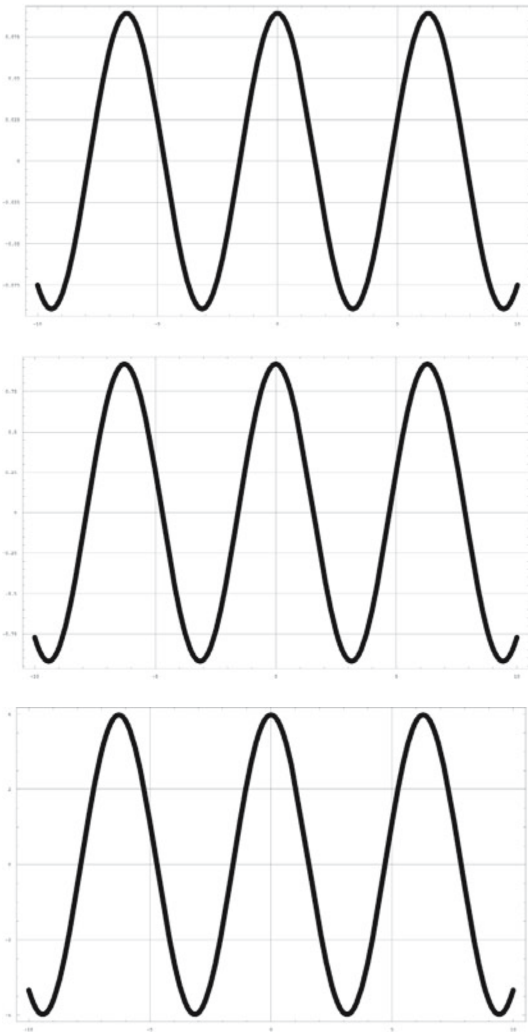

Fig. 6.1.3. The output autocorrelation functions $\gamma_Y(\tau)$ (6.1.11) of the integrating system (6.1.8) with $T = 0.3$, 1, and 3 (top to bottom) in the case of a simple random harmonic oscillation input with $\gamma_X(\tau) = \cos\tau$. Note the increasing amplitude of $\gamma_Y(\tau)$ as T increases.

In the case of the white noise input signal with $\gamma_X(\tau) = \delta(\tau)$, the output autocorrelation function, for $\tau > 0$, is

$$\begin{aligned}\gamma_Y(\tau) &= \int_0^\infty \int_0^\infty \delta(u-(s+\tau))h(u)h(s)\,du\,ds = \int_0^\infty h(s+\tau)h(s)\,ds \\ &= \int_0^\infty \frac{1}{RC}e^{s+\tau/RC}\cdot\frac{1}{RC}e^{s/RC}\,ds = \frac{1}{2RC}e^{-\tau/RC}.\end{aligned}$$

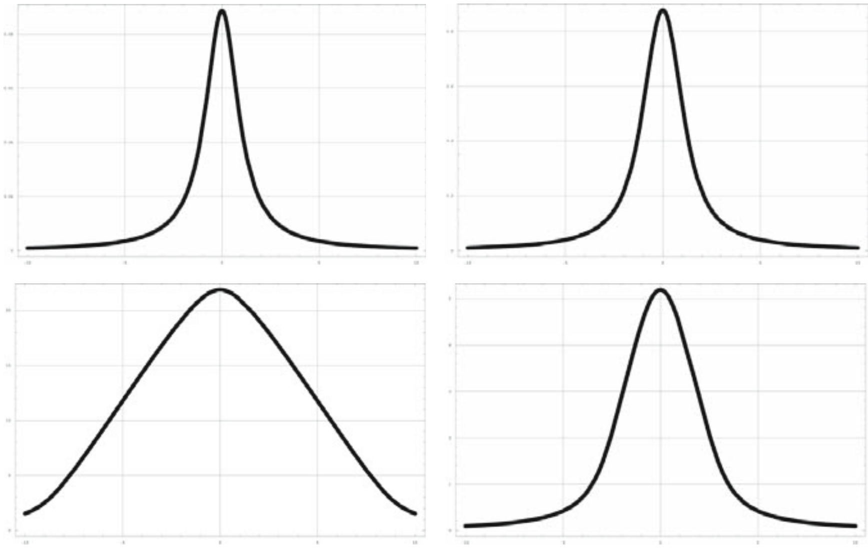

Fig. 6.1.4. The output autocorrelation functions $\gamma_Y(\tau)$ (6.1.13) of the integrating system (6.1.8) with $T = 0.3$, 1.3, and 9 (clockwise from top left corner), in the case of input with $\gamma_X(\tau) = \frac{1}{1+\tau^2}$. Note the growing maximum and spread of $\gamma_Y(\tau)$ as T increases.

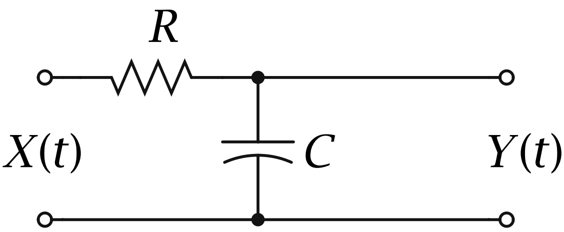

Fig. 6.1.5. A standard RC filter with the impulse response function $h(t) = \frac{1}{RC}\exp(-\frac{t}{RC}) \cdot U(t)$.

So

$$\gamma_Y(\tau) = \frac{1}{2RC}\exp\left(-\frac{|\tau|}{RC}\right). \tag{6.1.15}$$

The shape of the output autocorrelation function for small and large values of the RC constant is shown in Figure 6.1.6.

For the simple random harmonic oscillation with autocorrelation $\gamma_X(\tau) = \cos\tau$ as the input, the output autocorrelation is

$$\begin{aligned}\gamma_Y(\tau) &= \int_0^\infty\int_0^\infty \cos(\tau - u + s)\frac{1}{RC}\exp\left(\frac{-s}{RC}\right)\frac{1}{RC}\exp\left(-\frac{u}{RC}\right)ds\,du\\ &= \frac{\cos\tau}{1+(RC)^2}.\end{aligned}$$

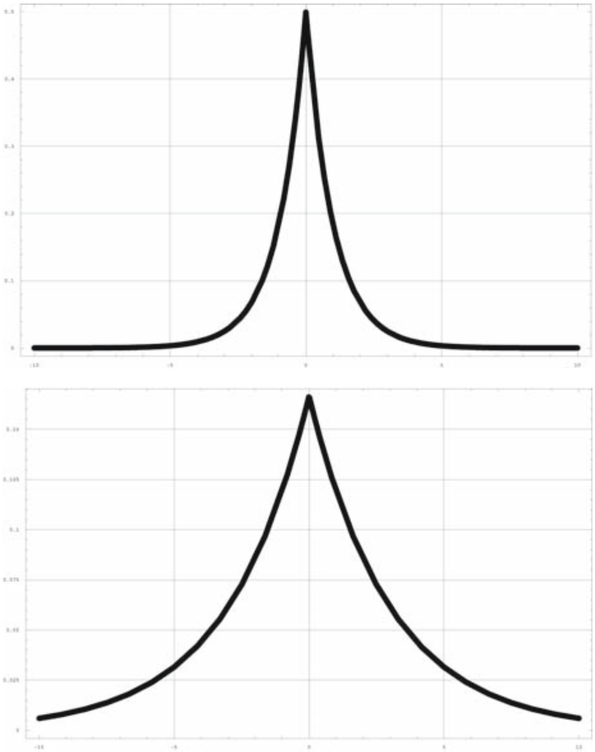

Fig. 6.1.6. The output autocorrelation function $\gamma_Y(\tau)$ for the RC filter (6.1.14) with a standard white noise input with $\gamma_X(\tau) = \delta(\tau)$. The top figure shows the case of small time constant $RC = 1$ and the bottom the case of the larger time constant $RC = 3$. Note the difference in the maximum and the spread of $\gamma_Y(\tau)$ in these two cases.

But a slightly more complex input autocorrelation function

$$\gamma_X(\tau) = e^{-2|\tau|},$$

corresponding to the switching input signal, produces the output autocorrelation function of the form

$$\begin{aligned}
\gamma_Y(\tau) &= \frac{1}{(RC)^2}\int_0^\infty\int_0^\infty e^{-|\tau-u+s|}e^{-(s+u)/(RC)}\,ds\,du \qquad (6.1.16)\\
&= \frac{1}{(RC)^2}\Bigg[\int_0^\tau\int_0^\infty e^{-(\tau-u+s)}e^{-(s+u)/(RC)}\,ds\,du\\
&\quad + \int_\tau^\infty\Bigg(\int_0^{u-\tau} e^{\tau-u+s}e^{-(s+u)/(RC)}\,ds\\
&\quad + \int_{u-\tau}^\infty e^{-(\tau-u+s)}e^{-(s+u)/(RC)}\,ds\Bigg)\,du\Bigg]
\end{aligned}$$

which, although doable (see Section 6.4, problems and exercises), is not fun to evaluate.

6.2 Frequency domain analysis and system bandwidth

Examples provided in the preceding section demonstrated analytic difficulties related to the time domain analysis of random stationary signals transmitted through linear systems. In many cases analysis becomes much simpler if it is carried out in the frequency domain. For this purpose, let us consider the Fourier transform $H(f)$ of the system's impulse response function $h(t)$:

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt, \tag{6.2.1}$$

which traditionally is called the system's *transfer function*.

Now the task is to calculate the power spectrum

$$S_Y(f) = \int_{-\infty}^{\infty} \gamma_Y(\tau)e^{-2\pi jf\tau}d\tau \tag{6.2.2}$$

of the output signal given the power spectrum

$$S_X(f) = \int_{-\infty}^{\infty} \gamma_X(\tau)e^{-2\pi jf\tau}d\tau$$

of the input signal. Since the output autocorrelation function $\gamma_Y(t)$ has been calculated in Section 6.1, substituting the expression obtained in (6.1.4) into (6.2.1), we get

$$\begin{aligned} S_Y(f) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma_X(\tau-s+u)h(s)h(u)dsdu\right)e^{-2\pi jf\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\gamma_X(\tau-s+u)e^{-2\pi jf(\tau-s+u)}d\tau\right)h(s)e^{-2\pi jfs}ds \\ &\quad\cdot h(u)e^{2\pi jfu}du. \end{aligned}$$

Making the substitution $\tau - s + u = w$ in the inner integral, we arrive at the final formula

$$S_Y(f) = S_X(f)\cdot H(f)\cdot H^*(f) = S_X(f)\cdot |H(f)|^2. \tag{6.2.3}$$

So the output power spectrum is obtained simply by multiplying the input power spectrum by a fixed factor $|H(f)|^2$, which is called the system's *power transfer function*.

The appearance of power transfer function $|H(f)|^2$ in formula (6.2.3) suggests introduction of the concept of the system's bandwidth. As in the case of signals (see Section 5.4) several choices are possible.

The *equivalent-noise bandwidth* BW_n is defined as the cutoff frequency $f_{\max}$ of the limited-band white noise with the amplitude equal to the value of the system's power transfer function at 0 and the mean power equal to the integral of the system's power transfer function, that is,

$$2\,\mathrm{BW}_n\,|H(0)|^2 = \int_{-\infty}^{\infty} |H(f)|^2 df,$$

which gives

$$\mathrm{BW}_n = \frac{1}{2|H(0)|^2}\int_{-\infty}^{\infty} |H(f)|^2 df. \tag{6.2.4}$$

The *half-power bandwidth* $\mathrm{BW}_{1/2}$ is defined as the frequency where the system's power transfer function declines to one-half of its maximum value which is always equal to $|H(0)|^2$. Thus it is obtained by solving, for $\mathrm{BW}_{1/2}$, the equation

$$|H(\mathrm{BW}_{1/2})|^2 = \frac{1}{2}|H(0)|^2. \tag{6.2.5}$$

Obviously, the above bandwidth concepts make the most sense for lowpass filters, that is, in the case when the system's power transfer function has a distinctive maximum at 0, dominating its values elsewhere. But for other systems such as band-pass filters, similar bandwidth definitions can be easily devised.

Example 6.2.1 (an RC filter). Recall that in this case the impulse response function is given by

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}} \cdot u(t).$$

So the transfer function is

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-2\pi jft}dt = \int_0^{\infty} \frac{1}{RC}e^{-\frac{t}{RC}}e^{-2\pi jft}dt = \frac{1}{1+2\pi jRCf},$$

and, consequently, the power transfer function is

$$|H(f)|^2 = \frac{1}{1+2\pi jRCf} \cdot \frac{1}{1-2\pi jRCf} = \frac{1}{1+(2\pi RCf)^2}. \tag{6.2.6}$$

The half-power bandwidth of the RC filter is easily computable from the equation

$$\frac{1}{1+(2\pi RC(\mathrm{BW}_{1/2}))^2} = \frac{1}{2},$$

which gives

$$\mathrm{BW}_{1/2} = \frac{1}{2\pi RC}.$$

The bandwidth decreases hyperbolically with the increase of the RC constant.

The output power spectra for an RC filter are thus easily evaluated. In the case of the standard white noise input with $S_X(f) \equiv 1$, the output power spectrum is

$$S_Y(f) = \frac{1}{1 + (2\pi RCf)^2}.$$

If the input signal is a random oscillation with the power spectrum

$$S_X(f) = \frac{A_0^2}{2}(\delta(f - f_0) + \delta(f + f_0)),$$

then the output power spectrum is

$$S_Y(f) = \frac{A_0^2}{2}(\delta(f - f_0) + \delta(f + f_0)) \cdot \frac{1}{1 + (2\pi RCf)^2}.$$

If the input is a switching signal with the power spectrum

$$S_X(f) = \frac{1}{1 + (af)^2},$$

then the output power spectrum is

$$S_Y(f) = \frac{1}{1 + (af)^2} \cdot \frac{1}{1 + (2\pi RCf)^2}.$$

Example 6.2.2 (bandwidth of the finite-time integrating circuit). Let us calculate the bandwidths BW_n and $\mathrm{BW}_{1/2}$ for the finite-time integrator with the impulse response function

$$h(t) = \begin{cases} 1 & \text{for } 0 \le t \le T; \\ 0 & \text{elsewhere.} \end{cases}$$

In this case, the transfer function is

$$H(f) = \int_0^T e^{-2\pi jft}\,dt = \frac{1}{2\pi jf}(1 - e^{-2\pi jfT}),$$

so that the power transfer function is

$$|H(f)|^2 = \frac{(1 - e^{-2\pi jfT})(1 - e^{2\pi jfT})}{(2\pi f)^2} = \frac{2(1 - \cos 2\pi fT)}{(2\pi f)^2}. \qquad (6.2.7)$$

Finding directly the integral of the power transfer function is a little tedious, but fortunately, by Parseval's formula,

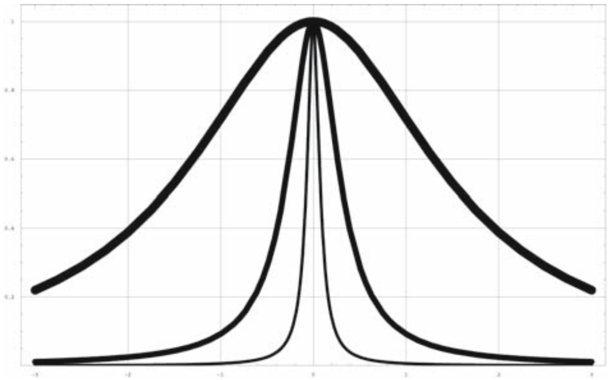

Fig. 6.2.1. Power transfer functions $|H(f)|^2 = \frac{1}{1+(2\pi RCf)^2}$ for the RC filter with the RC constants 0.1 (thick line), 0.5 (medium line), and 2.0 (thin line). The half-power bandwidths $\mathrm{BW}_{1/2}$ are, respectively, 1.6, 0.32, and 0.08.

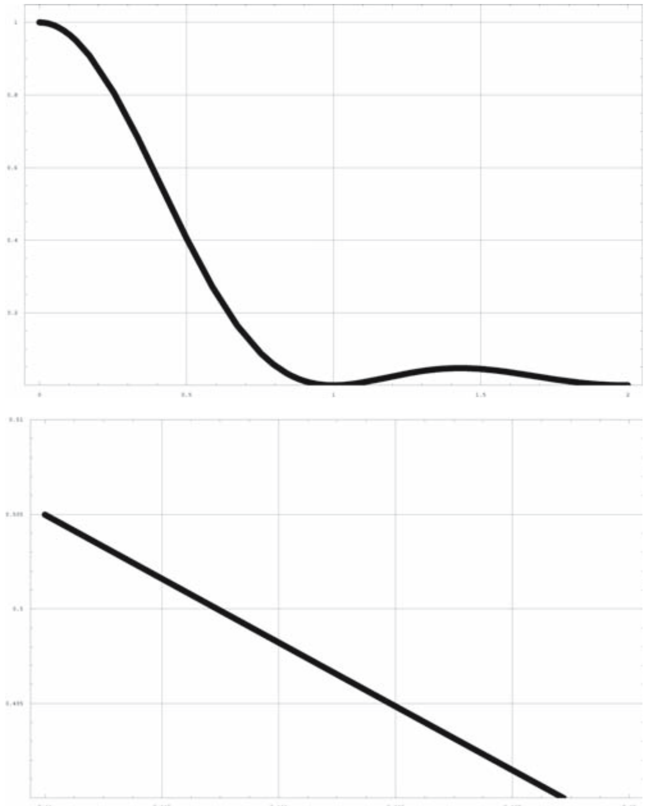

Fig. 6.2.2. *Top*: Power transfer function (6.2.7) of the finite-time integrating circuit with $T = 1$. *Bottom*: Magnified portion of the power transfer function for f between 0.44 and 0.45. This graphical analysis gives the half-power bandwidth $\mathrm{BW}_{1/2} = 0.443$.

$$\int_{-\infty}^{\infty} |H(f)|^2 df = \int_{-\infty}^{\infty} h^2(t)dt = \int_0^T dt = T,$$

and

$$H(0) = \int_0^T h(t)dt = T.$$

Thus the equivalent-noise bandwidth (6.2.4) is

$$\mathrm{BW}_n = \frac{1}{2T^2} \cdot T = \frac{1}{2T}.$$

Finding the half-power bandwidth requires solving equation (6.2.5):

$$\frac{2(1 - \cos 2\pi(\mathrm{BW}_{1/2})T)}{(2\pi(\mathrm{BW}_{1/2}))^2} = \frac{T^2}{2},$$

which can be done only numerically. Indeed, a quick graphical analysis (see Figure 6.2.1) for $T = 1$ gives the half-power bandwidth $\mathrm{BW}_{1/2} = 0.443$, slightly less than the corresponding equivalent-noise bandwidth $\mathrm{BW}_{\mathrm{eqn}} = 0.500$.

6.3 Digital signal, discrete-time sampling

In this section we will take a look at transmission of random stationary signals through linear systems when the signals are sampled at discrete times with the sampling interval T_s. The system can be schematically represented as follows:

$$X(nT_s) \longrightarrow \boxed{h(nT_s)} \longrightarrow Y(nT_s).$$

The input signal now forms a stationary random sequence

$$X(nT_s), \quad n = \cdots - 1, 0, 1, \ldots, \tag{6.3.1}$$

and the output signal

$$Y(nT_s), \quad n = \cdots - 1, 0, 1, \ldots, \tag{6.3.2}$$

is produced by discrete-time convolution of the input signal $X(nT_s)$ with the discrete-time impulse reponse sequence $h(nTs)$:

$$Y(nT_s) = \sum_{i=-\infty}^{\infty} X(iT_s)h(nT_s - iT_s)T_s. \tag{6.3.3}$$

In the discrete-time case, the realizability condition is

$$\sum_{n=-\infty}^{\infty} |h(nT_s)| < \infty,$$

and the causality condition means that

$$h(nT_s) = 0 \quad \text{for } n < 0.$$

With discrete-time inputs and outputs the autocorrelation functions are just discrete sequences and are defined by the formulas

$$\gamma_X(kT_s) = \mathbf{E}(X(nT_s)X(nT_s + kT_s)),$$
$$\gamma_Y(kT_s) = E(Y(nT_s)Y(nT_s + kT_s)).$$

Then after a direct application of (6.3.3), one obtains the following formula for the output autocorrelation sequence as a function of the input autocorrelation sequence and the impulse response sequence:

$$\gamma_Y(kT_s) = \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \gamma_X(kT_s - lT_s + iT_s)h(lT_s)h(iT_s)T_s^2. \tag{6.3.4}$$

To move into the frequency domain one can either directly apply the discrete or fast Fourier transforms or, as in Section 6.3, use the straight continuous-time Fourier transform technique assuming that both the signal and the impulse response function have been interpolated by constants between sampling points. We will follow the latter approach. Therefore, using formula (5.3.5), we get

$$S_X(f) = S_1(f) \cdot S_{2,X}(f), \tag{6.3.5}$$

with

$$S_{2,X}(f) = \sum_{m=-\infty}^{\infty} \gamma_X(mT_s)e^{-j2\pi mfT_s}T_s,$$

and

$$S_Y(f) = S_1(f) \cdot S_{2,Y}(f), \tag{6.3.6}$$

with

$$S_{2,Y}(f) = \sum_{m=-\infty}^{\infty} \gamma_Y(mT_s)e^{-j2\pi mfT_s}T_s,$$

and

$$S_1(f) = \frac{1 - \cos 2\pi fT_s}{2\pi^2 f^2 T_s^2}.$$

Remember that all the relevant information about the discrete sampled signal is contained in the frequency interval $(-\frac{f_s}{2}, \frac{f_s}{2})$ (see Remark 5.3.1). The transfer function of this system is

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = \sum_{k=-\infty}^{\infty} h(kT_s)\int_{kT_s}^{(k+1)T_s} e^{-j2\pi ft}dt$$
$$= \frac{1-e^{j2\pi fT_s}}{-j2\pi fT_s}\sum_{k=-\infty}^{\infty} h(kT_s)e^{-j2\pi fkT_s}T_s, \tag{6.3.7}$$

so that the power transfer function

$$|H(f)|^2 = \frac{1-\cos 2\pi fT_s}{2\pi^2 f^2 T_s^2}\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h(kT_s)h(nT_s)e^{-j2\pi f(k-n)T_s}T_s^2. \tag{6.3.8}$$

Again, all the relevant information about the discrete power transfer function contained in the frequency interval $(-\frac{f_s}{2},\frac{f_s}{2})$ (see Remark 5.3.1).

Finally, since we already know from Section 6.2 that

$$S_Y(f) = |H(f)|^2 S_X(f),$$

we also get from (6.3.5)–(6.3.6) that

$$S_{2,Y}(f) = |H(f)|^2 S_{2,X}(f). \tag{6.3.9}$$

or, equivalently,

$$\sum_{m=-\infty}^{\infty}\gamma_Y(mT_s)e^{-j2\pi mfT_s}T_s = |H(f)|^2\cdot\sum_{m=-\infty}^{\infty}\gamma_X(mT_s)e^{-j2\pi mfT_s}T_s. \tag{6.3.10}$$

Example 6.3.1 (autoregressive moving average system (ARMA)). We now take the sampling period $T_s = 1$ and the output $Y(n)$ determined from the input $X(n)$ via the autoregressive moving average scheme with parameters p and q (in brief, ARMA(p,q)):

$$Y(n) = \sum_{l=0}^{q} b(l)X(n-l) - \sum_{l=1}^{p} a(l)Y(n-l). \tag{6.3.11}$$

Defining $a(0) = 1$, we can then write

$$\sum_{l=0}^{p} a(l)Y(n-l) = \sum_{l=0}^{q} b(l)X(n-l).$$

Since the Fourier transform of the convolution is a product of Fourier transforms, we have

$$X(f)\sum_{l=0}^{q} b(l)e^{-2\pi jflT} = Y(f)\sum_{l=0}^{p} a(l)e^{-2\pi jflT},$$

so the transfer function

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{l=0}^{q} b(l)e^{-2\pi jflT}}{\sum_{l=0}^{p} a(l)e^{-2\pi jflT}}. \tag{6.3.12}$$

Example 6.3.2 (a solution of the stochastic difference equation). This example was considered in Chapter 4, but let us observe that it is a special case of Example 6.3.1, with parameters $p = 1$, $q = 0$, and the input signal being the standard discrete white noise $W(n)$ with $\sigma_W^2 = 1$. In other words,

$$Y(n) = -a_1 Y(n-1) + b_0 W(n).$$

In view of (6.3.12), the power transfer function is

$$|H(f)|^2 = \frac{b_0}{1 + a_1 e^{-2\pi jf}} \cdot \frac{b_0}{1 + a_1 s e^{2\pi jf}} = \frac{b_0^2}{1 + a_1^2 + 2a_1 \cos 2\pi f},$$

with, again, all the relevant information contained in the frequency interval $-\frac{1}{2} < f < \frac{1}{2}$.

Given that the input is the standard white noise, we have that

$$S_Y(f) = |H(f)|^2 \cdot 1 = \frac{b_0^2}{1 + a_1^2 + 2a_1 \cos 2\pi f}. \tag{6.3.13}$$

One way to find the output autocorrelation sequence $\gamma_Y(n)$ would be to take into account the relationship (6.3.10) and expand (6.3.13) into the Fourier series; its coefficients will form the desired autocorrelation sequence. This procedure is streightforward and requires only an application of the formula for the sum of a geometric series (see Section 6.4).

However, we would like to explore here a different route and employ a recursive procedure to find the output autocorrelation sequence. First, observe that

$$\begin{aligned}
\gamma_Y(k) &= \mathbf{E}(Y(n)Y(n+k)) \\
&= \mathbf{E}(-a_1 Y(n-1) + b_0 X(n)) \cdot (-a_1 Y(n+k-1) + b_0 X(n+k)) \\
&= a_1^2 \mathbf{E}(Y(n-1)Y(n+k-1)) - a_1 b_0 \mathbf{E}(Y(n-1)X(n+k))) \\
&\quad - a_1 b_0 \mathbf{E}(X(n)Y(n+k-1)) + b_0^2 \mathbf{E}(X(n)X(n+k)) \\
&= a_1^2 \gamma_Y(k) - a_1 b_0 \gamma_{XY}(k-1) + b_0)^2 \gamma_X(k),
\end{aligned}$$

where

$$\gamma_{XY}(k) = \mathbf{E}(X(n)Y(n+k)),$$

is the cross-correlation sequence of signals $X(n)$ and $Y(n)$. Thus

$$\gamma_Y(k) = \frac{b_0}{1 - a_1^2}(-a_1 \gamma_{XY}(k-1) + b_0 \gamma_X(k)).$$

For $k = 0$,

$$\gamma_Y(0) = \sigma_Y^2 = \frac{b_0}{1 - a_1^2}(-a_1 \mathbf{E}(X(n)Y(n-1)) + b_0 \gamma_X(0))$$

$$= \frac{b_0^2}{1-a_1^2}\gamma_X(0) = \frac{b_0^2}{1-a_1^2}.$$

For $k = 1$,

$$\begin{aligned}\gamma_Y(1) &= \frac{b_0}{1-a_1^2}(-a_1\gamma_{XY}(0) + b_0\gamma_X(1)) = \frac{b_0(-a_1)}{1-a_1^2}\mathbf{E}(X(0)Y(0)) \\ &= \frac{b_0(-a_1)}{1-a_1^2}\mathbf{E}(X(0)(a_1Y(-1) + b_0X(0))) = \frac{b_0^2(-a_1)}{1-a_1^2}.\end{aligned}$$

For a general $k > 1$,

$$\gamma_Y(k) = \frac{b_0}{1-a_1^2}(-a_1\gamma_{XY}(k-1) + b_0\gamma_X(k)),$$

and, as above,

$$\begin{aligned}\gamma_{XY}(k-1) &= \mathbf{E}(X(0)Y(k-1)) \\ &= \mathbf{E}(X(0)(-a_1Y(k-2) + b_0X(k-1))) \\ &= (-a_1)\mathbf{E}(X(0)Y(k-2)) \\ &= (-a_1)\gamma_{XY}(k-2) \\ &= \cdots = (-a_1)^{k-1}\gamma_{XY}(0) = b(0)(-a_1)^{k-1}.\end{aligned}$$

Since the autocorrelation sequence must be an even function of variable k, we finally get, for any $k = \ldots, -2, -1, 0, 1, 2, \ldots$,

$$\gamma_Y(k) = \frac{b_0^2}{1-a_1^2}(-a_1)^{|k|},$$

thus recovering the result from Chapter 4.

6.4 Problems and exercises

In the first three exercises, also try solving the problem by first finding the autocorrelation function of the output to see how hard the problem is in the time domain framework.

6.4.1. The impulse response function of a linear system is $h(t) = 1 - t$ for $0 \le t \le 1$ and 0 elsewhere.

(a) Produce a graph of $h(t)$.

(b) Assume that the input is the standard white noise. Find the autocorrelation function of the output.

(c) Find the power transfer function of the system, its equivalent-noise bandwidth and half-power bandwidth.

(d) Assume that the input has the autocorrelation function $\gamma_X(t) = \frac{3}{1+4t^2}$. Find the power spectrum of the output signal.

(e) Assume that the input has the autocorrelation function $\gamma_X(t) = \exp(-4|t|)$. Find the power spectrum of the output signal.

(f) Assume that the input has the autocorrelation function $\gamma_X(t) = 1 - |t|$ for $|t| < 1$ and 0 elsewhere. Find the power spectrum of the output signal.

6.4.2. The impulse response function of a linear system is $h(t) = e^{-2t}$ for $0 \le t \le 2$ and 0 elsewhere.

(a) Produce a graph of $h(t)$.

(b) Assume that the input is the standard white noise. Find the autocorrelation function of the output.

(c) Find the power transfer function of the system, its equivalent-noise bandwidth and half-power bandwidth.

(d) Assume that the input has the autocorrelation function $\gamma_X(t) = \frac{3}{1+4t^2}$. Find the power spectrum of the output signal.

(e) Assume that the input has the autocorrelation function $\gamma_X(t) = \exp(-4|t|)$. Find the power spectrum of the output signal.

(f) Assume that the input has the autocorrelation function $\gamma_X(t) = 1 - |t|$ for $|t| < 1$ and 0 elsewhere. Find the power spectrum of the output signal.

6.4.3. The impulse response function of a linear system is $h(t) = e^{-0.05t}$ for $t \ge 10$ and 0 elsewhere.

(a) Produce a graph of $h(t)$.

(b) Assume that the input is the standard white noise. Find the autocorrelation function of the output.

(c) Find the power transfer function of the system, its equivalent-noise bandwidth and half-power bandwidth.

(d) Assume that the input has the autocorrelation function $\gamma_X(t) = \frac{3}{1+4t^2}$. Find the power spectrum of the output signal.

(e) Assume that the input has the autocorrelation function $\gamma_X(t) = \exp(-4|t|)$. Find the power spectrum of the output signal.

(f) Assume that the input has the autocorrelation function $\gamma_X(t) = 1 - |t|$ for $|t| < 1$ and 0 elsewhere. Find the power spectrum of the output signal.

6.4.4. *Cross-correlation* ρ_{XY} and *cross-covariance* γ_{XY} for random signals $X(t)$ and $Y(t)$ are defined, respectively, as follows:

$$\rho_{XY}(t,s) = \mathbf{E}(X(t)Y(s)),$$
$$\gamma_{XY}(t,s) = \mathbf{E}((X(t) - \mu_X(t))(Y(s) - \mu_Y(s)).$$

Random signals $X(t)$ and $Y(t)$ are said to be *jointly stationary* if they are stationary and

$$\rho_{XY}(t,s) = \rho_{XY}(t-s,0).$$

Consider random signals

$$X(t) = a\cos(2\pi(f_0t + \Theta)), \qquad Y(t) = b\sin(2\pi(f_0t + \Theta)),$$

where a and b are nonrandom constants and Θ is uniformly distributed on $[0, 1]$. Find the cross-correlation function for X and Y. Are these signals jointly stationary?

6.4.5. Consider the circuit shown in Figure 6.4.1

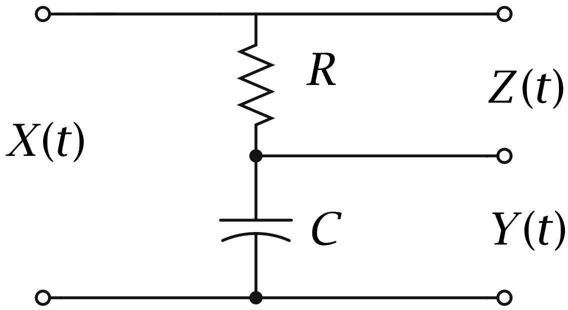

Fig. 6.4.1.

Assume that the input is the standard white noise.

(a) Find the power spectra $S_Y(f)$ and $S_Z(f)$ of the outputs $Y(t)$ and $Z(t)$.

(b) Find the cross-correlation

$$\gamma_{YZ}(\tau) = \mathbf{E}(Z(t)Y(t+\tau))$$

between those two outputs.

6.4.6. Find the output autocorrelation sequence for the discrete-time system representing a stochastic difference equation described in Example 6.3.2. Use the Fourier series expansion of formula (6.3.13).

6.4.7. Consider the circuit shown in Figure 6.4.2.

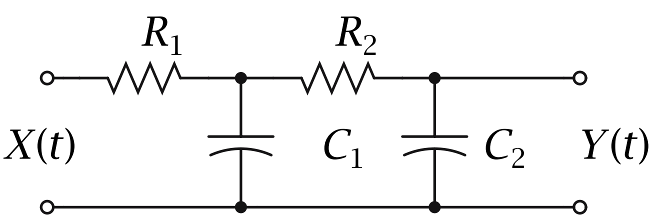

Fig. 6.4.2.

Assume that the input is the standard white noise. Find the power spectrum $S_Y(f)$ and the autocorrelation function $\gamma_Y(\tau)$ of the output $Y(t)$.

6.4.8. Find the half-power and equivalent-noise bandwidth of the system shown in Figure 6.4.2.