Chapter 8
Ring and Module Hulls

A motivation for the need to study ring and module hulls that are intermediate between a ring R and $Q(R)$ or $E(R)$, and between a module M and $E(M)$, respectively, can be seen from the following examples. Consider

$$R = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix}.$$

The ring R is neither right nor left Noetherian and its prime radical is nonzero. However, $Q(R) = \mathrm{Mat}_2(\mathbb{Q})$ is simple Artinian. Next, take R to be a domain which is not right Ore. Then $Q(R)$ is a simple regular right self-injective ring (see Theorem 2.1.31 and [262, Corollary 13.38′]) which is neither orthogonally finite nor with bounded index (of nilpotency). The disparity between R and $Q(R)$ in the preceding examples limits the transfer of information between R and $Q(R)$.

Although every module has an injective hull, it is generally hard to construct or explicitly describe it. However, certain known subsets of the injective hull or of the endomorphism ring of the injective hull of a given ring (or module) can be used to generate an overring (or an overmodule) in conjunction with the base ring (or module) to serve as a hull of the ring (module) with some desirable properties. For example, since $Q(R)$ can be constructed for a ring R by Utumi's method (see Theorem 1.3.13), $\mathcal{B}(Q(R))$ can also be determined. Hence, the set of all $f(1)$, where f is a central idempotent in $\mathrm{End}(E(R_R))$ is explicitly described via $\mathcal{B}(Q(R))$ (see Lemma 8.3.10). Therefore, rings or modules generated by such a known subset of the injective hull in conjunction with the base ring or module may provide hulls. Additionally, these hulls may possess properties of interest to us.

These examples and constructions illustrate a need to find overrings of a given ring that have some weaker versions of the properties traditionally associated with right rings of quotients (e.g., semisimple Artinian, or (regular) right self-injective, or right continuous, etc.). These overrings are close enough to the base ring to facilitate an effective exchange of information between the base ring and the overrings. Furthermore, this need is reinforced when one studies the classes of rings for which $R = Q(R)$ (e.g., right Kasch rings). For these classes, the theory of right rings of

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quotients does not apply as was seen in Chap. 7 (and now in Chap. 8). However, the results presented in Chap. 7 which deal with right essential overrings will still be applicable (as will also be seen for such results from this chapter).

Our goal is to find methods that enable us to describe all right essential overrings of a ring R in a selected class $\mathfrak{K}$ (or essential overmodules of a module M in a selected class $\mathfrak{M}$). For this, our focus is on the study of the following problems:

Problem I. Given a ring R and a class $\mathfrak{K}$ of rings, determine what information transfers between R and its right essential overrings in $\mathfrak{K}$.

Problem II. Assume that a ring R and a class of rings $\mathfrak{K}$ are given.

(i) Determine conditions to ensure the existence of right rings of quotients and that of right essential overrings of R, which are, in some sense, "minimal" with respect to belonging to the class $\mathfrak{K}$.

(ii) Characterize the right rings of quotients and the right essential overrings of R which are in the class $\mathfrak{K}$ possibly by using the "minimal" ones obtained in (i).

Problem III. Given classes of rings $\mathfrak{A}$ and $\mathfrak{B}$, determine those rings $R \in \mathfrak{B}$ such that $Q(R) \in \mathfrak{A}$.

Problem IV. Given a ring R and a class $\mathfrak{K}$ of rings, let $X(R)$ denote some standard type of extension of R (e.g., $X(R) = R[x]$ or $X(R) = \text{Mat}_n(R)$, etc.) and let $H(R)$ denote a right essential overring of R which is "minimal" with respect to belonging to the class $\mathfrak{K}$. Determine when $H(X(R))$ is comparable to $X(H(R))$.

Problems I and II will be discussed in Sects. 8.1–8.3, while Problems III and IV will be studied in Sects. 9.1 and 9.3, respectively. We shall see that the right essential overrings which are minimal with respect to belonging to a specific class of rings are important tools in these investigations. To accommodate various notions of minimality, three basic notions of hulls are included in our discussion (see Definition 8.2.1). Using these notions, we establish the existence and uniqueness of the FI-extending ring hull for a semiprime ring (which, in this case, coincides with the quasi-Baer ring hull). In another basic type of a ring hull, we shall use R and certain subsets of $E(R_R)$ to generate a right essential overring S, so that S is in $\mathfrak{K}$ in some minimal fashion (see Definition 8.2.8). This construction leads to the concept of a pseudo ring hull. Moreover, we show that there is an effective transfer of information between the aforementioned hulls and the base ring. The results we present in this chapter will be applied to the study of boundedly centrally closed C^*-algebras later in Chap. 10.

We will conclude the chapter with a discussion on module hulls in Sect. 8.4. In particular, we will discuss quasi-injective, continuous and quasi-continuous hulls of a module. Conditions for a continuous hull to exist will be shown. We will see that every finitely generated projective module over a semiprime ring has an FI-extending hull. Moreover, it will be shown that the extending and FI-extending properties transfer from a module M to its rational hull.

For the convenience of the reader, **Con**, **qCon**, **E**, and **FI** are used respectively to denote the class of right continuous rings (modules), the class of right quasi-continuous rings (modules), the class of right extending rings (modules), and the class of right FI-extending rings (modules) according to the context. Further, we let **B** and **qB** denote the class of Baer rings and the class of quasi-Baer rings, respectively.

In this chapter, in general, all rings are assumed to have an identity element. However, in Definition 8.1.5, Definition 8.2.1, and Sect. 8.3, we do not require that rings must have an identity element.

8.1 Background and Preliminaries

This section is devoted to background information and preliminary results. Various properties are presented which transfer from a ring to its right rings of quotients or to its right essential overrings.

Definition 8.1.1 A ring R is said to be *right essentially Baer* (resp., *right essentially quasi-Baer*) if the right annihilator of any nonempty subset (resp., ideal) of R is essential in a right ideal generated by an idempotent. Let **eB** (resp., **eqB**) denote the class of right essentially Baer (resp., right essentially quasi-Baer) rings.

It can be seen that **eB** properly contains **E** and **B**, while **eqB** properly contains **FI** and **qB**. If $S = A \oplus \mathbb{Z}_4$, where A is a domain which is not right Ore, then S is neither right extending nor Baer. But S is right essentially Baer. Next let R be the ring as in Example 7.1.13. Then the ring R is neither right FI-extending nor quasi-Baer. But R is right essentially quasi-Baer (see Exercise 8.1.10.1). In Theorem 3.2.37, we have seen that when a ring R is semiprime, R is quasi-Baer if and only if R is right essentially quasi-Baer. The next result shows that replacing semiprime with nonsingularity also yields this equivalence.

Proposition 8.1.2 *Assume that R is a right nonsingular ring.*

(i) *If $R \in$ **eB**, then $R \in$ **B**.*

(ii) *If $R \in$ **eqB**, then $R \in$ **qB**.*

Proof (i) Assume that R is right essentially Baer. Say $\emptyset \neq X \subseteq R$. Then $r_R(X)_R$ is essential in eR_R with $e^2 = e \in R$. As in the proof of Theorem 3.2.38, we obtain that $\ell_R(r_R(X)) = \ell_R(eR) = R(1-e)$, so $r_R(X) = r_R(\ell_R(r_R(X))) = eR$. Thus R is Baer.

(ii) Say R is right essentially quasi-Baer. Take X to be an ideal of R and follow the proof of part (i). □

Lemma 8.1.3 *Let T be a right ring of quotients of R. Then*:

(i) *For right ideals I and J of T, if $I_T \leq^{\text{ess}} J_T$, then $I_R \leq^{\text{ess}} J_R$.*

(ii) *If $A_R \trianglelefteq T_R$, then $A_R \leq^{\text{ess}} TAT_R$.*

Proof (i) Let $0 \neq y \in J$. Then there is $t \in T$ with $0 \neq yt \in I$. As $R_R \leq^{\text{den}} T_R$, there is $r \in R$ satisfying $tr \in R$ and $ytr \neq 0$. Now $ytr \in I$. So $I_R \leq^{\text{ess}} J_R$.

(ii) By Proposition 2.1.32, $\text{End}(T_R) = \text{End}(T_T) \cong T$. Thus $TA \subseteq A$ as $A_R \trianglelefteq T_R$. Let $0 \neq y \in TAT = AT$. Then $y = a_1t_1 + \cdots + a_nt_n$ where $a_i \in A$, $t_i \in T$ for each i, $1 \leq i \leq n$. Since $R_R \leq^{\text{den}} T_R$, there is $r_1 \in R$ with $t_1r_1 \in R$ and $yr_1 \neq 0$. Again there is $r_2 \in R$ with $t_2r_1r_2 \in R$ and $yr_1r_2 \neq 0$. Continuing this process, there is $r \in R$ with $0 \neq yr \in A$. So $A_R \leq^{\text{ess}} TAT_R$. □

Proposition 8.1.4 *Let T be a right ring of quotients of R. Then*:

(i) *T_T is FI-extending if and only if T_R is FI-extending.*
(ii) *T_T is extending if and only if T_R is extending.*

Proof (i) Let T_T be FI-extending. Say $A_R \trianglelefteq T_R$. Then $A_R \leq^{\text{ess}} TAT_R$ by Lemma 8.1.3(ii). There exists $e^2 = e \in T$ satisfying $TAT_T \leq^{\text{ess}} eT_T$. Thus $TAT_R \leq^{\text{ess}} eT_R$ by Lemma 8.1.3(i), so $A_R \leq^{\text{ess}} eT_R$. Hence, T_R is FI-extending.

Conversely, let T_R be FI-extending. Then $\text{End}(T_R) = \text{End}(T_T)$ by Proposition 2.1.32. Take $B \trianglelefteq T$. Then $B_R \trianglelefteq T_R$ because $\text{End}(T_R) = \text{End}(T_T) \cong T$. So there exists $e^2 = e \in \text{End}(T_R) = \text{End}(T_T)$ such that $B_R \leq^{\text{ess}} eT_R$. Hence, $B_T \leq^{\text{ess}} e(1)T_T$ and $e(1)^2 = e(1) \in T$. Therefore, T_T is FI-extending.

(ii) The proof is similar to that of part (i). □

The condition that T is a right ring of quotients of R in Proposition 8.1.4 cannot be replaced by the condition that T is a right essential overring of R (see Exercise 8.1.10.2). The concept of a **D-E** class is introduced in the next definition. Such a class has the advantage that its members have an abundance of idempotents for their "designated" right ideals.

Definition 8.1.5 Let $\mathfrak{K}$ be a class of rings not necessarily with identity and P be a property of right ideals. We say that $\mathfrak{K}$ is a *class determined by P* if:

(i) there exists an assignment $\mathfrak{D}_{\mathfrak{K}}$ on the class of all rings such that $\mathfrak{D}_{\mathfrak{K}}(R)$ is a set of right ideals of a ring R.

(ii) each element of $\mathfrak{D}_{\mathfrak{K}}(R)$ has the property P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is such a class where P is the property that a right ideal is essential in a right ideal generated by an idempotent, then we say that $\mathfrak{K}$ is a ***D-E** class* and use $\mathfrak{C}$ to denote a **D-E** class. Thus, a **D-E** class exhibits the extending property with respect to a designated set of right ideals of a ring in $\mathfrak{C}$. We note that any **D-E** class always contains the class of right extending rings.

Some examples illustrating Definition 8.1.5 are as follows.

(1) $\mathfrak{K}$ is the class of semisimple Artinian rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{I \mid I_R \leq R_R\}$, and P is the property that every right ideal is a direct summand.
(2) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{I \mid I_R \leq R_R\}$, and P is the property that every right ideal is finitely generated.
(3) $\mathfrak{K}$ is the class of regular rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{aR \mid a \in R\}$, and P is the property that every right ideal is generated by an idempotent.
(4) $\mathfrak{K}$ is the class of biregular rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{RaR \mid a \in R\}$, and P is the property that every ideal is generated by a central idempotent.
(5) $\mathfrak{K}$ is the class of right Rickart rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{aR \mid a \in R\}$, and P is the property that every right ideal is projective.

(6) $\mathfrak{K} = \mathbf{B}$, $\mathfrak{D}_{\mathbf{B}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$, and P is the property that every right ideal is generated by an idempotent.
(7) $\mathfrak{K} = \mathbf{qB}$, $\mathfrak{D}_{\mathbf{qB}}(R) = \{r_R(I) \mid I \trianglelefteq R\}$, and P is the property that every right ideal is generated by an idempotent.
(8) $\mathfrak{C} = \mathbf{E}$ and $\mathfrak{D}_{\mathbf{E}}(R) = \{I \mid I_R \leq R_R\}$.
(9) $\mathfrak{C} = \mathbf{eB}$ and $\mathfrak{D}_{\mathbf{eB}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$.
(10) $\mathfrak{C} = \mathbf{eqB}$ and $\mathfrak{D}_{\mathbf{eqB}}(R) = \{r_R(I) \mid I \trianglelefteq R\}$.
(11) $\mathfrak{C} = \mathbf{FI}$ and $\mathfrak{D}_{\mathbf{FI}}(R) = \{I \mid I \trianglelefteq R\}$.
(12) $\mathfrak{C} = \mathbf{pFI}$ (**pFI** is the class of right principally FI-extending rings), and $\mathfrak{D}_{\mathbf{pFI}}(R) = \{RaR \mid a \in R\}$.

We observe that the same class $\mathfrak{K}$ of rings can be determined by more than one $\mathfrak{D}_{\mathfrak{K}}$ and P. For example, with the class **E** we can also use the set of closed right ideals of R for $\mathfrak{D}_{\mathbf{E}}(R)$ and take P to be the property that every right ideal is either essential in a right ideal generated by an idempotent, or P to be the property that every right ideal generated by an idempotent. Also we note that the class of right Rickart rings can also be characterized by $\mathfrak{D}_{\mathfrak{K}}(R) = \{r_R(a) \mid a \in R\}$, and P is the property that every right ideal is generated by an idempotent.

Lemma 8.1.6 (i) *Assume that T is a right ring of quotients of R. If $J_R \leq T_R$, then $\ell_R(J) = \ell_R(J \cap R)$.*

(ii) *Assume that T is a left ring of quotients of R. If ${}_RJ \leq {}_RT$, then $r_R(J) = r_R(J \cap R)$.*

Proof (i) Clearly, $\ell_R(J) \subseteq \ell_R(J \cap R)$. Let $a \in \ell_R(J \cap R)$ and suppose that there is $y \in J$ such that $ay \neq 0$. Since $R_R \leq^{\text{den}} T_R$, there is $r \in R$ such that $yr \in J \cap R$ and $ayr \neq 0$, a contradiction. Thus, $\ell_R(J) = \ell_R(J \cap R)$.

(ii) The proof is similar to that of part (i). □

We say that an overring T of a ring R is a *right intrinsic* (*ideal intrinsic*) *extension* of R if every nonzero right ideal (ideal) of T has a nonzero intersection with R. Note that if T is a right essential overring of R, then T is a right intrinsic extension of R. See [162] and [64] for more details on right intrinsic extensions.

Proposition 8.1.7 (i) *Let $\mathfrak{C}$ be a **D-E** class of rings, and let T be a right intrinsic extension of a ring R. Assume that for each $J \in \mathfrak{D}_{\mathfrak{C}}(T)$ there exists $e^2 = e \in R$ such that $eJ \subseteq J$ and $(J \cap R)_R \leq^{\text{ess}} eR_R$. Then $T \in \mathfrak{C}$.*

(ii) *Let $\mathfrak{C}$ be a **D-E** class of rings, and T be a right ring of quotients of R. Assume that $R \in \mathfrak{C}$. If $J \in \mathfrak{D}_{\mathfrak{C}}(T)$ implies $J \cap R \in \mathfrak{D}_{\mathfrak{C}}(R)$, then $T \in \mathfrak{C}$.*

Proof (i) We first note that $J = eJ \oplus (1-e)J$. Suppose that $(1-e)J \neq 0$. Then $0 \neq (1-e)J \cap R \subseteq J \cap R \subseteq eR$, a contradiction. Hence, $J = eJ \subseteq eT$. To show that $J_T \leq^{\text{ess}} eT_T$, we take $0 \neq ev \in eT$ with $v \in T$. Then $evT \cap R \neq 0$, hence $0 \neq evu \in R$ for some $u \in T$, and so $0 \neq evu \in eR$. Thus, there is $r \in R$ such that $0 \neq euvr \in J \cap R \subseteq J$. So $J_T \leq^{\text{ess}} eT_T$, therefore $T \in \mathfrak{C}$.

(ii) Let $J \in \mathfrak{D}_{\mathfrak{C}}(T)$. Since $J \cap R \in \mathfrak{D}_{\mathfrak{C}}(R)$ by assumption, there exists $e^2 = e \in R$ with $(J \cap R)_R \leq^{\text{ess}} eR_R$. Because $1 - e \in \ell_R(J \cap R)$, $(1 - e)J = 0$ by Lemma 8.1.6. Hence, $J_R \leq eT_R$ and $eR_R \leq^{\text{ess}} eT_R$, so $J_R \leq^{\text{ess}} eT_R$. Thus, $J_T \leq^{\text{ess}} eT_T$ and therefore $T \in \mathfrak{C}$. □

A ring R is called *right finitely Σ-extending* if $R_R^{(n)}$ is extending for each positive integer n (cf. Exercise 6.1.18.1). A ring R is said to be *right uniform-extending* if every uniform right ideal of R is essential in a direct summand of R_R. The following result demonstrates that the right extending property transfers to right rings of quotients, while the right FI-extending property transfers to right intrinsic extensions.

Theorem 8.1.8 (i) *Assume that T is a right intrinsic extension of a ring R. If R_R is FI-extending, then so is T_T.*

(ii) *Let T be an ideal intrinsic extension of a ring R such that $\mathcal{B}(R) \subseteq \mathcal{B}(T)$. If R is semiprime and R is* (*right*) *FI-extending, then T is semiprime and* (*right*) *FI-extending.*

(iii) *Assume that T is a right ring of quotients of a ring R. If R_R is extending, then so is T_T.*

(iv) *Assume that T is a right ring of quotients of a ring R. If R_R is finitely Σ-extending, then so is T_T.*

(v) *Assume that T is a right ring of quotients of a ring R. If R_R is uniform-extending, then so is T_T.*

Proof (i) Let $J \trianglelefteq T$. Then $J \in \mathfrak{D}_{\mathbf{FI}}(T)$ and $J \cap R \in \mathfrak{D}_{\mathbf{FI}}(R)$. Because R_R is FI-extending, $(J \cap R)_R \leq^{\text{ess}} eR_R$ with $e^2 = e \in R$. From Proposition 8.1.7(i), T_T is FI-extending.

(ii) Clearly, T is semiprime. Let $0 \neq I \trianglelefteq T$. Then $(I \cap R)_R \leq^{\text{ess}} eR_R$ for some $e \in \mathcal{B}(R) \subseteq \mathcal{B}(T)$ by Theorem 3.2.37 and assumption. Similar to the proof of Proposition 8.1.7(i), $(1 - e)I = 0$, so $I = eI \subseteq eT$. We show that $I_T \leq^{\text{ess}} eT_T$. For this, we prove that $I_{eTe} \leq^{\text{ess}} eTe_{eTe}$. Say V is a nonzero ideal of eTe. Then V is an ideal of T, so $V \cap R \neq 0$. Hence $0 \neq V \cap R \subseteq eT \cap R = eR$, and thus $0 \neq (V \cap R) \cap (I \cap R) \subseteq V \cap I$ because $(I \cap R)_R \leq^{\text{ess}} eR_R$. Therefore, $I_{eTe} \leq^{\text{ess}} eTe_{eTe}$. As $e \in \mathcal{B}(T)$, $I_T \leq^{\text{ess}} eT_T$. So T is (right) FI-extending.

(iii) The proof follows from Proposition 8.1.7(ii) since the class $\mathbf{E}$ of right extending rings is a $\mathbf{D}$-$\mathbf{E}$ class and $\mathfrak{D}_{\mathbf{E}}(R)$ is the set of all right ideals of R.

(iv) Let T be a right ring of quotients of a ring R and assume that R_R is finitely Σ-extending. Note that $\text{Mat}_n(R)$ is a right extending ring for every positive integer n (Exercise 6.1.18.1). So $\text{Mat}_n(T)$ is a right extending ring by part (iii) as $\text{Mat}_n(T)$ is a right ring of quotients of $\text{Mat}_n(R)$. Thus, T_T is a finitely Σ-extending.

(v) Let T be a right ring of quotients of R and assume that R_R is a uniform-extending. Say J is a uniform right ideal of T. Let $I = J \cap R$, and take nonzero elements x and y in I. Then $xT \cap yT \neq 0$. Say $xs = yt \neq 0$ with $s, t \in T$. As $R_R \leq^{\text{den}} T_R$, there is $r \in R$ such that $sr \in R$ and $xsr = ytr \neq 0$. Again since $R_R \leq^{\text{den}} T_R$, there exists $a \in R$ with $tra \in R$ and $ytra \neq 0$. So $sra \in R$, $tra \in R$, and $0 \neq xsra = ytra \in xR \cap yR$. Thus, I is a uniform right ideal of R. Hence the proof follows directly from Proposition 8.1.7(ii). □

Theorem 8.1.9 (i) *Assume that T is a right and left essential overring of a ring R. If $R \in$ **qB**, then $T \in$ **qB**.*

(ii) *Assume that T is a right essential overring of a ring R which is also a left ring of quotients of R. If $R \in$ **eqB**, then $T \in$ **eqB**.*

(iii) *Assume that T is a right essential overring of a ring R which is also a left ring of quotients of R. If $R \in$ **B**, then $T \in$ **B**.*

(iv) *Assume that T is a right and left ring of quotients of a ring R. If $R \in$ **eB**, then $T \in$ **eB**.*

Proof (i) Let R be quasi-Baer. Say $J \trianglelefteq T$ and let $I = J \cap R$. There exists $e^2 = e \in R$ with $r_R(I) = eR$. Let $t \in r_T(J)$.

If $(1-e)t \neq 0$, then there is $r \in R$ with $0 \neq (1-e)tr \in R$ as $R_R \leq^{\text{ess}} T_R$. We see that $I(1-e)tr \subseteq Itr \subseteq Jtr = 0$. Hence $(1-e)tr \in r_R(I) = eR$, a contradiction. Therefore, $(1-e)\,r_T(J) = 0$. Thus $r_T(J) \subseteq eT$. To show that $eT \subseteq r_T(J)$, assume on the contrary that there is $y \in J$ such that $ye \neq 0$. As T is a left essential overring of R, there is $s \in R$ with $0 \neq sye \in R$. Hence, $sye \in J \cap R = I$.

But $sye \in Ie = 0$, a contradiction. Thus, $Je = 0$ and so $r_R(J) = eT$. Therefore, T is quasi-Baer.

(ii) Assume that R is right essentially quasi-Baer. Say $J \trianglelefteq T$ and $I = J \cap R$. There exists $e^2 = e \in R$ such that $r_R(I)_R \leq^{\text{ess}} eR_R$. As in the proof of part (i), we obtain $r_T(J) \subseteq eT$. By Lemma 8.1.6(ii), $r_R(J) = r_R(I)$. Thus we have that $r_R(J)_R \leq^{\text{ess}} eR_R$. Since $r_T(J) \subseteq eT$, $r_T(J)_R \leq^{\text{ess}} eT_R$. Thus $r_T(J)_T \leq^{\text{ess}} eT_T$, so T is right essentially quasi-Baer.

(iii) Let R be Baer. Take $\emptyset \neq X \subseteq T$ and $J = TX$. Then $r_T(X) = r_T(J)$. We now set $I = J \cap R$. Then there exists $e^2 = e \in R$ such that $r_R(I) = eR$. First to show that $r_T(J) \subseteq eT$, suppose that there is $t \in r_T(J)$ with $(1-e)t \neq 0$.

Since $R_R \leq^{\text{ess}} T_R$, there is $r \in R$ with $0 \neq (1-e)tr \in R$. So

$$I(1-e)tr = Itr = 0,$$

hence $0 \neq (1-e)tr \in r_R(I) = eR$, a contradiction. Thus $r_T(J) \subseteq eT$.

If $ye \neq 0$ for some $y \in J$, then there is $s \in R$ with $sy \in R$ and $sye \neq 0$ as ${}_RR$ is dense in ${}_RT$, So $sy \in I$. Hence $0 \neq sye \in Ie = 0$, a contradiction. Thus $ye = 0$ for all $y \in J$, hence $e \in r_T(J)$. Therefore, $eT \subseteq r_T(J)$ and thus $r_T(X) = r_T(J) = eT$. So T is Baer.

(iv) Assume that R is right essentially Baer. Let $\emptyset \neq X \subseteq T$ and $J = TX$. Then $r_T(X) = r_T(J)$. Take $I = J \cap R$. There exists $e^2 = e \in R$ such that $r_R(I)_R$ is essential in eR_R.

We show that $r_T(J)_T \leq^{\text{ess}} eT_T$. For this, say $t \in r_T(J)$. If $(1-e)t \neq 0$, then since $R_R \leq^{\text{den}} T_R$, there exists $r \in R$ with $tr \in R$ and $(1-e)tr \neq 0$. But because $Itr \subseteq Jtr = 0$, $tr \in r_R(I)$. Hence

$$(1-e)tr \in (1-e)r_R(I) \subseteq (1-e)eR = 0,$$

a contradiction. So $r_T(J) \subseteq eT$. To see that $r_T(J)_T \leq^{\text{ess}} eT_T$, use the corresponding part of the proof in part (ii). Therefore T is right essentially Baer. □

As an application of Theorems 8.1.8 and 8.1.9, note that (by direct computation) $T_n(R)_{T_n(R)} \leq^{\text{den}} \text{Mat}_n(R)_{T_n(R)}$ (see Exercise 8.1.10.5). Hence for various conditions in Theorems 8.1.8 and 8.1.9, if the condition holds for $T_n(R)$, then it holds for $\text{Mat}_n(R)$. Proposition 8.1.7, Theorems 8.1.8, and 8.1.9 show that if R is a ring which belongs to a certain class (of rings) and S is a right essential overring of R in that class, then every other right essential overring of R which contains S as a subring, also belongs to that certain class, under some conditions. These results provide information related to Problem I.

Exercise 8.1.10

1. ([89, Birkenmeier, Park, and Rizvi]) Show that the ring R as in Example 7.1.13 is neither quasi-Baer nor right FI-extending, but R is right essentially quasi-Baer.
2. ([89, Birkenmeier, Park, and Rizvi]) For a field K, as in Example 7.3.13(i), let $T = K[x]/x^4K[x]$ and $\overline{x}$ be the image of x in T. Put $T = K + K\overline{x} + K\overline{x}^2 + K\overline{x}^3$ and $R = K + K\overline{x}^2 + K\overline{x}^3$ which is a subring of T. Then T_T is injective. Also T is a right essential overring of R. Prove that T_R is not FI-extending. (Hint: check with $\overline{x}^3R_R \trianglelefteq T_R$.)
3. ([89, Birkenmeier, Park, and Rizvi]) Show that if a ring R is Abelian and right extending, then so is $Q(R)$.
4. ([89, Birkenmeier, Park, and Rizvi]) Let T be a right and left ring of quotients of R. Show that if R is right semihereditary and $\text{Mat}_n(R)$ is orthogonally finite for every positive integer n, then T is right and left semihereditary.
5. Let R be a ring and n a positive integer. Show that $\text{Mat}_n(R)$ is a right ring of quotients of $T_n(R)$. Hence if P is a property that transfers from a ring to its right rings of quotients, then P transfers from $T_n(R)$ to $\text{Mat}_n(R)$ (see Theorems 8.1.8, 8.1.9, [4, Theorem 1 and Corollary 2], and [67, Theorem 3.5 and Corollary 3.6]).

8.2 Ring Hulls and Pseudo Ring Hulls

Motivated by the results of Sect. 8.1 and Chap. 7, we shall introduce and develop ring hull concepts in this section. These enable us to study Problem II mentioned in introduction of this chapter. After illustrating the ring hull notions via examples, we shall discuss some technical machinery which enables us to verify the existence of hulls for various **D-E** classes.

As a standing assumption in our considerations on hulls, for a given ring R, all right essential overrings of R are assumed to be contained as right R-modules in a fixed injective hull $E(R_R)$ of R_R and all right rings of quotients of R are assumed to be subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

We begin with the following definition on various ring hulls.

Definition 8.2.1 Let R be a ring with $\ell_R(R) = 0$, but not necessarily with an identity element. Let $\mathfrak{K}$ denote a class of rings.

(i) The smallest right ring of quotients T of a ring R which belongs to $\mathfrak{K}$ is called the $\mathfrak{K}$ *absolute to* $Q(R)$ *right ring hull* of R (when it exists). We denote $T = \widehat{Q}_{\mathfrak{K}}(R)$.

(ii) The smallest right essential overring S of a ring R which belongs to $\mathfrak{K}$ is called the $\mathfrak{K}$ *absolute right ring hull* of R (when it exists). We denote $S = Q_{\mathfrak{K}}(R)$.

(iii) A minimal right essential overring of a ring R which belongs to $\mathfrak{K}$ is called a $\mathfrak{K}$ *right ring hull* of R (when it exists).

We remark that if R is a ring (not necessarily with identity), then any right R-module M_R has an injective hull $E(M_R)$ (see [153, Theorem 9, p. 19]). Further, if $Z(R_R) = 0$ for such a ring, then $Q(R) = E(R_R)$ (see [153, p. 69]). Next, we note that when $Q(R) = E(R_R)$, $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. In particular, from Theorem 2.1.25, $Q_{\mathbf{qCon}}(R)$ exists whenever $Q(R) = E(R_R)$ (e.g., $Z(R_R) = 0$).

Since we are mostly dealing with the right-sided notions, we will drop the word "right" (from the preceding definition) in the future to make it easier on the reader. Thus, when the context is clear, we will use "$\mathfrak{K}$ absolute to $Q(R)$ ring hull" of R instead of "$\mathfrak{K}$ absolute to $Q(R)$ *right* ring hull" of R, etc.

The next example, taken from Theorems 7.2.1, 7.2.2, and their proofs, illustrates some examples of ring hulls defined in Definition 8.2.1.

Example 8.2.2 Let R, V, S, U, and T be as in Theorem 7.2.1. Then:

(i) All right FI-extending ring hulls of R are precisely: $(S, +, \circ_{(1,0)})$, $(S, +, \circ_{(1,2)})$, $(U, +, \circledcirc_1)$, $(U, +, \circledcirc_2)$, $(T, +, \diamond_1)$, and $(T, +, \diamond_2)$.
(ii) All right extending ring hulls of R are precisely: $(V, +, \bullet_1)$, $(V, +, \bullet_2)$, $(V, +, \bullet_3)$, $(V, +, \bullet_4)$, $(S, +, \circ_{(1,0)})$, and $(S, +, \circ_{(1,2)})$.
(iii) All right quasi-continuous ring hulls of R are precisely: $(S, +, \circ_{(1,0)})$ and $(S, +, \circ_{(1,2)})$.
(iv) All right continuous ring hulls of R are precisely: $(S, +, \circ_{(1,0)})$ and $(S, +, \circ_{(1,2)})$.
(v) All right self-injective ring hulls of R are precisely: $(S, +, \circ_{(1,0)})$ and $(S, +, \circ_{(1,2)})$.

The following example also illustrates Definition 8.2.1. In fact, it exhibits a ring R which has several isomorphic right FI-extending ring hulls, but R does not have a quasi-Baer ring hull.

Example 8.2.3 Let A, R, $\mathbb{E} = \mathbb{E}_R$, and T be as in Example 7.3.18. Then from Theorem 7.3.17, $\mathbb{E}$ has exactly p^2 compatible ring structures $(\mathbb{E}, +, \bullet_{(\alpha,\beta)})$, where $\alpha, \beta \in \mathrm{Soc}(A)$. These ring structures on $\mathbb{E}_R$ are isomorphic and they are QF. Also by Example 7.3.18, on T there are exactly p distinct compatible ring structures $(T, +, \diamond_{(0,\beta)})$ where $\beta \in \mathrm{Soc}(A)$ and $\diamond_{(0,\beta)}$ is the restriction of $\bullet_{(\alpha,\beta)}$ to T. Further, all compatible ring structures $(T, +, \diamond_{(0,\beta)})$, $\beta \in \mathrm{Soc}(A)$, on T are isomorphic. The rings $(T, +, \diamond_{(0,\beta)})$ are right FI-extending ring hulls of R by Example 7.3.18. Say $I = \begin{bmatrix} J(A) & 0 \\ 0 & 0 \end{bmatrix}$. Then I is a right ideal of each of R, $(T, +, \diamond_{(0,0)})$, and $(\mathbb{E}, +, \bullet_{(0,0)})$, respectively. We see that $r_R(I)$ is not generated by an idempotent of R, so R is not quasi-Baer. Also the right annihilator of I in $(T, +, \diamond_{(0,0)})$ (resp., $(\mathbb{E}, +, \bullet_{(0,0)})$) is not generated by an idempotent in $(T, +, \diamond_{(0,0)})$ (resp., $(\mathbb{E}, +, \bullet_{(0,0)})$). Thus, neither $(T, +, \diamond_{(0,0)})$ nor $(\mathbb{E}, +, \bullet_{(0,0)})$ is quasi-Baer. So R does not have a quasi-Baer ring hull.

Recall that $\mathbf{I}(R)$ and $\mathcal{B}(R)$ denote the set of all idempotents and the set of all central idempotents of a ring R, respectively. Let R be a ring. Then $R\mathcal{B}(Q(R))$, the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$, has been called the *idempotent closure* of R by Beidar and Wisbauer [42, p. 65]. In the following result, the Baer ring hull $Q_{\mathbf{B}}(R)$ is $R\mathcal{B}(Q(R))$ for a commutative semiprime ring R, is due to Mewborn [298].

Theorem 8.2.4 *Assume that R is a commutative semiprime ring. Then $Q_{\mathbf{B}}(R) = Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$.*

Proof Say A is a commutative semiprime ring. Then A is reduced, so it is nonsingular by Theorem 1.2.20(ii). From Corollary 3.3.3, A is Baer if and only if A is extending. As A is commutative, A satisfies (C_3) condition. Thus, A is extending if and only if A is quasi-continuous. For the proof it is enough to show that $Q_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$. From Corollary 1.3.15, Theorem 2.1.25, and Proposition 2.1.32, $R\mathcal{B}(Q(R))$ is a quasi-continuous ring. Next, say S is a quasi-continuous (right) ring of quotients of R. Then again by Corollary 1.3.15, Theorem 2.1.25, and Proposition 2.1.32, $\mathcal{B}(Q(R)) \subseteq S$ as $Q(S) = Q(R)$. Thus $R\mathcal{B}(Q(R)) \subseteq S$. So $Q_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$. □

Theorem 8.2.5 *Assume that R is a regular right self-injective ring. Then $R = A \oplus B$ (ring direct sum), where A is a strongly regular ring and B is a ring generated by idempotents.*

Proof See [397, Theorems 2 and 4]. □

Theorem 8.2.6 *Let R be a right nonsingular ring and S the intersection of all right continuous right rings of quotients of R. Then $Q_{\mathbf{Con}}(R) = S$ and S is regular.*

Proof By Theorem 2.1.31 and Theorem 8.2.5, $Q(R) = A \oplus B$ (ring direct sum), where A is strongly regular and B is a ring generated by idempotents. Let T be a right continuous right ring of quotients of R. Since $Z(T_T) = 0$, T is regular by Corollary 2.1.30. Put $A = eQ(R)$ with $e \in \mathcal{B}(Q(R))$. From Theorem 2.1.25, $e \in T$ and $B \subseteq T$. So $T = (eQ(R) \cap T) \oplus B = eT \oplus B$ (ring direct sum).

Let $\{T_\alpha \mid \alpha \in \Lambda\}$ be the set of all right continuous right rings of quotients of R. Then $\cap T_\alpha = [\cap(eT_\alpha)] \oplus B$. In fact, note that $[\cap(eT_\alpha)] \oplus B \subseteq T_\alpha$ for each α as $T_\alpha = eT_\alpha \oplus B$. So $[\cap(eT_\alpha)] \oplus B \subseteq \cap T_\alpha$. Next, say $x \in \cap T_\alpha$ and $\beta \in \Lambda$. Then $x \in T_\beta = eT_\beta \oplus B$, hence $x = y + b$ with $y \in eT_\beta$ and $b \in B$. So $y = ey$ and $y = x - b \in (\cap T_\alpha) + B = \cap T_\alpha$ as $B \subseteq T_\alpha$ for every α. Hence, $y \in T_\alpha$

for every α, so $y = ey \in eT_\alpha$ for every α. Thus, $y \in \cap(eT_\alpha)$, and therefore $x = y + b \in [\cap(eT_\alpha)] \oplus B$. Hence, $\cap T_\alpha = [\cap(eT_\alpha)] \oplus B$.

Say $a \in \cap(eT_\alpha)$. There is a unique element $b \in eQ(R)$ with $a = aba$ and $b = bab$ as $eQ(R)$ is a strongly regular ring (see [264, Exercise 3, p. 36]). Also since each eT_α is strongly regular, there exists $b_\alpha \in eT_\alpha \subseteq eQ(R)$ such that $a = ab_\alpha a$ and $b_\alpha = b_\alpha ab_\alpha$, for each α. By the uniqueness of b, $b = b_\alpha \in eT_\alpha$ for each α. Hence, $b \in \cap(eT_\alpha)$, so $\cap(eT_\alpha)$ is a strongly regular ring.

As B is regular, $\cap T_\alpha = [\cap(eT_\alpha)] \oplus B$ is regular. From $\mathbf{I}(\cap T_\alpha) = \mathbf{I}(Q(R))$, $\cap T_\alpha$ is right quasi-continuous by Theorem 2.1.25. So, $\cap T_\alpha$ is right continuous. □

A ring is called *right duo* if every right ideal is an ideal. The next result shows the existence of the right duo absolute ring hull for a right Ore domain.

Proposition 8.2.7 *If R is a right Ore domain, then R has a right duo absolute ring hull.*

Proof Clearly, $Q(R)$ is right duo. Let S be the intersection of all right duo right rings of quotients of R. Let T and U be right duo right rings of quotients of R. Say $s, x \in S$ with $x \neq 0$. Then there are $t \in T$ and $u \in U$ with $sx = xt = xu$. Hence $x(t - u) = 0$, so $t = u$ and t (or u) $\in T \cap U$. As T and U are arbitrary right duo right rings of quotients of R, $t \in S$ and so $sx = xt \in xS$. Hence, S is the right duo absolute ring hull of R. □

Theorem 8.2.4 and the construction of $Q_{\mathbf{qCon}}(R)$ by Theorem 2.1.25 suggest how to design a "hull" of R by adjoining a certain subset of $E(R_R)$ to R. This leads to the notion of a pseudo ring hull which we define next. To define pseudo ring hulls in Definition 8.2.8, for a **D**-**E** class $\mathfrak{C}$, we *fix* $\mathfrak{D}_{\mathfrak{C}}(R)$ for the class $\mathfrak{C}$ (e.g., for **E**, we fix $\mathfrak{D}_{\mathbf{E}} = \{I \mid I_R \leq R_R\}$ rather than $\{J \mid J_R$ is closed in $R_R\}$). Define

$$\delta_{\mathfrak{C}}(R) = \{e \in \mathbf{I}(\mathrm{End}(E(R_R)) \mid I_R \leq^{\mathrm{ess}} eE(R_R) \text{ for some } I \in \mathfrak{D}_{\mathfrak{C}}(R)\}$$

and $\delta_{\mathfrak{C}}(R)(1) = \{e(1) \mid e \in \delta_{\mathfrak{C}}(R)\}$. For example,

$$\delta_{\mathbf{FI}}(R) = \{e \in \mathbf{I}(\mathrm{End}(E(R_R)) \mid I_R \leq^{\mathrm{ess}} eE(R_R) \text{ for some } I \trianglelefteq R\}$$

because $\mathfrak{D}_{\mathbf{FI}}(R)$ is the set of all ideals of R.

We next generate a right essential overring in a class $\mathfrak{C}$ from a base ring R and $\delta_{\mathfrak{C}}(R)$. By using an equivalence relation, say ρ on $\delta_{\mathfrak{C}}(R)$, we reduce the size of the subset of idempotents needed to generate a right essential overring of R in $\mathfrak{C}$. For this, we consider $\delta^{\rho}_{\mathfrak{C}}(R)$, which is a set of representatives of all equivalence classes of ρ, and let $\delta^{\rho}_{\mathfrak{C}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta^{\rho}_{\mathfrak{C}}(R)\}$.

Recall that $\langle X\rangle_A$ denotes the subring of a ring A generated by a subset X of A (see 1.1.2).

Definition 8.2.8 Let S be a right essential overring of R.

(i) If $\delta_{\mathfrak{C}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S \in \mathfrak{C}$, then we put

$$\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S = R(\mathfrak{C},\ S).$$

If $S = R(\mathfrak{C}, S)$, then S is called a $\mathfrak{C}$ *pseudo right ring hull of* R.

(ii) If $\delta^{\rho}_{\mathfrak{C}}(R)(1) \subseteq S$ and $\langle R \cup \delta^{\rho}_{\mathfrak{C}}(R)(1)\rangle_S \in \mathfrak{C}$, then we put

$$\langle R \cup \delta^{\rho}_{\mathfrak{C}}(R)(1)\rangle_S = R(\mathfrak{C}, \rho,\ S).$$

If $S = R(\mathfrak{C},\ \rho,\ S)$, then S is called a $\mathfrak{C}$ ρ *pseudo right ring hull of* R.

If $\delta_{\mathfrak{C}}(R)(1) \subseteq Q(R)$ and S is a right essential overring of R such that $R(\mathfrak{C}, S)$ exists, then $R(\mathfrak{C},\ S) = R(\mathfrak{C},\ Q(R))$ from Proposition 7.1.11.

For example, assume that $Q(R) = E(R_R)$. Then $Q_{\mathbf{qCon}}(R)$ exists, and we see that $Q_{\mathbf{qCon}}(R) = R(\mathbf{qCon}, Q(R))$.

As we are usually using the right-sided notions, we will drop the word "right" in the preceding definition. Thus we will call "$\mathfrak{C}$ pseudo *right* ring hull of R" just "$\mathfrak{C}$ pseudo ring hull of R", etc.

The next examples illustrate Definitions 8.2.1 and 8.2.8. They show that neither $\mathfrak{C}$ ring hulls nor $\mathfrak{C}$ ρ pseudo ring hulls are unique.

Example 8.2.9 In this example, we see that the intersection of all right FI-extending ring hulls is not necessarily a right FI-extending absolute ring hull. Further, it is shown that a right FI-extending ring hull may not be unique even up to isomorphism (cf. Example 8.2.3). Let F be a field and as in Example 3.2.39, we put

$$R = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & c \end{bmatrix} \mid a, c, x, y \in F \right\} \cong \begin{bmatrix} F & F \oplus F \\ 0 & F \end{bmatrix}.$$

Then by [262, Example 13.26(5)], R is right nonsingular and $Q(R) = \mathrm{Mat}_3(F)$.

(i) Let $H_1 = \left\{ \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\} \cong \begin{bmatrix} F \oplus F & F \oplus F \\ 0 & F \end{bmatrix}$, and let

$$H_2 = \left\{ \begin{bmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}.$$

Note that H_1 and H_2 are subrings of $\mathrm{Mat}_3(F)$. Define $\phi : H_1 \to H_2$ by

$$\phi \begin{bmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & a-b & x-y \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix}.$$

Then ϕ is a ring isomorphism. The ring R is not right FI-extending (see Example 3.2.39), but H_1 is right FI-extending by Corollary 5.6.11. Thus H_2 is right FI-extending because $H_1 \cong H_2$.

Let $F = \mathbb{Z}_2$. Then there is no proper intermediate ring between R and H_1, also between R and H_2. Thus, H_1 and H_2 are right FI-extending ring hulls of R. Since $H_1 \cap H_2 = R$, the intersection of right FI-extending ring hulls is not a right FI-extending absolute ring hull.

(ii) Assume that $F = \mathbb{Z}_2$. Consider

$$H_3 = \left\{ \begin{bmatrix} a+b & b & x \\ b & a & y \\ 0 & 0 & c \end{bmatrix} \mid a, b, c, x, y \in F \right\}.$$

The ring H_3 is right FI-extending from Corollary 5.6.11. Also H_3 is a right FI-extending ring hull of R because there is no proper intermediate ring between R and H_3. Further, $\mathrm{Tdim}(H_1) = 3$, but $\mathrm{Tdim}(H_3) = 2$. Thus $H_3 \not\cong H_1$.

(iii) From Theorem 5.6.5 (see also Example 3.2.39), $R = Q_{\mathbf{qB}}(R)$. Also we see that $R(\mathbf{FI}, Q(R)) = \begin{bmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{bmatrix} \neq \mathrm{Mat}_3(F) = Q_{\mathbf{qCon}}(R) = Q_{\mathbf{Con}}(R)$.

Example 8.2.10 There is a right nonsingular ring which has an infinite number of right FI-extending ρ pseudo ring hulls. Furthermore, none of these pseudo ring hulls is a right FI-extending ring hull, for some equivalence relation ρ on $\delta_{\mathbf{FI}}(R)$. Take $R = T_2(\mathbb{Z})$. Then R is right FI-extending from Theorem 5.6.19. Say e_{ij} is the matrix in R with 1 in (i, j)-position and 0 elsewhere. We note that $\{0, 1_R\} \cup \{e_{11} + qe_{12} \mid q \in \mathbb{Q}\} \subseteq \delta_{\mathbf{FI}}(R)$. Define an equivalence relation ρ on $\delta_{\mathbf{FI}}(R)$ such that: $e \rho f$ if and only if $e = fe$ and $f = ef$. Then each $\delta^{\rho}_{\mathbf{FI}}(R)$ contains $\{0, 1_R, e_{11} + qe_{12}\}$, where $q \in \mathbb{Q}$ is fixed.

Suppose that $q \notin \mathbb{Z}$. Then $\langle R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)\rangle_{Q(R)}$, the subring of $Q(R)$ generated by $R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)$, is a right FI-extending ρ pseudo ring hull of R because by Theorem 8.1.8(i) $\langle R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)\rangle_{Q(R)}$ is right FI-extending. Therefore, we obtain that $R(\mathbf{FI}, \rho, Q(R)) = \langle R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)\rangle_{Q(R)}$. But $\langle R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)\rangle_{Q(R)}$ is not a right FI-extending ring hull of R as $\langle R \cup \delta^{\rho}_{\mathbf{FI}}(R)(1)\rangle_{Q(R)} \neq R = Q_{\mathbf{FI}}(R)$.

We introduce two new equivalence relations which will be helpful.

Definition 8.2.11 (i) We define an equivalence relation α on $\delta_{\mathfrak{C}}(R)$ by $e \alpha f$ if $e = fe$ and $f = ef$.

(ii) We define an equivalence relation β on $\delta_{\mathfrak{C}}(R)$ by $e \beta f$ if there exists $I_R \leq R_R$ such that $I_R \leq^{\text{ess}} eE(R_R)$ and $I_R \leq^{\text{ess}} fE(R_R)$.

The equivalence relation α was used as ρ in Example 8.2.10. Note that for e, f in $\delta_{\mathfrak{C}}(R)$, $e \alpha f$ implies $e \beta f$. If $Z(R_R) = 0$, then $\alpha = \beta$.

Lemma 8.2.12 *Let R be a ring and $H = \mathrm{End}(E(R_R))$.*

(i) *If T is a right essential overring of R, then for $e \in \mathbf{I}(T)$, there exists $c \in \mathbf{I}(H)$ such that $c|_T \in \mathrm{End}(T_T)$ and $c(1) = e$.*

(ii) *For $b \in \mathbf{I}(H)$, if $b(1) \in Q(R)$, then $b(1) \in \mathbf{I}(Q(R))$.*

Proof (i) Note that $E(T_R) = E(eT_R) \oplus E((1-e)T_R)$. Let c be the canonical projection from $E(T_R)$ onto $E(eT_R)$. Then $c(t) = c(et) + c((1-e)t) = et$ for $t \in T$. Hence $c(1) = e$. If $s \in T$, then $c(ts) = ets = c(t)s$. So $c|_T \in \mathrm{End}(T_T)$.

(ii) As $E(R_R)$ is an $(H, Q(R))$-bimodule, each element of H is a $Q(R)$-homomorphism. So if $b(1) \in Q(R)$, then $b(1) = b(b(1)) = b(1b(1)) = b(1)b(1)$, thus $b(1) \in \mathbf{I}(Q(R))$. □

Proposition 8.2.13 *Let $\mathfrak{C}$ be a **D**-**E** class of rings, and let T be a right ring of quotients of R, δ be some $\delta^{\alpha}_{\mathfrak{C}}(R)$ such that $\delta(1) \subseteq T$. Take $S = \langle R \cup \delta(1)\rangle_T$. Suppose that for each $J \in \mathfrak{D}_{\mathfrak{C}}(S)$ there is $I \in \mathfrak{D}_{\mathfrak{C}}(R)$ with $I_R \leq^{\text{ess}} J_R$. Then $S = R(\mathfrak{C}, \alpha, T)$, which is a $\mathfrak{C}$ α pseudo ring hull of R.*

Proof Since $\delta(1) \subseteq Q(R)$, $\delta(1) \subseteq \mathbf{I}(S)$ by Lemma 8.2.12(ii). To show that $S = R(\mathfrak{C}, \alpha, T)$, we only need to see that $S \in \mathfrak{C}$. For this, let $J \in \mathfrak{D}_{\mathfrak{C}}(S)$. By assumption, there exists $I \in \mathfrak{D}_{\mathfrak{C}}(R)$ satisfying $I_R \leq^{\text{ess}} J_R$. Therefore we have that $I_R \leq^{\text{ess}} J_R \leq^{\text{ess}} E(J_R) = eE(R_R)$ for some $e \in \mathbf{I}(H)$, where $H = \mathrm{End}(E(R_R))$. Hence, $e \in \delta_{\mathfrak{C}}(R)$, so there exists $f \in \delta$ satisfying $eE(R_R) = fE(R_R)$. Thus we get $J_R \leq^{\text{ess}} fE(R_R)$ and so $J_R \leq^{\text{ess}} fS_R$.

Note that $f \in \mathrm{End}(E_R) = \mathrm{End}(E_{Q(R)})$ by the proof of Theorem 2.1.31, where $E = E(R_R)$. So $J_R \leq^{\text{ess}} fS_R = f(1)S_R$ because S is a subring of $Q(R)$. Hence, $J_S \leq^{\text{ess}} f(1)S_S$ and $f(1)^2 = f(1) \in S$, so $S \in \mathfrak{C}$. □

Proposition 8.2.14 *Let $\mathfrak{C}$ be a **D**-**E** class of rings, and let T be a right essential overring of R. Assume that for each $I \in \mathfrak{D}_{\mathfrak{C}}(R)$ there exists $e \in \mathbf{I}(T)$ satisfying $I_R \leq^{\text{ess}} eT_R$. Then there exists $\delta^{\beta}_{\mathfrak{C}}(R)$ such that, for each $c \in \delta^{\beta}_{\mathfrak{C}}(R)$, $c|_T \in \mathrm{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$.*

Proof Let $b \in \delta_{\mathfrak{C}}(R)$. Then there is $I \in \mathfrak{D}_{\mathfrak{C}}(R)$ with $I_R \leq^{\text{ess}} bE(R_R)$. By assumption, $I_R \leq^{\text{ess}} eT_R$ for some $e \in \mathbf{I}(T)$. From Lemma 8.2.12(i), there is $c^2 = c$ in $\mathrm{End}(E(R_R))$ such that $c|_T \in \mathrm{End}(T_T)$ and $c(1) = e$.

We note that $I_R \leq^{\text{ess}} eT_R = c(1)T_R = cT_R$, so $I_R \leq^{\text{ess}} cE(R_R)$. Thus, $b\,\beta\,c$. □

The next result will be used to find right extending right rings of quotients of certain rings in Sect. 9.1.

Theorem 8.2.15 *Let R be a ring such that $\alpha = \beta$ (e.g., $Z(R_R) = 0$), and let T be a right ring of quotients of R. Then the following are equivalent.*

(i) *T is right extending.*
(ii) *There exists a right extending α pseudo ring hull $R(\mathbf{E}, \alpha, Q(R))$ and it is a subring of T.*

Proof (i)⇒(ii) Assume that T is right extending. To apply Proposition 8.2.14, let $I \in \mathfrak{D}_{\mathbf{E}}(R)$, that is, $I_R \leq R_R$. By the proof of Lemma 8.1.3(ii), $I_R \leq^{\text{ess}} IT_R$. Take $J = IT$. Since T is right extending, there is $e \in \mathbf{I}(T)$ with $J_T \leq^{\text{ess}} eT_T$. Thus $J_R \leq^{\text{ess}} eT_R$ by Lemma 8.1.3(i), so $I_R \leq^{\text{ess}} J_R \leq^{\text{ess}} eT_R$.

By Proposition 8.2.14, there is $\delta_{\mathbf{E}}^{\beta}(R)$ with $c|_T \in \mathrm{End}(T_T)$ and $c(1) \in \mathbf{I}(T)$ for each $c \in \delta_{\mathbf{E}}^{\beta}(R)$. Take $S = \langle R \cup \delta_{\mathbf{E}}^{\beta}(R)(1)\rangle_T = \langle R \cup \delta_{\mathbf{E}}^{\beta}(R)(1)\rangle_{Q(R)}$. Now for each $K_S \leq S_S$, $(K \cap R)_R \leq R_R$ and $(K \cap R)_R \leq^{\mathrm{ess}} K_R$. Since $Z(R_R) = 0$, $\alpha = \beta$ (Exercise 8.2.16.3), and hence $S = \langle R \cup \delta_{\mathbf{E}}^{\alpha}(R)(1)\rangle_{Q(R)} = R(\mathbf{E}, \alpha, Q(R))$ by Proposition 8.2.13. Clearly S is a subring of T.

(ii)⇒(i) The proof follows from Theorem 8.1.8(iii). □

Exercise 8.2.16

1. ([89, Birkenmeier, Park, and Rizvi]) Assume that A is a commutative local QF-ring such that $J(A) \neq 0$. In this case, we take $S_0 = \begin{bmatrix} A & J(A) \\ 0 & A \end{bmatrix}$, $S_1 = T_2(A)$, $S_2 = \begin{bmatrix} A & A \\ J(A) & A \end{bmatrix}$, and $S_3 = \mathrm{Mat}_2(A)$. Prove that the following hold true.
   (i) $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3$ is a chain of subrings of S_3 where S_i, $1 \leq i \leq 3$ is a right essential overring of its predecessor.
   (ii) ${S_0}_{S_0} \leq^{\mathrm{ess}} {S_1}_{S_0}$, ${S_1}_{S_1} \leq^{\mathrm{den}} {S_3}_{S_1}$, but ${S_0}_{S_0}$ is not essential in ${S_2}_{S_0}$.
   (iii) S_1 is a right FI-extending ring hull of S_0, $S_2 = Q_{\mathbf{E}}(S_1)$, and also $S_3 = Q_{\mathbf{SI}}(S_1) = Q_{\mathbf{SI}}(S_2)$, where **SI** is the class of right self-injective rings.
2. ([89, Birkenmeier, Park, and Rizvi]) Assume that $\mathfrak{U}$ denotes the class of rings, $\{R \mid R \cap \mathbf{U}(Q(R)) = \mathbf{U}(R)\}$, where $\mathbf{U}(-)$ is the set of invertible elements of a ring. We let $R_1 = \langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R\}\rangle_{Q(R)}$. Let i and j be ordinal numbers. When $j = i + 1$, put
$$R_j = \langle R_i \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R_i\}\rangle_{Q(R)}.$$
   If j is a limit ordinal, let $R_j = \cup_{i<j} R_i$. Prove the following.
   (i) $\widehat{Q}_{\mathfrak{U}}(R)$ exists and $\widehat{Q}_{\mathfrak{U}}(R) = R_j$ for any j with $|j| > |Q(R)|$.
   (ii) If R is a right Ore ring, then $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$. Thus $\widehat{Q}_{\mathfrak{U}}(R)$ is a ring hull that coincides with $Q^r_{c\ell}(R)$ when R is right Ore.
3. Let α and β be as in Definition 8.2.11. Show that $\alpha = \beta$ if $Z(R_R) = 0$.
4. ([89, Birkenmeier, Park, and Rizvi]) Let R be the ring in Example 8.2.9. Show that $\cap_\alpha R(\mathbf{FI}, \alpha, Q(R)) = T_3(F)$.
5. ([89, Birkenmeier, Park, and Rizvi]) Let T be a right ring of quotients of a ring R and assume that $IT \trianglelefteq T$ for any $I \trianglelefteq R$. Prove that $T \in \mathbf{FI}$ if and only if there exists an $R(\mathbf{FI}, \beta, Q(R))$ which is a subring of T.
6. ([89, Birkenmeier, Park, and Rizvi]) Assume that W is a local ring and V is a subring of W with $J(W) \subseteq V$. Let $R = \begin{bmatrix} V & W \\ 0 & W \end{bmatrix}$, $S = \begin{bmatrix} V & W \\ J(W) & W \end{bmatrix}$, and $T = \mathrm{Mat}_2(W)$. Prove the following.
   (i) For each $e \in \mathbf{I}(T)$, there exists $f \in \mathbf{I}(S)$ such that $e \,\alpha\, f$.
   (ii) $S \in \mathbf{E}$ if and only if $T \in \mathbf{E}$ if and only if $S = R(\mathbf{E}, \rho, T)$ for some ρ.
   (iii) If W is right self-injective, then $S = R(\mathbf{E}, \alpha, T)$.
   (iv) If W is right self-injective, then $Q_{\mathbf{qCon}}(R) = R(\mathbf{E}, T) = T$.
   (v) $R \in \mathbf{FI}$ if and only if $W \in \mathbf{FI}$.

7. ([89, Birkenmeier, Park, and Rizvi]) Let W be a local ring and V be a subring of W. Take $R = \begin{bmatrix} V & W \\ 0 & W \end{bmatrix}$. Show that the following are equivalent.
   (i) R is right extending.
   (ii) $T_2(W)$ is right extending.
   (iii) W is a division ring.
8. ([89, Birkenmeier, Park, and Rizvi]) Assume that A is a right FI-extending ring and $W = \oplus_{i=1}^{n} A_i$, where $A_i = A$ for each i. Let D be the set of all $(a_1, \ldots, a_n) \in W$ such that $a_i = a \in A$ for all $i = 1, \ldots, n$. Say S is a subring of W containing D. Prove that the ring $H = \begin{bmatrix} W & W \\ 0 & A \end{bmatrix}$ is a right FI-extending ring hull of the ring $R = \begin{bmatrix} S & W \\ 0 & A \end{bmatrix}$.
9. ([89, Birkenmeier, Park, and Rizvi]) Assume that R is a ring such that $Q(R)$ is Abelian. Prove the following.
   (i) $\widehat{Q}_{\mathbf{E}}(R) = \widehat{Q}_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$ if and only if $Q(R)$ is right extending.
   (ii) Let R be a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$ and $Z(R_R)$ has finite right uniform dimension. Then $Q(R)$ is right extending if and only if $\widehat{Q}_{\mathbf{Con}}(R)$ exists and $\widehat{Q}_{\mathbf{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a right continuous strongly regular ring and H_2 is a direct sum of right continuous local rings.
10. ([89, Birkenmeier, Park, and Rizvi]) Let R be a commutative ring. Prove the following.
   (i) If R or $Q^{r}_{c\ell}(R)$ is extending, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^{r}_{c\ell}(R)$.
   (ii) If $Z(R_R) = 0$, then $\widehat{Q}_{\mathrm{Con}}(R)$ is the intersection of all regular right rings of quotients T of R such that $\mathcal{B}(Q(R)) \subseteq T$.

8.3 Idempotent Closure Classes and Ring Hulls

This section is mainly devoted to discussions and study of Problems I and II mentioned in the introduction of this chapter. As $E(R_R)$ is extending, for each right ideal I of R there exists $e^2 = e \in \mathrm{End}(E(R_R))$ such that $I_R \leq^{\mathrm{ess}} eE(R_R)$. Furthermore, in many cases $Q(R) = E(R_R)$ (e.g., when $Z(R_R) = 0$). So one may expect that $Q(R)$ would satisfy the extending property for a certain subset of the set of right ideals of R.

We let $\mathfrak{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0 \text{ and } \ell_R(I) \cap \ell_R(\ell_R(I)) = 0\}$. In Theorem 8.3.8, we show that $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ if and only if there exists e in $\mathcal{B}(Q(R))$ such that $I_R \leq^{\mathrm{den}} eQ(R)_R$. This result motivates the definition of the idempotent closure class of rings, we shall consider in this section, denoted by **IC**. This class of rings is a **D-E** class for which $\widehat{Q}_{\mathbf{IC}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ (see Theorem 8.3.11). Thus this hull exists for every ring (not necessarily with identity) for which $Q(R)$ exists (i.e., when $\ell_R(R) = 0$). The set $\mathfrak{D}_{\mathbf{IC}}(R)$ forms a sublattice of the lattice of ideals of R and is quite large, in general. In fact, if R is semiprime, then

$\mathfrak{D}_{\mathbf{IC}}(R)$ is the full lattice of ideals of R. From this if R is a semiprime ring, then $\widehat{Q}_{\mathbf{FI}}(R) = \widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{eqB}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$. Further, if R is a semiprime ring with identity, then $\widehat{Q}_{\mathbf{FI}}(R) = R(\mathbf{FI}, Q(R))$ and $\widehat{Q}_{\mathbf{eqB}}(R) = R(\mathbf{eqB}, Q(R))$ (see Theorem 8.3.17).

This result demonstrates that the semiprime condition of a ring R overcomes the somewhat chaotic situation we encountered in Examples 8.2.2, 8.2.3, 8.2.9, and 8.2.10 by providing a unique ring hull which agrees with its pseudo ring hulls. Next we consider the transfer of algebraic information between R and $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ in terms of prime ideals, various radicals, regularity conditions, and so on (see Problem I). We shall see that for a semiprime ring R with identity, $\widehat{Q}_{\mathbf{pqB}}(R)$, $\widehat{Q}_{\mathbf{pFI}}(R)$, and $\widehat{Q}_{\mathbf{fgFI}}(R)$ all exist and are equal to each other. Also the transfer of algebraic information between R and these various hulls will also be discussed. Finally, we shall apply these results to obtain a proper generalization of Rowen's well-known result: Let R be a semiprime PI-ring. Then $\mathrm{Cen}(R) \cap I \neq 0$ for any $0 \neq I \trianglelefteq R$ (Theorem 3.2.16).

Throughout this section, R does not necessarily have an identity unless mentioned otherwise. However, we assume that $\ell_R(R) = 0$ to guarantee the existence of $Q(R)$ which has an identity (*see* [395]).

Definition 8.3.1 (i) Let R be a ring. We recall that $\mathfrak{D}_{\mathbf{IC}}(R)$ is the set of all ideals of R such that $I \cap \ell_R(I) = 0$ and $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$.

(ii) A ring R is called an **IC**-*ring* if for each $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ there exists $e^2 = e \in R$ such that $I_R \leq^{\text{ess}} eR_R$. The class of **IC**-rings is denoted by **IC** and is called the *idempotent closure class*. Thus, **IC** is a **D-E** class.

If a ring R with identity is right FI-extending, then $R \in \mathbf{IC}$. The set $\mathfrak{D}_{\mathbf{IC}}(R)$ was studied by Johnson [236] and denoted by $\mathfrak{F}'(R)$. While Propositions 8.3.2 and 8.3.3 provide examples of $\mathfrak{D}_{\mathbf{IC}}(R)$ when R is right nonsingular or semiprime, we shall see from Theorem 8.3.8 that $R \cap eQ(R) \in \mathfrak{D}_{\mathbf{IC}}(R)$ for any $e \in \mathcal{B}(Q(R))$. Also Theorem 8.3.11(ii) characterizes the **IC** class of rings.

Proposition 8.3.2 *If $Z(R_R) = 0$, then $\mathfrak{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0\}$.*

Proof Assume that $I \trianglelefteq R$ such that $I \cap \ell_R(I) = 0$. Say J_R is a complement of I_R in R_R. Then $(I \oplus J)_R \leq^{\text{ess}} R_R$. Now $JI \subseteq J \cap I = 0$, thus $J \subseteq \ell_R(I)$. Therefore $(I \oplus \ell_R(I))_R \leq^{\text{ess}} R_R$. If $x(I \oplus \ell_R(I)) = 0$, then $x = 0$ because $Z(R_R) = 0$. Hence $\ell_R(I \oplus \ell_R(I)) = \ell_R(I) \cap \ell_R(\ell_R(I)) = 0$. □

Proposition 8.3.3 (i) *A ring R is semiprime if and only if $\mathfrak{D}_{\mathbf{IC}}(R)$ is precisely the set of all ideals of R.*

(ii) *If $e \in \mathbf{S}_\ell(R)$, then $eR \in \mathfrak{D}_{\mathbf{IC}}(R)$ if and only if $e \in \mathcal{B}(R)$.*

(iii) *Let P be a prime ideal of R. Then $P \in \mathfrak{D}_{\mathbf{IC}}(R)$ if and only if $P \cap \ell_R(P) = 0$.*

(iv) *Let P be a prime ideal of R and $P \in \mathfrak{D}_{\mathbf{IC}}(R)$. If $I \trianglelefteq R$ such that $P \subseteq I$, then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

(v) *If $I \trianglelefteq R$ such that $\ell_R(I) \cap P(R) = 0$, then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

(vi) *If $Z(R_R) = 0$ and $I \trianglelefteq R$ such that $I \cap P(R) = 0$, then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

Proof (i) Assume that R is a semiprime ring. Let $I \trianglelefteq R$. Since R is semiprime and $(I \cap \ell_R(I))^2 = 0$, $I \cap \ell_R(I) = 0$. Now $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$ because $\ell_R(I) \trianglelefteq R$. So $\mathfrak{D}_{\mathbf{IC}}(R)$ is the set of all ideals of R. Conversely, assume that $\mathfrak{D}_{\mathbf{IC}}(R)$ is the set of all ideals of R. Let $I \trianglelefteq R$ with $I^2 = 0$. Then $I \subseteq \ell_R(I)$. As $I \in \mathfrak{D}_{\mathbf{IC}}(R)$, $I \cap \ell_R(I) = 0$ and so $I = 0$. Hence, R is semiprime.

(ii)–(vi) Exercise. □

Let R be a ring and $I \trianglelefteq R$ with $I \cap \ell_R(I) = 0$. As $I\ell_R(I) \subseteq I \cap \ell_R(I) = 0$, so $I \subseteq \ell_R(\ell_R(I))$. The next lemma will be used in the sequel. We note that every ideal in a semiprime ring satisfies all of these conditions.

Lemma 8.3.4 *Assume that $I \trianglelefteq R$ with $I \cap \ell_R(I) = 0$. Then the following are equivalent.*

(i) $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$.
(ii) $\ell_R(I \oplus \ell_R(I)) = 0$.
(iii) $(I \oplus \ell_R(I))_R \leq^{\text{den}} R_R$.
(iv) $I_R \leq^{\text{den}} \ell_R(\ell_R(I))_R$.
(v) $I_R \leq^{\text{ess}} \ell_R(\ell_R(I))_R$.

Proof Exercise. □

Proposition 8.3.5 *Let R be a ring. Then $\mathfrak{D}_{\mathbf{IC}}(R)$ is the set of all ideals I of R such that there exists an ideal J of R with $I \cap J = 0$ and $(I \oplus J)_R \leq^{\text{den}} R_R$.*

Proof Let $\mathfrak{D}_1$ be the set of all ideals I of R such that there is an ideal J of R satisfying $I \cap J = 0$ and $(I \oplus J)_R \leq^{\text{den}} R_R$. Then we show that $\mathfrak{D}_{\mathbf{IC}}(R) = \mathfrak{D}_1$. Take $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $J = \ell_R(I)$. Then $I \cap J = 0$. Also, $\ell_R(I \oplus J) = \ell_R(I) \cap \ell_R(J) = \ell_R \cap \ell_R(\ell_R(I)) = 0$ as $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. By Lemma 8.3.4 or Proposition 1.3.11(iv), $(I \oplus J)_R \leq^{\text{den}} R_R$. Thus $I \in \mathfrak{D}_1$, and so $\mathfrak{D}_{\mathbf{IC}}(R) \subseteq \mathfrak{D}_1$.

Next, we take $I \in \mathfrak{D}_1$. Then there exists $J \trianglelefteq R$ satisfying that $I \cap J = 0$ and $(I \oplus J)_R \leq^{\text{den}} R_R$. We note that $J \subseteq \ell_R(I)$, $I \subseteq \ell_R(J)$, and by Proposition 1.3.11(iv) $\ell_R(I \oplus J) = \ell_R(I) \cap \ell_R(J) = 0$. Thus $I \cap \ell_R(I) = 0$. Since $J \subseteq \ell_R(I)$, $\ell_R(\ell_R(I)) \subseteq \ell_R(J)$. Hence $\ell_R(I) \cap \ell_R(\ell_R(I)) \subseteq \ell_R(I) \cap \ell_R(J) = 0$, and thus $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Therefore $\mathfrak{D}_1 \subseteq \mathfrak{D}_{\mathbf{IC}}(R)$. Whence $\mathfrak{D}_{\mathbf{IC}}(R) = \mathfrak{D}_1$. □

We note that $\mathfrak{D}_{\mathbf{IC}}(R)$ contains all ideals of R which are dense in R_R as right R-modules from Proposition 1.3.11(iv). Also if a ring R is semiprime, then by Proposition 8.3.3(i), $\mathfrak{D}_{\mathbf{IC}}(R)$ is precisely the set of all ideals. We provide an example of a nonsemiprime ring R where the cardinality of $\mathfrak{D}_{\mathbf{IC}}(R)$ is greater than or equal to the cardinality of its complement in the set of all ideals of R. Indeed, take $R = T_2(S)$, where S is a right nonsingular prime ring with identity. The set of all ideals of R is $\left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid A, B, C \trianglelefteq S \text{ with } A, C \subseteq B \right\}$. Since R is right nonsingular, by Proposition 8.3.2

$$\mathfrak{D}_{\mathbf{IC}}(R) = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \mid A, B, C \trianglelefteq S \text{ with } A, C \subseteq B \text{ and } C \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Hence, we see that the cardinality of $\mathfrak{D}_{\mathbf{IC}}(R)$ is greater than or equal to the cardinality of its complement.

Lemma 8.3.6 *Assume that T is a right ring of quotients of R and let $I \in \mathfrak{D}_{\mathbf{IC}}(T)$. Then $I \cap R \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

Proof Let $I \in \mathfrak{D}_{\mathbf{IC}}(T)$ and put $K = I \cap R$. Then $\ell_R(K) = \ell_R(I)$ from Lemma 8.1.6(i). Hence $K \cap \ell_R(K) = K \cap \ell_R(I) \subseteq I \cap \ell_R(I) \subseteq I \cap \ell_T(I) = 0$.

Say $a \in \ell_R(K \oplus \ell_R(K))$. Then $a \in \ell_R(K) = \ell_R(I)$, so $aI = 0$. We show that $a\,\ell_T(I) = 0$. For this, assume on the contrary that $at \neq 0$ for some $t \in \ell_T(I)$. Then there exists $r \in R$ satisfying $tr \in R$ and $atr \neq 0$ since $R_R \leq^{\text{den}} T_R$. Therefore

$$tr \in R \cap \ell_T(I) = \ell_R(I) = \ell_R(K).$$

Because $a \in \ell_R(K \oplus \ell_R(K))$, $a\,\ell_R(K) = 0$. So $atr = 0$, a contradiction. Hence we get $a\,\ell_T(I) = 0$ and $a \in \ell_T(I) \cap \ell_T(\ell_T(I)) = 0$. So $\ell_R(K \oplus \ell_R(K)) = 0$, as a consequence $K \in \mathfrak{D}_{\mathbf{IC}}(R)$. □

Lemma 8.3.7 *Let I and J be ideals of R.*

(i) *If $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $I_R \leq^{\text{ess}} J_R$, then $I_R \leq^{\text{den}} J_R$ and $J \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

(ii) *If $I_R \leq^{\text{den}} J_R$ and $J \in \mathfrak{D}_{\mathbf{IC}}(R)$, then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

(iii) *If $I \cap J = 0$ and $I \oplus J \in \mathfrak{D}_{\mathbf{IC}}(R)$, then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $J \in \mathfrak{D}_{\mathbf{IC}}(R)$.*

(iv) *$I \in \mathfrak{D}_{\mathbf{IC}}(R)$ if and only if $\ell_R(I) \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $I \cap \ell_R(I) = 0$.*

Proof (i) Assume that $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $I_R \leq^{\text{ess}} J_R$. From Proposition 8.3.5, there exists $K \trianglelefteq R$ such that $(I \oplus K)_R \leq^{\text{den}} R_R$. By the modular law,

$$(J \cap (I \oplus K))_R = (I \oplus (J \cap K))_R \leq^{\text{den}} J_R.$$

As $I_R \leq^{\text{ess}} J_R$ and $I \cap (J \cap K) = 0$, $J \cap K = 0$, so $I_R \leq^{\text{den}} J_R$. We show that $\ell_R(I) = \ell_R(J)$. For this, it suffices to see that $\ell_R(I) \subseteq \ell_R(J)$. Assume on the contrary that there is $x \in \ell_R(I)$ but $xJ \neq 0$. There is $y \in J$ with $xy \neq 0$. Since $I_R \leq^{\text{den}} J_R$, there is $r \in R$ such that $yr \in I$ and $xyr \neq 0$, which is a contradiction since $xI = 0$. Thus $\ell_R(I) = \ell_R(J)$.

So $I \cap \ell_R(J) = I \cap \ell_R(I) = 0$ and $J \cap \ell_R(J) = 0$. Now $I \oplus \ell_R(I) \subseteq J \oplus \ell_R(J)$. Thus, $(J \oplus \ell_R(J))_R \leq^{\text{den}} R_R$ as $(I \oplus \ell_R(I))_R \leq^{\text{den}} R_R$. Hence, $J \in \mathfrak{D}_{\mathbf{IC}}(R)$.

(ii) Let $J \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $I_R \leq^{\text{den}} J_R$. Then $\ell_R(I) = \ell_R(J)$ by the proof of part (i). From Lemma 8.3.4, $J_R \leq^{\text{ess}} \ell_R(\ell_R(J))_R = \ell_R(\ell_R(I))_R$. Therefore, it follows that $I_R \leq^{\text{ess}} \ell_R(\ell_R(I))_R$. Hence, we obtain $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$ from Lemma 8.3.4 because $I \cap \ell_R(I) \subseteq J \cap \ell_R(J) = 0$. Therefore, $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.

(iii) Suppose that $I \oplus J \in \mathfrak{D}_{\mathbf{IC}}(R)$. From Proposition 8.3.5, there is $V \trianglelefteq R$ with $((I \oplus J) \oplus V)_R \leq^{\text{den}} R_R$. Therefore, $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and $J \in \mathfrak{D}_{\mathbf{IC}}(R)$ again by Proposition 8.3.5.

(iv) Say $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Then $I \cap \ell_R(I) = 0$ and $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$. Since $I \subseteq \ell_R(\ell_R(I))$, we have that $I \oplus \ell_R(I) \subseteq \ell_R(\ell_R(I)) \oplus \ell_R(I)$. As a consequence $\ell_R[\ell_R(I) \oplus \ell_R(\ell_R(I))] \subseteq \ell_R(I \oplus \ell_R(I)) = 0$. So $\ell_R(I) \in \mathfrak{D}_{\mathbf{IC}}(R)$.

Conversely, $\ell_R(I) \in \mathfrak{D}_{\mathbf{IC}}(R)$ implies $\ell_R(I) \cap \ell_R(\ell_R(I)) = 0$. Therefore, $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ because $I \cap \ell_R(I) = 0$ by assumption. $\square$

Let R be a ring (not necessarily with identity) with $\ell_R(R) = 0$. Say U is a subring of R such that $U_U \leq^{\text{den}} R_U$ (i.e., for $x, y \in R$ with $y \neq 0$, there exists $u \in U$ satisfying $xu \in U$ and $yu \neq 0$). Then $\ell_U(U) = 0$. Indeed, let $x \in \ell_U(U)$. If $xr \neq 0$ for some $r \in R$, then there exists $u \in U$ such that $ru \in U$ and $xru \neq 0$, a contradiction. So $x \in \ell_R(R) = 0$, and hence $\ell_U(U) = 0$. Thus, $Q(U)$ exists. Therefore, $Q(U) = Q(R)$ as R is a right ring of quotients of U.

The following result characterizes the ideals of R which are dense as right R-modules in some ring direct summands of $Q(R)$ as precisely the elements of $\mathfrak{D}_{\mathbf{IC}}(R)$.

Theorem 8.3.8 *Assume that R is a ring and $I \trianglelefteq R$. Then the following are equivalent.*

(i) $I \in \mathfrak{D}_{\mathbf{IC}}(R)$.
(ii) *There exists $e \in \mathcal{B}(Q(R))$ such that $Q(I) = eQ(R)$.*
(iii) $I_R \leq^{\text{den}} eQ(R)_R$ *for some (unique) $e \in \mathcal{B}(Q(R))$.*

Proof (i)⇒(ii) Put $J = \ell_R(I)$. Since $I \in \mathfrak{D}_{\text{IC}}(R)$, $\ell_R(I \oplus J) = 0$ and hence $\ell_{I \oplus J}(I \oplus J) = 0$. Therefore $\ell_I(I) = 0$ and $\ell_J(J) = 0$, hence $Q(I)$ and $Q(J)$ exist. Put $U = I \oplus J$. For $U_U \leq^{\text{den}} R_U$, take $x, y \in R$ with $y \neq 0$. As $U_R \leq^{\text{den}} R_R$, there exists $r \in R$ such that $xr \in U$ and $yr \neq 0$. Again since $U_R \leq^{\text{den}} R_R$, there exists $a \in R$ satisfying that $ra \in U$ and $yra \neq 0$. Because $ra \in U$ and $xra \in U$, we see that $U_U \leq^{\text{den}} R_U$. So, $Q(R) = Q(U) = Q(I \oplus J) = Q(I) \oplus Q(J)$ by [395, (2.1)]. Consequently, $Q(I) = eQ(R)$ for some $e \in \mathcal{B}(Q(R))$.

(ii)⇒(iii) Say $Q(I) = eQ(R)$ for some $e \in \mathcal{B}(Q(R))$. Take $eq_1, eq_2 \in eQ(R)$ with $q_1, q_2 \in Q(R)$ and $eq_2 \neq 0$. As $I_I \leq^{\text{den}} Q(I)_I$, there exists $a \in I$ such that $eq_1a \in I$ and $eq_2a \neq 0$. Since $a \in R$, $I_R \leq^{\text{den}} eQ(R)_R$. If $f \in \mathcal{B}(Q(R))$ satisfying $I_R \leq^{\text{den}} fQ(R)_R$, then $e = f$ as $e \in \mathcal{B}(Q(R))$.

(iii)⇒(i) Let $I_R \leq^{\text{den}} eQ(R)_R$ for some $e \in \mathcal{B}(Q(R))$. Then we have that $I_R \leq^{\text{den}} (eQ(R) \cap R)_R$. Now Lemma 8.3.6 yields that $eQ(R) \cap R \in \mathfrak{D}_{\mathbf{IC}}(R)$ because $eQ(R) \in \mathfrak{D}_{\mathbf{IC}}(Q(R))$. From Lemma 8.3.7(ii), $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. $\square$

We note that if $I \in \mathfrak{D}_{\mathbf{IC}}(R)$, then from Lemma 8.3.4, Lemma 8.3.7(i), and Theorem 8.3.8, there exists $e \in \mathcal{B}(Q(R))$ such that $\ell_R(\ell_R(I))_R \leq^{\text{den}} eQ(R)_R$. Further, $\ell_R(\ell_R(I)) = eQ(R) \cap R$ and $\ell_R(\ell_R(I))$ is the unique closure of I_R in R_R (see Exercise 8.3.58.5).

Corollary 8.3.9 *Assume that $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ and T is a right ring of quotients of R. Then $(I) \in \mathfrak{D}_{\mathbf{IC}}(T)$ and $I_R \leq^{\text{den}} (I)_R$, where (I) is the ideal of T generated by I.*

Proof There exists $e \in \mathcal{B}(Q(R))$ with $I_R \leq^{\text{den}} eQ(R)$ from Theorem 8.3.8. Hence, $I_R \leq (I)_R \leq eQ(R)$ as $I = eI$. Therefore $(I)_R \leq^{\text{den}} eQ(R)_R$, and thus we see that $(I)_T \leq^{\text{den}} eQ(R)_T$. Because $Q(R) = Q(T)$, $(I)_T \leq^{\text{den}} eQ(T)_T$. Thus from Theorem 8.3.8, $(I) \in \mathfrak{D}_{\mathbf{IC}}(T)$. □

Say $A \in \mathfrak{D}_{\mathbf{IC}}(Q(R))$. Then $A_{Q(R)} \leq^{\text{den}} eQ(R)_{Q(R)}$ for some $e \in \mathcal{B}(Q(R))$ by Theorem 8.3.8. Thereby $Q(R)$ is an **IC**-ring and this suggests that there may be a smallest right ring of quotients of R which is an **IC**-ring. So one may naturally ask: *Does $\widehat{Q}_{\mathbf{IC}}(R)$ exist for every ring R when $\ell_R(R) = 0$?* For this question, we need the following lemma.

Lemma 8.3.10 *Assume that R is a ring with identity and $b \in \mathcal{B}(Q(R))$. Then there exists $\lambda \in \mathcal{B}(\text{End}(E(R_R)))$ such that $b = \lambda(1)$.*

Proof Note that $E(R_R)$ is an $(\text{End}(E(R_R)), Q(R))$-bimodule. Define a map

$$\lambda : E(R_R) \to E(R_R) \text{ by } \lambda(x) = xb$$

for $x \in E(R_R)$. Then $\lambda \in \text{End}(E(R_R))$ and $\lambda^2 = \lambda$ because $b \in \mathcal{B}(Q(R))$. Next, say $\varphi \in \text{End}(E(R_R))$. For $x \in E(R_R)$,

$$(\lambda\varphi)(x) = \varphi(x)b = \varphi(xb),$$

since $\text{End}(E(R_R)) = \text{End}(E(R_R)_{Q(R)})$ (see the proof of Theorem 2.1.31). Further, $(\varphi\lambda)(x) = \varphi(xb)$. So $\lambda\varphi(x) = \varphi\lambda(x)$ for all $x \in E(R_R)$, thus $\lambda\varphi = \varphi\lambda$. Hence $\lambda \in \mathcal{B}(\text{End}(E(R_R)))$ and $b = \lambda(1)$. □

Our next result shows that $\widehat{Q}_{\mathbf{IC}}(R)$ exists for all rings R with $\ell_R(R) = 0$ and it can be used to characterize **IC** right rings of quotients of R. When R is a ring with $\ell_R(R) = 0$, we recall from 1.1.2 that $\langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)}$ denotes the subring of $Q(R)$ generated by $R \cup \mathcal{B}(Q(R))$. Observe that if R has identity, then we see that $\langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)} = R\mathcal{B}(Q(R))$.

Theorem 8.3.11 *Assume that R is a ring.*

(i) *Let T be a right ring of quotients of R. Then $T \in$ **IC** if and only if $\mathcal{B}(Q(R)) \subseteq T$.*

(ii) *$R \in$ **IC** if and only if $\mathcal{B}(Q(R)) \subseteq R$. Hence, **IC**-rings have identity.*

(iii) *$\widehat{Q}_{\mathbf{IC}}(R) = \langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)}$.*

(iv) *If R has identity, then $\widehat{Q}_{\mathbf{IC}}(R) = R(\mathbf{IC}, Q(R))$.*

Proof (i) Say $T \in$ **IC**. Take $c \in \mathcal{B}(Q(R))$ and we let $I = R \cap cQ(R)$. Then $I_R \leq^{\text{ess}} cQ(R)_R$. We note that $cQ(R) \in \mathfrak{D}_{\mathbf{IC}}(Q(R))$. From Lemma 8.3.6, $Y := cQ(R) \cap T \in \mathfrak{D}_{\mathbf{IC}}(T)$ and $I_R \leq^{\text{ess}} Y_R$. Since $Y \in \mathfrak{D}_{\mathbf{IC}}(T)$ and $T \in$ **IC**, $Y_T \leq^{\text{ess}} eT_T$ for some $e \in \mathbf{I}(T)$. Thus, $Y_R \leq^{\text{ess}} eT_R$ by Lemma 8.1.3(i). Now $c = e \in T$, as

$I_R \leq^{\text{ess}} Y_R \leq^{\text{ess}} eT_R \leq^{\text{ess}} eQ(R)_R$ and $I_R \leq^{\text{ess}} Y_R \leq^{\text{ess}} cQ(R)_R$. So $\mathcal{B}(Q(R)) \subseteq T$. Conversely, let $\mathcal{B}(Q(R)) \subseteq T$. Take $I \in \mathfrak{D}_{\mathbf{IC}}(T)$. As $Q(R) = Q(T)$, Theorem 8.3.8 yields that there is $e \in \mathcal{B}(Q(T)) \subseteq T$ such that $I_T \leq^{\text{den}} eQ(T)_T$. Hence, we get that $I_T \leq^{\text{den}} eT_T$. Therefore, $T \in \mathbf{IC}$.

(ii) and (iii) These parts follows from part (i) immediately.

(iv) By part (iii) $\widehat{Q}_{\mathbf{IC}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$. Recall that

$$\delta_{\mathbf{IC}}(R) = \{e^2 = e \in \text{End}(E(R_R)) \mid I_R \leq^{\text{ess}} eE(R_R) \text{ for some } I \in \mathfrak{D}_{\mathbf{IC}}(R)\}$$

and $\delta_{\mathbf{IC}}(R)(1) = \{e(1) \mid e \in \delta_{\mathbf{IC}}(R)\}$.

We prove that $\mathcal{B}(Q(R)) = \delta_{\mathbf{IC}}(R)(1)$. For this, say $c \in \mathcal{B}(Q(R))$. Then it follows that $R \cap cQ(R) \trianglelefteq R$ and $(R \cap cQ(R))_R \leq^{\text{ess}} cQ(R)_R$. From Lemma 8.3.6, we get $R \cap cQ(R) \in \mathfrak{D}_{\mathbf{IC}}(R)$. Also, there exists $\lambda^2 = \lambda \in \mathcal{B}(\text{End}(E(R_R)))$ such that $c = \lambda(1)$ by Lemma 8.3.10.

We note that $(R \cap cQ(R))_R \leq^{\text{ess}} \lambda(1)Q(R)_R = \lambda Q(R)_R \leq^{\text{ess}} \lambda E(R_R)$ because $\lambda \in \text{End}(E(R_R)_{Q(R)})$. Thus $\lambda \in \delta_{\mathbf{IC}}(R)$, so $c = \lambda(1) \in \delta_{\mathbf{IC}}(R)(1)$. As a consequence, $\mathcal{B}(Q(R)) \subseteq \delta_{\mathbf{IC}}(R)(1)$.

Next, say $h \in \delta_{\mathbf{IC}}(R)$. Then there is $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ with $I_R \leq^{\text{ess}} hE(R_R)$. By Theorem 8.3.8, $I_R \leq^{\text{ess}} bQ(R)_R$ for some $b \in \mathcal{B}(Q(R))$. From Lemma 8.3.10, there exists $\gamma \in \mathcal{B}(\text{End}(E(R_R)))$ such that $b = \gamma(1)$. Sometimes we will use E_R for $E(R_R)$.

Observe that $I_R \leq^{\text{ess}} bQ(R)_R = \gamma(1)Q(R)_R = \gamma Q(R)_R \leq^{\text{ess}} \gamma E(R_R)$. So $hE(R_R) = \gamma E(R_R)$ because $\gamma \in \mathcal{B}(\text{End}(E_R))$. Therefore $h(1) = \gamma(x)$ for some $x \in E(R_R)$, and hence $\gamma h(1) = h(1)$. Also $\gamma(1) = h(y)$ with $y \in E(R_R)$. As a consequence, $h\gamma(1) = \gamma(1)$, so $h(1) = \gamma h(1) = h\gamma(1) = \gamma(1) = b$. Hence, it follows that $\delta_{\mathbf{IC}}(R)(1) \subseteq \mathcal{B}(Q(R))$. Therefore, $\mathcal{B}(Q(R)) = \delta_{\mathbf{IC}}(R)(1)$.

Now $\langle R \cup \delta_{\mathbf{IC}}(R)(1)\rangle_{Q(R)} = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$. By the definition of pseudo ring hulls, $R(\mathbf{IC}, Q(R)) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ since $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ is an $\mathbf{IC}$-ring by part (iii). □

From Theorems 8.3.8 and 8.3.11, we see that any intermediate ring T between $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ and $Q(R)$ satisfies that for every $I \in \mathfrak{D}_{\mathbf{IC}}(R)$, there exists $e^2 = e \in T$ such that $I_R \leq^{\text{ess}} eT_R$. Furthermore, we see that for every $J \in \mathfrak{D}_{\mathbf{IC}}(T)$, $J_T \leq^{\text{ess}} fT_T$ for some $f^2 = f \in T$.

Corollary 8.3.12 *Let R be an* **IC***-ring with $Z(R_R) = 0$. Then $R = R_1 \oplus R_2$ (ring direct sum), where R_1 is a semiprime FI-extending ring and $P(R)$ is ideal essential in R_2.*

Proof Exercise. □

The following result is on the lattice properties of $\mathfrak{D}_{\mathbf{IC}}(R)$ as suggested by earlier results.

Theorem 8.3.13 (i) *$\mathfrak{D}_{\mathbf{IC}}(R)$ is a sublattice of the lattice of ideals of R.*

(ii) *If $\mathfrak{D}_{\mathbf{IC}}(R)$ is a complete sublattice of the lattice of ideals of R, then $\mathcal{B}(Q(R))$ is a complete Boolean algebra.*

(iii) *Let* $R \in \mathbf{IC}$ *such that* $\mathfrak{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0\}$. *Then* $\mathfrak{D}_{\mathbf{IC}}(R)$ *is a complete sublattice of the lattice of ideals of* R.

(iv) *If* R *is right and left FI-extending, then* $\mathfrak{D}_{\mathbf{IC}}(R)$ *is a complete sublattice of the lattice of ideals of* R.

Proof (i) Assume that $I, J \in \mathfrak{D}_{\mathbf{IC}}(R)$. By Theorem 8.3.8 there are unique c_1, c_2 in $\mathcal{B}(Q(R))$ such that $I_R \leq^{\text{den}} c_1Q(R)_R$ and $J_R \leq^{\text{den}} c_2Q(R)_R$. Therefore

$$(I \cap J)_R \leq^{\text{den}} c_1Q(R)_R \cap c_2Q(R)_R = c_1c_2Q(R)_R \text{ and } c_1c_2 \in \mathcal{B}(Q(R)).$$

By Theorem 8.3.8, $I \cap J \in \mathfrak{D}_{\mathbf{IC}}(R)$.

Let $c = c_1 + c_2 - c_1c_2$. Then $(I + J)_R \leq (c_1Q(R) + c_2Q(R))_R = cQ(R)_R$ and $c \in \mathcal{B}(Q(R))$. Take $K = R \cap \ell_{cQ(R)}(I + J)$. Then $K \subseteq \ell_R(I) \cap \ell_R(J)$. As $I_R \leq^{\text{den}} (R \cap c_1Q(R))_R$ and $J_R \leq^{\text{den}} (R \cap c_2Q(R))_R$, it follows that $\ell_R(I) = \ell_R(R \cap c_1Q(R)) = \ell_R(c_1Q(R)) = R \cap (1 - c_1)Q(R)$ by Lemma 8.1.6(i) and the proof of Lemma 8.3.7(i). Also $\ell_R(J) = R \cap (1 - c_2)Q(R)$ similarly.

Since $K \subseteq \ell_R(I) \cap \ell_R(J) = R \cap (1 - c_1)Q(R) \cap (1 - c_2)Q(R)$, it follows that $Kc_1 = 0$ and $Kc_2 = 0$. So $Kc = 0$. But we see that $Kc = K$ because

$$K = R \cap \ell_{cQ(R)}(I + J) \subseteq cQ(R),$$

so $\ell_{cQ(R)\cap R}(I + J) = K = 0$.

Now since $I + J \trianglelefteq cQ(R) \cap R$, $(I + J)_{cQ(R)\cap R} \leq^{\text{den}} (cQ(R) \cap R)_{cQ(R)\cap R}$ from Proposition 1.3.11(iv), and hence $(I + J)_R \leq^{\text{den}} (R \cap cQ(R))_R$. Thus it follows that $(I + J)_R \leq^{\text{den}} cQ(R)_R$. By Theorem 8.3.8, $I + J \in \mathfrak{D}_{\mathbf{IC}}(R)$. Hence $\mathfrak{D}_{\mathbf{IC}}(R)$ is a sublattice of the lattice of ideals of R.

(ii) Let $\{e_i \mid i \in \Lambda\} \subseteq \mathcal{B}(Q(R))$. Then $I_i := e_iQ(R) \cap R \in \mathfrak{D}_{\mathbf{IC}}(R)$ for all $i \in \Lambda$ from Lemma 8.3.6. Put $I = \sum_{i\in\Lambda} I_i$. Then $I \in \mathfrak{D}_{\mathbf{IC}}(R)$ by assumption. From Theorem 8.3.8, there is $e \in \mathcal{B}(Q(R))$ with $I_R \leq^{\text{den}} eQ(R)_R$.

For each $i \in \Lambda$, $I_{iR} \leq^{\text{ess}} e_iQ(R)_R$. Because $I_{iR} \leq I_R \leq^{\text{ess}} eQ(R)_R$, we have that $I_{iR} \leq^{\text{ess}} ee_iQ(R)_R$. Thus, $e_i = ee_i$, so $e_i \leq e$ for all $i \in \Lambda$.

We claim that $e = \sup\{e_i \mid i \in \Lambda\}$. For this, say $f \in \mathcal{B}(Q(R))$ such that $e_i = fe_i$ (i.e., $e_i \leq f$) for all $i \in \Lambda$. By Lemma 8.3.6, $fQ(R) \cap R \in \mathfrak{D}_{\mathbf{IC}}(R)$. Since $I_i = e_iQ(R) \cap R \subseteq fQ(R) \cap R$ for all i, $I \subseteq fQ(R) \cap R \subseteq fQ(R)$. As $I_R \leq^{\text{ess}} eQ(R)_R$, $I_R \leq^{\text{ess}} (eQ(R) \cap fQ(R))_R = efQ(R)_R \leq^{\text{ess}} eQ(R)_R$, so $efQ(R) = eQ(R)$. Hence $e = ef = fe$ (i.e., $e \leq f$), so $e = \sup\{e_i \mid i \in \Lambda\}$. Therefore, $\mathcal{B}(Q(R))$ is a complete Boolean algebra.

(iii) Assume that $\{I_i \mid i \in \Lambda\} \subseteq \mathfrak{D}_{\mathbf{IC}}(R)$. Then from Theorem 8.3.8, there exists $\{e_i \mid i \in \Lambda\} \subseteq \mathcal{B}(Q(R))$ with $I_{iR} \leq^{\text{den}} e_iQ(R)_R$ for each $i \in \Lambda$.

Assume that F is a finite nonempty subset of Λ. First, say $F = \{1, 2\}$. Then $I_{1R} \leq^{\text{den}} e_1Q(R)_R$ and $I_{2R} \leq^{\text{den}} e_2Q(R)_R$. From the proof of part (i), $(I_1 + I_2)_R \leq^{\text{den}} eQ(R)_R$, where $e = e_1 + e_2 - e_1e_2$. Inductively, we can see that $\sum_{i\in F} I_{iR} \leq^{\text{den}} \sum_{i\in F} e_iQ(R)_R$. Next, we show that

$$\sum_{i\in\Lambda} I_{iR} \leq^{\text{den}} \sum_{i\in\Lambda} e_iQ(R)_R.$$

For this, let $x, y \in \sum_{i\in\Lambda} e_i Q(R)$ with $y \neq 0$. Then there is a nonempty finite subset F of Λ with $x, y \in \sum_{i\in F} e_i Q(R)$. As $\sum_{i\in F} I_{i\,R} \leq^{\text{den}} \sum_{i\in F} e_i Q(R)_R$ by the preceding argument, there is $r \in R$ with $xr \in \sum_{i\in F} I_{i\,R} \leq \sum_{i\in\Lambda} I_{i\,R}$ and $yr \neq 0$. Therefore, $\sum_{i\in\Lambda} I_{i\,R} \leq^{\text{den}} \sum_{i\in\Lambda} e_i Q(R)_R$.

From Theorem 8.3.11(ii), $\mathcal{B}(Q(R)) \subseteq R$, hence $e_i \in \mathcal{B}(R)$ for each $i \in \Lambda$. To see that $(\sum_{i\in\Lambda} e_i R) \cap \ell_R(\sum_{i\in\Lambda} e_i R) = (\sum_{i\in\Lambda} e_i R) \cap (\cap_{i\in\Lambda}(1-e_i)R) = 0$, it is enough to prove that

$$(\sum_{i\in F} e_i R) \cap (\cap_{i\in F}(1-e_i)R) = 0$$

for any nonempty finite subset F of Λ. If $F = \{1\}$, then we are done. Say $F = \{1, 2\}$. Then

$$(e_1R + e_2R) \cap ((1-e_1)R \cap (1-e_2)R) = (e_1R + e_2R) \cap (1-e_1)(1-e_2)R = 0.$$

So $(\sum_{i\in F} e_i R) \cap \ell_R(\sum_{i\in F} e_i R) = (\sum_{i\in F} e_i R) \cap (\cap_{i\in F}(1-e_i)R) = 0$ inductively. Thus, with the hypothesis $\mathfrak{D}_{\mathbf{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0\}$, it follows that $\sum_{i\in\Lambda} e_i R \in \mathfrak{D}_{\mathbf{IC}}(R)$. By Lemma 8.3.7(ii), $\sum_{i\in\Lambda} I_i \in \mathfrak{D}_{\mathbf{IC}}(R)$.

(iv) Let R be right and left FI-extending. Then R is an **IC**-ring, so $\mathcal{B}(Q(R)) \subseteq R$ by Theorem 8.3.11(ii). Let $\{I_i \mid i \in \Lambda\} \subseteq \mathfrak{D}_{\mathbf{IC}}(R)$. From Theorem 8.3.8, there exists a set $\{e_i \mid i \in \Lambda\} \subseteq \mathcal{B}(Q(R))$ with $I_{i\,R} \leq^{\text{den}} e_i Q(R)_R$ for each $i \in \Lambda$.

Now $(\sum_{i\in\Lambda} e_i R) \cap \ell_R(\sum_{i\in\Lambda} e_i R) = (\sum_{i\in\Lambda} e_i R) \cap r_R(\sum_{i\in\Lambda} e_i R) = 0$ by the preceding argument. From Theorem 2.3.15, there exists $c \in \mathcal{B}(R)$ such that $\ell_R(\sum_{i\in\Lambda} e_i R) = (1-c)R$. We recall that $\sum_{i\in\Lambda} I_{i\,R} \leq^{\text{den}} \sum_{i\in\Lambda} e_i R_R$ from the proof of part (iii). Therefore, the proof of Lemma 8.3.7(i) yields that

$$\ell_R(\sum_{i\in\Lambda} I_i) = \ell_R(\sum_{i\in\Lambda} e_i R) = (1-c)R$$

from the proof of Lemma 8.3.7(i). So $\ell_R(\sum_{i\in\Lambda} I_i) \in \mathfrak{D}_{\mathbf{IC}}(R)$. Also

$$\begin{aligned}(\sum_{i\in\Lambda} I_i) \cap \ell_R(\sum_{i\in\Lambda} I_i) &= (\sum_{i\in\Lambda} I_i) \cap (1-c)R \subseteq r_R(\ell_R(\sum_{i\in\Lambda} I_i)) \cap (1-c)R \\ &= cR \cap (1-c)R = 0.\end{aligned}$$

From Lemma 8.3.7(iv), $\sum_{i\in\Lambda} I_i \in \mathfrak{D}_{\mathbf{IC}}(R)$. Hence, $\mathfrak{D}_{\mathbf{IC}}(R)$ is a complete sublattice of the lattice of ideals of R. □

Corollary 8.3.14 *If $Q(R)$ is semiprime, then $\mathcal{B}(Q(R))$ is a complete Boolean algebra.*

Proof By Theorem 8.3.11(ii), $Q(R)$ is an **IC**-ring. As $Q(R)$ is semiprime, $Q(R)$ is right FI-extending from Proposition 8.3.3(i). Thus by Theorem 3.2.37, $Q(R)$ is also left FI-extending. So Theorem 8.3.13(ii) and (iv) yield that $\mathcal{B}(Q(R))$ is a complete Boolean algebra. □

Corollary 8.3.15 *If R is a right nonsingular* **IC**-*ring, then $\mathfrak{D}_{\mathbf{IC}}(R)$ is a complete sublattice of the lattice of ideals of R.*

Proof The proof follows from Proposition 8.3.2 and Theorem 8.3.13(iii). □

Proposition 8.3.16 *Assume that R is a semiprime ring. Then, for any ideal I of R, $r_{Q(R)}(Q(R)IQ(R)) = r_{Q(R)}(I)$.*

Proof Let $I \trianglelefteq R$. Clearly, $r_{Q(R)}(IQ(R)) \subseteq r_{Q(R)}(I)$. Let $\alpha \in r_{Q(R)}(I)$ and $\sum x_i q_i \in IQ(R)$ with $x_i \in I$ and $q_i \in Q(R)$. Assume that $(\sum x_i q_i)\alpha \neq 0$. Since $R_R \leq^{\text{den}} Q(R)_R$, there exists $r_1 \in R$ with $\alpha r_1 \in R$ and $(\sum x_i q_i)\alpha r_1 \neq 0$. Thus, $\alpha r_1 \in R \cap r_{Q(R)}(I) = r_R(I) = \ell_R(I)$ because R is semiprime. Also there is $r_2 \in R$ with $0 \neq (\sum x_i q_i)\alpha r_1 r_2 \in R$ since $R_R \leq^{\text{ess}} Q(R)_R$.

Let $y = (\sum x_i q_i)\alpha r_1 r_2$. As $\alpha r_1 \in \ell_R(I)$, $\alpha r_1 r_2 \in \ell_R(I)$ and so $\alpha r_1 r_2 I = 0$. Hence $yRI = (\sum x_i q_i)\alpha r_1 r_2 RI \subseteq (\sum x_i q_i)\alpha r_1 r_2 I = 0$. Further, note that $yR = (\sum x_i q_i \alpha r_1 r_2)R \subseteq IQ(R)$. So $(yR)^2 = (yR)(yR) \subseteq yRIQ(R) = 0$, which is a contradiction because R is semiprime. Therefore $\alpha \in r_{Q(R)}(IQ(R))$, and thus $r_{Q(R)}(I) = r_{Q(R)}(IQ(R)) = r_{Q(R)}(Q(R)IQ(R))$. □

The next result demonstrates the existence and uniqueness of the quasi-Baer and the right FI-extending ring hulls of a semiprime ring. It extends Mewborn's result (Theorem 8.2.4) as a commutative quasi-Baer ring is Baer.

Theorem 8.3.17 *Let R be a semiprime ring. Then*:

(i) $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{FI}}(R) = \widehat{Q}_{\mathbf{eqB}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$.
(ii) *If R has identity, then* $\widehat{Q}_{\mathbf{FI}}(R) = R(\mathbf{FI}, Q(R))$.
(iii) *If R has identity, then* $\widehat{Q}_{\mathbf{eqB}}(R) = R(\mathbf{eqB}, Q(R))$.

Proof (i) Note that $\widehat{Q}_{\mathbf{FI}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ by Proposition 8.3.3(i) and Theorem 8.3.11(iii). From Theorem 3.2.37, $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{eqB}}(R) = \widehat{Q}_{\mathbf{FI}}(R)$.

(ii) This part follows from Proposition 8.3.3(i) and Theorem 8.3.11(iv).

(iii) To prove that $R(\mathbf{eqB}, Q(R)) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$, we claim that $\mathcal{B}(Q(R)) = \delta_{\mathbf{eqB}}(R)(1)$. For this, let $a \in \mathcal{B}(Q(R))$ and $I = R \cap (1-a)Q(R)$. Then $I_R \leq^{\text{ess}} (1-a)Q(R)_R$, and so $Q(R)IQ(R)_R \leq^{\text{ess}} (1-a)Q(R)_R$. Thus $Q(R)IQ(R)_{Q(R)} \leq^{\text{ess}} (1-a)Q(R)_{Q(R)}$.

By Theorem 8.3.11(ii), $Q(R)$ is an **IC**-ring. As $Q(R)$ is semiprime, $Q(R)$ is a right FI-extending ring from Proposition 8.3.3(i). By Theorem 3.2.37, $Q(R)$ is quasi-Baer. So there is $k \in \mathcal{B}(Q(R))$ with $r_{Q(R)}(Q(R)IQ(R)) = kQ(R)$ by Proposition 1.2.6(ii). Now $Q(R)IQ(R)_{Q(R)} \leq^{\text{ess}} (1-k)Q(R)_{Q(R)}$ by Lemma 2.1.13. Thus $1-a = 1-k$, so $a = k$.

From Lemma 8.3.10, there is $\mu^2 = \mu \in \text{End}(E(R_R))$ such that $a = \mu(1)$. By Proposition 8.3.16, $r_{Q(R)}(I) = r_{Q(R)}(Q(R)IQ(R)) = kQ(R)$. Hence

$$\begin{aligned} r_R(I)_R &= (r_{Q(R)}(I) \cap R)_R = (kQ(R) \cap R)_R \\ &\leq^{\text{ess}} kQ(R)_R = aQ(R)_R = \mu(1)Q(R)_R = \mu Q(R)_R \\ &\leq^{\text{ess}} \mu E(R_R) \end{aligned}$$

because $\mu \in \operatorname{End}(E(R_R)) = \operatorname{End}(E(R_R)_{Q(R)})$. Thus $\mu \in \delta_{\mathbf{eqB}}(R)$, and therefore $a = \mu(1) \in \delta_{\mathbf{eqB}}(R)(1)$. Hence $\mathcal{B}(Q(R)) \subseteq \delta_{\mathbf{eqB}}(R)(1)$.

To show that $\delta_{\mathbf{eqB}}(R)(1) \subseteq \mathcal{B}(Q(R))$, let $\nu \in \delta_{\mathbf{eqB}}(R)$. Then there is $J \trianglelefteq R$ with $r_R(J)_R \leq^{\text{ess}} \nu E(R_R)$. By Proposition 8.3.3(i) and Theorem 8.3.8, $r_R(J)_R \leq^{\text{ess}} dQ(R)_R$ for some $d \in \mathcal{B}(Q(R))$. From Lemma 8.3.10, there exists ϕ in $\mathcal{B}(\operatorname{End}(E(R_R)))$ such that $d = \phi(1)$. Thus $\nu(1) = \phi(1) = d \in \mathcal{B}(Q(R))$ as in the proof of Theorem 8.3.11(iv). Hence, $\delta_{\mathbf{eqB}}(R)(1) \subseteq \mathcal{B}(Q(R))$. Therefore, $\mathcal{B}(Q(R)) = \delta_{\mathbf{eqB}}(R)(1)$. So $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)} = \langle R \cup \delta_{\mathbf{eqB}}(R)(1)\rangle_{Q(R)}$.

Consequently, $\langle R \cup \delta_{\mathbf{eqB}}(R)(1)\rangle_{Q(R)} = R(\mathbf{eqB}, Q(R))$ from the definition of pseudo ring hulls, since $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ is right essentially quasi-Baer by part (i). Hence, $\widehat{Q}_{\mathbf{eqB}}(R) = \langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)} = R(\mathbf{eqB}, Q(R))$. □

We note that from Theorems 3.2.37 and 8.3.17 when R is a semiprime ring, $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ is also the strongly FI-extending absolute to $Q(R)$ ring hull of R. The following example shows that the semiprimeness of R in Theorem 8.3.17 is not a superfluous condition.

Example 8.3.18 There is a right nonsingular ring R which is not semiprime and $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)} \neq \widehat{Q}_{\mathbf{qB}}(R)$. Let F be a field, and put

$$R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}.$$

Observe that $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)} = R\mathcal{B}(Q(R))$ since R has an identity. Also we see that R is quasi-Baer by Corollary 5.4.2 or Theorem 5.6.5. Therefore $\widehat{Q}_{\mathbf{qB}}(R) = R$. As R is right Artinian, $\operatorname{Soc}(R_R) \leq^{\text{ess}} R_R$. Since $\operatorname{Soc}(R_R)$ is the intersection of all essential right ideals of R, $\operatorname{Soc}(R_R)$ is the smallest essential right ideal of R. Also as R is right nonsingular, $\operatorname{Soc}(R_R)$ is the smallest dense right ideal of R from Proposition 1.3.14. If $q \in Q(R)$, then $q\operatorname{Soc}(R_R) \subseteq R$, and so $q\operatorname{Soc}(R_R) \subseteq \operatorname{Soc}(R_R)$. By Proposition 1.3.11(ii), $\ell_{Q(R)}(\operatorname{Soc}(R_R)) = 0$. Hence, $Q(R) \cong \operatorname{End}(\operatorname{Soc}(R_R))$. As $\operatorname{Soc}(R_R) = \ell_R(J(R))$, $\operatorname{Soc}(R_R) = M_R \oplus N_R$, where

$$M = \begin{bmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}.$$

So $Q(R) \cong \operatorname{End}(M_R \oplus N_R)$. In this case, by straightforward computation,

$$Q(R) \cong \operatorname{End}(M_R) \oplus \operatorname{End}(N_R) = \operatorname{End}(M_F) \oplus \operatorname{End}(N_F) \cong \operatorname{Mat}_2(F) \oplus \operatorname{Mat}_2(F).$$

Now $|\mathcal{B}(R)| = 2$. But $|\mathcal{B}(Q(R))| = 4$. Thus, $R = \widehat{Q}_{\mathbf{qB}}(R) \neq R\mathcal{B}(Q(R))$.

Since idempotents as well as various properties lift modulo the prime radical, Theorem 8.3.17 provides an effective mechanism for transferring information between an arbitrary ring R and $\widehat{Q}_{\mathbf{qB}}(R/P(R))$ (or $\widehat{Q}_{\mathbf{FI}}(R/P(R))$) via

$$R \xrightarrow{\mu} R/P(R) \xrightarrow{\iota} \widehat{Q}_{\mathbf{qB}}(R/P(R)),$$

where μ is the natural homomorphism and ι is the inclusion.

Corollary 8.3.19 *Let T be a semiprime right ring of quotients of a ring R. Then T is quasi-Baer* (*and right FI-extending*) *if and only if $\mathcal{B}(Q(R)) \subseteq T$.*

Proof Proposition 8.3.3(i), Theorems 3.2.37 and 8.3.17 yield the result. □

It is worth noting that if we modify the ring R in Example 8.2.9 and instead of a field take F to be a commutative domain which is not a field, then R is neither semiprime nor right FI-extending. Now, $T = \text{Mat}_3(F)$ is a semiprime quasi-Baer (and right FI-extending) right ring of quotients of R such that $\mathcal{B}(Q(R)) \subseteq T$. But observe that $T \neq Q(R) = \text{Mat}_3(K)$, where K is the field of fractions of F. If R is a semiprime ring, $Q^s(R)$, $Q^m(R)$, and $Q(R)$ are all semiprime rings. Also, they contain $\mathcal{B}(Q(R))$. If R is a semiprime ring with identity, then the central closure of R and the normal closure of R are semiprime and contain $\mathcal{B}(Q(R))$. So Theorem 8.3.17 or Corollary 8.3.19 yields the following consequence.

Corollary 8.3.20 (i) *If R is a semiprime ring, then $Q^s(R)$, $Q^m(R)$, and $Q(R)$ are quasi-Baer and right FI-extending.*

(ii) *If R is a semiprime ring with identity, then the central closure and the normal closure are quasi-Baer and right FI-extending.*

There is a semiprime ring R for which neither $Q^m(R)$ nor $Q^s(R)$ is Baer. In fact, there is a simple ring R which is not a domain and 0, 1 are its only idempotents (see Example 3.2.7(ii)). Then $Q^m(R) = R$ and $Q^s(R) = R$. So neither $Q^m(R)$ nor $Q^s(R)$ is Baer.

Corollary 8.3.21 *Let R be a right Osofsky compatible ring with identity. If R has a right FI-extending right essential overring which is a subring of $E(R_R)$, then $E(R_R)$ is right FI-extending. In particular, if $Q(R)$ is semiprime, then $E(R_R)$ is right FI-extending.*

Proof Let S be a right FI-extending right essential overring of R which is a subring of the ring $E(R_R)$. Then $E(R_R)$ is a right essential overring of S. Thus $E(R_R)$ is a right FI-extending ring by Theorem 8.1.8(i). If $Q(R)$ is semiprime, then from Corollary 8.3.20(i), $Q(R)$ is right FI-extending. By Proposition 7.1.11, $Q(R)$ is a subring of $E(R_R)$, so $E(R_R)$ is a right essential overring of $Q(R)$. Hence, Theorem 8.1.8(i) yields that $E(R_R)$ is a right FI-extending ring. □

We remark that the ring R in Example 7.3.6 is right FI-extending and right Osofsky compatible, so $E(R_R)$ is right FI-extending by Corollary 8.3.21.

A ring R is said to have *no nonzero n-torsion* (n is a positive integer) if $na = 0$ with $a \in R$ implies $a = 0$.

Theorem 8.3.22 *Let $R[G]$ be the group ring of a group G over a ring R with identity. Then $R[G]$ is semiprime if and only if R is semiprime and R has no $|N|$-torsion for any finite normal subgroup N of G.*

Proof See [264, Proposition 8, p. 162] or [341, Theorem 2.13, p. 131]. □

The next corollary is obtained from Theorems 8.3.22 and 8.3.17. It is of interest to compare this result with Theorem 6.3.10(ii).

Corollary 8.3.23 *Assume that $R[G]$ is the semiprime group ring of a group G over a ring R with identity. If $R[G]$ is quasi-Baer, then $|N|^{-1} \in R$ for any finite normal subgroup N of G.*

Proof Let N be a finite normal subgroup of G. Because $R[G]$ is semiprime, R has no $|N|$-torsion by Theorem 8.3.22. Let $e = |N|^{-1} \sum_{g \in N} g$. Then

$$e \in Q^m(R)[G] \subseteq Q^m(R[G]) \subseteq Q(R[G])$$

(see the proof of Theorem 9.3.1(i)). Further, we see that $e \in \mathcal{B}(Q(R[G]))$. From Theorem 8.3.17, $e \in R[G]$ since $R[G]$ is quasi-Baer. So $|N|^{-1} \in R$. □

The next example illustrates the existence of a right nonsingular ring R which is not semiprime such that $\mathcal{B}(Q(R)) \subseteq R$, but R is not quasi-Baer.

Example 8.3.24 For a field F, as in Example 3.2.9, let

$$R = \begin{bmatrix} F\mathbf{1} & \mathrm{Mat}_2(F) & \mathrm{Mat}_2(F) \\ 0 & F\mathbf{1} & \mathrm{Mat}_2(F) \\ 0 & 0 & F\mathbf{1} \end{bmatrix}$$

be a subring of $T_3(\mathrm{Mat}_2(F))$, where $\mathbf{1}$ is the identity matrix in $\mathrm{Mat}_2(F)$. Then we see that R is right nonsingular. However, $\langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)} (= R)$ is not quasi-Baer (see Example 3.2.9).

In contrast to Examples 8.3.18 and 8.3.24, there exists a nonsemiprime ring R for which Theorem 8.3.17(ii) holds true as in the next example.

Example 8.3.25 Let A be a QF-ring with $J(A) \neq 0$. Assume that A is right strongly FI-extending, and A has nontrivial central idempotents, while the subring of A generated by 1_A contains no nontrivial idempotents (e.g., $A = \mathbb{Q} \oplus \mathrm{Mat}_2(\mathbb{Z}_4)$). Let $\mathbf{1}$ denote the identity of $\prod_{i=1}^{\infty} A_i$, where $A_i = A$. Take R to be the subring of $\prod_{i=1}^{\infty} A_i$ generated by $\mathbf{1}$ and $\oplus_{i=1}^{\infty} A_i$. We note that $Q(R) = \prod_{i=1}^{\infty} A_i = E(R_R)$ and $\langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)} = R\mathcal{B}(Q(R))$.

In this case, we have the following:

(i) R is not right FI-extending and $R\mathcal{B}(Q(R))$ is not quasi-Baer.
(ii) $Q_{\mathbf{FI}}(R) = R(\mathbf{FI}, Q(R)) = R\mathcal{B}(Q(R))$.
(iii) R has no right and left essential overring which is quasi-Baer.

Let k be a nontrivial central idempotent of A. Let ι_i denote the i-th canonical injection, respectively of the direct product. Let K be the ideal of R generated by $\{\iota_i(k) \mid 1 \leq i < \infty\}$. Then there exists no $b^2 = b \in R$ such that $K_R \leq^{\mathrm{ess}} bR_R$. So R is not right FI-extending.

We claim that $R\mathcal{B}(Q(R))$ is not quasi-Baer. For this, first we observe that $\mathbf{S}_\ell(Q(R)) = \mathcal{B}(Q(R))$ as $\mathbf{S}_\ell(A_i) = \mathcal{B}(A_i)$ for each i by [262, Exercise 16, p. 421]. Suppose that $Q(R)$ is quasi-Baer. Take $q \in Q(R)$ such that $qQ(R)q = 0$. Now we note that $r_{Q(R)}(qQ(R)) = \alpha Q(R)$ such that $\alpha \in \mathbf{S}_\ell(Q(R)) = \mathcal{B}(Q(R))$. Since $q \in r_{Q(R)}(qQ(R))$, $q = \alpha q = q\alpha = 0$. Therefore $Q(R)$ is semiprime, a contradiction. So $Q(R)$ is not quasi-Baer.

Because A is QF, $Q(R) = Q^\ell(R) = E(R_R) = E({}_RR)$. Therefore the ring $R\mathcal{B}(Q(R))$ is not quasi-Baer by Theorem 8.1.9(i). Further, R has no right and left essential overring which is quasi-Baer from Theorem 8.1.9(i).

We prove that $\delta_{\mathbf{FI}}(R)(\mathbf{1}) = \mathcal{B}(Q(R))$. For this, let $f \in \delta_{\mathbf{FI}}(R)$. Then there exists $I \trianglelefteq R$ such that $I_R \le^{\text{ess}} fE(R_R) = fQ(R)_R = f(\mathbf{1})Q_R$, because $\text{End}(E(R_R)) = \text{End}(Q(R)_R) = \text{End}(Q(R)_{Q(R)})$.

Furthermore, we note that $f(\mathbf{1})^2 = f(\mathbf{1})f(\mathbf{1}) = f(\mathbf{1}f(\mathbf{1})) = f(f(\mathbf{1})) = f(\mathbf{1})$.

Let π_i be the canonical projection of the direct product. Then $\pi_i(I) \trianglelefteq A_i$. By [262, Exercise 16, p. 421], there is $e_i \in \mathcal{B}(A_i)$ such that $\pi_i(I)_{A_i} \le^{\text{ess}} e_iA_{i\,A_i}$, because A_i is right strongly FI-extending by assumption. Let $e \in Q(R)$ such that $\pi_i(e) = e_i$ for all i. Then we see that $I_R \le^{\text{ess}} eQ(R)_R$ and $e \in \mathcal{B}(Q(R))$. So $f(\mathbf{1}) = e$. Thus, $\delta_{\mathbf{FI}}(R)(\mathbf{1}) \subseteq \mathcal{B}(Q(R))$.

Next, say $b \in \mathcal{B}(Q(R))$. Then $(bR \cap R)_R \le^{\text{ess}} bR_R \le^{\text{ess}} bQ(R)_R$. There exists $\lambda \in \mathcal{B}(\text{End}(E(R_R)))$ such that $b = \lambda(\mathbf{1})$ from Lemma 8.3.10, and hence $bQ(R)_R = \lambda(\mathbf{1})Q(R)_R = \lambda Q(R)_R$. So $\lambda \in \delta_{\mathbf{FI}}(R)$ and $b = \lambda(\mathbf{1}) \in \delta_{\mathbf{FI}}(R)(\mathbf{1})$, thus $\mathcal{B}(Q(R)) \subseteq \delta_{\mathbf{FI}}(R)(\mathbf{1})$. Hence $\mathcal{B}(Q(R)) = \delta_{\mathbf{FI}}(R)(\mathbf{1})$. Therefore we have that $S := \langle R \cup \delta_{\mathbf{FI}}(R)(\mathbf{1})\rangle_{Q(R)} = R\mathcal{B}(Q(R))$.

To show that $S = R(\mathbf{FI}, Q(R))$, let $J \trianglelefteq R\mathcal{B}(Q(R))$. First, we note that $\text{End}(E(R_R)) = \text{End}(Q(R)_R) = \text{End}(Q(R)_{Q(R)}) \cong Q(R)$. Thus, it follows that $(J \cap R)_R \le^{\text{ess}} J_R \le^{\text{ess}} E(J_R) = hQ(R)_R$ with $h^2 = h \in Q(R)$. Since $J \cap R \trianglelefteq R$, there is $g \in \mathcal{B}(Q(R))$ with $(J \cap R)_R \le^{\text{ess}} gQ(R)_R$ from the preceding argument. Hence $h = g$, and thus $J_R \le^{\text{ess}} gQ(R)_R$. Therefore, $J = Jg \subseteq R\mathcal{B}(Q(R))$. Hence, we have that $J_R \le^{\text{ess}} gR\mathcal{B}(Q(R))_R$, and thus $J_{Q(R)} \le^{\text{ess}} gR\mathcal{B}(Q(R))_{Q(R)}$. Whence $R\mathcal{B}(Q(R))$ is right FI-extending, so $S = R(\mathbf{FI}, Q(R))$.

Next, we show that $S = Q_{\mathbf{FI}}(R)$. Let T be a right FI-extending right ring of quotients of R. Take $c \in \mathcal{B}(Q(R))$. Then $cQ(R) \cap T \trianglelefteq T$. Since T is right FI-extending, there is $s^2 = s \in T$ such that $(cQ(R) \cap T)_T \le^{\text{ess}} sT_T$.

Therefore $(cQ(R) \cap T)_R \le^{\text{ess}} sT_R$ from Lemma 8.1.3(i), and hence it follows that $(cQ(R) \cap T)_R \le^{\text{ess}} sQ(R)_R$, thus $(cQ(R) \cap R)_R \le^{\text{ess}} sQ(R)_R$. Also we see that $(cQ(R) \cap R)_R \le^{\text{ess}} cQ(R)_R$. So $c = s \in T$. Thus $\mathcal{B}(Q(R)) \subseteq T$, and hence S is a subring of T. Therefore, $S = Q_{\mathbf{FI}}(R)$.

Now from Theorems 8.3.11 and 8.3.17, we see that $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ is a ring hull for the **IC** class, as well as a ring hull for a semiprime ring R in the **qB** and **FI** classes. This motivates our interest in the transfer of information between R and the ring $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$.

Let S be an overring of a ring R. We consider the following properties between prime ideals of R and S (see [248, p. 28]).

(1) *Lying over* (LO). For any prime ideal P of R, there exists a prime ideal Q of S such that $P = Q \cap R$.

(2) *Going up* (GU). Given prime ideals $P_1 \subseteq P_2$ of R and Q_1 of S with $P_1 = Q_1 \cap R$, there exists a prime ideal Q_2 of S with $Q_1 \subseteq Q_2$ and $P_2 = Q_2 \cap R$.

(3) *Incomparable* (INC). Two different prime ideals of S with the same contraction in R are not comparable.

Lemma 8.3.26 *Let R be a subring of a ring T and $\emptyset \neq \mathbb{E} \subseteq \mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. Assume that S is the subring of T generated by R and $\mathbb{E}$.*

(i) *If K is a prime ideal of S, then $R/(K \cap R) \cong S/K$.*

(ii) *LO, GU, and INC hold between R and S. In particular, LO, GU, and INC hold between R and $\langle R \cup \mathcal{B}(Q(R)) \rangle_{Q(R)}$.*

Proof (i) Let $\overline{S} = S/K$. Assume that $e \in \mathbb{E}$ such that $e \notin K$. Then $e \in \mathbf{S}_\ell(T)$ or $e \in \mathbf{S}_r(T)$. First, we show that $\overline{e} = e + K \in S/K$ is an identity of S/K. Without loss of generality, assume that $e \in \mathbf{S}_\ell(T)$. Then $\overline{0} \neq \overline{e} \in \mathbf{S}_\ell(\overline{S})$, so $\overline{S} = \overline{e}\overline{S} \oplus r_{\overline{S}}(\overline{e})$. As $\overline{e} \in \mathbf{S}_\ell(\overline{S})$, $(r_{\overline{S}}(\overline{e}))(\overline{e}\overline{S}) = \overline{0}$. Thus, $r_{\overline{S}}(\overline{e}) = \overline{0}$ because $\overline{S}$ is a prime ring. So $\overline{e}$ is a left identity for $\overline{S}$. Also, $\overline{S} = \overline{S}\overline{e} \oplus \ell_{\overline{S}}(\overline{e})$. As $\overline{e} \in \mathbf{S}_\ell(\overline{S})$, $(\ell_{\overline{S}}(\overline{e}))(\overline{S}\overline{e}) = \overline{0}$. Thus, $\ell_{\overline{S}}(\overline{e}) = \overline{0}$ since $\overline{S}$ is a prime ring. So $\overline{S} = \overline{S}\overline{e}$. Therefore, $\overline{e}$ is an identity element for $\overline{S}$. A similar argument works if $e \in \mathbf{S}_r(T)$.

From the preceding argument, for $f \in \mathbb{E}$, either $f + K = \overline{0}$ or $f + K$ is an identity of S/K. We define $\varphi : R \to S/K$ by $\varphi(r) = r + K$. Because S is generated by R and $\mathbb{E}$, φ is a ring epimorphism. Also $\mathrm{Ker}(\varphi) = K \cap R$. Thus, $R/(K \cap R) \cong S/K$.

(ii) (LO) Assume that P is a prime ideal of R. By Zorn's lemma, there exists an ideal K of S maximal with respect to $K \cap R \subseteq P$. Then K is a prime ideal of S. By (i), $R/(K \cap R) \cong S/K$. Since $P/(K \cap R)$ is a prime ideal of $R/(K \cap R)$ ($\cong S/K$), there is a prime ideal K_0 of S with $K \subseteq K_0$, so K_0/K is a prime ideal of S/K, and $K_0/K = \overline{\varphi}(P/(K \cap R))$, where $\overline{\varphi}$ is the isomorphism from $R/(K \cap R)$ to S/K induced from φ in the proof of part (i). Therefore $K_0 = P + K$, hence we obtain that $K_0 \cap R = P + (K \cap R) = P$. Therefore, LO holds.

(GU) Suppose that $P_1 \subseteq P_2$ are prime ideals of R and K_1 is a prime ideal of S such that $K_1 \cap R = P_1$. Then by part (i), $R/P_1 \cong S/K_1$. By the same argument for LO, there is a prime ideal K_2 of S such that $K_1 \subseteq K_2$ and $K_2 \cap R = P_2$. Thus GU holds.

(INC) Suppose that K_1, K_2 are prime ideals of S and P is a prime ideal of R such that $K_1 \cap R = K_2 \cap R = P$. Assume that $K_1 \subseteq K_2$.

First, we show that $K_2/K_1 = \{r + K_1 \mid r \in K_2 \cap R\}$. For this, we observe that $S/K_1 = \{a + K_1 \mid a \in R\}$ by the argument in the proof of part (i). Let $r \in K_2 \cap R$. Then $r + K_1 \in K_2/K_1$, so $\{r + K_1 \mid r \in K_2 \cap R\} \subseteq K_2/K_1$.

Let $k_2 + K_1 \in K_2/K_1$. Then $k_2 + K_1 \in S/K_1$, so $k_2 + K_1 = a + K_1$ for some a in R. Thus, $k_2 = a + k_1$ for some $k_1 \in K_1$, hence

$$a = k_2 - k_1 \in K_2 + K_1 = K_2.$$

Therefore $a \in K_2 \cap R$. Thus $k_2 + K_1 \in \{r + K_1 \mid r \in K_2 \cap R\}$, so we have that

$$K_2/K_1 = \{r + K_1 \mid r \in K_2 \cap R\}.$$

As $P = K_1 \cap R = K_2 \cap R$, we see that $K_2/K_1 = 0$. Hence $K_2 = K_1$. □

The next theorem, due to Fisher and Snider [170], is a characterization of regular rings.

Theorem 8.3.27 *A ring R is regular if and only if the following hold*:

(i) *R is semiprime.*
(ii) *The union of any chain of semiprime ideals of R is semiprime.*
(iii) *Every prime factor ring of R is regular.*

Proof See [170, Theorem 1.1] or [183, Theorem 1.17]. □

A class ϱ of rings (not necessarily satisfying $\ell_R(R) = 0$) is called a *special class* if ϱ is a class of prime rings that is hereditary (i.e., closed with respect to ideals) and closed with respect to ideal essential extensions. That is, if I is in ϱ and $I \trianglelefteq R$ that is ideal essential in R, then R is in ϱ (see [176, p. 80]). Let ϱ be a special class of rings. The *special radical* $\varrho(R)$ for a ring R is the intersection of all ideals I of R such that R/I is a ring in the special class ϱ. Note that the class of special radicals includes most well-known radicals (e.g., the prime radical, the Jacobson radical, the Brown-McCoy radical, the nil radical, and the generalized nil radical, etc.). See [139] and [176] for more details.

For a ring R with identity, the *classical Krull dimension* $\mathrm{kdim}(R)$ is the supremum of all lengths of chains of prime ideals of R. We show that various types of information transfer between a ring R and $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$. The transference of information in Lemma 8.3.26 and Theorem 8.3.28 is used to study $\widehat{Q}_{\mathbf{qB}}(R)$ (or $\widehat{Q}_{\mathbf{FI}}(R)$) when R is a semiprime ring.

Theorem 8.3.28 *Let R be a subring of a ring T and $\emptyset \neq \mathbb{E} \subseteq \mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. Assume that S is the subring of T generated by R and $\mathbb{E}$. Then*:

(i) *$\varrho(R) = \varrho(S) \cap R$, where ϱ is a special radical. In particular, we have that $\varrho(R) = \varrho(\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}) \cap R$.*
(ii) *R is strongly π-regular if and only if S is strongly π-regular. Hence, R is strongly π-regular if and only if $\langle R \cup \mathcal{B}(Q(R))\rangle_{Q(R)}$ is strongly π-regular.*
(iii) *If S is regular, then so is R.*
(iv) *If the ring R has identity, then* $\mathrm{kdim}(R) = \mathrm{kdim}(S)$. *Thus, we have that* $\mathrm{kdim}(R) = \mathrm{kdim}(R\mathcal{B}(Q(R)))$.

Proof (i) Let K be a prime ideal of S such that S/K is in the special class of ρ. From Lemma 8.3.26, $R/(K \cap R)$ is in the special class of ϱ. Therefore $\varrho(R) \subseteq \varrho(S) \cap R$. As in the proof of LO in Lemma 8.3.26, $\varrho(S) \cap R \subseteq \varrho(R)$.

(ii) This part is a consequence of Lemma 8.3.26 and Theorem 1.2.18 (note that Theorem 1.2.18 holds for rings not necessarily with an identity).

(iii) Since S is regular, R is semiprime by part (i). Let $I_1 \subseteq I_2 \subseteq \dots$ be a chain of semiprime ideals of R. Let $\mathbf{U}_k$ be the set of all prime ideals of R containing I_k, for $k = 1, 2, \dots$. Then I_k is the intersection of all prime ideals in $\mathbf{U}_k$. By LO in Lemma 8.3.26, for each $P \in \mathbf{U}_1$, there exists a prime ideal K of S such that

$P = K \cap R$. Let $\mathbf{V}_1$ be the set of all prime ideals K of S such that $K \cap R \in \mathbf{U}_1$, and let J_1 be the intersection of all prime ideals K in $\mathbf{V}_1$. Then $J_1 \cap R = I_1$ by using Lemma 8.3.26.

Next, consider $\mathbf{U}_2$. Then $\mathbf{U}_2 \subseteq \mathbf{U}_1$ since $I_1 \subseteq I_2$. Let $\mathbf{V}_2$ be the set of prime ideals K such that $K \cap R \in \mathbf{U}_2$. Let J_2 be the intersection of all prime ideals in $\mathbf{V}_2$. Because $\mathbf{U}_2 \subseteq \mathbf{U}_1$, $\mathbf{V}_2 \subseteq \mathbf{V}_1$ and so $J_1 \subseteq J_2$. Again by Lemma 8.3.26, $J_2 \cap R = I_2$. Continuing this process, there exists a chain of semiprime ideals $J_1 \subseteq J_2 \subseteq \dots$, of S with $J_n \cap R = I_n$ for each n. So $(\cup J_n) \cap R = \cup I_n$.

Note that $S/(\cup J_n)$ is semiprime by Theorem 8.3.27. Since $\cup J_n$ is a semiprime ideal of S, $\cup J_n = \cap K_\alpha$ for some prime ideals K_α of S. Then each $K_\alpha \cap R$ is a prime ideal of R by Lemma 8.3.26(i). So $\cup I_n = (\cup J_n) \cap R = (\cap K_\alpha) \cap R = \cap(K_\alpha \cap R)$ is a semiprime ideal of R.

Finally, say P is a prime ideal of R. By LO in Lemma 8.3.26, there is a prime ideal K of S with $P = K \cap S$ and $R/P \cong S/K$. Since S/K is regular, so is R/P. By Theorem 8.3.27, the ring R is regular.

(iv) The proof follows immediately from Lemma 8.3.26. □

Lemma 8.3.29 *Assume that T is an overring with identity, of a ring R and $\{f_1, \dots, f_n\} \subseteq \mathcal{B}(T)$. Then there exists a set of orthogonal idempotents $\{e_1, \dots, e_m\} \subseteq \mathcal{B}(T)$ such that $\sum_{i=1}^n f_i R \subseteq \sum_{i=1}^m e_i R$.*

Proof We use induction on n. If $n = 1$, then we are done by taking $e_1 = f_1$. Assume that $n \geq 2$ and the lemma is true for $n = k - 1$, and let $n = k$.

By induction, there exists a set of orthogonal idempotents $\{e_1, \dots, e_\ell\} \subseteq \mathcal{B}(T)$ such that $\sum_{i=1}^{k-1} f_i R \subseteq \sum_{i=1}^{\ell} e_i R$. Hence,

$$\sum_{i=1}^{k} f_i R = \sum_{i=1}^{k-1} f_i R + f_k R \subseteq \sum_{i=1}^{\ell} e_i R + f_k R$$

$$\subseteq f_k(1 - \sum_{i=1}^{\ell} e_i)R \oplus (\oplus_{i=1}^{\ell}(1 - f_k)e_i R) \oplus (\oplus_{i=1}^{\ell} f_k e_i R).$$

This yields the result. □

Corollary 8.3.30 *For a ring R with identity, the following are equivalent.*

(i) *R is regular.*
(ii) *$R\mathcal{B}(Q(R))$ is regular.*
(iii) *R is semiprime and $\widehat{Q}_{\mathbf{qB}}(R)$ is regular.*

Proof Assume that R is regular. Take $q \in R\mathcal{B}(Q(R))$. From Lemma 8.3.29, $q = a_1 e_1 + \dots + a_m e_m \in R\mathcal{B}(Q(R))$, where $a_i \in R$, $e_i \in \mathcal{B}(Q(R))$, and e_i are orthogonal. Since R is regular, there is $b_i \in R$ with $a_i = a_i b_i a_i$ for each i. Let

$p = b_1e_1 + \cdots + b_me_m \in R\mathcal{B}(Q(R))$. Then $q = qpq$, so $R\mathcal{B}(Q(R))$ is regular. The rest of the proof follows by an easy application of Theorem 8.3.28(iii) and the fact that $\widehat{Q}_{\mathbf{qB}}(R) = R\mathcal{B}(Q(R))$ from Theorem 8.3.17 when R is semiprime. □

Lemma 8.3.26, Theorem 8.3.28, and Corollary 8.3.30 show the transference of some properties between R and $\widehat{Q}_{\mathbf{qB}}(R)$. Our next example indicates that in general these properties do not transfer between R and its right rings of quotients which properly contain $\widehat{Q}_{\mathbf{qB}}(R)$, in general.

Example 8.3.31 Let $R = \mathbb{Z}[C_2]$ be the group ring of the group $C_2 = \{1, g\}$ over the ring $\mathbb{Z}$. Then $\mathbb{Z}[C_2]$ is semiprime and $Q(\mathbb{Z}[C_2]) = \mathbb{Q}[C_2]$.

Note that $\mathcal{B}(\mathbb{Q}[C_2]) = \{0,\ 1,\ (1/2)(1+g),\ (1/2)(1-g)\}$. Thus, using Theorem 8.3.17, $\widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[C_2]) = \{(a + c/2 + d/2) + (b + c/2 - d/2)g \mid a, b, c, d \in \mathbb{Z}\}$. Therefore

$$\mathbb{Z}[C_2] \subsetneq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[C_2]) \subsetneq \mathbb{Z}[1/2][C_2] \subsetneq \mathbb{Q}[C_2],$$

where $\mathbb{Z}[1/2]$ is the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $1/2$.

Note that $\mathbb{Z}[C_2]/2\mathbb{Z}[C_2] \cong \mathbb{Z}_2[C_2]$, and $\mathbb{Z}_2[C_2]$ is a local ring. Thus there exists a prime ideal P (in fact, a maximal ideal) of $\mathbb{Z}[C_2]$ containing $2\mathbb{Z}[C_2]$. Also we note that $P \cap \mathbb{Z} = 2\mathbb{Z}$. Assume on the contrary that LO holds between $\mathbb{Z}[C_2]$ and $\mathbb{Z}[1/2][C_2]$. Then there exists a prime ideal K of $\mathbb{Z}[1/2][C_2]$ with $K \cap \mathbb{Z}[C_2] = P$. Now put $K_0 = K \cap \mathbb{Z}[1/2]$.

We see that $K_0 \cap \mathbb{Z} = K \cap \mathbb{Z}[1/2] \cap \mathbb{Z} = K \cap \mathbb{Z} = K \cap \mathbb{Z}[C_2] \cap \mathbb{Z} = P \cap \mathbb{Z} = 2\mathbb{Z}$. Thus $2 \in K_0$. But because K_0 is an ideal of $\mathbb{Z}[1/2]$, $1 = 2 \cdot (1/2) \in K_0$, hence $K = \mathbb{Z}[1/2][C_2]$, a contradiction. Thus, LO does not hold between $\mathbb{Z}[C_2]$ and $\mathbb{Z}[1/2][C_2]$.

Theorem 8.3.32 *Let R be a semiprime ring with identity. Then R has index of nilpotency at most n if and only if $Q_{\mathbf{qB}}(R)$ has index of nilpotency at most n. In particular, if R is reduced, then $Q_{\mathbf{qB}}(R) = Q_{\mathbf{B}}(R)$ and it is reduced.*

Proof Let R have index of nilpotency at most n. By Theorem 1.2.20(ii), R is right nonsingular. Hence $E(R_R) = Q(R)$ from Corollary 1.3.15. Therefore, we see that $\widehat{Q}_{\mathbf{qB}}(R) = Q_{\mathbf{qB}}(R)$. Now say $q \in Q_{\mathbf{qB}}(R)$. Then Lemma 8.3.29 yields that

$$q = a_1e_1 + \cdots + a_te_t,$$

where $a_i \in R$, $e_i \in \mathcal{B}(Q(R))$, and e_i are orthogonal.

Suppose that $q^k = 0$. We show that $q^n = 0$. If $k \le n$, then we are done. So assume that $k > n$. In this case, $q^k = a_1^ke_1 + \cdots + a_t^ke_t = 0$. Thus $a_i^ke_i = 0$ for all i. Note that $\mathcal{B}(Q(R)) = \mathcal{B}(Q^m(R))$ (recall that $Q^m(R)$ denotes the Martindale right ring of quotients of R). Hence, there is $I_i \trianglelefteq R$ with $\ell_R(I_i) = 0$ and $e_iI_i \subseteq R$. Therefore, $a_i^ke_iI_i = 0$ and $e_iI_i \subseteq r_R(a_i^k)$. Since R has index of nilpotency at most n, by Theorem 1.2.20(i) $r_R(a_i^k) = r_R(a_i^n)$, so $e_iI_i \subseteq r_R(a_i^n)$. Thus $a_i^ne_iI_i = 0$.

As $\ell_R(I_i) = 0$, $\ell_{Q(R)}(I_i) = 0$. Hence $a_i^ne_i = 0$ for each i. So

$$q^n = (a_1e_1 + \cdots + a_te_t)^n = a_1^ne_1 + \cdots + a_t^ne_t = 0.$$

Thus $Q_{\mathbf{qB}}(R)$ has index of nilpotency at most n. The converse is clear.

If R is reduced (so $Z(R_R) = 0$), then $Q_{\mathbf{qB}}(R)$ is a reduced quasi-Baer ring by the preceding argument, so it is a Baer ring (see Exercise 3.2.44.10(i)). Say T is a right ring of quotients of R and T is Baer. Then T is quasi-Baer. Hence, $Q_{\mathbf{qB}}(R) \subseteq T$ by Theorem 8.3.17. Therefore, $Q_{\mathbf{qB}}(R) = Q_{\mathbf{B}}(R)$. □

Recall that a ring R is called strongly regular if R is regular and reduced (see 1.1.12). Corollary 8.3.30 and Theorem 8.3.32 yield the next result.

Corollary 8.3.33 *A ring R with identity is strongly regular if and only if $R\mathcal{B}(Q(R))$ is strongly regular.*

If R is a domain with identity which is not right Ore, then $R = Q_{\mathbf{qB}}(R)$ has index of nilpotency 1, but $Q(R)$ does not have bounded index of nilpotency. So we cannot replace $Q_{\mathbf{qB}}(R)$ with $Q(R)$ in Theorem 8.3.32.

By Theorem 8.3.32, a reduced ring with identity always has a Baer absolute ring hull. However a Baer absolute ring hull does not exist even for prime PI-rings with index of nilpotency 2, as shown in the next example.

Example 8.3.34 Let $R = \mathrm{Mat}_k(F[x, y])$, where F is a field and k is an integer such that $k \geq 2$. Then R is a prime PI-ring with index of nilpotency k. (In particular, if $k = 2$, then R has index of nilpotency 2.) The ring R has the following properties. We note that $Q(R) = E(R_R)$, hence $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$ for any class $\mathfrak{K}$ of rings.

(i) The Baer absolute ring hull $Q_{\mathbf{B}}(R)$ does not exist.
(ii) The right extending absolute ring hull $Q_{\mathbf{E}}(R)$ does not exist.

As R is a prime ring, $R = Q_{\mathbf{qB}}(R) = Q_{\mathbf{FI}}(R)$. We claim that $Q_{\mathbf{B}}(R)$ does not exist (the same argument shows that $Q_{\mathbf{E}}(R)$ does not exist). Assume on the contrary that $Q_{\mathbf{B}}(R)$ exists. Note that $F(x)[y]$ and $F(y)[x]$ are Prüfer domains. So $\mathrm{Mat}_k(F(x)[y])$ and $\mathrm{Mat}_k(F(y)[x])$ are Baer rings by Theorem 6.1.4 (and right extending rings by Theorem 6.1.4). Since $Q(R) = \mathrm{Mat}_k(F(x, y))$,

$$Q_{\mathbf{B}}(R) \subseteq \mathrm{Mat}_k(F(x)[y]) \cap \mathrm{Mat}_k(F(y)[x]) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x]).$$

To see that $F(x)[y] \cap F(y)[x] = F[x, y]$, let

$$\begin{aligned}\gamma(x, y) &= f_0(x)/g_0(x) + (f_1(x)/g_1(x))y + \cdots + (f_m(x)/g_m(x))y^m \\ &= h_0(y)/k_0(y) + (h_1(y)/k_1(y))x + \cdots + (h_n(y)/k_n(y))x^n\end{aligned}$$

be in $F(x)[y] \cap F(y)[x]$, where $f_i(x), g_i(x) \in F[x]$, $h_j(y), k_j(y) \in F[y]$, and $g_i(x) \neq 0$, $k_j(y) \neq 0$ for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$. Let $\overline{F}$ be the algebraic closure of F. If $\deg(g_0(x)) \geq 1$, then there exists $\alpha \in \overline{F}$ such that $g_0(\alpha) = 0$. Therefore $\gamma(\alpha, y)$ cannot be defined. On the other hand, we observe that

$$\gamma(\alpha, y) = h_0(y)/k_0(y) + (h_1(y)/k_1(y))\alpha + \cdots + (h_n(y)/k_n(y))\alpha^n,$$

a contradiction. Thus $g_0(x) \in F$. Similarly, $g_1(x), \ldots, g_m(x) \in F$.

Hence $\gamma(x, y) \in F[x, y]$. Therefore $F(x)[y] \cap F(y)[x] = F[x, y]$, and so

$$Q_{\mathbf{B}}(R) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x]) = \mathrm{Mat}_k(F[x, y]).$$

Thus $\mathrm{Mat}_k(F[x, y])$ is a Baer ring, a contradiction because the commutative domain $F[x, y]$ is not Prüfer (see Theorem 6.1.4).

A ring R with identity is called *right Utumi* [382, p. 252] if it is both right nonsingular and right cononsingular. In the proof of Theorem 3.3.1 or by Lemma 4.1.16, every right extending ring is right cononsingular.

Proposition 8.3.35 *Let R be a reduced ring with identity. Then R is right Utumi if and only if $Q(R)$ is strongly regular.*

Proof See [382, Proposition 5.2, p. 254]. □

Proposition 8.3.36 *A reduced ring R with identity is right Utumi if and only if $Q_{\mathbf{qCon}}(R) = Q_{\mathbf{E}}(R) = R\mathcal{B}(Q(R))$.*

Proof Assume that R is right Utumi. Because R is reduced, $Z(R_R) = 0$ and from Theorem 8.3.32 $R\mathcal{B}(Q(R)) = Q_{\mathbf{qB}}(R) = Q_{\mathbf{B}}(R)$. Also, we observe that $Q(R) = Q(R\mathcal{B}(Q(R)))$ is strongly regular from Proposition 8.3.35. So $R\mathcal{B}(Q(R))$ is right Utumi, since $R\mathcal{B}(Q(R))$ is reduced by Theorem 8.3.32. Hence, $R\mathcal{B}(Q(R))$ is right cononsingular. As $R\mathcal{B}(Q(R))$ is Baer, $R\mathcal{B}(Q(R))$ is right extending by Theorem 3.3.1.

From Theorem 8.3.17, $R\mathcal{B}(Q(R)) = Q_{\mathbf{FI}}(R)$. If S is a right extending right ring of quotients of R, then S is right FI-extending, and hence $R\mathcal{B}(Q(R)) \subseteq S$. Thus, $R\mathcal{B}(Q(R)) = Q_{\mathbf{E}}(R)$. As $Q(R)$ is strongly regular, $\mathbf{I}(Q(R)) = \mathcal{B}(Q(R))$.

By Corollary 1.3.15, Theorem 2.1.25, and Proposition 2.1.32, $R\mathcal{B}(Q(R))$ is a right quasi-continuous ring. Let T be a right quasi-continuous right ring of quotients of R. Then again from Corollary 1.3.15, Theorem 2.1.25, and Proposition 2.1.32, $\mathcal{B}(Q(R)) = \mathcal{B}(Q(T)) \subseteq T$ as $Q(R) = Q(T)$. Thus $R\mathcal{B}(Q(R)) \subseteq T$, and hence $Q_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$. So $R\mathcal{B}(Q(R)) = Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R)$.

Conversely, if $R\mathcal{B}(Q(R)) = Q_{\mathbf{E}}(R)$, then $R\mathcal{B}(Q(R))$ is right cononsingular by Theorem 3.3.1. Hence, $R\mathcal{B}(Q(R))$ is right Utumi. Further, $R\mathcal{B}(Q(R))$ is reduced by Theorem 8.3.32, so $Q(R) = Q(R\mathcal{B}(Q(R)))$ is strongly regular and thus R is right Utumi from Proposition 8.3.35. □

There exists a nonreduced right Utumi ring R for which the equalities

$$Q_{\mathbf{qCon}}(R) = Q_{\mathbf{E}}(R) \text{ and } Q_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$$

in Proposition 8.3.36 do not hold true, as the next example shows.

Example 8.3.37 Let $R = \mathrm{Mat}_k(F[x])$, where F is a field and k is an integer such that $k > 1$. Then R is right Utumi by Proposition 3.3.2. Note that

$$E(R_R) = Q(R) = \mathrm{Mat}_k(F(x)),$$

where $F(x)$ is the field of fractions of $F[x]$.

There is $e^2 = e \in Q(R)$ such that $e \notin R$. By Theorem 2.1.25, R is not right quasi-continuous. Now $R\mathbf{B}(Q(R)) = R \neq Q_{\mathbf{qCon}}(R)$. From Theorem 6.1.4, R is right extending, so $R = Q_{\mathbf{E}}(R)$. Thus $Q_{\mathbf{E}}(R) \neq Q_{\mathbf{qCon}}(R)$.

For a semiprime ring R with identity, the notions of (right) FI-extending and quasi-Baer coincide by Theorem 3.2.37. Theorem 8.3.17 shows that the quasi-Baer ring hull of a semiprime ring exists and is precisely the same as its right FI-extending ring hull.

In view of this result, it is natural to ask: *Whether the right principally quasi-Baer ring hull and the right principally FI-extending ring hull exist for a semiprime ring and if so, are they equal?* In Theorem 8.3.39, an affirmative answer to these questions will be provided.

Burgess and Raphael [108] study ring extensions of regular rings with bounded index (of nilpotency). In particular, for a regular ring R with bounded index (of nilpotency), they obtain a unique closely related smallest overring, $R^{\#}$, which is "almost biregular" (see [108, p. 76 and Theorem 1.7]). Theorem 8.3.39 shows that their ring $R^{\#}$ is exactly the right principally FI-extending pseudo ring hull of a regular ring R with bounded index (of nilpotency). When R is commutative semiprime, the "weak Baer envelope" defined by Dobbs and Picavet in [141] is exactly the right p.q.-Baer ring hull $\widehat{Q}_{\mathbf{pqB}}(R)$ obtained in Theorem 8.3.39.

We use **pFI** and **fgFI** to denote the class of right principally FI-extending rings and the class of right finitely generated FI-extending rings, respectively (see Proposition 3.2.41 for **pFI** and **fgFI**). The following definition is useful for studying p.q.-Baer ring hulls.

Definition 8.3.38 For a ring R with identity, define

$$\mathcal{B}_p(Q(R)) = \{c \in \mathcal{B}(Q(R)) \mid \text{ there is } x \in R \text{ with } RxR_R \leq^{\text{ess}} cQ(R)_R\}.$$

The next Theorem 8.3.39 unifies the result by Burgess and Raphael [108] and that of Dobbs and Picavet [141].

Theorem 8.3.39 *Let R be a semiprime ring with identity. Then*:

(i) $\widehat{Q}_{\mathbf{pFI}}(R) = \langle R \cup \mathcal{B}_p(Q(R))\rangle_{Q(R)} = R(\mathbf{pFI},\ Q(R))$.
(ii) $\widehat{Q}_{\mathbf{pqB}}(R) = \langle R \cup \mathcal{B}_p(Q(R))\rangle_{Q(R)}$.
(iii) $\widehat{Q}_{\mathbf{fgFI}}(R) = \langle R \cup \mathcal{B}_p(Q(R))\rangle_{Q(R)}$.

Proof (i) Using a proof similar to that of Theorem 8.3.11(iv), we obtain that $\delta_{\mathbf{pFI}}(R)(1) = \mathcal{B}_p(Q(R))$. Let $S = \langle R \cup \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)}$. Then we have that $S = \langle R \cup \mathcal{B}_p(Q(R))\rangle_{Q(R)}$. We show that S is right principally FI-extending. For this, take $0 \neq s \in S$. From Lemma 8.3.29, $s = \sum_{i=1}^n r_i b_i$, where each $r_i \in R$ and the b_i are orthogonal idempotents in $\mathcal{B}(S)$. From Proposition 8.3.3(i) and Theorem 8.3.8, we see that there is $c_i \in \mathcal{B}(Q(R))$ with $Rr_iR_R \leq^{\text{ess}} c_iQ(R)_R$ for each i. So each $c_i \in \mathcal{B}_p(Q(R))$. Hence, $s = \sum_{i=1}^n r_i b_i = \sum_{i=1}^n r_i c_i b_i$. Put $e_i = c_i b_i$ for each i. Then $s = \sum_{i=1}^n r_i e_i$. We note that the e_i are orthogonal idempotents in $\mathcal{B}(S)$.

Put $D = \oplus_{i=1}^n e_i S$. To see that $SsS_S \leq^{\text{ess}} D_S$, say $0 \neq y \in D$. Then there exist $y_i \in S$ for $1 \leq i \leq n$ so that $y = \sum_{i=1}^n e_i y_i$. In this case, there exists $e_j y_j \neq 0$ for some j, $1 \leq j \leq n$, and $v \in R$ such that $0 \neq e_j y_j v \in R$. Because

$$ye_j v = e_j y_j v = c_j b_j y_j v \in c_j R \ \text{ and } \ Rr_j R_R \leq^{\text{ess}} c_j R_R,$$

there is $w \in R$ with $0 \neq ye_j vw \in Rr_j R$.

So $0 \neq y(e_j vw) = e_j y_j vw \in Rr_j e_j R = Rse_j R \subseteq SsS$ as $se_j = r_j e_j$. Hence $SsS_S \leq^{\text{ess}} D_S$. Let $f = \sum_{i=1}^{n} e_i \in \mathcal{B}(S)$. Then S is right principally FI-extending since $SsS_S \leq^{\text{ess}} D_S = \oplus_{i=1}^{n} e_i S_S = fS_S$. Therefore, $S = R(\mathbf{pFI}, Q(R))$.

Assume that T is a right ring of quotients of R and T is right principally FI-extending. Say $e \in \mathcal{B}_p(Q(R))$. Then there is $x \in R$ with $RxR_R \leq^{\text{ess}} eQ(R)_R$. Note that $TxT = T(RxR)T \subseteq T(eQ(R))T = eQ(R)$, so $TxT_R \leq^{\text{ess}} eQ(R)_R$. Hence $TxT_T \leq^{\text{ess}} eQ(R)_T$. Since T is right principally FI-extending, there exists $c^2 = c \in T$ such that $TxT_T \leq^{\text{ess}} cT_T \leq^{\text{ess}} cQ(R)_T$. Thus $e = c$ because $e \in \mathcal{B}(Q(R))$. Hence, $e \in T$ for each $e \in \mathcal{B}_p(Q(R))$. So S is a subring of T. Thus, $S = \widehat{Q}_{\mathbf{pFI}}(R)$ and $\widehat{Q}_{\mathbf{pFI}}(R) = \langle R \cup \mathcal{B}_p(Q(R)) \rangle_{Q(R)} = R(\mathbf{pFI}, Q(R))$.

Parts (ii) and (iii) follow from part (i) and Proposition 3.2.41. □

Corollary 8.3.40 *Let R be a semiprime ring with identity. Then R is right p.q.-Baer if and only if $\mathcal{B}_p(Q(R)) \subseteq R$.*

Corollary 8.3.41 *Let R be a semiprime ring with identity.*

(i) *If K is a prime ideal of $\widehat{Q}_{\mathbf{pqB}}(R)$, then $\widehat{Q}_{\mathbf{pqB}}(R)/K \cong R/(K \cap R)$.*
(ii) *LO, GU, and INC hold between R and $\widehat{Q}_{\mathbf{pqB}}(R)$.*

Proof Theorem 8.3.39 and Lemma 8.3.26 yield the result. □

Corollary 8.3.42 *Let R be a semiprime ring with identity. Then*:

(i) *$\varrho(R) = \varrho(\widehat{Q}_{\mathbf{pqB}}(R)) \cap R$, where $\varrho(-)$ is a special radical of a ring.*
(ii) *R is strongly π-regular if and only if $\widehat{Q}_{\mathbf{pqB}}(R)$ is strongly π-regular.*
(iii) $\text{kdim}\,(R) = \text{kdim}\,(\widehat{Q}_{\mathbf{pqB}}(R))$.

Proof The proof follows from Theorems 8.3.28 and 8.3.39. □

Corollary 8.3.43 *Let R be a semiprime ring with identity. Then*:

(i) *R is regular if and only if $\widehat{Q}_{\mathbf{pqB}}(R)$ is regular.*
(ii) *R has index of nilpotency at most n if and only if $\widehat{Q}_{\mathbf{pqB}}(R)$ has index of nilpotency at most n.*
(iii) *R is strongly regular if and only if $\widehat{Q}_{\mathbf{pqB}}(R)$ is strongly regular.*

Proof Put $S = \widehat{Q}_{\mathbf{pqB}}(R)$. Then S is semiprime and $\widehat{Q}_{\mathbf{qB}}(S) = \widehat{Q}_{\mathbf{qB}}(R)$ by Theorem 8.3.17.

(i) If R is regular, then $\widehat{Q}_{\mathbf{qB}}(S)$ is regular by Corollary 8.3.30. Since S is semiprime, again by Corollary 8.3.30 S is regular. Conversely, if S is regular, then from Corollary 8.3.30 $\widehat{Q}_{\mathbf{qB}}(S) = \widehat{Q}_{\mathbf{qB}}(R)$ is regular, so R is regular.

(ii) and (iii) The proof follows immediately from Theorem 8.3.32, Corollary 8.3.33, and the argument used for the proof of part (i). □

Theorem 8.3.44 *Let R be a reduced ring with identity. Then the p.q.-Baer absolute ring hull $Q_{\mathbf{pqB}}(R)$ is the Rickart absolute ring hull of R.*

Proof Because R is reduced, $Z(R_R) = 0$. Hence, Corollary 1.3.15 yields that $Q(R) = E(R_R)$. By Theorem 8.3.39, $S := Q_{\mathbf{pqB}}(R)$ exists. From Corollary 8.3.43, S is reduced and so S is Rickart (see Exercise 3.2.44.10(ii)).

Let T be a (right) Rickart right ring of quotients of R. Take $e \in \mathcal{B}_p(Q(R))$. Then $e \in S$ and there exists $x \in R$ such that $RxR_R \leq^{\text{ess}} eQ(R)_R$. Hence $SxS_S \leq^{\text{ess}} eS_S$. As S is right nonsingular, $SxS_S \leq^{\text{den}} eS_S$ by Proposition 1.3.14, as a consequence $\ell_S(SxS) = \ell_S(eS) = S(1-e)$ from the proof of Lemma 8.3.7(i). Since S is semiprime, $r_S(SxS) = \ell_S(SxS)$. So $r_S(SxS) = S(1-e) = (1-e)S$. Further, as S is reduced, $r_S(x) = r_S(SxS) = (1-e)S$.

Because T is right Rickart, $r_T(x) = fT$ for some $f^2 = f \in T$. Observe that $r_R(x) = (1-e)S \cap R$ and $r_R(x) = r_T(x) \cap R$. Therefore, we have that

$$r_R(x)_R \leq^{\text{ess}} (1-e)S_R \leq^{\text{ess}} (1-e)Q(R)_R \quad \text{and} \quad r_R(x)_R \leq^{\text{ess}} fT_R \leq^{\text{ess}} fQ(R)_R.$$

Thus $1-e = f$ as $1-e$ is central in $Q(R)$. Hence $e = 1-f \in T$, so $\mathcal{B}_p(Q(R)) \subseteq T$. From Theorem 8.3.39, $S \subseteq T$. Whence $Q_{\mathbf{pqB}}(R)$ is the Rickart absolute ring hull of R. □

When R is a semiprime ring with identity, $\widehat{Q}_{\mathbf{pqB}}(R) \subseteq \widehat{Q}_{\mathbf{qB}}(R)$. However, in the following example, we see that there exists a semiprime ring R with identity such that $\widehat{Q}_{\mathbf{pqB}}(R) \subsetneq \widehat{Q}_{\mathbf{qB}}(R)$.

Example 8.3.45 Let R be the ring as in Example 4.5.5. Then R is (right) p.q.-Baer, so $R = \widehat{Q}_{\mathbf{pqB}}(R)$. But R is not quasi-Baer. By Theorem 8.3.17,

$$\widehat{Q}_{\mathbf{qB}}(R) = R\mathcal{B}(Q(R)), \quad \text{therefore} \quad \widehat{Q}_{\mathbf{qB}}(R) = Q(R) = \prod_{n=1}^{\infty} F_n,$$

where $F_n = \mathbb{Z}_2$ for $n = 1, 2, \ldots$. Thus, $\widehat{Q}_{\mathbf{pqB}}(R) \subsetneq \widehat{Q}_{\mathbf{qB}}(R)$ (further, we observe that $\widehat{Q}_{\mathbf{qB}}(R) = Q_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{pqB}}(R) = Q_{\mathbf{pqB}}(R)$ as R is right nonsingular).

In Theorem 8.3.47, we will see that there is a connection between the right FI-extending ring hulls of semiprime homomorphic images of R and the right FI-extending right rings of quotients of R. For this, we need the next lemma.

Lemma 8.3.46 *Assume that I is a proper ideal of a ring R with identity such that I is a complement of a right ideal of R. If $P(R) \subseteq I$, then R/I is a semiprime ring.*

Proof Let J be a right ideal of R such that I is a complement of J. First we show that $(I \oplus J)/I$ is essential in R/I as a right R/I-module. To see this, assume on the contrary that there exists a nonzero right R/I-submodule K/I of R/I such that

$[(I \oplus J)/I] \cap (K/I) = 0$. There is $y \in K$ with $y \notin I$. Then $(I + yR) \cap J \neq 0$. So there exist $c \in I$, $r \in R$, and $0 \neq x \in J$ such that $c + yr = x$. Then

$$yr = -c + x \in (I \oplus J) \cap K \subseteq I.$$

Hence $x \in I \cap J = 0$, a contradiction. So $(I \oplus J)/I$ is essential in R/I as a right R/I-module.

Next, let $0 \neq B/I \trianglelefteq R/I$ such that $(B/I)^2 = 0$. Then $B^2 \subseteq I$. Note that

$$(B/I) \cap [(I \oplus J)/I] \neq 0$$

because $(I \oplus J)/I$ is essential in R/I as a right R/I-module.

From the modular law, $B \cap (I \oplus J) = I \oplus (B \cap J)$. As $B \cap (I \oplus J) \nsubseteq I$, $I \oplus (B \cap J) \nsubseteq I$, and thus $B \cap J \neq 0$. But $(B \cap J)^2 \subseteq I \cap J = 0$ as $B^2 \subseteq I$. Hence $B \cap J \subseteq J \cap P(R) \subseteq J \cap I = 0$, which is a contradiction. Therefore, R/I is a semiprime ring. □

Theorem 8.3.47 *Assume that R is a ring with identity which is either semiprime or $Q(R) = E(R_R)$. Let I be a proper ideal of R such that I_R is closed in R_R. Then*:

(i) *There exists $e \in \mathbf{I}(Q(R))$ such that $I = (1 - e)Q(R) \cap R$.*
(ii) *$eR = eRe$ and $R(1 - e) = (1 - e)R(1 - e)$.*
(iii) *R/I is ring isomorphic to eRe.*
(iv) *If R is semiprime, then $eQ(R)e \subseteq Q(eRe)$.*
(v) *If $E(R_R) = Q(R)$, then $E(eRe_{eRe}) = eQ(R)e$ and $eQ(R)e = Q(eRe)$.*
(vi) *If $P(R) \subseteq I$, then R/I is semiprime and $\widehat{Q}_{\mathbf{FI}}(R/I) \cong \widehat{Q}_{\mathbf{FI}}(eRe)$.*
(vii) *Suppose that R is semiprime (resp., right nonsingular and semiprime). Then $\widehat{Q}_{\mathbf{FI}}(R/I) \cong e\widehat{Q}_{\mathbf{FI}}(R)e$ (resp., $Q_{\mathbf{FI}}(R/I) \cong eQ_{\mathbf{FI}}(R)e$).*

Proof (i) If R is semiprime, use Proposition 8.3.3(i) and Theorem 8.3.8. In this case, we observe that $e \in \mathcal{B}(Q(R))$. If $Q(R) = E(R_R)$, then the proof is routine.

(ii) If R is semiprime, the proof of this part is clear since $e \in \mathcal{B}(Q(R))$. For $Q(R) = E(R_R)$, let $r \in R$ with $er(1 - e) \neq 0$. Since R_R is dense in $Q(R)_R$, there exists $s \in R$ such that $(1 - e)s \in R$ and $er(1 - e)s \neq 0$. Then

$$(1 - e)s \in R \cap (1 - e)Q(R) = I.$$

Hence $0 \neq er(1 - e)s \in eI = 0$, a contradiction. So $eR(1 - e) = 0$. Consequently, $eR = eRe$ and $R(1 - e) = (1 - e)R(1 - e)$.

(iii) Define $f : R/I \to eRe$ by $f(r + I) = er$. As $eI = 0$, f is well defined. Clearly, f is a ring epimorphism. If $x + I \in \mathrm{Ker}(f)$, then $x \in (1 - e)Q(R) \cap R$. By part (i), $x \in I$. Hence $\mathrm{Ker}(f) = 0$. Thus, f is a ring isomorphism.

(iv) As $e \in \mathcal{B}(Q(R))$, $eRe_{eRe} \leq^{\mathrm{den}} eQ(R)e_{eRe}$. So $eQ(R)e \subseteq Q(eRe)$.

(v) Let K be a right ideal of eRe and let $g : K \to eQ(R)e$ be an eRe-homomorphism. From part (ii) K, eRe, and $eQ(R)e$ are right R-modules, and g is an R-homomorphism. As $eQ(R)e \subseteq eQ(R)$ and $eQ(R)$ is the injective hull of eR_R, g can be extended to an R-homomorphism $\overline{g} : eR \to eQ(R)$. Now $\overline{g}$ can be extended to an R-homomorphism $\widetilde{g} : eQ(R) \to eQ(R)$. Therefore, $\widetilde{g}$ is a $Q(R)$-homomorphism as in the proof of Proposition 2.1.32.

As $eR = eRe$, $\overline{g}(eR) = \widetilde{g}(eRe) = \widetilde{g}(eRe)e = \widetilde{g}(eR)e \subseteq eQ(R)e$. By Baer's Criterion, $eQ(R)e$ is an injective right eRe-module. Further, we observe that $eRe_{eRe} \leq^{\text{den}} eQ(R)e_{eRe}$. Hence, $eQ(R)e$ is the injective hull of eRe as a right eRe-module and $eQ(R)e = Q(eRe)$.

(vi) Note that a closed right ideal of R is a complement of some right ideal of R (see Exercise 2.1.37.3). Hence this part is a consequence of part (iii), Lemma 8.3.46, and Theorem 8.3.17.

(vii) Let R be semiprime. Then $1-e \in \mathcal{B}(Q(R))$ by Proposition 8.3.3(i) and Theorem 8.3.8, so $e \in \mathcal{B}(Q(R))$. Hence $\mathcal{B}(eQ(R)e) = e\mathcal{B}(Q(R))e$. Thus we have that $\widehat{Q}_{\mathbf{FI}}(R/I) \cong \langle eRe \cup \mathcal{B}(eQ(R)e)\rangle_{eQ(R)e} = eR\mathcal{B}(Q(R))e = e\widehat{Q}_{\mathbf{FI}}(R)e$ from Theorem 8.3.17. If additionally $Z(R_R) = 0$, then eR_R is nonsingular, so $(R/I)_R$ is right nonsingular since $(R/I)_R \cong eR_R$ by modifying the proof of part (iii). Thus, R/I is a right nonsingular ring by [180, Proposition 1.28] and so eRe is a right nonsingular ring. The result follows from the fact that for any right nonsingular ring T, $\widehat{Q}_{\mathbf{FI}}(T) = Q_{\mathbf{FI}}(T)$ since $Q(T) = E(T_T)$. □

Corollary 8.3.48 *Let R be a semiprime ring with identity, S a ring with identity, and $\theta : R \to S$ a ring epimorphism such that* $\mathrm{Ker}(\theta)$ *is a nonessential ideal of R. Then there exists a nonzero ring homomorphism $h : S \to \widehat{Q}_{\mathbf{FI}}(R)$.*

Proof Let $K = \mathrm{Ker}(\theta)$ and $I = \ell_R(\ell_R(K))$. Then $K \in \mathfrak{D}_{\mathbf{IC}}(R)$ by Proposition 8.3.3(i) since R is semiprime. So I is the unique closure of K_R in R_R (see Exercise 8.3.58.5(i)). From Theorem 8.3.47(i), there exists $e \in \mathcal{B}(Q(R))$ such that $I = (1-e)Q(R) \cap R$. As K is not essential and R is semiprime, $\ell_R(K) \neq 0$ by Proposition 1.3.16, so $I \neq R$. We have the following sequence of ring homomorphisms $S \overset{\alpha}{\to} R/K \overset{\beta}{\to} R/I \overset{\lambda}{\to} \widehat{Q}_{\mathbf{FI}}(R/I) \overset{\delta}{\to} e\widehat{Q}_{\mathbf{FI}}(R)e \overset{\iota}{\to} \widehat{Q}_{\mathbf{FI}}(R)$, using Theorem 8.3.47, where α and δ are ring isomorphisms, β is a ring epimorphism, and λ and ι are inclusions. Take $h = \iota\delta\lambda\beta\alpha$. □

Proposition 8.3.49 *Let $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Then* $\mathrm{Cen}(I) = I \cap \mathrm{Cen}(R)$.

Proof Let $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Then $Q(I) = eQ(R)$ with $e \in \mathcal{B}(Q(R))$ by Theorem 8.3.8. So $\mathrm{Cen}(I) \subseteq \mathrm{Cen}(Q(I)) = \mathrm{Cen}(eQ(R)) \subseteq \mathrm{Cen}(Q(R))$. Therefore we have that $\mathrm{Cen}(I) = I \cap \mathrm{Cen}(R)$. □

A nonempty subset M of a ring R is called an *m-system* if $0 \notin M$ and for any $a, b \in M$ there exists $x \in R$ such that $axb \in M$ (see [296]). We note that an ideal P of a ring R maximal with respect to $P \cap M = \emptyset$, where M is an m-system, is always a prime ideal.

Theorem 8.3.50 *Let R be a semiprime ring with a descending chain of essential ideals $K_1 \supseteq K_2 \supseteq \dots$ such that $\bigcap_{i \geq 1} K_i = 0$. Then R has a prime ideal P such that $K_i \nsubseteq P$ for all $i \geq 1$.*

Proof We use the condition on $\{K_i\}_{i=1}^{\infty}$ to find a properly descending subsequence $\{L_i\}_{i=1}^{\infty}$ and nonzero elements $\{a_i\}$, $\{x_i\}$ such that $a_{i+1} = a_i x_i a_i$, $a_{i+1} \in L_i$ and $a_{i+1} \notin L_{i+1}$ for $i \geq 1$.

Let $L_1 = K_1$ and choose $0 \neq a_1 \in L_1$. Then we show that $a_1 K_2 a_1 \neq 0$. For this, assume on the contrary that $a_1 K_2 a_1 = 0$. Then $(K_2 a_1 K_2)(K_2 a_1 K_2) = 0$, so $K_2 a_1 K_2 = 0$ because R is semiprime. Now $\ell_R(K_2) = r_R(K_2) = 0$ since K_2 is essential in R, and hence $K_2 a_1 = 0$. Again since $r_R(K_2) = 0$, $a_1 = 0$, a contradiction. Thus, $a_1 K_2 a_1 \neq 0$. From $\cap_{i \geq 1} K_i = 0$, there exists K_j with j minimal, such that $a_1 K_2 a_1 \not\subseteq K_j$, and hence there is $x_1 \in K_2$ such that $a_1 x_1 a_1 \notin K_j$. Let $L_2 = K_j$ and $a_2 = a_1 x_1 a_1$; then $a_2 \in L_1$ and $a_2 \notin L_2$.

Next, $a_2 L_2 a_2 \neq 0$ by the preceding argument. Choose L_3 such that $a_2 L_2 a_2 \not\subseteq L_3$. So there is $x_2 \in L_2$ with $a_3 := a_2 x_2 a_2 \notin L_3$. Note that $a_3 \in L_2$. Continue this procedure to get L_{i+1} and $a_{i+1} = a_i x_i a_i \in L_i$ but $a_{i+1} \notin L_{i+1}$ as needed. The sequence $\{a_i\}$ constitutes an m-system. In fact, let $a_\ell, a_n \in \{a_i\}$. If $\ell = n$, then $a_\ell x_n a_n = a_{n+1}$. So without of loss of generality, we may assume that $n > \ell$. Then $a_{n+1} = a_\ell[(x_\ell a_\ell)(x_{\ell+1} a_{\ell+1}) \cdots (x_{n-1} a_{n-1}) x_n] a_n$. Hence, an ideal P maximal with respect to $\{a_i\} \cap P = \emptyset$ is a prime ideal. By construction, $K_i \not\subseteq P$ for all $i \geq 1$. □

Lemma 8.3.51 *Let R be a semiprime ring and $I \trianglelefteq R$. Then*:

(i) *$\ell_R(I)$ is a semiprime ideal of R.*
(ii) *$(I \oplus \ell_R(I))/\ell_R(I)$ is an essential ideal of $R/\ell_R(I)$.*

Proof (i) To show that $\ell_R(I)$ is a semiprime ideal, let $a \in R$ such that $aRa \subseteq \ell_R(I)$. Then $aRaI = 0$, so $(aI)R(aI) = 0$. Thus, $aI = 0$ because R is semiprime. Hence, $a \in \ell_R(I)$, so $\ell_R(I)$ is a semiprime ideal.

(ii) Let $S = R/\ell_R(I)$. By part (i), S is a semiprime ring. To show that $V := (I \oplus \ell_R(I))/\ell_R(I)$ is essential in S, it suffices to see that $\ell_S(V) = 0$ by Proposition 1.3.16. Say $a + \ell_R(I) \in \ell_S(V)$, where $a \in R$. Then $aI \subseteq \ell_R(I)$, so $aI^2 = 0$. Hence, $(aI)^2 = 0$. Thus, $aI = 0$ because R is semiprime. Therefore, $a \in \ell_R(I)$, hence $a + \ell_R(I) = 0$. □

The following theorem is well known (see [366, Remark 1.2.14, Theorems 1.4.1 and 1.6.27]).

Theorem 8.3.52 *Let R be a semiprime PI-ring. Then R satisfies a standard identity $f_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}$, where S_n is the symmetric group of degree n and $\mathrm{sgn}(\sigma)$ is the signature of $\sigma \in S_n$. Further, R satisfies $f_m(x_1, \ldots, x_m)$ for $m \geq n$.*

An ideal I of a ring is called a *PI-ideal* if I is a PI-ring as a ring by itself.

Theorem 8.3.53 *Let R be a semiprime ring such that R/P is a PI-ring for each prime ideal P of R. Then R contains a nonzero PI-ideal, and the sum of all PI-ideals of R is an essential ideal of R.*

Proof Put $\mathcal{F}_n = \{P \mid P \text{ is a prime ideal and } R/P \text{ satisfies } f_n(x_1, \ldots, x_n)\}$ for $n \geq 2$, and let $K_n = \cap_{P \in \mathcal{F}_n} P$. Since $\mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \ldots$ from Theorem 8.3.52, the sequence of ideals $\{K_j\}$ is a descending sequence of semiprime ideals with $\cap_{i\geq 2} K_i = 0$ since R is semiprime and $\cup_{i\geq 2} \mathcal{F}_i$ is the set of all prime ideals. We note that R/K_n embeds in $\prod_{P\in\mathcal{F}_n} R/P$, hence it satisfies a PI. If each K_i is essential, Theorem 8.3.50 yields a prime ideal P which contains none of the K_i. However $P \in \mathcal{F}_m$ for some $m \geq 2$ and so $K_m \subseteq P$, a contradiction. Thus there exists some K_n which is not essential. Hence, $\ell_R(K_n) \neq 0$ by Proposition 1.3.16. As R is semiprime, $\ell_R(K_n) \cap K_n = 0$ and so $\ell_R(K_n)$ embeds in R/K_n. Therefore, $\ell_R(K_n)$ is an PI-ideal.

Let S be the sum of all PI-ideals of R and let $A = \ell_R(S)$. Then $B := \ell_R(A)$ is a semiprime ideal by Lemma 8.3.51(i) and $A \cap B = 0$. Since all prime factor rings of R are PI-rings, all prime factor rings of the semiprime ring R/B are PI-rings. If $B = R$, then $R = \ell_R(A)$, so $A = 0$ because R is semiprime. Thus $\ell_R(S) = 0$, hence by Proposition 1.3.16, S is essential in R.

Next, we assume that $B \neq R$. Then R/B contains a nonzero PI-ideal by the previous argument. To see that S is an essential ideal of R, we need to show that $A = 0$ from Proposition 1.3.16. If $A \neq 0$, then $(A + B)/B$ is essential in R/B by Lemma 8.3.51(ii). So $(A + B)/B$ contains a nonzero PI-ideal, say V/B of R/B. Put

$$K = \{a \in A \mid a + B \in V/B\}.$$

Then $K \trianglelefteq R$ and $K \cong V/B$ as rings since $A \cap B = 0$. So K is a nonzero PI-ideal of R and $K \subseteq A$. Hence $S \cap A \neq 0$, which is a contradiction because $A = \ell_R(S)$. So $A = 0$. Therefore, S is essential in R. □

The next lemma, known as Andrunakievic's lemma, is useful for studying the relationship between the ideal structure of a given ideal of a ring R and that of R (see [9, Lemma 4]).

Lemma 8.3.54 *Let R be a ring and $V \trianglelefteq R$. Assume that $I \trianglelefteq V$ and W is the ideal of R generated by I. Then $W^3 \subseteq I$.*

Proof Since $V \trianglelefteq R$ and $I \trianglelefteq V$, we get $W = I + IR + RI + RIR$. Therefore it follows that $W^3 \subseteq VWV = V(I + IR + RI + RIR)V \subseteq I$. □

Proposition 8.3.55 *Let R be a ring and $V \trianglelefteq R$.*

(i) *If R is a semiprime ring, then V is a semiprime ring.*

(ii) *If R is a prime ring, then V is a prime ring.*

Proof (i) To show that V is a semiprime ring, let $I \trianglelefteq V$ with $I^2 = 0$. Say W is the ideal of R generated by I. By Lemma 8.3.54, $W^3 \subseteq I$. So $W^6 \subseteq I^2 = 0$. As R is semiprime, $W = 0$ and so $I = 0$. Hence, V is a semiprime ring.

(ii) Similarly, we see that V is a prime ring if R is a prime ring. □

Every semiprime PI-ring satisfies the hypothesis of our next result. Example 8.3.57 illustrates that Theorem 8.3.56 is a proper generalization of Theorem 3.2.16.

Theorem 8.3.56 *Let R be a semiprime ring with R/P a PI-ring for each prime ideal P of R. If $0 \neq I \trianglelefteq R$, then $I \cap \mathrm{Cen}(R) \neq 0$.*

Proof From Theorem 8.3.53, there exists $V \trianglelefteq R$ such that

$$V_R \leq^{\mathrm{ess}} R_R \quad \text{and} \quad V = \sum_{\lambda \in \Lambda} V_\lambda,$$

where each V_λ is a nonzero PI-ideal. If $I \cap V_\lambda = 0$ for all $\lambda \in \Lambda$, then $IV = 0$, and hence $I \cap V = 0$, contrary to $V_R \leq^{\mathrm{ess}} R_R$.

So there is $\beta \in \Lambda$ with $0 \neq I \cap V_\beta \trianglelefteq V_\beta$. By Theorem 3.2.16 and Proposition 8.3.55, $I \cap \mathrm{Cen}(V_\beta) = I \cap V_\beta \cap \mathrm{Cen}(V_\beta) \neq 0$ since V_β is a semiprime PI-ring. Propositions 8.3.3(i) and 8.3.49 yield that $\mathrm{Cen}(V_\beta) = V_\beta \cap \mathrm{Cen}(R)$. As a consequence, $I \cap \mathrm{Cen}(V_\beta) = I \cap V_\beta \cap \mathrm{Cen}(R) \neq 0$. Therefore, $I \cap \mathrm{Cen}(R) \neq 0$. □

Example 8.3.57 There is a semiprime ring R which does not satisfy a PI, but R/P is a PI-ring for every prime ideal P of R. For a field F, let

$$R = \{(A_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathrm{Mat}_n(F) \mid A_n \text{ is a scalar matrix eventually}\},$$

which is a subring of $\prod_{n=1}^{\infty} \mathrm{Mat}_n(F)$. Then R is a semiprime ring which does not satisfy a PI. Let P be a prime ideal of R.

Case 1. Assume that the k-th component of all elements of P is zero for some k. Let $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the k-th component. Take $x \in R$ such that x has zero in its k-th component. Then $e_k R x = 0$ and so $x \in P$. Therefore $P = \{(A_n)_{n=1}^{\infty} \in R \mid A_k = 0\}$. Hence $R/P \cong \mathrm{Mat}_k(F)$.

Case 2. Assume that for any k, there is an element of P with a nonzero entry in its k-th component. Then for any k, there is $0 \neq \alpha \in \mathrm{Mat}_k(F)$ such that $\mu_k := (0, 0, \ldots, 0, \alpha, 0, \ldots) \in P$, where α is in the k-th component. Thus $R\mu_k R \subseteq P$, so $\oplus_{k=1}^{\infty} \mathrm{Mat}_k(F) \subseteq P$. As $R/\oplus_{k=1}^{\infty} \mathrm{Mat}_k(F)$ is commutative, and R/P is a ring homomorphic image of $R/\oplus_{k=1}^{\infty} \mathrm{Mat}_k(F)$, R/P is commutative.

By Cases 1 and 2, R/P is a PI-ring for every prime ideal P of R.

Exercise 8.3.58

1. Finish the proof of Proposition 8.3.3 and prove Lemma 8.3.4.
2. Let $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Prove the following.
   (i) $\ell_R(I) \subseteq r_R(I)$.
   (ii) $\ell_R(I) = r_R(I)$ if and only if $r_R(I) \cap I = 0$.
3. Assume that R is a ring.
   (i) Show that $\mathfrak{D}_{\mathbf{IC}}(R)$ contains no nonzero nilpotent ideals of R.
   (ii) Find an example of a right nonsingular quasi-Baer ring R such that $0 \neq P(R) \in \mathfrak{D}_{\mathbf{IC}}(R)$ (see [232]).
4. Let R be a ring. Show that the following are equivalent.
   (i) $R \in \mathbf{IC}$.

(ii) For each $K \in \mathfrak{D}_{\mathbf{IC}}(R)$ with K_R closed in R_R, there exists $e^2 = e \in R$ such that $K = eR$.
(iii) For each $K \in \mathfrak{D}_{\mathbf{IC}}(R)$ with K_R closed in R_R, there is $c \in \mathcal{B}(R)$ satisfying $K = cR$.

5. Let R be a ring with identity and $I \in \mathfrak{D}_{\mathbf{IC}}(R)$. Prove the following.
   (i) There exists $e \in \mathcal{B}(Q(R))$ such that $\ell_R(\ell_R(I)) = eQ(R) \cap R$ and $\ell_R(\ell_R(I))$ is the unique closure of I_R in R_R.
   (ii) Let $K = \ell_R(\ell_R(I))$. Then $R/K \cong (1-e)R(1-e)$ as rings.
6. Prove Corollary 8.3.12.
7. Show that in Lemma 8.3.26 and in Theorem 8.3.28, the set $\mathbb{E}$ can be a set of idempotents each taken from some set of left or right triangulating idempotents (see [97, Example 2.3]).
8. ([42, Beidar and Wisbauer]) Show that a ring R with identity is biregular if and only if R is semiprime and $R\mathcal{B}(Q(R))$ is biregular.
9. Let R be a ring (not necessarily with identity) and $S = \langle R \cup 1_{Q(R)} \rangle_{Q(R)}$. Show that $Q(R) = Q(S) \subseteq E(S_S) \subseteq E(S_R) = E(R_R)$.

8.4 Module Hulls

It is well known that for every module M, there always exists a unique (up to isomorphism) minimal injective extension (overmodule) which is called its injective hull and is denoted by $E(M)$. While the injective hull has been studied and used extensively, in some instances it is difficult for a fruitful transfer of information to take place between M and $E(M)$. For example, take M to be the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$, where p is a prime integer. Then $H = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^3}$ is an extending hull of M. We observe that both M and H are finite, but $E(M)$ is infinite.

The studies on module hulls have been rather limited. In this section, we discuss module hulls satisfying some generalizations of injectivity. One may expect that such minimal overmodules will allow for a rich transfer of information similar to the case of rings. This is because each of these hulls, with more general properties than injectivity, sits in between M and a fixed injective hull $E(M)$ of M; and hence it generally lies closer to the module M than $E(M)$.

Definition 8.4.1 Let M be a module. We fix an injective hull $E(M)$ of M. Let $\mathfrak{M}$ be a class of modules. We call, when it exists, a module $H_{\mathfrak{M}}(M)$ the $\mathfrak{M}$ *hull* of M if $H_{\mathfrak{M}}(M)$ is the smallest extension of M in $E(M)$ that belongs to $\mathfrak{M}$ (i.e., $H_{\mathfrak{M}}(M)$ is the $\mathfrak{M}$ absolute hull of M).

We begin this section with a description of a quasi-injective hull of a module M (i.e., $H_{\mathbf{qI}}(M)$, where $\mathbf{qI}$ is the class of quasi-injective modules). We recall that an R-module M is quasi-injective if and only if $f(M) \subseteq M$, for all $f \in \mathrm{End}(E(M))$ (see Theorem 2.1.9). The next result about the existence of quasi-injective hulls is due to Johnson and Wong [238].

Theorem 8.4.2 *Let M be a right R-module and let $S = \mathrm{End}(E(M))$. Then SM is the quasi-injective hull of M.*

Proof We put $U = SM$. Then $M \le U \le E(M)$ and $E(U) = E(M)$. Now take $\phi \in \mathrm{End}(E(U)) = \mathrm{End}(E(M))$. Then $\phi(U) \subseteq U$. By Theorem 2.1.9, U is quasi-injective. Next we assume that $M \le N \le E(M)$ and N is quasi-injective. Then $\varphi(N) \subseteq N$ for any $\varphi \in \mathrm{End}(E(N)) = \mathrm{End}(E(M))$ by Theorem 2.1.9. Thus, $SN \subseteq N$ and so $SM \subseteq SN \subseteq N$. Therefore SM is the quasi-injective hull of M (i.e., $SM = H_{\mathbf{qI}}(M)$). □

The following result for the existence of the quasi-continuous hull of a module is obtained by Goel and Jain [177].

Theorem 8.4.3 *Let M be a right R-module and $S = \mathrm{End}(E(M))$. Let Ω be the subring of S generated by the set of all idempotents of S. Then ΩM is the quasi-continuous hull of M.*

Proof As $E(\Omega M) = E(M)$, Ω is also the subring of $\mathrm{End}(E(\Omega M))$ generated by the set of all idempotents. As $\Omega(\Omega M) = \Omega M$, ΩM is quasi-continuous by Theorem 2.1.25. Say $M \le N \le E(M)$ and N is quasi-continuous. Then $E(N) = E(M)$, so Ω is the subring of $\mathrm{End}(E(N))$ generated by the set of all idempotents. From Theorem 2.1.25, $\Omega N \subseteq N$. Thus, $\Omega M \subseteq \Omega N \subseteq N$. So ΩM is the quasi-continuous hull of M (i.e., $\Omega M = H_{\mathbf{qCon}}(M)$). □

In contrast to Theorems 8.4.2 and 8.4.3, for the case of continuous hulls, there exists a nonsingular uniform cyclic module over a noncommutative ring which does not have an absolute continuous hull as follows.

Example 8.4.4 Let V be a vector space over a field F with basis elements v_m, w_k $(m, k = 0, 1, 2, \dots)$. We denote by V_n the subspace generated by the v_m $(m \ge n)$ and all the w_k. Also we denote by W_n the subspace generated by the w_k $(k \ge n)$. We write $\mathcal{S}$ for the shift operator such that $\mathcal{S}(w_k) = w_{k+1}$ and $\mathcal{S}(v_i) = 0$ for all k, i. Let R be the set of all $\rho \in \mathrm{End}_F(V)$ with $\rho(v_m) \in V_m$, $\rho(w_0) \in W_0$ and $\rho(w_k) = \mathcal{S}^k \rho(w_0)$, for $m, k = 0, 1, 2, \dots$.

Note that $\tau\rho(w_k) = \mathcal{S}^k \tau\rho(w_0)$, for $\rho, \tau \in R$, and so $\tau\rho \in R$. Thus, it is routine to check that R is a subring of $\mathrm{End}_F(V)$. Further, we see that $V_n = Rv_n$, $W_n = Rw_n$, and $V_{n+1} \subseteq V_n$ for all n. (When $f \in R$ and $v \in V$, we also use fv for the image $f(v)$ of v under f.)

Consider the left R-module $M = W_0$. First, we show that $M = Rw_0$ is uniform. For this, take $fw_0 \ne 0$, $gw_0 \ne 0$ in M, where $f, g \in R$. We need to find $h_1, h_2 \in R$ such that $h_1 f w_0 = h_2 g w_0 \ne 0$. Let

$$fw_0 = b_0 w_0 + b_1 w_1 + \cdots + b_m w_m \in Rw_0$$

and

$$gw_0 = c_0 w_0 + c_1 w_1 + \cdots + c_m w_m \in Rw_0,$$

where $b_i, c_j \in F$, $i, j = 0, 1, \dots, m$, and some terms of b_i and c_j may be zero.

Put $h_1 w_0 = x_0 w_0 + x_1 w_1 + \cdots + x_\ell w_\ell$ and $h_2 w_0 = y_0 w_0 + y_1 w_1 + \cdots + y_\ell w_\ell$, where $x_i, y_i \in F$, $i = 0, 1, \ldots, \ell$ (also some terms of x_i and y_j may be zero). Since $h_1(w_k) = \mathcal{S}^k h_1(w_0)$ and $h_2(w_k) = \mathcal{S}^k h_2(w_0)$ for $k = 0, 1, 2 \ldots,$ we need to find such $x_i, y_i \in F$, $0 \leq i \leq \ell$ so that $h_1 f w_0 = h_2 g w_0 \neq 0$ from the following equations:

$$b_0 x_0 = c_0 y_0, \ b_0 x_1 + b_1 x_0 = c_0 y_1 + c_1 y_0,$$

$$b_0 x_2 + b_1 x_1 + b_2 x_0 = c_0 y_2 + c_1 y_1 + c_2 y_0,$$

$$b_0 x_3 + b_1 x_2 + b_2 x_1 + b_3 x_0 = c_0 y_3 + c_2 y_1 + c_2 y_1 + c_3 y_0,$$

and so on. Now say $\alpha(t) = b_0 + \cdots + b_m t^m \neq 0$ and $\beta(t) = c_0 + \cdots + c_m t^m \neq 0$ in the polynomial ring $F[t]$. Then $\alpha(t)F[t] \cap \beta(t)F[t] \neq 0$. We may note that finding such $x_0, x_1 \ldots, x_\ell, y_0, y_1 \ldots, y_\ell$ in F above is the same job for finding $x_0, x_1 \ldots, x_\ell, y_0, y_1 \ldots, y_\ell$ such that

$$\alpha(t)(x_0 + x_1 t + \cdots + x_\ell t^\ell) = \beta(t)(y_0 + y_1 t + \cdots + y_\ell t^\ell) \neq 0$$

in the polynomial ring $F[t]$. Observing that $0 \neq \alpha(t)\beta(t) \in \alpha(t)F[t] \cap \beta(t)F[t]$, take $h_1 w_0 = c_0 w_0 + c_1 w_1 + \cdots + c_m w_m$ by putting $\ell = m$, $x_i = c_i$ for $0 \leq i \leq m$, and $h_2 w_0 = b_0 w_0 + b_1 w_1 + \cdots + b_m w_m$ by putting $\ell = m$, $y_i = b_i$ for $0 \leq i \leq m$. As $\alpha(t)\beta(t) \neq 0$, $0 \neq h_1 f w_0 = h_2 g w_0 \in Rf w_0 \cap Rg w_0$. So M is uniform.

Next, we show that each V_n is an essential extension of M (hence each V_n is uniform). Indeed, let $0 \neq \mu v_n \in R v_n = V_n$, where $\mu \in R$. Say

$$\mu v_n = a_{n+k} v_{n+k} + \cdots + a_{n+k+\ell} v_{n+k+\ell} + b_s w_s + \cdots + b_{s+m} w_{k+m}.$$

If $a_{n+k} = \cdots = a_{n+k+\ell} = 0$, then $\mu v_n \in W_0$. Otherwise, we assume that $a_{n+k} \neq 0$. Let $\omega \in R$ such that $\omega(v_{n+k}) = w_0$ and $\omega(v_i) = 0$ for $i \neq n+k$ and $\omega(w_j) = 0$ for all j. Then $0 \neq \omega\mu v_n = a_{n+k} w_0 \in W_0$. Thus $M = W_0$ is essential in V_n. Since M is uniform, V_n is also uniform for all n.

We prove that ${}_R M$ is nonsingular. For this, assume that $u \in Z({}_R M)$ and let $K = \{\alpha \in R \mid \alpha u = 0\}$. Then K is an essential left ideal of R. So $K \cap R\mathcal{S}^2 \neq 0$. Thus there exists $\rho \in R$ such that $\rho\mathcal{S}^2 \neq 0$ and $\rho\mathcal{S}^2(u) = 0$. Say

$$u = a_k w_k + a_{k+1} w_{k+1} + \cdots + a_n w_n \ \text{ with } \ a_k, a_{k+1}, \ldots, a_n \in F.$$

Assume on the contrary that $u \neq 0$. Then we may suppose that $a_k \neq 0$. Because $\rho(w_n) = \mathcal{S}^n \rho(w_0)$ for $n = 0, 1, 2, \ldots,$

$$\begin{aligned} 0 = \rho\mathcal{S}^2(u) &= a_k \rho\mathcal{S}^2(w_k) + a_{k+1}\rho\mathcal{S}^2(w_{k+1}) + \cdots + a_n \rho\mathcal{S}^2(w_n) \\ &= a_k \mathcal{S}^{k+2}\rho(w_0) + a_{k+1}\mathcal{S}^{k+3}\rho(w_0) + \cdots + a_n \mathcal{S}^{n+2}\rho(w_0). \end{aligned}$$

Here we put $\rho(w_0) = b_\ell w_\ell + b_{\ell+1} w_{\ell+1} + \cdots + b_t w_t$. If $\rho(w_0) = 0$, then we see that $\rho\mathcal{S}^2(w_0) = \rho(w_2) = \mathcal{S}^2\rho(w_0) = 0$. Also, $\rho\mathcal{S}^2(w_m) = 0$ for all $m = 1, 2, \ldots,$ and

$\rho\mathcal{S}^2(v_i) = 0$ for all $i = 0, 1, \ldots$. Thus $\rho\mathcal{S}^2 = 0$, a contradiction. Hence $\rho(w_0) \neq 0$, and so we may assume that $b_\ell \neq 0$. Note that

$$\mathcal{S}^{k+2}\rho(w_0) = b_\ell w_{\ell+k+2} + b_{\ell+1} w_{\ell+k+3} + \cdots + b_t w_{t+k+2},$$

$$\mathcal{S}^{k+3}\rho(w_0) = b_\ell w_{\ell+k+3} + b_{\ell+1} w_{\ell+k+4} + \cdots + b_t w_{t+k+3},$$

and so on. Thus $0 = \rho\mathcal{S}^2(u) = a_k b_\ell w_{\ell+k+2} + (a_k b_{\ell+1} + a_{k+1} b_\ell) w_{\ell+k+3} + \cdots$, and hence $a_k b_\ell = 0$, which is a contradiction because $a_k \neq 0$ and $b_\ell \neq 0$. Therefore $u = 0$, and so M is nonsingular.

We show now that V_n is continuous. Note that V_n is uniform. So clearly, V_n has (C_1) condition. Thus, to show that V_n is continuous, it suffices to prove that every R-monomorphism of V_n is onto for V_n to satisfy (C_2) condition.

Let $\varphi : V_n \to V_n$ be an R-monomorphism. We put

$$\varphi(v_n) = \rho v_n \in R v_n = V_n, \quad \text{where} \quad \rho \in R.$$

We claim that $\rho v_n \notin V_{n+1}$. For this, assume on the contrary that $\rho v_n \in V_{n+1}$. Now we let $\lambda \in R$ such that $\lambda v_n = v_n$, $\lambda v_k = 0$ for $k \neq n$, and $\lambda w_m = 0$ for all m. Then $\varphi(\lambda v_n) = \lambda(\rho v_n) = 0$ since $\rho(v_n) \in V_{n+1}$. But $\lambda v_n = v_n \neq 0$. Thus φ is not one-to-one, a contradiction. Therefore $\rho v_n \notin V_{n+1}$.

As $\rho v_n \in V_n$, write

$$\rho v_n = a_n v_n + a_{n+1} v_{n+1} + \cdots + a_{n+\ell} v_{n+\ell} + b_0 w_0 + \cdots + b_h w_h,$$

where $a_n, a_{n+1}, \ldots, a_{n+\ell}, b_0, b_1, \ldots, b_h \in F$, and $a_n \neq 0$.

Take $\nu \in R$ such that $\nu v_n = a_n^{-1} v_n$, $\nu v_k = 0$ for $k \neq n$ and $\nu w_m = 0$ for all m. Then we see that $v_n = \nu\rho v_n \in R\rho v_n$. So $R v_n \subseteq R\rho v_n$, hence $V_n = R v_n = R\rho v_n$. Thus $\varphi(R v_n) = R\varphi(v_n) = R\rho v_n = V_n$, so φ is onto. Therefore each V_n is continuous.

Finally, note that the uniform nonsingular module $M = R w_0$ is not continuous, since the shifting operator $\mathcal{S}$ provides an R-monomorphism which is not onto. Hence, M does not have a continuous hull (in $E(M) = E(V)$), because such a hull would have to be contained in each V_n, and hence in $M = \cap_n V_n$.

Despite Example 8.4.4, we will show that continuous hulls do exist for certain classes of modules over a commutative ring as shown in the next several results. We start with a lemma.

Lemma 8.4.5 *Assume that R is a commutative ring and M is a nonsingular cyclic R-module. Let $E = E(M_R)$ and T be a subring of* End(E_R). *Then*:

(i) $E\, r_R(M) = 0$.
(ii) *There exists a smallest continuous module V such that $M \leq V \leq E$ and $TV \subseteq V$.*

Proof Let $I = r_R(M) \trianglelefteq R$. Put $\overline{R} = R/I$. Then $M \cong \overline{R}_R$.

(i) Note that E_R is nonsingular because M_R is nonsingular. Let $x \in E(\overline{R}_R)$. Then there is an essential ideal L of R with $xL \subseteq R/I$. Hence $(xI)L = xLI = 0$, so $xI \subseteq Z(E_R) = 0$. Thus, $xI = 0$. Therefore, $EI = 0$.

(ii) *Step 1.* By part (i), E has an $\overline{R}$-module structure induced from the R-module E_R. To see that E is the injective hull of the $\overline{R}$-module M, note that E is an essential extension of M as an $\overline{R}$-module. Let K/I be an ideal of R/I and $\alpha \in \mathrm{Hom}((K/I)_{\overline{R}}, E_{\overline{R}})$. Then $\alpha \in \mathrm{Hom}((K/I)_R, E_R)$ and so there exists an extension $\beta \in \mathrm{Hom}((R/I)_R, E_R)$ of α. We see that $\beta \in \mathrm{Hom}((R/I)_{\overline{R}}, E_{\overline{R}})$. Hence E is an injective $\overline{R}$-module. Therefore, E is an injective hull of M as an $\overline{R}$-module. Further, M is nonsingular as an $\overline{R}$-module by routine arguments.

By Theorem 2.1.31, $E = Q(\overline{R})$, which is a commutative regular ring. Also from Proposition 2.1.32, $E = \mathrm{End}(E_E) = \mathrm{End}(E_{\overline{R}})$ $(= \mathrm{End}(E_R))$. Thus T is a subring of E. Also $\overline{R}$ is a subring of E.

Let P be the subring of E generated by all idempotents of E. We claim that any regular subring A of E satisfying $\overline{R}P \subseteq A$ is continuous as an R-module (or equivalently, as an $\overline{R}$-module).

First, by Theorem 2.1.25 or Theorem 8.4.3, A is a quasi-continuous R-module because $PA = A$. We show that A_R has (C_2) condition. For this, let $A = A_1 \oplus A_2$, which is an R-module decomposition, and let $\varphi : A_1 \to N$ be an R-isomorphism, where $N_R \leq A_R$. Note that $\mathrm{Hom}_R(A, A_1) = \mathrm{Hom}_{\overline{R}}(A, A_1)$. Further, from the proof of Proposition 2.1.32, $\mathrm{Hom}_{\overline{R}}(A, A_1) = \mathrm{Hom}_A(A, A_1)$, because A is a ring of quotients of $\overline{R}$. Thus $\mathrm{Hom}_R(A, A_1) = \mathrm{Hom}_A(A, A_1)$.

We let $\pi_1 : A \to A_1$ be the canonical projection of R-modules. Then we see that π_1 is an A-homomorphism. Therefore $A_1 = \pi_1(A) = \pi_1(1)A$. Similarly, we observe that $\varphi \in \mathrm{Hom}_R(A_1, N) \subseteq \mathrm{Hom}_R(A_1, A) = \mathrm{Hom}_A(A_1, A)$.

So we have that $N = \varphi(A_1) = \varphi(\pi_1(1)A) = \varphi\pi_1(1)A$ is a principal (right) ideal of A. Hence $N_A \leq^{\oplus} A_A$ because A is a regular ring, and so $N_R \leq^{\oplus} A_R$. Thus A_R satisfies (C_2) condition. Therefore, A_R is a continuous module.

Let V be the intersection of all regular subrings V_i of E with $\overline{R}PT \subseteq V_i$. Then as in the proof of Theorem 8.2.6, V is a regular ring. Also $\overline{R}PT \subseteq V$. Thus by the preceding consideration, V_R is continuous. Clearly, $\overline{R} \subseteq V \subseteq E$. Moreover, we obtain $TV \subseteq V$ since $T \subseteq \overline{R}PT \subseteq V$.

Step 2. Let Y be a continuous R-module such that $\overline{R}_R \leq Y_R \leq E_R$ and $TY \subseteq Y$. Put $B = \{b \in E \mid bY \subseteq Y\}$. Then B is a subring of E. Further, $\overline{R} \subseteq B$ and $T \subseteq B$. Since Y is a continuous R-module and $E(Y_R) = E$, $PY = Y$ by Theorem 2.1.25 or Theorem 8.4.3 (recall that P is the subring of E generated by the set of all idempotents of E). So $P \subseteq B$. Thus $\overline{R}PT \subseteq B \subseteq E$.

We claim that B is regular. For this, take $b \in B$. Since E is commutative regular, there exists $c \in E$ such that $b = bcb$ and $c = cbc$ (see [264, Exercise 3, p. 36]). Note that $(cb)^2 = cb \in E$ and so $cb \in P$. Hence, $cbY \subseteq Y$ and $cbY_R \leq^{\oplus} Y_R$. Define

$$\phi : bY \to cbY \ \text{ by } \ \phi(by) = cby,$$

where $y \in Y$. Then ϕ is an R-isomorphism because $b = bcb$. Hence by (C_2) condition of Y, there is $g^2 = g \in \mathrm{End}(Y_R)$ such that $bY = gY$. Also there exists $f \in E$, which is an extension of g. Then we have that $bY = gY = fY$ and $(f - f^2)(Y) = (g - g^2)(Y) = 0$.

We show that $(f - f^2)(E) = 0$. Assume on the contrary that there exists $x \in E$ such that $(f - f^2)(x) \neq 0$. Since $Y_R \leq^{\text{ess}} E_R$ and E_R is nonsingular, $Y_R \leq^{\text{den}} E_R$ by Proposition 1.3.14. Thus there exists $r \in R$ such that $xr \in Y$ and $(f - f^2)(x)r \neq 0$. Therefore, $0 \neq (f - f^2)(x)r = (f - f^2)(xr)$, which is a contradiction because $xr \in Y$ and $(f - f^2)(Y) = 0$. Hence $(f - f^2)(E) = 0$, so

$$f^2 = f \in P \quad \text{and} \quad bY = gY = fY \subseteq Y$$

as $b \in B$. Thus $(1 - f)Y \subseteq Y$ and $Y = fY \oplus (1 - f)Y$. Therefore

$$cY = cfY \oplus c(1 - f)Y = cbY \oplus cbc(1 - f)Y = cbY \oplus c^2(1 - f)bY.$$

As $bY = fY$, $c^2(1 - f)bY = c^2(1 - f)fY = 0$, and hence $cY = cbY \subseteq PY = Y$. Thus $c \in B$, and so B is a regular ring. As $\overline{R}\,PT \subseteq B$ and B is a regular ring, $V \subseteq B$ by the definition of V. So $V = V\overline{R} \subseteq B\,\overline{R} \subseteq BY \subseteq Y$. □

We remark that, if R is a commutative semiprime ring, then by Lemma 8.4.5 and Theorem 8.4.6 the continuous hull of R_R is the intersection of all intermediate continuous regular rings between R and $Q(R)$. Thus, the continuous hull of R_R is exactly the continuous absolute ring hull $Q_{\mathbf{Con}}(R)$ of R (see Theorem 8.2.6).

Theorem 8.4.6 *Every nonsingular cyclic module over a commutative ring has a continuous hull* (*which is a regular ring*).

Proof Assume that M be a nonsingular cyclic module over a commutative ring R and $I = r_R(M)$. Put $\overline{R} = R/I$. Then $M \cong \overline{R}_R$. Let $E = E(M_R)$.

From Lemma 8.4.5(i), $EI = 0$. Thus, $T := R/I$ can be considered as a subring of $\text{End}_R(E)$. By Lemma 8.4.5(ii), there exists a smallest continuous module V such that $M \leq V \leq E$ and $TV \subseteq V$. So V is a continuous hull of M. □

The next example shows that quasi-continuous hulls (even for commutative semiprime rings) are distinct from continuous hulls which are, in turn, distinct from (quasi-)injective hulls.

Example 8.4.7 Let $F_n = \mathbb{R}$ for $n = 1, 2, \ldots$ and R the subring of $\prod_{n=1}^{\infty} F_n$ generated by $\oplus_{n=1}^{\infty} F_n$ and $1_{\prod_{n=1}^{\infty} F_n}$. Then $E(R_R) = Q(R) = \prod_{n=1}^{\infty} F_n$. In this case, we see that

$$U = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid a_n \in \mathbb{Z} \text{ eventually}\}$$

is the quasi-continuous hull of R_R (see Theorem 8.4.3). By Lemma 8.4.5,

$$V = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid a_n \in \mathbb{Q} \text{ eventually}\}$$

is the continuous hull of R_R because V is the smallest continuous regular ring between R and $Q(R)$ (therefore V is the intersection of all intermediate continuous regular rings between R and $Q(R)$).

Consider an arbitrary cyclic R-module $M = \overline{R} = R/r_R(M)$ over a commutative ring R. We fix the following notations: $E = E(\overline{R}_R)$, $E = E_1 \oplus E_2$, where $E_1 = Z_2(E)$ (note that since E_R is injective, $Z_2(E) \leq^{\oplus} E$ by Proposition 2.3.10). Write $1_{\overline{R}} = e_1 + e_2$ (where $e_1 \in E_1$, and $e_2 \in E_2$) be the corresponding decomposition. Then $E_1 = E(e_1 R)$ and $E_2 = E(e_2 R)$.

Proposition 8.4.8 *Let M be a cyclic module over a commutative ring R and let $I = r_R(e_2 R)$. Then the following conditions are equivalent.*

(i) *$e_1 R + \ell_{E_1}(I)$ has a continuous hull.*
(ii) *M has a continuous hull.*

Proof Note that $e_2 R_R$ is a nonsingular cyclic R-module. Say $\pi_2 : E \to E_2$ is the canonical projection onto E_2. Let T be the subring of $\text{End}_R(E_2)$ generated by the set $\{\pi_2 \pi|_{E_2}\}$, where $\pi^2 = \pi \in \text{End}_R(E)$. By Lemma 8.4.5(i), $E_2 I = 0$. Also from Lemma 8.4.5(ii), there exists a smallest continuous module V_2 with $e_2 R \leq V_2 \leq E_2$ and $T V_2 \subseteq V_2$.

(i)⇒(ii) Assume that there exists a continuous hull V_1 of $e_1 R + \ell_{E_1}(I)$. We claim that $V = V_1 \oplus V_2$ is continuous. For this, first we prove that V is quasi-continuous. Let $\pi^2 = \pi \in \text{End}_R(E)$. Then $\pi|_{E_1} \in \text{End}_R(E_1)$ because $E_1 = Z_2(E) \trianglelefteq E$. Therefore, $\pi(V_1) = \pi|_{E_1}(V_1) \subseteq V_1$ by Theorem 2.1.25 since V_1 is continuous. Let $\pi_1 : E \to E_1$ be the canonical projection onto E_1 and put $\phi = \pi_1 \pi|_{E_2}$. Then $\phi \in \text{Hom}_R(E_2, E_1)$. Also, $\phi(V_2)I = \phi(V_2 I) \subseteq \phi(E_2 I) = 0$, so $\phi(V_2) \subseteq \ell_{E_1}(I) \subseteq V_1$. Hence, $\pi_1 \pi(V_2) \subseteq V_1$.

Next $\pi_2 \pi|_{E_2} \in T$, and hence $\pi_2 \pi(V_2) \subseteq T V_2 \subseteq V_2$. Therefore, we have that $\pi(V) = \pi(V_1) + \pi(V_2) = \pi(V_1) + \pi_1 \pi(V_2) + \pi_2 \pi(V_2) \subseteq V_1 + V_2 = V$. Thus V is quasi-continuous by Theorem 2.1.25.

By Lemma 2.2.4, V_1 and V_2 are relatively injective. Since V_1 and V_2 are continuous, $V = V_1 \oplus V_2$ is continuous by Theorem 2.2.16. Next, we show that $V = V_1 \oplus V_2$ is a continuous hull of $M = \overline{R}_R$. For this, say Y is a continuous module such that $\overline{R} \leq Y \leq E = E_1 \oplus E_2$. Then since $E(Y) = E$, $Y = Y_1 \oplus Y_2$ from Theorem 2.1.25, where $Y_1 = Y \cap E_1$ and $Y_2 = Y \cap E_2$. Observe that $e_1 = \pi_1(1_{\overline{R}}) \in \pi_1(Y) = Y_1$ and $e_2 = \pi_2(1_{\overline{R}}) \in \pi_2(Y) = Y_2$. So $e_1 R \subseteq Y_1$ and $e_2 R \subseteq Y_2$. Since Y is continuous, $\pi(Y) \subseteq Y$ by Theorem 2.1.25 and so $\pi_2 \pi(Y_2) \subseteq \pi_2 \pi(Y) \subseteq \pi_2(Y) = Y_2$. Hence, $T Y_2 \subseteq Y_2$. Note that Y_2 is continuous by Theorem 2.2.16. Thus $V_2 \subseteq Y_2$ since V_2 is the smallest continuous module such that $e_2 R \leq V_2 \leq E_2$ and $T V_2 \subseteq V_2$.

To show that $\ell_{E_1}(I) \subseteq Y_1$ so that $e_1 R + \ell_{E_1}(I) \subseteq Y_1$, take $a \in \ell_{E_1}(I)$. Then the map $f : e_2 R \to aR$ defined by $f(e_2 r) = ar$ for $r \in R$ is an R-homomorphism. Thus, there is $\varphi \in \text{Hom}_R(E_2, E_1)$ with $\varphi|_{e_2 R} = f$. Note that $E_1 = E(Y_1)$ and $E_2 = E(Y_2)$. Since $Y = Y_1 \oplus Y_2$ is continuous, Y_1 is Y_2-injective by Lemma 2.2.4. Thus, $\varphi(Y_2) \subseteq Y_1$ from Theorem 2.1.2. Whence $a = f(e_2) = \varphi(e_2) \in \varphi(Y_2) \subseteq Y_1$.

Therefore, $\ell_{E_1}(I) \subseteq Y_1$, so $e_1R + \ell_{E_1}(I) \subseteq Y_1$. Hence $V_1 \subseteq Y_1$ because Y_1 is continuous by Theorem 2.2.16. This yields that $V = V_1 \oplus V_2 \subseteq Y_1 \oplus Y_2 = Y$. Therefore V is a continuous hull of $\overline{R}_R$.

(ii)⇒(i) Assume that there exists a continuous hull W of $\overline{R}_R$. Then as in the argument used in the proof of (i)⇒(ii), we have that

$$W = W_1 \oplus W_2,\ e_1R + \ell_{E_1}(I) \subseteq W_1 \subseteq E_1,\ e_2R \subseteq W_2 \subseteq E_2,$$

and $TW_2 \subseteq W_2$.

Let $e_1R + \ell_{E_1}(I) \leq U \leq E_1$ with U a continuous module. We see that $U \oplus W_2$ is quasi-continuous exactly as in the proof of (i)⇒(ii) for showing that $V = V_1 \oplus V_2$ is quasi-continuous. Thus U and W_2 are relatively injective by Lemma 2.2.4. So $U \oplus W_2$ is continuous by Theorem 2.2.16 as both U and W_2 are continuous. Hence, $W = W_1 \oplus W_2 \leq U \oplus W_2$. Therefore, $W_1 \leq U$. Thus W_1 is a continuous hull of $e_1R + \ell_{E_1}(I)$. □

An element $a \in R$ is said to *act regularly* on an R-module M, if $ma = 0$ implies $m = 0$ for $m \in M$. Motivated by the condition in Proposition 8.4.8, we now obtain the following result.

Lemma 8.4.9 *Let E be an indecomposable injective module over a commutative ring R. Assume that $f \in E$ and $I \trianglelefteq R$. Then $fR + \ell_E(I)$ has a continuous hull.*

Proof Let C be the multiplicatively closed set of those elements of R which act regularly on E, and let RC^{-1} be the corresponding right ring of fractions of R (see Proposition 5.5.4). For $c \in C$, we see that $E \cong Ec \leq E$. Since E is indecomposable and injective, $E = Ec$. Take $y \in E$ and $rc^{-1} \in RC^{-1}$, where $r \in R$ and $c \in C$. From $E = Ec$, there exists uniquely $y_1 \in E$ such that $y = y_1c$. Define $yrc^{-1} = y_1r$. Then E becomes an RC^{-1}-module.

Say V is a continuous R-submodule of E. Then each $c \in C$ defines an R-monomorphism $V \to V$. Thus $V \cong Vc \leq V$. Since V is continuous and uniform, $Vc = V$. As in the previous argument, V becomes an RC^{-1}-module.

Let $A = \ell_E(I)$. To see that A is an RC^{-1}-module, we first prove that A is quasi-injective. For this, take $h \in \mathrm{End}(E)$ and let $x \in A$. Then $xI = 0$ and thus $h(x)I = h(xI) = 0$. Therefore, $h(x) \in A$. Thus, $A \trianglelefteq E$. If $A = 0$, then A is quasi-injective. Suppose that $A \neq 0$. As E is indecomposable injective, $E = E(A)$ and so A is quasi-injective by Theorem 2.1.9. Thus A is an RC^{-1}-module by the preceding argument.

We show that $fRC^{-1} + A$ is a continuous R-module. If $f \in A$, then we obtain $fRC^{-1} + A = A$, and therefore $fRC^{-1} + A$ is a continuous R-module. Next, assume that $f \notin A$. We let $\varphi : fRC^{-1} + A \to fRC^{-1} + A$ be an R-monomorphism. Then φ can be extended to an isomorphism $\overline{\varphi}$ of E because $fRC^{-1} + A$ is essential in E, and E is indecomposable and injective.

Write $\varphi(f) = ft + a$, where $t \in RC^{-1}$ and $a \in A$. We note that $\varphi(f) \notin A$. For, if $\varphi(f) = \overline{\varphi}(f) \in A$, then $f \in \overline{\varphi}^{-1}(A) \subseteq A$ as $A \trianglelefteq E$, which is a contradiction. Hence $ft \neq 0$, so $t \neq 0$.

Put $t = rc^{-1}$ with $r \in R$ and $c \in C$. We show that t is invertible in RC^{-1}. Let $\mu \in \mathrm{End}(E)$ such that $\mu(y) = yt$, where $y \in E$. If μ is one-to-one, then r acts regularly on E, thus $r \in C$. Therefore, $t = rc^{-1}$ is invertible in RC^{-1}.

Assume that μ is not one-to-one. Then $\mu \in J(\mathrm{End}(E))$ as $\mathrm{End}(E)$ is a local ring. Thus, $\overline{\varphi} - \mu$ is an isomorphism because $\overline{\varphi}$ is an isomorphism. Put $\psi = \overline{\varphi} - \mu$. By Theorem 2.1.9, $\psi(A) \subseteq A$ because A is quasi-injective.

Next, for $w \in A$, there exists $v \in E$ such that $\psi(v) = w$ as ψ is an isomorphism. Whence $\psi(vI) = \psi(v)I = wI = 0$. Hence $vI = 0$, so $v \in A$. Thus $w \in \psi(A)$. As a consequence, $A = \psi(A) = (\overline{\varphi} - \mu)(A)$.

In particular, $a = (\overline{\varphi} - \mu)(b)$ with $b \in A$. So $\varphi(b) - bt = a$. Let $f' = f - b$. Then $fR + A = f'R + A$. As $f \notin A$, $f' \neq 0$. Recall that $\varphi(f) = ft + a$. Therefore, $\varphi(f') = \varphi(f - b) = \varphi(f) - \varphi(b) = (ft + a) - (a + bt) = ft - bt = f't$. Take $0 \neq x \in E$. Since E is indecomposable injective and $f' \neq 0$, $f'R$ is essential in E. So there exist $r, r' \in R$ with $xr = f'r' \neq 0$.

If $xt = 0$, then $\varphi(f'r') = \varphi(f')r' = f'tr' = xtr = 0$. Hence $f'r' = 0$ as φ is a monomorphism, a contradiction. Thus $xt \neq 0$, so t acts regularly on E. Hence $t \in C$, and thus t is invertible in RC^{-1}.

From $\varphi(f) = ft + a$, $\varphi(f) - a = ft$. Therefore

$$f = (\varphi(f) - a)t^{-1} = \varphi(f)t^{-1} - at^{-1} \in \varphi(f)RC^{-1} + A$$

because A is an RC^{-1}-module. Hence $fRC^{-1} + A \subseteq \varphi(f)RC^{-1} + A$. As $\overline{\varphi}$ is an isomorphism, $A = \overline{\varphi}(A)$ by the preceding argument. Hence $A = \varphi(A)$.

Note that $\varphi \in \mathrm{End}_{RC^{-1}}(fRC^{-1} + A)$. Indeed, for $\alpha \in fRC^{-1} + A$ and $c \in C$, $\varphi(\alpha c^{-1})c = \varphi(\alpha c^{-1}c) = \varphi(\alpha)$ and so $\varphi(\alpha c^{-1}) = \varphi(\alpha)c^{-1}$. Thus we have that $fRC^{-1} + A \subseteq \varphi(f)RC^{-1} + A \subseteq \varphi(f)RC^{-1} + \varphi(A) = \varphi(fRC^{-1} + A)$. Hence φ is onto. From this fact, every R-monomorphism from $fRC^{-1} + A$ to $fRC^{-1} + A$ is onto. Therefore, $fRC^{-1} + A$ is a continuous R-module because $fRC^{-1} + A$ is uniform.

Finally, assume that N is a continuous R-module with $fR + A \subseteq N \subseteq E$. By the preceding argument, N is an RC^{-1}-module (also note that A is an RC^{-1}-module). Thus, $fRC^{-1} + A \subseteq N$. So $fRC^{-1} + A$ is a continuous hull of $fR + A$. □

The following result is an extension of Theorem 8.4.6.

Theorem 8.4.10 *Let R be a commutative ring. Then every cyclic module M with $Z(M)$ uniform, has a continuous hull.*

Proof Let $E = E(M)$. Then $E = E_1 \oplus E_2$, where $E_1 = Z_2(E)$. We observe that $E_1 = E(Z_2(M)) = E(Z(M))$ as $Z(M)$ is essential in $Z_2(M)$. Since $Z(M)$ is uniform, E_1 is indecomposable injective. Let $I = r_R(e_2R)$. By Lemma 8.4.9, $e_1R + \ell_{E_1}(I)$ has a continuous hull. Hence, Proposition 8.4.8 yields that M has a continuous hull. □

When M is a uniform cyclic module over a commutative ring, M has a continuous hull by Theorem 8.4.10. This continuous hull is described explicitly in the next theorem.

Theorem 8.4.11 *Let R be a commutative ring, and $M = fR$ a uniform cyclic R-module. Then $MC^{-1} = fRC^{-1}$ is a continuous hull of M, where C is the multiplicatively closed set of those elements of R which act regularly on M.*

Proof Take $I = R$ in Lemma 8.4.9. Then $\ell_E(I) = 0$. By the proof of Lemma 8.4.9, $MC^{-1} = fRC^{-1}$ is a continuous hull of M. □

The following is an example of a continuous hull of a uniform cyclic module over a commutative ring, which is distinct from its quasi-continuous and injective hulls.

Example 8.4.12 Consider the ring

$$A = \{ \sum_{i\in[0,\infty)} \alpha_i x^i \mid \alpha_i \in \mathbb{Z} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\}.$$

Let $R = A/I$, where I is the ideal of A generated by x. Then R_R is uniform and nonsingular. Thus $Q(R) = E(R_R)$ by Corollary 1.3.15, and $Q(R)$ is regular by Theorem 2.1.31. Since R_R is uniform, $Q(R)$ has only 0 and 1 as its idempotents (hence $Q(R)$ is a field). So the quasi-continuous hull of R_R is R_R itself by Theorem 8.4.3 or Theorem 2.1.25. Next, consider

$$B = \{ \sum_{i\in[0,\infty)} \alpha_i x^i \mid \alpha_i \in \mathbb{Q} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\}.$$

Take $Q = B/K$, where K is the ideal of B generated by x. Let C be the set of all non zero-divisors of R. Then $Q = RC^{-1}$, which becomes the classical ring of quotients of R. By Theorem 8.4.11, Q_R is the continuous hull of R_R.

We claim that Q_R is not injective. For this, consider the ideal $\cup_{n=1}^{\infty} x^{1/n}R$ of R and the map $\phi : \cup_{n=1}^{\infty} x^{1/n}R \to Q$, where $\phi|_{x^{1/n}R} = \phi_n$ is given by the multiplication by $1 + x^{1/2} + \cdots + x^{(n-2)/(n-1)} + x^{(n-1)/n}$. Then ϕ is well-defined since $\phi_{n+1}|_{x^{1/n}R} = \phi_n$. Also, ϕ is an R-homomorphism. However, there is no element $q \in Q$, for which $\phi(x) = qx$ for all x in $\cup_{n=1}^{\infty} x^{1/n}R$. Since, in that case, q would have to be an infinite sum, and such q does not lie in Q. Consequently, ϕ cannot be extended to R. Thus, Q_R is not injective.

In the next example, we exhibit a free module of finite rank over a commutative domain, which does not have an extending hull.

Example 8.4.13 Let $R = \mathbb{Z}[x, y]$, the polynomial ring. Put $M = R \oplus R$. Then the R-module M is not extending by Theorem 6.1.4 and Exercise 6.1.18.1 because the commutative domain R is not Prüfer. Let $F = \mathbb{Q}(x, y)$, the field of fractions of R. Note that $E(M) = F \oplus F$.

Let $U = F \oplus R$ and $S = \text{End}(U_R)$. As $\text{Hom}(F_R, R_R) = 0$,

$$S = \begin{bmatrix} \text{End}(F_R) & \text{Hom}(R_R, F_R) \\ 0 & \text{End}(R_R) \end{bmatrix}.$$

By Theorem 4.2.18, U_R is a Baer module. We claim that U_R is a $\mathcal{K}$-cononsingular. For this, say $N_R \leq U_R$ such that $\ell_S(N) = 0$. If $N \subseteq F \oplus 0$, then $\ell_S(N) \neq 0$. Also, if $N \subseteq 0 \oplus R$, then $\ell_S(N) \neq 0$. Thus, there are $0 \neq q_0 \in F$ and $0 \neq r_0 \in R$ such that $\alpha := \begin{bmatrix} q_0 \\ r_0 \end{bmatrix} \in N$. Let $f \in \text{Hom}(R_R, F_R)$ defined by $f(r) = (-q_0/r_0)r$ for $r \in R$. Put

$$\varphi = \begin{bmatrix} 1 & f \\ 0 & 0 \end{bmatrix} \in S.$$

Then $\varphi(\alpha) = 0$, and so $\ell_S(\alpha) \neq 0$. If $N = \alpha R$, then $\ell_S(N) = \ell_S(\alpha R) \neq 0$, a contradiction. Therefore, $\alpha R \subsetneq N$. Assume that $\alpha R \cap \beta R \neq 0$ for each $\beta \in N \setminus \alpha R$. Then there are $a, b \in R$ with $\alpha a = \beta b \neq 0$. For $s \in S$, note that $s\alpha = 0$ if and only if $s\alpha a = 0$ if and only if $s\beta b = 0$ if and only if $s\beta = 0$. Thus $\ell_S(\alpha) = \ell_S(\beta)$ for all $\beta \in N \setminus \alpha R$. Take $0 \neq s_0 \in \ell_S(\alpha)$. Then $s_0 \in \ell_S(N)$, which contradicts $\ell_S(N) = 0$. Thus, there exists $\beta \in N \setminus \alpha R$ such that $\alpha R \cap \beta R = 0$.

So $\alpha F \cap \beta F = 0$, hence α and β are linearly independent vectors in the vector space $F \oplus F$ over F. Thus, $\alpha F \oplus \beta F = F \oplus F$. Therefore, we have that $(\alpha R \oplus \beta R)_R \leq^{\text{ess}} (\alpha F \oplus \beta F)_R = (F \oplus F)_R$. So $N_R \leq^{\text{ess}} (F \oplus R)_R$ because $(\alpha R \oplus \beta R)_R \leq N_R \leq (F \oplus R)_R \leq (F \oplus F)_R$. Hence, U_R is $\mathcal{K}$-cononsingular.

By Theorem 4.1.15, U_R is extending. Similarly, $W_R = (R \oplus F)_R$ is extending. Because $U \cap W = M$ and M is not extending, M cannot have an extending hull.

We use **SFI** to denote the class of strongly FI-extending right modules (or the class of right strongly FI-extending rings according to the context). In contrast to Example 8.4.13, we show that over a semiprime ring R, every finitely generated projective module P_R has the FI-extending module hull $H_{\mathbf{FI}}(P_R)$ (see Definition 8.4.1). This module hull $H_{\mathbf{FI}}(P_R)$ is explicitly described in Theorem 8.4.15. As a consequence, it will be seen that a finitely generated projective module P_R over a semiprime ring R is FI-extending if and only if it is a quasi-Baer module if and only if $\text{End}(P_R)$ is a quasi-Baer ring. This result will also be applied to C^*-algebras in Chap. 10.

Lemma 8.4.14 *Assume that M_R is an FI-extending module. Then $fM \subseteq M$ for any $f \in \mathcal{B}(\text{End}(E(M_R)))$.*

Proof Say $f \in \mathcal{B}(\text{End}(E(M_R)))$. Then $fE(M_R) \cap M \trianglelefteq M$. Because M is FI-extending, there exists $g^2 = g \in \text{End}(M_R)$ satisfying

$$fE(M_R) \cap M \leq^{\text{ess}} gM \leq^{\text{ess}} \overline{g}E(M_R),$$

where $\overline{g}$ is the canonical projection from $E(M_R) = E(gM_R) \oplus E((1-g)M_R)$ to $E(gM_R)$. Now we note that $fE(M_R) \cap M_R \leq^{\text{ess}} fE(M_R)$. Thus $f = \overline{g}$ as f is central in $\text{End}(E(M_R))$. So $fM = \overline{g}M = gM \subseteq M$. □

We observe that Lemma 8.4.14 shows connections to Theorem 2.1.25 (and also Lemma 9.3.12). The next result shows and explicitly describes the unique (up to isomorphism) FI-extending hull for every finitely generated projective module over a semiprime ring.

Theorem 8.4.15 *Every finitely generated projective module P_R over a semiprime ring R has the FI-extending hull $H_{\mathbf{FI}}(P_R)$. Indeed,*

$$H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R),$$

where $P \cong e(\oplus^n R_R)$ for some positive integer n and $e^2 = e \in \mathrm{End}(\oplus^n R_R)$.

Proof Step 1. $\widehat{Q}_{\mathbf{FI}}(R)_R$ is strongly FI-extending. From Theorems 3.2.37 and 8.3.17, $\widehat{Q}_{\mathbf{FI}}(R) = \widehat{Q}_{\mathbf{qB}}(R) = R\mathcal{B}(Q(R))$ is quasi-Baer, right strongly FI-extending, and semiprime. To show that $\widehat{Q}_{\mathbf{FI}}(R)_R$ is strongly FI-extending, take $U_R \trianglelefteq \widehat{Q}_{\mathbf{FI}}(R)_R$. Then by Lemma 8.1.3(ii), $U_R \leq^{\mathrm{ess}} \widehat{Q}_{\mathbf{FI}}(R)U\widehat{Q}_{\mathbf{FI}}(R)_R$. Theorem 3.2.37 yields that $\widehat{Q}_{\mathbf{FI}}(R)U\widehat{Q}_{\mathbf{FI}}(R)_{\widehat{Q}_{\mathbf{FI}}(R)} \leq^{\mathrm{ess}} h\widehat{Q}_{\mathbf{FI}}(R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ for some $h \in \mathcal{B}(\widehat{Q}_{\mathbf{FI}}(R))$. By Lemma 8.1.3(i), $\widehat{Q}_{\mathbf{FI}}(R)U\widehat{Q}_{\mathbf{FI}}(R)_R \leq^{\mathrm{ess}} h\widehat{Q}_{\mathbf{FI}}(R)_R$.

Now $\mathrm{End}(\widehat{Q}_{\mathbf{FI}}(R)_R) = \mathrm{End}(\widehat{Q}_{\mathbf{FI}}(R)_{\widehat{Q}_{\mathbf{FI}(R)}}) \cong \widehat{Q}_{\mathbf{FI}}(R)$ from Proposition 2.1.32. Therefore, $\lambda(h\widehat{Q}_{\mathbf{FI}}(R)) = h(\lambda\widehat{Q}_{\mathbf{FI}}(R))$ for any $\lambda \in \mathrm{End}(\widehat{Q}_{\mathbf{FI}}(R)_R)$. Thus $h\widehat{Q}_{\mathbf{FI}}(R)_R \trianglelefteq \widehat{Q}_{\mathbf{FI}}(R)_R$, so $\widehat{Q}_{\mathbf{FI}}(R)_R$ is strongly FI-extending because $U_R \leq^{\mathrm{ess}} h\widehat{Q}_{\mathbf{FI}}(R)_R$.

Step 2. $H_{\mathbf{FI}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R$. Note that $\widehat{Q}_{\mathbf{FI}}(R)_R$ is FI-extending by Step 1, so $\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R$ is FI-extending by Theorem 2.3.5. Suppose that N_R is FI-extending such that $\oplus^n R_R \leq N_R \leq E(\oplus^n R_R) = \oplus^n E(R_R)$.

Take $f \in \mathcal{B}(Q(R))$. Then $f = \lambda(1)$ for some $\lambda \in \mathcal{B}(\mathrm{End}(E(R_R)))$ from Lemma 8.3.10. Let $\lambda\mathbf{1}$, which is the $n \times n$ diagonal matrix with λ on the diagonal, where $\mathbf{1}$ is the identity matrix in $\mathrm{End}(\oplus^n E(R_R)) = \mathrm{Mat}_n(\mathrm{End}(E(R_R)))$. Then because $\lambda\mathbf{1} \in \mathcal{B}(\mathrm{End}(\oplus^n E(R_R)))$, $\lambda\mathbf{1}N \subseteq N$ by Lemma 8.4.14, and so

$$\lambda\mathbf{1}\begin{bmatrix} R \\ \vdots \\ R \end{bmatrix} = \begin{bmatrix} fR \\ \vdots \\ fR \end{bmatrix} \subseteq N, \text{ where } \begin{bmatrix} R \\ \vdots \\ R \end{bmatrix} = \oplus^n R_R.$$

As $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$ by Theorem 8.3.17, we have that $\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R \leq N_R$, hence $H_{\mathbf{FI}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R$.

Step 3. $H_{\mathbf{FI}}(e(\oplus^n R_R)) = e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$. For this, we first observe that $\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R = e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) \oplus (1-e)(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$. As $\widehat{Q}_{\mathbf{FI}}(R)_R$ is strongly FI-extending by Step 1, $\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R$ is strongly FI-extending by Theorem 2.3.23. So $e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$ is strongly FI-extending from Theorem 2.3.19.

Let V_R be FI-extending such that $e(\oplus^n R_R) \leq V_R \leq E(e(\oplus^n R_R))$. Then

$$\begin{aligned}\oplus^n R_R &= e(\oplus^n R_R) \oplus (1-e)(\oplus^n R_R) \leq V_R \oplus (1-e)(\oplus^n R_R) \\ &\leq V_R \oplus E[(1-e)(\oplus^n R_R)].\end{aligned}$$

Since V_R is FI-extending and $E[(1-e)(\oplus^n R_R)]$ is injective, Theorem 2.3.5 yields that $V_R \oplus E[(1-e)(\oplus^n R_R)]$ is FI-extending. Therefore by Step 2,

$$H_{\mathbf{FI}}(\oplus^n R_R) = \oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R \leq V_R \oplus E[(1-e)(\oplus^n R_R)].$$

To prove that $e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) \leq V_R$, we take

$$e\alpha \in e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R), \quad \text{where } \alpha \in \oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R.$$

Since $e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) \le V_R \oplus E[(1-e)(\oplus^n R_R)]$, $e\alpha = v + y$ for some $v \in V$ and $y \in E[(1-e)(\oplus^n R_R)]$. Thus,

$$e\alpha - v = y \in [e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) + V] \cap E[(1-e)(\oplus^n R_R)].$$

Since $e(\oplus^n R_R) \le^{\text{ess}} e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$, $E[e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)] = E[e(\oplus^n R_R)]$. So $[e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) + V] \cap E((1-e)(\oplus^n R_R)] \le E[e(\oplus^n R_R)] \cap E[(1-e)(\oplus^n R_R)]$. Hence, $e\alpha - v = y = 0$, so $e\alpha = v \in V$. Therefore, $e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R) \le V_R$. Consequently, $H_{\mathbf{FI}}(e(\oplus^n R_R)) = e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$.

Step 4. $H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$. Let $\sigma : P_R \to e(\oplus^n R_R)$ be an isomorphism. Then σ can be extended to an isomorphism $\overline{\sigma} : E(P_R) \to E(e(\oplus^n R_R))$. We see that $H_{\mathbf{FI}}(P_R) = \overline{\sigma}^{-1}(e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$. □

Remark 8.4.16 By the proof of Theorem 8.4.15, the strongly FI-extending hull and the FI-extending hull of a finitely generated projective module P_R coincide when R is semiprime.

If R is not semiprime, the above remark does not hold. For example, let $R = \mathbb{Z}_3[S_3]$, the group algebra of S_3 over the field $\mathbb{Z}_3$, where S_3 is the symmetric group on $\{1, 2, 3\}$. By Example 2.3.18, R_R is not strongly FI-extending. Thus $H_{\mathbf{SFI}}(R_R)$ does not exist because R_R is injective.

The existence of an FI-extending hull of a module is not always guaranteed, even in the presence of nonsingularity, as the next example shows.

Example 8.4.17 Let R be the ring of Example 8.2.9. Then $H_{\mathbf{FI}}(R_R)$ does not exist. Indeed, let H_1 and H_2 be rings as in Example 8.2.9, which are right FI-extending rings. Since H_1 and H_2 are right rings of quotients of R, H_1 and H_2 are FI-extending right R-modules by Proposition 8.1.4(i). Suppose $H_{\mathbf{FI}}(R_R)$ exists. Then it follows that $H_{\mathbf{FI}}(R_R) \subseteq H_1 \cap H_2 = R$, so $H_{\mathbf{FI}}(R_R) = R_R$. But, R_R is not FI-extending, a contradiction.

Corollary 8.4.18 *Assume that R is a semiprime ring and P_R is a finitely generated projective module. Then $\widehat{Q}_{\mathbf{FI}}(\text{End}(P_R)) \cong \text{End}(H_{\mathbf{FI}}(P_R))$.*

Proof Since $P_R \cong e(\oplus^n R_R)$ with $e^2 = e \in \text{Mat}_n(R)$, $\text{End}(P_R) \cong e\text{Mat}_n(R)e$. Also by Theorem 8.4.15, $H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R))$. Thus it follows that

$$\text{End}(H_{\mathbf{FI}}(P_R)) \cong e\text{Mat}_n(\text{End}(\widehat{Q}_{\mathbf{FI}}(R)_R)e.$$

Now $\text{End}(\widehat{Q}_{\mathbf{FI}}(R)_R) \cong \widehat{Q}_{\mathbf{FI}}(R)$ by Proposition 2.1.32.

Hence $\text{End}(H_{\mathbf{FI}}(P_R)) \cong e\text{Mat}_n(\text{End}_R(\widehat{Q}_{\mathbf{FI}}(R)_R))e \cong e\text{Mat}_n(\widehat{Q}_{\mathbf{FI}}(R))e$. Next, we observe that $\widehat{Q}_{\mathbf{FI}}(e\text{Mat}_n(R)e) = e\widehat{Q}_{\mathbf{FI}}(\text{Mat}_n(R))e$ since $\text{Mat}_n(R)$ is semiprime and $0 \neq e^2 = e \in \text{Mat}_n(R)$ (see Theorem 3.2.37 and Lemma 9.3.9).

So $\text{End}(H_{\mathbf{FI}}(P_R)) \cong e\widehat{Q}_{\mathbf{FI}}(\text{Mat}_n(R))e = \widehat{Q}_{\mathbf{FI}}(e\text{Mat}_n(R)e) \cong \widehat{Q}_{\mathbf{FI}}(\text{End}(P_R))$. □

When P_R is a progenerator, we have the following.

Corollary 8.4.19 *Let R be a semiprime ring. If P_R is a progenerator of the category Mod-R of right R-modules, then $H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathbf{FI}}(R)$ of right $\widehat{Q}_{\mathbf{FI}}(R)$-modules.*

Proof Assume that P_R is a progenerator for Mod-R. Let $P_R \cong e(\oplus^n R_R)$ with $e^2 = e \in \text{Mat}_n(R)$ and let $S = \text{End}(P_R)$. Then R is Morita equivalent to S and

$$S \cong e\text{Mat}_n(R)e \ \text{ with } \ \text{Mat}_n(R)e\text{Mat}_n(R) = \text{Mat}_n(R).$$

Now $\text{Mat}_n(\widehat{Q}_{\mathbf{FI}}(R))e\text{Mat}_n(\widehat{Q}_{\mathbf{FI}}(R)) = \text{Mat}_n(R\mathcal{B}(Q(R))) = \text{Mat}_n(\widehat{Q}_{\mathbf{FI}}(R))$ by observing that $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$ from Theorem 8.3.17.

Since $H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R))$, $\text{End}(H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}) \cong e\text{Mat}_n(\widehat{Q}_{\mathbf{FI}}(R))e$. Thus, we get that $H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathbf{FI}}(R)$ of right $\widehat{Q}_{\mathbf{FI}}(R)$-modules. □

A connection between FI-extending modules and quasi-Baer modules can be seen in the next result.

Theorem 8.4.20 *Assume that P_R is a finitely generated projective module over a semiprime ring R. Then the following are equivalent.*

(i) *P_R is (strongly) FI-extending.*
(ii) *P_R is a quasi-Baer module.*
(iii) End(P_R) *is a quasi-Baer ring.*
(iv) End(P_R) *is a right FI-extending ring.*

Proof Let $P_R \cong e(\oplus^n R_R)$, where $e^2 = e \in \text{End}(\oplus^n R_R) \cong \text{Mat}_n(R)$ and n is a positive integer.

(i)⇒(ii) If P_R is FI-extending, then $P_R = H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{qB}}(R)_R)$ by Theorems 3.2.37, 8.3.17, and 8.4.15. Note that $\text{End}(\widehat{Q}_{\mathbf{qB}}(R)_R) \cong \widehat{Q}_{\mathbf{qB}}(R)$ from Proposition 2.1.32. By Theorems 3.2.37, 8.3.17, and Proposition 8.1.4(i), $\widehat{Q}_{\mathbf{qB}}(R)_R$ is FI-extending. Next, we show that $\widehat{Q}_{\mathbf{qB}}(R)_R$ is quasi-Baer. For this, take $N_R \trianglelefteq \widehat{Q}_{\mathbf{qB}}(R)_R$. As $\text{End}(\widehat{Q}_{\mathbf{qB}}(R)_R) \cong \widehat{Q}_{\mathbf{qB}}(R)$, N is a left ideal of $\widehat{Q}_{\mathbf{qB}}(R)$. Thus $\ell_{\widehat{Q}_{\mathbf{qB}}(R)}(N) = \widehat{Q}_{\mathbf{qB}}(R)g$ for some $g^2 = g \in \widehat{Q}_{\mathbf{qB}}(R)$. So $\widehat{Q}_{\mathbf{qB}}(R)_R$ is a quasi-Baer module. By Theorem 4.6.15 $\oplus^n \widehat{Q}_{\mathbf{qB}}(R)_R$ is a quasi-Baer module. Hence $e(\oplus^n \widehat{Q}_{\mathbf{qB}}(R)_R)$ is a quasi-Baer module by Theorem 4.6.14. So P_R is quasi-Baer.

(ii)⇒(iii) It follows from Theorem 4.6.16.

(iii)⇒(i) Let $\text{End}(P_R)$ be quasi-Baer. Because $\text{End}(P_R) \cong e\text{Mat}_n(R)e$, $e\text{Mat}_n(R)e = \widehat{Q}_{\mathbf{qB}}(e\text{Mat}_n(R)e) = e\widehat{Q}_{\mathbf{qB}}(\text{Mat}_n(R))e = e\text{Mat}_n(\widehat{Q}_{\mathbf{qB}}(R))e$ (see Proposition 9.3.7 and Lemma 9.3.9). Next, let $f \in \mathcal{B}(Q(R))$. Then we have that $f\mathbf{1} \in \mathcal{B}(\text{Mat}_n(Q(R)))$, where $\mathbf{1}$ is the identity matrix of $\text{Mat}_n(R)$. Thus

$$e(f\mathbf{1})e \in e\text{Mat}_n(\widehat{Q}_{\mathbf{qB}}(R))e = e\text{Mat}_n(R)e.$$

Take $e(f\mathbf{1})e = [\alpha_{ij}] \in e\mathrm{Mat}_n(R)e$. Then

$$e\begin{bmatrix} fR \\ \vdots \\ fR \end{bmatrix} = e(f\mathbf{1})e \begin{bmatrix} R \\ \vdots \\ R \end{bmatrix} = e[\alpha_{ij}]e \begin{bmatrix} R \\ \vdots \\ R \end{bmatrix} \subseteq e \begin{bmatrix} R \\ \vdots \\ R \end{bmatrix}.$$

So $e(\oplus^n \widehat{Q}_{\mathbf{qB}}(R)_R) = e(\oplus^n R_R)$ because $\widehat{Q}_{\mathbf{qB}}(R) = R\mathcal{B}(Q(R))$ by Theorem 8.3.17. From Theorems 8.4.15 and 8.3.17, $H_{\mathbf{FI}}(e(\oplus^n R_R)) = e(\oplus^n R_R)$ since $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{FI}}(R)$, and so $e(\oplus^n R_R)$ is (strongly) FI-extending. Therefore, P_R is (strongly) FI-extending.

(iii)⇔(iv) Since $\mathrm{End}(P_R)$ is semiprime, Theorem 3.2.37 yields the equivalence. □

We observe that the rational hull $\widetilde{E}(M)$ of a module M is an $\mathfrak{M}$ hull of M, where $\mathfrak{M}$ is the class of rationally complete modules (see Definition 8.4.1 and [262, p. 277]). Consider $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ and $N = \mathbb{Z}_p \oplus p\mathbb{Z}_{p^3}$, where p is a prime integer. Then $N_{\mathbb{Z}} \leq^{\text{ess}} M_{\mathbb{Z}}$ and $N_{\mathbb{Z}}$ is extending (by direct calculation or [301, p. 19]). But recall from Example 2.2.1(ii) that $M_{\mathbb{Z}}$ is not extending. So the extending property does not, in general, transfer to essential extensions of modules. However, Theorem 8.1.8 motivates one to ask: *Does the* (*FI-*)*extending property transfer to rational extensions in modules?* Our next result shows this to be the case for rational hulls.

Theorem 8.4.21 *Let M be an* (*FI-*)*extending module. Then $\widetilde{E}(M)$ is an* (*FI-*) *extending module.*

Proof First, we assume that M is extending. Let $K \leq \widetilde{E}(M)$ and $N = K \cap M$. Then $N \leq^{\text{ess}} eM$ for some $e^2 = e \in \mathrm{End}(M)$. By Proposition 1.3.6 and [262, Theorem 8.24], there exists $f \in \mathrm{End}(\widetilde{E}(M))$ such that $f|_M = e$. As $E(M)$ is injective, there is $g \in \mathrm{End}(E(M))$ satisfying $g|_{\widetilde{E}(M)} = f$.

Let $m \in M$. Then $(g^2 - g)(m) = (e^2 - e)(m) = 0$. From the definition of $\widetilde{E}(M)$ (see the definition of $\widetilde{E}(M)$ after Proposition 1.3.6), $(g^2 - g)(y) = 0$ for all y in $\widetilde{E}(M)$. Hence $f^2 = f$. Assume that there exists $k \in K$ such that $f(k) - k \neq 0$. As $M \leq^{\text{den}} \widetilde{E}(M)$, there exists $r \in R$ satisfying $kr \in M$ and $(f(k) - k)r \neq 0$. Then $kr \in N$, so $(f(k) - k)r = f(kr) - kr = e(kr) - kr = 0$, a contradiction. Hence, $K \leq f\widetilde{E}(M)$. Let $0 \neq f(v) \in f\widetilde{E}(M)$ with $v \in \widetilde{E}(M)$. Then there is $s \in R$ such that $vs \in M$ and $f(v)s \neq 0$. Now we see that $0 \neq f(v)s = f(vs) = e(vs) \in M$. So $0 \neq f(v)st \in N \leq K$ for some $t \in R$. Therefore, $K \leq^{\text{ess}} f\widetilde{E}(M)$, so $\widetilde{E}(M)$ is extending.

Next, assume that M is FI-extending and that $K \trianglelefteq \widetilde{E}(M)$. Put $N = K \cap M$. We claim that $N \trianglelefteq M$. For this, take $h \in \mathrm{End}(M)$. From Proposition 1.3.6 and [262, Theorem 8.24], there exists $f \in \mathrm{End}(\widetilde{E}(M))$ such that $f|_M = h$.

So $h(N) = f(N) \subseteq K \cap M = N$. Thus, $N \trianglelefteq M$. From the proof similar to the case when M is extending, we obtain that $\widetilde{E}(M)$ is FI-extending. □

For an example illustrating Theorem 8.4.21, consider $M = \mathbb{Z} \oplus \mathbb{Z}_p$, where p is a prime integer (see [262, Example 8.21]). By Theorem 2.3.5, $M_{\mathbb{Z}}$ is FI-extending, but not extending (see [301, p. 19]). Now $\widetilde{E}(M_{\mathbb{Z}}) = \mathbb{Z}_P \oplus \mathbb{Z}_p$ is FI-extending from Theorem 8.4.21 or Theorem 2.3.5, but not extending (see [301, p. 19]), where $\mathbb{Z}_P$ is the localization of $\mathbb{Z}$ at $P = p\mathbb{Z}$.

Exercise 8.4.22

1. Let R be the ring in Example 8.4.12. Prove that R_R is uniform and nonsingular.
2. ([98, Birkenmeier, Park, and Rizvi]) Assume that R is a semiprime ring and P_R is a finitely generated projective module. Show that
   (i) $\text{Rad}(H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}) \cap P = \text{Rad}(P_R)$.
   (ii) $H_{\mathbf{FI}}(P_R) \cong P \otimes_R \widehat{Q}_{\mathbf{FI}}(R)$ as $\widehat{Q}_{\mathbf{FI}}(R)$-modules.
   (iii) $H_{\mathbf{FI}}(P_R)$ is also a finitely generated projective $\widehat{Q}_{\mathbf{FI}}(R)$-module.
3. Let M be a bounded Abelian group. Prove that $M_{\mathbb{Z}}$ has an extending hull. (Hint: see [172, p. 88] and [301, p. 19].)
4. Let M be a continuous module. Show that $\widetilde{E}(M)$ is quasi-continuous.

Historical Notes Results of Sect. 8.1 are obtained by Birkenmeier, Park, and Rizvi in [89]. The concept of a $\mathfrak{K}$ absolute ring hull in Definition 8.2.1 was already implicit in the paper [307] by Müller and Rizvi from their definition of a type III continuous module hull (see also Definition 8.4.1). Theorem 8.2.6 from [89], is an adaptation of [354, Theorem 4.25]. Other results of Sect. 8.2 appear in [89].

Many results in Sect. 8.3, which were originally stated and proved for a ring with identity, have been extended to rings R with $\ell_R(R) = 0$. Definition 8.3.1 was provided in [96]. Proposition 8.3.2 is due to Johnson [236]. Results 8.3.3–8.3.8 are due to Birkenmeier, Park, and Rizvi in [96]. Theorem 8.3.8(ii) is an unpublished new characterization. Theorem 8.3.11(i), (ii), and (iii) appear in [96]. Corollary 8.3.12 is an unpublished new result. Also Theorem 8.3.13(i), (ii), and (iv) were shown in [96]. Proposition 8.3.16 and Theorem 8.3.17 are due to Birkenmeier, Park, and Rizvi [97]. In [163], Ferrero has shown that $Q^s(R)$ is quasi-Baer for any semiprime ring R. Example 8.3.18 is taken from [262, Example 13.26(4)]. Results 8.3.20, 8.3.21 and 8.3.23 appear in [97].

Theorem 8.3.22 is due to Passman [340] and Connell [131]. Example 8.3.25 appears in [102]. Lemma 8.3.26 and Theorem 8.3.28 are obtained in [97]. In [42], it is shown that LO does hold between R and $R\mathcal{B}(Q(R))$. Lemma 8.3.29 is from [322]. Beidar and Wisbauer [42] show that R is biregular if and only if R is semiprime and $R\mathcal{B}(Q(R))$ is biregular (see Exercise 8.3.58.8). Also, they show that R is regular and biregular if and only if $R\mathcal{B}(Q(R))$ is regular and biregular [42]. Corollary 8.3.30 from [97], complements their results.

Results 8.3.31–8.3.37 appear in [97]. Let R be a semiprime PI-ring. Then so is $Q(R)$ by a result of Martindale [292]. Also by a result of Fisher [168], a semiprime PI-ring R is right nonsingular. Thus $Q(R)$ is a regular right self-injective PI-ring from Theorem 2.1.31. So $Q(R)$ has bounded index (of nilpotency) (see [221, Corollary, p. 226]). Therefore, any semiprime PI-ring R has bounded index (of nilpotency). Also any semiprime right Goldie ring has bounded index (of nilpotency).

Results in [160] show that a semiprime right FPF ring has bounded index (of nilpotency).

For a commutative semiprime ring R, Storrer [386] called the intersection of all regular rings of $Q(R)$ containing R the epimorphic hull of R. By showing this intersection was regular, he showed that every commutative semiprime ring has a smallest regular ring of quotients. The existence of Baer ring hulls shown in [298] for the case of commutative semiprime rings (see also Theorem 8.2.4) and in [208] for the case of reduced Utumi rings, now follow directly from Proposition 8.3.36 (see [323] for the existence of Baer ring hulls of commutative regular rings by a sheaf theoretic method). Results 8.3.39–8.3.44 appear in [101]. Theorem 8.3.44 shows that when R is a commutative semiprime ring, $Q_{\mathbf{pqB}}(R)$ is related to the Baer extension considered in [254]. Lemma 8.3.46 and Theorem 8.3.47 appear in [94]. Theorem 8.3.50, Theorem 8.3.53, and Example 8.3.57 were obtained by Armendariz, Birkenmeier, and Park [29], while Proposition 8.3.49 and Theorem 8.3.56 appear in [96].

Results 8.4.4–8.4.12 are taken from [307], while Results 8.4.14–8.4.20 appear in [98]. Theorem 8.4.20 is a module theoretic version of Theorem 3.2.37 for a finitely generated projective module over a semiprime ring. The proof of Theorem 8.4.21 when M is extending corrects the proof of [1, Theorem 5.3]. We include some more related references such as [43, 86, 87, 90, 133, 143, 146, 197, 225, 257, 258, 337, 351], and [370].