1

Description of Signals

Signals are everywhere, literally. The universe is bathed in the background radiation, the remnant of the original Big Bang, and as your eyes scan this page, a signal is being transmitted to your brain where different sets of neurons analyze it and process it. All human activities are based on processing and analysis of sensory signals, but the goal of this book is somewhat narrower. The signals we will be mainly interested in can be described as *data* resulting from quantitative measurements of some physical phenomena, and our emphasis will be on data that display *randomness* that may be due to different causes, such as errors of measurements, algorithmic complexity, or the chaotic behavior of the underlying physical system itself.

1.1 Types of random signals

For the purposes of this book, signals will be functions of real variable t interpreted as time. To describe and analyze signals we will adopt the functional notation: $x(t)$ will denote the value of a nonrandom signal at time t. The values themselves can be real or complex numbers, in which case we will symbolically write $x(t) \in \mathbf{R}$, or, respectively, $x(t) \in \mathbf{C}$. In certain situations it is necessary to consider vector-valued signals with $x(t) \in \mathbf{R}^d$, where d stands for the dimension of the vector $x(t)$ with d real components.

Signals can be classified into different categories depending on their features. For example, there are the following:

- *Analog signals* are functions of continuous time and their values form a continuum. *Digital signals* are functions of discrete time dictated by the computer's clock, and their values are also discrete and dictated by the resolution of the system. Of course, one can also encounter mixed-type signals which are sampled at discrete times but whose values are not restricted to any discrete set of numbers.

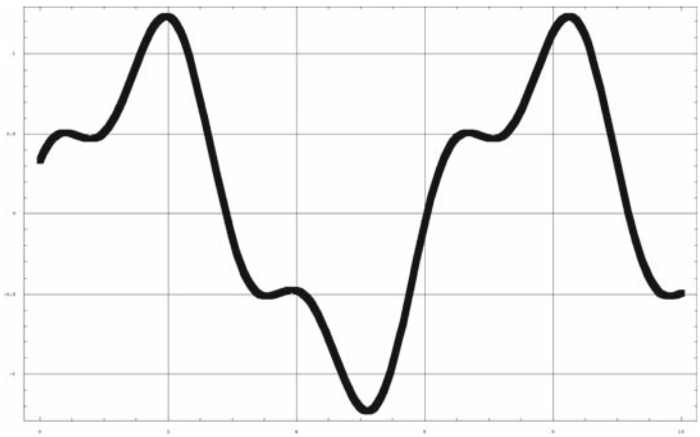

Fig. 1.1.1. Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] is analog and periodic with period $P = 2\pi$ [s]. It is also deterministic.

- *Periodic signals* are functions whose values are periodically repeated. In other words, for a certain number $P > 0$, we have $x(t + P) = x(t)$ for any t. The number P is called the *period of the signal*. *Aperiodic signals* are signals that are not periodic.
- *Deterministic signals* are signals not affected by random noise; there is no uncertainty about their values. *Stochastic* or *random signals* include an element of uncertainty; their analysis requires use of statistical tools, and providing such tools is the principal goal of this book.

For example, signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] shown in Figure 1.1.1 is deterministic, analog, and periodic with period $P = 2\pi$ [s]. The same signal, digitally sampled during the first five seconds at time intervals equal to 0.5 s, with resolution 0.01 V, gives tabulated values:

t	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$x(t)$	0.50	0.51	0.93	1.23	0.71	−0.16	0.51	−0.48	−0.78	−1.21

This sampling process is called the *analog-to-digital conversion*: given the *sampling period* T and the *resolution* R, the digitized signal $x_d(t)$ is of the form

$$x_d(t) = R\left\lfloor \frac{x(t)}{R} \right\rfloor \quad \text{for } t = T, 2T, \ldots, \tag{1.1.1}$$

where the (convenient to introduce here) "floor" function $\lfloor a \rfloor$ is defined as the largest integer not exceeding real number a. For example, $\lfloor 5.7 \rfloor = 5$, but $\lfloor 5.0 \rfloor = 5$ as well.

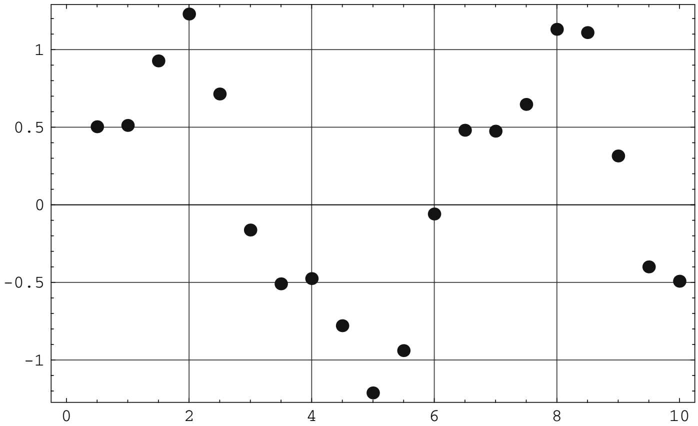

Fig. 1.1.2. Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] digitally sampled at time intervals equal to 0.5 s with resolution 0.01 V.

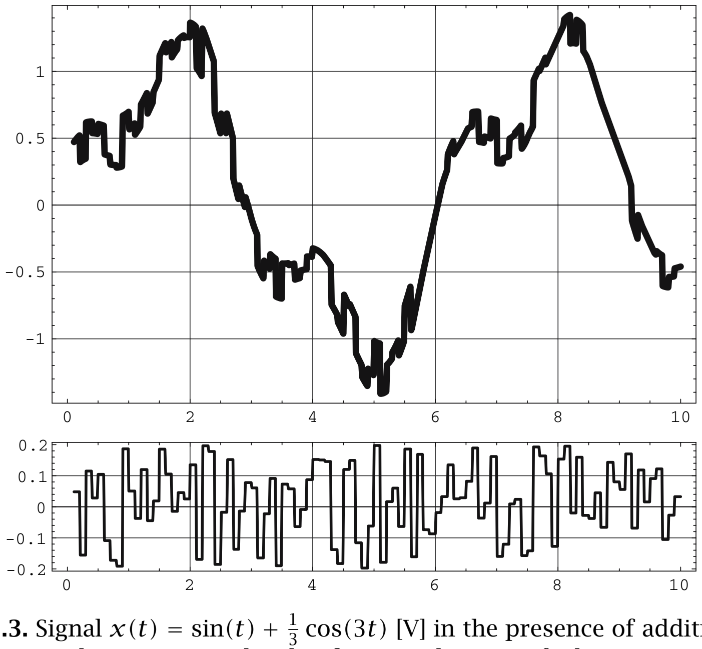

Fig. 1.1.3. Signal $x(t) = \sin(t) + \frac{1}{3}\cos(3t)$ [V] in the presence of additive random noise with average amplitude of 0.2 V. The magnified noise component itself is pictured underneath the graph of the signal.

Note the role the resolution R plays in the above formula. Take, for example, $R = 0.01$. If the signal $x(t)$ takes all the continuum of values between $m = \min_t x(t)$ and $M = \max_t x(t)$, then $\frac{x(t)}{0.01}$ takes all the continuum of values between $100m$ and $100M$, but $\lfloor \frac{x(t)}{0.01} \rfloor$ takes

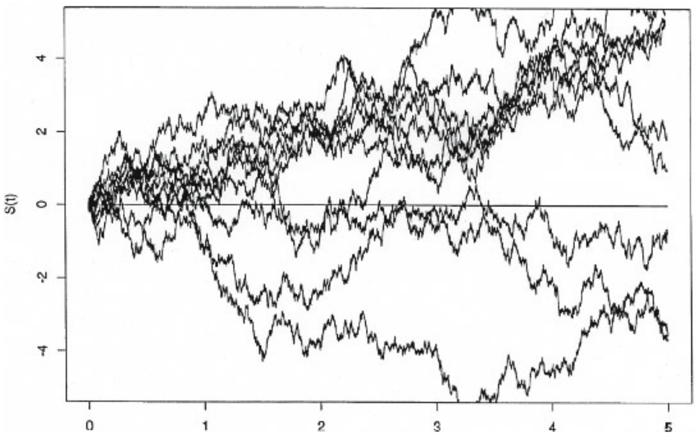

Fig. 1.1.4. Several computer-generated trajectories (sample paths) of a random signal called the *Brownian motion* stochastic process or the *Wiener stochastic process*. Its trajectories, although very rough, are continuous. It is often used as a simple model of *diffusion*. The random mechanism that created different trajectories was the same.

only integer values between $100m$ and $100M$. Finally, $0.01\lfloor \frac{x(t)}{0.01} \rfloor$ takes as its values only all the discrete numbers between m and M that are 0.01 apart.

Randomness of signals can have different origins, such as the quantum *uncertainty principle*, the *computational complexity* of algorithms, the *chaotic behavior* in dynamical systems, or the random fluctuations and errors in measurement of outcomes of independently repeated experiments.[1] The usual way to study them is via their aggregated statistical properties. The main purpose of this book is to introduce some of the basic mathematical and statistical tools useful in the analysis of random signals that are produced under *stationary conditions*, that is, in situations where the measured signal may be stochastic and contain random fluctuations, but the basic underlying random mechanism producing it does not change over time; think here about outcomes of independently repeated experiments, each consisting of tossing a single coin.

At this point, to help the reader visualize the great variety of random signals appearing in the physical sciences and engineering, it is worthwhile to review a gallery of pictures of random signals, both experimental and simulated, presented in Figures 1.1.4–1.1.8. The captions explain the context in each case.

[1] See, e.g., M. Denker and W. A. Woyczyński, *Introductory Statistics and Random Phenomena: Uncertainty, Complexity, and Chaotic Behavior in Engineering and Science*, Birkhäuser Boston, Cambridge, MA, 1998.

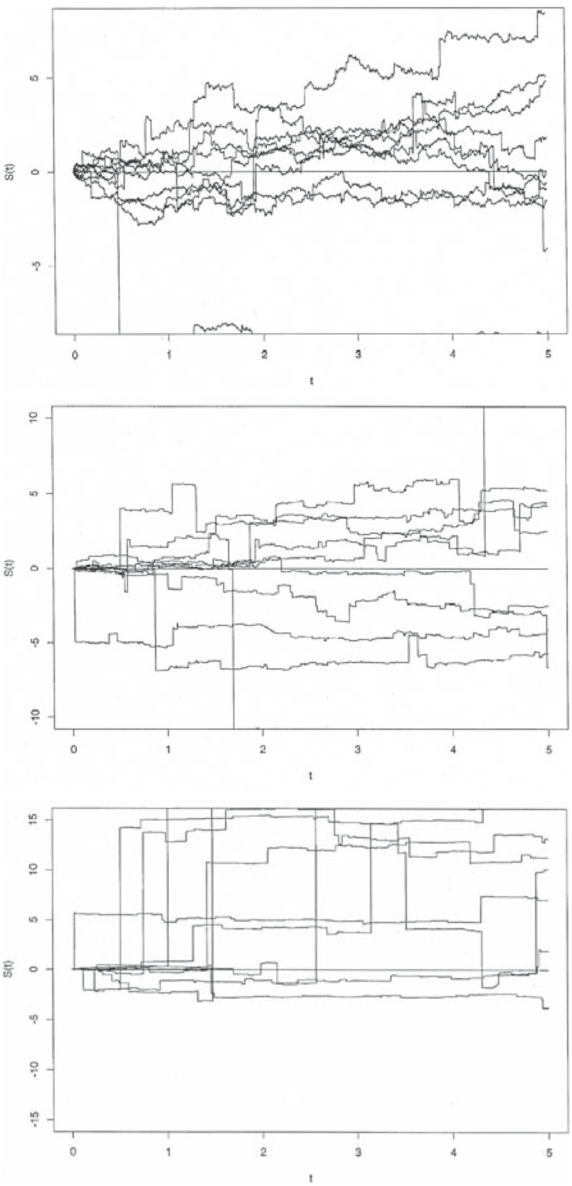

Fig. 1.1.5. Several computer-generated trajectories (sample paths) of random signals called *Lévy stochastic processes* with parameter $\alpha = 1.5$, 1, and 0.75, respectively (from top to bottom). They are often used to model anomalous diffusion processes wherein diffusing particles are also permitted to change their position by jumping. Parameter α indicates the intensity of jumps of different sizes. Parameter value $\alpha = 2$ corresponds to the Wiener process with trajectories that have no jumps. In each figure, the random mechanism that created different trajectories was the same. However, different random mechanisms led to trajectories presented in different figures.

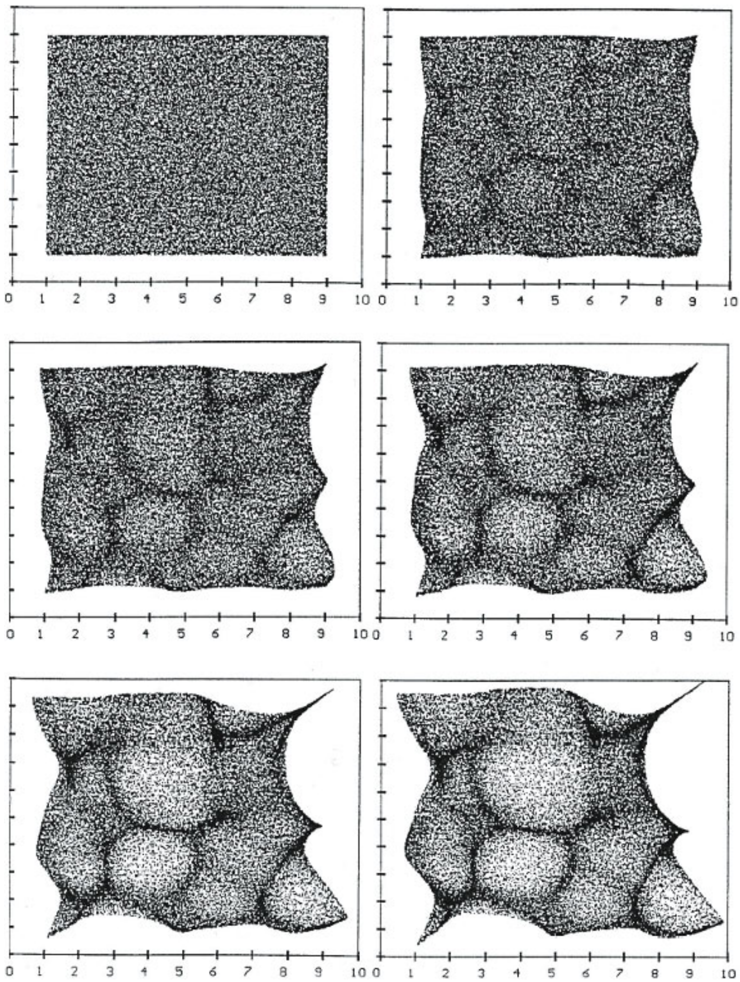

Fig. 1.1.6. Computer simulation of the evolution of a passive tracer density in a turbulent Burgers velocity field with random initial distribution and random "shot-noise" initial velocity data. The simulation was performed for 100,000 particles. The consecutive frames show the location of passive tracer particles at times $t = 0.0, 0.3, 0.6, 1.0, 2.0, 3.0$.

The signals shown in Figures 1.1.4–1.1.5 are, obviously, not stationary and have a diffusive character. However, their increments (differentials) are stationary and, in Chapter 9, they will play an important role in constructing the spectral representation of stationary signals themselves. The signal shown in Figure 1.1.4 can be interpreted as a *trajectory*, or *sample path*, of a *random walker* moving in discrete time steps up or down a certain distance with equal probabilities $\frac{1}{2}$ and $\frac{1}{2}$.

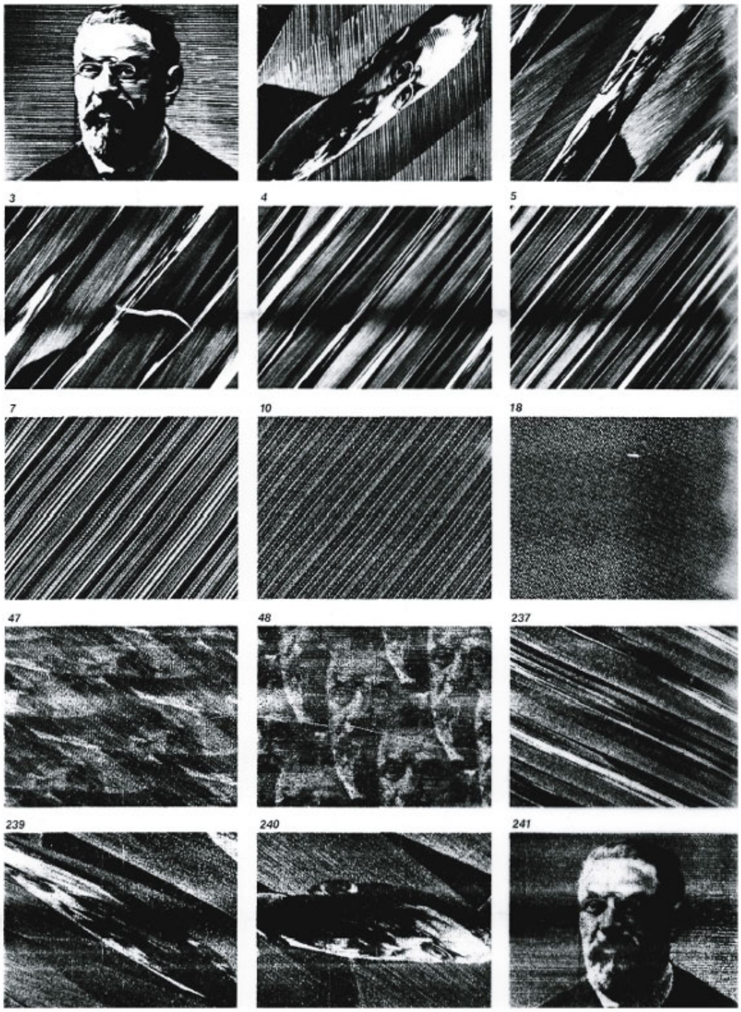

Fig. 1.1.7. Some deterministic signals (in this case, the images) transformed by deterministic systems can appear random. The above picture shows a series of iterated transformations of the original image via a fixed linear 2D mapping (matrix). The number of iterations applied is indicated in the top left corner of each image. The curious behavior of iterations, the original image first dissolving into seeming randomness only to return later to an almost original condition, is related to the so-called *ergodic* behavior. Thus irreverently transformed is Professor Henri Poincaré (1854–1912) of the University of Paris, the pioneer of ergodic theory of stationary phenomena. (From *Scientific American*; reproduced with permission. Copyright 1986 James P. Crutchfield.)

However, in the picture these trajectories are viewed from far away, and in accelerated time, so that both time and space appear continuous.

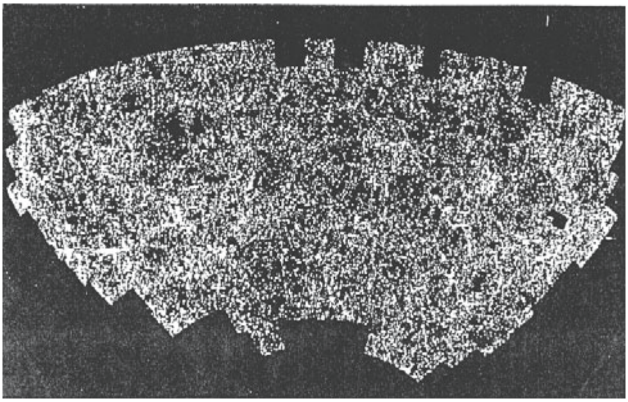

Fig. 1.1.8. A signal (again, an image) representing the large-scale and apparently random distribution of mass in the universe. The data come from the APM galaxy survey and shows more than 2 million galaxies in a section of sky centered on the South Galactic Pole. The so-called *adhesion model* of the large-scale mass distribution in the universe uses the Burgers equation to model the relevant velocity fields.

In certain situations the randomness of the signal is due to uncertainty about initial conditions of the underlying phenomenon which otherwise can be described by perfectly deterministic models such as partial differential equations. A sequence of pictures in Figure 1.1.6 shows the evolution of the system of particles with an initially random (and homogeneous in space) spatial distribution. The particles are then driven by the velocity field $\vec{v}(t,\vec{x}) \in \mathbf{R}^2$ governed by the so-called 2*D* *Burgers equation*[2]

$$\frac{\partial \vec{v}(t,\vec{x})}{\partial t} + (\vec{v}(t,\vec{x}) \cdot \nabla)\vec{v}(t,\vec{x}) = D\left(\frac{\partial^2 \vec{v}(t,\vec{x})}{\partial x_1} + \frac{\partial^2 \vec{v}(t,\vec{x})}{\partial x_2}\right), \quad (1.1.2)$$

where $\vec{x} = (x_1, x_2)$, the *nabla* operator $\nabla = \frac{\partial}{\partial x_1}\vec{i} + \frac{\partial}{\partial x_2}\vec{j}$, and the positive constant D is the coefficient of diffusivity. The inital velocity field is also assumed to be random.

1.2 Time domain and frequency domain descriptions

A periodic signal with period P (measured, say, in seconds [s]) can be written in the form of an infinite series

$$x(t) = c_0 + \sum_{m=1}^{\infty} c_m \cos(2\pi m f_0 t + \theta_m), \quad (1.2.1)$$

[2] See, e.g., W. A. Woyczyński, *Burgers–KPZ Turbulence–Göttingen Lectures*, Springer-Verlag, Berlin, New York, 1998.

where $f_0 = \frac{1}{P}$ [Hz] is the fundamental frequency of the signal. This expansion, called the *Fourier expansion* of the signal, is the basic tool in the analysis of random signals; it will be reviewed in detail in Chapter 2. The components

$$c_m \cos(2\pi m f_0 t + \theta_m), \quad m = 2, 3, \ldots,$$

are called higher harmonics of the signal with the amplitudes c_m, higher frequencies mf_0, and the corresponding phase shifts θ_m. In the case of zero phase shifts, $\theta_m = 0$, the collection of pairs

$$(mf_0, c_m), \quad m = 1, 2, \ldots,$$

or, equivalently, their graphical representation, is called the *frequency spectrum* of the signal. Note that, for a periodic signal, the spectrum is always concentrated on a discrete set of frequencies, namely, the multiplicities of the fundamental frequency f_0. For example, the signal

$$x(t) = \sum_{m=1}^{12} \frac{1}{m^2} \cos(2\pi m t), \tag{1.2.2}$$

shown in Figure 1.2.1, has the fundamental frequency 1 Hertz (Hz), i.e., one cycle per second, and the frequency spectrum

$$c_m = \begin{cases} m^{-2} & \text{for } m = 1, 2, \ldots, 12, \\ 0 & \text{for } m = 13, 14, \ldots \end{cases} \tag{1.2.3}$$

shown in Figure 1.2.2.

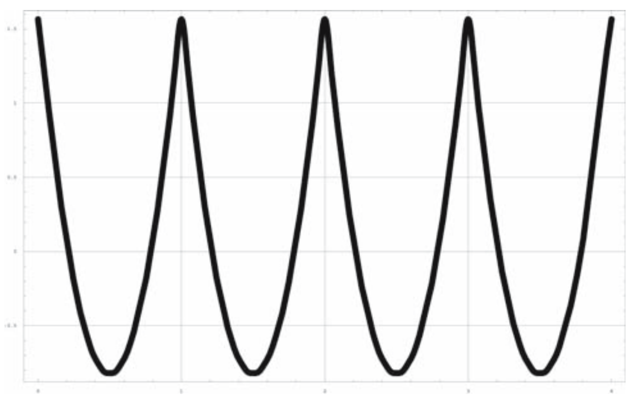

Fig. 1.2.1. Signal $x(t) = \sum_{m=1}^{12} m^{-2} \cos(2\pi m t)$ in its time domain representation.

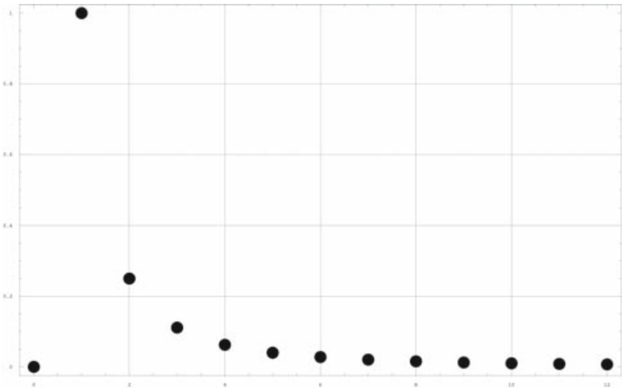

Fig. 1.2.2. Signal $x(t) = \sum_{m=1}^{12} m^{-2}\cos(2\pi m t)$ in its frequency domain (spectral) representation.

If the signal is studied only in a finite time interval $[0, P]$, it can always be treated as a periodic signal with period P since one can extend its definition periodically to the whole time line by copying its waveform from the interval $[0, P]$ to intervals $[P, 2P]$, $[2P, 3P]$, and so on.

Given the familar trigonometric formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and de Moivre's formulas

$$e^{j\alpha} = \cos\alpha + j\sin\alpha, \quad j = \sqrt{-1}, \quad \cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}),$$

which tie together the trigonometric functions of the real variable α with exponential functions of the imaginary variable $j\alpha$, the spectral representation of the signal can be rewritten either in the real phaseless form

$$x(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos(2\pi m f_0 t) + \sum_{m=1}^{\infty} b_m \sin(2\pi m f_0 t), \qquad (1.2.4)$$

with coefficients in representations (1.2.4) and (1.2.1) connected by the formulas

$$a_0 = c_0, \qquad a_m = c_m\cos\theta_m, \qquad b_m = -c_m\sin\theta_m, \quad m = 1, 2, \ldots,$$

or in the complex exponential form

$$x(t) = \sum_{m=-\infty}^{\infty} z_m e^{j2\pi m f_0 t}, \qquad (1.2.5)$$

Table 1.2.1. Trigonometric formulas and complex numbers.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha;$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta;$$
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2};$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2};$$
$$\sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha = \sin(\alpha+\beta)\sin(\alpha-\beta);$$
$$\cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha = \cos(\alpha+\beta)\cos(\alpha-\beta);$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)];$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)];$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)];$$
$$j = \sqrt{-1}, \qquad j^{4m} = 1, \qquad j^{4m+1} = j, \qquad j^{4m+2} = -1, \qquad j^{4m+3} = -j,$$

where m is an integer;

$$z = a + jb, \quad a = \operatorname{Re} z, \quad b = \operatorname{Im} z, \quad z^* = a - jb;$$
$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot z^*};$$
$$\operatorname{Re} z = \frac{z + z*}{2} = |z|\cos\theta, \qquad \operatorname{Im} z = \frac{z - z*}{2j} = |z|\sin\theta,$$

where

$$\theta = \operatorname{Arg} z = \arctan\frac{\operatorname{Im} z}{\operatorname{Re} z}$$

is the *argument* of z.

Table 1.2.2. De Moivre formulas.

$$e^{\beta + j\alpha} = e^{\beta}(\cos\alpha + j\sin\alpha),$$
$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j},$$
$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta.$$

with coefficients (amplitudes) in representations (1.2.5) and (1.2.4) connected by the formulas

$$a_0 = z_0, \qquad a_m = z_m + z_{-m}, \qquad b_m = j(z_m - z_{-m}), \quad m = 1, 2, \ldots .$$

For the complex exponential form (1.2.5) to represent a real-valued signal, that is, for a_0, a_m, b_m, given by the above formulas to be real, the condition $z_{-m} = z_m^*$, where the asterisk denotes the complex conjugate, must be satisfied.

Nonperiodic signals can also be analyzed in terms of their spectra, but those spectra are not discrete. We will study them later on.

At the first sight, the above introduction of complex numbers and functions of complex numbers may seem as an unnecessary complication in the analysis of signals. However, as we will see in subsequent chapters, the calculations within the theory of random signals actually become simpler and more transparent if one operates in the complex domain. The book assumes familiarity with elementary properties of trigonometric functions and complex numbers. However, for the reader's peace of mind, and by popular demand of the readers of the preliminary versions of this book, we summarize the basic formulas in this area in the table below and include a few exercises in Section 1.4 to review basic operational procedures on complex numbers.

1.3 Characteristics of signals

Several physical characteristics of signals are of primary interest.

- *The time average of the signal*: For analog, continuous-time signals, the time average is defined by the formula

$$x_{\text{av}} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt, \tag{1.3.1}$$

and for digital, discrete-time signals which are defined only for the time instants $t = n$, $n = 0, 1, 2, \ldots, N-1$, it is defined by the formula

$$x_{\text{av}} = \frac{1}{N}\sum_{n=0}^{N-1} x(nT). \tag{1.3.2}$$

For periodic signals, it follows from (1.3.1) that

$$x_{\text{av}} = \frac{1}{P}\int_0^P x(t)\,dt, \tag{1.3.3}$$

so that, for signals described by their Fourier expansions, (1.2.1) and (1.2.4)–(1.2.5), the time averages are

$$x_{\text{av}} = c_0 = a_0 = z_0,$$

because the integral of the sine and cosine functions over the full period is 0.

- *Energy of the signal*: For an analog signal $x(t)$, the energy is

$$E_x = \int_0^\infty |x(t)|^2 dt, \tag{1.3.4}$$

and for digital signals,

$$E_x = T \sum_{n=0}^{\infty} |x(nT)|^2. \tag{1.3.5}$$

Remember that, since in what follows it will be convenient to consider complex-valued signals, the above formulas include notation for the square of the modulus of a complex number: $|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = z \cdot z^*$.

- *Power of the signal*: Again, for an analog signal, the power is

$$\text{PW}_x = \lim_{T\to\infty} \frac{1}{T} \int_0^T |x(t)|^2 dt, \tag{1.3.6}$$

and for a digital signal,

$$\text{PW}_x = \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} |x(nT)|^2 \cdot T. \tag{1.3.7}$$

As a consequence, for a periodic signal with period P,

$$\text{PW}_x = \frac{1}{P} \int_0^P |x(t)|^2 dt. \tag{1.3.8}$$

Sometimes it is convenient to consider signals defined for all time instants t, $-\infty < t < +\infty$, rather than just for positive t. In such cases, all of the above definitions have to be adjusted in obvious ways, replacing the one-sided integrals and sums by two-sides integrals and sums, and adjusting the averaging constants correspondingly.

1.4 Problems and exercises

1.4.1. Find the real and imaginary parts of $\frac{j+3}{j-3}$; $(1 + j\sqrt{2})^3$; $\frac{1}{2-j}$; $\frac{2-3j}{3j+2}$.

1.4.2. Find the moduli $|z|$ and arguments θ of complex numbers $z = 5$; $z = -2j$; $z = -1 + j$; $z = 3 + 4j$.

1.4.3. Find the real and imaginary components of complex numbers $z = 5e^{j\pi/4}$; $z = -2e^{j(8\pi+1.27)}$; $z = -1e^{j}$; $z = 3e^{je}$.

1.4.4. Show that

$$\frac{5}{(1-j)(2-j)(3-j)} = \frac{j}{2} \quad \text{and} \quad (1-j)^4 = -4.$$

1.4.5. Sketch sets of points in the complex plane (x, y), $z = x + jy$, such that $|z - 1 + j| = 1$; $|z + j| \le 3$; $\mathrm{Re}(z^* - j) = 2$; $|2z - j| = 4$; $z^2 + (z^*)^2 = 2$.

1.4.6. Using de Moivre's formulas, find $(-2j)^{1/2}$ and $\mathrm{Re}(1 - j\sqrt{3})^{77}$. Are these complex numbers uniquely defined?

1.4.7. Write the signal $x(t) = \sin t + \cos\frac{3t}{3}$ from Figure 1.1.1 in the pure cosine form (1.2.1). Use the fact that sine can be written as a cosine with a phase shift.

1.4.8. Using de Moivre's formulas, write the signal $x(t) = \sin t + \cos\frac{3t}{3}$ from Figure 1.1.1 in the complex exponential form (1.2.5).

1.4.9. Find the time average and power of the signal $x(t) = \sin t + \cos\frac{3t}{3}$ from Figure 1.1.1.

1.4.10. Using de Moivre's formula, derive the complex exponential representation (1.2.5) of the signal $x(t)$ given by the cosine series representation (1.2.1). Then apply this procedure to obtain the complex exponential representation of the signal given by formula (1.2.2) and shown in Figure 1.2.1.

1.4.11. Find the time average and power of the signal $x(t)$ from Figure 1.2.1. Use a symbolic manipulation language such as *Mathematica* or MATLAB if you like.

1.4.12. Verify that for the signal $x(t)$ in (1.2.5) to be real valued, condition $z_{-m} = z_m^*$ has to be satisfied for all integers m.

1.4.13. Using a computing platform such as *Mathematica*, MAPLE, or MATLAB, produce plots of the signals

$$x_n(t) = \frac{\pi}{4} + \sum_{m=1}^{M}\left[\frac{(-1)^m - 1}{\pi m^2}\cos mt - \frac{(-1)^m}{m}\sin mt\right]$$

for $M = 0, 1, 2, 3, \ldots, 9$ and $-2\pi < t < 2\pi$. Then produce their plots in the frequency domain representation. Calculate their power (again, using *Mathematica*, MAPLE, or MATLAB if you wish). Write down your observations. What is likely to happen with the plots of these signals as we take more and more terms of the above series, that is, as $M \to \infty$?

1.4.14. Use the analog-to-digital conversion formula (1.1.1) to digitize signals from Problem 1.4.13 for a variety of sampling periods and resolutions. Plot the results.

1.4.15. Use your computing platform to produce a discrete-time signal consisting of a string of random numbers uniformly distributed on the interal $[0, 1]$. For example, in *Mathematica*, the command

```
Table[Random[], {20}]
```

may produce the following string of 20 random numbers between 0 and 1:

```
{0.175245, 0.552172, 0.471142, 0.910891, 0.219577,
 0.198173, 0.667358, 0.226071, 0.151935, 0.42048,
 0.264864, 0.330096, 0.346093, 0.673217, 0.409135,
 0.265374, 0.732021, 0.887106, 0.697428, 0.7723}
```

Use the "random numbers" string as additive noise to produce random versions of the digitized signals from Problem 1.4.14. Follow the example described in Figure 1.1.3. Experiment with different string lengths and various noise amplitudes. Then center the noise around zero and repeat your experiments.