Chapter 5
Triangular Matrix Representations and Triangular Matrix Extensions

A ring R is said to have a *generalized triangular matrix representation* if R is ring isomorphic to a generalized triangular matrix ring

$$\begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix},$$

where each R_i is a ring and R_{ij} is an (R_i, R_j)-bimodule for $i < j$, and the matrices obey the usual rules for matrix addition and multiplication. Generalized triangular matrix representations provide an effective tool in the investigation of the structures of a wide range of rings. In this chapter, these representations, in an abstract setting, are discussed by introducing the concept of a *set of left triangulating idempotents*.

The importance and applicability of the concept of a generalized triangular matrix representation can be seen from: (1) for any right R-module M, the generalized triangular matrix ring

$$\begin{bmatrix} S & M \\ 0 & R \end{bmatrix},$$

where $S = \text{End}(M)$, completely encodes the algebraic information of M into a single ring; (2) a ring R is ring isomorphic to

$$\begin{bmatrix} R_1 & R_{12} \\ 0 & R_2 \end{bmatrix},$$

where $R_1 \neq 0$ and $R_2 \neq 0$ if and only if there exists $e \in \mathbf{S}_\ell(R)$ with $e \neq 0$ and $e \neq 1$. From (2), we see that there is a natural connection between quasi-Baer rings and modules and generalized triangular matrix representation, since the "e" in Proposition 3.2.4(ii) is in $\mathbf{S}_\ell(R)$ and the "f" in Proposition 4.6.3(ii) is in $\mathbf{S}_\ell(\text{End}(M))$.

In a manner somewhat analogous to determining a matrix ring by a set of matrix units (see 1.1.16), a generalized triangular matrix ring is determined by a set

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of left (or right) triangulating idempotents. The existence of a set of left triangulating idempotents does not depend on any specific conditions on a ring (e.g., $\{1\}$ is a set of left triangulating idempotents); however, if the ring satisfies a mild finiteness condition, then such a set can be refined to a certain set of left triangulating idempotents in which each diagonal ring R_i has no nontrivial generalized triangular matrix representation. When this occurs, the generalized triangular matrix representation is said to be *complete*.

Complete triangular matrix representations and left triangulating idempotents are applied to get a structure theorem for a certain class of quasi-Baer rings (see Theorem 5.4.12). A number of well known results follow as consequences of this structure theorem. These include Levy's decomposition theorem of semiprime right Goldie rings, Faith's characterization of semiprime right FPF rings with no infinite set of central orthogonal idempotents, Gordon and Small's characterization of piecewise domains, and Chatters' decomposition theorem of hereditary Noetherian rings.

Further, a sheaf representation of quasi-Baer rings is studied as another application of our results of this chapter. Also the Baer, the quasi-Baer, the FI-extending, and the strongly FI-extending properties of (generalized) triangular matrix rings are discussed. Most results of Sects. 5.1, 5.2, and 5.3 are applicable to an algebra over a commutative ring.

5.1 Triangulating Idempotents

In this section, some basic properties of triangulating idempotents are discussed. Then a result showing the connection between triangulating idempotents and generalized triangular matrix rings is presented.

Definition 5.1.1 Let R be a ring. An ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in R is called a *set of left triangulating idempotents* of R if the following conditions hold:

(i) $1 = b_1 + \cdots + b_n$;
(ii) $b_1 \in \mathbf{S}_\ell(R)$;
(iii) $b_{k+1} \in \mathbf{S}_\ell(c_k R c_k)$, where $c_k = 1 - (b_1 + \cdots + b_k)$, for $1 \le k \le n-1$.

Similarly, we define a *set of right triangulating idempotents* of R by using part (i) in the preceding, $b_1 \in \mathbf{S}_r(R)$, and $b_{k+1} \in \mathbf{S}_r(c_k R c_k)$. By condition (iii) of Definition 5.1.1, a set of left (right) triangulating idempotents is a set of orthogonal idempotents.

Definition 5.1.2 A set $\{b_1, \ldots, b_n\}$ of left (right) triangulating idempotents of R is said to be *complete* if each b_i is semicentral reduced.

Theorem 5.1.3 *Let $\{b_1, \ldots, b_n\}$ be an ordered set of nonzero idempotents of R. Then the following are equivalent.*

(i) $\{b_1, \ldots, b_n\}$ *is a set of left triangulating idempotents.*
(ii) $b_1 + \cdots + b_n = 1$ *and* $b_j R b_i = 0$, *for all* $i < j \leq n$.

Proof (i)⇒(ii) By definition, $b_1 + \cdots + b_n = 1$. As $b_2 \in (1 - b_1)R(1 - b_1)$ and $b_1 \in \mathbf{S}_\ell(R)$, $b_2 b_1 = 0$ and $b_2 R b_1 = b_2 b_1 R b_1 = 0$. Similarly we obtain $b_j R b_1 = 0$, for all $j > 1$. By assumption $b_2 \in \mathbf{S}_\ell((1 - b_1)R(1 - b_1))$ and $\{b_1, \ldots, b_n\}$ is orthogonal, thus for $j > 2$,

$$\begin{aligned} b_j R b_2 &= b_j R(1 - b_1)b_2 = b_j(b_1 R + (1 - b_1)R)(1 - b_1)b_2 \\ &= b_j(1 - b_1)R(1 - b_1)b_2 = b_j b_2(1 - b_1)R(1 - b_1)b_2 \\ &= 0. \end{aligned}$$

Continue the process, using $(1 - b_1 - b_2)R(1 - b_1 - b_2)$ in the next step, and so on, to get $b_j R b_i = 0$ for all $i < j \leq n$.

(ii)⇒(i) Note that $(1 - b_1)Rb_1 = (b_2 + \cdots + b_n)Rb_1 = 0$. So $b_1 \in \mathbf{S}_\ell(R)$ by Proposition 1.2.2. Now $b_2 \in (1 - b_1)R(1 - b_1)$ as $b_2(1 - b_1) = b_2 - b_2 b_1 = b_2$ and $(1 - b_1)b_2 = b_2$. Also $(1 - b_1 - b_2)(1 - b_1) = b_3 + b_4 + \cdots + b_n$. Therefore $(1 - b_1 - b_2)[(1 - b_1)R(1 - b_1)]b_2 = \sum_{i=3}^{n} b_i R(1 - b_1)b_2 = \sum_{i=3}^{n} b_i R b_2 = 0$. So $b_2 \in \mathbf{S}_\ell((1 - b_1)R(1 - b_1))$ by Proposition 1.2.2. Continuing this process yields the desired result. □

Theorem 5.1.4 *R has a* (*resp*., *complete*) *set of left triangulating idempotents if and only if R has a* (*resp*., *complete*) *generalized triangular matrix representation.*

Proof Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R. Using Theorem 5.1.3 and a routine argument shows that the map

$$\theta : R \to \begin{bmatrix} b_1 R b_1 & b_1 R b_2 & \cdots & b_1 R b_n \\ 0 & b_2 R b_2 & \cdots & b_2 R b_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n R b_n \end{bmatrix}$$

defined by $\theta(r) = [b_i r b_j]$ is a ring isomorphism, where $[b_i r b_j]$ is the matrix whose (i, j)-position is $b_i r b_j$. Conversely, assume that

$$\phi : R \to \begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}$$

is a ring isomorphism. Then $\{\phi^{-1}(e_{11}), \ldots, \phi^{-1}(e_{nn})\}$ is a set of left triangulating idempotents of R by a routine calculation, where e_{ii} is the matrix with 1_{R_i} in the (i, i)-position and 0 elsewhere. □

Lemma 5.1.5 (i) $\mathbf{S}_\ell(eRe) \subseteq \mathbf{S}_\ell(R)$ *for* $e \in \mathbf{S}_\ell(R)$.

(ii) $f\mathbf{S}_\ell(R)f \subseteq \mathbf{S}_\ell(fRf)$ *for* $f^2 = f \in R$.

(iii) *Let* $e \in \mathbf{S}_\ell(R)$. *If* f *is a primitive idempotent of* R *such that* $efe \neq 0$, *then* efe *is a primitive idempotent in* eRe *and* $fef = f$.

Proof (i) For $g \in \mathbf{S}_\ell(eRe)$, $gRg = geReg = eReg = Rg$. So $g \in \mathbf{S}_\ell(R)$.

(ii) Let $g \in \mathbf{S}_\ell(R)$ and $r \in R$. Then $(fgf)(frf)(fgf) = (ff)(frf)(fgf)$. Thus $(fgf)(frf)(fgf) = (frf)(fgf)$. So $fgf \in \mathbf{S}_\ell(fRf)$.

(iii) Note that $0 \neq efe = fe = fefe$, so $fef \neq 0$ and $(fef)^2 = fef$. As f is primitive, $fef = f$. To show that efe is a primitive idempotent of eRe, we note that $(efe)(efe) = e(fef)e = efe$. Let $0 \neq h^2 = h \in (efe)(eRe)(efe)$. Since $e \in \mathbf{S}_\ell(R)$, $he = h$, $fh = h$, so $hf = fhf$, and thus $(hf)(hf) = hf$. As $hf = 0$ implies that $h = hefe = hfe = 0$, hf is a nonzero idempotent in fRf. Thus, $hf = f$ since f is a primitive idempotent. Note that $(fe)^2 = fe$ and $h \in (efe)(eRe)(efe)$, so $h = hefe = hfe = fe = efe$. Thus, efe is a primitive idempotent eRe. □

Lemma 5.1.6 (i) *If* h *is a ring homomorphism from a ring* R *to a ring* A, *then* $h(\mathbf{S}_\ell(R)) \subseteq \mathbf{S}_\ell(h(R))$.

(ii) *Assume that* $e \in \mathbf{S}_\ell(R) \cup \mathbf{S}_r(R)$ *and* $f \in \mathbf{S}_\ell(eRe) \cup \mathbf{S}_r(eRe)$. *Then the map* $h : R \to fRf$, *defined by* $h(r) = frf$ *for* $r \in R$, *is a ring epimorphism.*

Proof (i) The proof is routine.

(ii) Say $x, y \in R$. Since $e \in \mathbf{S}_\ell(R) \cup \mathbf{S}_r(R)$ and $f \in \mathbf{S}_\ell(eRe) \cup \mathbf{S}_r(eRe)$,

$$fxyf = fexyef = fexeyef = fexefeyef = fxfyf.$$

Therefore, $h(xy) = h(x)h(y)$. □

Proposition 5.1.7 *Let* $\{b_1, \ldots, b_n\}$ *be a set of left triangulating idempotents of* R. *Then*:

(i) $c_k \in \mathbf{S}_r(R)$, $k = 1, \ldots, n-1$, *where* $c_k = 1 - (b_1 + \cdots + b_k)$.
(ii) $b_1 + \cdots + b_k \in \mathbf{S}_\ell(R)$, $k = 1, \ldots, n$.
(iii) *The map* $h_j : R \to b_jRb_j$, *defined by* $h_j(r) = b_jrb_j$ *for all* $r \in R$, *is a ring epimorphism.*

Proof (i) Recall that $b_1 \in \mathbf{S}_\ell(R)$ implies $c_1 = 1 - b_1 \in \mathbf{S}_r(R)$ by Proposition 1.2.2. As $b_2 \in \mathbf{S}_\ell(c_1Rc_1)$, $c_2 = 1 - b_1 - b_2 \in \mathbf{S}_r(c_1Rc_1)$ by Proposition 1.2.2. Therefore $c_2 \in \mathbf{S}_r(R)$ by the right-sided version of Lemma 5.1.5(i). Using this procedure, an induction proof completes the argument.

(ii) It is a direct consequence of part (i) and Proposition 1.2.2.

(iii) Put $e = c_k$ and $f = b_{k+1}$. By part (i), $e \in \mathbf{S}_r(R)$, so $f \in \mathbf{S}_\ell(eRe)$. From Lemma 5.1.6(ii), the map $r \to frf$ is a ring epimorphism. □

Corollary 5.1.8 *The ordered set* $\{b_1, \ldots, b_n\}$ *is a* (*complete*) *set of left triangulating idempotents of* R *if and only if the ordered set* $\{b_n, \ldots, b_1\}$ *is a* (*complete*) *set of right triangulating idempotents.*

Proof Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R. Then by Proposition 5.1.7(i), $1 - (b_1 + \cdots + b_{n-1}) = b_n \in \mathbf{S}_r(R)$. We next show that $b_{n-1} \in \mathbf{S}_r((1-b_n)R(1-b_n))$. For this, first it can be checked that $\{b_1, \ldots, b_{n-1}\}$ is a set of left triangulating idempotents of $(1-b_n)R(1-b_n)$ and $1-b_n$ is the identity of $(1-b_n)R(1-b_n)$. By Proposition 5.1.7(ii), $b_1 + \cdots + b_{n-2} \in \mathbf{S}_\ell(R)$, and hence $b_1 + \cdots + b_{n-2} \in \mathbf{S}_\ell((1-b_n)R(1-b_n))$. Therefore by Proposition 1.2.2,

$$(1-b_n) - (b_1 + b_2 + \cdots + b_{n-2}) = b_{n-1} \in \mathbf{S}_r((1-b_n)R(1-b_n))$$

and so on. By this argument, the ordered set $\{b_n, \ldots, b_1\}$ is a set of right triangulating idempotents. Also, if $\{b_1, \ldots, b_n\}$ is complete, then so is $\{b_n, \ldots, b_1\}$.

The converse is proved similarly. Further, completeness is left-right symmetric since $\mathbf{S}_\ell(b_i R b_i) = \{0, b_i\}$ if and only if $\mathbf{S}_r(b_i R b_i) = \{0, b_i\}$ (see Proposition 1.2.11). □

Exercise 5.1.9

1. Let R be a subdirectly irreducible ring (i.e., the intersection of all nonzero ideals of R is nonzero) and $\{b_1, \ldots, b_n\}$ a set of left triangulating idempotents. Prove the following.
   (i) For each $i \neq 1$ there exists $j < i$ such that $b_j R b_i \neq 0$.
   (ii) For each $i \neq n$ there exists $j > i$ such that $b_i R b_j \neq 0$.
   (iii) The heart of R (i.e., the intersection of all nonzero ideals of R) is contained in $b_1 R b_n$.
2. Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of a ring R. Prove the following.
   (i) $b_i \in \mathbf{S}_\ell(R)$ if and only if $b_j R b_i = 0$ for all $j < i$.
   (ii) $b_i \in \mathbf{S}_r(R)$ if and only if $b_i R b_j = 0$ for all $j > i$.

5.2 Generalized Triangular Matrix Representations

Rings with a complete generalized triangular matrix representation will be characterized. Then the uniqueness of a complete set of triangulating idempotents will be discussed. We shall see that if a ring R satisfies some mild finiteness conditions, then R has a generalized triangular matrix representation with semicentral reduced rings on the diagonal which satisfy the same finiteness condition as R. Thereby reducing the study of such rings to those which are semicentral reduced. Further, it will be shown that the condition of having a complete set of left triangulating idempotents is strictly between that of having a complete set of primitive idempotents and that of having a complete set of centrally primitive idempotents.

Lemma 5.2.1 *Let* $0 \neq f^2 = f \in R$. *If* $fR = eR$ *for every* $0 \neq e \in \mathbf{S}_\ell(fRf)$, *then* f *is semicentral reduced.*

Proof Let $0 \neq e \in \mathbf{S}_\ell(fRf)$. Then since $fR = eR$, $f = ex$ for some $x \in R$, and so $e = ef = eex = ex = f$. Thus, f is semicentral reduced. □

Lemma 5.2.2 (i) *A ring R has DCC on $\{bR \mid b \in \mathbf{S}_\ell(R)\}$ if and only if R has ACC on $\{Rc \mid c \in \mathbf{S}_r(R)\}$.*

(ii) *A ring R has ACC on $\{bR \mid b \in \mathbf{S}_\ell(R)\}$ if and only if R has DCC on $\{Rc \mid c \in \mathbf{S}_r(R)\}$.*

(iii) *If a ring R has DCC on $\{Rc \mid c \in \mathbf{S}_r(R)\}$, then R has DCC on $\{cR \mid c \in \mathbf{S}_r(R)\}$.*

Proof (i) Assume that R has DCC on $\{bR \mid b \in \mathbf{S}_\ell(R)\}$. Consider a chain $Rc_1 \subseteq Rc_2 \subseteq \dots$, where $c_i \in \mathbf{S}_r(R)$. Then $(1-c_1)R \supseteq (1-c_2)R \supseteq \dots$ with $1-c_i \in \mathbf{S}_\ell(R)$ (see Proposition 1.2.2). This descending chain becomes stationary, say with $(1-c_n)R = (1-c_{n+j})R$ for each $j \geq 1$. Then we have that $\ell_R((1-c_n)R) = \ell_R((1-c_{n+j})R)$ for each $j > 1$. Thus, $Rc_n = Rc_{n+j}$ for each $j > 1$. The converse is proved similarly.

(ii) The proof is similar to that of part (i).

(iii) Assume that R has DCC on $\{Rc \mid c \in \mathbf{S}_r(R)\}$. Let $c_1R \supseteq c_2R \supseteq \dots$ be a descending chain with $c_i \in \mathbf{S}_r(R)$. Then $c_{i+1} = c_ic_{i+1}$. So it follows that $c_{i+1}c_i = c_ic_{i+1}c_i = c_ic_{i+1} = c_{i+1}$ because $c_i \in \mathbf{S}_r(R)$. Therefore $Rc_i \supseteq Rc_{i+1}$ for each i. Thus we have a descending chain $Rc_1 \supseteq Rc_2 \supseteq \dots$, so there is n with $Rc_n = Rc_{n+1} = \dots$. Therefore, $(1-c_n)R = (1-c_{n+1})R$. Hence, we obtain that $(1-c_n)Rc_n = (1-c_{n+1})Rc_n = (1-c_{n+1})Rc_{n+1}$.

We observe that $Rc_n = c_nRc_n + (1-c_n)Rc_n = c_nR + (1-c_n)Rc_n$ and

$$Rc_{n+1} = c_{n+1}Rc_{n+1} + (1-c_{n+1})Rc_{n+1} = c_{n+1}R + (1-c_n)Rc_n$$

because $c_n, c_{n+1} \in \mathbf{S}_r(R)$ and $(1-c_n)Rc_n = (1-c_{n+1})Rc_{n+1}$. Therefore, we have that $c_nR + (1-c_n)Rc_n = c_{n+1}R + (1-c_n)Rc_n$ as $Rc_n = Rc_{n+1}$.

To show that $c_nR = c_{n+1}R$, it suffices to check that $c_nR \subseteq c_{n+1}R$ because $c_{n+1}R \subseteq c_nR$. Now $c_n = c_{n+1}y + \alpha$, where $y \in R$ and $\alpha \in (1-c_n)Rc_n$, as $c_nR + (1-c_n)Rc_n = c_{n+1}R + (1-c_n)Rc_n$. Since $c_n\alpha = 0$ and $c_{n+1} = c_nc_{n+1}$ from $c_{n+1}R \subseteq c_nR$, $c_n = c_n^2 = c_nc_{n+1}y + c_n\alpha = c_{n+1}y \in c_{n+1}R$. Therefore $c_nR \subseteq c_{n+1}R$, and hence $c_nR = c_{n+1}R = \dots$. We conclude that R satisfies DCC on $\{cR \mid c \in \mathbf{S}_r(R)\}$. □

Lemma 5.2.3 *Let $e \in \mathbf{S}_r(R)$. If R has DCC on $\{bR \mid b \in \mathbf{S}_\ell(R)\}$, then eRe has DCC on $\{d(eRe) \mid d \in \mathbf{S}_\ell(eRe)\}$.*

Proof First, we show that $\{(eRe)c \mid c \in \mathbf{S}_r(eRe)\}$ has ACC. For this, assume that $(eRe)c_1 \subseteq (eRe)c_2 \subseteq \dots$ is an ascending chain, where $c_i \in \mathbf{S}_r(eRe)$ for $i = 1, 2, \dots$. By the right-sided version of Lemma 5.1.5(i), each $c_i \in \mathbf{S}_r(R)$. Note that $ec_ie \in (eRe)ec_ie \subseteq (eRe)ec_{i+1}e$.

So there exists $x \in eRe$ such that $ec_ie = xec_{i+1}e$. Thus,

$$\begin{aligned}(1-e)Rc_i &= (1-e)Rec_ie = (1-e)Rxec_{i+1}e \\ &\subseteq (1-e)Rec_{i+1}e = (1-e)Rc_{i+1}.\end{aligned}$$

Therefore, for each i,

$$Rc_i = eRc_i + (1-e)Rc_i = (eRe)ec_ie + (1-e)Rc_i$$

$$\subseteq (eRe)ec_{i+1}e + (1-e)Rc_{i+1} = eRc_{i+1} + (1-e)Rc_{i+1}$$
$$= Rc_{i+1}.$$

By assumption and Lemma 5.2.2(i), $Rc_n = Rc_{n+1} = \dots$ for some n as each c_i is in $\mathbf{S}_r(R)$. Therefore, $eRc_n = eRc_{n+1} = \dots$, so $(eRe)c_n = (eRe)c_{n+1} = \dots$. From Lemma 5.2.2(i), eRe has DCC on $\{d(eRe) \mid d \in \mathbf{S}_\ell(eRe)\}$. □

Lemma 5.2.4 *Let $\{b_1, \dots, b_n\}$ be a complete set of left triangulating idempotents of R. If $e \in \mathbf{S}_\ell(R)$, then $eR = \bigoplus_i b_i R$, where the sum runs over a subset of $\{1, \dots, n\}$. Thus, $|\{eR \mid e \in \mathbf{S}_\ell(R)\}| \le 2^n$.*

Proof Assume that $0 \ne e \in \mathbf{S}_\ell(R)$. Consider i such that $b_i e \ne 0$. We show that $b_i eR = b_i R$. For this, note that $b_i e b_i e = b_i e \ne 0$, so $b_i e b_i \ne 0$. From Lemma 5.1.5(ii), $b_i \mathbf{S}_\ell(R) b_i \subseteq \mathbf{S}_\ell(b_i R b_i)$. Hence $b_i e b_i \in \mathbf{S}_\ell(b_i R b_i)$, but by hypothesis $\mathbf{S}_\ell(b_i R b_i) = \{0, b_i\}$. So $b_i e b_i = b_i$. Also $b_i R = b_i e b_i R \subseteq b_i eR \subseteq b_i R$, and thus $b_i eR = b_i R$. Recall that b_i are orthogonal. Hence, $b_i e b_j e = b_i b_j e = 0$ yields that $b_1 e, \dots, b_n e$ are orthogonal idempotents. Let $I = \{i \mid 1 \le i \le n \text{ and } b_i e \ne 0\}$. Then $eR = \oplus_{i \in I} b_i eR = \oplus_{i \in I} b_i R$. □

The next result characterizes rings with a complete generalized triangular matrix representation.

Theorem 5.2.5 *The following are equivalent for a ring R.*

(i) *R has a complete set of left triangulating idempotents.*
(ii) *$\{bR \mid b \in \mathbf{S}_\ell(R)\}$ is a finite set.*
(iii) *$\{bR \mid b \in \mathbf{S}_\ell(R)\}$ satisfies ACC and DCC.*
(iv) *$\{bR \mid b \in \mathbf{S}_\ell(R)\}$ and $\{Rc \mid c \in \mathbf{S}_r(R)\}$ satisfy ACC.*
(v) *$\{bR \mid b \in \mathbf{S}_\ell(R)\}$ and $\{Rc \mid c \in \mathbf{S}_r(R)\}$ satisfy DCC.*
(vi) *$\{bR \mid b \in \mathbf{S}_\ell(R)\}$ and $\{cR \mid c \in \mathbf{S}_r(R)\}$ satisfy DCC.*
(vii) *R has a complete set of right triangulating idempotents.*
(viii) *R has a complete generalized triangular matrix representation.*

Proof Lemma 5.2.4 yields (i)⇒(ii), and (ii)⇒(iii) is trivial. From Lemma 5.2.2, (iii)⇒(iv)⇒(v)⇒(vi) follows immediately.

We show that (vi)⇒(i). If $\mathbf{S}_\ell(R) = \{0, 1\}$, then we are finished. Otherwise take e_1 to be a nontrivial element of $\mathbf{S}_\ell(R)$.

If e_1 is not semicentral reduced, then there exists $0 \ne e_2 \in \mathbf{S}_\ell(e_1 R e_1)$ such that $e_1 R \ne e_2 R$ by Lemma 5.2.1, and so $e_1 R \supsetneq e_2 R$. From Lemma 5.1.5(i), $e_2 \in \mathbf{S}_\ell(R)$. If e_2 is not semicentral reduced, then by Lemmas 5.2.1 and 5.1.5(i) again there exists $0 \ne e_3 \in \mathbf{S}_\ell(e_2 R e_2) \subseteq \mathbf{S}_\ell(R)$ such that $e_2 R \ne e_3 R$. So we have that $e_2 R \supsetneq e_3 R$. This process should be stopped within a finite steps. Thus, we obtain a semicentral reduced idempotent $e_n \in \mathbf{S}_\ell(R)$ for some positive integer n because $\{eR \mid e \in \mathbf{S}_\ell(R)\}$ has DCC.

Starting a new process, let $b_1 = e_n$. Then $\mathbf{S}_\ell(b_1Rb_1) = \{0, b_1\}$. From Proposition 1.2.2, $1 - b_1 \in \mathbf{S}_r(R)$. If $1 - b_1$ is semicentral reduced, then we see that $\{b_1, 1 - b_1\}$ is a complete set of left triangulating idempotents.

Otherwise, we consider $R_1 = (1 - b_1)R(1 - b_1)$. Note that by Lemma 5.2.3, R_1 has DCC on $\{dR_1 \mid d \in \mathbf{S}_\ell(R_1)\}$. By a similar argument to that used to get b_1, we obtain $b_2 \in \mathbf{S}_\ell(R_1)$ such that $\mathbf{S}_\ell(b_2R_1b_2) = \{0, b_2\}$.

As $1 - b_1$ is the identity of R_1 and $b_2 \in R_1$, it follows that $b_2R_1b_2 = b_2Rb_2$, so $\mathbf{S}_\ell(b_2Rb_2) = \{0, b_2\}$. Also, $(1 - b_1) - b_2 \in \mathbf{S}_r(R_1)$. The right-sided version of Lemma 5.1.5(i) yields that $\mathbf{S}_r(R_1) \subseteq \mathbf{S}_r(R)$. Therefore, $1 - b_1 - b_2 \in \mathbf{S}_r(R)$. If $1 - b_1 - b_2$ is semicentral reduced in R, then $\{b_1, b_2, 1 - b_1 - b_2\}$ is a complete set of left triangulating idempotents.

We continue the process to obtain a descending chain in $\{cR \mid c \in \mathbf{S}_r(R)\}$, which is $(1 - b_1)R \supseteq (1 - b_1 - b_2)R \supseteq (1 - b_1 - b_2 - b_3)R \supseteq \dots$. By the DCC hypothesis of $\{cR \mid c \in \mathbf{S}_r(R)\}$, this chain becomes stationary after a finite steps, yielding a complete set of left triangulating idempotents.

The equivalence (vii)$\Leftrightarrow$(i) follows from Corollary 5.1.8, while the equivalence (i)$\Leftrightarrow$(viii) follows from Theorem 5.1.4. □

Corollary 5.2.6 *Let R be a ring with a complete set of left triangulating idempotents. Then for any $0 \neq e \in \mathbf{S}_\ell(R)$ (resp., $0 \neq e \in \mathbf{S}_r(R)$), the ring eRe also has a complete set of left (resp., right) triangulating idempotents.*

Proof Say $0 \neq e \in \mathbf{S}_\ell(R)$. Define

$$\lambda : \{bR \mid b \in \mathbf{S}_\ell(R)\} \to \{d(eRe) \mid d \in \mathbf{S}_\ell(eRe)\}$$

by $\lambda(bR) = (ebe)(eRe)$. From Lemma 5.1.5(ii), $ebe \in \mathbf{S}_\ell(eRe)$ for $b \in \mathbf{S}_\ell(R)$. If $bR = b_1R$ with $b, b_1 \in \mathbf{S}_\ell(R)$, then $bRe = b_1Re$, and so $ebeRe = eb_1eRe$ since $e \in \mathbf{S}_\ell(R)$. Thus λ is well-defined. As $\mathbf{S}_\ell(eRe) \subseteq \mathbf{S}_\ell(R)$ by Lemma 5.1.5(i), λ is onto. From Theorem 5.2.5, it follows that $\{bR \mid b \in \mathbf{S}_\ell(R)\}$ is finite. Furthermore, we get that $\{d(eRe) \mid d \in \mathbf{S}_\ell(eRe)\}$ is also finite. Again by Theorem 5.2.5, eRe has a complete set of left triangulating idempotents. Similarly, if $0 \neq e \in \mathbf{S}_r(R)$, then eRe has also a complete set of right triangulating idempotents. □

In Theorem 5.2.8, the uniqueness of a complete generalized triangular matrix representation will be established. For the proof of this theorem, we need the following result due to Azumaya [32, Theorem 3].

Lemma 5.2.7 *Let I be a quasi-regular ideal of a ring R. If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ are two sets of orthogonal idempotents of R such that $\overline{e}_i = \overline{f}_i$ for each i with images $\overline{e}_i$ and $\overline{f}_i$ in R/I, then there is an invertible element $\alpha \in R$ with $f_i = \alpha^{-1}e_i\alpha$ for each i.*

Proof Let $e = \sum_{i=1}^n e_i$ and $f = \sum_{i=1}^n f_i$. Put $\beta = e + f - ef - \sum_{i=1}^n e_i f_i$. Then $\alpha = 1 - \beta$ is invertible and $f_i = \alpha^{-1}e_i\alpha$ for each i. □

A nonzero central idempotent e of R is said to be *centrally primitive* if 0 and e are the only central idempotents in eRe. Let g be a nonzero central idempotent in R such that $g = g_1 + \cdots + g_t$, where $\{g_i \mid 1 \le i \le t\}$ is a set of centrally primitive orthogonal idempotents of R. Then t is uniquely determined (see Exercise 5.2.21.1). A ring R is said to have a *complete set of centrally primitive idempotents* if there exists a finite set of centrally primitive orthogonal idempotents whose sum is 1. It is routine to check that R has a complete set of centrally primitive idempotents if and only if R is a ring direct sum of indecomposable rings.

Theorem 5.2.8 (Uniqueness) *Let $\{b_1, \ldots, b_n\}$ and $\{c_1, \ldots, c_k\}$ each be a complete set of left triangulating idempotents of R. Then $n = k$ and there exist an invertible element $\alpha \in R$ and a permutation σ on $\{1, \ldots, n\}$ such that $b_{\sigma(i)} = \alpha^{-1} c_i \alpha$ for each i. Thus for each i, $c_i R \cong b_{\sigma(i)} R$, as R-modules, and $c_i R c_i \cong b_{\sigma(i)} R b_{\sigma(i)}$, as rings.*

Proof Let $U = \sum_{i<j} b_i R b_j$. Then $U \trianglelefteq R$ and $U^n = 0$. Let $\overline{R} = R/U$ and denote by $\overline{x}$ the image of $x \in R$ in R/U. Since $b_i R b_i \cap U = 0$, for $i = 1, \ldots, n$, $b_i R b_i \cong \overline{b_i}\, \overline{R}\, \overline{b_i}$ as rings. So $\overline{R}$ is a direct sum of the $\overline{b}_i\, \overline{R}\, \overline{b}_i$, and consequently $\{\overline{b}_1, \ldots, \overline{b}_n\}$ is a complete set of centrally primitive idempotents of $\overline{R}$.

Clearly, $\overline{c}_1 \in \mathbf{S}_\ell(\overline{R})$. Further, $\overline{c}_1 \ne \overline{0}$. Indeed, if $\overline{c}_1 = \overline{0}$, then $c_1 \in U$, and so $c_1 = c_1^n \in U^n = 0$, a contradiction. Because $\overline{b}_i$ is semicentral reduced, $\overline{c}_1 \overline{b}_i \in \{\overline{0}, \overline{b}_i\}$. Therefore $\overline{c}_1 = \sum_{i=1}^n \overline{c}_1 \overline{b}_i = \sum \overline{b}_k$ for which $\overline{c}_1 \overline{b}_k \ne \overline{0}$. So $\overline{c}_1 \in \mathcal{B}(\overline{R})$. Now we note that $\overline{c}_2 \in \mathbf{S}_\ell((\overline{1} - \overline{c}_1)\, \overline{R}\, (\overline{1} - \overline{c}_1))$. As $\overline{1} - \overline{c}_1 \in \mathcal{B}(\overline{R})$, $\overline{c}_2 \in \mathbf{S}_\ell(\overline{R})$ by Lemma 5.1.5(i). Using the preceding argument, with $\overline{c}_2$ in place of $\overline{c}_1$, we obtain $\overline{c}_2 \in \mathcal{B}(\overline{R})$.

Continuing this procedure, we obtain that $\{\overline{c}_1, \ldots, \overline{c}_k\}$ is a set of orthogonal nonzero central idempotents in $\overline{R}$. Hence $\overline{c}_i\, \overline{R}\, \overline{c}_j = \overline{0}$ for $i < j$. Thus $c_i R c_j \subseteq U$ for all $1 \le i < j \le k$.

Let $V = \sum_{i<j} c_i R c_j$. Then $V^k = 0$. By the preceding argument, $b_i R b_j \subseteq V$ for all $1 \le i < j \le n$. Hence, $U = V$ and so $\{\overline{b}_1, \ldots, \overline{b}_n\}$ and $\{\overline{c}_1, \ldots, \overline{c}_k\}$ are both complete sets of centrally primitive idempotents for $\overline{R}$. It is well known that for such sets of centrally primitive idempotents, $n = k$ and there is a permutation σ on $\{1, \ldots, n\}$ such that $\overline{c}_i = \overline{b}_{\sigma(i)}$ (Exercises 5.2.21.1 and 5.2.21.2). As $U^n = 0$, U is a quasi-regular ideal of R.

From Lemma 5.2.7, there exists an invertible element $\alpha \in R$ such that $b_{\sigma(i)} = \alpha^{-1} c_i \alpha$ for every i. Thus, $c_i R \cong b_{\sigma(i)} R$ as R-modules. We observe that $\mathrm{End}(c_i R_R) \cong c_i R c_i$ and $\mathrm{End}(b_j R_R) \cong b_j R b_j$. So $c_i R c_i \cong b_{\sigma(i)} R b_{\sigma(i)}$. □

The following example shows that the isomorphism $c_i R \cong b_{\sigma(i)} R$, given in Theorem 5.2.8, cannot be sharpened to equality. This is in contrast to the result for a complete set of centrally primitive idempotents.

Example 5.2.9 Let $R = T_2(\mathbb{R})$. Consider

$$b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\; b_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and let

$$c_1 = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, \ c_2 = \begin{bmatrix} 0 & -a \\ 0 & 1 \end{bmatrix}, \ 0 \neq a \in \mathbb{R}.$$

Then $\{b_1, b_2\}$ and $\{c_1, c_2\}$ are complete sets of left triangulating idempotents for R. In this case, $b_1 R = c_1 R$ and $b_2 R \cong c_2 R$, but $b_2 R \neq c_2 R$.

Kaplansky raised the following question: Let A and B be two rings. If $\mathrm{Mat}_n(A) \cong \mathrm{Mat}_n(B)$ as rings, does it follow that $A \cong B$ as rings? (See [261, p. 35].) It is known that there are nonisomorphic semicentral reduced rings (e.g., simple Noetherian domains) which have isomorphic matrix rings (see [260] and [378]). The next result shows that this cannot happen for $n \times n$ $(n > 1)$ upper triangular matrix rings over semicentral reduced rings.

Corollary 5.2.10 *Let A and B be semicentral reduced rings. If $T_m(A) \cong T_n(B)$ as rings, then $m = n$ and $A \cong B$ as rings.*

Proof Let e_{ii} be the matrix in $T_m(A)$ with 1_A in the (i, i)-position and 0 elsewhere. As A is semicentral reduced, $\{e_{11}, \ldots, e_{mm}\}$ is a complete set of left triangulating idempotents for $T_m(A)$. A similar fact holds for $T_n(B)$. Because $T_m(A) \cong T_n(B)$, $m = n$ by Theorem 5.2.8.

Next, say $\lambda : T_n(A) \to T_n(B)$ is an isomorphism. Then $\{\lambda(e_{11}), \ldots, \lambda(e_{nn})\}$ is a complete set of left triangulating idempotents of $T_n(B)$. Let f_{ii} be the matrix in $T_n(B)$ with 1_B in the (i, i)-position and 0 elsewhere. Then because B is semicentral reduced, $\{f_{11}, \ldots, f_{nn}\}$ is also a complete set of left triangulating idempotents of $T_n(B)$.

By Theorem 5.2.8, $f_{11} T_n(B) f_{11} \cong \lambda(e_{jj}) T_n(B) \lambda(e_{jj})$ for some j. Therefore, $B \cong f_{11} T_n(B) f_{11} \cong \lambda(e_{jj}) T_n(B) \lambda(e_{jj}) \cong e_{jj} T_n(A) e_{jj} \cong A$. □

From Theorem 5.2.8, the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set). This is also the number of elements in any complete set of right triangulating idempotents of R by Corollary 5.1.8. So we are motivated to give the following definition.

Definition 5.2.11 A ring R is said to have *triangulating dimension n*, written $\mathrm{Tdim}(R) = n$, if R has a complete set of left triangulating idempotents with n elements. Note that R is semicentral reduced if and only if $\mathrm{Tdim}(R) = 1$. If R has no complete set of left triangulating idempotents, then we say that R has *infinite triangulating dimension*, denoted $\mathrm{Tdim}\,(R) = \infty$.

Lemma 5.2.12 *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents of R. If $0 \neq b \in \mathbf{S}_\ell(R) \cup \mathbf{S}_r(R)$, then there exists a nonempty subset P of $\{e_1, \ldots, e_n\}$ such that $\{b e_j b \mid e_j \in P\}$ forms a complete set of primitive idempotents of bRb.*

Proof Assume that $b \in \mathbf{S}_\ell(R)$. From $b = b(e_1 + \cdots + e_n)b = be_1b + \cdots + be_nb$, some $be_kb \neq 0$. Let P be the set of all e_j such that the elements be_jb are nonzero. Without loss of generality, let $P = \{e_1, \dots, e_m\}$.

By Lemma 5.1.5(iii), the be_jb, $j = 1, \dots, m$, are primitive idempotents in bRb. From $b = be_1b + \cdots + be_nb = be_1b + \cdots + be_mb$, $\{be_jb \mid 1 \leq j \leq m\}$ is a complete set of primitive idempotents for bRb. The proof for $b \in \mathbf{S}_r(R)$ is a right-sided version of the preceding proof. □

The next two results may be useful for studying many well known classes of rings via complete generalized triangular matrix representations and semicentral reduced rings from the same respective class.

Proposition 5.2.13 *Let a ring R satisfy any one of the following conditions.*

(i) *R has a complete set of primitive idempotents.*
(ii) *R is orthogonally finite.*
(iii) *R has DCC on idempotent generated (resp., principal, or finitely generated) ideals.*
(iv) *R has ACC on idempotent generated (resp., principal, or finitely generated) ideals.*
(v) *R has DCC on idempotent generated (resp., principal, or finitely generated) right ideals.*
(vi) *R has ACC on idempotent generated (resp., principal, or finitely generated) right ideals.*
(vii) *R is a semilocal ring.*
(viii) *R is a semiperfect ring.*
(ix) *R is a right perfect ring.*
(x) *R is a semiprimary ring.*

Then $\operatorname{Tdim}(R) < \infty$ *and*

$$R \cong \begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix},$$

where $n = \operatorname{Tdim}(R)$, *each R_i is semicentral reduced, and satisfies the same condition as R. Further, each R_{ij} is an (R_i, R_j)-bimodule, and the rings $R_1, \dots, R_n$ are uniquely determined by R up to isomorphism and permutation.*

Proof (i) Let $\{f_1, \dots, f_k\}$ be a complete set of primitive idempotents of R. Then for any $0 \neq b \in \mathbf{S}_\ell(R)$, $b = f_1b + \cdots + f_kb$. Each f_ib is an idempotent, as $b \in \mathbf{S}_\ell(R)$. Assume that $j = 1, \dots, m$ is the set of all indices for which $f_jb \neq 0$.

Now we have that $bR \subseteq f_1bR + \cdots + f_mbR = bf_1bR + \cdots + bf_mbR \subseteq bR$, hence $bR = f_1bR + \cdots + f_mbR$. Primitivity of f_j implies that $f_jbR = f_jR$, whenever $f_jb \neq 0$. Hence, the total number of right ideals of the form bR, $b \in \mathbf{S}_\ell(R)$ cannot exceed 2^k. Thus, by Theorem 5.2.5, R has a complete set of left triangulating idempotents. So $\operatorname{Tdim}(R) < \infty$.

Let $\{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R. Take $R_i = e_i R e_i$ and $R_{ij} = e_i R e_j$ for $i < j$. Then R_{ij} is an (R_i, R_j)-bimodule for $i < j$. Since $e_1 \in \mathbf{S}_\ell(R)$, $R_1 = e_1 R e_1$ has a complete set of primitive idempotents from Lemma 5.2.12. Also $1 - e_1 \in \mathbf{S}_r(R)$ by Proposition 1.2.2, $(1 - e_1)R(1 - e_1)$ has a complete set of primitive idempotents by Lemma 5.2.12. Next we see that $e_2 \in \mathbf{S}_\ell((1 - e_1)R(1 - e_1))$, again Lemma 5.2.12 yields that

$$R_2 = e_2 R e_2 = e_2((1 - e_1)R(1 - e_1))e_2$$

has a complete set of primitive idempotents, and so on. The uniqueness of the R_i follows from Theorem 5.2.8.

(ii) By part (i) and Proposition 1.2.15, we have a unique generalized triangular matrix representation. Further, each R_i is orthogonally finite.

(iii) Assume that R has DCC on idempotent generated (resp., principal, or finitely generated) ideals. Then $\{eR \mid e \in \mathbf{S}_\ell(R)\}$ has DCC since $eR = ReR$ for each e in $\mathbf{S}_\ell(R)$. Consider $\{Rf \mid f \in \mathbf{S}_r(R)\}$. Then $Rf = RfR$ for each $f \in \mathbf{S}_r(R)$. Thus $\{Rf \mid f \in \mathbf{S}_r(R)\}$ also has DCC. By Theorem 5.2.5, R has a complete set of left triangulating idempotents. So $\text{Tdim}(R) < \infty$.

Now say $h^2 = h \in R$. Then hRh has DCC on idempotent generated (resp., principal, or finitely generated) ideals by using [259, Theorem 21.11].

(iv) By assumption, $\{eR \mid e \in \mathbf{S}_\ell(R)\}$ has ACC as $eR = ReR$. Also since $Rf = RfR$ for any $f \in \mathbf{S}_r(R)$, $\{Rf \mid f \in \mathbf{S}_r(R)\}$ has ACC. From Theorem 5.2.5, R has a complete set of left triangulating idempotents, so $\text{Tdim} < \infty$. Say $h^2 = h \in R$. By using [259, Theorem 21.11], hRh has ACC on idempotent generated (resp., principal, or finitely generated) ideals.

(v) By Proposition 1.2.13, R is orthogonally finite. By part (ii), R has a complete set of left triangulating idempotents, so $\text{Tdim}(R) < \infty$. Next, let $h^2 = h \in R$. Then hRh has DCC on idempotent generated (resp., principal or finitely generated) right ideals by using [259, Theorem 21.11].

(vi) The proof is similar to that of part (v) by Proposition 1.2.13 and using [259, Theorem 21.11].

(vii) and (viii) We note that, for each of these conditions, R is orthogonally finite. By part (ii), $\text{Tdim}(R) < \infty$. Homomorphic images of a semilocal ring and a semiperfect ring are semilocal and semiperfect, respectively (see [259, Proposition 20.7] and [8, Corollary 27.9]). By Proposition 5.1.7(iii), if R is semilocal (resp., semiperfect), then each R_i is semilocal (resp., semiperfect).

(ix) If R is right perfect, then R is orthogonally finite. Thus part (ii) yields that $\text{Tdim}(R) < \infty$. By 1.1.14, R has DCC on principal left ideals. Say $h^2 = h \in R$. Then by the left-sided version of the proof for part (v), hRh also has DCC on principal left ideals. So hRh is right perfect, and hence each R_i is right perfect.

(x) If R is semiprimary, then also R is orthogonally finite. Hence by part (ii), $\text{Tdim}(R) < \infty$. Say $h^2 = h \in R$. It is well known that $J(hRh) = hJ(R)h$ (see [259, Theorem 21.10]). Hence if R is semiprimary, then so is hRh. Thus each R_i is semiprimary. □

Proposition 5.2.14 *Let P be a property of rings such that whenever a ring A satisfies P, then A/I ($I \trianglelefteq A$) or eAe ($e^2 = e \in A$) also satisfies P. Assume that R is a* *ring with* Tdim$(R) = n < \infty$ *and satisfies P. Then*

$$R \cong \begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix},$$

where each R_i *is semicentral reduced and satisfies the property P. Further, each* R_{ij} *is an* (R_i, R_j)*-bimodule, and the rings* $R_1, \ldots, R_n$ *are uniquely determined by* R *up to isomorphism and permutation.*

Proof Since Tdim$(R) = n < \infty$, R has the indicated unique generalized triangular matrix representation by Theorems 5.1.4 and 5.2.8. Rings R_i have the form eRe, where $e^2 = e \in R$, also R_i are ring homomorphic images of R by Proposition 5.1.7(iii). By assumption each R_i has the property P. □

We remark that the following classes of rings determined by property P indicated in Proposition 5.2.14: Baer rings, right Rickart rings, quasi-Baer rings, right p.q.-Baer rings, right hereditary rings, right semihereditary rings, π-regular rings, PI-rings, and rings with bounded index (of nilpotency), etc.

By the next result, if Tdim$(R) < \infty$, central idempotents can be written as sums of elements in a complete set of left triangulating idempotents.

Proposition 5.2.15 *Assume that* $\{b_1, \ldots, b_n\}$ *is a complete set of left triangulating idempotents for a ring* R. *If* $c \in \mathcal{B}(R) \setminus \{0, 1\}$, *then there exists* $\emptyset \neq \Lambda \subsetneq \{1, \ldots, n\}$ *such that* $c = \sum_{i \in \Lambda} b_i$.

Proof Let $c \in \mathcal{B}(R) \setminus \{0, 1\}$. Then $c = c(b_1 + \cdots + b_n) = cb_1 + \cdots + cb_n$. We note that $cb_i \in \mathbf{S}_\ell(b_i R b_i)$ and $\mathbf{S}_\ell(b_i R b_i) = \{0, b_i\}$ for each i. Therefore, there exists $\emptyset \neq \Lambda \subsetneq \{1, \ldots, n\}$ such that $c = \sum_{i \in \Lambda} b_i$. □

Theorem 5.2.16 *Let* R *be a ring. Consider the following conditions.*

(i) *R has a complete set of primitive idempotents.*
(ii) *R has a complete set of left triangulating idempotents.*
(iii) *R has a complete set of centrally primitive idempotents.*

Then (i)⇒(ii)⇒(iii).

Proof Proposition 5.2.13(i) yields the implication (i)⇒(ii). For (ii)⇒(iii), assume that R has a complete set of left triangulating idempotents for R. By Proposition 5.2.15, $\mathcal{B}(R)$ is a finite set. Now a standard argument yields that R has a complete set of centrally primitive idempotents. □

We remark that when R is commutative, conditions (i), (ii), and (iii) of Theorem 5.2.16 are equivalent. The next example shows that the converse of each of the implications in Theorem 5.2.16 does not hold.

Example 5.2.17 (i) There is a ring R with a complete set of left triangulating idempotents (i.e., $\mathrm{Tdim}\,(R) < \infty$), but R does not have a complete set of primitive idempotents. Indeed, let V be an infinite dimensional right vector space over a field F and let $R = \mathrm{End}_F(V)$. Then R is a prime ring, so $\mathrm{Tdim}(R) = 1$. Since R is a regular ring which is not semisimple Artinian, R cannot have a complete set of primitive idempotents.

(ii) There is a ring R with a complete set of centrally primitive idempotents, but R does not have a complete set of left triangulating idempotents. For this, let R be the $\aleph_0 \times \aleph_0$ upper triangular row finite matrix ring over a field. Then $\{\mathbf{1}\}$ is a complete set of centrally primitive idempotents of R, where $\mathbf{1}$ is the identity of R. Let e_{ii} be the matrix in R with 1 in the (i, i)-position and 0 elsewhere. Then for any positive integer n, $e_{11} + \cdots + e_{nn} \in \mathbf{S}_\ell(R)$. As

$$(e_{11} + \cdots + e_{nn})R \subsetneq (e_{11} + \cdots + e_{nn} + e_{n+1\,n+1})R$$

for each n, Theorem 5.2.5 yields that R cannot have a complete set of left triangulating idempotents.

We need the next lemma for investigating $\mathrm{Tdim}(R)$ of a ring R.

Lemma 5.2.18 *Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of a ring R and $\{b_{(i,1)}, \ldots, b_{(i,k_i)}\}$ a set of left triangulating idempotents of b_iRb_i. Then $\{b_{(1,1)}, \ldots, b_{(1,k_1)}, b_{(2,1)}, \ldots, b_{(2,k_2)}, \ldots, b_{(n,1)}, \ldots, b_{(n,k_n)}\}$ is a set of left triangulating idempotents of R.*

Proof Clearly $1 = \sum_{i=1}^{k_1} b_{(1,i)} + \cdots + \sum_{i=1}^{k_n} b_{(n,i)}$. Also $b_{(1,1)} \in \mathbf{S}_\ell(R)$ by Lemma 5.1.5(i). Let $c_{(i,j)} = 1 - \sum_{\alpha=1}^{i-1} b_\alpha - \sum_{\gamma=1}^{j} b_{(i,\gamma)}$, where $1 \le j < k_i$. Then $b_{(i,j+1)}(\sum_{\alpha=1}^{i-1} b_\alpha + \sum_{\gamma=1}^{j} b_{(i,\gamma)}) = 0$, and so $b_{(i,j+1)}c_{(i,j)} = b_{(i,j+1)}$. Similarly, $c_{(i,j)}b_{(i,j+1)} = b_{(i,j+1)}$. So $b_{(i,j+1)} \in c_{(i,j)}Rc_{(i,j)}$. Note that $c_{(i,\,j)}^2 = c_{(i,\,j)}$.

We claim that $b_{(i,j+1)} \in \mathbf{S}_\ell(c_{(i,j)}Rc_{(i,j)})$. Put $c_j = b_i - \sum_{\gamma=1}^{j} b_{(i,\gamma)}$. Then $b_{(i,j+1)} \in \mathbf{S}_\ell(c_j(b_iRb_i)c_j) = \mathbf{S}_\ell(c_jRc_j)$ and $c_{(i,j)} = 1 - \sum_{\alpha=1}^{i} b_\alpha + c_j$. Note that $b_{(i,j+1)} \in b_iRb_i$, $(\sum_{\alpha=1}^{i-1} b_\alpha)b_{(i,j+1)} = 0$, and $\{b_1, \ldots, b_n\}$ is a set of orthogonal idempotents. Hence,

$$\begin{aligned} b_{(i,j+1)} = c_{(i,j)}b_{(i,j+1)} &= (1 - \sum_{\alpha=1}^{i} b_\alpha + c_j)b_{(i,j+1)} \\ &= b_{(i,j+1)} - b_ib_{(i,j+1)} + c_jb_{(i,j+1)} = c_jb_{(i,j+1)} \end{aligned}$$

as $b_ib_{(i,j+1)} = b_{(i,j+1)}$. Similarly, $b_{(i,j+1)} = b_{(i,j+1)}c_{(i,j)} = b_{(i,j+1)}c_j$. For $r \in R$,

$$\begin{aligned} (c_{(i,j)}rc_{(i,j)})b_{(i,j+1)} &= (1 - \sum_{\alpha=1}^{i} b_\alpha + c_j)rc_jb_{(i,j+1)} \\ &= (1 - \sum_{\alpha=1}^{i} b_\alpha)rc_jb_{(i,j+1)} + c_jrc_jb_{(i,j+1)}. \end{aligned}$$

From Proposition 5.1.7(i), $1 - \sum_{\alpha=1}^{i} b_\alpha \in \mathbf{S}_r(R)$. Therefore, we now obtain that $(1 - \sum_{\alpha=1}^{i} b_\alpha) r c_j b_{(i,j+1)} = (1 - \sum_{\alpha=1}^{i} b_\alpha) r (1 - \sum_{\alpha=1}^{i} b_\alpha) c_j b_{(i,j+1)} = 0$ since

$$\begin{aligned}(1 - \sum_{\alpha=1}^{i} b_\alpha) c_j b_{(i,j+1)} &= (1 - \sum_{\alpha=1}^{i} b_\alpha) b_{(i,j+1)} = (1 - \sum_{\alpha=1}^{i-1} b_\alpha - b_i) b_{(i,j+1)} \\ &= b_{(i,j+1)} - b_i b_{(i,j+1)} = 0.\end{aligned}$$

Thus,

$$\begin{aligned}(c_{(i,j)} r c_{(i,j)}) b_{(i,j+1)} &= (c_j r c_j) b_{(i,j+1)} = b_{(i,j+1)} (c_j r c_j) b_{(i,j+1)} \\ &= b_{(i,j+1)} (c_{(i,j)} r c_{(i,j)}) b_{(i,j+1)}.\end{aligned}$$

So $b_{(i,j+1)} \in \mathbf{S}_\ell(c_{(i,j)} R c_{(i,j)})$. Now routinely we obtain the desired result. □

Theorem 5.2.19 *Let* $\{b_1, \ldots, b_n\}$ *be a set of left triangulating idempotents of a ring* R. *Then* $\mathrm{Tdim}(R) = \sum_{i=1}^{n} \mathrm{Tdim}(b_i R b_i)$.

Proof If $\mathrm{Tdim}(R) = \infty$, then $\mathrm{Tdim}(b_j R b_j) = \infty$ for some $1 \le j \le n$, otherwise Lemma 5.2.18 yields a contradiction.

Let $\mathrm{Tdim}(R) < \infty$. By Corollary 5.2.6, $\mathrm{Tdim}(b_1 R b_1) < \infty$. From Proposition 1.2.2, $1 - b_1 \in \mathbf{S}_r(R)$. By Corollary 5.2.6, $\mathrm{Tdim}((1 - b_1) R (1 - b_1)) < \infty$. We see that $b_2 \in \mathbf{S}_\ell((1 - b_1) R (1 - b_1))$. Hence, Corollary 5.2.6 yields that $\mathrm{Tdim}(b_2 R b_2) < \infty$. This procedure, by using Corollary 5.2.6, can be continued to show that $\mathrm{Tdim}(b_i R b_i) < \infty$ for all $1 \le i \le n$. Lemma 5.2.18 yields that $\mathrm{Tdim}(R) = \sum_{i=1}^{n} \mathrm{Tdim}(b_i R b_i)$. □

Corollary 5.2.20 *Let* R *be a ring with a generalized triangular matrix representation*

$$\begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix}.$$

Then $\mathrm{Tdim}(R) = \sum_{i=1}^{n} \mathrm{Tdim}(R_i)$. *So*, $\mathrm{Tdim}(T_n(A)) = n\mathrm{Tdim}(A)$, *where* A *is a ring and* n *is a positive integer.*

Exercise 5.2.21

1. Assume that R is a ring and $0 \ne g \in \mathcal{B}(R)$ such that $g = g_1 + \cdots + g_t$, where $\{g_i \mid 1 \le i \le t\}$ is a set of orthogonal centrally primitive idempotents in R. Show that t is uniquely determined.
2. Let R be a ring, and let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be two complete sets of centrally primitive idempotents of R. Show that $m = n$ and there exists a permutation σ on $\{1, \ldots, n\}$ such that $e_i = f_{\sigma(i)}$.

3. Assume that M_R is a right R-module and $S = \mathrm{End}(M_R)$. Show that the following are equivalent.
   (i) S has a complete set of left triangulating idempotents.
   (ii) There exists a positive integer n such that:
      (1) $M = M_1 \oplus \cdots \oplus M_n$.
      (2) $\mathrm{Hom}(M_i, M_j) = 0$ for $i < j$.
      (3) Each M_i has no nontrivial fully invariant direct summands.
4. ([93, Birkenmeier, Park and Rizvi]) Assume that S is an overring of a ring R such that $R_R \leq^{\mathrm{ess}} S_R$. (The ring S is called a right essential overing of R. See Chap. 7 for right essential overrings for more details.) Show that if R is right FI-extending, then $\mathrm{Tdim}(S) \leq \mathrm{Tdim}(R)$.
5. ([93, Birkenmeier, Park and Rizvi]) Let S be an overring of a ring R such that $R_R \leq^{\mathrm{ess}} S_R$. Prove that if R is right extending and $\{e_1, \ldots, e_n\}$ is a complete set of primitive idempotents for R, then $\{e_1, \ldots, e_n\}$ is a complete set of primitive idempotents for S.
6. ([79, Birkenmeier, Kim, and Park]) Show that a ring R is left perfect if and only if R has a complete generalized triangular matrix representation, where each diagonal ring R_i is simple Artinian or left perfect with $(\mathrm{Soc}({R_i}_{R_i}))^2 = 0$.

5.3 Canonical Representations

We show that if a ring R has a set of left triangulating idempotents, then it has a canonical generalized triangular matrix representation, where the diagonal subrings are organized into blocks of square diagonal matrix rings. This canonical representation is then used to obtain a result on the right global dimension of rings with a set of left triangulating idempotents.

Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of R. If J is a subset of $\{1, \ldots, n\}$, we denote $\sigma_J = \sum_{i \in J} b_i$. Our first result shows that under certain conditions the ordering in a set of left triangulating idempotents can be changed to obtain a new set of left triangulating idempotents.

Proposition 5.3.1 *Let j and m be in $\{1, \ldots, n\}$ with $j < m \leq n$. If $\{b_1, \ldots, b_n\}$ is a set of left triangulating idempotents of a ring R such that $b_i R b_m = 0$ for each i with $j \leq i < m$, then*

$$\{b_1, \ldots, b_{j-1}, b_m, b_j, b_{j+1}, \ldots, b_{m-1}, b_{m+1}, \ldots, b_n\}$$

is a set of left triangulating idempotents of R.

Proof The proof follows routinely from Theorem 5.1.3. □

Proposition 5.3.1 is applied to obtain a canonical form for a generalized triangular matrix representation of R. Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents. Recursively define the sets I_k and $J(k)$ as follows:

$$I_1 = \{i \mid b_i \in \mathbf{S}_\ell(R)\} \text{ and } J(1) = I_1;$$

and let

$$I_{k+1} = \{i \mid b_i \in \mathbf{S}_\ell((1-\sigma_{J(k)})R(1-\sigma_{J(k)}))\} \text{ and } J(k+1) = J(k) \cup I_{k+1},$$

whenever I_k and $J(k)$ are defined. This process terminates within n steps.

Let $S_j = \{b_i \mid i \in I_j\}$. Then $S_1, \ldots, S_q$ is a partition for $\{b_1, \ldots, b_n\}$ (we will show in the proof of Theorem 5.3.2 that this always occurs). Then reorder $\{1, \ldots, n\}$ so that each I_j has any (fixed) ordering and so that elements of I_j always precede elements in I_{j+1}. This can be thought of in terms of a permutation ψ on $\{1, \ldots, n\}$. Then the ordered set $\{b_{\psi(1)}, \ldots, b_{\psi(n)}\}$ is called a *canonical form* for $\{b_1, \ldots, b_n\}$.

Theorem 5.3.2 *Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents. Then a canonical form for $\{b_1, \ldots, b_n\}$ exists, and any such canonical form is a set of left triangulating idempotents of R.*

Proof The proof involves repeated use of Propositions 5.3.1, as in the following discussion. We note that $b_1 \in S_1 = \mathbf{S}_\ell(R)$. If $b_m \in S_1$ and $m \neq 1$, then $b_i R b_m = b_i b_m R b_m = 0$ for all $i \neq m$. We use Proposition 5.3.1 to get that $\{b_m, b_1, \ldots, b_{m-1}, b_{m+1}, \ldots, b_n\}$ is a set of left triangulating idempotents of R. Continue this process using elements of S_1 until they are exhausted.

Following the procedure given in Proposition 5.3.1, there exists a permutation α on $\{1, \ldots, n\}$ such that $S_1 = \{b_{\alpha(1)}, \ldots, b_{\alpha(n_1)}\}$. Also, the ordered set $\{b_{\alpha(1)}, b_{\alpha(2)}, \ldots, b_{\alpha(n)}\}$ is a set of left triangulating idempotents of R.

If $n_1 = n$, then we are finished. So consider $n_1 < n$ and let $q = \alpha(n_1 + 1)$, where $\alpha(n_1 + 1)$ is the smallest positive integer i such that $b_i \notin S_1$. Observe that b_q is the first element in this new ordering which is not in S_1.

We show that $b_q \in S_2$. For this, let y be the sum of all elements in S_1. Thus, $y = b_{\alpha(1)} + \cdots + b_{\alpha(n_1)}$. Let g be the sum of all elements in $\{b_{\alpha(1)}, \ldots, b_{\alpha(n)}\}$ which are not in $\{b_q, b_{\alpha(1)}, \ldots, b_{\alpha(n_1)}\}$. Then $1 = y + b_q + g$. Thus $1 - y = b_q + g$, and therefore $b_q \in (1-y)R(1-y)$. Now for every $a \in R$, we can see that

$$\begin{aligned}(1-y)a(1-y)b_q &= (1-y)ab_q = (b_q + g)ab_q \\ &= b_q a b_q = b_q(1-y)a(1-y)b_q\end{aligned}$$

as $b_q(1-y) = b_q$, $(1-y)b_q = b_q$, and $gab_q = 0$. So $b_q \in \mathbf{S}_\ell((1-y)R(1-y))$.

Consequently, $q \in I_2$ and hence $b_q \in S_2$. Either this exhausts the elements in S_2 or (in the ordering given by α) there is an element $b_p \in S_2$ beyond b_q. Use Proposition 5.3.1 as before to obtain a set of left triangulating idempotents of R of the form $\{b_{\alpha(1)}, \ldots, b_{\alpha(n_1)}, b_p, b_q, b_{\alpha(n_1+2)}, \ldots, b_{\alpha(n)}\}$.

Repeat this process using elements of S_2 until they are exhausted. Then there exists a permutation γ on $\{1, \ldots, n\}$ such that

$$\{b_{\gamma(1)}, \ldots, b_{\gamma(n_1)}, b_{\gamma(n_1+1)}, \ldots, b_{\gamma(n_2)}, \ldots, b_{\gamma(n)}\}$$

forms a set of left triangulating idempotents, where $\gamma(i) = \alpha(i)$ for $1 \leq i \leq n_1$, $b_{\gamma(n_2)} = b_q$, and $\{b_{\gamma(n_1+1)}, \ldots, b_{\gamma(n_2)}\} = S_2$.

Now either $S_1 \cup S_2 = \{b_1, \ldots, b_n\}$ or we can continue the process on S_3, and so on. After k steps, $k \leq n$, the process terminates in a set of left triangulating idempotents of R in a canonical form. So we obtain a permutation ψ so that $S_1, \ldots, S_k$ is our desired partition of $\{b_1 \ldots, b_n\}$. □

Theorems 5.1.4 and 5.3.2 provide a tool for a generalized triangular matrix representation of R in a special canonical form, which we give next.

Corollary 5.3.3 (Canonical Representation) *Let* $\{b_1, \ldots, b_n\}$, $S_1, \ldots, S_k$, *and* ψ *be as before. Then using* $0 = n_0 < n_1 < \cdots < n_k$, *we have that* $S_{j+1} = \{b_{\psi(n_j+1)}, \ldots, b_{\psi(n_{j+1})}\}$, $j = 0, 1, \ldots, k-1$, *and* R *is isomorphic to the* $n \times n$ *matrix* $[A(i, j)]$, *where the* $A(i, j)$ *are* $n_i \times n_j$ *block matrices*

$$A(i+1, i+1) = \begin{bmatrix} b_{\psi(n_i+1)} R b_{\psi(n_i+1)} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{\psi(n_{i+1})} R b_{\psi(n_{i+1})} \end{bmatrix};$$

$$A(i+1, j+1) = \begin{bmatrix} b_{\psi(n_i+1)} R b_{\psi(n_j+1)} & \cdots & b_{\psi(n_i+1)} R b_{\psi(n_{j+1})} \\ \vdots & \ddots & \vdots \\ b_{\psi(n_{i+1})} R b_{\psi(n_j+1)} & \cdots & b_{\psi(n_{i+1})} R b_{\psi(n_{j+1})} \end{bmatrix},$$

for $i < j$; *and* $A(i, j) = 0$ *for* $i > j$, *where* $i, j = 0, 1, \ldots, k-1$.

For the proof of Theorem 5.3.5, we need the following lemma.

Lemma 5.3.4 *Let* A *and* B *be rings, and let* M *be an* (A, B)*-bimodule. Set* $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, *a generalized triangular matrix ring. Then*

$$\max\{\text{r.gl.dim}(A), \text{r.gl.dim}(B)\} \leq \text{r.gl.dim}(R)$$
$$\leq \max\{\text{r.gl.dim}(A) + \text{pd}(M_B) + 1, \ \text{r.gl.dim}(B)\},$$

where $\text{pd}(M_B)$ *is the projective dimension of* M_B.

Proof See [295, Proposition 7.5.1] for the proof. □

In Lemma 5.3.4, if $M = 0$, then $R = A \oplus B$ (ring direct sum). Also

$$\text{r.gl.dim}(R) \leq \max\{\text{r.gl.dim}(A) + \text{pd}(A_R), \text{r.gl.dim}(B) + \text{pd}(B_R)\}$$

from the proof of [295, Proposition 7.5.1]. As A_R and B_R are projective, it follows that $\text{pd}(A_R) = 0$ and $\text{pd}(B_R) = 0$, so

$$\text{r.gl.dim}(R) \leq \max\{\text{r.gl.dim}(A), \text{r.gl.dim}(B)\}.$$

Thus, $\text{r.gl.dim}(A \oplus B) = \max\{\text{r.gl.dim}(A), \text{r.gl.dim}(B)\}$ by Lemma 5.3.4.

As an application of canonical representation, we discuss the following result which exhibits a connection between the right global dimension of R and that of the sum of diagonal subrings.

Theorem 5.3.5 *Let* $\{b_1, \ldots, b_n\}$ *be a set of left triangulating idempotents of* R, *and* $S_1, \ldots, S_k$ *be as in Corollary 5.3.3. Then*

$$\text{r.gl.dim}(D) \le \text{r.gl.dim}(R) \le k\,(\text{r.gl.dim}(D)) + k - 1,$$

where $D = b_1 R b_1 + \cdots + b_n R b_n$. *Thereby,* $\text{r.gl.dim}(R) < \infty$ *if and only if* $\text{r.gl.dim}(D) < \infty$.

Proof The proof is given by induction on k. If $k = 1$, then $R = D$ by Theorem 5.3.2 and we are finished. Assume that $k \ge 2$. We take $A = \sum_{b_i \in S_1} b_i R b_i$, $M = \sum_{b_i \in S_1, b_j \in S_2 \cup \cdots \cup S_k} b_i R b_j$, and $B = (1 - \sum_{b_i \in S_1} b_i)\, R\, (1 - \sum_{b_i \in S_1} b_i)$. Then obviously $B = (\sum_{b_j \in S_2 \cup \cdots \cup S_k} b_j)\, R\, (\sum_{b_j \in S_2 \cup \cdots \cup S_k} b_j)$.

We note that $S_2 \cup \cdots \cup S_k$ is a set of left triangulating idempotents of B and $\{S_2, \ldots, S_k\}$ is a partition which establishes a canonical generalized triangular matrix representation for B. Let $D_1 = \sum_{b_j \in S_2 \cup \cdots \cup S_k} b_j R b_j$. Then by induction $\text{r.gl.dim}(D_1) \le \text{r.gl.dim}(B) \le (k-1)(\text{r.gl.dim}(D_1)) + k - 2$.

Because $D = A \oplus D_1$ from Theorem 5.3.2 or Corollary 5.3.3, it follows that $\text{r.gl.dim}(D) = \max\{\text{r.gl.dim}(A), \text{r.gl.dim}(D_1)\}$. Observe that $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ and M is an (A, B)-bimodule. Hence,

$$\max\{\text{r.gl.dim}(A), \text{r.gl.dim}(B)\} \le \text{r.gl.dim}(R)$$
$$\le \max\{\text{r.gl.dim}(A) + \text{pd}(M_B) + 1,\ \text{r.gl.dim}(B)\}$$

from Lemma 5.3.4. Because $\text{r.gl.dim}(D_1) \le \text{r.gl.dim}(B)$,

$$\begin{aligned} \text{r.gl.dim}(D) &= \max\{\text{r.gl.dim}(A),\ \text{r.gl.dim}(D_1)\} \\ &\le \max\{\text{r.gl.dim}(A),\ \ \text{r.gl.dim}(B)\} \\ &\le \text{r.gl.dim}(R). \end{aligned}$$

We observe that $\text{pd}(M_B) \le \text{r.gl.dim}(B)$. Therefore,

$$\begin{aligned} \text{r.gl.dim}(R) &\le \max\{\text{r.gl.dim}(A) + \text{pd}(M_B) + 1,\ \text{r.gl.dim}(B)\} \\ &\le \max\{\text{r.gl.dim}(A) + \text{r.gl.dim}(B) + 1,\ \text{r.gl.dim}(B)\} \\ &= \text{r.gl.dim}(A) + \text{r.gl.dim}(B) + 1 \\ &\le \text{r.gl.dim}(D) + [(k-1)\,(\text{r.gl.dim}(D_1)) + (k-2)] + 1 \\ &\le \text{r.gl.dim}(D) + (k-1)\,(\text{r.gl.dim}(D)) + k - 1 \\ &= k\,(\text{r.gl.dim}(D)) + k - 1. \end{aligned}$$

Therefore, $\text{r.gl.dim}(D) \le \text{r.gl.dim}(R) \le k\,(\text{r.gl.dim}(D)) + k - 1$. Thereby, $\text{r.gl.dim}(R) < \infty$ if and only if $\text{r.gl.dim}(D) < \infty$. □

5.4 Piecewise Prime Rings and Piecewise Domains

In this section, a criterion for a ring with a complete set of triangulating idempotents to be quasi-Baer is provided. Also a structure theorem for a quasi-Baer ring with a complete set of triangulating idempotents is shown. Among the applications of this structure theorem, several well-known results are obtained as its consequences. These include Levy's decomposition theorem of semiprime right Goldie rings, Faith's characterization of semiprime right FPF rings with no infinite set of central orthogonal idempotents, Gordon and Small's characterization of piecewise domains, and Chatters' decomposition theorem of hereditary Noetherian rings. A result related to Michler's splitting theorem for right hereditary right Noetherian rings is also obtained as an application.

The next result provides a criterion for a ring with a complete set of left triangulating idempotents to be quasi-Baer.

Theorem 5.4.1 *Assume that a ring R has a complete set of left triangulating idempotents with* $\text{Tdim}(R) = n$. *Then the following are equivalent.*

(i) *R is quasi-Baer.*
(ii) *For any complete set of left triangulating idempotents $\{b_1, \ldots, b_n\}$ of R, if $b_i x b_j R b_j y b_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$, then either $b_i x b_j = 0$ or $b_j y b_k = 0$.*
(iii) *There is a complete set of left triangulating idempotents $\{c_1, \ldots, c_n\}$ of R such that if $c_i x c_j R c_j y c_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$, then either $c_i x c_j = 0$ or $c_j y c_k = 0$.*
(iv) *For any complete set of left triangulating idempotents $\{b_1, \ldots, b_n\}$, assume that $K b_j V = 0$ for some ideals K and V of R and some b_j, $1 \le j \le n$. Then either $K b_j = 0$ or $b_j V = 0$.*

Proof (i)⇒(ii) Assume that $b_i x b_j R b_j y b_k = 0$ for some $x, y \in R$ and some $1 \le i, j, k \le n$. Since R is quasi-Baer, $r_R(b_i x b_j R) = fR$ for some $f \in \mathbf{S}_\ell(R)$. By Lemma 5.1.5(ii), $b_j f b_j \in \mathbf{S}_\ell(b_j R b_j)$. As $\{b_1, \ldots, b_n\}$ is a complete set of left triangulating idempotents, $\mathbf{S}_\ell(b_j R b_j) = \{0, b_j\}$. So either $b_j f b_j = 0$ or $b_j f b_j = b_j$. If $b_j f b_j = 0$, then since $b_j y b_k \in r_R(b_i x b_j R) = fR$, we have that $b_j y b_k = f b_j y b_k$. So $b_j y b_k = b_j f b_j y b_k = 0$. On the other hand, if $b_j f b_j = b_j$, then $b_i x b_j = b_i x b_j f b_j = 0$ as $b_i x b_j f = 0$.

(ii)⇒(iii) It follows immediately because R has a complete set of left triangulating idempotents.

(iii)⇒(i) Say L is a left ideal of R. First, assume that $Rc_i \cap \ell_R(L) \neq 0$ for some i. Then we may assume that

$$Rc_1 \cap \ell_R(L) \neq 0, \ldots, Rc_m \cap \ell_R(L) \neq 0,$$

and

$$Rc_{m+1} \cap \ell_R(L) = 0, \ldots, Rc_n \cap \ell_R(L) = 0.$$

Thus $\ell_R(L) Rc_{m+1} = 0, \ldots$, and $\ell_R(L) Rc_n = 0$. Put $T = Rc_1 + \cdots + Rc_m$.

Say $v \in \ell_R(L)$. Then $v = v(c_1 + \cdots + c_n) = vc_1 + \cdots + vc_m \in T$. Therefore, $\ell_R(L) \subseteq T$. To show that $c_1 \in \ell_R(L)$, take $y \in L$. Since $Rc_1 \cap \ell_R(L) \neq 0$, there exists $x \in R$ such that $0 \neq xc_1 \in Rc_1 \cap \ell_R(L)$. So $xc_1Rc_1y = 0$. Now there is $c_kxc_1 \neq 0$ for some c_k because $1 = c_1 + \cdots + c_n$. Thus, $c_kxc_1Rc_1yc_j = 0$ for all j. Therefore $c_1yc_j = 0$ for all j, and so $c_1y = 0$. Hence, $c_1 \in \ell_R(L)$. Thus, $Rc_1 \subseteq \ell_R(L)$. Similarly, $Rc_2, \dots, Rc_m \subseteq \ell_R(L)$. So $T \subseteq \ell_R(L)$. Therefore, $\ell_R(L) = T = Rc_1 + \cdots + Rc_m = R(c_1 + \cdots + c_m)$. Put $e = c_1 + \cdots + c_m$. Then $e^2 = e \in R$ and so $\ell_R(L) = Re$.

Next, assume that $Rc_i \cap \ell_R(L) = 0$ for all i. Then $\ell_R(L)Rc_i = 0$ for all i. So $\ell_R(L) = \ell_R(L)(Rc_1 + \cdots + Rc_n) = 0$. Therefore, R is quasi-Baer.

(ii)⇒(iv) Let $Kb_jV = 0$ and $b_jV \neq 0$ for some b_j. Say $y \in V$ with $b_jy \neq 0$. So $0 \neq b_jy = \sum_{t=1}^{n} b_jyb_t$, hence $b_jyb_k \neq 0$ for some b_k.

Let $x \in K$. Then $xb_jRb_jy = 0$. Hence $b_ixb_jRb_jyb_k = 0$ for each b_i. As $b_jyb_k \neq 0$, $b_ixb_j = 0$ for all b_i. Thus $xb_j = \sum_{i=1}^{n} b_ixb_j = 0$, so $Kb_j = 0$. If $Kb_j \neq 0$, similarly $b_jV = 0$.

(iv)⇒(ii) If $b_ixb_jRb_jyb_k = 0$, then $(Rb_ixb_jR)b_j(Rb_jyb_kR) = 0$. By assumption $Rb_ixb_jR = 0$ or $Rb_jyb_kR = 0$, so $b_ixb_j = 0$ or $b_jyb_k = 0$. □

Corollary 5.4.2 *If R has a complete set of primitive idempotents, then the following are equivalent.*

(i) *R is quasi-Baer.*
(ii) *For any given complete set of primitive idempotents $\{e_1, \dots, e_n\}$, if $e_ixe_jRe_jye_k = 0$ for some $x, y \in R$ and some $1 \leq i, j, k \leq n$, then either $e_ixe_j = 0$ or $e_jye_k = 0$.*
(iii) *There is a complete set of primitive idempotents $\{f_1 \dots, f_m\}$ of R such that if $f_ixf_jRf_jyf_k = 0$ for some $x, y \in R$ and some $1 \leq i, j, k \leq m$, then either $f_ixf_j = 0$ or $f_jyf_k = 0$.*
(iv) *For any complete set of primitive idempotents $\{g_1, \dots, g_\ell\}$, assume that $Kg_jV = 0$ for some ideals K and V of R and for some g_j, $1 \leq j \leq \ell$. Then either $Kg_j = 0$ or $g_jV = 0$.*

Proof Let $f \in \mathbf{S}_\ell(R)$ and $0 \neq e^2 = e \in R$. Then $efe \in \mathbf{S}_\ell(eRe)$ by Lemma 5.1.5(ii). In particular, if e is primitive, then $\mathbf{S}_\ell(eRe) = \{0, e\}$. So either $efe = 0$ or $efe = e$. The proof can then be completed by using a similar argument as in the proof of Theorem 5.4.1. □

Definition 5.4.3 A ring R is called a *piecewise domain* (or simply, *PWD*) if there is a complete set of primitive idempotents $\{e_1, \dots, e_n\}$ such that $xy = 0$ implies $x = 0$ or $y = 0$ whenever $x \in e_iRe_j$ and $y \in e_jRe_k$, for $1 \leq i,\ j,\ k \leq n$.

To avoid ambiguity, we sometimes say that R is a PWD with respect to a complete set $\{e_i\}_{i=1}^{n}$ of primitive idempotents. In light of Theorem 5.4.1 and Corollary 5.4.2, it is interesting to compare quasi-Baer rings having a complete set of left triangulating (or primitive) idempotents with PWDs. In fact, Definition 5.4.3 and the equivalence of (i) and (iii) in Theorem 5.4.1 and Corollary 5.4.2 suggest the following definition.

Definition 5.4.4 A quasi-Baer ring with a complete set of triangulating idempotents is called a *piecewise prime ring* (or simply, *PWP ring*).

The following result is somewhat of a right p.q.-Baer analogue of Theorem 3.1.25.

Proposition 5.4.5 *Let R be a right p.q.-Baer ring with* $\mathrm{Tdim}(R) < \infty$. *Then R is a PWP ring.*

Proof Let I be a right ideal of R, and say $I = \sum_{i \in \Lambda} x_i R$ with $x_i \in R$. Then $r_R(I) = \cap_{i \in \Lambda} r_R(x_i R) = \cap_{i \in \Lambda} e_i R$ with $e_i \in \mathbf{S}_\ell(R)$ for each $i \in \Lambda$ because R is right p.q.-Baer. By Theorem 5.2.5 and Proposition 1.2.4(i), there exists $e \in \mathbf{S}_\ell(R)$ such that $\sum_{i \in \Lambda} e_i R = eR$. Therefore R is a PWP ring. □

The next question was posed by Gordon and Small (see [187, p. 554]): *Can a PWD R possess a complete set $\{f_i\}_{i=1}^m$ of primitive idempotents for which it is not true that $xy = 0$ implies $x = 0$ or $y = 0$ for some $x \in f_i R f_k$ and $y \in f_k R f_j$?* Theorem 5.4.1 and Corollary 5.4.2 show that if R is a PWP ring, then it is a PWP ring with respect to *any* complete set of left triangulating idempotents. Thereby for the case of PWP rings it provides an answer to the above question.

Proposition 5.4.6 *Any PWD is a PWP ring.*

Proof The result follows from Proposition 5.2.13 and Corollary 5.4.2. □

The following example illustrates that the converse of Proposition 5.4.6 does not hold true.

Example 5.4.7 (i) Let R be the ring in Example 3.2.7(ii). Then R is a PWP ring, but it is not a PWD.

(ii) Let R be the ring of Example 5.2.17(i). Then R is a prime ring, so it is a PWP ring. But R does not have a complete set of primitive idempotents. Thus, R is not a PWD.

Example 5.4.8 There is a PWD which is not Baer. Let R be a commutative domain which is not semihereditary (e.g., $\mathbb{Z}[x]$). Then $\mathrm{Mat}_n(R)$ is a PWD for any positive integer $n > 1$, but it is not a Baer ring (see Theorem 6.1.4).

Proposition 5.4.9 *Let R be a ring and $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents of R. Then the following are equivalent.*

(i) *R is a PWD with respect to $\{e_1, \ldots, e_n\}$.*

(ii) *Every nonzero element of* $\mathrm{Hom}(e_iR_R, e_jR_R)$ *is a monomorphism for all* i, j, $1 \le i, j \le n$.

(iii) *Every nonzero element of* $\mathrm{Hom}(e_iR_R, R_R)$ *is a monomorphism for all* i, $1 \le i \le n$.

Proof Exercise. □

Example 5.4.10 (i) It is routine to check that the ring of $n \times n$ matrices over a PWD is a PWD.

(ii) The polynomial ring over a PWD is a PWD. Indeed, say R is a PWD with respect to a complete set of primitive idempotent $\{e_1, \dots, e_n\}$. Then $\{e_1, \dots, e_n\}$ is a complete set of primitive idempotents of $R[x]$, and $R[x]$ is a PWD with respect to $\{e_1, \dots, e_n\}$.

(iii) A right Rickart ring with a complete set of primitive idempotents is a PWD. In fact, say R is a right Rickart ring with a complete set $\{e_1, \dots, e_n\}$ of primitive idempotents.

Suppose that $e_ixe_je_jye_k = 0$, where $x, y \in R$ and $1 \le i, j, k \le n$. Since R is right Rickart, $r_R(e_ixe_j) = fR$ for some $f^2 = f \in R$. So $1 - e_j = f(1 - e_j)$ since $1 - e_j \in r_R(e_ixe_j)$. Note that $1 - e_j = \sum_{k \neq j}^n e_k$, thus

$$\sum_{k \neq j} e_k = 1 - e_j = f(1 - e_j) = \sum_{k \neq j} fe_k.$$

Hence $e_k = fe_k$ for $k \neq j$ and $1 \le k \le n$. Therefore,

$$f = \sum_{k=1}^{n} fe_k = \sum_{k \neq j}^{n} fe_k + fe_j = \sum_{k \neq j}^{n} e_k + fe_j.$$

Thus $1 - f = 1 - \sum_{k \neq j}^n e_k - fe_j = e_j - fe_j = (1 - f)e_j$, so $R(1 - f) \subseteq Re_j$. Hence, it follows that $R(1 - f) = Re_j$ or $R(1 - f) = 0$ as e_j is a primitive idempotent.

If $R(1 - f) = Re_j$, then $e_jf = 0$. Because $e_ixe_je_jye_k = e_ixe_jye_k = 0$, we get that $ye_k \in r_R(e_ixe_j) = fR$, and $ye_k = fye_k$. Hence, $e_jye_k = e_jfye_k = 0$. Finally, assume that $R(1 - f) = 0$. Then $f = 1$, and thus $e_ixe_j = 0$. So R is a PWD.

If $R(1 - f) = Re_j$, then $e_jf = 0$. Because $e_ixe_je_jye_k = e_ixe_jye_k = 0$, we get $ye_k \in r_R(e_ixe_j) = fR$, and therefore $ye_k = fye_k$. Hence $e_jye_k = e_jfye_k = 0$.

Further, if $R(1 - f) = 0$, then $f = 1$, and thus $e_ixe_j = 0$. So R is a PWD.

(iv) There exists a PWD which is not right Rickart. Let $R = \mathrm{Mat}_2(\mathbb{Z}[x])$. Then R is a PWD by part (i), but R is not (right) Rickart (see Example 3.1.28).

(v) A right nonsingular ring which is a direct sum of uniform right ideals is a PWD. Indeed, let R be a right nonsingular ring such that $R = \oplus_{i=1}^n I_i$, where each I_i is a uniform right ideal of R. Then there is a complete set of primitive idempotents $\{e_1, \dots, e_n\}$ with $I_i = e_iR$ for each i. As $Z(R_R) = 0$, by Corollary 1.3.15 $E(R_R) = Q(R)$. Now $Q(R)$ is a regular ring from Theorem 2.1.31 and $Q(R) = e_1Q(R) \oplus \cdots \oplus e_nQ(R)$. Also each $e_iQ(R)_{Q(R)}$ is uniform, so $\{e_1, \dots, e_n\}$

is a complete set of primitive idempotents in $Q(R)$. Thus, $Q(R)$ is semisimple Artinian. Say $e_i x e_j e_j y e_k = 0$, where $x, y \in R$ and $1 \le i, j, k \le n$. Then since $Q(R)$ is a PWD with respect to $\{e_1, \ldots, e_n\}$ by part (iii), either $e_i x e_j = 0$ or $e_j y e_k = 0$. So R is a PWD.

Proposition 5.4.11 *Let $\{b_1, \ldots, b_n\}$ be a set of left triangulating idempotents of a ring R. Then the following are equivalent.*

(i) *P is a (minimal) prime ideal of R.*
(ii) *There exist m, $1 \le m \le n$, and a (minimal) prime ideal P_m of the ring $b_m R b_m$ such that $P = P_m + \sum_{k \neq m} b_k R b_k + \sum_{i \neq j} b_i R b_j$.*

Proof The proof is routine. □

Theorem 5.4.12 *Let R be a PWP ring with* $\mathrm{Tdim}(R) = n$. *Then $R = A \bigoplus B$ (ring direct sum) such that:*

(i) *$A = \bigoplus_{i=1}^{k} A_i$ is a direct sum of prime rings A_i.*
(ii) *There exists a ring isomorphism*

$$B \cong \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1m} \\ 0 & B_2 & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_m \end{bmatrix},$$

where each B_i is a prime ring, and B_{ij} is a (B_i, B_j)-bimodule.
(iii) *$n = k + m$.*
(iv) *For each $i \in \{1, \ldots, m\}$ there is $j \in \{1, \ldots, m\}$ such that $B_{ij} \neq 0$ or $B_{ji} \neq 0$.*
(v) *The rings $B_1, \ldots, B_m$ are uniquely determined by B up to isomorphism and permutation.*
(vi) *B has exactly m minimal prime ideals $P_1, \ldots, P_m$, R has exactly n minimal prime ideals of the form $A \oplus P_i$ or $C_i \oplus B$ where $C_i = \bigoplus_{j \neq i} A_j$. Further, $P_1, \ldots, P_m$ are comaximal, $P(R) = P(B)$, and $P(R)^m = 0$.*

Proof Say $E = \{b_1, b_2, \ldots, b_n\}$ is a complete set of left triangulating idempotents of R.

(i) Let $\{e_1, \ldots, e_k\} = E \cap \mathcal{B}(R)$. Take $A_i = e_i R$. By Proposition 3.2.5 and Theorem 3.2.10, each A_i is a prime ring.

(ii) Let $\{f_1, \ldots, f_m\} = E \setminus \{e_1, \ldots, e_k\}$, where the f_i are maintained in the same relative order as they were in E. Let $B_i = f_i B f_i$ and $B_{ij} = f_i B f_j$. Then each B_i is a prime ring by Proposition 3.2.5 and Theorem 3.2.10. Define ϕ by $\phi(b) = [f_i b f_j]$ for $b \in B$, as in the proof of Theorem 5.1.4. Then ϕ is a ring isomorphism.

(iii) The proof follows immediately from the proof of part (ii).

(iv) It is evident since $\{f_1, \ldots, f_m\} = E \setminus \{e_1, , \ldots, e_k\}$.

(v) This is a consequence of Theorem 5.2.8.

(vi) The proof follows from a routine argument using Lemma 5.4.11. □

Corollary 5.4.13 (i) *Any semiprime PWP ring is a finite direct sum of prime rings.*
(ii) *Any biregular ring R with* $\mathrm{Tdim}(R) < \infty$ *is a finite direct sum of simple rings.*

Proof The proof follows from Theorems 5.4.12 and 3.2.22(ii). □

The next corollary is related to Michler's splitting theorem [299, Theorem 2.2] for right hereditary right Noetherian rings.

Corollary 5.4.14 *Let R be a right hereditary right Noetherian ring. Then*

$$R \cong \begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix},$$

where each R_i is a prime right hereditary, right Noetherian ring, and each R_{ij} is an (R_i, R_j)-bimodule.

Proof As R is right hereditary right Noetherian, R is Baer by Theorem 3.1.25. Thus the proof follows from Theorem 5.4.12 and Proposition 5.2.14. □

We will now see that Levy's decomposition theorem [279] for semiprime right Goldie right hereditary rings, follows as a consequence of Theorem 5.4.12.

Corollary 5.4.15 *Any semiprime right Goldie, right hereditary ring is a finite direct sum of prime right Goldie, right hereditary rings.*

Proof Let R be a semiprime right Goldie, right hereditary ring. Then R is orthogonally finite, so R is Baer by Theorem 3.1.25 and $\mathrm{Tdim}(R) < \infty$ from Proposition 5.2.13(ii). Corollary 5.4.13(i) and a routine verification yield that R is a finite direct sum prime right Goldie, right hereditary rings. □

A ring R is called *right FPF* if every faithful finitely generated right R-module generates the category Mod-R of right R-modules (see [156]). We may note that a semiprime right FPF ring is quasi-Baer (see [78, Corollary 1.19]). By Theorem 5.4.12, Faith's characterization of semiprime right FPF rings with no infinite set of central orthogonal idempotents (see [156, Theorem I.4]) is provided as follows.

Corollary 5.4.16 *Let R be a ring with no infinite set of central orthogonal idempotents. Then R is semiprime right FPF if and only if R is a finite direct sum of prime right FPF rings.*

Proof Let R be a semiprime right FPF ring with no infinite set of central orthogonal idempotents. Because R is semiprime, $\mathcal{B}(R) = \mathbf{S}_\ell(R)$ by Proposition 1.2.6(ii). Since

R has no infinite set of central orthogonal idempotents, we see that

$$\{eR \mid e \in \mathbf{S}_\ell(R)\} = \{eR \mid e \in \mathcal{B}(R)\}$$

has ACC and DCC. By Theorem 5.2.5, $\mathrm{Tdim}(R) < \infty$, so R is a PWP ring. By Corollary 5.4.13(i), R is a finite direct sum of prime rings. Since ring direct summands of right FPF rings are right FPF, these prime rings are right FPF. The converse is immediate. □

A ring R for which the diagonal rings R_i in a complete generalized triangular matrix representation are simple Artinian, is called a *TSA ring*. Recall from 1.1.14 that if R is a right (or left) perfect ring, then $J(R) = P(R)$. Thus any prime right (or left) perfect ring is simple Artinian.

By Theorem 5.4.12, every quasi-Baer right (or left) perfect ring is a TSA ring. So Teply's result [391] given next follows from Theorem 5.4.12 since an orthogonally finite right Rickart ring is Baer by Theorem 3.1.25.

Corollary 5.4.17 *A right (or left) perfect right Rickart ring is a semiprimary TSA ring.*

For a π-regular Baer ring with only countably many idempotents, we obtain the following.

Corollary 5.4.18 *A π-regular Baer ring with only countably many idempotents is a semiprimary TSA ring.*

Proof Theorems 3.1.11, 3.1.26, and 5.4.12 yield the result. □

Corollary 5.4.19 *Assume that R is a PWP ring with $\mathrm{Tdim}(R) = n$. Then the following are equivalent.*

(i) $\mathrm{r.gl.dim}(R) < \infty$.
(ii) $\mathrm{r.gl.dim}(R/P(R)) < \infty$.
(iii) $\mathrm{r.gl.dim}(R_1 + \cdots + R_n) < \infty$, *where the R_i are the diagonal rings in the complete generalized triangular matrix representation of R.*

Proof (i)⇔(iii) is a direct consequence of Theorem 5.3.5. From Theorem 5.4.12, $R/P(R) \cong R_1 \oplus \cdots \oplus R_n$. Hence, (ii)⇔(iii) follows immediately. □

Theorem 5.4.20 *Let R be a right p.q.-Baer ring. Then $\mathrm{Tdim}(R) = n$ if and only if R has exactly n minimal prime ideals.*

Proof Assume that $\mathrm{Tdim}(R) = n$. By Proposition 5.4.5, R is a PWP ring. Thus from Theorem 5.4.12, R has exactly n minimal prime ideals.

Conversely, let R have exactly n minimal prime ideals. We proceed by induction on n. First, say $n = 1$. If $\mathrm{Tdim}(R) \neq 1$, then R is not semicentral reduced. So there is $0 \neq b \in \mathbf{S}_\ell(R)$ with $b \neq 1$. Then bRb and $(1-b)R(1-b)$ each have at least one minimal prime ideal. Note that $\{b, 1-b\}$ is a set of left triangulating idempotents of R. Thus, by Proposition 5.4.11, R has at least two minimal prime ideals, a contradiction. Hence, $\mathrm{Tdim}(R) = 1$.

Suppose that $n > 1$. If R is semicentral reduced, then R is prime by Proposition 3.2.25. So $n = 1$, a contradiction. Thus R is not semicentral reduced, hence there is $0 \neq d \in \mathbf{S}_\ell(R)$ and $d \neq 1$. By Theorem 3.2.34(i), both dRd and $(1-d)R(1-d)$ are right p.q.-Baer rings. We note that $\{d, 1-d\}$ is a set of left triangulating idempotents. From Proposition 5.4.11, there are some positive integers k_1 and k_2 such that dRd and $(1-d)R(1-d)$ have exactly k_1 and k_2 number of minimal prime ideals, respectively, where $k_1 + k_2 = n$.

By induction, $\mathrm{Tdim}(dRd) + \mathrm{Tdim}((1-d)R(1-d)) = k_1 + k_2 = n$. From Theorem 5.2.19, $\mathrm{Tdim}(R) = n$. □

Corollary 5.4.21 *The PWP property is Morita invariant.*

Proof Assume that R and S are Morita equivalent rings. Suppose that R is a PWP ring and let $\mathrm{Tdim}(R) = n$. By Theorem 5.4.20, R has exactly n minimal prime ideals. Since R is quasi-Baer, S is also quasi-Baer from Theorem 3.2.11. Now S has also exactly n minimal prime ideals because R and S are Morita equivalent (see [262, Proposition 18.44 and Corollary 18.45]). Thus $\mathrm{Tdim}(S) = n$ by Theorem 5.4.20, so S is also a PWP ring. □

The next example illustrates that the right p.q.-Baer condition is not superfluous in Theorem 5.4.20.

Example 5.4.22 There exists a ring R such that:

(i) R has only two minimal prime ideals.
(ii) $\mathrm{Tdim}(R) = 1$.

Indeed, we let $F\{X, Y\}$ be the free algebra over a field F, and we put $R = F\{X, Y\}/I$, where I is the ideal of $F\{X, Y\}$ generated by YX. Say $x = X + I$ and $y = Y + I$ in R. Then $R/RxR \cong F[y]$ and $R/RyR \cong F[x]$, so RxR and RyR are prime ideals of R. As $yx = 0$, we see that $(RyR)(RxR) = 0$. So, if P is a prime ideal, then either $RyR \subseteq P$ or $RxR \subseteq P$. Thus RxR and RyR are the only two minimal prime ideals of R. We can verify that all idempotents of R are only 0 and 1. In particular, R is semicentral reduced, so $\mathrm{Tdim}(R) = 1$.

Let R be a quasi-Baer (resp., Baer) ring with $\mathrm{Tdim}\,(R) < \infty$. Then $P(R)$ is nilpotent and $R/P(R)$ is a finite direct sum of prime (resp., Baer) rings from Theorem 5.4.12, so $R/P(R)$ is a quasi-Baer (resp., Baer) ring (cf. Example 3.2.42). There is a quasi-Baer ring R with $P(R)$ nilpotent, but $\mathrm{Tdim}(R)$ is infinite. Let $R = T_2(\prod_{n=1}^{\infty} F_n)$, where F is a field, and $F_n = F, n = 1, 2, \ldots$. In this case, $P(R)^2 = 0$, but $\mathrm{Tdim}(R) = \infty$.

An R-module M is said to satisfy the *restricted minimum condition* if, for every essential submodule N of M, the module M/N is Artinian.

Lemma 5.4.23 *Let R be a hereditary Noetherian ring. Then both R_R and ${}_RR$ satisfy the restricted minimum condition.*

Proof Assume that $J_R \leq^{\text{ess}} R_R$. Then J_R is finitely generated projective because R is right hereditary and right Noetherian. From Dual Basis lemma (see [262, Lemma 2.9]), there are $a_1, \dots, a_n \in J$ and $f_1, \dots, f_n \in \text{Hom}(J_R, R_R)$ such that $x = a_1 f_1(x) + \cdots + a_n f_n(x)$ for each $x \in J$. Because $Z(R_R) = 0$ from Proposition 3.1.18, $J_R \leq^{\text{den}} R_R$ by Proposition 1.3.14. Thus, it follows that $f_i \in Q(R)$ for $i = 1, \dots, n$, so $a_1 f_1 + \cdots + a_n f_n \in Q(R)$. We note that $a_1 f_1 + \cdots + a_n f_n = 1$ in $Q(R)$ as $a_1 f_1 + \cdots + a_n f_n$ is the identity map of J.

Put $D(J) = \text{Hom}\,(J_R, R_R)$. Then $Rf_1 + \cdots + Rf_n \subseteq D(J)$ because $D(J)$ is a left R-module. Let $q \in D(J)$. Then $qJ \subseteq R$ and so

$$q = q(a_1 f_1 + \cdots + a_n f_n) = qa_1 f_1 + \cdots + qa_n f_n \in Rf_1 + \cdots + Rf_n$$

since each $a_i \in J$. So $D(J) = Rf_1 + \cdots + Rf_n$.

Furthermore, $J = \{r \in R \mid D(J)r \subseteq R\}$. Indeed, first obviously we have that $J \subseteq \{r \in R \mid D(J)r \subseteq R\}$. Next, we take $r \in R$ such that $D(J)r \subseteq R$. Then

$$r = a_1 f_1 r + \cdots + a_n f_n r \in a_1 D(J)r + \cdots + a_n D(J)r \subseteq JR \subseteq J$$

since $1 = a_1 f_1 + \cdots + a_n f_n$ in $Q(R)$. So $J = \{r \in R \mid D(J)r \subseteq R\}$.

We show that R_R satisfies the restricted minimum condition. For this, we now let $I_1 \supseteq I_2 \supseteq \dots$ be a descending chain of right ideals of R all containing a fixed essential right ideal I of R. Then $D(I_1) \subseteq D(I_2) \subseteq \dots$ and all $D(I_i)$ are contained in the left R-module $D(I)$. By the preceding argument, $D(I)$ is finitely generated as a left R-module.

Since R is left Noetherian, $D(I)$ is Noetherian as a left R-module. So there exists a positive integer n such that $D(I_n) = D(I_{n+1}) = \dots$. Therefore, we have that $\{r \in R \mid D(I_n)r \subseteq R\} = \{r \in R \mid D(I_{n+1})r \subseteq R\} = \dots$. Hence $I_n = I_{n+1} = \dots$, so R_R satisfies the restricted minimum condition. Similarly, ${}_RR$ has the restricted minimum condition. □

As another application of Theorem 5.4.12, Chatters' decomposition theorem [117] for hereditary Noetherian rings is shown as follows.

Theorem 5.4.24 *If R is a hereditary Noetherian ring, then $R = A \oplus B$ (ring direct sum), where A is a finite direct sum of prime rings and B is an Artinian TSA ring.*

Proof Note that a hereditary Noetherian ring is Baer by Theorem 3.1.25. Thus R is a PWP ring. Therefore, $R = A \oplus B$ as in Theorem 5.4.12.

We claim that B is an Artinian TSA ring. For this, say $\{f_1, \dots, f_m\}$ is a complete set of left triangulating idempotents of B as in the proof of Theorem 5.4.12. We need to show that each B_i is simple Artinian. By Theorem 5.4.12, for given i, $1 \leq i \leq m$ there exists j, $1 \leq j \leq m$ such that either $B_{ij} \neq 0$ or $B_{ji} \neq 0$. We may assume that $B_{ij} \neq 0$ and $i < j$. Now $B_i = f_i B f_i$, $B_{ij} = f_i B f_j$, and $B_j = f_j B f_j$. Consider

$$S = (f_i + f_j)B(f_i + f_j) \cong \begin{bmatrix} B_i & B_{ij} \\ 0 & B_j \end{bmatrix}.$$

Then S is a hereditary Noetherian ring. Also $\{f_i, f_j\}$ is a complete set of left triangulating idempotents of S. Since B is Baer, so is S by Theorem 3.1.8. Therefore, S is a PWP ring.

We show that B_{ij} is a faithful left B_i-module. For this, let $f_i b f_i \in B_i$ with $b \in B$ such that $f_i b f_i B_{ij} = 0$. Since $f_i B f_j = B_{ij} \neq 0$, there exists $y \in B$ such that $f_i y f_j \neq 0$. Now $(f_i b f_i)(f_i B f_i y f_j) \subseteq (f_i b f_i)(f_i B f_j) = 0$, and so we have that $f_i b f_i B f_i y f_j = (f_i b f_i)(f_i B f_i y f_j) = 0$. Since $f_i y f_j \neq 0$, $f_i b f_i = 0$ from Theorem 5.4.1. Therefore, B_{ij} is a faithful left B_i-module. Similarly, B_{ij} is a faithful right B_j-module. Let

$$V_1 = \begin{bmatrix} 0 & B_{ij} \\ 0 & B_j \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} B_i & B_{ij} \\ 0 & 0 \end{bmatrix}.$$

The ideal V_1 of S is right essential in S since B_{ij} is a faithful left B_i-module. Also the ideal V_2 of S is left essential in S. Since both S_S and ${}_SS$ satisfy the restricted minimum condition by Lemma 5.4.23, S/V_1 is a right Artinian S-module, while S/V_2 is a left Artinian S-module. Now to show that B_i is a right Artinian ring, we let $I_1 \supseteq I_2 \supseteq \dots$ be a descending chain of right ideals of B_i. Put

$$K_\ell = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + V_1 \in S/V_1 \mid \alpha \in I_\ell \right\}$$

for $\ell = 1, 2, \dots$. Then we see that each K_ℓ is a right S-submodule of $(S/V_1)_S$ and $K_1 \supseteq K_2 \supseteq \dots$. Since $(S/V_1)_S$ is Artinian, $K_t = K_{t+1} = \dots$ for some positive integer t. So $I_t = I_{t+1} = \dots$. Therefore, B_i is a right Artinian ring. Similarly, B_j is a left Artinian ring. Since B_i and B_j are prime rings by Theorem 5.4.12, B_i and B_j are simple Artinian rings.

The preceding argument is applied to show that all B_i are simple Artinian rings. Now $J(B) = \sum_{i \neq j} B_{ij}$ is nilpotent and $B/J(B) = B_1 \oplus \cdots \oplus B_m$. Hence, B is semiprimary Noetherian. So B is an Artinian TSA ring. □

To obtain a structure theorem for PWDs, we need the next lemma.

Lemma 5.4.25 *If R is a PWD and $0 \neq e \in \mathbf{S}_\ell(R) \cup \mathbf{S}_r(R)$, then the ring eRe is also a PWD.*

Proof Say $e \in \mathbf{S}_\ell(R)$. Let R be a PWD with respect to a complete set of primitive idempotents $\{e_1, \dots, e_n\}$. Since $e \in \mathbf{S}_\ell(R)$, $e_i e = e e_i e$ is an idempotent for each i. As e_i is primitive and $e_i e R \subseteq e_i R$, either $e_i e = 0$ or $e_i e R = e_i R$. If necessary, rearrange $\{e_1, \dots, e_n\}$ so that $J = \{1, \dots, r\}$ is the set of all indices such that $e_i e \neq 0$ for all $i \in J$. Then $e = (e_1 + \cdots + e_n)e = e_1 e + \cdots + e_r e$ and

$$eR = e_1 e R + \cdots + e_r e R = e_1 R + \cdots + e_r R.$$

Further, by Lemma 5.2.12, $\{e e_1 e, \dots, e e_r e\}$ is a complete set of primitive idempotents in eRe.

Assume that $x \in (ee_ie)(eRe)(ee_je)$ and $y \in (ee_je)(eRe)(ee_ke)$ with $xy = 0$ for $1 \le i, j, k \le r$. Put $x = (ee_ie)(eae)(ee_je)$ and $y = (ee_je)(ebe)(ee_ke)$ with $a, b \in R$. Then $x = e_iae_je$ since $e \in \mathbf{S}_\ell(R)$. Similarly, $y = e_jbe_ke$. Thus $xy = e_iae_jee_jbe_ke = e_iae_jbe_ke = 0$. So $e_iae_je_jbe_keR = e_iae_je_jbe_kR = 0$ since $e_keR = e_kR$. Hence $(e_iae_j)(e_jbe_k) = 0$, so $e_iae_j = 0$ or $e_jbe_k = 0$ as R is a PWD. Thus $x = 0$ or $y = 0$. Therefore, eRe is a PWD with respect to the complete set of primitive idempotents $\{ee_1e, \ldots, ee_re\}$. Similarly, when $e \in \mathbf{S}_r(R)$, we see that eRe is a PWD. □

As yet another application of Theorem 5.4.12, we obtain the next theorem, due to Gordon and Small [187], which describes the structure of a PWD.

Theorem 5.4.26 *Assume that R is a PWD. Then*

$$R \cong \begin{bmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix},$$

where each R_i is a prime PWD and each R_{ij} is an (R_i, R_j)-bimodule. The integer n is unique and the ring R_i is unique up to isomorphism. Furthermore,

$$R_i \cong \begin{bmatrix} D_1 & \cdots & D_{1n_i} \\ \vdots & \ddots & \vdots \\ D_{n_i1} & \cdots & D_{n_i} \end{bmatrix},$$

where each D_i is a domain and each D_{jk} is isomorphic as a right D_k-module to a nonzero right ideal in D_k, and as a left D_j-module to a nonzero left ideal in D_j.

Proof Let R be a PWD. By Proposition 5.4.6, R is a PWP ring. The uniqueness of n and that of the ring R_i up to isomorphism follow from Theorem 5.2.8 or Theorem 5.4.12.

Say $\{b_1, \ldots, b_n\}$ is a complete set of left triangulating idempotents of R. By Theorem 5.4.12, each $R_i = b_iRb_i$ is a prime ring. From Lemma 5.4.25, $R_1 = b_1Rb_1$ and $(1 - b_1)R(1 - b_1)$ are PWDs.

We observe that $0 \ne b_2 \in \mathbf{S}_\ell((1 - b_1)R(1 - b_1))$. Thus, Lemma 5.4.25 yields that $R_2 = b_2Rb_2 = b_2(1 - b_1)R(1 - b_1)b_2$ is a PWD. By the same method, we see that each $R_i = b_iRb_i$ is a PWD. Hence, there exists a complete set of primitive idempotents $\{c_1, \ldots, c_{n_i}\}$ for R_i such that $c_jxc_kyc_q = 0$ implies that $c_jxc_k = 0$ or $c_kyc_q = 0$, for $x, y \in R_i$. Put $D_{jk} = c_jR_ic_k$ and $D_i = D_{ii}$. Then each D_i is a domain.

As R_i is a prime ring and $0 \ne c_k, 0 \ne c_j \in R_i$, it follows that $c_kR_ic_j \ne 0$. We let $0 \ne x \in c_kR_ic_j$. Then $c_jR_ic_k$ is isomorphic to a nonzero right ideal $xc_jR_ic_k$ of $c_kR_ic_k$ as a right $c_kR_ic_k$-module since R_i is a PWD with respect to the complete set of primitive idempotents $\{c_1, \ldots, c_{n_i}\}$. Similarly $c_jR_ic_k$ is isomorphic to a nonzero left ideal of $c_jR_ic_j$ as a left $c_jR_ic_j$-module. □

Exercise 5.4.27

1. Prove Propositions 5.4.9 and 5.4.11.
2. Show that if R is a PWD, then $\mathrm{Mat}_n(R)$ is a PWD for every positive integer n (see Example 5.4.10(i)).
3. ([66, Birkenmeier and Park]) Assume that R is a ring and X is a nonempty set of not necessarily commuting indeterminates. Show that R is quasi-Baer with $\mathrm{Tdim}(R) = n$ if and only if Γ is quasi-Baer with $\mathrm{Tdim}(\Gamma) = n$, where Γ is any of the following ring extensions of R.
   (i) $R[X]$. (ii) $R[x, x^{-1}]$. (iii) $R[[x, x^{-1}]]$. (iv) $\mathrm{Mat}_k(R)$ for every positive integer k.
4. ([82, Birkenmeier, Kim, and Park]) Prove that the following conditions are equivalent for a ring R.
   (i) R is a TSA ring.
   (ii) R is a left perfect ring such that there exists a numbering of all the distinct prime ideals $P_1, P_2, \ldots, P_n$ of R such that $P_1 P_2 \cdots P_n = 0$.
   (iii) R is a left perfect ring such that some product of distinct prime ideals, without repetition, is zero.
5. Let R be a quasi-Baer ring such that $\mathbf{S}_\ell(R)$ is a countable set. Show that R is a PWP ring. Additionally, if R is also biregular, then R is a direct sum of simple rings (cf. Corollary 5.4.13(ii)).

5.5 A Sheaf Representation of Piecewise Prime Rings

After a brief discussion on certain ideals in a quasi-Baer ring, PWP rings with a sheaf representation will be studied in this section. Quasi-Baer rings with a nontrivial subdirect product representation will also be discussed.

The set of all prime ideals and the set of all minimal prime ideals of a ring R is denoted by *Spec* (R) and *MinSpec*(R), respectively. For a subset X of R, let $\mathrm{supp}(X) = \{P \in \mathrm{Spec}(R) \mid X \not\subseteq P\}$, which is called the support of X. In case, $X = \{s\}$, we write $\mathrm{supp}(s)$.

For any $P \in \mathrm{Spec}(R)$, there is $s \in R \setminus P$ and so $P \in \mathrm{supp}(s)$. Thus the family $\{\mathrm{supp}(s) \mid s \in R\}$ covers $\mathrm{Spec}(R)$. Also for $P \in \mathrm{supp}(x) \cap \mathrm{supp}(y)$, $d = xcy \notin P$ for some $c \in R$. So $P \in \mathrm{supp}(d) \subseteq \mathrm{supp}(x) \cap \mathrm{supp}(y)$. Therefore, $\{\mathrm{supp}(s) \mid s \in R\}$ forms a base (for open sets) on $\mathrm{Spec}(R)$. This induced topology on $\mathrm{Spec}(R)$ is called the hull-kernel topology on $\mathrm{Spec}(R)$.

For $P \in \mathrm{Spec}(R)$, let $O(P) = \{a \in R \mid aRs = 0 \text{ for some } s \in R \setminus P\}$. Then $O(P)$ is an ideal of R, $O(P) = \sum_{s \in R \setminus P} \ell_R(Rs)$, and $O(P) \subseteq P$. We let

$$\mathfrak{K}(R) = \bigcup_{P \in \mathrm{Spec}(R)} R/O(P)$$

be the *disjoint* union of the rings $R/O(P)$, where P ranges through $\mathrm{Spec}(R)$.

For $a \in R$, define $\widehat{a} : \mathrm{Spec}(R) \to \mathfrak{K}(R)$ by $\widehat{a}(P) = a + O(P)$. Then it can be verified that $\mathfrak{K}(R)$ is a sheaf of rings over $\mathrm{Spec}(R)$ with the topology on $\mathfrak{K}(R)$ generated by $\{\widehat{a}(\mathrm{supp}(s)) \mid a, s \in R\}$. By a *sheaf representation* of a ring R, we mean a sheaf representation whose base space is $\text{Spec}(R)$ and whose stalks are the $R/O(P)$, where $P \in \text{Spec}(R)$. Let $\Gamma(\text{Spec}(R), \mathfrak{K}(R))$ be the set of all global sections. We remark that $\Gamma(\text{Spec}(R), \mathfrak{K}(R))$ becomes a ring (see [345, 3.1], [209], and [369] for more details).

It is well-known that $\widehat{a}$ is a global section for $a \in R$. Next, for $a, b \in R$ and $P \in \text{Spec}(R)$, $(\widehat{a} + \widehat{b})(P) = a + b + O(P)$ and $(\widehat{a}\,\widehat{b})(P) = ab + O(P)$. Therefore we see that the map

$$\theta : R \to \Gamma(\text{Spec}(R), \mathfrak{K}(R))$$

defined by $\theta(a) = \widehat{a}$ is a ring homomorphism, which is called the Gelfand homomorphism. Furthermore, $\text{Ker}(\theta) = \bigcap_{P \in \text{Spec}(R)} O(P)$, which is 0 (see Proposition 5.5.7). Thus θ is a monomorphism.

We discuss some relevant properties of $O(P)$ and $R/O(P)$ for the previously mentioned sheaf representation of PWP rings.

Proposition 5.5.1 *Let R be a quasi-Baer ring and P a prime ideal of R. Then $O(P) = \sum Rf$, where the sum is taken for all $f \in \mathbf{S}_r(R) \cap P$.*

Proof Note that $O(P) = \sum_{s \in R \setminus P} \ell_R(Rs)$. As R is quasi-Baer, $\ell_R(Rs) = Rf$ with $f \in \mathbf{S}_r(R)$. Then $f \in P$ because $fRs = 0$ and $s \notin P$. Next let $f \in \mathbf{S}_r(R) \cap P$. Then $f \in O(P)$ since $fR(1 - f) = 0$ (Proposition 1.2.2) and $1 - f \in R \setminus P$. Thus, we get the desired result. □

Corollary 5.5.2 *Let R be a quasi-Baer ring. If P and Q are prime ideals such that $P \subseteq Q$, then $O(P) = O(Q)$.*

Proof From the definition, we see that $O(Q) \subseteq O(P)$. Proposition 5.5.1 yields that $O(P) \subseteq O(Q)$, so $O(P) = O(Q)$. □

We remark that Proposition 5.5.1 and Corollary 5.5.2 hold true when R is a left p.q.-Baer ring.

Proposition 5.5.3 *Assume that R is a PWP ring and P is a prime ideal. Then $O(P) = Re$ for some $e \in \mathbf{S}_r(R)$.*

Proof As R has a complete set of triangulating idempotents, $\{Rb \mid b \in \mathbf{S}_r(R)\}$ is a finite set by the left-sided version of Theorem 5.2.5. From Proposition 5.5.1, $O(P) = \sum Rf$, where the sum is taken for all $f \in \mathbf{S}_r(R) \cap P$. Therefore, $O(P) = Rf_1 + \cdots + Rf_k$ with $f_i \in \mathbf{S}_r(R)$. By Proposition 1.2.4(ii), $O(P) = Re$ for some $e \in \mathbf{S}_r(R)$. □

Let R be a ring and S be a multiplicatively closed subset of R (i.e., $1 \in S$ and $s, t \in S$ implies $st \in S$). A ring RS^{-1} is called a *right ring of fractions* of R with respect to S together with a ring homomorphism $\phi : R \to RS^{-1}$ if the following are satisfied:

(i) $\phi(s)$ is invertible for every $s \in S$.

(ii) Each element in RS^{-1} has the form $\phi(a)\phi(s)^{-1}$ with $a \in R$ and $s \in S$.
(iii) $\phi(a) = 0$ with $a \in R$ if and only if $as = 0$ for some $s \in S$.

Proposition 5.5.4 *Let R be a ring and S a multiplicatively closed subset of R. Then RS^{-1} exists if and only if S satisfies*:

S1. *If $s \in S$ and $a \in R$, then there exist $t \in S$ and $b \in R$ with $sb = at$.*
S2. *If $sa = 0$ with $a \in R$ and $s \in S$, then $at = 0$ for some $t \in S$.*

Proof See [382, Proposition 1.4, p. 51] for the proof. □

When RS^{-1} exists, it has the form $RS^{-1} = (R \times S)/\sim$, where $\sim$ is the equivalence relation defined as $(a, s) \sim (b, t)$ if there exist $c, d \in R$ such that $sc = td \in S$ and $ac = bd$. A multiplicatively closed subset with S1 and S2 is called a *right denominator set*. In particular, if R is a right Ore ring and S is the set of all nonzero-divisors in R, then S is a right denominator set. Thus RS^{-1} exists by Proposition 5.5.4 and $Q^r_{c\ell}(R) = RS^{-1}$ (see 1.1.17).

Proposition 5.5.5 *Assume that P is a prime ideal of a ring R and let $S_P = \{e \in \mathbf{S}_\ell(R) \mid e \notin P\}$. Then RS_P^{-1} exists.*

Proof Obviously $1 \in S_P$. To see that S_P is a multiplicatively closed subset, let $e, f \in S_P$. Then $ef \in \mathbf{S}_\ell(R)$ by Proposition 1.2.4(i). If $ef \in P$, then $efRf \subseteq P$. Therefore $eRf = efRf \subseteq P$, a contradiction. Thus, $ef \notin P$. So $ef \in S_P$ and hence S_P is a multiplicatively closed subset of R.

For $e \in S_P$ and $a \in R$, we have that $e(ae) = ae$. So the condition S1 is satisfied. Next for S2, take $e \in S_P$ and $a \in R$ such that $ea = 0$. Then

$$ae = (1 - e)ae = (1 - e)eae = 0,$$

so the condition S2 is satisfied. Hence S_P is a denominator set. Thus, RS_P^{-1} exists from Proposition 5.5.4. □

When R is a quasi-Baer ring, we obtain the next result for stalks $R/O(P)$.

Theorem 5.5.6 *Assume that R is a quasi-Baer ring and P is a prime ideal of R. Then $RS_P^{-1} \cong R/O(P)$.*

Proof First we show that $O(P) = \{a \in R \mid ae = 0 \text{ for some } e \in S_P\}$. Indeed, if $a \in R$ such that $ae = 0$ with $e \in S_P$, then $aRe = aeRe = 0$ and so $a \in O(P)$. Thus $I := \{a \in R \mid ae = 0 \text{ for some } e \in S_P\} \subseteq O(P)$. To see that $O(P) \subseteq I$, first we prove that $I \trianglelefteq R$. For this, say $a_1, a_2 \in I$ with $a_1e_1 = 0$ and $a_2e_2 = 0$ for some $e_1, e_2 \in S_P$. Then $(a_1 + a_2)e_1e_2 = a_2e_1e_2 = a_2e_2e_1e_2 = 0$. By Proposition 5.5.5, S_P is a multiplicatively closed set, hence $e_1e_2 \in S_P$. So $a_1 + a_2 \in I$. Let $a \in I$ and $r \in R$. Clearly $ra \in I$. Say $e \in S_P$ such that $ae = 0$. Then $are = aere = 0$, so $ar \in I$. Therefore $I \trianglelefteq R$.

Now say $f \in \mathbf{S}_r(R) \cap P$. Then $1 - f \notin P$ and $1 - f \in \mathbf{S}_\ell(R)$. Hence $1 - f \in S_P$, so $f \in I$. By Proposition 5.5.1, $O(P) \subseteq I$. Thus $O(P) = I$.

From Proposition 5.5.5, RS_P^{-1} exists and there is a ring homomorphism ϕ from R to RS_P^{-1}, where $RS_P^{-1} = \{\phi(a)\phi(e)^{-1} \mid a \in R \text{ and } e \in S_P\}$. Now we observe that $O(P) = I$, so $\mathrm{Ker}(\phi) = O(P)$.

Further, for each $e \in S_P$, note that $\phi(e)^2 = \phi(e) \in RS_P^{-1}$, which is invertible. Thus $\phi(e) = 1$ for every $e \in S_P$. So $RS_P^{-1} = \phi(R)$ and $\mathrm{Ker}(\phi) = O(P)$. Hence we get that $RS_P^{-1} \cong R/O(P)$. □

Recall that a ring R is a subdirect product of rings S_i, $i \in \Lambda$, if $S_i \cong R/K_i$, where $K_i \trianglelefteq R$ and $\cap_{i\in\Lambda} K_i = 0$. A subdirect product is *nontrivial* if $K_i \neq 0$ for all $i \in \Lambda$. Otherwise, it is *trivial*.

Proposition 5.5.7 *Let R be a ring. Then $\bigcap_{P\in\mathrm{Spec}(R)} O(P) = 0$. Thus R has a subdirect product representation of $\{R/O(P) \mid P \in \mathrm{Spec}(R)\}$.*

Proof Assume that $\cap_{P\in\mathrm{Spec}(R)} O(P) \neq 0$. Let $0 \neq a \in \cap_{P\in\mathrm{Spec}(R)} O(P)$. Then $r_R(aR)$ is a proper ideal of R. Let P_0 be a prime ideal such that $r_R(aR) \subseteq P_0$. Because $a \in \cap_{P\in\mathrm{Spec}(R)} O(P) \subseteq O(P_0)$, $aRs = 0$ with $s \in R \setminus P_0$. Therefore $s \in r_R(aR) \subseteq P_0$, a contradiction. So $\cap_{P\in\mathrm{Spec}(R)} O(P) = 0$. □

The following example shows that the subdirect product representation in Proposition 5.5.7 may be trivial.

Example 5.5.8 For a field F, let $R = T_2(F)$. Then R is quasi-Baer. Let $e_{ij} \in T_2(F)$ be the matrix with 1 in the (i, j)-position and 0 elsewhere. Put $P = Fe_{11} + Fe_{12}$ and $Q = Fe_{12} + Fe_{22}$. Then we see that R has only two prime ideals which are P and Q (see Proposition 5.4.11). Hence, $O(P) = 0$ and $O(Q) = Q$ by using Proposition 5.5.1.

Next, we consider the subdirect product representation of Proposition 5.5.7 for quasi-Baer rings. Corollary 5.5.2 suggests that we may be able to improve the subdirect product representation by reducing the number of components through using only the minimal prime ideals. So it is natural to consider suitable conditions under which $\cap_{P\in\mathrm{MinSpec}(R)} O(P) = 0$. The next example illustrates that there is a ring R such that $\cap_{P\in\mathrm{MinSpec}(R)} O(P) \neq 0$.

Example 5.5.9 Assume that R is the Dorroh extension of $S = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ by $\mathbb{Z}$ (i.e., the ring formed from $S \times \mathbb{Z}$ with componentwise addition and multiplication given by $(x, k)(y, m) = (xy + mx + ky, km)$). Let e_{ij} be the matrix in S with 1 in the (i, j)-position and 0 elsewhere.

Put $e = (e_{11}, 0) \in R$. Then $e \in \mathbf{S}_\ell(R)$, so $(1 - e)Re = 0$ by Proposition 1.2.2. Also $eRe = (\mathbb{Z}_2 e_{11}, 0)$, $(1_R - e)R(1_R - e) = \{(me_{11}, m) \mid m \in \mathbb{Z}\}$, and $P(R) = eR(1 - e) = (\mathbb{Z}_2 e_{12}, 0)$ (note that $1 := 1_R = (0, 1) \in R$). Since

$$R \cong \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix},$$

all the minimal prime ideals of R are $P_1 := Q_1 + eR(1-e) + (1-e)R(1-e)$ and $P_2 := eRe + eR(1-e) + Q_2$, where Q_1 and Q_2 are minimal prime ideals of eRe and $(1-e)R(1-e)$, respectively by Proposition 5.4.11.

As $eRe \cong \mathbb{Z}_2$ and $(1-e)R(1-e) \cong \mathbb{Z}$, $Q_1 = 0$ and $Q_2 = 0$. So

$$P_1 = \{(me_{11} + ne_{12}, m) \mid m, n \in \mathbb{Z}\} \text{ and } P_2 = (\mathbb{Z}_2 e_{11} + \mathbb{Z}_2 e_{12}, 0).$$

Take $\alpha = (e_{12}, 0) \in R$. Then $\alpha R = (\mathbb{Z}_2 e_{12}, 0)$. Now say $s_1 = e = (e_{11}, 0)$ and $s_2 = (0, 2)$. Then $\alpha R s_1 = 0$ with $s_1 \in R \setminus P_1$, and $\alpha R s_2 = 0$ with $s_2 \in R \setminus P_2$. Hence, $0 \neq \alpha \in O(P_1) \cap O(P_2) = \cap_{P \in \text{MinSpec}(R)} O(P)$.

In spite of Example 5.5.9, we have the following.

Lemma 5.5.10 *If R is a quasi-Baer ring, then $\bigcap_{P \in \text{MinSpec}(R)} O(P) = 0$.*

Proof For a minimal prime ideal P of R, $O(P) = O(Q)$ for every prime ideal Q of R containing P by Corollary 5.5.2. Thus, $\cap_{P \in \text{MinSpec}(R)} O(P) = 0$ by Proposition 5.5.7. □

Theorem 5.5.11 *Let R be a semiprime ring, which is not prime. If R is quasi-Baer, then R has a nontrivial representation as a subdirect product of $R/O(P)$, where P ranges through all minimal prime ideals.*

Proof As R is a nonprime quasi-Baer ring, R is not semicentral reduced by Proposition 3.2.5. So there is $e \in \mathbf{S}_\ell(R)$ with $e \neq 0$ and $e \neq 1$. By Proposition 1.2.6(ii), $e \in \mathcal{B}(R)$ since R is semiprime. Suppose that there exists a minimal prime ideal P with $O(P) = 0$. Since R is not prime, $P \neq 0$. As $(1-e)Re = 0$, $e \in P$ or $1-e \in P$. If $e \in P$, then $1-e \notin P$ and $eR(1-e) = 0$, so $e \in O(P)$, a contradiction. Similarly, if $1-e \in P$, then we get a contradiction. Thus $O(P) \neq 0$ for every minimal prime ideal P of R. Lemma 5.5.10 yields the desired result. □

Corollary 5.5.12 *Let R be a semiprime ring, which is not prime. If R is quasi-Baer, then R has a nontrivial representation as a subdirect product of RS_P^{-1}, where P ranges through all minimal prime ideals.*

Proof It is a direct consequence of Theorems 5.5.6 and 5.5.11. □

Definition 5.5.13 For a ring R, a left (resp., right) semicentral idempotent $e\, (\neq 1)$ is called *maximal* if $eR \subseteq fR$ (resp., $Re \subseteq Rf$) with $f \in \mathbf{S}_\ell(R)$ (resp., $f \in \mathbf{S}_r(R)$), then $fR = eR$ or $fR = R$ (resp., $Rf = Re$ or $Rf = R$).

Hofmann showed in [209, Theorem 1.17] that $\theta : R \cong \Gamma(\text{Spec}(R), \mathfrak{K}(R))$ when R is a semiprime ring. This result motivates the following question: *If a quasi-Baer ring R has such the sheaf representation, then is R semiprime?* Theorem 5.5.14 provides an affirmative partial answer to the question by giving a characterization of a certain class of quasi-Baer rings having such the sheaf representation.

Theorem 5.5.14 *The following are equivalent for a ring R.*

(i) *R is a PWP ring and $\theta : R \cong \Gamma(\mathrm{Spec}(R), \mathfrak{K}(R))$.*
(ii) *R is a finite direct sum of prime rings.*
(iii) *R is a semiprime PWP ring.*

Proof (i)⇒(ii) Let $\mathrm{Tdim}(R) = n$. If $n = 1$, then R is semicentral reduced, so R is prime by Proposition 3.2.5, and hence we are done. So suppose that $n \geq 2$. By Theorem 5.4.20, there are exactly n minimal prime ideals of R, say $P_1, P_2, \ldots, P_n$ and from Theorem 5.4.12 these are comaximal (i.e., $P_i + P_j = R$ for $i \neq j$).

For each $i = 1, 2, \ldots, n$, we let $\mathfrak{A}_i = \{P \in \mathrm{Spec}(R) \mid P_i \subseteq P\}$. Then it follows that $\mathrm{Spec}(R) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_n$ since $\{P_1, P_2, \ldots, P_n\}$ is the set of all minimal prime ideals. Also because $P_i + P_j = R$ for $i \neq j$, $\mathfrak{A}_i \cap \mathfrak{A}_j = \emptyset$ for $i \neq j$. By the hull-kernel topology on $\mathrm{Spec}(R)$, each $\mathfrak{A}_i$ is a closed subset of $\mathrm{Spec}(R)$. Hence for $i = 1, 2, \ldots, n$, $\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_{i-1} \cup \mathfrak{A}_{i+1} \cup \cdots \cup \mathfrak{A}_n$ is closed, and so each $\mathfrak{A}_i$ is open.

Define $f : \mathrm{Spec}(R) \to \mathfrak{K}(R)$ such that $f(P) = 1 + O(P)$ for $P \in \mathfrak{A}_1$, and $f(P) = 0 + O(P)$ for $P \in \mathfrak{A}_k$ with $k \neq 1$. We claim that f is a continuous function. For this, first take $P \in \mathfrak{A}_1$. Then $f(P) = 1 + O(P) \in \mathfrak{K}(R)$. Consider a basic neighborhood $\widehat{r}(\mathrm{supp}(s))$ (with $r, s \in R$) containing $f(P) = 1 + O(P)$ in $\mathfrak{K}(R)$. Then $\mathrm{supp}(s) \cap \mathfrak{A}_1$ is an open subset of $\mathrm{Spec}(R)$ with $P \in \mathrm{supp}(s) \cap \mathfrak{A}_1$.

For $M \in \mathrm{supp}(s) \cap \mathfrak{A}_1$, $f(M) = 1 + O(M) \in R/O(M)$. Hence we obtain that $1 + O(P) = r + O(P)$ and so $r - 1 \in O(P)$ as $1 + O(P) \in \widehat{r}(\mathrm{supp}(s))$. Now we note that $O(P_1) = O(P) = O(M)$ from Corollary 5.5.2, hence $r - 1 \in O(M)$. Thus,

$$f(M) = 1 + O(M) = r + O(M) \in \widehat{r}(\mathrm{supp}(s)).$$

So $f(\mathrm{supp}(s) \cap \mathfrak{A}_1) \subseteq \widehat{r}(\mathrm{supp}(s))$.

For $P \in \mathfrak{A}_k$ with $k \neq 1$, assume that $f(P) = 0 + O(P) \in \widehat{r}(\mathrm{supp}(s))$ for some $r, s \in R$. Then we also see that $f(\mathrm{supp}(s) \cap \mathfrak{A}_k) \subseteq \widehat{r}(\mathrm{supp}(s))$. Therefore, f is a continuous function.

Next, consider $\pi : \mathfrak{K}(R) \to \mathrm{Spec}(R)$ defined by $\pi(r + O(P)) = P$ for $r \in R$ and $P \in \mathrm{Spec}(R)$. Then we see that $\pi(f(P)) = P$ for all $P \in \mathrm{Spec}(R)$. Thus, it follows that $f \in \Gamma(\mathrm{Spec}(R), \mathfrak{K}(R))$ as f is a continuous function.

Since $R \cong \Gamma(\mathrm{Spec}(R), \mathfrak{K}(R))$, there exists $a \in R$ with $f = \widehat{a}$. Therefore

$$a + O(P_1) = 1 + O(P_1) \text{ and } a + O(P_k) = 0 + O(P_k) \text{ for each } k \neq 1.$$

So $1 - a \in O(P_1)$ and $a \in O(P_k)$ for each $k \neq 1$. Thus $O(P_1) + O(P_k) = R$ for each $k \neq 1$. Similarly, $O(P_i) + O(P_j) = R$ for $i \neq j$, $1 \leq i, j \leq n$. By Lemma 5.5.10, we obtain that $O(P_1) \cap \cdots \cap O(P_n) = 0$, hence

$$R \cong R/O(P_1) \oplus \cdots \oplus R/O(P_n)$$

by Chinese Remainder Theorem. From Proposition 5.5.3, $O(P_1) = Re$ with $e \in \mathbf{S}_r(R)$, so $eR(1 - e) = 0$. Hence $R/O(P_1) \cong (1 - e)R(1 - e)$.

Our claim is that $(1-e)R(1-e)$ is semicentral reduced. For this, assume on the contrary that $(1-e)R(1-e)$ is not semicentral reduced. By Theorem 3.2.10, $(1-e)R(1-e)$ is a quasi-Baer ring. Hence, $(1-e)R(1-e)$ is a PWP ring by Theorem 5.2.19.

From Theorem 5.2.5, there is a maximal right semicentral idempotent in the ring $(1-e)R(1-e)$, say $(1-e)b(1-e)$. Because $(1-e)R(1-e)$ is not semicentral reduced,

$$[(1-e)R(1-e)](1-e)b(1-e)$$

is a nonzero proper ideal of $(1-e)R(1-e)$. Since $e \in \mathbf{S}_r(R)$,

$$e+(1-e)b(1-e) \in \mathbf{S}_r(R).$$

Put $g = e+(1-e)b(1-e)$. We show that g is a maximal right semicentral idempotent of R. Take $\alpha \in \mathbf{S}_r(R)$ such that $Rg \subseteq R\alpha$ and $\alpha \neq 1$. Because

$$R = \begin{bmatrix} eRe & 0 \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$$

and $g = e+(1-e)b(1-e)$, we have that $\alpha = e+k+h$ with $k \in (1-e)Re$ and $h \in \mathbf{S}_r((1-e)R(1-e))$.

Since $Rg \subseteq R\alpha$, $(1-e)R(1-e)(1-e)b(1-e) \subseteq (1-e)R(1-e)h$. From the maximality of $(1-e)b(1-e)$ and $h \neq 1-e$ (because $\alpha \neq 1$), we have that $(1-e)R(1-e)(1-e)b(1-e) = (1-e)R(1-e)h$, and thus $h(1-e)b(1-e) = h$. Further, $ke = k$ since $k \in (1-e)Re$. Hence,

$$\alpha g = \begin{bmatrix} e & 0 \\ k & h \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & (1-e)b(1-e) \end{bmatrix} = \begin{bmatrix} e & 0 \\ k & h \end{bmatrix} = \alpha.$$

Thus, $R\alpha \subseteq Rg$. Therefore, g is a maximal right semicentral idempotent of R.

Next, note that $\{1, 1-g\}$ forms a multiplicatively closed subset of R. By Zorn's lemma, there is an ideal Q of R maximal with respect to being disjoint with $\{1, 1-g\}$. Then Q is a prime ideal of R. Since $gR(1-g) = 0$ and $1-g \notin Q$, it follows $g \in O(Q)$. Also, since g is a maximal right semicentral idempotent of R and $g \in O(Q)$, $O(Q) = Rg$ from Proposition 5.5.3. We observe that $O(P_1) = Re \neq Rg$ as $(1-e)b(1-e) \neq 0$. Hence, $Q \notin \mathfrak{A}_1$ by Corollary 5.5.2. So $Q \in \mathfrak{A}_k$ for some $k \neq 1$. So $O(Q) = O(P_k)$ from Corollary 5.5.2. Now $R = O(P_1)+O(P_k) = Re+Rg$, a contradiction since $(1-e)b(1-e) \neq 1-e$. Thus, the ring $(1-e)R(1-e)$ is a semicentral reduced quasi-Baer ring. So $(1-e)R(1-e)$ is a prime ring by Proposition 3.2.5, thus $R/O(P_1)$ is a prime ring because $R/O(P_1) \cong (1-e)R(1-e)$. Similarly, $R/O(P_i)$ is a prime ring for each $i = 2, \ldots, n$. Therefore $R \cong R/O(P_1) \oplus \cdots \oplus R/O(P_n)$, which is a finite direct sum of prime rings. Further, note that $O(P_i) = P_i$ for each $i = 1, \ldots, n$, so $R \cong R/P_1 \oplus \cdots \oplus R/P_n$.

(ii)⇒(iii) It is evident.

(iii)⇒(i) The proof follows from [209, Theorem 1.17]. □

We obtain the next corollary from Proposition 5.4.6, Lemma 5.4.25, and Theorem 5.5.14.

Corollary 5.5.15 *The following are equivalent.*

(i) *R is a PWD with $\theta : R \cong \Gamma(\mathrm{Spec}(R), \mathfrak{K}(R))$.*
(ii) *R is a finite direct sum of prime PWDs.*
(iii) *R is a semiprime PWD.*

Exercise 5.5.16

1. ([74, Birkenmeier, Kim, and Park]) Assume that R is a (quasi-)Baer ring with $\mathrm{Tdim}(R) < \infty$ and P is a prime ideal of R. Prove that $R/O(P)$ is a (quasi-)Baer ring.
2. ([74, Birkenmeier, Kim, and Park]) Let R be a Baer ring and P be a prime ideal of R. Show that $R/O(P)$ is a right Rickart ring.
3. ([74, Birkenmeier, Kim, and Park]) Assume that R is a quasi-Baer ring and P is a prime ideal of R. Prove that $\mathrm{r.gl.dim}(R/O(P)) \le \mathrm{r.gl.dim}(R)$.

5.6 Triangular Matrix Ring Extensions

Our focus in this section is the study of the Baer, the quasi-Baer, and the (strongly) FI-extending properties of upper triangular and generalized triangular matrix ring extensions. The study of full matrix ring extensions will be considered in Chap. 6.

Theorem 5.6.1 *Let R be a ring. Then the following are equivalent.*

(i) *R is regular and right self-injective.*
(ii) *$T_n(R)$ is right nonsingular right extending for every positive integer n.*
(iii) *$T_k(R)$ is right nonsingular right extending for some integer $k > 1$.*
(iv) *$T_2(R)$ is right nonsingular right extending.*

Proof (i)⇒(ii) The proof follows from [3, Corollary 2.8(3)] and [1, Proposition 1.8(ii)].

(ii)⇒(iii) It is evident.

(iii)⇒(i) [3, Corollary 2.8(2) and Proposition 1.6(2)] yield this implication.

(i)⇔(iv) This equivalence follows from [393, Theorem 3.4] (see also Theorem 5.6.9). □

Theorem 5.6.2 *Let R be an orthogonally finite Abelian ring. Then the following are equivalent.*

(i) *R is a direct sum of division rings.*
(ii) *$T_n(R)$ is a Baer (resp., right Rickart) ring for every positive integer n.*
(iii) *$T_k(R)$ is a Baer (resp., right Rickart) ring for some integer $k > 1$.*

(iv) $T_2(R)$ *is a Baer* (*resp*., *right Rickart*) *ring*.

Proof (i)⇒(ii) The proof follows from Theorems 5.6.1, 3.3.1, and 3.1.25.

(ii)⇒(iii) It is evident.

(iii)⇒(iv) The proof follows from Theorems 3.1.8 and 3.1.22(i).

(iv)⇒(i) Let $T_2(R)$ be Baer (resp., right Rickart). By Proposition 1.2.15, R has a complete set of primitive idempotents. As R is Abelian, $R = \oplus_{i=1}^{m} R_i$ (ring direct sum), for some positive integer m, where each R_i is indecomposable as a ring. Then each $T_2(R_i)$ is a Baer (resp., right Rickart) ring by Proposition 3.1.5(i) (resp., Proposition 3.1.21). From Theorem 3.1.8 (resp., Theorem 3.1.22(i)), each R_i is a Baer (resp., right Rickart) ring. If R_i is Baer or right Rickart, R_i is a domain (see Example 3.1.4(ii)). From [246, Exercise 2, p. 16] or [262, Exercise 25, p. 271], each R_i is a division ring. □

Notation 5.6.3 Let S and R be rings, and let ${}_SM_R$ be an (S, R)-bimodule. For the remainder of this section, we let

$$T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$$

denote a generalized triangular matrix ring.

Lemma 5.6.4 *Let T be the ring as in Notation 5.6.3. Say*

$$e = \begin{bmatrix} e_1 & k \\ 0 & e_2 \end{bmatrix} \in \mathbf{S}_\ell(T) \text{ and } f = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}.$$

Then we have the following.

(i) $e_1 \in \mathbf{S}_\ell(S)$, $e_2 \in \mathbf{S}_\ell(R)$, *and* $f \in \mathbf{S}_\ell(T)$
(ii) $eT = fT$.

Proof (i) It can be easily checked that $e_1 \in \mathbf{S}_\ell(S)$ and $e_2 \in \mathbf{S}_\ell(R)$. Also we see that $e_1me_2 = me_2$ for all $m \in M$. Thus, $f \in \mathbf{S}_\ell(T)$.

(ii) Since $e_1me_2 = me_2$ for all $m \in M$, in particular $e_1ke_2 = ke_2$ and so $f = e\begin{bmatrix} e_1 & -ke_2 \\ 0 & e_2 \end{bmatrix}$. Hence $fT \subseteq eT$. As $e \in \mathbf{S}_\ell(T)$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e = e \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e$, so $k = e_1k$. Thus, $e = f\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \in fT$. Therefore $eT \subseteq fT$, and so $eT = fT$. □

Next, we characterize the quasi-Baer property for the ring T.

Theorem 5.6.5 *Let T be the ring as in Notation 5.6.3. Then the following are equivalent*.

(i) T *is a quasi-Baer ring*.
(ii) (1) R *and* S *are quasi-Baer rings*.

(2) $r_M(I) = r_S(I)M$ *for all* $I \trianglelefteq S$.
(3) *For any* ${}_S N_R \le {}_S M_R$, $r_R(N) = gR$ *for some* $g^2 = g \in R$.

Proof (i)⇒(ii) By Theorem 3.2.10, R and S are quasi-Baer. Let $I \trianglelefteq S$. Then $A := \begin{bmatrix} I & M \\ 0 & 0 \end{bmatrix} \trianglelefteq T$. Hence, $r_T(A) = eT$ for some $e^2 = e \in T$. Because $A \trianglelefteq T$, $e \in \mathbf{S}_\ell(T)$ by Proposition 1.2.2. Put $e = \begin{bmatrix} e_1 & k \\ 0 & e_2 \end{bmatrix}$ and $f = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$. From Lemma 5.6.4, $e_1 \in \mathbf{S}_\ell(S)$, $e_2 \in \mathbf{S}_\ell(R)$, $f \in \mathbf{S}_\ell(T)$, and $eT = fT$. Thus it is routine to check that $r_S(I) = e_1 S$ and $r_M(I) = e_1 M = e_1 SM = r_S(I)M$.

Next, let ${}_S N_R \le {}_S M_R$. Then $K := \begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \trianglelefteq T$. So $r_T(K) = hT$ for some $h \in \mathbf{S}_\ell(T)$. Say $h = \begin{bmatrix} g_1 & m \\ 0 & g_2 \end{bmatrix}$. Then $r_R(N) = g_2 R$, where $g_2 \in \mathbf{S}_\ell(R)$. Take $g = g_2$. Then $r_R(N) = gR$ and $g^2 = g \in R$.

(ii)⇒(i) Let $K \trianglelefteq T$. Then we see that $K = \begin{bmatrix} I & N \\ 0 & J \end{bmatrix}$, where $I \trianglelefteq S$, $J \trianglelefteq R$, ${}_S N_R \le {}_S M_R$, and $IM + MJ \subseteq N$. Because S and R are quasi-Baer, there are $e_1 \in \mathbf{S}_\ell(S)$, $f \in \mathbf{S}_\ell(R)$ satisfying $r_S(I) = e_1 S$ and $r_R(J) = fR$. By assumption, $r_M(I) = r_S(I)M = e_1 M$ and $r_R(N) = gR$ for some $g^2 = g \in R$. As $r_R(N) = gR \trianglelefteq R$, $g \in \mathbf{S}_\ell(R)$ by Proposition 1.2.2. From Proposition 1.2.4(i), $gf \in \mathbf{S}_\ell(R)$. Put $e = \begin{bmatrix} e_1 & 0 \\ 0 & gf \end{bmatrix} \in T$. Then $e^2 = e$ and $r_T(K) = eT$. Thus, T is quasi-Baer. □

Corollary 5.6.6 *Let* $S = \mathrm{End}(M_R)$ *and let* T *be the ring as in Notation 5.6.3. Then the following are equivalent.*

(i) T *is a quasi-Baer ring.*
(ii) (1) R *is a quasi-Baer ring.*
(2) M_R *is a quasi-Baer module.*
(3) *If* $N_R \trianglelefteq M_R$, *then* $r_R(N) = gR$ *for some* $g^2 = g \in R$.

Proof (i)⇒(ii) Assume that T is a quasi-Baer ring. Then M_R is a quasi-Baer module by Proposition 4.6.3 and Theorem 5.6.5. So we get (ii).

(ii)⇒(i) As M_R is a quasi-Baer module, S is a quasi-Baer ring by Theorem 4.6.16. Let $I \trianglelefteq S$. Then $r_S(I) = fS$ for some $f^2 = f \in S$. Also $r_M(I) = hM$ for some $h^2 = h \in S$ by Proposition 4.6.3. Since $If = 0$, $IfM = 0$, and so $fM \subseteq r_M(I) = hM$. As $IhM = 0$, $Ih = 0$, and hence $h \in r_S(I) = fS$. Thus, $hM \subseteq fSM = fM$. Therefore $hM = fM = fSM = r_S(I)M$. So T is a quasi-Baer ring by Theorem 5.6.5. □

We observe that in contrast to Theorem 5.6.2, the next two results hold true without any additional assumption on R.

Theorem 5.6.7 *The following are equivalent for a ring* R.

(i) *R is a quasi-Baer ring.*
(ii) $T_n(R)$ *is a quasi-Baer ring for every positive integer* n.
(iii) $T_k(R)$ *is a quasi-Baer ring for some integer* $k > 1$.
(iv) $T_2(R)$ *is a quasi-Baer ring.*

Proof (i)⇒(ii) We use induction on n. As R is quasi-Baer, $T_2(R)$ is quasi-Baer by applying Corollary 5.6.6.

Let $T_n(R)$ be quasi-Baer. We show that $T_{n+1}(R)$ is quasi-Baer. Write

$$T_{n+1}(R) = \begin{bmatrix} R & M \\ 0 & T_n(R) \end{bmatrix},$$

where $M = [R, \ldots, R]$ (n-tuple). To apply Theorem 5.6.5, let $I \trianglelefteq R$. Then $r_R(I) = eR$ for some $e^2 = e \in R$. Also $r_M(I) = eM = r_R(I)M$.

Next, say ${}_RN_{T_n(R)} \leq {}_RM_{T_n(R)}$. Note that $\begin{bmatrix} 0 & N \\ 0 & 0 \end{bmatrix} \trianglelefteq T_{n+1}(R)$. Therefore, we have that $N = [N_1, \ldots, N_n]$, where $N_i \trianglelefteq R$ for each i and $N_1 \subseteq \cdots \subseteq N_n$. As R is quasi-Baer, $r_R(N_i) = f_iR$ with $f_i^2 = f_i \in R$ for each i.

Let $e_{ij} \in T_n(R)$ be the matrix with 1 in the (i, j)-position and 0 elsewhere. Put $g = f_1e_{11} + \cdots + f_ne_{nn} \in T_n(R)$. Then $g^2 = g$ and $r_{T_n(R)}(N) = gT_n(R)$. By Theorem 5.6.5, $T_{n+1}(R)$ is a quasi-Baer.

(ii)⇒(iii) is obvious. For (iii)⇒(iv), let $e_{ij} \in T_k(R)$ be the matrix with 1 in the (i, j)-position and 0 elsewhere. Set $f = e_{11} + e_{22}$. Then $f^2 = f \in T_k(R)$ and $T_2(R) \cong fT_k(R)f$. By Theorem 3.2.10, $T_2(R)$ is quasi-Baer. Similarly, (iv)⇒(i) follows from Theorem 3.2.10. □

Proposition 5.6.8 *The following are equivalent for a ring* R.

(i) *R is a right p.q.-Baer ring.*
(ii) $T_n(R)$ *is a right p.q.-Baer ring for every positive integer* n.
(iii) $T_k(R)$ *is a right p.q.-Baer ring for some integer* $k > 1$.
(iv) $T_2(R)$ *is a right p.q.-Baer ring.*

Proof (i)⇒(ii) Put $T = T_n(R)$. Let e_{ij} be the matrix in T with 1 in the (i, j)-position and 0 elsewhere. Say $[a_{ij}] \in T$ and consider the right ideal $[a_{ij}]T$. Take $\alpha = [\alpha_{ij}] \in r_T([a_{ij}]T)$. Since R is right p.q.-Baer, for $i \leq j$, $r_R(a_{ij}R) = f_{ij}R$ with $f_{ij}^2 = f_{ij} \in R$. Then $f_{ij} \in \mathbf{S}_\ell(R)$ from Proposition 1.2.2 because $f_{ij}R \trianglelefteq R$.

Now observe that $\alpha_{1\ell} \in r_R(a_{11}R) = f_{11}R$ for $\ell = 1, \ldots, n$. Also we see that $\alpha_{2\ell} \in r_R(a_{11}R) \cap r_R(a_{12}R) \cap r_R(a_{22}R) = f_{11}R \cap f_{12}R \cap f_{22}R = f_{11}f_{12}f_{22}R$ for $\ell = 2, \ldots, n$, and $f_{11}f_{12}f_{22} \in \mathbf{S}_\ell(R)$ (see Proposition 1.2.4(i)). In general, $\alpha_{k\ell} \in (f_{11} \cdots f_{1k})(f_{22} \cdots f_{2k}) \cdots (f_{k-1k-1}f_{k-1k})f_{kk}R$ for $\ell = k, \ldots, n$.

Put $g_k = (f_{11} \cdots f_{1k})(f_{22} \cdots f_{2k}) \cdots (f_{k-1k-1}f_{k-1k})f_{kk}$ for $k = 1, \ldots, n$. Then $g_k \in \mathbf{S}_\ell(R)$ by Proposition 1.2.4(i). Note that $g_k\alpha_{k\ell} = \alpha_{k\ell}$ for $\ell = k, \ldots, n$.

Let $e = g_1e_{11} + \cdots + g_ne_{nn} \in T$. Then $e^2 = e$ and $r_R([a_{ij}]T) = eT$. Therefore, $T = T_n(R)$ is right p.q.-Baer.

(ii)⇒(iii) It is evident.

(iii)⇒(iv) Let $f = e_{11} + e_{22} \in T_k(R)$. Then we see that $f^2 = f \in T_k(R)$ and $T_2(R) \cong fT_k(R)f$, so $T_2(R)$ is right p.q.-Baer by Theorem 3.2.34(i).

(iv)⇒(i) It follows also from Theorem 3.2.34(i). □

The following result, due to Tercan in [393], characterizes the generalized triangular matrix ring T (see Notation 5.6.3) to be a right nonsingular right extending ring (hence T is Baer and right cononsingular by Theorem 3.3.1) when ${}_SM$ is faithful.

Theorem 5.6.9 *Let T be the ring as in Notation 5.6.3 and ${}_SM$ be faithful. Then the following are equivalent.*

(i) *T is right nonsingular and right extending.*
(ii) (1) *For each complement K_R in M_R there is $e^2 = e \in S$ with $K = eM$.*
 (2) *R is right nonsingular and right extending.*
 (3) *M_R is nonsingular and injective.*

In the next result, a characterization for T to be right FI-extending is presented. This will be used to consider the FI-extending triangular matrix ring extensions.

Theorem 5.6.10 *Let T be the ring as in Notation 5.6.3. Then the following are equivalent.*

(i) *T_T is FI-extending.*
(ii) (1) *For ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ with $IM \subseteq N$, there is $f^2 = f \in S$ such that $I \subseteq fS$, $N_R \leq^{\text{ess}} fM_R$, and $(I \cap \ell_S(M))_S \leq^{\text{ess}} (fS \cap \ell_S(M))_S$.*
 (2) *R_R is FI-extending.*

Proof Throughout the proof, we let $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T$.

(i)⇒(ii) First, we claim that $\ell_S(M) = eS$ for some $e^2 = e \in S$. Observe that $T_T = e_{11}T_T \oplus (1 - e_{11})T_T$ and $e_{11} \in \mathbf{S}_\ell(T)$. From Proposition 2.3.11(i), $e_{11}T_T = \begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}_T$ is FI-extending. First, to see that $\ell_S(M) = eS$ for some $e^2 = e \in S$, put $U = \begin{bmatrix} \ell_S(M) & 0 \\ 0 & 0 \end{bmatrix}$. Then $U_T \trianglelefteq e_{11}T_T$ because $\ell_S(M) \trianglelefteq S$ and $\text{End}(e_{11}T_T) \cong e_{11}Te_{11} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$. Because $e_{11}T_T$ is FI-extending, we have that $U_T \leq^{\text{ess}} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} e_{11}T_T$ for some $e^2 = e \in S$. So $\begin{bmatrix} \ell_S(M) & 0 \\ 0 & 0 \end{bmatrix}_T \leq^{\text{ess}} \begin{bmatrix} eS & eM \\ 0 & 0 \end{bmatrix}_T$. Thus, $\ell_S(M) \subseteq eS$. For any $m \in M$, $em = 0$ because $U \cap \begin{bmatrix} 0 & em \\ 0 & 0 \end{bmatrix} T = 0$. Hence $eM = 0$, so $e \in \ell_S(M)$. Thus $eS \subseteq \ell_S(M)$, and hence $\ell_S(M) = eS$.

For condition (1), let ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ such that $IM \subseteq N$. Then $V := \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}_T \trianglelefteq e_{11}T_T = \begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}_T$. Since $e_{11}T_T$ is FI-extending, we have that

$\begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}_T \leq^{\text{ess}} \begin{bmatrix} fS & fM \\ 0 & 0 \end{bmatrix}_T$ for some $f^2 = f \in S$, therefore $I \subseteq fS$ and $N_R \leq^{\text{ess}} fM_R$. Next, for $0 \neq fs \in fS \cap eS = fS \cap \ell_S(M)$ with $s \in S$, we see that $V \cap \begin{bmatrix} fs & 0 \\ 0 & 0 \end{bmatrix} T = V \cap \begin{bmatrix} fsS & 0 \\ 0 & 0 \end{bmatrix} \neq 0$. Hence, $fsS \cap (I \cap eS) = fsS \cap I \neq 0$ because $fsS \subseteq eS$. Therefore, we have that $(I \cap eS)_S \leq^{\text{ess}} (fS \cap eS)_S$.

Since $e_{11} \in \mathbf{S}_\ell(T)$, Proposition 2.3.11(ii) yields condition (2) immediately.

(ii)⇒(i) By condition (2), $(1 - e_{11})T_T$ is FI-extending. To show that $e_{11}T_T$ is FI-extending, let $V_T \trianglelefteq e_{11}T_T$. Since $e_{11} \in \mathbf{S}_\ell(T)$, $e_{11}T_T \trianglelefteq T_T$ from Proposition 1.2.2, and so $V_T \trianglelefteq T_T$ by Proposition 2.3.3(ii). Thus $V = \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}$ with $I \trianglelefteq S$, ${}_SN_R \leq {}_SM_R$, and $IM \subseteq N$. By condition (1), there is $f^2 = f \in S$ such that $I \subseteq fS$, $N_R \leq^{\text{ess}} fM_R$, and $(I \cap \ell_S(M))_S \leq^{\text{ess}} (fS \cap \ell_S(M))_S$. Thus, it follows that $V \subseteq \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} fS & fM \\ 0 & 0 \end{bmatrix}$. Let $W = \begin{bmatrix} fS & fM \\ 0 & 0 \end{bmatrix}$. Then W_T is a direct summand of $e_{11}T_T$ because $f^2 = f \in S \cong \text{End}(e_{11}T_T)$.

We prove that $V_T \leq^{\text{ess}} W_T$. For this, take $0 \neq w = \begin{bmatrix} fs & fm \\ 0 & 0 \end{bmatrix} \in W$, where $s \in S$ and $m \in M$. If $fm \neq 0$, then $V \cap wT \neq 0$ since $N_R \leq^{\text{ess}} fM_R$. Next, assume that $fm = 0$. Then $fs \neq 0$. Hence $wT = \begin{bmatrix} fsS & fsM \\ 0 & 0 \end{bmatrix}$.

If $fsM \neq 0$, clearly $V \cap wT \neq 0$ since $N_R \leq^{\text{ess}} fM_R$. If $fsM = 0$, then

$$fs \in \ell_S(M), \quad \text{so} \quad 0 \neq fs \in fS \cap \ell_S(M).$$

Since $(I \cap \ell_S(M))_S \leq^{\text{ess}} (fS \cap \ell_S(M))_S$, $fsS \cap (I \cap \ell_S(M)) \neq 0$, so $V \cap wT \neq 0$. Therefore $V_T \leq^{\text{ess}} W_T$, thus $e_{11}T_T$ is FI-extending. Hence T_T is FI-extending by Theorem 2.3.5. □

Corollary 5.6.11 *Let T be the ring as in Notation 5.6.3. Assume that ${}_SM$ is faithful. Then the following are equivalent.*

(i) *T_T is FI-extending.*
(ii) (1) *For ${}_SN_R \leq {}_SM_R$, there is $f^2 = f \in S$ with $N_R \leq^{\text{ess}} fM_R$.*
  (2) *R_R is FI-extending.*

Proof (i)⇒(ii) Assume that T_T is FI-extending. As ${}_SM$ is faithful, $\ell_S(M) = 0$. By taking $I = 0$ in Theorem 5.6.10, we obtain part (ii).

(ii)⇒(i) Let ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ such that $IM \subseteq N$. By (1), there exists $f^2 = f \in S$ such that $N_R \leq^{\text{ess}} fM_R$. Since $IM \subseteq N \subseteq fM$, $fn = n$ for all $n \in N$, in particular $fsm = sm$ for any $s \in I$ and $m \in M$. Therefore, $(s - fs)M = 0$, so $s - fs = 0$ for any $s \in I$ because ${}_SM$ is faithful. Hence, $I = fI \subseteq fS$. Thus, T_T is FI-extending by Theorem 5.6.10. □

Corollary 5.6.12 *Let M_R be a right R-module. Then the ring*

$$T = \begin{bmatrix} \mathrm{End}_R(M) & M \\ 0 & R \end{bmatrix}$$

is right FI-extending if and only if M_R and R_R are FI-extending.

Proof It follows immediately from Corollary 5.6.11. □

We remark that if R is a right FI-extending ring, then $T_2(R)$ is right FI-extending by taking $M = R_R$ in Corollary 5.6.12. When $n \geq 2$, we obtain the FI-extending property of $T_n(R)$ in Theorem 5.6.19 precisely when R is right FI-extending. By our previous results, we establish a class of rings which are right FI-extending, but not left FI-extending as the next example illustrates.

Example 5.6.13 Let R be a right self-injective ring with $J(R) \neq 0$. Put

$$T = \begin{bmatrix} R/J(R) & R/J(R) \\ 0 & R \end{bmatrix}.$$

Then the ring $R/J(R)$ is right self-injective by Corollary 2.1.30. Further, $\mathrm{End}_R(R/J(R)) \cong R/J(R)$. Also $R/J(R)$ is an FI-extending right R-module. Thus the ring T is right FI-extending by Corollary 5.6.12. If T is left FI-extending, then $r_R((R/J(R))_R) = J(R) = Rf$ for some $f \in \mathbf{S}_r(R)$ from the left-sided version of the proof for (i)⇒(ii) of Theorem 5.6.10. Thus $f = 0$ and hence $J(R) = 0$, a contradiction. Thus, T cannot be left FI-extending.

Definition 5.6.14 Let $N_R \leq M_R$. We say that N_R has a *direct summand cover* $\mathcal{D}(N_R)$ if there is $e^2 = e \in \mathrm{End}_R(M)$ with $N_R \leq^{\mathrm{ess}} eM_R = \mathcal{D}(N_R)$.

If M_R is a strongly FI-extending module, then every fully invariant submodule has a unique direct summand cover from Lemma 2.3.22. For $N_R \leq M_R$, let $(N_R : M_R) = \{a \in R \mid Ma \subseteq N\}$. Then $(N_R : M_R) \trianglelefteq R$.

We use $\mathcal{D}[(N_R : M_R)_R]$ to denote a direct summand cover of the right ideal $(N_R : M_R)$ in R_R. Let M be an (S, R)-bimodule and ${}_SN_R \leq {}_SM_R$. If there exists $e^2 = e \in \mathbf{S}_\ell(S)$ such that $N_R \leq^{\mathrm{ess}} eM_R$, then we write $\mathcal{D}_S(N_R) = eM$.

In the next result, we obtain a necessary and sufficient condition for a 2×2 generalized triangular matrix ring to be right strongly FI-extending. Some applications of this characterization will also be presented.

Theorem 5.6.15 *Let T be as in Notation 5.6.3. Then the following are equivalent.*

(i) *T_T is strongly FI-extending.*
(ii) (1) *For ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ with $IM \subseteq N$, there is $e \in \mathbf{S}_\ell(S)$ such that $I \subseteq eS$, $N_R \leq^{\mathrm{ess}} eM_R$ and $(I \cap \ell_S(M))_S \leq^{\mathrm{ess}} (eS \cap \ell_S(M))_S$.*
  (2) *R_R is strongly FI-extending.*

(3) $\mathcal{D}_S(N_R)\mathcal{D}[(N_R : M_R)_R] = M\mathcal{D}[(N_R : M_R)_R]$ *for* ${}_SN_R \leq {}_SM_R$.

Proof (i)⇒(ii) We let $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in T$. Assume that T_T is strongly FI-extending. By Theorem 2.3.19, $(1 - e_{11})T_T$ is strongly FI-extending, so R_R is strongly FI-extending, which is condition (2).

For condition (1), let ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ with $IM \subseteq N$. Then $V := \begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}_T \trianglelefteq e_{11}T_T = \begin{bmatrix} S & M \\ 0 & 0 \end{bmatrix}_T$. Since $e_{11}T_T$ is strongly FI-extending, there exists $e^2 = e \in \mathbf{S}_\ell(S)$ such that $V_T \leq^{\text{ess}} \begin{bmatrix} eS & eM \\ 0 & 0 \end{bmatrix}_T$. So $I \subseteq eS$ and $N_R \leq^{\text{ess}} eM_R$.

Next, say $0 \neq es \in eS \cap \ell_S(M)$ with $s \in S$. There is $\begin{bmatrix} s_1 & m_1 \\ 0 & r_1 \end{bmatrix} \in T$ such that

$$0 \neq \begin{bmatrix} es & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s_1 & m_1 \\ 0 & r_1 \end{bmatrix} = \begin{bmatrix} ess_1 & 0 \\ 0 & 0 \end{bmatrix} \in V.$$

Thus $0 \neq ess_1 \in I \cap \ell_S(M)$. Therefore $(I \cap \ell_S(M))_S \leq^{\text{ess}} (eS \cap \ell_S(M))_S$.

For condition (3), let ${}_SN_R \leq {}_SM_R$ and put $A = (N_R : M_R)$. Take $I = 0$ in condition (1). There exists $e \in \mathbf{S}_\ell(S)$ with $\mathcal{D}_S(N_R) = eM$. By condition (2), $\mathcal{D}(A_R) = fR$ for some $f \in \mathbf{S}_\ell(R)$. Since $MA \subseteq N$, $W := \begin{bmatrix} 0 & N \\ 0 & A \end{bmatrix} \trianglelefteq T$, and $W_T \leq^{\text{ess}} wT_T$ for some $w \in \mathbf{S}_\ell(T)$. By Lemma 5.6.4, there exist $e_0 \in \mathbf{S}_\ell(S)$ and $f_0 \in \mathbf{S}_\ell(R)$ such that $wT = \begin{bmatrix} e_0 & 0 \\ 0 & f_0 \end{bmatrix} T$. We put $w_0 = \begin{bmatrix} e_0 & 0 \\ 0 & f_0 \end{bmatrix} \in \mathbf{S}_\ell(T)$. Hence $N_R \leq^{\text{ess}} e_0M_R$ and $A_R \leq^{\text{ess}} f_0R_R$. So $\mathcal{D}_S(N_R) = eM = e_0M$ by Lemma 2.3.22 as $e_0 \in \mathbf{S}_\ell(S)$. Also $\mathcal{D}(A_R) = fR = f_0R$.

Note that $Mf_0 = e_0Mf_0$ since $w_0 \in \mathbf{S}_\ell(T)$. Thus, $e_0Mf_0R = Mf_0R$. Therefore, $\mathcal{D}_S(N_R)\mathcal{D}[(N_R : M_R)_R] = M\mathcal{D}[(N_R : M_R)_R]$.

(ii)⇒(i) Assume that $K \trianglelefteq T$. Then

$$K = \begin{bmatrix} I & N \\ 0 & B \end{bmatrix} \trianglelefteq T,$$

where ${}_SN_R \leq {}_SM_R$, $I \trianglelefteq S$, $IM + MB \subseteq N$, and $B \trianglelefteq R$.

From condition (1), there exists $e \in \mathbf{S}_\ell(S)$ with

$$I \subseteq eS,\ \ \mathcal{D}_S(N_R) = eM,\ \ \text{and}\ \ (I \cap \ell_S(M))_S \leq^{\text{ess}} (eS \cap \ell_S(M))_S.$$

Since $B \trianglelefteq R$, by condition (2), there exists $f \in \mathbf{S}_\ell(R)$ with $\mathcal{D}(B_R) = fR$. Also, from condition (2), $\mathcal{D}[(N_R : M_R)_R] = f_0R$ for some $f_0 \in \mathbf{S}_\ell(R)$.

As $MB \subseteq N$, $B \subseteq (N_R : M_R)$. Thus,

$$B_R \leq^{\text{ess}} (fR \cap f_0R)_R = f_0fR$$

with $f_0f \in \mathbf{S}_\ell(R)$ (see Proposition 1.2.4(i)). So $\mathcal{D}(B_R) = f_0fR$. By Lemma 2.3.22, we get that $fR = f_0fR$.

By condition (3), $eMf_0R = Mf_0R$. Because $f \in \mathbf{S}_\ell(R)$ and $f_0fR = fR$, $eMf_0Rf = eMf_0fRf = eMfRf = eMRf = eMf$. Similarly, we have that $Mf_0Rf = Mf$. As $eMf_0R = Mf_0R$, $eMf_0Rf = Mf_0Rf$ and so $eMf = Mf$.

Since $(I \cap \ell_S(M))_S \leq^{\text{ess}} (eS \cap \ell_S(M))_S$ and $N_R \leq^{\text{ess}} eM_R$, we see that $\begin{bmatrix} I & N \\ 0 & 0 \end{bmatrix}_T \leq^{\text{ess}} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} T_T$. So $\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}_T \leq^{\text{ess}} \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} T_T$ because $B_R \leq^{\text{ess}} fR_R$.

Thus $K_T \leq^{\text{ess}} \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} T_T$. As $Mf = eMf$, $mf = emf$ for each $m \in M$. Hence $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in \mathbf{S}_\ell(T)$. Therefore, T_T is strongly FI-extending. □

Corollary 5.6.16 *Let T be the ring as in Notation 5.6.3 with ${}_SM$ faithful. Then the following are equivalent.*

(i) *T_T is strongly FI-extending.*
(ii) (1) *For ${}_SN_R \leq {}_SM_R$, there is $e \in \mathbf{S}_\ell(S)$ with $N_R \leq^{\text{ess}} eM_R$.*
 (2) *R_R is strongly FI-extending.*
 (3) *$\mathcal{D}_S(N_R)\mathcal{D}[(N_R : M_R)_R] = M\mathcal{D}[(N_R : M_R)_R]$ for ${}_SN_R \leq {}_SM_R$.*

Proof (i)⇒(ii) The proof follows from Theorem 5.6.15 by taking $I = 0$. For (ii)⇒(i), let ${}_SN_R \leq {}_SM_R$ and $I \trianglelefteq S$ such that $IM \subseteq N$. By condition (1), there is $e \in \mathbf{S}_\ell(S)$ with $N_R \leq^{\text{ess}} eM_R$. As $IM \subseteq N \subseteq eM$, $n = en$ for all $n \in N$, in particular $sm = esm$ for any $s \in I$ and $m \in M$. Thus $(s - es)M = 0$, so $s - es = 0$ for any $s \in I$, as ${}_SM$ is faithful. So $I = eI \subseteq eS$. Thus T_T is strongly FI-extending by Theorem 5.6.15. □

Corollary 5.6.17 *Let M_R be a right R-module and $T = \begin{bmatrix} \text{End}_R(M) & M \\ 0 & R \end{bmatrix}$. Then the following are equivalent.*

(i) *T_T is strongly FI-extending.*
(ii) (1) *M_R is strongly FI-extending.*
 (2) *R_R is strongly FI-extending.*
 (3) *For any $N_R \trianglelefteq M_R$, $\mathcal{D}(N_R)\mathcal{D}[(N_R : M_R)_R] = M\mathcal{D}[(N_R : M_R)_R]$.*

Proof It follows immediately from Corollary 5.6.16. □

Theorem 5.6.18 *Let R be a ring. Then the following are equivalent.*

(i) *R is right strongly FI-extending.*
(ii) *$T_n(R)$ is right strongly FI-extending for every positive integer n.*
(iii) *$T_k(R)$ is right strongly FI-extending for some integer $k > 1$.*
(iv) *$T_2(R)$ is right strongly FI-extending.*

Proof (i)⇒(ii) Assume that R is right strongly FI-extending. We proceed by induction on n. Let $n = 2$. Take $M = R$ in Corollary 5.6.17. Let $N_R \trianglelefteq M_R$. Since R_R

is strongly FI-extending, there exists $e^2 = e \in \mathbf{S}_\ell(R)$ such that $N_R \leq^{\text{ess}} eM_R$. We observe that $(N_R : M_R) = N_R \leq^{\text{ess}} eR_R$. Therefore we have that

$$\mathcal{D}_R(N_R)\mathcal{D}[(N_R : M_R)_R] = eReR = ReR = M\mathcal{D}[(N_R : M_R)_R].$$

Hence, $T_2(R)$ is a right strongly FI-extending ring by Corollary 5.6.17.

Assume that $T_n(R)$ is right strongly FI-extending. Then we show that $T_{n+1}(R)$ is right strongly FI-extending. Now

$$T_{n+1}(R) = \begin{bmatrix} R & M \\ 0 & T_n(R) \end{bmatrix},$$

where $M = [R, \ldots, R]$ (n-tuple). Let ${}_R N_{T_n(R)} \leq {}_R M_{T_n(R)}$. As in the proof of Theorem 5.6.7, $N = [N_1, \ldots, N_n]$, where $N_i \trianglelefteq R$ for each i and $N_1 \subseteq \cdots \subseteq N_n$. As R_R is strongly FI-extending, there is $e \in \mathbf{S}_\ell(R)$ with $N_{nR} \leq^{\text{ess}} eR_R$, so $N = [N_1, \ldots, N_n]_{T_n(R)} \leq^{\text{ess}} e[R, \ldots, R]_{T_n(R)} = eM$. Thus,

$$(N_{T_n(R)} : M_{T_n(R)}) = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \\ 0 & N_2 & \cdots & N_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_n \end{bmatrix}_{T_n(R)} \leq^{\text{ess}} (e\mathbf{1})T_n(R)_{T_n(R)},$$

where $\mathbf{1}$ is the identity matrix in $T_n(R)$. Hence, we have that

$$\mathcal{D}_R(N_{T_n(R)})\mathcal{D}[(N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}] = eM(e\mathbf{1})T_n(R) = M(e\mathbf{1})T_n(R),$$

since $e \in \mathbf{S}_\ell(R)$. Note that $M\mathcal{D}[(N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}] = M(e\mathbf{1})T_n(R)$. So $M\mathcal{D}[(N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}] = \mathcal{D}_R(N_{T_n(R)})\mathcal{D}[(N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}]$. Thus by Corollary 5.6.16, $T_{n+1}(R)$ is a right strongly FI-extending ring.

(ii)⇒(iii) is obvious, and (iii)⇒(i) is a consequence of Theorem 5.6.15.

(i)⇒(iv) follows from the proof of (i)⇒(ii) for the case when $n = 2$, and (iv)⇒(i) follows from Theorem 5.6.15. □

Theorem 5.6.19 *Let R be a ring. Then the following are equivalent.*

(i) *R is right FI-extending.*
(ii) *$T_n(R)$ is right FI-extending for every positive integer n.*
(iii) *$T_k(R)$ is right FI-extending for some integer $k > 1$.*
(iv) *$T_2(R)$ is right FI-extending.*

Proof The proof follows by using Corollary 5.6.11 and an argument similar to that used in the proof of Theorem 5.6.18. □

Theorem 5.6.19 provides a full characterization of $T_n(R)$ to be right FI-extending for any positive integer n. Let R be a commutative domain which is not a field. Say n is an integer such that $n > 1$. Then $T_n(R)$ is right strongly FI-extending (hence right FI-extending) by Theorem 5.6.18. Observe that $T_n(R)$ is not Baer from Theorem 5.6.2. Thus by Corollary 3.3.3, $T_n(R)$ is neither right nor left extending. Corollary 5.6.16 and Theorem 5.6.18 are now applied to show that the strongly FI-extending property for rings is not left-right symmetric.

Example 5.6.20 Let R be a commutative domain and let $M = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix}$. Then naturally M can be considered as an $(R, T_2(R))$-bimodule. We show that the generalized triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & T_2(R) \end{bmatrix}$ is right strongly FI-extending, but it is not left strongly FI-extending. For this, note that ${}_RM$ is faithful. Because R is right strongly FI-extending, $T_2(R)$ is right strongly FI-extending from Theorem 5.6.18. Say ${}_RN_{T_2(R)} \leq {}_RM_{T_2(R)}$. If $N = 0$, then $\mathcal{D}_R(N_{T_2(R)})\mathcal{D}[(N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}] = 0 = M\mathcal{D}[(N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}]$. So assume that $N \neq 0$. Then there is $0 \neq I \trianglelefteq R$ with $N = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$. Then $I_R \leq^{\text{ess}} R_R$, hence $\mathcal{D}_R(N_{T_2(R)}) = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix} = M$. Therefore,

$$\mathcal{D}_R(N_{T_2(R)})\mathcal{D}[(N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}] = M\mathcal{D}[(N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}].$$

Thus, T_T is strongly FI-extending by Corollary 5.6.16.

We may note that $r_{T_2(R)}(M)$ is not generated, as a left ideal, by an idempotent in $T_2(R)$. Thus, ${}_TT$ is not FI-extending by the left-sided version of the proof for (i)⇒(ii) of Theorem 5.6.10. So ${}_TT$ is not strongly FI-extending.

Exercise 5.6.21

1. Assume that R is a PWP ring. Show that $T_n(R)$ is a PWP ring for each positive integer n.
2. ([85, Birkenmeier, Park, and Rizvi]) Let R be a prime ring with P a nonzero prime ideal. Prove that the ring $\begin{bmatrix} R/P & R/P \\ 0 & R \end{bmatrix}$ is right FI-extending, but not left FI-extending.
3. ([85, Birkenmeier, Park, and Rizvi]) Let R be a commutative PID and let I be a nonzero proper ideal of R. Show that the ring $\begin{bmatrix} R/I & R/I \\ 0 & R \end{bmatrix}$ is right FI-extending, but not left FI-extending.
4. ([64, Birkenmeier and Lennon]) Let T be the ring as in Notation 5.6.3. Prove that T_T is FI-extending if and only if the following conditions hold.
   (1) $\ell_S(M) = eS$, where $e \in \mathbf{S}_\ell(S)$, and eS_S is FI-extending.
   (2) For ${}_SN_R \leq {}_SM_R$, there is $f^2 = f \in S$ with $N_R \leq^{\text{ess}} fM_R$.
   (3) R_R is FI-extending.
5. Let T be the ring as in Notation 5.6.3. Characterize T being right p.q.-Baer in terms of conditions on S, M, and R. (Hint: see [78, Birkenmeier, Kim, and Park].)

Historical Notes Some of the diverse applications associated with generalized triangular matrix representations appear in the study of operator theory [212], quasitriangular Hopf algebras [113], and various Lie algebras [303]. Also many authors have studied a variety of conditions on generalized triangular matrix rings (e.g., [37, 189–191, 196, 228, 280], and [416]). Most results from Sects. 5.1, 5.2, and 5.3 are due to Birkenmeier, Heatherly, Kim, and Park [70]. Results 5.2.18–5.2.20 appear in [66]. Some of the motivating ideas for defining triangulating idempotents originated with [55]. Lemma 5.3.4 is due to Fields [164].

Theorem 5.4.1, Corollary 5.4.2, and Definition 5.4.4 appear in [70]. Piecewise domains (PWDs) were defined and investigated by Gordon and Small [187]. Proposition 5.4.6 is in [70]. Proposition 5.4.9 and Example 5.4.10(i)–(iii) and (v) are taken from [187]. Theorem 5.4.12 from [70] is a structure theorem for a PWP ring. Results 5.4.13–5.4.16 and Corollary 5.4.19 appear in [70]. Theorem 5.4.20 and Corollary 5.4.21 are taken from [66]. Examples 5.4.22 appears in [103] and [68]. In [118], Theorem 5.4.24 has been improved to the case when R is a Noetherian Rickart ring. Lemma 5.4.25 is in [70].

Results 5.5.1–5.5.3, Proposition 5.5.5, and Theorem 5.5.6 appear in [74]. Proposition 5.5.7 is in [369]. Examples 5.5.8, 5.5.9, Results 5.5.10–5.5.12 are taken from [74]. Theorem 5.5.14 is due to Birkenmeier, Kim, and Park [74]. Koh ([255] and [256]), Lambek [265], Shin [369], and Sun [388] showed that the Gelfand homomorphism θ is an isomorphism for various classes of rings.

Theorem 5.6.1 is due to Akalan, Birkenmeier, and Tercan (see [1, 3], and [393]). Theorem 5.6.2 appears to be a new result which is due to the authors. Results 5.6.4–5.6.6 appear in [85]. Theorem 5.6.7 was obtained by Pollingher and Zaks in [347], but we give the proof in a different way by applying Theorem 5.6.5. Proposition 5.6.8 is from [78]. Theorem 5.6.9 is completely generalized in [3]. Results 5.6.10–5.6.13 and Definition 5.6.14 appear in [85]. A characterization of generalized triangular right FI-extending rings are also considered in [64] (see Exercise 5.6.21.4). Results 5.6.15–5.6.18 appear in [85]. Theorem 5.6.19 was shown in [83], while Example 5.6.20 was given in [85]. Further related references include [51, 81, 91, 116, 122, 125, 135, 160], and [387].