Chapter 4
Using Concept Maps and Vee Diagrams to Analyse the "Fractions" Strand in Primary Mathematics

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The chapter presents data from Ken, a post-graduate student who participated in a case study to examine the value of concept maps and vee diagrams as means of communicating his conceptual analyses and developing understanding of the "Fractions" content strand of a primary mathematics syllabus. Ken's work required that he analysed syllabus outcomes and related mathematics problems and to display the results on concept maps and vee diagrams (maps/diagrams) to illustrate the interconnectedness of key and subsidiary concepts and their applications in solving problems. Ken's progressive maps/diagrams illustrated how his pedagogical understanding of fractions evolved over the semester as a consequence of social critiques and further revision. Progressive vee diagrams also illustrated his growing confidence to justify methods of solutions in terms of mathematical principles underlying the main steps.

Introduction

As mathematics teachers, it is incumbent upon us to ensure that we have a deep understanding of the content of the syllabus we are going to teach, that we can pedagogically and effectively mediate the development of school students' understanding and meaningful learning of mathematical concepts and processes by providing students with support as they engage with appropriately designed learning activities that challenge their mathematical thinking and reasoning. Further, teachers should have the capacity to diagnose instances of significant learning and, when it is not occurring, to provide appropriate support to assist students along their developmental learning trajectories (e.g., AAMT, 2007; NCTM, 2007). This requires that we are, not only familiar with the psychology and epistemology of learning to empower us to appropriately assist individuals coming to know, understand and learn new ideas meaningfully, but that we are also familiar with the range of socio-cultural factors that impact on learning in a social milieu so that we can support students'

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interactions and exchange of ideas to further the development of their conceptual understanding of mathematical situations, concepts and processes.

Shulman's taxonomy of knowledges for effective quality teaching includes knowledge of the content of the discipline, i.e., subject matter knowledge (SMK), and knowledge of teaching mathematics, i.e., pedagogical content knowledge (PCK). SMK is further defined as consisting of substantive knowledge (i.e., knowledge of principles and concepts of the discipline) and syntactic knowledge (i.e., knowledge of the discipline's methods of generating and validating knowledge) (Shulman, 1986). While knowledge has the properties of a commodity – that is, it is categorical, codifiable, and can be traded or exchanged (Lyotard, 1979, as cited in Feldman (1996)), understanding is the result of meaning-making in situations (Bruner, 1990) and requires that students actively organize knowledge hierarchically to show interconnections between relevant concepts, with the most general and most inclusive concepts superordinate to less general and most specific concepts (Ausubel, 2000; Novak & Gowin, 1984; Novak, 2002). That the conceptual interconnections may be described, in accordance with the discipline knowledge, to generate propositions that are mathematically correct, indicate the occurrence of learning that is meaningful and conceptually based.

Through social interactions in a classroom setting, students and teacher collaboratively negotiate meaning and shared understanding as they deliberate about, and engage with, a learning activity. According to Ausubel's theory of meaningful learning, students' understanding is developed through the construction of their own patterns of meanings and through participation in social interactions and critiques. When new knowledge is meaningfully learnt, the student decides which established ideas in his/her cognitive structure of meanings are most relevant to it. If there are discrepancies and conflicts, the student reorganizes and reconstructs existing patterns of meanings, reformulates propositions, or forms new patterns to allow for the effective assimilation of new meaning. For example, if the student could not reconcile the apparent contradictory ideas, then a degree of synthesis (integrative reconciliation) or reorganization of existing knowledge under more inclusive and broadly explanatory principles would be attempted (progressive differentiation). In contrast, rote learning is learning where students tend to accumulate isolated propositions rather than developing integrated, interconnected hierarchical frameworks of concepts (Ausubel, 2000; Novak & Cañas, 2006). Students' conceptual understanding of a domain may be displayed on hierarchical concept maps and vee diagrams.

Definitions of Concept Maps and Vee Diagrams

Concept maps are hierarchical graphs of interconnecting concepts (i.e., nodes) with linking words on connecting lines to form meaningful propositions. Concepts are arranged with the most general and most inclusive concepts at the top with less general and less inclusive concepts towards the bottom. Vee diagrams, in contrast, are vee structures with its vee tip situated in the problem or event to be analysed with its left side displaying the conceptual aspects (i.e., theory, principles, and concepts) of the problem/activity while the methodological aspects (i.e., records (given information), transformations and knowledge claims) are on the left side. A completed vee diagram represents a record of the conceptual and methodological information of solving a problem to answer some focus questions. The theoretical basis of these meta-cognitive tools is Ausubel's theory of meaningful learning (Ausubel, 2000; Novak & Gowin, 1984). Examples of maps/diagram are provided later.

Case Study

The case study of a post-graduate student, Ken, presented here, is about his concept maps and vee diagrams (maps/diagrams) constructed to illustrate and communicate his analysis for key and subsidiary concepts (knowledge), comprehension and critical interpretation (meaning) of the "Fractions" syllabus outcomes holistically at the macro level, from early primary to early secondary (i.e., Early Stage 1 to Stage 4 of the *NSWBOS K-10 Mathematics Syllabus*) (NSWBOS, 2002), and then at the micro level in the context of solving fraction problems. Over the semester, Ken transformed his evolving understanding of the relevant conceptual and methodological interconnections of key and subsidiary concepts of fractions and its progressive development across the primary years of schooling (NSWBOS, 2002) into visual displays of hierarchical conceptual interconnections on concept maps on one hand while on the other, of the synthesis of conceptual and methodological information in solving problems on vee diagrams.

Draft maps/diagrams were presented to the lecturer-researcher on a weekly basis in 1-hour workshops for 12 weeks. Through interactive discussions and negotiations of meaning in a social setting, Ken explained, justified and elaborated his constructed maps/diagrams while the lecturer-researcher challenged his explanations and critiqued his presented maps/diagrams. In between meetings, Ken revised his maps/diagrams to accommodate critical comments of the previous workshop in preparation for the next presentation.

Context

Ken was enrolled in a post-graduate mathematics education one-semester course which introduced the meta-cognitive strategies of concept mapping and vee diagramming to analyse and explicate the conceptual structure of a domain, in terms of the hierarchical interconnections of its key and subsidiary concepts as an abstract overview displayed on concept maps and to make visible the connections between methods of solving mathematics problems and the relevant concepts and principles underpinning the methods on vee diagrams. Ken practiced constructing maps/diagrams of different syllabus outcomes and mathematics problems. The set of maps/diagrams presented here constituted part of his assignments for the course. Only data from two tasks are presented.

Task 1 was for Ken to analyse the treatment of fractions in the NSW K-10 Mathematics Syllabus, specifically, the developmental learning trend of "fractions of the form $\frac{a}{b}$, equivalence and operations with fractions" before conducting a small pilot study to examine some students' conceptions of fractions/equivalence/operations (i.e., pilot study content). The students were selected from Years 4, 6 and 8 of a local school to correspond to the end years of Stages 2, 3 and 4 of the *K-10 NSW Mathematics Syllabus* (NSWBOS, 2002). Whilst the results of this pilot study is reported elsewhere (Jiygel & Afamasaga-Fuata'i, 2007), Task 1 focused on Ken's conceptual analyses of the "Fractions" content most relevant to his pilot study.

Task 2 required Ken to construct (a) an overview concept map of the "Fractions" content strand and (b) vee diagrams of fraction problems to demonstrate the application of some of the mapped conceptual interconnections in (a).

Ken was an international student enrolled in the Master of Education program in a regional Australian university. Although, he was an experienced primary teacher from his own country, it was important that he fully understood the development of the "Fractions" content strand in the primary mathematics and early secondary mathematics (PESM) syllabus implemented at the local school. Ultimately, his concept mapping and vee diagram tasks were intended to make explicit, for discussion and evaluation, his conceptual analyses and pedagogical understanding of (a) the content specific to his pilot study and (b) the overall "Fractions" content strand of the PESM syllabus. Therefore, the focus question of this chapter is: *In what ways do concept maps and vee diagrams facilitate the conceptual analyses and pedagogical understanding of syllabus outcomes*?

Data Collected and Analysis

The following sections present Ken's concept map and vee diagram data as required for Tasks 1 and 2 followed by a discussion of the results. Concept maps are analysed by considering the *propositions* formed by strings of connected nodes and linking words, the presence of *cross-links* between concept hierarchies which indicate integrative reconciliation between groups or systems of concepts and *multiple branching* nodes which indicate progressive differentiation between more general and less general concepts. Further, Ken's pedagogical content knowledge and understanding of fractions in accordance with the requirements of Tasks 1 and 2 (and as displayed on maps/diagrams) are compared to the relevant syllabus outcomes of the *K-10 NSW Mathematics Syllabus* (NSWBOS, 2002).

Task 1 Data and Analysis

Early Stage 1 and Stage 1 Concept Maps

Ken's analysis of Early Stage 1 (Kindergarten) syllabus outcomes in Fig. 4.1 shows that "half" or "halves" are introduced concretely from everyday context by the

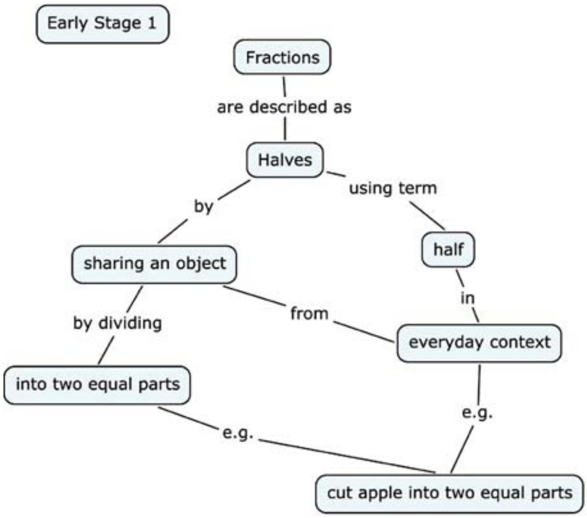

Fig. 4.1 Early Stage 1 *Fractions* concept map

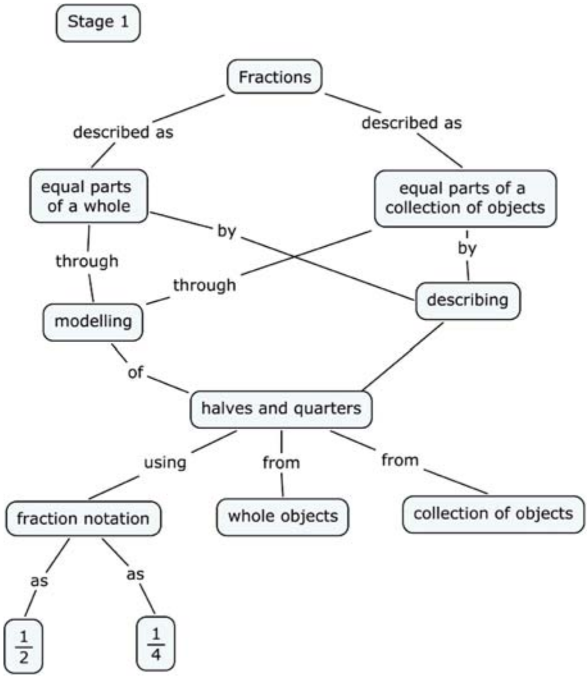

Fig. 4.2 Stage 1 *Fractions* concept map

sharing of an object and by dividing it into "two equal parts". The emphasis at this stage is that the two parts are equal to ensure fairness.

Figure 4.2 indicates that, at Stage 1 (Years 1–2), fractions are used in two different ways, to describe "equal parts of a whole" and to describe "equal parts of a collection of objects". Fractions also expand to include "quarters". Modelling and describing halves and quarters using whole objects and collection of objects continues and the notations "$\frac{1}{2}$" and "$\frac{1}{4}$" are introduced to represent "half" and "quarter" respectively. According to the syllabus (NSWBOS, 2002, p. 61), it is not necessary for students at this stage to distinguish between the roles of the numerator and denominator. Subsequently, students may use the symbol "$\frac{1}{2}$" as an entity to mean "one-half" or "a half" and similarly for "$\frac{1}{4}$". These last two points (Stage 1) and the "fairness" basis of Early Stage 1 were not included in Ken's maps.

Stage 2 Concept Map

Figure 4.3 shows that, at Stage 2 (Years 3–4), fractions are described in different ways such as "equal parts of a whole" and "equal parts of collection of objects" and students' repertoire of fractions increase to include those with denominators 8, 5, 10

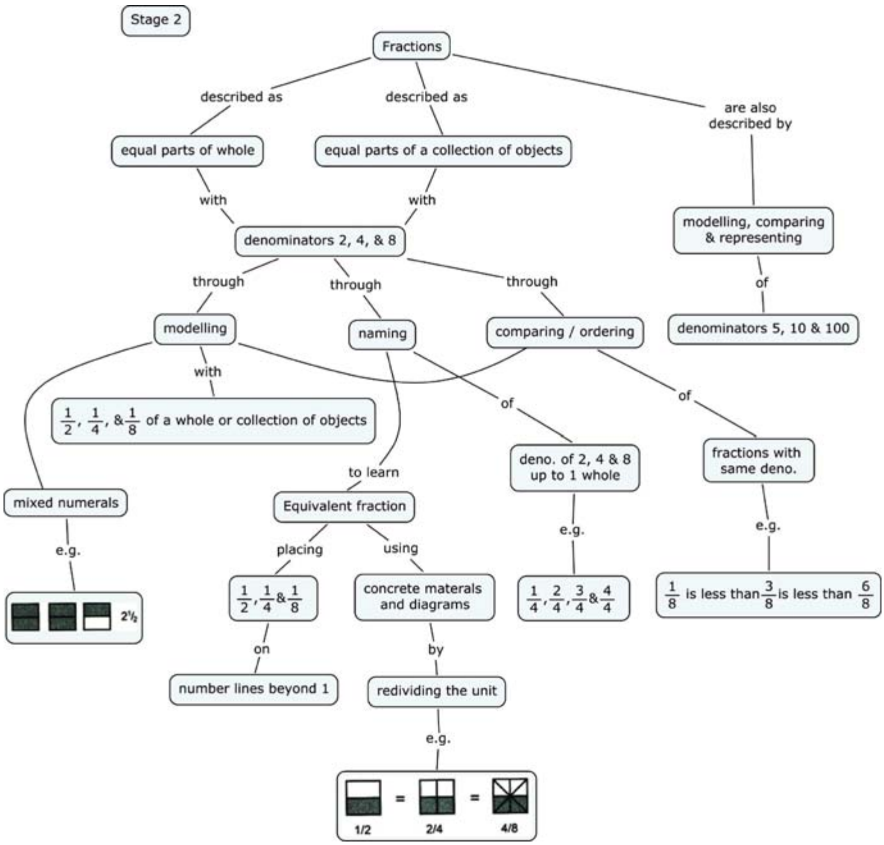

Fig. 4.3 Stage 2 *Fractions* concept map

and 100. Students learn about "modelling", "naming", and "comparing/ordering" fractions with "denominators 2, 4 & 8".

These fractions are modelled with "$\frac{1}{2}$, $\frac{1}{4}$ & $\frac{1}{8}$" of a whole or collection of objects and "mixed numerals" are modelled using diagrams as shown for $2\frac{1}{2}$. Naming fractions with "(denominators) 2, 4, & 8 up to 1 whole" is also developed, for example, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and $\frac{4}{4}$ for quarters. Naming is also used to learn about "equivalent fraction" (e.g., between half, quarters and eighths) by placing "$\frac{1}{2}$, $\frac{2}{4}$ & $\frac{4}{8}$" on "number lines beyond 1" and using "concrete materials and diagrams" by "redividing the unit" as shown diagrammatically for $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$. Furthermore, fractions with the same denominators are compared and ordered such as "$\frac{1}{8}$ is less than $\frac{3}{8}$ is less than $\frac{6}{8}$".

The rightmost branch illustrated that the "modelling, comparing & representing" of fractions with "denominators 5, 10 and 100" (i.e., fifths, tenths and hundredths) is also developed by extending the knowledge and skills illustrated by the branches to the left with halves, quarters and eighths. Whilst this last point is explicitly mentioned in the syllabus outcomes, Ken did not explicate this on his map, either by cross-linking to the left branches or extending by adding more nodes. Alternatively, it is a pedagogical concern that could be specifically addressed when planning lessons and designing classroom activities (not covered here).

Omitted also from the concept map is the notion of "numerator" although "denominator" is explicitly mentioned. However, syllabus notes caution that "(a)t this Stage, it is not intended that students necessarily use the terms 'numerator' and 'denominator'" (NSWBOS, 2002, p. 62). Other missing ideas are the use of fractions as operators related to division and the term "commonly used fractions" to refer to those with denominators 2, 4, 8, 5, 10 and 100 as recommended in the syllabus documentation. Although the introduction of decimals (to two decimal places), place value, money (as an application of decimals to two decimal places), and simple percentage are to occur at this stage, Ken did not include them in his concept map in Fig. 4.3. However, given the specific content of the pilot study and focus of Task 1, the omission was to be expected. It was nonetheless, a point that needed consideration for Task 2.

Stage 3 Concept Map

Ken's analysis of Stage 3 (Years 5–6) syllabus outcomes as concept mapped in Fig. 4.4 shows that the new fractions introduced are the thirds and sixths to supplement the halves, quarters, eighths, fifths, tenths and hundredths from previous stages. The "mixed numerals" branch illustrated fractions may be expressed with "mixed numerals" as "improper fractions" through the use of diagrams and number lines leading to a mental strategy. Adjacent to the right of this branch, are interconnections indicating that the "modelling" of thirds, sixths and eighths are to be done with "whole/collection of objects" and by "placing" them on a "number line between 0 and 1" to "develop equivalence" as illustrated by the 3 number lines provided in the middle of the map. The rightmost branch of the map (Fig. 4.4), which is

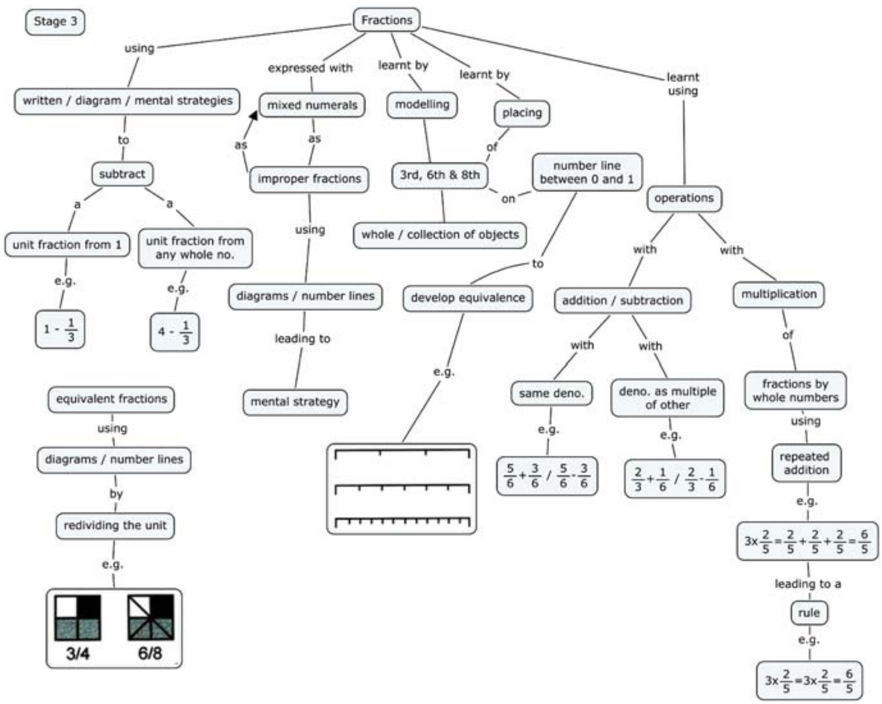

Fig. 4.4 Stage 3 *Fractions* concept map

inclusive under the node "operations" are two concept hierarchies (or sub-branches). Whereas the left hierarchy displayed "addition/subtraction" of fractions with the "same denominator" (e.g., $\frac{5}{6} + \frac{3}{6}/\frac{5}{6} - \frac{3}{6}$) and with "(denominators) as a multiple of the other" (e.g., $\frac{2}{3} + \frac{1}{6}/\frac{2}{3} - \frac{1}{6}$), the right hierarchy illustrated "multiplication" of "fractions by whole numbers" using "repeated addition" (as shown by the illustrative example) which led to a "rule" as shown by the last node.

In comparison, the leftmost branch inclusive under the "written/diagram/mental strategies" node are illustrative examples for subtraction of a "unit fraction from 1" (e.g., $1 - \frac{1}{3}$) and "unit fraction from any whole number" (e.g., $4 - \frac{1}{3}$). A concept sequence subsumed by the "equivalent fractions" node is isolated from the rest of the map. It depicted the "redividing the unit" idea that was viewed in Fig. 4.3 (for the equivalence of half, two-quarters and four-eighths) but is now illustrating the case of three-quarters and six-eighths.

Twelfths were not explicitly mentioned in the upper hierarchical levels of Fig. 4.4 but it is diagrammed on the number line shown in the middle. In this Stage, the label "simple fractions" referred to those with denominators 2, 3, 4, 5, 6, 10, 12, and 100 but was missing from the map. Also missing were "mental strategies" for finding equivalent fractions and for reducing them to lowest term and calculating unit fractions of a collection (e.g., $\frac{1}{3}$ of 30). Decimal and percentage coverage for this stage is omitted from the concept map. Instead it focused specifically on the case of fractions of the form $\frac{a}{b}$ and equivalence given the emphasis of the pilot study (e.g., Task 1).

Stage 4 Concept Map

Ken's analysis of the Stage 4 (Years 6–7) fraction outcomes in Fig. 4.5 shows a focus on "operations" involving "addition" and "multiplication/division" of "fractions & mixed numerals" and "substraction" of "fractions from a whole number". Illustrative examples are shown for the four operations. The rightmost branch illustrated that fractions may be expressed as "improper fraction" and as mixed numerals shown by a cross-link to the more general "fractions & mixed numerals" node. The leftmost branch, in comparison, depicted the idea that "equivalent fractions" may be reduced to its "lowest term".

The Stage 4 outcomes about decimals, percentages, ratios and rates are omitted from the concept map at this early stage of his mapping experiences. Instead, Ken focussed on the development of fractions of the form $\frac{a}{b}$, equivalent fractions and operations with fractions $\frac{a}{b}$ where the denominators are 2, 3, 4, 5, 6, 8, 10, 12 and 100 (defined in the syllabus as "simple fractions") most relevant to the focus of his pilot study.

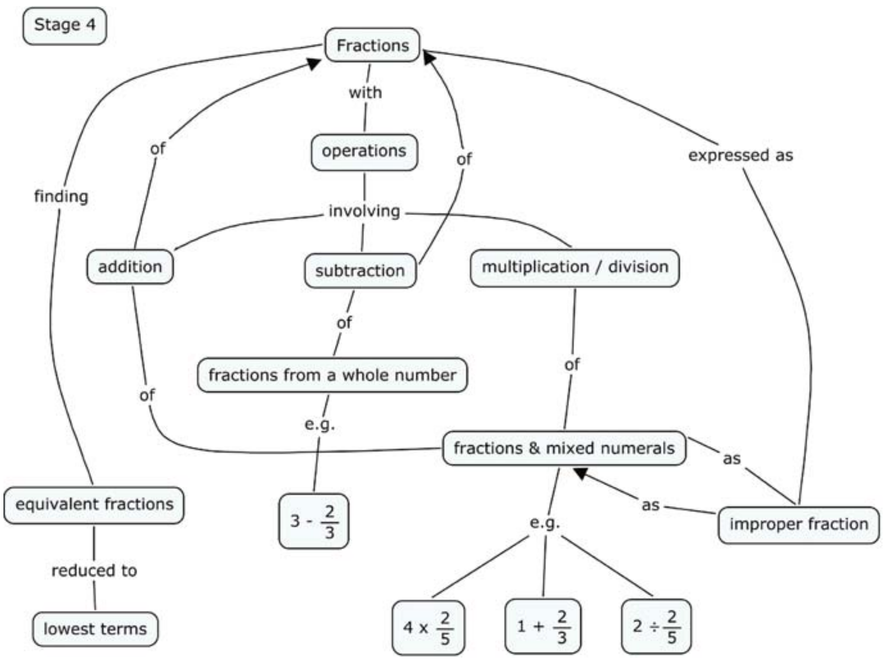

Fig. 4.5 Stage 4 *Fractions* concept map

Overall, Ken's five concept maps traced the introduction, development, and consolidation of (a) the views of simple fractions as "part of a whole", "part of a collection of objects" and a "number" on the number line; (b) equivalence between, and ordering of, simple fractions; and (c) operations with fractions and mixed numerals using concrete materials, illustratively with diagrams and the $\frac{a}{b}$ notation.

Whilst his objective for Task 1 is met with the presentation of the 5 concept maps (Figs. 4.1–4.5), it was incumbent upon him as a teacher to also develop a big picture overview of the coverage of fractions across the four stages of the PESM syllabus (NSWBOS, 2002), as described for Task 2.

Task 2 Data and Analysis

Overview "Fractions" Concept Map

Ken's overview concept map of the "Fractions" content strand, provided in Fig. 4.6, had over 100 nodes. Close-up views of the top, middle and bottom sections of the full map are shown in Figs. 4.7, 8 and 9. Structurally, all three sections are interconnected. For example, integratively reconciled links from the "Fractions" (Level 1) node and Level two nodes ("part of a whole", "part of collection of objects", "ratio", "decimals", "quotients", "percents", "probability", and "rates") all merge at the "$\frac{a}{b}$" node (Figs. 4.6 and 4.7), collectively illustrating the interconnections of the top section to the "$\frac{a}{b}$" node of the middle section (Fig. 4.8).

Interconnections between the middle (Fig. 4.8) and bottom sections (Fig. 4.9) are through the two progressive differentiation links from the node "computation", at the bottom of the middle section (Fig. 4.8), to link to "single operations" at the top of the bottom section and "mixed operations" towards the bottom of the bottom section (Fig. 4.9). Hence, the three sections (although split up for legibility and ease of discussion) are appropriately linked to provide a single overview concept map as requested for Task 2. In contrast to the early maps in Task 1, this overview concept map was completed towards the end of the study period, and as such, it was expected that Ken would accommodate some of the syllabus omissions raised during the presentations of Figs. 4.1 to 4.5. Each section is further examined below.

Top Section – Inclusive under the "Fractions" node (Fig. 4.7) are 8 branches subsumed under 8 Level 2 nodes, which, collectively, represented different ways of describing or using fractions. While the 2-leftmost branches was the focus of Task 1, Fig. 4.7 displays the full range of forms and applications of fractions based on Ken's critical analysis of the relevant syllabus outcomes. An inspection of the interconnections within each branch revealed Ken's attempts at elaborating, representing and communicating his understanding of the meaning and/or application of each of the Level 2 nodes. For example, the "part of a whole" branch included the extended proposition (P1): "Fractions" represent "part of a whole" which can be represented by "area models" such as "squares", "rectangles" and "circles". Further linear linking from each of the last 3 three nodes (i.e., "squares", "rectangles" and "circles"), showed illustrative examples descriptively and diagrammatically.

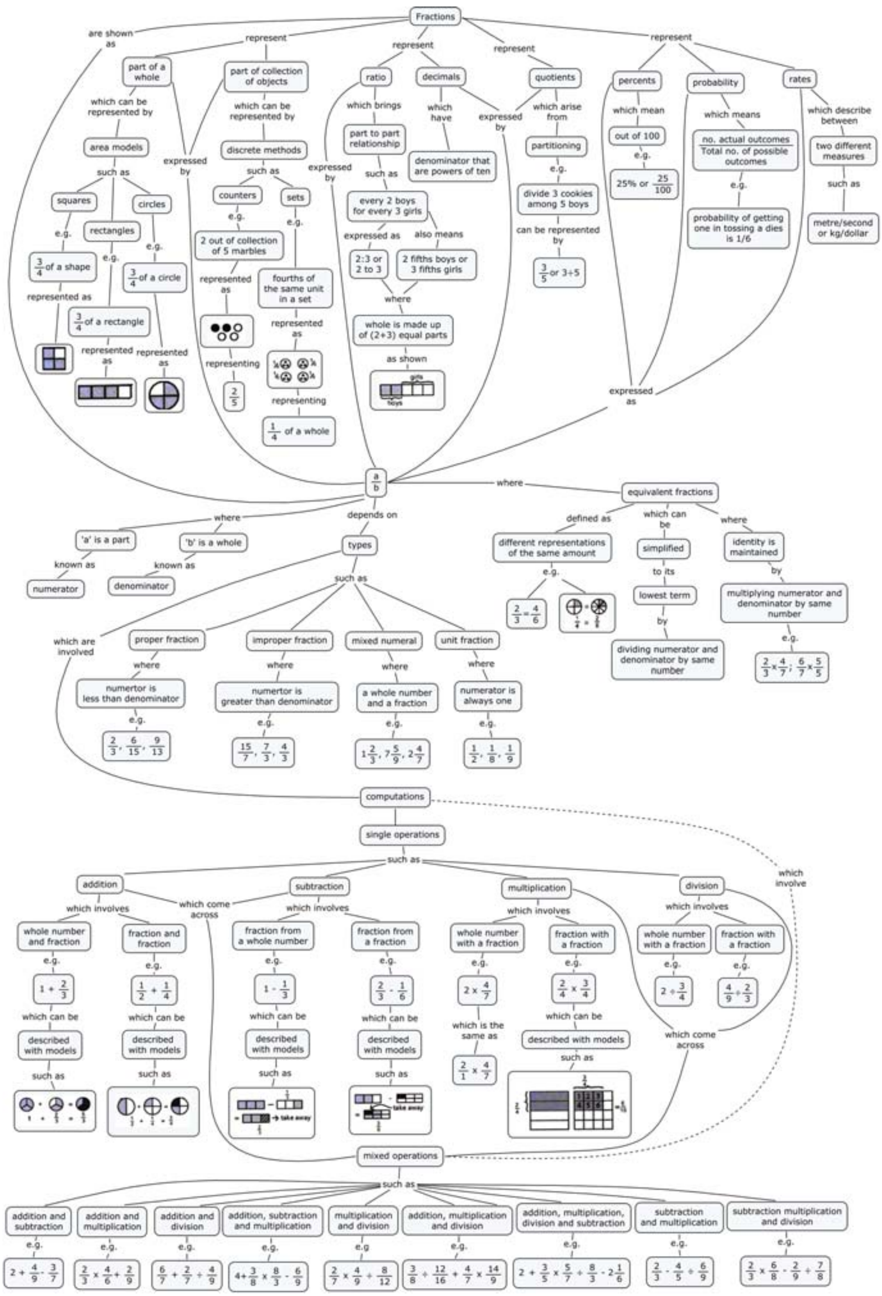

Fig. 4.6 Ken's overview *Fractions* concept map

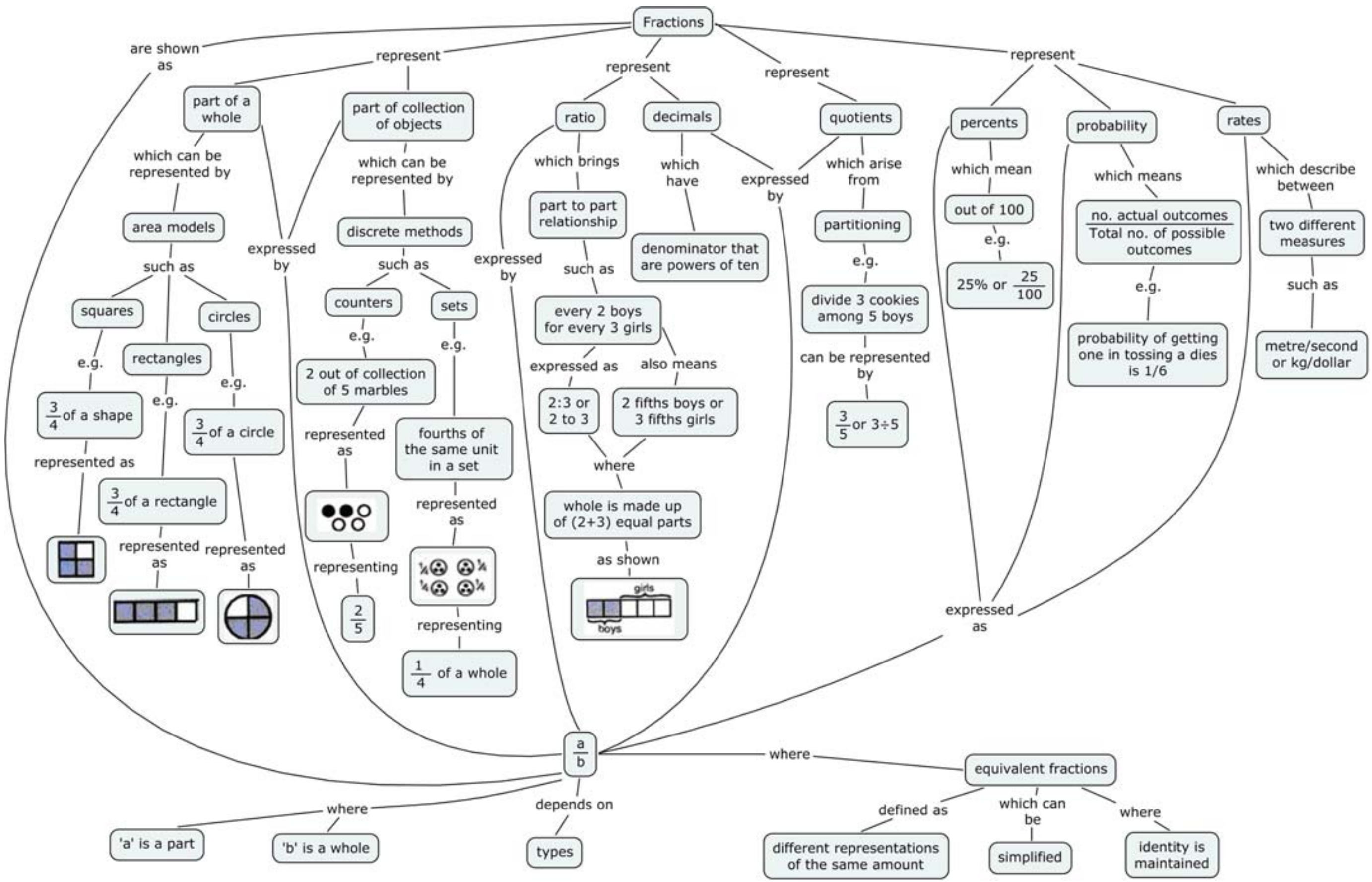

Fig. 4.7 Top section – Ken's overview *Fractions* concept map

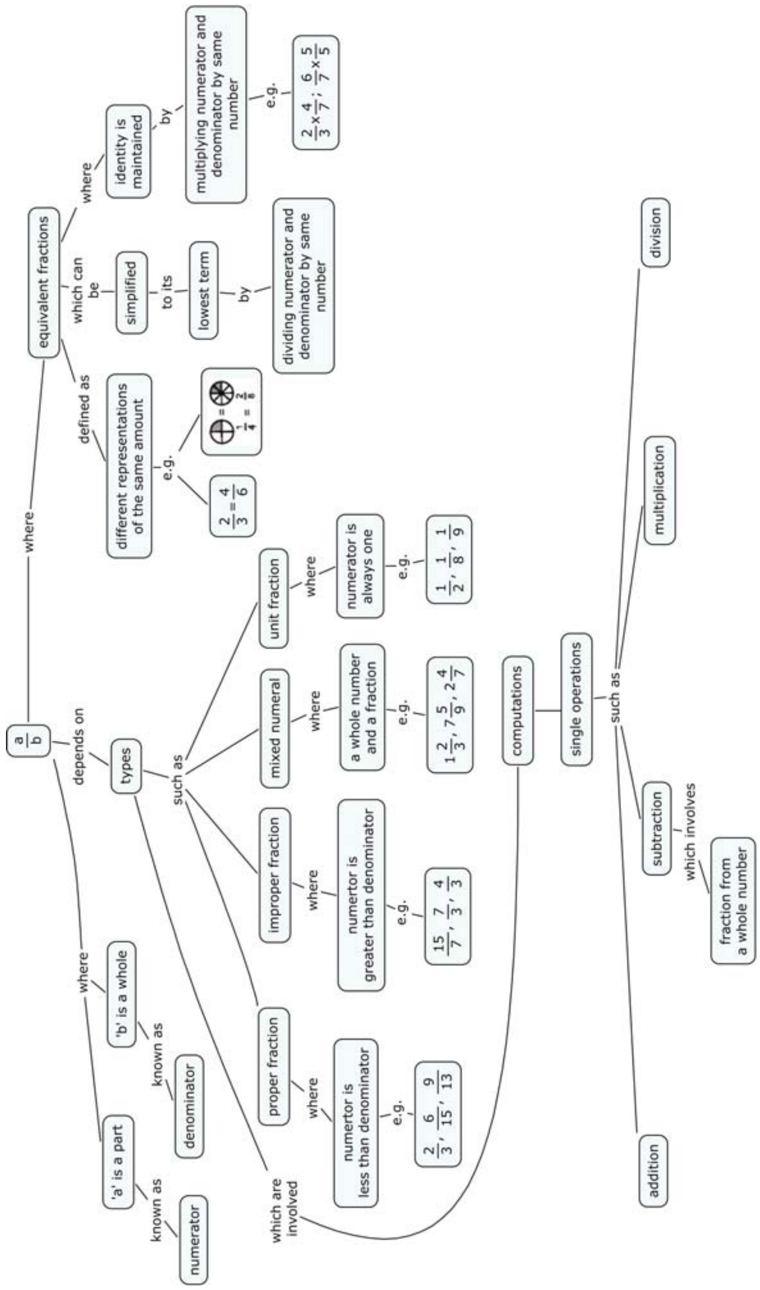

Fig. 4.8 Middle section – Ken's overview *Fractions* concept map

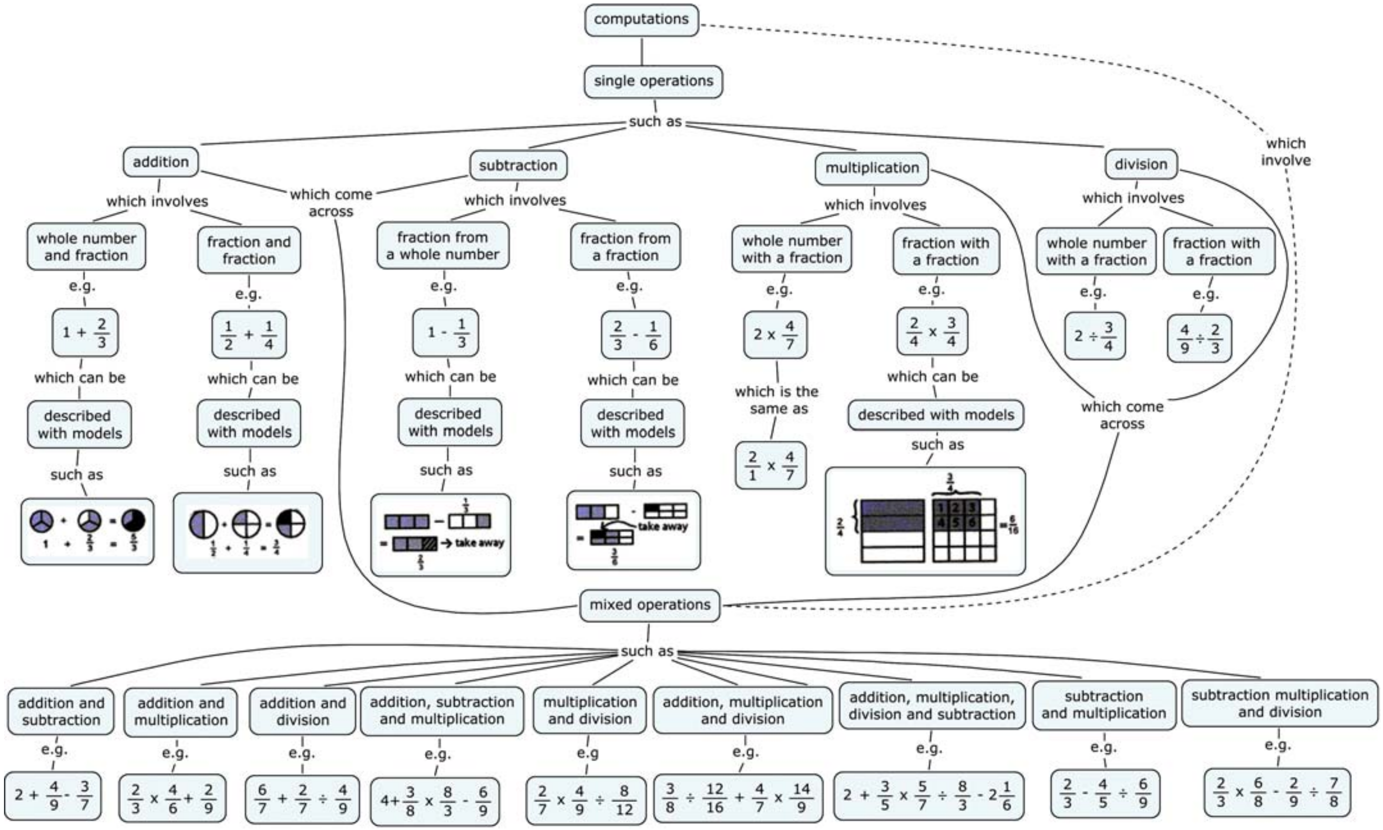

Fig. 4.9 Bottom section – Ken's overview *Fractions* concept map

For the next branch to the right, the extended proposition (P2) is: “Fractions” represent “part of collection of objects” which can be represented as “discrete methods” such as “counters” and “sets”. Using counters, an illustrative example is (P2a): “2 out of (a) collection of 5 marbles” represented as (diagrammatically shown) representing “$\frac{2}{5}$”. The illustrative example for “sets” is: (P2b): “fourths of the same unit in a set” represented as (diagrammatically shown) representing “$\frac{1}{4}$ of a whole”.

The extended proposition (P3a) for the “ratio” branch is: “Fractions” represent “ratio” which brings “part to part relationship” such as “every 2 boys for 3 girls” expressed as “2:3 or 2 to 3” where “whole is made up of (2+3) equal parts” as shown (diagrammatically). A second extended proposition (P3b) of the “ratio” branch is the result of the progressive differentiating links at the node “every 2 boys for 3 girls” node and integrative reconciliation links merging at the node “whole is made up of (2+3) equal parts”. Whereas the proposition (P4) for the “decimals” branch is: “Fractions” represent “decimals” which have “denominator(s) that are powers of ten” such as “$\frac{1}{10} = 0.1$”, that for the “quotients” branch is: (P5) “Fractions” represent “quotients” which arise from “partitioning”, for example, “divide 3 cookies among 5 boys” can be represented as “$\frac{3}{5}$ or $3 \div 5$”.

Towards the right, the extended proposition (P6) for the “percents” branch is: “Fractions” represent “percents” which means “out of 100”, for example, “25% or $\frac{25}{100}$” while that for the “probability” branch is: (P7) “Fractions” represent “probability”, which means “$\frac{\text{no actual outcomes}}{\text{Total no. of possible outcomes}}$”, for example, “probability of getting one in tossing a die is $\frac{1}{6}$”. Lastly, the rightmost branch displayed the extended proposition (P8): “Fractions” represent “rates” which describe relationship between “two different measures” such as “metre/second or kg/dollar”.

A single link at the extreme left of the map resulted in the proposition (P9) “Fractions” are shown as “$\frac{a}{b}$”. Each of the Level 2 nodes cross-linked to the “$\frac{a}{b}$” node resulting in 8 more propositions. Some examples are (P10): “Fractions” represent “part of a whole” expressed by “$\frac{a}{b}$”; (P11): “Fractions” represent “quotients” expressed by “$\frac{a}{b}$”; and (P12): “Fractions” represent “rates” expressed by “$\frac{a}{b}$”. Overall, the top section presented a connected web of knowledge displaying the different meanings, uses, applications and notation $\frac{a}{b}$ of fractions. These are more comprehensive and reflective of the “Fractions” syllabus outcomes in the PESM syllabus than Figs. 4.1 to 4.5.

Middle Section – Inclusive under the node “$\frac{a}{b}$” (Fig. 4.8) are four progressive differentiating links to nodes: “‘a’ is part”, “‘b’ is a whole”, “types” and “equivalent fractions”. The first proposition (P13) is: “$\frac{a}{b}$” where “‘a’ is part” known as “numerator” while the adjacent proposition (P14) is: “$\frac{a}{b}$” where “‘b’ is a whole” known as “denominator”. These two sub-branches addressed the information that was missing from Ken’s analysis of Stage 2 in Fig. 4.3.

The extended propositions from the “Fractions” node of Level 1 that are inclusive under “types”, from left-to-right, are (P15): “Fractions” are shown as “$\frac{a}{b}$” depends on “types” such as “proper fraction” where “numerator is less than denominator”, for example, “$\frac{2}{3}$, $\frac{6}{15}$, $\frac{9}{13}$”; (P16): “Fractions” are shown as “$\frac{a}{b}$” depends on “types” such as “improper fraction” where “numerator is greater than denominator”, for example, "$\frac{15}{7}$, $\frac{7}{3}$, $\frac{4}{3}$"; (P17): "Fractions" are shown as "$\frac{a}{b}$" depends on "types" such as "mixed numeral" where "a whole number and a fraction", for example, "$1\frac{2}{3}$, $7\frac{5}{9}$, $2\frac{4}{7}$"; and (P18): "Fractions" are shown as "$\frac{a}{b}$" depends on "types" such as "unit fraction" where "numerator is always one", for example, "$\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{9}$".

These propositions provided some concepts and definitions that were missing from Figs. 4.1 to 4.5 such as numerator and unit fraction. Inclusive within the "equivalent fractions" branch are the rightmost sub-branches of the middle section (Fig. 4.8). The relevant propositions are: (P19a): "Equivalent fractions" (are) defined as "different representations of the same amount", for example, "$\frac{2}{3} = \frac{4}{6}$"; (P19b): "Equivalent fractions" (are) defined as "different representations of the same amount", for example, (as shown diagrammatically for $\frac{1}{4} = \frac{2}{8}$); (P20): "Equivalent fractions" which can be "simplified" to its "lowest term" by "dividing the numerator and denominator by same number"; and (P21): "Equivalent fractions" where "identity is maintained" by "multiplying numerator and denominator by same number", for example, "$\frac{2}{3} \times \frac{4}{4}$; $\frac{6}{7} \times \frac{5}{5}$". Overall, this section defined the notation $\frac{a}{b}$ as well as defined and illustrated the different types of fractions.

Bottom Section – Two extended propositions from the top of the overview concept map (Fig. 4.6) connected all three sections (Figs. 4.7, 4.8 and 4.9). These are (P22): (1) "Fractions" are shown as " $\frac{a}{b}$" depends on "types" which are involved in "computation" using "single operations" such as "addition", "substraction", "multiplication", and "division" and (2) (P23): "Fractions" are shown as "$\frac{a}{b}$" depends on "types" which are involved in "computation" which involves "mixed operations" such as listed at the second to last level of the bottom section (see Fig. 4.9) from left to right including illustrative examples for each type.

Four branches particular to the bottom section (Fig. 4.9) are inclusive under the node "single operations" and subsumed under the nodes: "addition", "subtraction", "multiplication", and "division". Inclusive under the "addition" node of proposition P22 are two concept hierarchies. Reading from left-to-right, the left extended proposition (P22a) is: "single operations" such as "addition" which involves "whole number and fraction", for example, "$1 + \frac{2}{3}$" which can be "described with models" such as (diagrammatically shown); and the right one (P22b) is: "single operations" such as "addition" which involves "fraction and fraction", for example, "$\frac{1}{2}+\frac{1}{4}$" which can be "described with models" such as (shown diagrammatically) for "$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$". For the "subtraction" branch, the extended propositions are (P22c): "single operations" such as "subtraction" which involves "fraction from a whole number", for example, "$1 - \frac{1}{3}$" which can be "described with models" such as (shown diagrammatically) and (P22d): "single operations" such as "subtraction" which involves "fraction from a fraction", for example, "$\frac{2}{3} - \frac{1}{6}$" which can be "described with models" such as (shown diagrammatically). This pattern of propositional links continued all the way across the map to the rightmost node "division".

The second half of the bottom section is the result of merging the integratively reconciled links from the 4 operation nodes ("addition", "subtraction", "multiplication" and "division") and a progressively differentiating link from the "computation" node (bottom of the middle section) at the "mixed operations" node.

Some propositions are (P24): "single operations" such as "multiplication" which come across "mixed operations" and part of P23, namely, "computation" which involves "mixed operations". Emanating from the "mixed operations" node are multiple progressive differentiating links to describe and illustrate the different types of mixed operations at the bottom of Fig. 4.9. These 10 concept hierarchies (inclusive under "mixed operations") represented the final branch of the overview concept map.

Overall, the overview concept map showed hierarchical networks of concepts with the most general concepts at the top (e.g., Level 2 nodes) and progressively less general ones (e.g., "single operations") towards the middle with the more specific ones towards the bottom (e.g., "mixed operations"). Most (sub-)branches terminate with illustrative examples at the bottom. This generality pattern (i.e., most general -> most specific -> illustrative example) was consistently evident with most concept hierarchies.

Concept Maps and Vee Diagrams of "Fraction" Problems

Two "Fractions" problems are presented to illustrate Ken's use of maps/diagrams to communicate his thinking and reasoning when solving problems. His conceptual and process analysis results are provided in Fig. 4.10 for the first problem: "*How many $3\frac{1}{2}$ m lengths of rope can be cut from a length of 35 m?*" (Problem 1).

Concept Maps – Ken envisaged that a useful proposition (P1) is: "Fractions" (should be) understood as "Part of a whole" and "Whole" is "1" helps in solving "Word Problems", for example, (problem 1) while a second proposition (P2) is: "Fractions" involve "operations" involving "Fractions & numerals" for example, "$1 - \frac{2}{5}$" and "$2 \times \frac{3}{4}$." A multi-branched proposition (P3) is: "Fractions" involve "operations" like "Addition", "Subtraction", "Multiplication", "Division" which are involved in "Problem Solving" of "Quantities" expressed as "Word problems", for example, (problem 1). Proposition P3 demonstrated examples of a progressive differentiating link from the "Operations" node and integrative reconciliation links from the four operations before linking to the "Problem Solving" node. A short proposition (P4) is: "Quantities", for example, "$\frac{3}{4}$of 40 cm". Displayed at the "Word Problems" node is a proposition (P5): "Word Problems" (are) solved using "Strategies" of "Problem Solving" which also illustrated an example of an uplink from a less general concept to a more inclusive one towards the top of the map. Critiques concerned the possibility of elaborating further on what is meant by strategies.

In subsequent workshops, Ken considered a second fraction problem, namely, "*If $\frac{1}{4}$ of a post is below ground level, and 150 cm remains above the ground then find the total length of the post*" (Problem 2). This time, instead of constructing a new concept map, he revised and expanded his previous draft map (Fig. 4.10) to incorporate his thinking and reasoning about the two problems. The revised version combined the main concepts and strategies for the two problems (see Fig. 4.11).

Reading from left-to-right (Fig. 4.11), the leftmost branch displayed the proposition (P6): "Fractions" should be understood as "part of the whole" visualised as (shown diagrammatically) and where the "whole" implies "1" which is evidently a

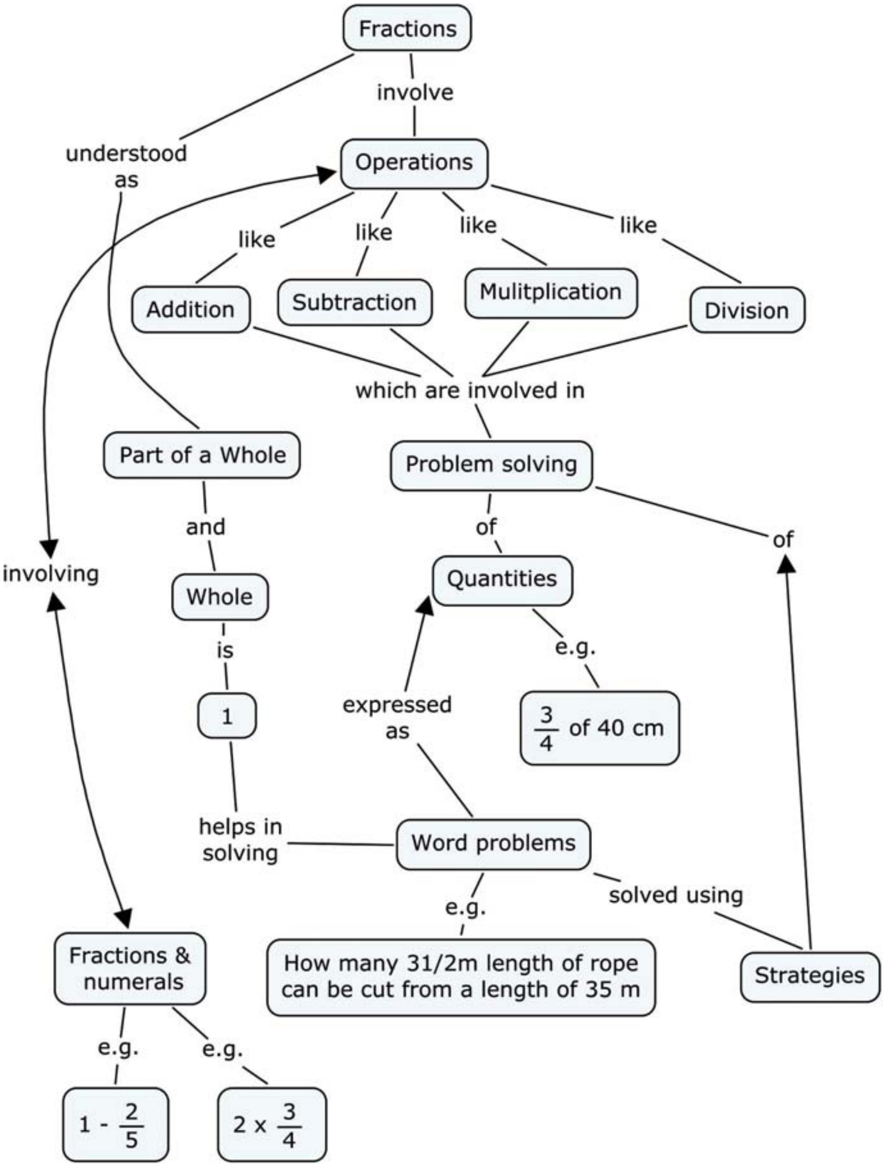

Fig. 4.10 Ken's draft concept map of the rope problem

better and expanded revision of proposition P1 of the previous map (Fig. 4.10) with the inclusion of a diagram to illustrate the relationship between "part" and "whole". A cross-link from this branch at node "1" connected to the adjacent "word problems" branch. The next proposition (P7) is: "Fractions" are used in "word problems" involving "quantities", for example, "$\frac{3}{4}$ of \$40". Some of the extended propositions,

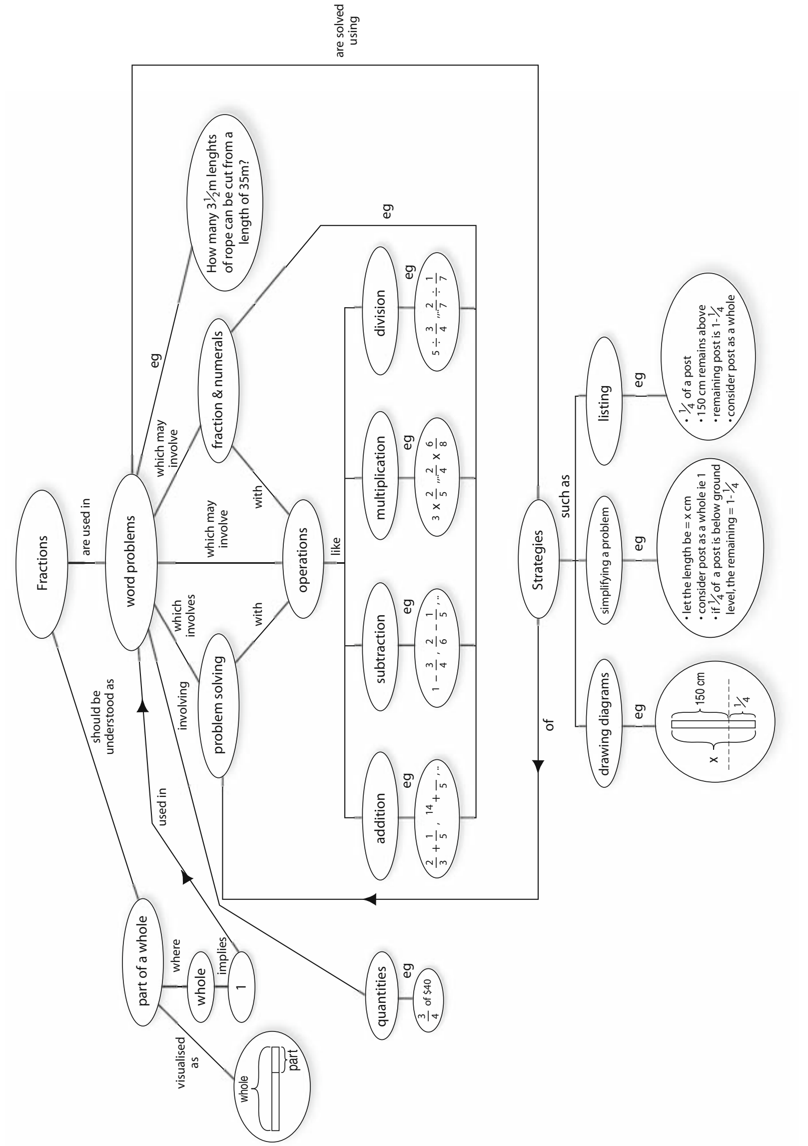

Fig. 4.11 Ken's revised concept map of the two problems

displayed in the middle, emanate from the multi-branching node "word problems". For example, (P8): "Fractions" are used in "word problems" which involves "problem solving" with "operations" like "addition", "subtraction", "multiplication", and "division" with each of the operation node linking to an illustrative example as shown; (P9): "Fractions" are used in "word problems" which may involve "operations"; (P10): "Fractions" are used in "word problems", for example, (problem 1); (P11): "Fractions" are used in "word problems" which may involve "fractions & numerals" with cross-links to the illustrative examples of proposition P8; (P12): "Fractions" are used in "word problems" which may involve "fractions & numerals" with "operations" like "addition", "subtraction", "multiplication", and "division".

Propositions P8, P9 and P12 are integrated at the "operations" node with P11 cross-linking to the propositions' illustrative examples. A number of integrative reconciliation links to, and progressive differentiating links from, nodes: "operations" and "strategies" are displayed.

The rightmost proposition (P13) involving the "strategies" node is: "word problems" are solved using "strategies" of problem solving such as "drawing diagrams" (P13a), "simplifying a problem" (P13b), and "listing" (P13c).

The three sub-branches inclusive under the "strategies" node represented the main additions in this revised map. Proposition P13a is an illustration of the "drawing diagrams" strategy for problem 1 while the second one (P13b) illustrated the "simplifying a problem" strategy. The latter demonstrated a meaningful interpretation and transformation of the given information as situated in the problem's context.

The third concept hierarchy (P13c) illustrated the "listing" strategy using the given information of the problem (i.e., first two bullet points) and application of fraction knowledge to the given information (i.e., last two bullet points). Taken together, the three concept hierarchies (P13a, b & c) depicted the results of his processes of representing, transforming and listing/interpreting the given problem. Overall, the revised and combined concept map for the two problems displayed the key ideas that were applied (e.g., P6, P7 and P8) with the "word problems" node shifting to a more general more inclusive level than the case was in Fig. 4.10, the results of the thinking and reasoning from given information and the key strategies applied as evidenced by propositions P13, and P13a, b, & c respectively. The diversity of propositions delineated above directly resulted from progressive differentiation such as at nodes "word problems", "operations" and "strategies" and integrative reconciliation at the last two nodes.

Vee Diagrams – Only one set of draft and revised vee diagrams are presented here, namely, that for problem 2. Figure 4.12 showed the given problem statement listed in the "Activity (Event)" section at the tip of the vee with the focus question (*What is the total length of the post?*) listed at the top of the vee. On the "Conceptual (Thinking) Side" on the left are the conceptual aspects relevant to the problem. For example, Ken identified three relevant outcomes (as listed under "Outcomes") with the appropriate "Prior Knowledge (Principles)" as shown and 5 relevant concepts listed under "Language (Concepts)". On the "Methodological (Doing) Side" on the right are the given information under "Data (Records)", interpretations and transformations of given information guided by the principles and as displayed under

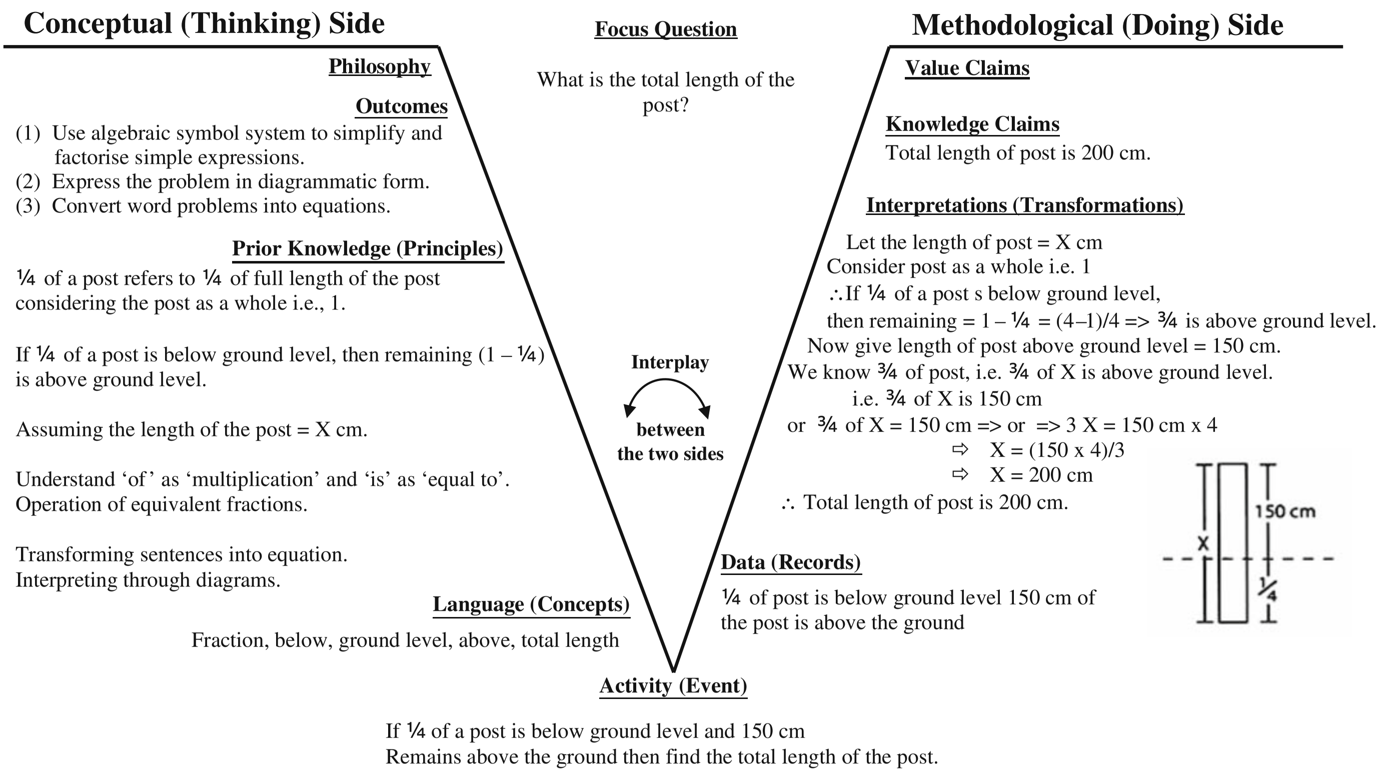

Fig. 4.12 Ken's first vee diagram of the *post* problem

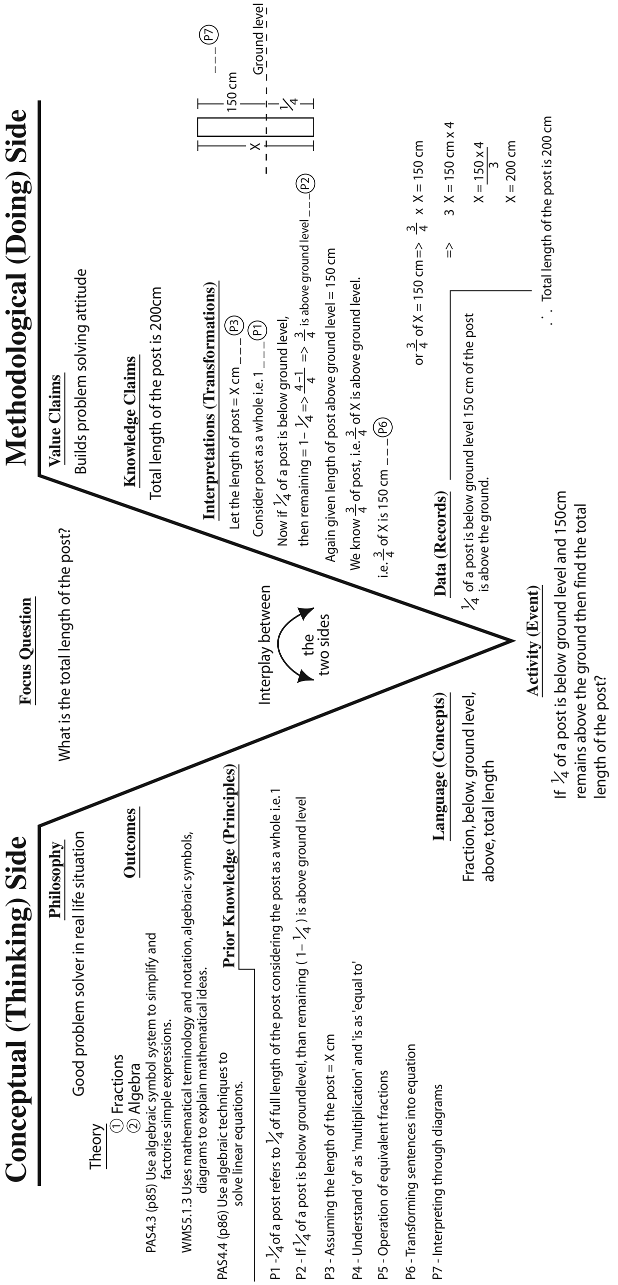

Fig. 4.13 Ken's revised vee diagram of the *post* problem

"Interpretations (Transformations)" with the answer to the focus question under "Knowledge Claims". Ken left the sections "Philosophy" and "Value Claims" blank.

Critiques in class challenged the relevance and appropriateness of the statements of "Prior Knowledge (Principles)" as conceptual statements, and the need to provide philosophical and value statements given the context of the problem. Other comments emphasised the need to ensure that there was a one-to-one correspondence between the listed principles and main steps of the methods. For example, which principles justify which main steps?

The revised vee diagram (Fig. 4.13) showed a number of additions such as the inclusion of references to the relevant syllabus outcomes of the *K-10 NSW Mathematics Syllabus* (NSWBOS, 2002) and reference labels for the listed principles. The "Philosophy" and "Value Claims" sections now included entries (albeit they could be better statements) with the addition of a "Theory" section to display the two main topics most relevant to the problem. For the "Prior Knowledge (Principles)" section, the principle list was now organised and labeled from P1 to P7. While the actual (P1 to P7) statements remained more or less the same as in Fig. 4.12, they would require further elaboration into more suitable theoretical justifications for each main step of the solution (displayed on the right side of the vee). For example, Principles P1 and P2 are not general statements about fractions, which would be more consistent with propositions P6 (Fig. 4.11) and P1 (Fig. 4.6). Instead, they represented Ken's interpretation of the given situation and application of proposition P6 (Fig. 4.11). As stated, these would be more suitable on the right side of the vee in the "Interpretations (Transformations)" section. Each principle from P4 to P7 would need to be rephrased and elaborated more fully so that they are more conceptual (i.e., general statements of relationships between concepts) and less procedural (e.g., P4, P5 and P6) to make them more suitable mathematical justifications as principles or formal statements of conceptual relationships.

Discussion and Implications

The presented concept maps and vee diagrams displayed the results of Ken's conceptual analyses, comprehension and pedagogical understanding of the "Fractions" content strand of the PESM syllabus as required for the two tasks. Whilst Figs. 4.1 to 4.5 represented his early interpretations of a subset of the content strand and early attempts at concept mapping, Fig. 4.6 provided a more macro level and comprehensive, summative concept map which evolved throughout the semester as a result of multiple cycles of presentations, social critiques and revisions. Figure 4.11, in contrast, provided a more situated view of fractions in the context of two problems. Taken together, the 3 sets of concept maps were qualitatively different in terms of their purpose, situation and therefore focus.

For Task 1, the situation was the pilot study and the purpose was for Ken to conceptually analyse the set of syllabus outcomes most relevant to the content of his pilot study. Subsequently, the focus was to make explicit the results of his conceptual analyses and to explicate his comprehension and depth of pedagogical understanding of the interconnections between, and meanings of the identified key and subsidiary ideas visually on concept maps. The quality of his conceptual analyses and pedagogical understanding was assessed by considering the hierarchical levels of organization of the selected concepts, grouping of concepts into coherent hierarchies and their interconnections, and richness of linking words to describe the interrelationships, which collectively formed networks of propositions. In addition, the accuracy of his conceptual analyses and pedagogical understanding was judged by whether or not he had analysed all of the relevant syllabus outcomes most pertinent to the identified situation based on syllabus documentation. The resulting concept maps for Task 1 collectively explicated the progressive development of the fraction concept using the different stages of cognitive development, (i.e., the concrete, iconic, and symbolic ($\frac{a}{b}$)) as part of a whole, part of a collection and a number across the different stages of the PESM syllabus. The demonstrated increasing structural complexity of the five maps not only reflected the incremental depth and breadth of content coverage (according to Ken's interpretations of syllabus documentation) but also his understanding of the interconnectedness between fraction concepts and its multiple models, representations and examples as evidenced by the progressive differentiation of concepts between more inclusive and less general ones and integrative reconciliation between coherent groups of ideas. As expected at primary level, a number of illustrative examples were selected from every day contexts especially for Early Stage 1, and increasingly more use, in subsequent stages, of concrete objects, the $\frac{a}{b}$ notation with selective denominators, diagrams, and number line to model, represent and demonstrate fraction types, equivalence, order and simple operations. Individually, each of the five concept maps illustrated the extent and depth of coverage to be expected for each stage in terms of the meanings to be developed using multiple models and increasingly more sophisticated development of order, equivalence and operations towards the upper stages (i.e., Stages 3 and 4). As Hierbert and Wearne (1986) argue, "conceptual knowledge grows as additional connections are made via assimilation and integration (as) . . . related bits of knowledge are related to earlier ideas" (p. 200). Although a number of key ideas were omitted as irrelevant to the situation for Task 1, there were still some significant relevant concepts missing especially towards the upper stages.

For Task 2, the situation for the first sub-task was the entire "Fractions" content strand in the PESM syllabus and the purpose was for Ken to conceptually analyse the content to be covered up to Stage 4 and the focus was to construct an overview concept map to include key and subsidiary ideas. Subsequently, over the semester, as a result of multiple cycles of presentations, social critiques and revisions, the Task 2 overview concept map evolved into a map that was organised around three main sections, each with its own particular emphasis. Focussing on the different definitions and applications of fractions in the top section, the middle section was on the definition of the notation $\frac{a}{b}$ and fraction types while the bottom section elaborated computations with fractions. The hierarchical levels of generality within each sub-branch appeared clearly defined, each following a basic sequence that constituted a (i) concept label, (ii) brief description of concept meaning, (iii) example description, and (iv) illustrative model/representation using diagrams, pictures, word descriptions and/or $\frac{a}{b}$ notation as demonstrated by propositions P1 to P12 of Fig. 4.7, P13 to P21 of Fig. 4.8 and P22 to 24 of Fig. 4.9. In addition, the overview concept map illustrated integration and interconnectedness between the 3-sections as demonstrated by the extended propositions P22 and P23. Demonstrating Ken's growth of comprehension and pedagogical understanding of the interconnectedness of "Fractions" syllabus outcomes were multiple occurrences of progressive differentiation nodes and integrative reconciliation links which produced a more comprehensive overview of fraction concepts, computation types and illustrative examples with over a hundred nodes, and structurally more complex concept map than the earlier ones.

For the second sub-task of Task 2 (i.e., Figs. 4.10 and 4.11), the situation was mathematical problems while the purpose was for Ken to conceptually analyse the problems and the focus was to construct a concept map of the key and subsidiary ideas pertinent to solving the particular problems. In contrast to the more abstract (i.e., general) concept maps provided in Figs. 4.1 to 4.5, the situation for this subtask was more contextualised. The most significant difference between the general and contextualised concept map was the inclusion of the "Problem Solving" concept hierarchy in Fig. 4.10, which introduced the concept of "strategies" for solving word problems, while the rest of the Fig. 4.10 nodes were similar to those previously viewed in the earlier abstract maps. Furthermore, although the "computation" branches of Fig. 4.6 displayed various single and mixed operations complete with illustrative examples, the distinction was that the "word problems" branch in the revised map (i.e., Fig. 4.11), in contrast, conveyed both the relevant conceptual propositions and problem solving strategies including the critical synthesis and application of these (hence providing the evidence of the critical thinking and reasoning involved) in the particular situation of problem 2.

Overall, the three sets of concept maps varied in the extent of their selection of concepts, as determined by the purpose, situation and focus of the task, and the hierarchical organisation and structural complexity of the interconnectedness of concepts and richness of its propositions. The various nodes indicating progressive differentiation and integrative reconciliation, hierarchical networks of concepts from most general to most specific, and richness of the resulting propositions evidenced the interconnectedness of Ken's knowledge and growth in conceptual and pedagogical understanding. The conceptual details (labels and meanings) and linking relationships apparent in the final overview concept map were substantively more enriched than the initial attempts of Figs. 4.1 to 4.5. Figure 4.6 is not only relatively more comprehensive conceptually, but it is also organisationally and structurally more differentiated and integrated than Figs. 4.1 to 4.5. In contrast to the general maps (Figs. 4.1 to 4.6), Fig. 4.11 captured the essential synthesis of, and interplay between, concepts, principles, generalisations and strategies most relevant in solving the two problems. Collectively, the concept maps displayed the evidence of Ken's conceptual analyses and his pedagogical content knowledge in terms of the substantive (or conceptual) knowledge of the PESM "Fractions" syllabus. As defined by Hierbert and Lefevre (1986), conceptual knowledge is "knowledge that is rich in relationships … a connected web of knowledge, in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network" (p. 3–4).

Interestingly, a comparison of Figs. 4.6 and 4.11 suggested a difference in the cognitive loading and processing, in terms of the critical thinking and reasoning involved for "computations" and "problem solving" as encapsulated by the propositions inclusive under "computation" in Fig. 4.6 in contrast to those under "word problems" and in particular under "strategies" in Fig. 4.11. More importantly, while proficiency in computations is desirable and equally important, Fig. 4.11 highlighted that being exposed to problem solving demands a much greater level of cognitive processing and critical thinking. Such higher level of reasoning would be required for constructing a vee diagram especially when completing the "Prior Knowledge (Principles)" and "Interpretations (Transformations)" sections not only to ensure a one-to-one correspondence between the listed principles on the left and the main steps of the solutions on the right but that the listed principles were appropriate general statements of relationships between concepts as mathematical justifications for the steps. According to Blanton and Kaput (2000), justification in any form is a significant part of algebraic (or mathematical) reasoning because it induces a habit of mind whereby one naturally questions and conjectures to establish a generalisation, or in the case of Ken, to establish a principle that underlies a main step in the solution. In addition, they argued that a classroom focus on justification could encourage students to conjecture in order to establish generalisations. The same can also be said for justification using principles to make explicit the conceptual bases of methods on vee diagrams and similarly to creatively structure concepts and linking words to form propositions on concept maps. Thus the data presented in this chapter demonstrated how, through the routine use of concept maps and vee diagrams, a student can develop habits of mind to conceptually and critically analyse mathematical situations, thinking and reasoning from situations and justifying interpretations and transformations in terms of the relevant substantive and syntactic knowledge of the discipline. In so doing, teachers and student teachers can develop a deeper more conceptual understanding of the structure of the relevant mathematics to pedagogically mediate meaning in an educational context.

Overall, the richness of the linking words on the connecting lines and consequently the conceptual richness of the propositions in Figs. 4.6 and 4.11 could be further improved to convey more enriched descriptions of interrelationships than had been shown. As Baroody, Feil, and Johnson (2007) proposed, "depth of understanding entails both the degree to which procedural and conceptual knowledge are interconnected and the extent to which that knowledge is otherwise complete, well-constructed, abstract and accurate" (p. 123). Similarly for further improvement are the statements of principles in Fig. 4.13 to make them more theoretical, less contextualised and less procedural statements, as principled mathematical justifications for the main steps. As Ellis (2007) pointed out for justifications and generalisations, (which is equally viable for justifications and methods of solutions), "learning mathematics in an environment in which providing justifications for one's generalisations (or methods of solutions) is regularly expected can promote the careful development of generalisations (or methods of solutions) that make sense and can therefore be explained" (p. 196). Furthermore, "a focus on justification may help students not only to better establish conviction in their generalisations (or methods of solutions) but also aid in the development of subsequent, more powerful generalisations" (Ellis, 2007, p. 196) (or more powerful methods of solutions).

Findings from this case study contributes empirical data to the literature on the use of concept maps and vee diagrams as viable tools that, through their routine construction, can encourage students to engage in the processes of critical analysis and synthesis, organising, thinking and reasoning, justifying and explaining their knowledge and understanding of a situation publicly for social critiques, discussion and evaluation. Further research is necessary to examine how these ideas could be implemented in a whole class situation in the classroom.

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