2

Spectral Representation of Deterministic Signals: Fourier Series and Transforms

In this chapter we will take a closer look at the spectral, or frequency domain, representation of deterministic (nonrandom) signals which was already mentioned in Chapter 1. The tools introduced below, usually called *Fourier* or *harmonic analysis*, will play a fundamental role later on in our study of random signals. Almost all of the calculations will be conducted in the complex form. Compared with working in the real domain, manipulation of formulas written in the complex form turns out to be simpler and all the tedium of remembering various trigonometric formulas is avoided. All of the results written in the complex form can be translated quickly into results for real trigonometric series expressed in terms of sines and cosines via de Moivre's formula $e^{jt} = \cos t + j\sin t$, familiar from Chapter 1.

2.1 Complex Fourier series for periodic signals

A complex-valued signal $x(t)$ that is periodic with period P (say, seconds) can be written in the form of an infinite complex Fourier series

$$x(t) = \sum_{m=-\infty}^{\infty} z_m e^{j2\pi m f_0 t} = \sum_{m=-\infty}^{\infty} z_m e^{jm\omega_0 t}, \tag{2.1.1}$$

where $f_0 = \frac{1}{P}$ is the *fundamental frequency* of the signal (measured in Hz $= \frac{1}{\mathrm{s}}$), and $\omega_0 = 2\pi f_0$ is called the fundamental *angular velocity* (measured in radians/s). The complex number z_m, where m can take values $\dots, -2, -1, 0, 1, 2, \dots$, is called the mth Fourier coefficient of signal $x(t)$.

In this text, we will carry out our calculations exclusively in terms of the fundamental frequency f_0, although one can find in the printed and software signal processing literature sources where all the work is done in terms of ω_0. It is an arbitrary choice, and transition from one system to the other is easily accomplished by adjusting various constants appearing in the formulas.

The infinite Fourier series representation (2.1.1) is unique in the sense that two different signals will have two different sequences of Fourier coefficients. The uniqueness is a result of the fundamental property of complex exponentials

$$e_m(t) := e^{j2\pi m f_0 t}, \quad m = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{2.1.2}$$

called *orthonormality*:

The scalar product (sometimes also called inner, or dot, product) of two complex exponentials e_n and e_m is 0 if the exponentials are different, and it is 1 if they are the same. Indeed,

$$\begin{aligned} \langle e_n, e_m \rangle &= \frac{1}{P}\int_0^P e_n(t) e_m^*(t)\,dt \\ &= \frac{1}{P}\int_0^P e^{j2\pi(n-m)f_0 t}\,dt = \begin{cases} 0 & \textit{if } n \neq m; \\ 1 & \textit{if } n = m. \end{cases} \end{aligned} \tag{2.1.3}$$

Recall that, for a complex number $z = a + jb = |z|e^{j\theta}$ with real component a and imaginary component b, the complex conjugate $z^* = a - jb = |z|e^{-j\theta}$. Sometimes it is convenient to describe the orthonormality using the so-called *Kronecker delta* notation:

$$\delta_{mn} = \begin{cases} 0 & \text{if } n \neq m; \\ 1 & \text{if } n = m. \end{cases}$$

Then, simply,

$$\langle e_m, e_n \rangle = \delta_{mn}.$$

Using the orthonormality property we can directly evaluate the coefficients z_m in the Fourier series (2.1.1) of signal $x(t)$ by formally calculating the scalar product of $x(t)$ and $e_m(t)$:

$$\begin{aligned} \langle x, e_m \rangle &= \frac{1}{P}\int_0^P \left(\sum_{n=-\infty}^{\infty} z_n e_n(t) \right) \cdot e_m^*(t)\,dt \\ &= \sum_{n=-\infty}^{\infty} z_n \frac{1}{P}\int_0^P e_n(t) e_m^*(t)\,dt = z_m, \end{aligned} \tag{2.1.4}$$

so that we get an explicit formula for the Fourier coefficent of signal $x(t)$,

$$z_m = \langle x, e_m \rangle = \frac{1}{P}\int_0^P x(t)e^{-j2\pi m f_0 t}\,dt. \tag{2.1.5}$$

Thus the basic Fourier expansion (2.1.1) can now be rewritten in the form of a formal identity

$$x(t) = \sum_{n=-\infty}^{\infty} \langle x, e_n \rangle e_n(t). \tag{2.1.6}$$

It is worthwhile to observe that the above calculations on infinite series and interchanges of the order of integration and infinite summations were purely formal, that is, the soundness of the limit procedures was not rigorously established. The missing steps can be found in the mathematical literature devoted to Fourier analysis.[3] For our purposes, it suffices to say that if a periodic signal $x(t)$ has finite power

$$\mathrm{PW}_x = \frac{1}{P}\int_0^P |x(t)|^2 dt < \infty, \tag{2.1.7}$$

and the concept of convergence of the functional infinite series (2.1.1) is defined in the right way, then all of the above formal manipulations can be rigorously justified. We will return to this issue at the end of this section. In what follows, we will usually consider signals with finite power.

Real-valued signals. Signal $x(t)$ is real-valued if and only if the coefficients z_m satisfy the obvious algebraic condition

$$z_{-m} = z_m^*, \tag{2.1.8}$$

in which case cancellation of the imaginary parts in the Fourier series (2.1.1) occurs. Indeed, under assumption (2.1.8),

$$z_m = |z_m|e^{j\theta_m}, \quad \theta_{-m} = -\theta_m, \tag{2.1.9}$$

and since

$$\frac{e^{j\alpha} + e^{-j\alpha}}{2} = \cos\alpha,$$

we get

$$x(t) = c_0 + \sum_{m=1}^{\infty} c_m \cos(2\pi m f_0 t + \theta_m), \tag{2.1.10}$$

where

[3] See, e.g., A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, UK, 1959.

$$c_0 = z_0 \quad \text{and} \quad c_m = 2|z_m|, \quad m = 1, 2, \ldots. \tag{2.1.11}$$

The power PW_x of a periodic signal $x(t)$ can also be directly calculated from its Fourier coefficient z_m. Indeed, again calculating formally, we obtain that

$$\begin{aligned}
\mathrm{PW}_x &= \frac{1}{P}\int_0^P |x(t)|^2 dt = \frac{1}{P}\int_0^P x(t)x^*(t)dt \\
&= \frac{1}{P}\int_0^P \left(\sum_{k=-\infty}^{\infty} z_k e_k(t)\right)\cdot\left(\sum_{m=-\infty}^{\infty} z_m e_m(t)\right)^* dt \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} z_k z_m^* \frac{1}{P}\int_0^P e_k(t)e_m^*(t)dt = \sum_{m=-\infty}^{\infty} z_m z_m^*,
\end{aligned}$$

in view of the orthonormality (2.1.3) of the complex exponentials. The multiplication of the two infinite series was carried out term by term. The resulting relationship,

$$\mathrm{PW}_x = \frac{1}{P}\int_0^P |x(t)|^2 dt = \sum_{m=-\infty}^{\infty} |z_m|^2, \tag{2.1.12}$$

is known as the *Parseval formula*. A similar calculation for the scalar product $\frac{1}{P}\int_0^P x(t)y^*(t)dt$ of two different periodic signals, $x(t)$ and $y(t)$, gives an *extended Parseval formula* listed in Table 2.1.1.

Analogy between the orthonormal basis of vectors in the 3D space $\mathbf{R}^3$ and the complex exponentials. It is useful to think about the complex exponentials $e_m(t) = e^{2\pi jmf_0t}$, $m = \ldots, -1, 0, 1, \ldots$, as an infinite-dimensional version of the orthonormal basics vectors in $\mathbf{R}^3$. In this mental picture the periodic signal $x(t)$ is now thought of as an infinite-dimensional "vector" uniquely expandable into an infinite linear combination of the complex exponentials in the same way a 3D vector is uniquely expandable into a finite linear combination of the three unit coordinate vectors. Table 2.1.1 describes this analogy more fully. Note that the Parseval formula can now be seen just as an infinite-dimensional extension of the familiar Pythagorean theorem.

Recall that a signal is called *even* if it is symmmetric under the change of the direction of time, i.e., if $x(t) = x(-t)$; it is called *odd* if it is antisymmetric under the change of the direction of time, i.e., if $x(t) = -x(-t)$. The real Fourier expansion of an even real-valued signal $x(t) = x(-t)$ will contain only cosine functions, and the real Fourier expansion of an odd real-valued signal $x(t) = -x(-t)$ will contain only sine functions. This phenomenon will be illustrated in the following examples. Of course, if one is only interested in the signal $x(t)$ for positive times $t > 0$, then one can arbitrarily extend the signal's values to

Table 2.1.1. Analogy between orthonormal expansions in 3D and in the space of periodic signals.

Objects	
3D vectors	Signals with finite power
$\vec{x} = (x_1, x_2, x_3)$ $\vec{y} = (y_1, y_2, y_3)$	$x(t) = \sum_{m=-\infty}^{\infty} z_m e_m(t)$, $\mathrm{PW}_x < \infty$ $y(t) = \sum_{m=-\infty}^{\infty} w_m e_m(t)$, $\mathrm{PW}_y < \infty$
Bases	
Unit coordinate vectors	Complex exponentials
$\vec{e_1} = (1, 0, 0)$ $\vec{e_2} = (0, 1, 0)$ $\vec{e_3} = (0, 0, 1)$	$\vdots$ $e_1(t) = e^{j2\pi f_0 t}$ $e_2(t) = e^{j2\pi(2f_0)t}$ $e_3(t) = e^{j2\pi(3f_0)t}$ $\vdots$
Scalar products	
$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{3} x_i y_i$	$\langle x(t), y(t) \rangle$ $= \frac{1}{P} \int_0^P x(t) y^*(t) dt$
Orthonormality	
$\langle \vec{e_m}, \vec{e_m} \rangle = \delta_{mn}$	$\langle e_m(t), e_n(t) \rangle = \delta_{mn}$
Expansions	
Basis	Fourier
$\vec{x} = \sum_{m=1}^{3} \langle \vec{x}, \vec{e_m} \rangle \vec{e_m}$	$x(t) = \sum_{i=-\infty}^{\infty} \langle x, e_m \rangle e_m(t)$
Formulas	
Pythagoras'	Parseval's
$\|\|\vec{x}\|\|^2 = \sum_{m=1}^{3} x_m^2$	$\mathrm{PW}_x = \frac{1}{P} \int_0^P \|x(t)\|^2 dt$ $= \sum_{m=-\infty}^{\infty} \|z_m\|^2$
Scalar product	Extended Parseval's
$\langle \vec{x}, \vec{y} \rangle = \sum_{m=1}^{3} x_m y_m$	$\frac{1}{P} \int_0^P x(t) y^*(t) dt$ $= \sum_{m=-\infty}^{\infty} z_m w_m^*$

the negative timeline to form either an odd or an even signal, and thus obtain either its pure sine or its pure cosine expansion.

Example 2.1.1 (pure cosine expansion of an even rectangular waveform). Consider a rectangular waveform with period P and amplitude

$a > 0$, defined by the formula

$$x(t) = \begin{cases} a & \text{for } 0 \le t < \frac{P}{4}; \\ 0 & \text{for } \frac{P}{4} \le t < \frac{3P}{4}; \\ a & \text{for } \frac{3P}{4} \le t < P. \end{cases}$$

The signal is pictured in Figure 2.1.1 for particular values $P = 1$ and $a = 1$.

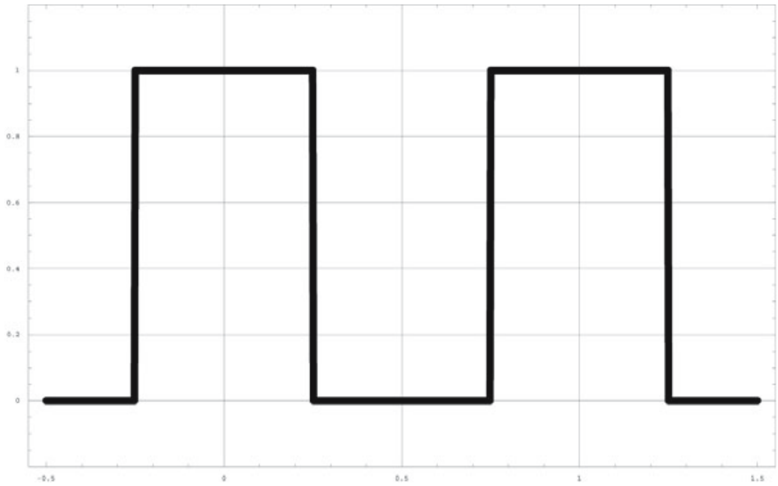

Fig. 2.1.1. An even rectangular waveform signal from Example 2.1.1. The period $P = 1$ and the amplitude $a = 1$.

Calculation of coefficients z_m in the expansion of the signal $x(t)$ into a complex Fourier series is here straightforward: For $m = 0$,

$$z_0 = \frac{1}{P}\int_0^P x(t)e^{-j2\pi 0t/P}dt = \frac{a}{P}\left(\frac{P}{4} - 0 + P - \frac{3P}{4}\right) = \frac{a}{2}.$$

In the case $m \neq 0$,

$$\begin{aligned} z_m &= \frac{1}{P}\int_0^P x(t)e^{-j2\pi mt/P}dt \\ &= \frac{a}{P}\left(\int_0^{P/4} e^{-j2\pi mt/P}dt + \int_{3P/4}^{P} e^{-j2\pi mt/P}dt\right) \\ &= \frac{a}{P}\left(\frac{P}{-j2\pi m}e^{-j2\pi mt/P}\Big|_0^{P/4} + \frac{P}{-j2\pi m}e^{-j2\pi mt/P}\Big|_{3P/4}^{P}\right) \\ &= \frac{a}{-j2\pi m}\left(e^{-j(\pi/2)m} - 1 - e^{-j(3\pi/2)m} + 1\right) \end{aligned}$$

$$= -\frac{a}{\pi m} e^{-j(2\pi/2)m} \left(\frac{e^{j(\pi/2)m} - e^{-j(\pi/2)m}}{2j} \right)$$
$$= -\frac{a}{\pi m} \cos \pi m \sin \frac{\pi}{2} m = -\frac{a}{\pi m} (-1)^m \sin \frac{\pi}{2} m.$$

If $m = 2k$, then $\sin \frac{\pi}{2} m = 0$, and if $m = 2k+1$, $k = 0, \pm 1, \pm 2, \dots$, then $\sin \frac{\pi}{2} m = (-1)^k$, which gives, for $k = \pm 1, \pm 2, \dots$,

$$z_{2k} = 0,$$

and

$$z_{2k+1} = \frac{-a}{\pi(2k+1)} (-1)^{2k+1} (-1)^k = \frac{(-1)^k a}{\pi(2k+1)}.$$

Thus the complex Fourier expansion of the signal $x(t)$ is

$$x(t) = \frac{a}{2} + \frac{a}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{2k+1} e^{j2\pi(2k+1)t/P}.$$

Observe that for any $m = \dots, -1, 0, 1, \dots$, we have $z_m = z_{-m}$. Pairing up complex exponentials with the exponents of opposite signs, and using de Moivre's formula, we arrive at the real Fourier expansion that contains only cosine functions:

$$x(t) = \frac{a}{2} + \frac{a}{\pi} \left(2 \cos \frac{2\pi t}{P} - \frac{2}{3} \cos \frac{2\pi 3t}{P}) + \cdots \right).$$

Example 2.1.2 (pure sine expansion of an odd rectangular waveform). Consider a periodic rectangular waveform of period P which is defined by the formula

$$x(t) = \begin{cases} a & \text{for } 0 \le t < \frac{P}{4}; \\ 0 & \text{for } \frac{P}{4} \le t < \frac{3P}{4}; \\ -a & \text{for } \frac{3P}{4} \le t < P. \end{cases}$$

The signal is pictured in Figure 2.1.2 for particular values $P = 1$ and $a = 1$.

For $m = 0$,

$$z_0 = \frac{1}{P} \int_0^P x(t) dt = 0,$$

and, for $m \neq 0$,

$$z_m = \frac{a}{P} \left(\int_0^{P/4} e^{-j2\pi m t/P} dt - \int_{3P/4}^{P} e^{-j2\pi m t/P} dt \right)$$
$$= \frac{-a}{j2\pi m} (e^{-j(\pi/2)m} - 1 - 1 + e^{-j(3\pi/2)m})$$

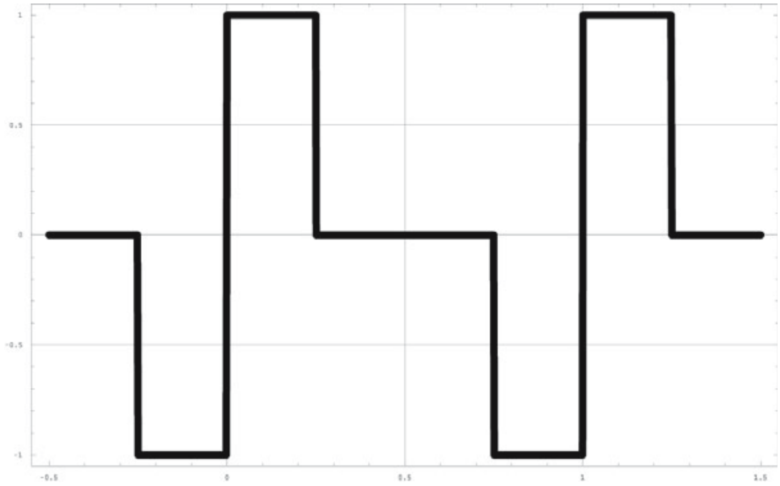

Fig. 2.1.2. An odd rectangular waveform signal from Example 2.1.2. The period $P = 1$ and the amplitude $a = 1$.

$$
\begin{aligned}
&= -\frac{aj}{2\pi m}[e^{-j(2\pi/2)m}(e^{j(\pi/2)m} + e^{-j(\pi/2)m}) - 2] \\
&= -\frac{aj}{\pi m}\left(\cos \pi m \cdot \cos \frac{\pi}{2} m - 1\right).
\end{aligned}
$$

Since $\cos \pi m = (-1)^m$, and since $\cos(\frac{\pi}{2})m = 0$ if m is odd and $= (-1)^k$ when $m = 2k$ is even, we get that

$$
z_m = \begin{cases} 0 & \text{for odd } m = 2k+1; \\ \frac{aj[(-1)^k - 1]}{2\pi k} & \text{for even } m = 2k. \end{cases}
$$

Thus the complex Fourier series of the signal $x(t)$ is of the form

$$
x(t) = \frac{a}{\pi} \sum_{k=-\infty}^{\infty} \frac{j[(-1)^k - 1]}{2k} e^{j2\pi 2kt/P}.
$$

Observe that in this case, for any $m = \dots, -1, 0, 1, \dots$, we have $z_m = -z_{-m}$, so pairing up the exponentials with opposite signs in the exponents and using de Moivre's formula, we get a real Fourier series expansion for $x(t)$ that contains only sine functions:

$$
x(t) = \frac{a}{\pi}\left[2 \sin\left(\frac{4\pi t}{P}\right) + \frac{2}{3} \sin\left(\frac{12\pi t}{P}\right) + \cdots\right].
$$

Example 2.1.3 (a general expansion for a rectangular waveform which is neither odd nor even). Consider a periodic rectangular waveform of period P which is defined by the formula

$$x(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{P}{4}; \\ a & \text{for } \frac{P}{4} \le t < \frac{P}{2}; \\ 0 & \text{for } \frac{P}{2} \le t < P. \end{cases}$$

The signal is pictured in Figure 2.1.3 for parameter values $P = 1$ and $a = 1$, and for simplicity's sake, we will carry out our calculations only in that case.

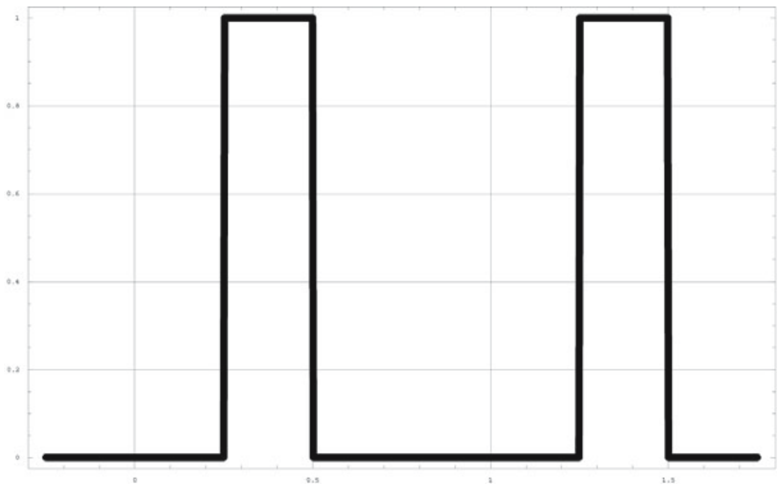

Fig. 2.1.3. A neither odd nor even rectangular waveform signal from Example 2.1.3. The period $P = 1$, and the amplitude $a = 1$.

For $m = 0$,

$$z_0 = \int_{1/4}^{1/2} = \frac{1}{4}.$$

For $m \neq 0$,

$$z_m = |z_m|e^{i\theta m} = \int_{1/4}^{1/2} e^{-j2\pi mt} dt = \frac{1}{-j2\pi m}[e^{-j2\pi m/2} - e^{-j2\pi m/4}]$$
$$= \frac{1}{\pi m} e^{-j3\pi m/4} \left(\frac{e^{j\pi m/4} - e^{-j\pi m/4}}{2j} \right) = \frac{1}{\pi m} \sin\left(\frac{\pi}{4} m\right) e^{-j3\pi m/4}.$$

Thus

$$|z_m| = \frac{1}{\pi m} \sin\left(\frac{\pi m}{4}\right) \quad \text{and} \quad \theta_m = -\frac{j3\pi m}{4},$$

and the complex Fourier series for $x(t)$ is

$$x(t) = \frac{1}{4} + \sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{\pi m} \sin\left(\frac{\pi m}{4}\right) e^{-j3\pi m/4} e^{j2\pi mt}.$$

Again, pairing up the complex exponentials with opposite signs in the exponents, we obtain the real expansion in terms of the cosines, but this time with phase shifts that depend on m:

$$x(t) = \frac{1}{4} + \sum_{m=1}^{\infty} \frac{2}{\pi m} \sin\left(\frac{\pi m}{4}\right) \cos\left(2\pi m t - \frac{3\pi m}{4}\right),$$

which, using the trigonometric formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, can be written as a general real Fourier series

$$x(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos(2\pi m t) + b_m \sin(2\pi m t),$$

with

$$a_0 = \frac{1}{4}, \qquad a_m = \frac{2}{\pi m} \sin\frac{\pi m}{4} \cos\frac{3\pi m}{4},$$
$$b_m = \frac{2}{\pi m} \sin\frac{\pi m}{4} \sin\frac{3\pi m}{4}.$$

2.2 Approximation of periodic signals by finite Fourier sums

Up to this point the equality in the Fourier series representation

$$x(t) = \sum_{m=-\infty}^{\infty} \langle x, e_m \rangle e_m(t)$$

for periodic signals, or its real version in terms of sine and/or cosine functions, was understood only formally. But, of course, the usefulness of such an expansion will depend on whether we can show that the signal $x(t)$ can be well approximated by a finite cutoff of the infinite Fourier series, that is, on whether we can prove that

$$x(t) \approx s_M(t) := \sum_{m=-M}^{M} \langle x, e_m \rangle e_m(t) \tag{2.2.1}$$

for M large enough, with the error in the above approximate equality $\approx$ rigorously estimated.

One can pursue here several options:

Approximation in power: Mean-square error. If the error of approximation is measured as the power of the difference between the signal $x(t)$ and the finite Fourier sum $s_M(t)$ in (2.2.1), then the calculation is relatively simple and the error is often called the *mean-square error*. Indeed, using the Parseval formula, we find that

$$
\begin{aligned}
\mathrm{PW}_{x-s_M} &= \frac{1}{P}\int_0^P |x(t)-s_M(t)|^2 dt \\
&= \frac{1}{P}\int_0^P \left| \sum_{m=-\infty}^{\infty} \langle x, e_m\rangle e_m(t) - s_M(t) \right|^2 dt \\
&= \frac{1}{P}\int_0^P \left| \sum_{|m|>M} \langle x, e_m\rangle e_m(t) \right|^2 dt = \sum_{|m|>M} |\langle x, e_m\rangle|^2,
\end{aligned}
$$

which converges to 0, as $M \to \infty$, because we assumed that the power of the signal is finite:

$$
\mathrm{PW}_x = \sum_{m=-\infty}^{\infty} |\langle x, e_m\rangle|^2 < \infty.
$$

Note that the unspoken assumption here is that the orthonormal system $e_n(t)$, $n = 0, \pm 1, \pm 2, \ldots$, is rich enough to make the Fourier representation possible for any finite power signal. This assumption, often called *completeness* of the above orthonormal system, can actually be rigorously proven.

Approximation at each time instant *t* separately. This type of approximation is often called the *pointwise approximation* and the goal is to verify that, for each time instant t,

$$
\lim_{M\to\infty} s_M(t) = x(t). \tag{2.2.2}
$$

Here the situation is delicate, as examples at the end of this section will show, and the assumption that signal $x(t)$ has finite power is not sufficient to guarantee the pointwise approximation. Neither is a stronger assumption that the signal is continuous. However,

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if the signal is continuous and has a bounded continuous derivative, except, possibly, at a finite number of points, then the pointwise approximation (2.2.2) *holds true.*

Uniform approximation in time *t*. If one wants to control the error of approximation simultaneously (uniformly) for all times t, then more stringent assumptions on the signal are necessary. Namely, we have the following theorem:[4]

[4] Proofs of these two mathematical theorems and other results quoted in this section can be found in, e.g., T. W. Körner, *Fourier Analysis*, Cambridge University Press, Cambridge, UK, 1988.

If the signal is continuous everywhere and has a bounded continuous derivative except at a finite number of points, then

$$\max_{0\le t\le P} |x(t) - s_M(t)| \to 0 \quad \textit{as } M \to \infty. \tag{2.2.3}$$

Note that the above statements do not resolve the question of what happens with the finite Fourier sums at discontinuity points of a signal, like those encountered in the rectangular waveforms in Examples 2.1.1–2.1.3. It turns out that under the assumptions of the above-quoted theorems, the points of discontinuity of the signal $x(t)$ are necessarily jumps, that is the left and right limits

$$x(t_-) = \lim_{s\uparrow t} x(s) \quad \text{and} \quad x(t_+) = \lim_{s\downarrow t} x(s) \tag{2.2.4}$$

exist, and the finite Fourier sums $s_M(x)$ of $x(t)$ converge, as $M \to \infty$, to the average value of the signal at the jump:

$$\lim_{M\to\infty} s_M(t) = \frac{x(t_-) + x(t_+)}{2}. \tag{2.2.5}$$

Example 2.2.1. For the signal $x(t)$ in Example 2.1.1, the first three nonzero terms of its cosine expansion were

$$x(t) = \frac{a}{2} + \frac{a}{\pi}\left(2\cos\left(2\pi\frac{t}{P}\right) - \frac{2}{3}\cos\left(2\pi\frac{3t}{P}\right) + \cdots\right).$$

Hence, in the case of period $P = 1$ and amplitude $a = 1$, the first four approximating sums are as follows:

$$s_0(t) = \frac{1}{2}, \qquad s_1(t) = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t,$$
$$s_2(t) = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t, \qquad s_3(t) = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi t - \frac{2}{3\pi}\cos 6\pi t.$$

The graphs of $s_1(t)$ and $s_3(t)$ are compared with the original signal $x(t)$ in Figures 2.2.1–2.2.2. Note the behavior of the Fourier sums at the signal's discontinuities where the Fourier sums converge to the average value of the signal on both sides of the jump according to formula (2.2.5).

Remark. A word of warning is appropriate here. Abandoning the assumptions in the above two theorems leads very quickly to difficulties with approximating the signal by its Fourier series. For example, there are continuous signals for which, at some time instants, their finite

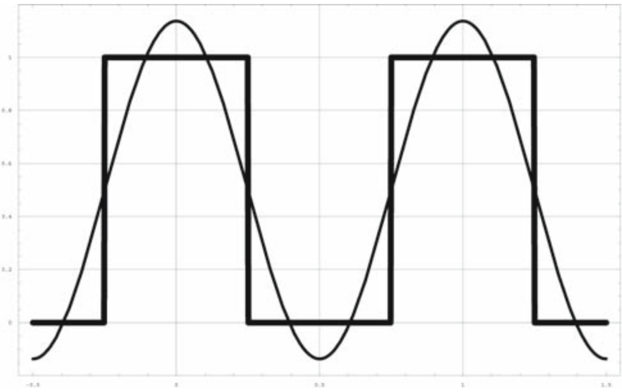

Fig. 2.2.1. Graph of the Fourier sum $s_1(t)$ for the rectangular waveform signal $x(t)$ from Example 2.1.1, plotted against the original signal $x(t)$.

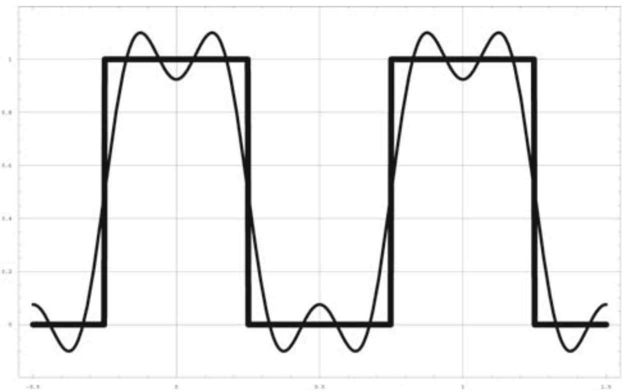

Fig. 2.2.2. Graph of the Fourier sum $s_3(t)$ for the rectangular waveform signal $x(t)$ from Example 2.1.1, plotted against the original signal $x(t)$. Note the behavior of the Fourier sum $s_3(t)$ at the signal's discontinuities, where it matches the average value of the signal at both sides of the jump, reflecting the asymptotics of formula (2.2.5).

Fourier sums diverge to infinity. However, even for them, one can guarantee that the averages of consecutive Fourier sums converge to the signal for each t:

$$\frac{s_0(t) + s_1(t) + \cdots + s_M(t)}{M+1} \to x(t) \quad \text{as } M \to \infty.$$

The expression on the left-hand side of the above formula is called the Mth *Césaro average* of the Fourier series. If one only assumes that the signal $x(t)$ is integrable, that is $\int_0^P |x(t)|dt < \infty$, which is the minimum assumption assuring that the Fourier coefficients $z_m = \langle x, e_m \rangle$ make

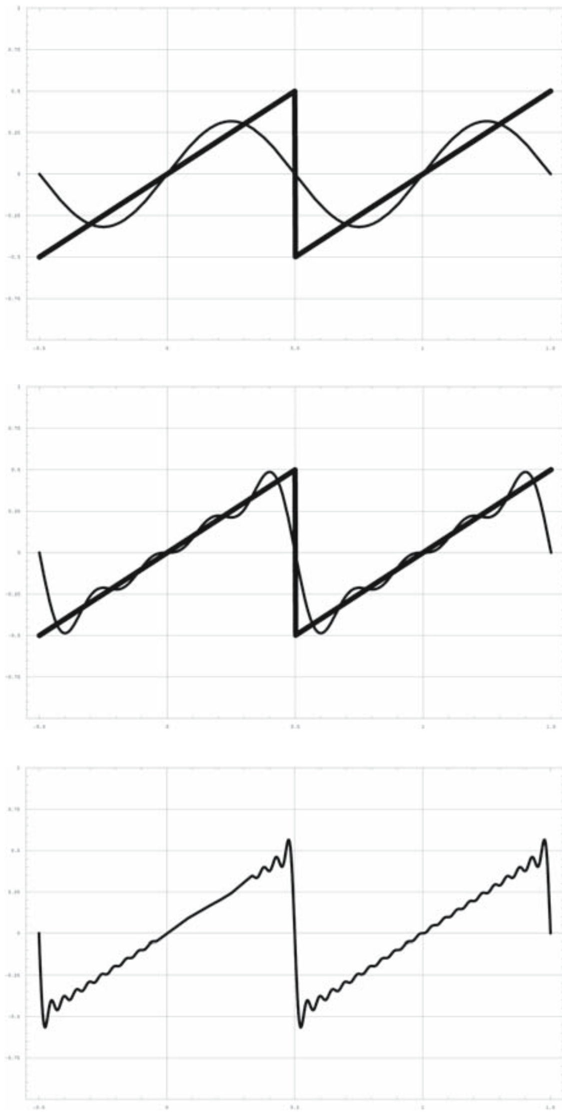

Fig. 2.2.3. Approximation of the periodic signal $x(t)$ from Example 2.2.2 by Fourier sums $s_1(t)$, $s_4(t)$, and $s_{20}(t)$. Visible is the Gibbs phenomenon demonstrating that the shape of the Fourier sum near a point of discontinuity of the signal does not necessarily resemble the shape of the signal itself.

sense, then one can find signals whose Fourier sums diverge to infinity, for all time instants t.

The Gibbs phenomenon. Another observation is that the finite Fourier sums of a signal satisfying the assumptions of the above-quoted statements, despite being convergent to the signal, may have shapes that are very unlike the signal itself.

Example 2.2.2. Consider the signal $x(t)$, with period $P = 1$, defined by the formula

$$x(t) = t \quad \text{for } -\frac{1}{2} \le t < \frac{1}{2}.$$

Clearly, it is an odd signal, so $z_0 = 0$. For $m \neq 0$, integrating by parts,

$$\begin{aligned} z_m &= \int_{-1/2}^{1/2} te^{-j2\pi mt} dt \\ &= t\frac{-1}{j2\pi m} e^{-j2\pi mt} \Big|_{-1/2}^{1/2} - \frac{-1}{j2\pi m} \int_{-1/2}^{1/2} e^{-j2\pi mt} dt \\ &= -\frac{1}{j2\pi m}(-1)^m \end{aligned}$$

because the last integral is zero. The complex Fourier expansion of $x(t)$ is

$$x(t) = \sum_{m=-\infty, m\neq 0}^{\infty} -\frac{1}{j2\pi m}(-1)^m e^{j2\pi mt},$$

which yields a pure sine real Fourier expansion

$$\begin{aligned} x(t) &= \sum_{m=1}^{\infty} \left(-\frac{1}{j2\pi m}(-1)^m e^{j2\pi mt} + -\frac{1}{j2\pi(-m)}(-1)^{-m} e^{j2\pi(-m)t} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\pi m} \sin(2\pi mt). \end{aligned}$$

Figure 2.2.3 shows the approximation of the periodic signal $x(t)$ from Example 2.2.2 by Fourier sums $s_1(t)$, $s_4(t)$, and $s_{20}(t)$. Visible is the so-called *Gibbs phenomenon* demonstrating that the shape of the Fourier sum near a point of discontinuity of the signal does not necessarily resemble the shape of the signal itself. Yet, as the order M of the approximation increases, the oscillations move closer to the jump so that the mean-square convergence of finite Fourier sums to the signal $x(t)$ still obtains.

2.3 Aperiodic signals and Fourier transforms

Periodic signals with increasing period: From Fourier series to Fourier transform. Consider a signal $x_P(t)$ of period P and fundamental frequency $f_0 = \frac{1}{P}$. We already know that such signals can be represented by this Fourier series

$$x_P(t) = \sum_{m=-\infty}^{\infty} \left[\frac{1}{P} \int_{-P/2}^{P/2} x(s) e^{-j2\pi m f_0 s} ds \right] \cdot e^{j2\pi m f_0 t}. \tag{2.3.1}$$

Notice that, for the purposes of this section, we have written the formula for the Fourier coefficients of $x_P(t)$ as an integral over a symmetric interval $(-\frac{P}{2}, \frac{P}{2}]$ rather than the usual interval of periodicity $(0, P]$. Since both the signal $x_P(t)$ and complex exponentials

$$\exp(-j2\pi m f_0 s) = \cos(2\pi m f_0 s) + j\sin(2\pi m f_0 s)$$

are periodic with period P, any interval of length P will do.

Instead of considering aperiodic signals right off the bat, we will make a gradual transition from the analysis of periodic to aperiodic signals by considering what happens with the Fourier series if in the above representation (2.3.1) period P increases to ∞; the limit case of infinite period $P = \infty$ would then correspond to the case of an aperiodic signal.

To see the limit behavior of the Fourier series (2.3.1), we shall introduce the following notation:

(1) The multiplicities of the fundamental frequency will become a running discrete variable f_m:

$$f_m = m \cdot f_0;$$

(2) The increments of the new running variable will be denoted by

$$\Delta f_m = f_m - f_{m-1} = f_0 = \frac{1}{P}.$$

In this notation the Fourier expansion (2.2.1) can be rewritten in the form

$$x_P(t) = \sum_{m=-\infty}^{\infty} \left[\int_{-P/2}^{P/2} x(s) e^{-j2\pi f_m s} ds \right] e^{j2\pi f_m t} \Delta f_m \tag{2.3.2}$$

because $\Delta f_m = f_0 = \frac{1}{P}$. Now, if the period $P \to \infty$, which is the same as assuming that the fundamental frequency $f_0 = \Delta f_m \to 0$, the sum on the right-hand side of the formula (2.3.2) converges to the integral so that our Fourier representation (2.3.2) of a periodic signal $x_P(t)$ becomes the following integral identity for the aperiodic signal:

$$x_\infty(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_\infty(s) e^{-j2\pi f s} ds \right] e^{j2\pi f t} df. \tag{2.3.3}$$

The inner transformation

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \tag{2.3.4}$$

is called the *Fourier transform* of signal $x(t)$, and the outer transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{2.3.5}$$

is called the *inverse Fourier transform* of (complex in general) function $X(f)$. The variable in the Fourier transform is the frequency f.

Note that since $|e^{-j2\pi ft}| = 1$, the necessary condition for the existence of the Fourier transform in the usual sense is the absolute integrability of the signal:

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty. \tag{2.3.6}$$

Later on we will try to extend its definition to some important nonintegrable signals.

Example 2.3.1. Let us trace the above limit procedure in the case of an aperiodic signal $x_\infty(t) = e^{-|t|}$. If this signal is approximated by periodic signals with period P obtained by truncating $x(t)$ to the interval $[-\frac{P}{2}, \frac{P}{2})$ and extending it periodically, i.e.,

$$x_P(t) = e^{-|t|} \quad \text{for } -\frac{P}{2} \le t < \frac{P}{2},$$

then the Fourier coefficients of the latter are, remembering that $P = \frac{1}{f_0}$,

$$\begin{aligned} z_{m,P} &= \frac{1}{P}\int_{-P/2}^{P/2} e^{-|t|}e^{-j2\pi mt/P}dt \\ &= \frac{2f_0}{1+(2\pi mf_0)^2}(1 - e^{-1/(2f_0)}(\cos(2\pi mf_0) \\ &\qquad + 2\pi mf_0\sin(2\pi mf_0))). \end{aligned}$$

Since the original periodic signal $x_P(t)$ was even, the Fourier coefficients $z_m = z_{-m}$, so that the discrete spectrum of $x_P(t)$ is symmetric. Now, as $P \to \infty$, that is $f_0 = \frac{1}{P} \to 0$, the exponentional term $e^{-1/(2f_0)} \to 0$, and with $f_0 = \Delta f$, $mf_0 = f$, we get that

$$z_{m,P} \to \frac{2}{1+(2\pi f)^2}df.$$

Thus the Fourier transform of the aperiodic signal $x_\infty(t)$ is

$$X_\infty(f) = \frac{2}{1+(2\pi f)^2}.$$

Taking the inverse Fourier transform, we verify[5] that

$$\int_{-\infty}^{\infty} \frac{2}{1+(2\pi f)^2}e^{j2\pi ft}df = e^{-|t|}.$$

[5] When faced with integrals of this sort, the reader is advised to consult a book of integrals, or a computer package such as *Mathematica* or MAPLE.

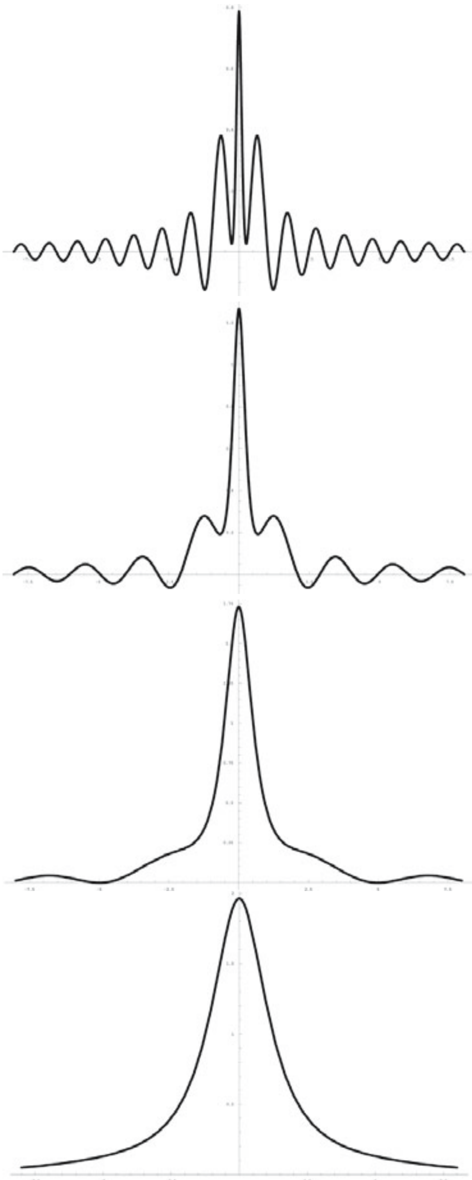

Fig. 2.3.1. Adjusted Fourier coefficients $Z_{m,P} \cdot P$, shown as functions of continuous parameter m for graphical convenience, approach the Fourier transform $X_\infty(f)$ of the aperiodic signal $x_\infty(t) = e^{-|t|}$. The values of P, from top to bottom, are 1, 2, 4, 8.

Figure 2.3.1 illustrates the convergence, as period P increases, of Fourier coefficients $z_{m,P}$ to the Fourier transform $X_\infty(f)$.

2.4 Basic properties of the Fourier transform

The property that makes the Fourier transform of signals so useful is its *linearity*, that is the Fourier transform of a linear composition $\alpha x(t) + \beta y(t)$ of signals $x(t)$ and $y(t)$ is the same linear composition $\alpha X(f) + \beta Y(f)$ of their Fourier transforms. To facilitate notation we will often denote the fact that $X(f)$ is the Fourier transform of signal $x(t)$ by writing $x(t) \mapsto X(f)$. So

$$\alpha x(t) + \beta y(t) \longmapsto \alpha X(f) + \beta Y(f). \tag{2.4.1}$$

The proof is instantaneous using linearity of the integral.

The familiar Parseval formula for periodic signals carries over in the form

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df. \tag{2.4.2}$$

That is, the total energy of the signal can be calculated as the integral of the square of the modulus of its Fourier transform. An observant reader will see immediately that integrability of the signal necessary to define the Fourier transform is not sufficient for the validity of the Parseval formula (2.4.2) as the finiteness of the integral $\int_{-\infty}^{\infty} |x(t)|dt$ does not imply that the signal has finite energy E_x.

Parseval's formula also has the following useful extension:

$$\int_{-\infty}^{\infty} x(t) \cdot y(t)dt = \int_{-\infty}^{\infty} X(f) \cdot Y^*(f)df. \tag{2.4.3}$$

In the context of transmission of signals through linear systems the critical property of the Fourier transform is that the *convolution* $[x * y](t)$ of signals $x(t)$ and $y(t)$,

$$[x * y](t) = \int_{-\infty}^{\infty} x(s)y(t-s)ds, \tag{2.4.4}$$

a fairly complex operation, has the Fourier transform that is simply the product of the corresponding Fourier transforms

$$[x * y](t) \longmapsto X(f) \cdot Y(f). \tag{2.4.5}$$

Indeed,

Table 2.4.1. Fourier transform properties.

Signal		Fourier Transform
Linearity		
$\alpha x(t)+\beta y(t)$	$\longmapsto$	$\alpha X(f)+\beta Y(f)$
Convolution		
$[x*y](t)$	$\longmapsto$	$X(f)\cdot Y(f)$
Differentiation		
$x^{(n)}(t)$	$\longmapsto$	$(j2\pi f)^n X(f)$
Time reversal		
$x(-t)$	$\longmapsto$	$X(-f)$
Time delay		
$x(t-t_0)$	$\longmapsto$	$X(f)\cdot e^{-j2\pi t_0 f}$
Frequency translation		
$x(t)\cdot e^{j2\pi f_0 t}$	$\longmapsto$	$X(f-f_0)$
Frequency differentiation		
$(-j)^n t^n x(t)$	$\longmapsto$	$(2\pi)^{-1}X^{(n)}(f)$
Frequency convolution		
$x(t)y(t)$	$\longmapsto$	$[X*Y](f)$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}[x*y](t)e^{-j2\pi ft}dt\\
&\quad=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(s)y(t-s)ds\right]e^{-j2\pi ft}dt\\
&\quad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y(t-s)e^{-j2\pi f(t-s)}x(s)e^{-j2\pi fs}dsdt\\
&\quad=\int_{-\infty}^{\infty}y(u)e^{-j2\pi fu}du\cdot\int_{-\infty}^{\infty}x(s)e^{-j2\pi fs}ds=X(f)\cdot Y(f),
\end{aligned}
$$

where the penultimate equality resulted from the substitution $t-s=u$.

Since many electrical circuits are described by differential equations, the behavior of the Fourier transform under differentiation of the signal is another important issue. Here the calculation is also direct:

$$\begin{aligned}\int_{-\infty}^{\infty} x'(t)e^{-j2\pi ft}dt &= x(t)e^{-j2\pi ft}|_{-\infty}^{\infty} + j2\pi f\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= 0 + j2\pi f X_Z(f).\end{aligned}$$

The first term is 0 because the signal's absolute integrability (remember, we have to assume it to guarantee the existence of the Fourier transform) implies that $x(\infty) = x(-\infty) = 0$. Thus we have a rule

$$x'(t) \longmapsto (j2\pi f)\cdot X(f). \tag{2.4.6}$$

The above and other, simple-to-derive rules are summarized in Table 2.4.1.

Example 2.4.1. Consider the signal $x(t) = e^{-\pi t^2}$, which has the familiar bell shape. Its Fourier transform is

$$X(f) = \int_{-\infty}^{\infty} e^{-\pi t^2 - j2\pi ft}dt = \int_{-\infty}^{\infty} e^{-\pi(t+jf)^2}e^{-\pi f^2}dt = e^{-\pi f^2},$$

because $\int_{-\infty}^{\infty} e^{-\pi(t+jf)^2}dt = \int_{-\infty}^{\infty} e^{-\pi t^2}dt = 1$. Indeed, changing to polar coordinates r, θ, we can evaluate easily that

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\pi t^2}dt\right)^2 &= \int_{-\infty}^{\infty} e^{-\pi t^2}dt\cdot\int_{-\infty}^{\infty} e^{-\pi s^2}ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(t^2+s^2)}dt ds = \int_0^{2\pi} d\theta\int_0^{\infty} e^{-\pi r^2}r dr = 1.\end{aligned}$$

Thus the signal $x(t) = e^{-\pi t^2}$ has the remarkable property of having the Fourier transform of exactly the same functional shape. This fact has profound consequences in mathematical physics and Fourier analysis.

2.5 Fourier transforms of some nonintegrable signals; Dirac delta impulse

There exist important nonintegrable signals, such as $x(t) =$ constant, or $x(t) = \cos t$ that are not absolutely integrable over the whole timeline, and the usual calculus does not permit us to define their Fourier transforms. However, to cover these and other important cases, one can extend the standard calculus by introduction of the so-called *Dirac delta "function"* $\delta(f)$ which is an infinitely high but infinitely narrow spike located at $f = 0$ which, very importantly, has the "area," that is the "integral," equal to 1.[6]

Intuitively, but one can also make this approach rigorous, the best way to think about the Dirac delta is as a limit

$$\delta(f) = \lim_{\epsilon \to 0} r_\epsilon(f), \tag{2.5.1}$$

where

$$r_\epsilon(f) = \begin{cases} \frac{1}{2\epsilon} & \text{for } -\epsilon \le f \le +\epsilon; \\ 0 & \text{elsewhere} \end{cases}$$

is a family, indexed by ϵ, of rectangular functions all of which have area 1 underneath; see Figure 2.5.1.

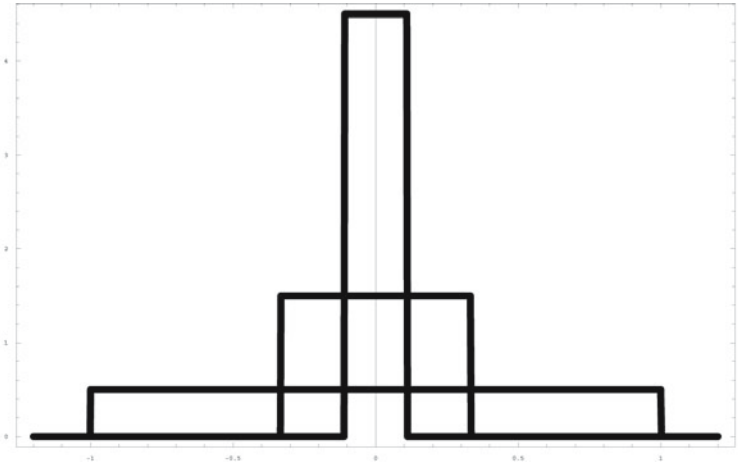

Fig. 2.5.1. Approximation of the Dirac delta $\delta(f)$ by rectangular functions $r_\epsilon(f)$ for $\epsilon = 1, \frac{1}{3}$, and $\frac{1}{9}$.

Obviously the choice of the rectangular functions is not unique here. Any sequence of nonnegative functions which integrate to 1 over the whole real line and converge to zero pointwise at every point different from the origin would do. For example, as approximations to the Dirac delta we can also take the family of double-sided exponential functions of variable x,

$$\frac{1}{2a} \exp\left(\frac{|f|}{a}\right),$$

indexed by parameter $a \to 0+$. Three functions of this family, for parameter values $a = 1, \frac{1}{3}, \frac{1}{9}$, are pictured in Figure 2.5.2.

[6] Of course, one can similarly introduce the time domain Dirac delta $\delta(t)$, in which case it will be called the *Dirac delta impulse*.

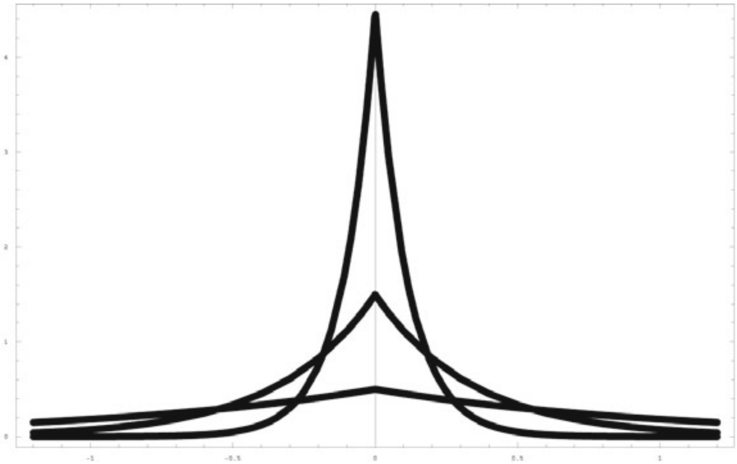

Fig. 2.5.2. Approximation of the Dirac delta $\delta(f)$ by two-sided exponential functions $(\frac{1}{2a})\exp(-\frac{|f|}{a})$ for $a = 1, \frac{1}{3}$, and $\frac{1}{9}$.

The Dirac delta is characterized by its "probing property" (also known as the "sifting property"):

$$\int_{-\infty}^{\infty} \delta(f)X(f)df = X(0); \tag{2.5.2}$$

integrating a function against the Dirac delta produces a value of the function at $f = 0$. Operationally, all we need is the formula (2.5.2), which can actually be taken as a formal definition of the Dirac delta.

The "probing" formula (2.5.2) can be justified by remembering our intuitive definition (2.5.1): Indeed, if function $X(f)$ is regular enough, then

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(f)X(f)df &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} r_\epsilon(f)X(f)df\\ &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} X(f)df = X(0)\end{aligned}$$

in view of the fundamental theorem of calculus.

Other properties of the Dirac delta follow immediately:

$$\int_{-\infty}^{\infty} \delta(f-f_0)X(f)df = X(f_0), \tag{2.5.3}$$

$$\int_{-\epsilon}^{\epsilon} \delta(f)df = 1, \tag{2.5.4}$$

and

$$\int_{-\infty}^{\infty} \delta(f)X(f)df = 0 \quad \text{if } X(0) = 0, \tag{2.5.5}$$

The last property is often intuitively stated as

$$\delta(f) = 0 \quad \text{for } f \neq 0. \tag{2.5.6}$$

Equipped with the Dirac delta technique, we can immediately obtain the Fourier transform of some nonintegrable signals.

Example 2.5.1. Finding the Fourier transform of the harmonic oscillation signal $x(t) = e^{j2\pi f_0 t}$ is impossible by direct integration of

$$\int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{-j2\pi f t} dt.$$

But one immediately notices that the inverse transform of the shifted Dirac delta is, by (2.5.2),

$$\int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} df = e^{j2\pi f_0 t}.$$

Thus the Fourier transform of $x(t) = e^{j2\pi f_0 t}$ is $\delta(f - f_0)$. In particular, the Fourier transform of a constant 1 is $\delta(f)$ itself.

Table 2.5.1 lists Fourier transforms of some common signals. Here and subsequently, $u(t)$ denotes Heaviside's unit step function, equal to 0 for $t < 0$ and 1 for $t \geq 0$.

Example 2.5.2. The Fourier transform of the signal $x(t) = \cos 2\pi t$ can be found in a similar fashion, as direct integration of

$$\int_{-\infty}^{\infty} \cos(2\pi t) e^{-j2\pi f t} dt$$

is impossible. But one observes that the inverse transform

$$\int_{-\infty}^{\infty} \frac{1}{2}(\delta(f-1) + \delta(f+1)) e^{j2\pi f t} df = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} = \cos 2\pi t,$$

so the Fourier transform of $\cos 2\pi t$ is $\frac{\delta(f-1)+\delta(f+1)}{2}$.

A sample of the calculus of Dirac delta "functions." There exists a large theory of Dirac delta "functions," and of similar mathematical objects called distributions (in the sense of Schwartz),[7] which develops

[7] For a more complete exposition of the theory and applications of the Dirac delta and related "distributions," see A. I. Saichev and W. A. Woyczyński, *Distributions in the Physical and Engineering Sciences, Vol.* 1: *Distributional Calculus, Integral Transforms, and Wavelets*, Birkhäuser Boston, Cambridge, MA, 1998.

Table 2.5.1. Common Fourier transforms.

Signal		Fourier Transform
$e^{-a\|t\|}$	$\longmapsto$	$\dfrac{2a}{a^2+(2\pi f)^2}, \quad a>0$
$e^{-\pi t^2}$	$\longmapsto$	$e^{-\pi f^2}$
$\begin{cases} 1 & \text{for } \|t\| \le \frac{1}{2}; \\ 0 & \text{for } \|t\| > \frac{1}{2}. \end{cases}$	$\longmapsto$	$\dfrac{\sin \pi f}{\pi f}$
$\begin{cases} 1-\|t\| & \text{for } \|t\| \le 1; \\ 0 & \text{for } \|t\| > 1. \end{cases}$	$\longmapsto$	$\dfrac{\sin^2 \pi f}{\pi^2 f^2}$
$e^{j2\pi f_0 t}$	$\longmapsto$	$\delta(f-f_0)$
$\delta(t)$	$\longmapsto$	1
$\cos 2\pi f_0 t$	$\longmapsto$	$\dfrac{\delta(f+f_0)+\delta(f-f_0)}{2}$
$\sin 2\pi f_0 t$	$\longmapsto$	$\dfrac{j\delta(f+f_0)-\delta(f-f_0)}{2}$
$u(t) = \begin{cases} 0 & \text{for } t<0; \\ 1 & \text{for } t \ge 0. \end{cases}$	$\longmapsto$	$\frac{1}{2}\delta(f) + \dfrac{1}{j2\pi f}$
$t \cdot u(t)$	$\longmapsto$	$\dfrac{j}{4\pi}\delta'(f) - \dfrac{1}{4\pi^2 f^2}$
$e^{-at} \cdot u(t)$	$\longmapsto$	$\dfrac{1}{a+j2\pi f}, \quad a>0$

tools that help carry out operations such as distributional differentiation, distributional multiplication, etc. To give the reader a little taste of it let us start here with the classical integration-by-parts formula which, for usual, vanishing at $f = \pm\infty$ functions $X(f)$ and $Y(f)$, states that

$$\int_{-\infty}^{\infty} X(f) \cdot Y'(f)df = -\int_{-\infty}^{\infty} X'(f) \cdot Y(f)df. \tag{2.5.7}$$

This identity, applied formally, can be used as the *definition* of the derivative $\delta'(f)$ of the Dirac delta by assigning to it the following probing property:

$$\int_{-\infty}^{\infty} X(f) \cdot \delta'(f)df = -\int_{-\infty}^{\infty} X'(f) \cdot \delta(f)df = -X'(0). \tag{2.5.8}$$

Symbolically, we can write

$$X(f) \cdot \delta'(f) = -X'(f) \cdot \delta(f).$$

In the particular case $X(f) = f$ (here, the function has to be thought of as a limit of functions vanishing at $\pm\infty$), we get

$$f \cdot \delta(f) = -\delta(f),$$

a useful computational formula which can be employed, for example, to justify the next to the last entry in the above table of common Fourier transforms.

2.6 Discrete and fast Fourier transforms

In practice, for many signals we only obtain the value of the signal at discrete times, but we can imagine that the signal continues between these times. Thus we can approximate the integrals involved in calculation of the Fourier transforms in the same way as one does in numerical integration in calculus, using left-handed rectangles, trapezoids, Simpson's rule, etc. We use the simplest approximation, which is equivalent to assuming that the signal is constant between the times at which we sample (and rectangular approximations of the area under the function).

Therefore, suppose that the sampling period is T_s, with the sampling frequency $f_s = \frac{1}{T_s}$, so that the signal's sample is given in the form of a sequence

$$x_k = x(kT_s), \quad k = 0, 1, 2, \ldots, N-1, \tag{2.6.1}$$

and we interpret it as a periodic signal with period

$$P = \frac{1}{f_0} = NT_s = \frac{N}{f_s}, \tag{2.6.2}$$

The integral in formula (2.3.1) approximating the Fourier transform of the signal $x(t)$ at discrete frequencies mf_0, $m = 0, 1, 2, \ldots, N-1$, can now, in turn, be approximated by the sum

$$\begin{aligned} X_m = X(mf_0) &= \frac{1}{P} \sum_{k=0}^{N-1} x(kT_s) e^{-j2\pi m f_0 k T_s} \cdot T_s \\ &= \frac{1}{N} \sum_{k=0}^{N-1} x_k e^{-j2\pi mk/N} \end{aligned} \tag{2.6.3}$$

in view of the relationships (2.6.2). The sequence

$$X_m, \quad m = 0, 1, 2, \ldots, N-1, \tag{2.6.4}$$

is traditionally called the *discrete Fourier transform* (*DFT*) of the signal sample x_k, $k = 0, 1, 2, \ldots, N-1$, described in (2.6.1).

Note that the calculation of the DFT via formula (2.6.3) calls for N^2 multiplications $x_k \cdot e^{-j2\pi mk/N}$, $m, k = 0, 1, 2, \ldots, N-1$. One often says that the formula's *computational* (*algorithmic*) *complexity* is of the order N^2. This computational complexity, however, can be dramatically reduced by cleverly grouping terms in the sum (2.6.3). The technique, which usually is called the *Fast Fourier Transform* (*FFT*), was known to Carl Friedrich Gauss at the beginning of the 19th century, but was rediscovered and popularized by Cooley and Tukey in 1965.[8] We will explain it in the special case when the signal's sample size is a power of 2.

So assume that $N = 2^n$, and let $\omega_N = e^{-j2\pi/N}$. It is called a complex Nth root of unity because $\omega_N^N = 1$. Obviously, for $M = \frac{N}{2}$, we have

$$\omega_{2M}^{(2k)m} = \omega_M^{km}, \qquad \omega_M^{M+m} = \omega_M^m, \quad \text{and} \quad \omega_{2M}^{M+m} = -\omega_{2M}^m. \tag{2.6.5}$$

The crucial observation is to recognize that the sum (2.6.3) can be split into two pieces

$$X_m = \frac{1}{2}(X_m^{\text{even}} + X_m^{\text{odd}} \cdot \omega_{2M}^m), \tag{2.6.6}$$

where

$$X_m^{\text{even}} = \frac{1}{M}\sum_{k=0}^{M-1} x_{2k}\omega_M^{km} \quad \text{and} \quad X_m^{\text{odd}} = \frac{1}{M}\sum_{k=0}^{M-1} x_{2k+1}\omega_M^{km}, \tag{2.6.7}$$

and that, in view of (2.6.5),

$$X_{m+M} = \frac{1}{2}(X_m^{\text{even}} - X_m^{\text{odd}} \cdot \omega_{2M}^m). \tag{2.6.8}$$

As a result, only values X_m, $m = 0, 1, 2, \ldots, M-1 = \frac{N}{2-1}$, have to be calculated by laborious multiplication. The values X_m, $m = M, M+1, \ldots, 2M-1 = N-1$, are simply obtained by formula (2.6.8). The above trick is then repeated at levels $\frac{N}{2^2}, \frac{N}{2^3}, \ldots, 2$. If we denote by $\mathrm{CC}(n)$ the *computational complexity* of the above scheme, that is the number of multiplications required, we see that

$$\mathrm{CC}(n) = 2\,\mathrm{CC}(n-1) + 2^{n-1},$$

with the first term on the right being the result of halving the size of the sample at each step, and the second term resulting from multiplications of X_m^{odd} by ω_{2M}^m in (2.6.6) and (2.6.8). Iterating the above recursive relation, one obtains that

[8] J. W. Cooley and O. W. Tukey, An algorithm for the machine calculation of complex Fourier series, *Math. Comput.*, **19** (1965), 297–301.

$$CC(n) = 2^{n-1} \log_2 2^n = \frac{1}{2} N \log_2 N, \tag{2.6.9}$$

a major improvement over the N^2 order of the computational complexity of the straightforward calculation of DFT.

2.7 Problems and exercises

2.7.1. Prove that the system of real harmonic oscillations

$$\sin(2\pi m f_0 t), \qquad \cos(2\pi m f_0 t), \quad m = 0, 1, 2, \dots,$$

form an orthogonal system. Is the system normalized? Use the above information to derive formulas for coefficients a_m, b_m, in the expansion (1.2.4). Model this derivation on (2.1.4).

2.7.2. Using the results from Problem 2.7.1, find formulas for amplitudes c_m and phases θ_m in the expansion (1.2.1).

2.7.3. Find a general formula for the coefficients c_m in the cosine Fourier expansion for the even rectangular waveform $x(t)$ from Example 2.1.1.

2.7.4. Find a general formula for the coefficients b_m in the sine Fourier expansion for the odd rectangular waveform $x(t)$ from Example 2.1.2.

2.7.5. Carry out calculations of Example 2.1.3 in the case of arbitrary period P and amplitude a.

2.7.6. Find three consecutive approximations by finite Fourier sums of the signal $x(t)$ from Example 2.1.3. Graph them and compare the graphs with the graph of the original signal.

2.7.7. Find the complex and real Fourier series for the periodic signal with period P defined by the formula

$$x(t) = \begin{cases} a & \text{for } 0 \le t < \frac{P}{2}; \\ -a & \text{for } \frac{P}{2} \le t < P. \end{cases}$$

In the case $P = \pi$ and $a = 2.5$ produce graphs comparing the signal $x(t)$ and its finite Fourier sums of order 1, 3, and 6.

2.7.8. Find the complex and real Fourier series for the periodic signal with period $P = 1$ defined by the formula

$$x(t) = \begin{cases} 1 - \frac{t}{2} & \text{for } 0 \le t < \frac{1}{2}; \\ 0 & \text{for } \frac{1}{2} \le t < 1. \end{cases}$$

Produce graphs comparing the signal $x(t)$ and its finite Fourier sums of order 1, 3, and 6.

2.7.9. Find the complex and real Fourier series for the periodic signal $x(t) = |\sin t|$. Produce graphs comparing the signal $x(t)$ and its finite Fourier sums of order 1, 3, and 6.

2.7.10. Find the complex and real Fourier series for the periodic signal with period $P = \pi$ defined by the formula

$$x(t) = e^t \quad \text{for } -\frac{\pi}{2} < t \le \frac{\pi}{2}.$$

Produce graphs comparing the signal $x(t)$ and its finite Fourier sums of order 1, 3, and 6.

2.7.11. Find an example of a signal $x(t)$ that is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x(t)|dt < \infty$ but has infinite energy $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$, and conversely, find an example of a signal which has finite energy but is not absolutely integrable.

2.7.12. Provide a detailed verification of Fourier transform properties listed in Table 2.4.1.

2.7.13. Provide a detailed verification of the Fourier transforms table (Table 2.5.1). Utilize the fact that the derivative $\delta'(f)$ of the Dirac delta impulse $\delta(f)$ is defined by the integration-by-parts formula

$$\int_{-\infty}^{\infty} \delta'(f)X(f)df = -\int_{-\infty}^{\infty} \delta(f)X'(f)df$$

for any smooth function $X(f)$.

2.7.14. Find the Fourier transform of the periodic signal

$$x(t) = \sum_{m=-\infty}^{\infty} z_m e^{j2\pi m f_0 t}.$$

2.7.15. Find the Fourier transform of the signal $x(t) = tu(t)$, where $u(t)$ is the unit step function equal to 0 for $t < 0$ and 1 for $t \ge 0$.

2.7.16. Find the Fourier transform of the signals given below. Graph both the signal and its Fourier transform:

(a) $$x(t) = \frac{1}{1+t^2}, \quad -\infty < t < \infty,$$

(b) $$e^{-t^2/2}, \quad -\infty < t < \infty,$$

(c) $$x(t) = \begin{cases} \sin t \cdot e^{-t} & \text{for } t \ge 0; \\ 0 & \text{for } t < 0. \end{cases}$$

(d) $x(t) = y * z(t), \qquad y(t) = u(t) - u(t-1), \qquad z(t) = e^{-|t|},$

where $u(t)$ is the unit step signal = 0 for negative t and = 1 for $t \ge 0$.

2.7.17. Find the convolution $(x * x)(t)$ if $x(t) = u(t) - u(t-1)$, where $u(t)$ is the unit step function. First, use the original definition of the convolution and then verify your result using the Fourier transform method.

2.7.18. Utilize the Fourier transform (in the space variable z) to find a solution of the diffusion (heat) partial differential equation

$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial z^2},$$

for a function $u(t, z)$ satisfying the initial condition $u(0, z) = \delta(z)$. The solution of the above equation is often used to describe the temporal evolution of the density of a diffusing substance.[9]

2.7.19. Assuming the validity of the Parseval formula $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$, prove its extended version $\int_{-\infty}^{\infty} x(t) \cdot y^*(t) dt = \int_{-\infty}^{\infty} X(f) \cdot Y^*(f) df$. *Hint*: In the case of real-valued $x(t)$, $y(t)$, $X(f)$, and $Y(f)$, it suffices to utilize the obvious identity $4xy = (x+y)^2 - (x-y)^2$, but in the general, complex case, first verify, and then apply the following *polarization identity*:

$$4xy = |x + y|^2 - |x - y|^2 + j(|x + jy|^2 - |x - jy|^2).$$

Remember that the modulus square $|z|^2 = zz^*$.

[9] It was the search for solutions to this problem that induced Jean-Baptiste Fourier (born March 21, 1768, in Auxerre, France; died May 16, 1830, in Paris) to introduce in his treatise *Théorie analytique de la chaleur* (*The Analytical Theory of Heat*; 1822), the tools of infinite functional series and integral transforms now known under the names of Fourier series and transforms. Fourier was also known as an Egyptologist and administrator. The modern author of research papers, impatient with delays in publication of his/her work, should find solace in the fact that the appearance of Fourier's great memoir was held up by the referees for 15 years; it was first presented to the Institut de France on December 21, 1807.