Chapter VIII

Brownian Motion

This chapter is on Brownian motions on the real line $\mathbb{R}$ with a few asides on those in $\mathbb{R}^d$. We concentrate on the Wiener process, the standard Brownian motion.

Section 1 introduces Brownian motions, indicates their connections to martingales, Lévy processes, and Gaussian processes, and gives several examples of Markov processes closely related to Brownian motions. Section 2 is on the distributions of hitting times and on the arcsine law for the probability of avoiding the origin. Section 3 treats the hitting times as a process; the process turns out to be an increasing pure-jump Lévy process that is stable with index 1/2.

The Wiener process W and its running maximum M are studied jointly in Section 4; it is shown that $M - W$ is a reflected Brownian motion and that $2M - W$ is a Bessel process. The relationship of M to $M - W$ is used to introduce the local time process for W; this is put in Section 5 along with the features of the zero-set for W. Brownian excursions are taken up in Section 6; the Poisson random measure of excursions is described, and the major arcsine law (on time spent on the positive half-line) is derived as an application.

Section 7 is on the fine properties of Brownian paths: total variation, quadratic variation, Hölder continuity, and the law of the iterated logarithm. Finally, in Section 8, we close the circle by showing that Brownian motions do exist; we give two constructions, one due to Lévy and one using Kolmogorov's theorem on continuous modifications.

1 Introduction

The aim is to introduce Brownian motions and Wiener processes. We start with an elementary definition and enhance it to its modern version. We shall also consolidate some results from the chapters on martingales

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and Lévy processes. Finally we describe several Markov processes which are closely related to Brownian motions. Throughout, $(\Omega, \mathcal{H}, \mathbb{P})$ is the probability space in the background.

1.1 DEFINITION. *A stochastic process* $X = (X_t)_{t\in\mathbb{R}_+}$ *with state space* $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ *is called a Brownian motion if it is continuous and has stationary independent increments. A process* $W = (W_t)_{t\in\mathbb{R}_+}$ *is called a Wiener process if it is a Brownian motion with*

1.2 $$W_0 = 0, \qquad \mathbb{E}\, W_t = 0, \qquad \text{Var } W_t = t, \quad t \in \mathbb{R}_+.$$

Let X be a Brownian motion. Then, $(X_t - X_0)_{t\in\mathbb{R}_+}$ is a continuous Lévy process. It follows from the characterization of such processes (see Theorem VII.4.2) that X has the form

1.3 $$X_t = X_0 + at + bW_t, \qquad t \in \mathbb{R}_+,$$

where a and b are constants in $\mathbb{R}$ and $W = (W_t)$ is a Wiener process independent of X_0. The constant a is called the *drift* rate, and b the *volatility* coefficient. The case $b = 0$ is degenerate and is excluded from further consideration.

Gaussian connection

Let W be a Wiener process. Its every increment $W_{s+t} - W_s$ has the Gaussian distribution with mean 0 and variance t:

1.4 $$\mathbb{P}\{W_{s+t} - W_s \in B\} = \mathbb{P}\{W_t \in B\} = \int_B dx \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}, \quad t > 0;$$

see Theorem VII.4.2 *et seq.* This implies, via the independence of the increments over disjoint intervals, that the random vector $(W_{t_1}, \ldots, W_{t_n})$ has the n-dimensional Gaussian distribution with

$$\mathbb{E}\, W_{t_i} = 0, \qquad \text{Cov}(W_{t_i}, W_{t_j}) = t_i, \qquad 1 \le i \le j \le n,$$

for arbitrary integers $n \ge 1$ and times $0 \le t_1 < \cdots < t_n$. Conversely, if $(W_{t_1}, \ldots, W_{t_n})$ has the n-dimensional Gaussian distribution described, then the increments $W_{t_1},\ W_{t_2} - W_{t_1},\ \ldots,\ W_{t_n} - W_{t_{n-1}}$ are independent Gaussian variables with mean 0 and respective variances $t_1,\ t_2 - t_1,\ \ldots,\ t_n - t_{n-1}$. These remarks prove the following.

1.5 THEOREM. *Let* $W = (W_t)$ *be a process with state space* $\mathbb{R}$. *It is a Wiener process if and only if it is continuous and is a Gaussian process with mean 0 and*

$$\text{Cov } (W_s, W_t) = s \wedge t, \qquad s, t \in \mathbb{R}_+.$$

The preceding theorem is often useful in showing that a given process is Wiener; see the next theorem for an instance of its use. It also raises an interesting question: Does Brownian motion exist? After all, the probability law of a Gaussian process is determined completely by its mean and covariance functions; how do we know that we can satisfy the further condition that its paths be continuous? We shall give two proofs of its existence in Section 8.

Symmetry, scaling, time inversion

1.6 THEOREM. *Let W be a Wiener process. Then, the following hold:*

a) Symmetry. The process $(-W_t)_{t\in\mathbb{R}_+}$ is again a Wiener process.

b) Scaling. $\hat{W} = (c^{-1/2}W_{ct})_{t\in\mathbb{R}_+}$ is a Wiener process for each fixed c in $(0,\infty)$, that is, W is stable with index 2.

c) Time inversion. Putting $\tilde{W}_0 = 0$ and $\tilde{W}_t = tW_{1/t}$ for $t > 0$ yields a Wiener process $\tilde{W} = (\tilde{W}_t)_{t\in\mathbb{R}_+}$.

Proof. Symmetry and scaling properties are immediate from Definition 1.1 for Wiener processes. To show (c), we start by noting that $\{\tilde{W}_t : t > 0\}$ is a continuous Gaussian process with mean 0 and $\text{Cov}(\tilde{W}_s,\ \tilde{W}_t) = s \wedge t$ for $s, t > 0$. Thus, the claim (c) will follow from Theorem 1.5 once we show that $\tilde{W}$ is continuous at time 0, that is, almost surely,

1.7 $$\lim_{t\downarrow 0} t\ W_{1/t} = 0.$$

Equivalently, we shall show that $W_t/t \to 0$ almost surely as $t \to \infty$. To this end, we start by noting that, if $n \geq 0$ is an integer and $n < t \leq n+1$,

1.8 $$\left|\frac{1}{t}\ W_t\right| \leq \frac{1}{n}\left|W_n + (W_t - W_n)\right| \leq \left|\frac{1}{n}W_n\right| + \frac{1}{n}\sup_{0\leq s\leq 1}\left|W_{n+s} - W_n\right|.$$

By the strong law of large numbers, $W_n/n \to 0$ almost surely, since W_n is the sum of n independent copies of W_1, and $\mathbb{E}\,W_1 = 0$. On the other hand, by Kolmogorov's inequality in continuous time (Lemma VII.1.39),

$$\mathbb{P}\left\{\frac{1}{n}\sup_{0\leq s\leq 1}\left|W_{n+s} - W_n\right| > \varepsilon\right\} \leq \frac{1}{n^2\varepsilon^2}\mathbb{E}\left|W_{n+1} - W_n\right|^2 = \frac{1}{n^2\varepsilon^2}$$

for each fixed $\varepsilon > 0$. Since $\Sigma\ 1/n^2$ is finite, Borel–Cantelli lemma (III.2.6) applies to show that, as $n \to \infty$, the very last term in 1.8 goes to 0 almost surely. Hence, $W_t/t \to 0$ almost surely as $t \to \infty$, and the proof is complete. □

In connection with the stability property 1.6b, we recall from Exercise VII.2.36 the following converse: if a continuous Lévy process is stable with index 2, then it necessarily has the form cW for some fixed constant c and some Wiener process W. As to the property 1.6c, time inversion, we remark at least two of its uses: first, the oscillatory behavior of a Wiener process near the time origin can be translated to its behavior for large times; second, conditioning on future values can be translated to become conditioning on the past. The following illustrates the latter point.

1.9 EXAMPLE. Let W be a Wiener process. For $0 < s < t$, consider the conditional distribution of W_s given that $W_t = x$. Instead of the direct approach, it is easier to use the time inversion property: The conditional distribution sought is that of $sW_{1/s}$ given that $tW_{1/t} = x$, which is the same as the distribution of $s(W_{1/s} - W_{1/t}) + \frac{sx}{t}$, which is Gaussian with mean sx/t and variance $s^2(1/s - 1/t) = s(1 - s/t)$. See Exercise 1.29 also.

Martingale connection

Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be a filtration over $(\Omega, \mathcal{H}, \mathbb{P})$, and let $W = (W_t)_{t\in\mathbb{R}_+}$ be a continuous process, adapted to $\mathcal{F}$, and having $W_0 = 0$. Recall Definition V.2.15: the process W is Wiener with respect to $\mathcal{F}$ if, for every t and u in $\mathbb{R}_+$, the increment $W_{t+u} - W_t$ is independent of $\mathcal{F}_t$ and has the Gaussian distribution with mean 0 and variance u.

If W is Wiener with respect to $\mathcal{F}$, then it is such in the sense of Definition 1.1 as well. Conversely, if W is Wiener in the sense of 1.1, then it is Wiener with respect to the filtration $\mathcal{G}^o$ generated by itself, and also with respect to the filtration $\mathcal{G}$, the augmentation of $\mathcal{G}^o$.

The following collects together characterizations in Proposition V.2.17 and Theorem V.2.19; see also Proposition V.6.21, Lemma V.6.22, and all the proofs. Recall that W is continuous, has $W_0 = 0$, and is adapted to the filtration $\mathcal{F}$.

1.10 THEOREM. *The following are equivalent:*

a) W is a Wiener process with respect to $\mathcal{F}$.

b) For each r in $\mathbb{R}$, the process $\{\exp(rW_t - \tfrac{1}{2}r^2t) : t \in \mathbb{R}_+\}$ is an $\mathcal{F}$-martingale.

c) The processes W and $\{W_t^2 - t : t \in \mathbb{R}_+\}$ are $\mathcal{F}$-martingales.

d) For every twice-differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ that is bounded along with its first derivative f' and second derivative f'', the process

$$M_t = f \circ W_t - \frac{1}{2}\int_0^t ds\ f'' \circ W_s, \qquad t < \mathbb{R}_+,$$

is an $\mathcal{F}$-martingale.

The preceding theorem is on the characterization of Wiener processes as martingales. Indeed, the connections between them run deep in both directions. In particular, it is known that every continuous martingale is obtained from a Wiener process by a random time change.

Wiener on a stochastic base

This is to re-introduce Wiener processes in the modern setup for Lévy processes; this is a repetition of Definitions VII.3.1 and VII.3.3 *et seq.* for this particular case.

Recall that a *stochastic base* is a collection $(\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$, where $(\Omega, \mathcal{H}, \mathbb{P})$ is a complete probability space, $\mathcal{F} = (\mathcal{F}_t)$ is an augmented right-continuous filtration, and $\theta = (\theta_t)$ is a semigroup of shift operators on Ω (each θ_t maps Ω into Ω, we have $\theta_0\omega = \omega$ for all ω, and $\theta_u \circ \theta_t = \theta_{t+u}$ for all t and u in $\mathbb{R}_+$).

1.11 DEFINITION. *A process $W = (W_t)$ is said to be Wiener on a stochastic base $(\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$ if it is a Wiener process with respect to $\mathcal{F}$ and is additive with respect to θ, the latter meaning that*

$$W_{t+u} = W_t + W_u \circ \theta_t, \qquad t, u \in \mathbb{R}_+.$$

The shift operators and additivity are useful for turning heuristic feelings into rigorous statements; for instance, $W_u \circ \theta_t$ is the increment over the future interval of length u when the present time is t, and the future is totally independent of the past. The right-continuity of $\mathcal{F}$ is essential for certain times to be $\mathcal{F}$-stopping times; augmentedness is for technical comfort. There is no loss of generality in all this: Every Wiener process in the sense of Definition 1.1 is equivalent to one in the sense of the preceding definition.

Brownian motions X on a stochastic base are defined similarly, except for the way the shifts work:

1.12 $$X_u \circ \theta_t = X_{t+u}, \qquad t, u \in \mathbb{R}_+.$$

This is equivalent to the additivity of W in the characterization 1.3 for X. See Figure 9 on page 341 for additivity.

Strong Markov property

Let $(\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$ be a stochastic base, and W a Wiener process over it. Let $\mathcal{G}^o = (\mathcal{G}^o_t)$ be the filtration generated by W, and $\mathcal{G}$ the augmentation of $\mathcal{G}^o$. Recall from Theorem VII.3.20 that $\mathcal{G}$ is right-continuous in addition to being augmented; it can replace $\mathcal{F}$ if needed. In particular, Blumenthal's zero-one law holds: every event in $\mathcal{G}_0$ has probability zero or one.

The following is the strong Markov property, Theorem VII.3.10, for the special Lévy process W, we re-state it here for reasons of convenience. As usual, we write $\mathbb{E}_T$ for $\mathbb{E}(\cdot|\mathcal{F}_T)$.

1.13 THEOREM. *Let T be an $\mathcal{F}$-stopping time. Then, for every bounded variable V in $\mathcal{G}_\infty$,*

$$\mathbb{E}_T(V \circ \theta_T)1_{\{T<\infty\}} = (\mathbb{E}V)1_{\{T<\infty\}}.$$

In particular, if $T < \infty$, the process $W \circ \theta_T = (W_{T+u} - W_T)_{u \in \mathbb{R}_+}$ is independent of $\mathcal{F}_T$ and is again a Wiener process.

Let U be a random time determined by the past $\mathcal{F}_T$ and consider $W_{T+U} - W_T$. Since $W \circ \theta_T$ is independent of $\mathcal{F}_T$, we may treat U as if it is fixed. We list the result next and give a direct proof. The heuristic idea is simpler, but requires some sophistication in its execution; see Exercise 1.31.

1.14 THEOREM. *Let T be an $\mathcal{F}$-stopping time, and let U be a positive real-valued variable belonging to $\mathcal{F}_T$. Let f be a bounded Borel function on $\mathbb{R}$, and define $g(u) = \mathbb{E} f \circ W_u,\ u \in \mathbb{R}_+$. Then,*

1.15 $$\mathbb{E}_T\ f\,(W_{T+U} - W_T)\ 1_{\{T<\infty\}} = g(U)\ 1_{\{T<\infty\}}.$$

Proof. a) The collection of f for which 1.15 holds is a monotone class. Thus, it is enough to show 1.15 for f that are bounded continuous. Fix f such, and note that the corresponding g is bounded and continuous in view of the continuity of W and the bounded convergence theorem for expectations.

b) Suppose that U is simple, say, with values in a finite subset D of $\mathbb{R}_+$. Since U is $\mathcal{F}_T$-measurable, $\{U = u\}$ is in $\mathcal{F}_T$ for each u in D. Thus,

$$\begin{aligned} &\mathbb{E}_T f\,(W_{T+U} - W_T)\, 1_{\{U=u,T<\infty\}} \\ &\quad = \mathbb{E}_T\ 1_{\{U=u,T<\infty\}}\ f(W_{T+u} - W_T) = g(u) 1_{\{U=u\}} 1_{\{T<\infty\}}, \end{aligned}$$

where we used the strong Markov property 1.13 at the last step. Summing both sides over all u in D yields 1.15. So, 1.15 holds for simple U.

c) In general, U is the limit of an increasing sequence (U_n) of simple variables in $\mathcal{F}_T$. Write 1.15 for U_n and take limits on both sides as $n \to \infty$. On the right side, the continuity of g shows that the limit is the right side of 1.15. On the left side, the continuity of W and f, together with the boundedness of f, imply that the limit is the left side of 1.15. □

Wiener and Brownian motion in $\mathbb{R}^d$

Let $W = (W_t)$ be a process with state space $\mathbb{R}^d$. It is called a *d-dimensional Wiener* process, or a *Wiener process in* $\mathbb{R}^d$, if its components $W^{(1)}, \ldots, W^{(d)}$ are independent Wiener processes. Then, W is a continuous Lévy process in $\mathbb{R}^d$ whose every increment $W_{s+t} - W_s$ has the d-dimensional Gaussian distribution with mean 0 and covariance matrix tI, the matrix I being the identity matrix in d-dimensions. So, for Borel subsets B of $\mathbb{R}^d$,

1.16 $$\mathbb{P}\,\{W_{s+t} - W_s \in B\} = \int_B\ dx\ \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}},$$

where $|x|$ is the length of the vector x in $\mathbb{R}^d$, and the integral is with respect to the Lebesgue measure on $\mathbb{R}^d$.

The properties of symmetry, 2-stability, and time inversion remain true for the d-dimensional case. Moreover, symmetry is extended to *isotropy*, invariance of the law of W under rotations and reflections: the probability laws of W and gW are the same for every orthogonal matrix g.

Brownian motions in $\mathbb{R}^d$ are defined as the ones in $\mathbb{R}$, except that the state space is $\mathbb{R}^d$ now. Every Brownian motion X in $\mathbb{R}^d$ is related to a Wiener process in $\mathbb{R}^d$ by the formula 1.3, but here a is a fixed vector in $\mathbb{R}^d$ and b is a fixed $d \times d$ matrix.

Markov processes

Brownian motions are the fundamental objects from which all continuous Markov processes are constructed. Several examples occur naturally as parts of the theory of Brownian motions. It will be convenient to provide a working definition for our current purposes and give several examples; see the next chapter for more.

Over some probability space $(\Omega, \mathcal{H}, \mathbb{P})$, let $X = (X_t)_{t \in \mathbb{R}_+}$ be a stochastic process with some state space $(E, \mathcal{E})$ and suppose that it is adapted to some filtration $\mathcal{F}^o = (\mathcal{F}^o_t)$. For each t, let P_t be a markovian kernel on $(E, \mathcal{E})$, that is, a transition kernel from $(E, \mathcal{E})$ into $(E, \mathcal{E})$ with $P_t(x, E) = 1$ for every x in E. Then, X is said to be an $\mathcal{F}^o$-*Markov process* with *transition semigroup* $(P_t)_{t \in \mathbb{R}_+}$ if

1.17 $$\mathbb{P}\{X_{s+t} \in B \mid \mathcal{F}^o_s\} = P_t(X_s, B), \qquad s, t \in \mathbb{R}_t,\ B \in \mathcal{E}.$$

The term "Markov process" without the mention of a filtration refers to the case where $\mathcal{F}^o$ is the filtration generated by the process itself.

The condition 1.17 implies that the Markovian kernels $P_t,\ t \in \mathbb{R}_+$, do indeed form a *semigroup*: $P_s\ P_t = P_{s+t}$ for s, t in $\mathbb{R}_+$, or, more explicitly,

1.18 $$P_{s+t}(x, B) = \int_E P_s(x, dy) P_t(y, B), \qquad s, t \in \mathbb{R}_+,\ x \in E,\ B \in \mathcal{E}.$$

Imagine a particle whose motion in E is represented by the process X. The defining property 1.17 means, in particular, that

$$P_t(x, B) = \mathbb{P}\{X_{s+t} \in B \mid X_s = x\}, \qquad x \in E,\ B \in \mathcal{E}.$$

The independence of this conditional probability from the time parameter s is referred to as *time-homogeneity* for X. Repeated use of 1.17 implies that, given the past $\mathcal{F}^o_s$, the conditional law of the future motion $\{X_{s+t} : t \in \mathbb{R}_+\}$ depends only on the present state X_s. A similar reasoning shows that the probability law of the process X is determined by its transition semigroup and its initial distribution (the distribution of X_0).

Examples

1.19 *Brownian motion in* $\mathbb{R}^d$. Let $X_t = X_0 + W_t,\ t \in \mathbb{R}_+$, where W is a Wiener process in $\mathbb{R}^d$ independent of X_0. Then, X is a Markov process with state space $\mathbb{R}^d$. Its transition semigroup is given as (see 1.16)

1.20 $$P_t(x, dy) = dy \frac{e^{-|y-x|^2/2t}}{(2\pi t)^{d/2}}, \qquad x, y \in \mathbb{R}^d.$$

In particular, W is a Markov process (with initial state $W_0 = 0$) with the same transition semigroup.

1.21 *Reflected Brownian motion.* Let $X = X_0 + W$ be a standard Brownian motion in $\mathbb{R}$, with initial state X_0. Define $R = |X|$, that is, R_t is the absolute value of X_t. Then, R is a Markov process with state space $\mathbb{R}_+$. To compute its transition semigroup (P_t), we start by noting that (see 1.20 with $d = 1$)

$$\mathbb{P}\left\{R_{s+t} \in dy | X_s = x\right\} = dy \left[\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} + \frac{e^{-(-y-x)^2/2t}}{\sqrt{2\pi t}}\right]$$

for x in $\mathbb{R}$ and y in $\mathbb{R}_+$. The right side remains the same whether x is positive or negative. Thus,

$$P_t(x, dy) = dy \left[\frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} + \frac{e^{-(y+x)^2/2t}}{\sqrt{2\pi t}}\right], \quad x,\ y \in \mathbb{R}_+.$$

1.22 *Bessel processes of index d.* This is the generalization of the preceding to higher dimensional Brownian motions. Let W be a Wiener process in $\mathbb{R}^d$ and define $R = |W|$, that is,

$$R_t = \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(d)}\right)^2}, \qquad t + \mathbb{R}_+.$$

Then, we call R a *Bessel process of index* d; some authors call it a *Bessel process of order* $\nu = d/2 - 1$, or *radial Brownian motion* in $\mathbb{R}^d$. It is a Markov process with state space $\mathbb{R}_+$; we shall show this. The case $d = 3$ plays an interesting role in describing the excursions of the one-dimensional Wiener away from the origin; we shall compute its transition semigroup explicitly.

For arbitrary dimension d, fixed, let B denote the closed unit ball in $\mathbb{R}^d$ and S its boundary, the unit sphere. For r in $\mathbb{R}_+$, then, $Br = \{xr :\ x \in B\}$ is the closed ball of radius r centered at the origin. From 1.16, we get

$$\mathbb{P}\left\{W_{s+t} \in Br \,|\, W_s = x\right\} = \int_{Br} dy\ \frac{e^{-|y-x|^2/2t}}{(2\pi t)^{d/2}}.$$

The left side remains unchanged if x, B, W are replaced with gx, gB, gW respectively, where g is some orthogonal transformation. But $gB = B$ since B is a ball centered at the origin, and gW has the same law as W by isotropy.

Hence, if $|x| = q$, choosing g such that $gx = (q,\ 0, \ldots, 0)$, we see that the left side is a function of $|x| = q$ only. Since $|W_s| = R_s$ and $\{W_{s+t} \in Br\} = \{R_{s+t} \le r\}$, we have shown that

1.23 $$\mathbb{P}\{R_{s+t} \le r\,|R_s = q\} = \int_{Br} dy\ \frac{e^{-|y-x|^2/2t}}{(2\pi t)^{d/2}}, \qquad q, r \in \mathbb{R}_+,$$

with $x = (q, 0, \ldots, 0)$ on the right side. Moreover, R_{s+t} is conditionally independent of $(W_u)_{u \le s}$ given W_s, and $(W_u)_{u \le s}$ determines $(R_u)_{u \le s}$. Thus, R_{s+t} is conditionally independent of $(R_u)_{u \le s}$ given W_s, and we have just seen that the conditional distribution of R_{s+t} given W_s is determined by $|W_s| = R_s$. Hence, R is Markov.

To evaluate the integral on the right side of 1.23, we turn to spherical coordinates. Write $y = ru$ with $u = (u_1, \ldots, u_d)$ on the unit sphere S. For $x = (q, 0, \ldots, 0)$, then, $|y - x|^2 = q^2 + r^2 - 2qr\ u_1$. Hence,

1.24 $$P_t\,(q, dr) = dr \quad r^{d-1}\ \frac{e^{-\left(q^2+r^2\right)/2t}}{(2\pi t)^{d/2}} \int_S \sigma\,(du)\, e^{qru_1/t},$$

where σ is the surface measure on S. The integral over S can be expressed in terms of modified Bessel functions (see Exercises 1.33 and 1.34), and hence the term Bessel process for R.

The surface integral is easy to evaluate when $d = 3$. We recall a result from elementary geometry: For spherical zones between two parallel planes that cut through S, the area is proportional to the distance h between the planes. So,

$$\int_S \sigma\,(du)\, e^{pu_1} = 2\pi \int_{-1}^{1} dh\ e^{ph} = \frac{2\pi}{p}\left(e^p - e^{-p}\right)$$

for $p > 0$, and the integral is the surface area 4π for $p = 0$. Putting this into 1.24 with $p = qr/t$, we see that, when $d = 3$,

1.25 $$P_t\,(q, dr) = dr \frac{r}{q}\left[\frac{e^{-(r-q)^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-(r+q)^2/2t}}{\sqrt{2\pi t}}\right] \qquad \textit{if}\ q > 0,\ r \ge 0,$$

and

1.26 $$P_t\,(q, dr) = dr \cdot \frac{2r^2 e^{-r^2/2t}}{\sqrt{2\pi t^3}} \qquad \textit{if}\ q = 0,\ r \ge 0,$$

We shall see later that, for almost every ω, we have $R_t(\omega) > 0$ for all $t > 0$; see 4.17 and thereabouts.

Exercises and complements

1.27 *Time reversal.* Let W be a Wiener process (on $\mathbb{R}$). Show that the probability laws of $\{W_t : 0 \le t \le 1\}$ and $\{W_1 - W_{1-t} :\ 0 \le t \le 1\}$ are the same. Hint: They are both Gaussian processes.

1.28 *Brownian bridge.* Let W be a Wiener process and define

$$X_t = W_t - tW_1, \qquad 0 \le t \le 1.$$

Observe that $X_0 = X_1 = 0$ and hence the name for the process $X = \{X_t : 0 \le t \le 1\}$. Obviously, X is a continuous Gaussian process. Compute its covariance function.

1.29 *Continuation.* Show that the probability law of X is the same as the conditional law of $\{W_t : 0 \le t \le 1\}$ given that $W_1 = 0$. In other words, show that, for $0 < t_1 < \ldots < t_n < 1$,

$$\mathbb{P}\left\{W_{t_1} \in dx_1, \ldots, W_{t_n} \in dx_n | W_1 = 0\right\} = \mathbb{P}\left\{X_{t_1} \in dx_1, \cdots, X_{t_n} \in dx_n\right\}.$$

Hint: Use time inversion (see Example 1.9) to show that the left side is an n-dimensional Gaussian distribution just as the right side, and compare their covariance matrices.

1.30 *Wiener space.* This is a special case of Exercise VII.3.24. Let W be a Wiener process on some probability space $(\Omega, \mathcal{H}, \mathbb{P})$. Let $C = C(\mathbb{R}_+ \mapsto \mathbb{R})$, the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. On it, we put the topology of uniform convergence on compacts: a sequence (w_n) in C converges to w in C in this topology if $\sup_{s \le t} |w_n(s) - w(s)| \to 0$ as $n \to \infty$ for every $t < \infty$. It can be shown that the Borel σ-algebra $\mathcal{B}_C$ corresponding to this topology is the same as the σ-algebra generated by the coordinate process $\{X_t : t \in \mathbb{R}_+\}$, where $X_t(w) = w(t)$ for every w in C. Let $\mathcal{G}^0_\infty$ be the σ-algebra generated by $\{W_t : t \in \mathbb{R}_+\}$.

For each ω in Ω, the path $W(\omega) : t \mapsto W_t(\omega)$ is a point in C. Show that the mapping $\omega \mapsto W(\omega)$ is measurable with respect to $\mathcal{G}^o_\infty$ and $\mathcal{B}_C$.

Let $\mathbb{Q} = \mathbb{P} \circ W^{-1}$, the distribution of W, where W is regarded as a random variable taking values in $(C, \mathcal{B}_C)$. Then, $\mathbb{Q}$ is the probability law of the Wiener process W. The probability space $(C, \mathcal{B}_C, \mathbb{Q})$ is called the Wiener space, and $\mathbb{Q}$ the Wiener measure. Finally, X is a Wiener process on $(C, \mathcal{B}_C, \mathbb{Q})$ and is called the *canonical Wiener process.*

1.31 *Alternative proof for Theorem 1.14.* Assume that $T < \infty$. Define $Y_t = W_t \circ \theta_T = W_{T+t} - W_T$. By the strong Markov property, the process $Y = (Y_t)$ is independent of $\mathcal{F}_T$ and is a Wiener process. Regard Y as a random variable taking values in $(C, \mathcal{B}_C)$, and consider $Y_U = W_{T+U} - W_T$. Since U is in $\mathcal{F}_T$ and Y is independent of $\mathcal{F}_T$, Exercise IV.2.27 is applicable. Conclude that 1.15 holds since

$$g(u) = \mathbb{E}\ f(W_u) = \mathbb{E}\ f(Y_u).$$

1.32 *Geometric Brownian motion.* Let W be a Wiener process and put

$$X_t = X_0 \exp(at + bW_t), \qquad t \in \mathbb{R}_+,$$

for fixed constants a and b in $\mathbb{R}$. Show that X is a Markov process.

1.33 *Bessel process of index $d = 2$.* Let R be as in Example 1.22 but with $d = 2$. It is a Markov process with state space $\mathbb{R}_+$. To compute its semigroup (P_t), we use 1.24 with $d = 2$, in which case S becomes the unit circle in $\mathbb{R}^2$. Since

$$\int_S \sigma(du) e^{pu_1} = \int_0^{2\pi} da\; e^{p\cos a} = 2\pi \sum_{k=0}^{\infty} \frac{(p/2)^{2k}}{(k!)^2} = 2\pi\; I_0(p),$$

one obtains

$$P_t(q, dr) = dr\; \frac{r}{t}\; e^{-(q^2+r^2)/2t}\; I_0\left(\frac{qr}{t}\right), \qquad q, r \le 0.$$

Here, I_0 is called the modified Bessel function of order 0, and hence the alternative name "Bessel process of order 0" for this R.

1.34 *Bessel processes.* Let R be as in Example 1.22 with arbitrary index $d \ge 2$. For $q > 0$ and $r \ge 0$, the formula 1.24 yields

$$P_t(q, dr) = dr \cdot \frac{q}{t} \left(\frac{r}{q}\right)^{d/2} e^{-\left(q^2+r^2\right)/2t}\; I_{d/2-1}\left(\frac{qr}{t}\right),$$

where I_ν is the modified Bessel function of order ν:

$$I_\nu\,(p) = \sum_{k=0}^{\infty} \frac{(p/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}, \qquad p \ge 0.$$

1.35 *Ornstein-Uhlenbeck process.* Let W be a Wiener process and write $W(t)$ for W_t. Let a and b be strictly positive constants, and define

1.36 $$X_t = X_0\; e^{-at} + b\; e^{-at}\; W\left(e^{2at} - 1\right),\; t \in \mathbb{R}_+,$$

where X_0 is independent of W.

a) Show that X defined by 1.36 is a Markov process with state space $\mathbb{R}$. It is also a Gaussian process if $X_0 = x$ fixed, or if X_0 is Gaussian.

b) Show that, as $t \to \infty$, the distribution of X_t converges weakly to the Gaussian distribution with mean 0 and variance b^2. If X_0 is Gaussian with mean 0 and variance b^2, and X_0 is independent of W, then X_t has the same distribution as X_0 for all t.

2 Hitting Times and Recurrence Times

Let $(\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$ be a stochastic base, and W a Wiener process on it; see Definition 1.11. By redefining $t \mapsto W_t(\omega)$ for a negligible set of ω if necessary, we may and do assume that $W_0(\omega) = 0$ and $t \mapsto W_t(\omega)$ is continuous for every ω in Ω. As before, we let $\mathcal{G}^o$ be the filtration generated by W, and $\mathcal{G}$ its augmentation. We are interested in the hitting times

2.1 $$T_a(\omega) = \inf\{t > 0 : \ W_t(\omega) > a\}, \qquad a \in \mathbb{R}_+, \omega \in \Omega.$$

It follows from general theorems that each T_a is a stopping time of $\mathcal{G}$, and its Laplace transform can be obtained by martingale techniques; see Chapter V for these. However, it is enjoyable to do the treatment once more and obtain the distribution directly by Markovian techniques.

Fix a in $\mathbb{R}_+$. For ω in Ω and $t > 0$, we have $T_a(\omega) < t$ if and only if $W_r(\omega) > a$ for some rational number r in $(0,t)$; this is because W is continuous and $W_0 = 0$. Since W_r is in $\mathcal{G}^0_t$ for each such r, it follows that the event $\{T_a < t\}$ belongs to $\mathcal{G}^0_t$. Hence, by Theorem V.7.4, T_a is a stopping time of the filtration $\left(\mathcal{G}^0_{t+}\right)$ and, therefore, of the finer filtrations $\mathcal{G} = (\mathcal{G}_t)$ and $\mathcal{F} = (\mathcal{F}_t)$.

Behavior at the origin

According to Blumenthal's zero-one law, every event in $\mathcal{G}_0$ has probability zero or one. The following is an application of it.

2.2 Proposition. *Almost surely, $T_0 = 0$.*

Proof. The event $\{T_0 = 0\}$ belongs to $\mathcal{G}_0$ and, thus, has probability 0 or 1. To decide which, note that $\{W_t > 0\}$ has probability $1/2$ and implies the event $\{T_0 < t\}$ for every $t > 0$. Thus, $\mathbb{P}\{T_0 < t\} \geq 1/2$ for every $t > 0$, and letting $t \to 0$ concludes the proof. □

The preceding proposition is deeper than it appears. Considering the definition 2.1 for $a = 0$ carefully, we see that the following picture holds for almost every ω: For every $\varepsilon > 0$ there is $u < \varepsilon$ such that $W_u(\omega) > 0$; there is also $s < \varepsilon$ such that $W_s(\omega) < 0$, this being by symmetry (see 1.6a). Taking ε of the second phrase to be the time u of the preceding one, and recalling the continuity of the paths, we conclude that for every $\varepsilon > 0$ there are $0 < s < t < u < \varepsilon$ such that $W_s(\omega) < 0$, $W_t(\omega) = 0$, $W_u(\omega) > 0$. Iterating the argument with s replacing ε yields the following.

2.3 Corollary. *For almost every ω, there are times $u_1 > t_1 > s_1 > u_2 > t_2 > s_2 > \ldots$ with limit 0 such that, for each n,*

$$W_{u_n}(\omega) > 0, \quad W_{t_n}(\omega) = 0, \ W_{s_n}(\omega) < 0.$$

Thus, the Wiener path $W(\omega)$ is highly oscillatory. Starting with $W_0\,(\omega) = 0$, the path spends no time at 0; it crosses over and under 0 at least infinitely many times during the time interval $(0, \varepsilon)$, however small $\varepsilon > 0$ may be. This statement has an interesting counterpart for large times obtained by time inversion, by applying 2.3 to the Wiener process of 1.6c.

2.4 COROLLARY. *For almost every ω there exist times $u_1 < t_1 < s_1 < u_2 < t_2 < s_2 < \ldots$ with limit $+\infty$ such that*

$$\lim W_{s_n}(\omega) = -\infty, \quad \lim W_{u_n}(\omega) = +\infty,$$

and $W_{t_n}(\omega) = 0$ for every n; in particular, the set $\{t \in \mathbb{R}_+ : W_t(\omega) = 0\}$ is unbounded.

We shall see shortly that the Wiener particle touches every point a in $\mathbb{R}$, and its path oscillates in the vicinity of a just as it does in the vicinity of the point 0.

Distribution of T_a

We start with a useful formula based on the strong Markov property and, more particularly, on Theorem 1.14. For its statement, it will be convenient to introduce the *Gaussian kernel*

2.5 $$G(t, B) = \mathbb{P}\{W_t \in B\} = \int_B dx\, \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}, \qquad t \in \mathbb{R}_+,\ B \in \mathcal{B}_{\mathbb{R}},$$

with $G(0, B)$ interpreted as $I(0, B)$ since $W_0 = 0$. Recall that $x + yB$ is the set of points $x + yz$ in $\mathbb{R}$ with z in B.

2.6 LEMMA. *For t and a in $\mathbb{R}_+$, and B a Borel subset of $\mathbb{R}$,*

$$\mathbb{P}\{T_a \le t, W_t \in B\} = \mathbb{E}\, G(t - T_a, B - a)\, 1_{\{T_a \le t\}}.$$

Proof. The case $a = 0$ follows from Proposition 2.2; the case $a > 0$ and $t = 0$ is trivially true. Fix $a > 0$ and $t > 0$ and B Borel, and write T for T_a. On the event $\{T \le t\}$, we have $W_T = a$ by the continuity of W and, thus,

$$W_t = W_{T+U} - W_T + a, \qquad \text{where } U = (t - T)1_{\{T \le t\}}.$$

Hence, by Theorem 1.14 with $f = 1_{B-a}$ and, therefore, $g(u) = G(u,\ B - a)$,

$$\begin{aligned}\mathbb{E}_T 1_{\{T \le t\}}\, 1_B(W_t) &= \mathbb{E}_T\, 1_{B-a}\,(W_{T+U} - W_T)\, 1_{\{T \le t\}} \\ &= G\,(t - T, B - a)\, 1_{\{T \le t\}}.\end{aligned}$$

Taking expectations on both sides completes the proof. □

2.7 PROPOSITION. *For a and t in $\mathbb{R}_+$, and B Borel,*

$$\mathbb{P}\{T_a \le t, W_t \in B\} = G(t, 2a - B), \qquad B \subset (-\infty, a)\,.$$

Proof. Since W is symmetric, we have $G(u, B - a) = G(u, a - B) = G(u,\ (2a - B) - a)$; thus, by the preceding lemma,

$$\mathbb{P}\{T_a \le t, W_t \in B\} = \mathbb{P}\{T_a \le t, W_t \in 2a - B\}\,.$$

If $B \subset (-\infty, a)$, then $2a - B \subset (a, \infty)$, and $W_t(\omega) > a$ implies that $T_a(\omega) \le t$. So, for $B \subset (-\infty, a)$, the right side becomes $\mathbb{P}\{W_t \in 2a - B\} = G(t,\ 2a - B)$. □

The preceding proposition is the basic computational formula. The restriction of B to subsets of $(-\infty, a)$ is without harm: we may re-state the result as

2.8 $$\mathbb{P}\{T_a > t, W_t \in B\} = G(t, B) - G(t, 2a - B), \qquad B \subset (-\infty, a),$$

and now the restriction on B is entirely logical, since the left side vanishes for subsets B of $[a, \infty)$.

In particular, taking $B = (-\infty, a)$ in 2.8, the event on the left side becomes $\{T_a > t\}$. So, since $2a - B = (a, \infty)$ then, 2.8 becomes

2.9 $$\mathbb{P}\{T_a > t\} = \mathbb{P}\{|W_t| \le a\} = 2\int_0^{a/\sqrt{t}} dx\, \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

The following collects together various interpretations of this formula.

2.10 Proposition. *Let $a > 0$. Then, $0 < T_a < \infty$ almost surely, but $\mathbb{E}T_a = +\infty$. The distribution of T_a is the same as that of a^2/Z^2, where Z is standard Gaussian. The distribution admits a continuous density function:*

2.11 $$\mathbb{P}\{T_a \in dt\} = dt\, \frac{ae^{-a^2/2t}}{\sqrt{2\pi t^3}}, \qquad t > 0.$$

Proof. Let Z have the standard Gaussian distribution. Then, W_t has the same distribution as $\sqrt{t}\, Z$. So, from 2.9,

$$\mathbb{P}\{T_a > t\} = \mathbb{P}\left\{\sqrt{t}\,|Z| \le a\right\} = \mathbb{P}\left\{\left(\frac{a}{Z}\right)^2 \ge t\right\} = \mathbb{P}\left\{\frac{a^2}{Z^2} > t\right\},$$

which means that T_a and a^2/Z^2 have the same distribution. Since $Z \in \mathbb{R}\backslash\{0\}$ almost surely, it follows that $T_a \in (0, \infty)$ almost surely. The density function in 2.11 is obtained by differentiating the last member of 2.9. It is seen from 2.11 that $\mathbb{E}T_a = +\infty$, since the integral of $1/\sqrt{t}$ over $(1, \infty)$ is infinity. □

The distribution in 2.11 appeared before in connection with stable processes with index 1/2; see VI.4.10 and also Chapter VII. Indeed, we shall see in the next section that $(T_a)_{a\in\mathbb{R}_+}$ is a stable Lévy process with index 1/2. For the present we note the corresponding Laplace transform (see Exercise 2.23 for one method, and 3.9 for a painless computation):

2.12 $$\mathbb{E}\, e^{-pT_a} = e^{-a\sqrt{2p}}, \qquad p \in \mathbb{R}_+.$$

Hitting times of points

The preceding Laplace transform appeared earlier, in Proposition V.5.20, for the time of entrance to $[a, \infty)$. The following is the reason for coincidence.

2.13 Proposition. *Fix a in $(0,\infty)$; define*

$$T_{a-} = \inf\{t > 0 : W_t \geq a\} = \inf\{t > 0 : W_t = a\}.$$

Then, T_{a-} is a stopping time of $\mathcal{G}^o$, and $T_{a-} = T_a$ almost surely.

Proof. Write T for T_{a-}. It is obviously a $\mathcal{G}^o$-stopping time. Clearly, $T \leq T_a$. By Proposition 2.10, $T_a < \infty$ almost surely. Thus, $T < \infty$ almost surely, and $W \circ \theta_T$ is again Wiener by the strong Markov property at T. Thus, by Proposition 2.2, we have $T_0 \circ \theta_T = 0$ almost surely, which completes the proof since $T_a = T + T_0 \circ \theta_T$. □

Indeed, as the notation indicates, T_{a-} is the left-limit at a of the increasing process $b \mapsto T_b$. To see this, let (a_n) be a strictly increasing sequence with limit a. For each n, then, $T_{a_n} < \infty$ almost surely and W is at the point a_n at time T_{a_n}. Since W is continuous, it must be at the point a at the time $T = \lim\ T_{a_n}$. So, the limit T is equal to T_{a-}.

Hitting times of negative points

All the results above extend, by the symmetry of W, to hitting times of $(-\infty, a)$ with negative a:

2.14 $$T_a = \inf\{t > 0 : W_t < a\}, \qquad a \leq 0.$$

For $a = 0$, the hitting times of $(0,\infty)$ and $(-\infty, 0)$ are both equal to 0 almost surely, and T_0 acquires an unambiguous double-meaning.

By the symmetry of W, each T_a has the same distribution as $T_{|a|}$. Thus, T_a has the same distribution as a^2/Z^2, where Z is standard Gaussian; this is for every a in $\mathbb{R}$.

Arcsine laws

We recall some elementary facts. Let X and Y be independent standard Gaussian variables. Then, X^2 and Y^2 are independent gamma distributed with shape index ½ and scale index ½. It follows that $A = X^2/(X^2 + Y^2)$ has the beta distribution with index pair (½, ½). This particular beta is called the *arcsine distribution*, because

2.15 $$\mathbb{P}\{A \leq u\} = \int_0^u dv \frac{1}{\pi\sqrt{v(1-v)}} = \frac{2}{\pi}\arcsin\sqrt{u}, \qquad 0 \leq u \leq 1.$$

Since $C = Y/X$ has the Cauchy distribution, we also have the connection to Cauchy distribution via $A = 1/(1 + C^2)$. Another connection can be noted by recalling that C has the same distribution as $\tan B$, where the angle B has the uniform distribution on $(0, 2\pi)$; thus, $A = (\sin\ B)^2$ where B is uniform on $(0, 2\pi)$, which explains 2.15 above.

The following arcsine law for the Wiener process is about the probability that W does not touch 0 during the time interval $[s, u]$. A more interesting arcsine law will be given later as Theorem 6.22. For $0 \le s < u < \infty$,

2.16 $$\mathbb{P}\{W_t \in \mathbb{R}\backslash\{0\} \text{ for all } t \in [s,u]\} = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{u}}.$$

We shall show this as a consequence of results on the recurrence times for the point 0; see Remark 2.22 below.

Backward and forward recurrence times

Thinking of the Wiener particle, let G_t be the last time before t, and D_t the first time after t, that the particle is at the origin: for t in $\mathbb{R}_+$,

2.17 $$G_t = \sup\{s \in [0,t] : W_s = 0\}, \quad D_t = \inf\{u \in (t,\infty) : W_u = 0\}.$$

For $t > 0$ fixed, W_t differs from 0 almost surely, which implies that $G_t < t < D_t$ almost surely. Also, in view of Corollaries 2.3 and 2.4 on the zeros of W for small and large times, it is evident that $0 < G_t$ and $D_t < \infty$ almost surely. Finally, note that D_t is a stopping time, but G_t is not; see Exercise 2.27 also.

2.18 Proposition. *Let A have the arcsine distribution as in 2.15. For each t in $\mathbb{R}_+$, then, G_t has the same distribution as tA, and D_t has the same distribution as t/A.*

Proof. Let X and Y be independent standard Gaussian variables. Recall that $T_a \approx a^2/Y^2$ for every a, where the symbol "$\approx$" stands for "has the same distribution as".

Consider $R_t = D_t - t$. If $W_t(\omega) = x$, then $R_t(\omega)$ is the hitting time of the point $-x$ by the path $W(\theta_t\omega)$. Since $W \circ \theta_t$ is Wiener independent of $\mathcal{F}_t$, and similarly for $(-W) \circ \theta_t$ by symmetry, we conclude that $R_t \approx {W_t}^2/Y^2$, where Y is independent of W_t. Thus, we may replace W_t with $\sqrt{t}X$; we obtain $R_t \approx t\, X^2/Y^2$. Hence,

$$D_t = t + R_t \approx t\left(X^2 + Y^2\right)/Y^2 \approx t/A$$

as claimed. Finally, $G_t \approx t\,A$ since, for s in $(0, t)$,

$$\mathbb{P}\{G_t < s\} = \mathbb{P}\{D_s > t\} = \mathbb{P}\left\{\frac{s}{A} > t\right\} = \mathbb{P}\{tA < s\}. \qquad \square$$

The terms forward and backward recurrence times refer to the variables

2.19 $$R_t = D_t - t, \qquad Q_t = t - G_t.$$

within the proof, it is shown that $R_t \approx t\,X^2/Y^2 = t\,C^2$, where C has the standard Cauchy distribution. The distribution of Q_t is the same as that of G_t:

2.20 $$\mathbb{P}\{G_t \le s\} = \mathbb{P}\{Q_t \le s\} = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}, \quad 0 \le s \le t;$$

this is because A and $1 - A$ have the same distribution. Various joint distributions can be obtained from the observation that

2.21 $$\{G_u < s\} = \{D_s > u\} = \{G_t < s, D_t > u\}\,, \qquad 0 \le s < t < u.$$

We put some such as exercises.

2.22 REMARK. Arcsine law 2.16 is a consequence of the arcsine distribution for G_t, because the event on the left side of 2.16 is the same as $\{G_u < s\}$.

Exercises

2.23 *Laplace transform for T_a.* This is to avoid a direct computation using the distribution 2.11. First, use 2.9 to show that

$$\mathbb{E}\, e^{-pT_a} = \int_0^\infty dt\; pe^{-pt}\; \mathbb{P}\{T_a \le t\} = \mathbb{P}\{|W_S| > a\}\,, \quad p \ge 0,$$

where S is independent of W and has the exponential distribution with parameter p. Recall that, then, W_S has the same distribution as $S_1 - S_2$, where S_1 and S_2 are independent exponential variables with parameter $\sqrt{2p}$. Conclude that 2.12 holds.

2.24 *Potentials.* Let $X = X_0 + W$ be the standard Brownian motion with initial state X_0. Write $\mathbb{E}^x$ for the expectation operator given that $X_0 = x$. For Borel $f : \mathbb{R} \mapsto \mathbb{R}_+$, define

$$\begin{aligned} U_p f(x) &= \mathbb{E}^x \int_0^\infty dt\; e^{-pt}\; f \circ X_t \\ &= \mathbb{E} \int_0^\infty dt\; e^{-pt}\; f(x + W_t), \qquad p \in \mathbb{R}_+,\; x \in \mathbb{R}. \end{aligned}$$

The function $U_p f$ is called the p-potential of f. Show that, for $p > 0$,

$$U_p f(x) = \int_{\mathbb{R}} dy\; u_p(x - y)\; f(y)\,,$$

where

$$u_p(x) = \int_0^\infty dt\; e^{-pt}\; \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} = \frac{1}{\sqrt{2p}}\; e^{-\sqrt{2px^2}},\; x \in \mathbb{R}.$$

2.25 *Zeros to left and right.* With G_t and D_t defined by 2.17, show that, for $0 < s < t < u$,

$$\mathbb{P}\{G_t \in ds\} = ds\; \tfrac{1}{\pi\sqrt{s(t-s)}}, \quad \mathbb{P}\{D_t \in du\} = du\; \tfrac{t}{\pi u\sqrt{t(u-t)}},$$
$$\mathbb{P}\{G_t \in ds,\; D_t \in du\} = ds\; du\; \tfrac{1}{2\pi\sqrt{s(u-s)^3}}.$$

2.26 *Recurrence times.* For R_t and Q_t defined by 2.19, show that, for $0 < q < t$ and $r \geq 0$,

$$\mathbb{P}\{Q_t \in dq\} = dq \, \frac{1}{\pi\sqrt{q(t-q)}}, \ \mathbb{P}\{R_t \in dr | Q_t = q\} = \frac{1}{2}\sqrt{\frac{q}{(q+r)^3}}.$$

2.27 *No stopping at G_t.* Of course, G_t is not a stopping time. This is to show that, moreover, G_t has no chance of coinciding with a stopping time: Let S be a stopping time of $\mathcal{F}$. We shall show that

$$\mathbb{P}\{S = G_t\} = 0.$$

By replacing S with $S \wedge t$, we may assume that $S \leq t$.

a) Show that $T_0 \circ \theta_S = 0$ almost surely; this is by the strong Markov property coupled with Proposition 2.2.

b) Show that, for almost every ω and every $\varepsilon > 0$, there is u in the interval $(S(\omega), \ S(\omega) + \varepsilon)$ such that $W_u(\omega) = 0$.

c) Show that the preceding statement is incompatible with the definition of G_t for ω in $\{S = G_t\}$.

3 Hitting Times and Running Maximum

The setup is as in the preceding section. We are interested in the process $T = (T_a)_{a \in \mathbb{R}_+}$ of hitting times and its relationship to the process $M = (M_t)_{t \in \mathbb{R}_+}$ of *running maximum*, where

3.1 $$M_t(\omega) = \max\nolimits_{0 \leq s \leq t} W_s(\omega), \qquad t \in \mathbb{R}_+, \ \omega \in \Omega.$$

The definition 2.1 of $T_a(\omega)$ remains true when $W_t(\omega)$ there is replaced with $M_t(\omega)$. Indeed, the paths $a \mapsto T_a(\omega)$ and $t \mapsto M_t(\omega)$ are functional inverses of each other:

3.2 $$T_a(\omega) = \inf\{t > 0 : M_t(\omega) > a\}, \ M_t(\omega) = \inf\{a > 0 : T_a(\omega) > t\}.$$

This relationship, together with the previous results on the T_a, shows that the following holds; see Figure 11 below as well. No further proof seems needed.

3.3 Lemma. *For almost every ω, the path $a \mapsto T_a(\omega)$ is right-continuous, strictly increasing, real-valued, and with $T_o(\omega) = 0$ and $\lim_{a \to \infty} T_a(\omega) = +\infty$. For almost every ω, the path $t \mapsto M_t(\omega)$ is increasing, continuous, real-valued, and with $M_o(\omega) = 0$ and $\lim_{t \to \infty} M_t(\omega) = +\infty$.*

In particular, $T_a(\omega) < t$ if and only if $M_t(\omega) > a$, this being true for every a and t in $\mathbb{R}_+$. Thus, the formula 2.9 may be re-stated as follows.

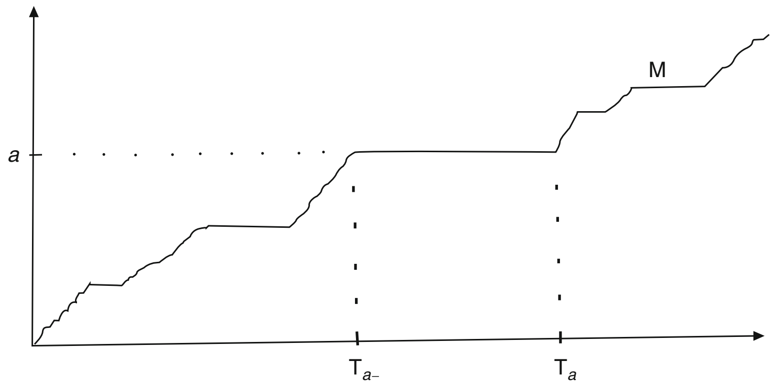

Figure 11: The path M is increasing and continuous; T_a is the time it hits the interval (a, ∞).

3.4 PROPOSITION. *For every a and t in $\mathbb{R}_+$,*

$$\mathbb{P}\{T_a < t\} = \mathbb{P}\{M_t > a\} = \mathbb{P}\{|W_t| > a\}.$$

3.5 REMARK. The preceding implies that M_t has the same distribution as $|W_t|$ for each t; thus, $\mathbb{E}\, M_t = \sqrt{2t/\pi}$ and $\mathbb{E}\, M_t^2 = t$ in particular. The probability law of the process M, however, is very different from that of $|W|$. The law of M is specified by the relationship 3.2 and the law of the process (T_a).

Hitting time process is stable Lévy

3.6 THEOREM. *The process $T = (T_a)_{a \in \mathbb{R}_+}$ is a strictly increasing pure-jump Lévy process. It is stable with index 1/2, and its Lévy measure is*

3.7 $$\lambda(dt) = dt \frac{1}{\sqrt{2\pi t^3}}, \qquad t > 0.$$

Proof. Fix a and b in $(0, \infty)$. In order for the process W to hit the interval $(a+b, \infty)$, it must hit (a, ∞) first, and, then, the future process $W \circ \theta_{T_a}$ must hit (b, ∞); in short,

$$T_{a+b} = T_a + T_b \circ \theta_{T_a}.$$

Since $T_a < \infty$ almost surely, the process $W \circ \theta_{T_a}$ is independent of $\mathcal{F}_{T_a}$ and is again a Wiener process; this is by the strong Marker property at T_a. Thus, $T_{a+b} - T_a = T_b \circ \theta_{T_a}$ is independent of $\mathcal{F}_{T_a}$ and has the same distribution as T_b. Together with Lemma 3.3, this shows that the process T is a strictly increasing Lévy process over the stochastic base $(\Omega, \mathcal{H}, \hat{\mathcal{F}}, \hat{\theta}, \mathbb{P})$, where $\hat{\mathcal{F}}_a = \mathcal{F}_{T_a}$ and $\hat{\theta}_a = \theta_{T_a}$; see Definition VII.3.3.

The distribution of T_a is the same as that of $a^2 T_1$; this is by Proposition 2.10. Thus, the Lévy process T is stable with index $1/2$. Every such process is of the pure-jump type, and its Lévy measure has the form $\lambda(dt) = dt\ c/t^{3/2}$; see Example VII.2.1. Finally, the constant c must be equal to $1/\sqrt{2\pi}$ in this case, since VII.2.1 and 2.12 imply

$$\mathbb{E}\ e^{-pT_a} = \exp\_\ a \int_{\mathbb{R}_+} \lambda(dt)\left(1 - e^{-pt}\right) = \exp\_\ a\sqrt{2p}.$$

□

Poisson jump measure

We use the preceding theorem to clarify the fine structure of the processes T and M. Recall the Itô-Lévy decomposition for Lévy processes; see Theorem VII.5.2 and VII.5.14 and *et seq.* The following needs no further proof.

3.8 THEOREM. *Let N be the random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ defined by*

$$N(\omega, B) = \sum_a 1_B\left(a, T_a(\omega) - T_{a\_}(\omega)\right), \qquad \omega \in \Omega,\ B \in \mathcal{B}\left(\mathbb{R}_+ \times \mathbb{R}_+\right),$$

where the sum is over all a for which $T_a(\omega) > T_{a-}(\omega)$. Then, N is Poisson with mean measure Leb $\times\, \lambda$*, where λ is as given by 3.7. Conversely,*

$$T_a(\omega) = \int_{(0,a]\times\mathbb{R}_+} N(\omega; db, du)\ u, \qquad a \in \mathbb{R}_+,\ \omega \in \Omega.$$

The relationship between the random measure N and the processes M and T are shown in the Figure 12 below. We describe some of the features:

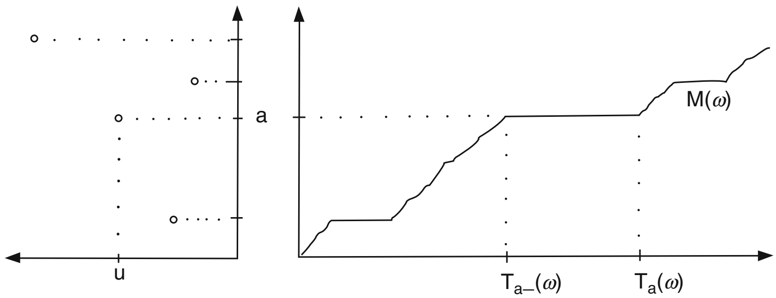

Figure 12: Big sized atoms of $N(\omega, \cdot)$ are marked with little circles on the graph left. Corresponding to the atom (a, u), there is a jump of size u from $T_{a-}(\omega)$ to $T_{a-}(\omega) + u = T_a(\omega)$, the path $M(\omega)$ stays constant at level a during the time interval $[T_{a-}(\omega),\ T_a(\omega)]$.

The following holds for almost every ω: A point (a, u) is an atom of the counting measure $N(\omega, \cdot)$ if and only if the path $M(\omega)$ has a flat stretch of length u at the level a, and then, the hitting time $T_a(\omega)$ of the interval (a, ∞) is exactly u time units later than the hitting time $T_{a-}(\omega)$ of the point a. Since $N(\omega, \cdot)$ has only countably many atoms, this situation occurs at countably many a only. Since there are infinitely many atoms in the set $(a, b) \times (0, \infty)$, the path $M(\omega)$ stays flat at infinitely many levels on its way from a to b; however, for $\varepsilon > 0$ however small, only finitely many of those sojourns exceed ε in duration.

The situation at a fixed level a is simpler. For $a > 0$ fixed, almost surely, there are no atoms on the line $\{a\} \times \mathbb{R}_+$; therefore, $T_a = T_{a-}$ almost surely.

Exercises

3.9 *Time change.* Show that, for every p in $\mathbb{R}_+$,

$$\int_{\mathbb{R}_+} e^{-pt}\, dM_t = \int_{\mathbb{R}_+} da\; e^{-pT_a}.$$

This suggests a painless way of computing the Laplace transform for T_a. Since (T_a) is Lévy, the Laplace transform has the form $e^{-a\varphi(p)}$. Hence, the expected value of the right side above is equal to $1/\varphi(p)$. Whereas, the expected value of the left side is easy to compute using $\mathbb{E}\, M_t = \sqrt{2t/\pi}$; the result is $1/\sqrt{2p}$. So, $\varphi(p) = \sqrt{2p}$, confirming 2.12 once more.

3.10 *Cauchy connection.* Let X be a Wiener process independent of W and, thus, independent of (T_a). Show that $(X_{T_a})_{a \in \mathbb{R}_+}$ is a Cauchy process; see Example VII.2.14 for Cauchy.

3.11 *Continuation.* Let $(X_t,\ Y_t)_{t \in \mathbb{R}_+}$ be a standard Brownian motion in $\mathbb{R}^2$ with initial state $(X_0,\ Y_0) = (0,\ y)$ for some fixed $y > 0$. Let S be the first time that the motion (X, Y) touches the x-axis. Find the distribution of X_S, the point touched on the x-axis.

4 Wiener and its Maximum

The setup and notations are as before in Sections 2 and 3. Our aim is to examine the joint law of the Wiener process W and its running maximum M defined by 3.1. We shall see that $M - W$ is a reflected Brownian motion and that it determines both M and W. As a supplement, we mention that $2M - W$ is a Bessel process of index 3, that is, it has the same law as the radial Brownian motion in dimension 3. These results will lead to excursions and local times in the next sections.

Distribution of M and W at a fixed time

4.1 PROPOSITION. *For fixed times $t > 0$,*

$$\mathbb{P}\{M_t \in da,\ M_t - W_t \in db\} = da\ db\ \frac{2(a+b)e^{-(a+b)^2/2t}}{\sqrt{2\pi t^3}},\ a, b \in \mathbb{R}_+.$$

Proof. Recall that $T_a(\omega) < t$ if and only if $M_t(\omega) > a$, and that the distribution of T_a is diffuse. Thus, we may re-write Proposition 2.7 in the form

$$\mathbb{P}\{M_t > a, W_t \le x\} = \mathbb{P}\{W_t > 2a - x\} = \int_{2a-x}^{\infty} dy\ \frac{e^{-y^2/2t}}{\sqrt{2\pi t}},\ x \le a,$$

Differentiating this with respect to a and x, and putting $a - x = b$, we see that the claimed expression holds. □

In the preceding proposition, it is worth noting the symmetry with respect to the arguments a and b. It follows that $M_t - W_t$ and M_t have the same marginal distribution, and the distribution of the latter is the same as that of $|W_t|$; see 3.4. This proves the following.

4.2 COROLLARY. *For fixed t, the variables M_t, $|W_t|$, and $M_t - W_t$ have the same distribution.*

As a process, M is very different from $|W|$ and $M - W$. But, the latter two are alike: they have the same law; see 4.6 below.

Construction of M from the zeros of $M - W$

Fix an outcome ω. The set

4.3 $$D_\omega = \{t \in \mathbb{R}_+ : \quad M_t(\omega) - W_t(\omega) > 0\}$$

is open, since it is the inverse image of the open set $(0, \infty)$ under the continuous mapping $t \mapsto M_t(\omega) - W_t(\omega)$. Thus, D_ω is a countable union of disjoint open intervals. For $\varepsilon > 0$, let $N_t(\omega, \varepsilon)$ be the number of those open intervals contained in $[0, t]$ and having lengths exceeding ε.

4.4 THEOREM. *For almost every ω,*

$$\lim_{\varepsilon \downarrow 0} \sqrt{2\pi\varepsilon}\ N_t(\omega, \varepsilon) = 2M_t(\omega), \quad t \in \mathbb{R}_+.$$

REMARK. This shows that M is determined by N, which is in turn determined by the zero-set of $M - W$. Interestingly, thus, $M - W$ determines both M and W.

Proof. In terms of the Poisson random measure N described by Theorem 3.8,

$$N_t(\omega, \varepsilon) = N(\omega, (0, M_t(\omega)) \times (\varepsilon, \infty)).$$

Thus, it is sufficient to show that, for each a in $\mathbb{R}_+$, almost surely,

4.5 $$\lim_{\varepsilon \downarrow 0} \sqrt{2\pi\varepsilon}\, N((0,a) \times (\varepsilon, \infty)) = 2a.$$

Recalling the mean measure of N (see 3.7 and 3.8), we have

$$\mathbb{E}\, N\left((0,a) \times \left(\frac{1}{k^2}, \infty\right)\right) = a \int_{1/k^2}^{\infty} dt \frac{1}{\sqrt{2\pi t^3}} = \frac{2\,a}{\sqrt{2\pi}}\, k.$$

Thus, since N is Poisson, the right side of the expression

$$N\left((0,a) \times \left(\frac{1}{n^2}, \infty\right)\right) = \sum_{k=1}^{n} N\left((0,a) \times \left(\frac{1}{k^2}, \frac{1}{(k-1)^2}\right]\right)$$

is the sum of n independent and identically distributed random variables with mean $2a/\sqrt{2\pi}$ each. Hence, by the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} N\left((0,a) \times \left(\frac{1}{n^2}, \infty\right)\right) = \frac{2a}{\sqrt{2\pi}}$$

almost surely. This proves 4.5 and completes the proof of 4.4. □

The preceding theorem can be strengthened: For almost every ω, the convergence shown is indeed uniform in t over compacts.

Process $M - W$ is a reflected Wiener

The following is the main result of this section. We shall prove it by constructing a Wiener process V such that $|V| = M - W$.

4.6 THEOREM. *The processes $M - W$ and $|W|$ have the same law.*

4.7 REMARK. This theorem is a corollary to Theorem 4.8 below, where we show the existence of a Wiener process V such that $M - W = |V|$. We start by analyzing the problem of constructing V.

Observing $M_t(\omega)$ and $W_t(\omega)$ yields only the absolute value $|V_t(\omega)|$; to obtain $V_t(\omega)$ we need to supply the sign. To see how this should be done, we examine Figure 13 below. Note that the path $W(\omega)$ coincides with the path $M(\omega)$ at all times except those belonging to the open set D_ω defined

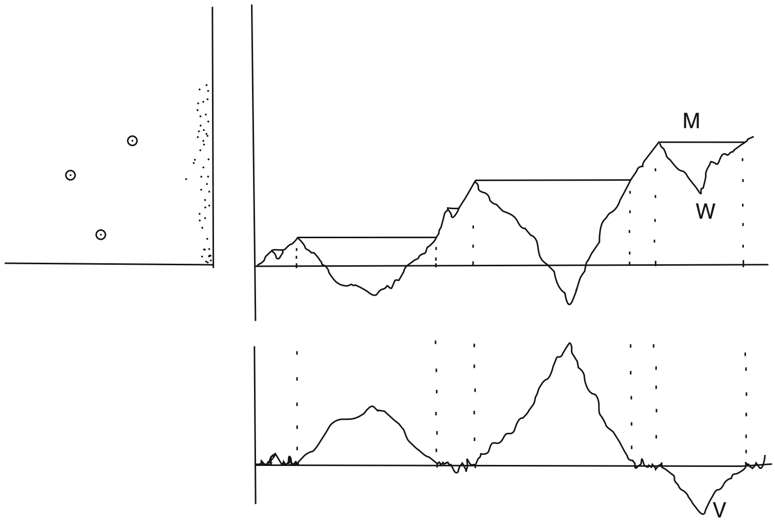

Figure 13: The path $M(\omega)$ is increasing continuous; $W(\omega)$ hangs like a stalactite from each flat stretch of $M(\omega)$. To construct $V(\omega)$, each stalactite is made to stand up or hang down from the time axis, the two choices being equally likely.

by 4.3. Over each component of D_ω, the path $M(\omega)$ stays flat and the path $W(\omega)$ hangs from $M(\omega)$ like a stalactite. Over the same interval, then, $V(\omega)$ will have to be either a stalactite hanging from the time axis, or a stalagmite standing up, the two possibilities being equally likely. Thus, we need to assign a sign, either positive or negative, to each stalactite hanging from $M(\omega)$.

To provide the needed signs, we need, independent of W, a countable independency of Bernoulli variables taking the values $+1$ and -1 with equal probabilities. If $(\Omega, \mathcal{H}, \mathbb{P})$ does not support such a sequence $(B_i)_{i\in\mathbb{N}}$, we enlarge it as follows: Let $D = \{+1, -1\}$, $\mathcal{D} = 2^D$, $\mu = \frac{1}{2}\,\delta_1 + \frac{1}{2}\,\delta_{-1}$ with δ_x being Dirac at x as usual; replace $(\Omega, \mathcal{H}, \mathbb{P})$ with

$$\left(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{P}}\right) = (\Omega, \mathcal{H}, \mathbb{P}) \times (D, \mathcal{D}, \mu)^{\mathbb{N}},$$

and, for $\hat{\omega} = (\omega, \omega')$ in $\hat{\Omega}$, define $\hat{W}_t(\hat{\omega}) = W_t(\omega)$ and let $B_i(\hat{\omega})$ be the i-coordinate of ω'. In the next theorem, we shall assume that this enlargement, if needed, is done already. Theorem 4.6 is a corollary of the next theorem.

4.8 Theorem. *There exists (on a possibly enlarged probability space) a Wiener process V such that $M - W = |V|$.*

Proof. We may and do assume that there is, independent of W, an independency $(B_i)_{i\in\mathbb{N}}$ of variables taking the values $+1$ and -1 with equal probabilities.

a) Let N be the Poisson random measure described by Theorem 3.8, and let $(A_i, U_i)_{i\in\mathbb{N}}$ be a labeling of its atoms. Then, the triplets (A_i, U_i, B_i) are the atoms of a Poisson random measure $\hat{N}$; see Corollary VI.3.5.

Fix ω, and let (a, u, b) be an atom of $\hat{N}(\omega, \cdot)$. Corresponding to that atom, $M(\omega)$ remains equal to a over the time interval $(s, s+u) = (T_{a-}(\omega),\ T_a(\omega))$; we define

4.9 $$V_t(\omega) = (M_t(\omega) - W_t(\omega))\, b, \qquad t \in (s, s+u)\,.$$

Doing this for every atom, we obtain $V_t(\omega)$ for every t for which $M_t(\omega) \neq W_t(\omega)$; for all other t, we define $V_t(\omega) = M_t(\omega) - W_t(\omega) = 0$.

For fixed t, we remarked in Corollary 4.2 that $M_t - W_t$ has the same distribution as $|W_t|$. Thus, in view of 4.9 and the independence of the Bernoulli variables B_i from W,

4.10 $$\mathbb{P}\{V_t \in A\} = \mathbb{P}\{W_t \in A\} = G\,(t, A)\,, \qquad A \in \mathcal{B}_{\mathbb{R}},$$

with the same notation 2.5 for the Gaussian kernel G.

b) It is obvious that V is continuous and starts from $V_0 = 0$. To show that it is Wiener, we shall show that

4.11 $$\mathbb{P}\left\{V_{s+t} - V_s \in A \,|\, \hat{\mathcal{F}}_s\right\} = G(t, A), \qquad s, t \in \mathbb{R}_+,\ A \in \mathcal{B}_{\mathbb{R}},$$

where $\hat{\mathcal{F}}_s$ is the σ-algebra generated by the union of $\mathcal{F}_s$ and $\sigma\{V_r : r \leq s\}$. This is obvious if $s = 0$ or $t = 0$. For the remainder of the proof, we fix $s > 0$ and $t > 0$ and define

$$D = \inf\{u > s : W_u = M_s\}\,, \quad R = D - s.$$

Observe that D is a stopping time of $\mathcal{F}$ and, thus, of $\hat{\mathcal{F}}$; moreover, almost surely, $D < \infty,\ W_D = M_D = M_s,\ V_D = 0$. It is clear that, in view of 4.10,

4.12 $$\mathbb{P}\{V_{D+u} - V_D \in A\} = \mathbb{P}\{V_u \in A\} = G(u, A).$$

c) On the event $\{R \leq t,\ V_s = x\}$, we have $s < D \leq s+t$ and $V_D = 0$ and

$$V_{s+t} = V_{D+(t-R)} - V_D.$$

Thus, as in Theorem 1.14, it follows from 4.12 that

$$\mathbb{P}\left\{R \leq t, V_{s+t} - V_s \in A \,|\, \hat{\mathcal{F}}_D\right\} = G(t-R, A+x)1_{\{R\leq t\}}$$

on $\{V_s = x\}$, on which we also have $R = T_a \circ \theta_s$ with $a = |x|$. Hence, conditioning both sides on $\hat{\mathcal{F}}_s$, since $\hat{\mathcal{F}}_s \subset \hat{\mathcal{F}}_D$ and $T_a \circ \theta_s$ is independent of $\hat{\mathcal{F}}_s$ and has the same distribution as T_a, we get

4.13 $$\mathbb{P}\left\{R \leq t, V_{s+t} - V_s \in A \,|\, \hat{\mathcal{F}}_s\right\} = f \circ V_s$$

where

4.14 $$f(x) = \mathbb{E}\, G\,(t - T_a, A + x)\, 1_{\{T_a \le t\}}, \quad x \in \mathbb{R},\ a = |x|\,.$$

d) On $\{R > t,\ V_s = x\}$, the variable V_{s+t} has the same sign as x, and

$$R = T_a \circ \theta_s,\ V_{s+t} - V_s = -b\, W_t \circ \theta_s \quad \text{with} \quad a = |x|\,,\ b = \text{sgn}\ x.$$

Thus, by the Markov property of W,

4.15 $$\mathbb{P}\left\{R > t, V_{s+t} - V_s \in A \,|\, \hat{\mathfrak{F}}_s\right\} = g \circ V_s,$$

where

$$g(x) = \mathbb{P}\left\{T_a > t, \quad -bW_t \in A\right\}, \qquad x \in \mathbb{R},\ a = |x|,\ b = \text{sgn}\ x.$$

We use Lemma 2.6 and the symmetry of $G(u, \cdot)$ to evaluate $g(x)$:

4.16 $$\begin{aligned} g(x) &= \mathbb{P}\{W_t \in -bA\} - \mathbb{P}\{T_a \le t, W_t \in -bA\} \\ &= G(t, A) - \mathbb{E}\, G(t - T_a, -bA - a) 1_{\{T_a \le t\}} \\ &= G(t, A) - \mathbb{E}\, G(t - T_a, A + x) 1_{\{T_a \le t\}}, \end{aligned}$$

where the last equality is justified by noting that $-bA - a$ is equal to $-A - x$ if $x \ge 0$ and to $A + x$ if $x \le 0$.

e) It follows from 4.14 and 4.16 that $f(x) + g(x) = G(t,\ A)$, and we obtain 4.11 by putting 4.13 and 4.15 together. □

Process $2M - W$ is a Bessel with index $d = 3$

Let $R = 2M - W$. Since $M - W \ge 0$, we have $R \ge M$. Recalling that M is increasing and strictly positive on $(0, \infty)$ and with limit equal to $+\infty$ as $t \to \infty$, we conclude the following: For almost every ω, we have

4.17 $R_0(\omega) = 0, \quad R_t(\omega) > 0 \quad \text{for every } t > 0,\ \lim_{t\to\infty} R_t(\omega) = +\infty.$

For each ω, the path $R(\omega)$ is obtained by reflecting the path $W(\omega)$ at its running maximum $M(\omega)$, that is, each stalactite of $W(\omega)$ hanging from $M(\omega)$ is made into a stalagmite sitting on $M(\omega)$. From this picture, it is now evident that

4.18 $$M_t(\omega) = \inf_{u \ge t} R_u(\omega), \qquad t \in \mathbb{R}_+.$$

Thus, the path $R(\omega)$ defines the path $M(\omega)$ and, hence, the path $W(\omega) = 2M(\omega) - R(\omega)$.

Recall from Example 1.22 that a Bessel process of index $d = 3$ is a Markov process whose law is identical to that of $|X|$, where X is a 3-dimensional Wiener process. The proof of the following will be sketched in Exercises 4.27.

4.19 THEOREM. *The process $R = 2M - W$ is a Bessel process of index $d = 3$.*

The preceding clarifies the recurrence properties of Wiener processes in $\mathbb{R}^d$ for $d \geq 3$. Let X be a Wiener process in $\mathbb{R}^3$. According to the preceding theorem, its radial part $|X|$ has the same law as $R = 2M - W$. It follows from 4.17 that, for almost every ω,

4.20 $$|X_0(\omega)| = 0,\ |X_t(\omega)| > 0 \text{ for every } t,\ \lim_{t\to\infty} |X_t(\omega)| = +\infty.$$

Thus, the Wiener particle X in $\mathbb{R}^3$ starts from the origin, and never returns to the origin, and the set of times spent in a bounded Borel set B is bounded. The process X is transient in this sense. The same statements are true for a Wiener process X in $\mathbb{R}^d$ with $d > 3$, since every choice of three components of X define a Wiener process in $\mathbb{R}^3$.

Exercises

4.21 *Arcsine law for M.*

a) Fix t. Show that, for almost every ω, $W_t(\omega) = M_t(\omega)$ if and only if $M_{t+\varepsilon}(\omega) > M_t(\omega)$ for every $\varepsilon > 0$.

b) Show that, for $0 < s < t$, the event $\{M_s = M_t\}$ and the event $\{W_u < M_u,\ s < u < t\}$ have the same probability.

c) Show that $\mathbb{P}\{M_s = M_t\} = \frac{2}{\pi} \arcsin \sqrt{s/t}$.

4.22 *Continuation.* For $t > 0$, let $\hat{G}_t = \sup\{s \leq t : W_s = M_s\}$. Compute the distribution of $\hat{G}_t$.

4.23 *Some joint distributions.* It will be convenient to introduce

$$h_t(a) = \frac{a\ e^{-a^2/2t}}{\sqrt{2\pi t^3}}, \qquad k_t(x,y) = \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-(x+y)^2/2t}}{\sqrt{2\pi t}}$$

for $t > 0,\ a > 0$, and x and y real. Note that $t \mapsto h_t(a)$ is the density for the distribution of T_a, the hitting time of a; thus

$$\int_0^t ds \quad h_s(a)\ h_{t-s}(b) = h_t(a+b), \qquad a, b > 0.$$

a) Show that, for a and b in $(0, \infty)$,

$$\int_0^t ds\ \frac{a\ e^{-a^2/2s}}{\sqrt{2\pi s^3}} \cdot \frac{e^{-b^2/2(t-s)}}{\sqrt{2\pi(t-s)}} = \frac{e^{-(a+b)^2/2t}}{\sqrt{2\pi t}}.$$

b) Show that, for a and x in $(0,\infty)$,

$$\mathbb{P}\{T_a > t,\ W_t \in a - dx\} = dx\ k_t(a,x),$$
$$\mathbb{P}\{M_t \in da, W_t \in a - dx\} = da\ dx\ 2\ h_t(a+x).$$

c) Another interpretation for $k_t(x,y)$: show that

$$\mathbb{P}\{W_{s+t} \in dy; \quad W_u \neq 0 \text{ for } u\ \in (s,s+t) | W_s = x\} = dy\ k_t(x,y)$$

provided that x and y are either both positive or both negative.

4.24 *The process* $Y = (M,\ M-W)$. This is clearly a Markov process with state space $\mathbb{R}_+ \times \mathbb{R}_+$: for s, t in $\mathbb{R}_+$ and B in $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_+)$

$$\mathbb{P}\{Y_{s+t} \in B | \mathcal{F}_s\} = P_t(Y_s, B).$$

The transition kernel P_t can be computed explicitly: in terms of h_t and k_t introduced above, for a, b, y in $(0,\infty)$ and $x \geq a$,

$$\begin{aligned}
&P_t\,(a,b;dx,\ dy)\\
&\quad = \mathbb{P}\{M_{s+t} \in dx, M_{s+t} - W_{s+t} \in dy \mid M_s = a, M_s - W_s = b\}\\
&\quad = \mathbb{P}\{T_b > t, b - W_t \in dy\}\, I(a,dx)\\
&\qquad + \int_0^t \mathbb{P}\{T_b \in t - du\}\ \mathbb{P}\{a + M_u \in dx, M_u - W_u \in dy\}\\
&\quad = I(a,dx)\ dy\ k_t(b,y) + dx\ dy\ 2h_t(x - a + b + y).
\end{aligned}$$

4.25 *A martingale*. For fixed p in $\mathbb{R}_+$,

$$Z_t = e^{-pM_t}\left[1 + p(M_t - W_t)\right], \qquad t \in \mathbb{R}_+,$$

is an $\mathcal{F}$-martingale. Show this via the following steps.

a) Use 4.1 to show directly that $\mathbb{E}\, Z_t = 1$ for every t.

b) In terms of the process Y of 4.24, note that $Z_t = f \circ Y_t$, where $f(x,\ y) = e^{-px}(1+py)$ for x, y in $\mathbb{R}_+$. Use the Markov property for Y to show that Z is a martingale if $P_t f = f$, that is, if f is harmonic for Y.

c) Use part (a) here and some of the stages in 4.24 to show that

$$\begin{aligned}
P_t f(a,b) &= e^{-pa}\ \mathbb{E}(1 + p(b - W_t))1_{\{T_b > t\}}\\
&\qquad + \int_0^t \mathbb{P}\{T_b \in t{-}du\}\ \mathbb{E}\ e^{-p(a+M_u)}(1 + p(M_u - W_u))\\
&= e^{-pa} + pe^{-pa}\ \mathbb{E}(b - W_t)1_{\{T_b > t\}}.
\end{aligned}$$

d) To conclude that $P_t f(a,\ b) = f(a,\ b)$, show that

$$\mathbb{E}(b - W_t)1_{\{T_b > t\}} = \mathbb{E}(b - W_t) + \mathbb{E}(W_t - b)1_{\{T_b \le t\}} = b + 0 = b.$$

4.26 *The process* $(M, 2M - W)$. Define $R = 2M - W$ as in Theorem 4.19. In preparation for the proof of 4.19, we consider the process $(M,\ R)$. It is obvious that $(M,\ R)$ is a Markov process whose state space is the region of $\mathbb{R}_+ \times \mathbb{R}_+$ above the diagonal. Of course, $M_0 = R_0 = 0$.

Show that, for $t > 0$,

$$\begin{aligned} \mu_t(dx, dy) &= \mathbb{P}\{M_t \in dx, R_t \in dy\} = dx\ dy\ 2h_t(y),\ 0 \le x \le y. \\ Q_t(a, b; dx, dy) &= \mathbb{P}\{M_{s+t} \in dx, R_{s+t} \in dy | M_s = a, R_s = b\} \\ &= I(a, dx)\ dy\ k_t(b-a, y-x) + dx\ dy\ 2h_t(b + y - 2a) \end{aligned}$$

for $0 < a \le b,\ a \le x \le y$. In particular, given that $R_t = y$, the conditional distribution of M_t is uniform on $(0,\ y)$.

4.27 *Proof of Theorem* 4.19. For the process $R = 2M - W$, the results of the preceding exercise can be used to compute that

$$\begin{aligned} \nu_t(dx) &= \mathbb{P}\{R_t \in dx\} = dx\ 2\ x\ h_t(x) \\ P_t(x, dy) &= \mathbb{P}\{R_{s+t} \in dy | R_s = x\} = dy\ \frac{y}{x}\ k_t(x, y) \end{aligned}$$

for $t > 0$ and $x,\ y > 0$, of course, $R_0 = 0$. These results coincide with their counterparts in Example 1.22 (see 1.25 and 1.26) for the Bessel process with index $d = 3$. To show that $R = 2M - W$ is a Bessel process with index 3, there remains to show that R is a Markov process. There does not seem to be an elegant proof. A direct proof, elementary but computationally intensive, can be obtained as follows.

Fix an integer $n \ge 2$, and a positive Borel function on $\mathbb{R}_+^n$. For times $0 < t_1 < t_2 < \ldots < t_n$, by the Markov property of (M, R), we have

$$\begin{aligned} &\mathbb{E}f(R_{t_1}, \ldots, R_{t_n}) \\ &\quad = \int \mu_{t_1}(dx_1, dy_1) \int Q_{t_2 - t_1}(x_1, y_1; dx_2, dy_2) \int \cdots \\ &\qquad\quad \int Q_{t_n - t_{n-1}}(x_{n-1}, y_{n-1}; dx_n, dy_n)\, f(y_1, y_2, \ldots, y_n) \end{aligned}$$

We need to show that the right side is as it should be, that is, that the right side is equal to

$$\int \nu_{t_1}(dy_1) \int P_{t_2 - t_1}(y_1, dy_2) \int \cdots \int P_{t_n - t_{n-1}}(y_{n-1}, dy_n)\, f(y_1, \ldots, y_n).$$

5 Zeros, Local Times

We keep the setup and notations of the previous sections: $W = (W_t)$ is a Wiener process, $M = (M_t)$ is its running maximum, and $T = (T_a)$ is the process of hitting times. We are interested in the Cantor set like features of the set C of times at which W is at 0, and in the existence of a random measure whose support is C, called the *local time* measure.

Closed and perfect sets

This is to review some terminology. Let C be a closed subset of $\mathbb{R}_+$. Then, its complement $\mathbb{R}_+ \backslash C$ is open and, therefore, is a countable union of disjoint open intervals. Those open intervals are said to be *contiguous* to C. A point of C is *isolated* if it is the common end point of two distinct contiguous intervals, or, if it is zero and is the left-end point of a contiguous interval. The set C is *dense in itself* if it has no isolated points, that is, if every point of C is a limit point of C.

A *perfect* set is a closed set with no isolated points. The simplest example is a union of finitely many disjoint closed intervals. Another example, closer to our present concerns, is the Cantor set. Every perfect set has the power of the continuum, that is, there exists an injection of $\mathbb{R}_+$ into C; see I.5.22 for this with the Cantor set.

Zeros of W

We are interested in the qualitative features of the set

5.1 $$C_\omega = \{t \in \mathbb{R}_+ : W_t(\omega) = 0\}, \quad \omega \in \Omega,$$

the set of zeros of W. For fixed ω, it is the inverse image of the closed set $\{0\}$ under the continuous mapping $t \mapsto W_t(\omega)$ from $\mathbb{R}_+$ into $\mathbb{R}$; thus, it is closed, and its complement is the union of a countable collection of disjoint open intervals, called *contiguous intervals.*

Fix the integers m and n in $\mathbb{N}^*$. Consider those intervals contiguous to C_ω whose lengths belong to the interval $\left[\frac{1}{m}, \frac{1}{m-1}\right)$. Going from left to right, let $(G_{m,n}(\omega),\ D_{m,n}(\omega))$ be the n^{th} such interval if it exists; otherwise, put $G_{m,n}(\omega) = D_{m,n}(\omega) = +\infty$ and note that the interval becomes empty. Finally, to lighten the notation, use a bijection $(m, n) \mapsto i$ from $\mathbb{N}^* \times \mathbb{N}^*$ onto $\mathbb{N}$ to re-label these intervals as $(G_i(\omega),\ D_i(\omega))$. Thus,

5.2 $$\mathbb{R}_+ \backslash C_\omega = \bigcup_{i \in \mathbb{N}} (G_i(\omega), D_i(\omega)), \qquad \omega \in \Omega.$$

Clearly, each D_i is a stopping time. Stability and recurrence properties of W imply that each D_i is almost surely finite. Incidentally, each G_i is a random variable but not a stopping time; see Exercise 2.27 for the reasoning. The following shows the Cantor set like features of the zero-set C.

5.3 THEOREM. *For almost every ω, the set C_ω is perfect and unbounded, its interior is empty, its Lebesgue measure is zero, and it has the power of the continuum.*

Proof. We have already seen that C_ω is closed. It is unbounded for almost every ω in view of Corollary 2.4. Its Lebesgue measure is zero for almost every ω, since

$$\mathbb{E}\ \text{Leb}\ C = \mathbb{E} \int_{\mathbb{R}_+} dt\ 1_{\{0\}} \circ W_t = \int_{\mathbb{R}_+} dt\ \mathbb{P}\{W_t = 0\} = 0.$$

This implies that the interior of C_ω is empty for almost every ω, because no set of zero Lebesgue measure can contain an open interval. To complete the proof, there remains to show that, for almost every ω, the set C_ω has no isolated points; then, the closed set C_ω is perfect and has the power of the continuum necessarily.

We start by recalling that $T_0 = 0$ almost surely. Thus, as mentioned in Corollary 2.3, there is an almost sure set Ω_{00} such that, for every ω in it, there is a strictly decreasing sequence (t_k) in C_ω with limit 0, that is, the point 0 of C_ω is a limit point of C_ω for every ω in Ω_{00}.

Similarly, for each i in $\mathbb{N}$, the stopping time D_i is almost surely finite, and the strong Markov property yields that

$$T_0 \circ D_i = 0 \qquad \text{almost surely.}$$

Thus, there is an almost sure event Ω_i such that $D_i(\omega)$ is a limit point of C_ω for every ω in Ω_i. Consider, finally, the intersection Ω' of the events Ω_{00}, Ω_0, Ω_1, ... For ω in it, neither 0 nor any $D_i(\omega)$ is isolated. In view of 5.2, then, C_ω is perfect for every ω in the almost sure event Ω'. □

5.4 REMARKS. a) It will be convenient to introduce here the almost sure event $\Omega^* = \Omega' \cap \Omega''$, where Ω' is as in the proof above, and where Ω'' is the set of ω for which the claims of the preceding theorem hold in addition to the regularity properties of the path $W(\omega)$.

b) We shall see in Corollary 5.11 below that there is a strictly increasing function $a \mapsto S_a(\omega)$ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $S_a(\omega)$ belongs to C_ω for every a. This shows, directly, that C_ω has at least "as many points" as $\mathbb{R}_+$.

Local time at zero

Imagine a clock whose mechanism is so rigged that the clock advances when and only when the Wiener particle is at the origin. We shall show that such a clock exists; it will be called the *local time at zero.*

First, some generalities. Let $c: \mathbb{R}_+ \mapsto \mathbb{R}_+$ be increasing and continuous with $c(0) = 0$. Think of it as a clock: when the standard time is t, the clock shows $c(t)$. The clock may remain stationary during some periods of time, that is, the function is not forced to be strictly increasing. The *set of times of increase for c* is defined to be

5.5 $$\text{Incr}\ c = \{t \in \mathbb{R}_+ : c(t-\varepsilon) < c(t+\varepsilon) \quad \text{for every } \varepsilon > 0\},$$

where we use the convention that $c(t-\varepsilon)=0$ for $t<\varepsilon$. Corresponding to c there is a unique measure on $\mathbb{R}_+$ whose "distribution" function is c. The set Incr c is also called the *support* of this measure, namely, the smallest closed set whose complement has measure zero. We shall show next the existence of a random measure on $\mathbb{R}_+$ whose support is the zero-set C defined by 5.1. This is interesting especially since C has zero as its Lebesgue measure.

Consider Figure 13 on page 402, and concentrate on the relationship of M to the Wiener process V there; recall that $|V|=M-W$. For every ω, the path $M(\omega)$ is increasing and continuous; and it increases at a time t if only if $V_t(\omega)=0$, more precisely,

5.6 $$\text{Incr } M(\omega)=\{t\in\mathbb{R}_+ : V_t(\omega)=0\}\,.$$

Moreover, it follows from Theorem 4.4 that the time-set on the right side determines the path $M(\omega)$.

Since W is a Wiener process just as V, there must be a process L that is related to W just as M is to V. We state this conclusion next; there is nothing new to prove. The process $L=(L_t)_{t\in\mathbb{R}_+}$ is called the *local time* of W at zero.

5.7 THEOREM. *There exists an increasing continuous process L that has the same law as M and is such that*

$$\text{Incr } L(\omega)=C_\omega=\{t\in R_+ : W_t(\omega)=0\}\,,\qquad \omega\in\Omega.$$

Inverse of the local time is a stable Lévy process

Heuristically, the local time process L is a random clock that advances when and only when W is at the point 0. When the standard time is t, the local time at 0 is L_t; conversely,

5.8 $$S_a=\inf\{t\in\mathbb{R}_+ : L_t>a\}$$

is the standard time when the local time is just about to pass a.

5.9 THEOREM. *The process $S=(S_a)_{a\in\mathbb{R}_+}$ has the same law as the hitting time process $T=(T_a)_{a\in\mathbb{R}_+}$. It is a strictly increasing pure-jump Lévy process; it is stable with index $1/2$; its Lévy measure is*

$$\lambda(ds)=ds\frac{1}{\sqrt{2\pi s^3}},\qquad s>0,$$

Proof. By comparing 5.8 and 3.2, we note that S bears the same relation to L as T does to M. By the last theorem, L and M have the same law. Hence, S and T have the same law. The statement about S as a Lévy process is the same as Theorem 3.6 about T. □

In terms of S, the local time L is defined as the functional inverse of S:

5.10 $$L_t = \inf\{a : S_a > t\}, \qquad t \in \mathbb{R}_+;$$

This is immediate from 5.8. The following is to clarify some further relationships.

5.11 COROLLARY. *For almost every ω, with $G_i(\omega)$ as in 5.2,*

$$\{S_a(\omega) : a \in \mathbb{R}_+\} = C_\omega \setminus \{G_i(\omega) : i \in \mathbb{N}\}.$$

Proof. Take ω such that $L(\omega)$ is continuous and $S(\omega)$ strictly increasing. Fix a in $\mathbb{R}_+$ and let $S_a(\omega) = t$. Then, 5.8 and the continuity of $L(\omega)$ imply that

5.12 $$S_a(\omega) = t \Leftrightarrow L_t(\omega) = a, \quad L_{t+\varepsilon}(\omega) > a \quad \text{for every } \varepsilon > 0.$$

Next, note that the set of t for which the right side holds for some a is exactly the set $(\text{Incr } L(\omega))\setminus\hat{G}_\omega$, where $\hat{G}_\omega$ is the countable set consisting of the left-end-points of the intervals contiguous to Incr $L(\omega)$. The proof is complete, since $C_\omega = \text{Incr } L(\omega)$ by Theorem 5.7 and, thus, $\hat{G}_\omega = \{G_i(\omega) = i \in \mathbb{N}\}$ by 5.2. □

Local times elsewhere

Fix a point x in $\mathbb{R}$. Consider the hitting time T_x defined by 2.1 or 2.14. It is almost rurely finite; the Wiener particle is at x at that time; and the point x becomes the point 0 of the new Wiener process $W \circ \theta_{T_x}$. With L as defined earlier, $L \circ \theta_{T_x}$ is the local time at 0 for $W \circ \theta_{T_x}$; using it, we introduce the following definition for every outcome ω and time t:

5.13 $$L_t^x(\omega) = \begin{cases} 0 & \text{if} \quad t < T_x(\omega), \\ \\ L_{t-s}(\theta_s\omega) & \text{if} \quad t \geq s = T_x(\omega). \end{cases}$$

It is immediate from Theorem 5.7 that

5.14 $$\text{Incr } L^x(\omega) = \{t \in \mathbb{R}_+ : W_t(\omega) = x\}.$$

Thus, the process $L^x = ({L^x}_t)_{t\in\mathbb{R}_+}$ is called the local time at x for W. Note that $L^0 = L$.

For $x \neq 0$, the path $L^x(\omega)$ stays at 0 during $[0,\ T_x]$ and, then, starts increasing just as L did at 0. All computations regarding L^x can be reduced to computations about L, but with special consideration for the delay at the start; see Exercise 5.19 for an example.

Master theorem on equivalence

The essential argument underlying the results of this section is that L bears the same relationship to W as M does to V. We put this observation next and supplement it by recalling that $|V| = M - W$. This is a summary of the results above; there is nothing new to prove.

5.15 THEOREM. *The three-dimensional process $(W,\ L,\ S)$ has the same law as $(V,\ M,\ T)$. Further, $(|W|,\ L,\ S)$ has the same law as $(M - W,\ M,\ T)$, and $L - |W|$ has the same law as W.*

Exercises

5.16 *Minimum of W.* Define $m_t(\omega) = \min_{0 \le s \le t} W_s(\omega)$. Obviously, the process $(-m_t)$ has the same law of (M_t). Show that $|W|$ has the same law as $W - m$.

5.17 *Local time measure.* For each ω, let $A \mapsto L(\omega, A)$ be the unique measure on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ whose distribution function is the increasing continuous function $t \mapsto L_t(\omega)$. Show that the support of that measure is exactly the set C_ω of zeros of $W(\omega)$–the support of a measure μ on $\mathbb{R}_+$ is the smallest closed subset of $\mathbb{R}_+$ whose complement has μ-measure 0. Obviously, $L(\omega, \cdot)$ is singular with respect to the Lebesgue measure, and

$$L(\omega, [0,t]) = L(\omega, C_\omega \cap [0,t]) = L_t(\omega).$$

5.18 *Computing the local time.* This is to relate L to the zero-set C. For every ω and every $\varepsilon > 0$, let $\hat{N}_t(\omega, \varepsilon)$ be the number of intervals that are contiguous to C_ω, are contained in $[0,t]$, and whose lengths exceed ε. Show that, for almost every ω,

$$\lim_{\varepsilon \downarrow 0} \sqrt{2\pi\varepsilon}\ \hat{N}_t(\omega, \varepsilon) = 2\ L_t(\omega).$$

5.19 *Local time at x.* Note that L^x and L^{-x} have the same law for every x in $\mathbb{R}$. Fix $x > 0$. Compute $\mathbb{P}\{L_t^x = 0\}$. Show that, for $a > 0$,

$$\mathbb{P}\{L_t^x \in da\} = \int_{[0,t]} \mathbb{P}\{T_x \in ds\}\, \mathbb{P}\{L_{t-s} \in da\} = da \frac{2e^{-(x+a)^2/2t}}{\sqrt{2\pi t}}.$$

5.20 *Inverse of the local time at x.* Fix x in $\mathbb{R}$. Define S_a^x from L^x as S_a is defined from L in 5.8. Show that the process $(S_a^x)_{a \in \mathbb{R}_+}$ has the same probability law as $(T_x + \hat{S}_a)_{a \in \mathbb{R}_+}$, where $(\hat{S}_a)$ is independent of T_x and has the same law as the stable Lévy process (S_a) described in Theorem 5.9.

5.21 *Occupation times.* For x in $\mathbb{R}$ and t in $\mathbb{R}_+$, define

$$A_t(\omega, x) = \int_0^t ds\ 1_{(-\infty, x]} \circ W_s(\omega),$$

the amount of time spent in $(-\infty,\ x]$ by $W(\omega)$ during $[0,\ t]$. Show that $x \mapsto A_t(\omega,\ x)$ is equal to 0 on $(-\infty,\ m_t(\omega)]$, and to t on $[M_t(\omega), +\infty)$, and is continuous and strictly increasing on $[m_t(\omega),\ M_t(\omega)]$; what are $m_t(\omega)$ and $M_t(\omega)$?

5.22 *Local times as derivatives.* It can be shown that, for almost every ω, $x \mapsto A_t(\omega,\ x)$ is differentiable, and its derivative at the point x is $L_t^x(\omega)$, that is,

$$L_t^x(\omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t ds\ 1_{(x-\varepsilon, x+\varepsilon)} \circ W_s(\omega).$$

5.23 *Occupation measure.* This is the name for the measure on $\mathbb{R}$ whose distribution function is $x \mapsto A_t(x)$. Letting it be denoted by K_t, we see from 5.21 and 5.22 that

$$K_t(\omega, B) = \int_0^t ds\ 1_B \circ W_s(\omega) = \int_B dx\ L_t^x(\omega), \quad B \in \mathcal{B}_{\mathbb{R}},$$

or, for f positive Borel on $\mathbb{R}$,

$$K_t f(\omega) = \int_0^t ds\ f \circ W_s(\omega) = \int_{\mathbb{R}} dx\ f(x)\ L_t^x(\omega).$$

5.24 *Continuity of local times.* It is known that, for almost every ω, the mapping

$$(x, t) \mapsto L_t^x(\omega)$$

from $\mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}_+$ is continuous.

6 Excursions

We continue with the setup and notations of the previous sections: W is the Wiener process under consideration, C is its set of zeros, L is its local time process at 0, and S is the inverse local time. Recall the almost sure event introduced in Remark 5.4a; we take it to be the new Ω in order to avoid boring repetitions of "almost every." We are interested in the excursions of W outside the point 0, that is, basically, in the path segments over the intervals contiguous to C.

Excursion space

The path segments in question are continuous functions that start from 0, stay away from 0 for some strictly positive time, and return to 0 some finite time later. It is convenient to let each such function remain at zero forever after the return to 0. The following is the space of such functions.

We define E to be the collection of all continuous functions $x : \mathbb{R}_+ \mapsto \mathbb{R}$ such that

6.1 $$\zeta(x) = \inf\{t > 0 : x(t) = 0\}$$

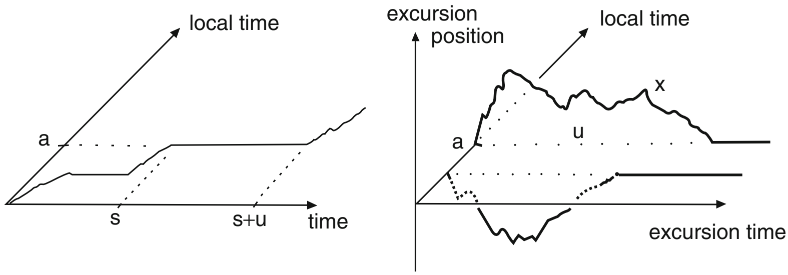

Figure 14: At the local time a there is an excursion x of duration u. There are infinitely many excursions, one for each flat stretch of the local time clock.

is a strictly positive real number, and x vanishes outside $(0, \zeta(x))$. Each element x of E is called an *excursion*; $\zeta(x)$ is its *duration*; note that x is either a positive function or negative. We let $\mathcal{E}$ be the Borel σ-algebra on E corresponding to the topology of uniform convergence on compacts. Then, $(E, \mathcal{E})$ is called the *excursion space.*

Excursions of W

Fix an outcome ω. Let a be a point on the local time axis (see Figure 14 below) at which $S(\omega)$ has a jump, say, from $s = S_{a-}(\omega)$ to $s+u = S_a(\omega)$, with $u > 0$. During the interval $[s,\ s+u]$ the local time $L(\omega)$ stays constant at the value a, and the Wiener path $W(\omega)$ has an excursion x defined formally by

6.2 $$x(t) = \begin{cases} W_{s+t}(\omega) & \text{if} \quad 0 \le t \le u \\ 0 & \text{if} \quad t > u. \end{cases}$$

This x is called the excursion of $W(\omega)$ at the local time a. It is an element of E, and its duration is

6.3 $$\zeta(x) = u = S_a(\omega) - S_{a-}(\omega),$$

which is strictly positive and finite by the way a is chosen. Each excursion corresponds to a local time at which $S(\omega)$ jumps.

Poisson random measure for excursions

The next theorem is fundamental. The measure ν on $(E, \mathcal{E})$ describing the mean here is called the *Itô measure* of excursions. We shall specify it later.

6.4 Theorem. *For every ω, let $N(\omega, \cdot)$ be the counting measure on $(\mathbb{R}_+ \times E, \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{E})$ whose atoms are the pairs (a, x), where x is an excursion of $W(\omega)$, and a the corresponding local time. Then, N is a Poisson random measure whose mean has the form* Leb $\times\, \nu$, *where ν is a σ-finite measure on the excursion space $(E,\ \mathcal{E})$.*

Proof. Fix $\varepsilon > 0$. Let $A_1,\ A_2,\ \ldots$ be the successive points of jump for $a \mapsto S_a$ with jump sizes exceeding ε. By Theorem 5.9, these A_i form a Poisson random measure on $\mathbb{R}_+$ with mean c_ε Leb, where $c_\varepsilon = \lambda(\varepsilon, \infty) = 2/\sqrt{2\pi\varepsilon}$.

Corresponding to the local time A_i, let X_i be the excursion, and $D_i = S_{A_i}$ the right-end point of the contiguous interval over which the local time is A_i. Each X_i is a random variable taking values in $(E, \mathcal{E})$. Each D_i is a finite stopping time with $W_{D_i} = 0$. Since $0 < A_1 < A_2 < \ldots$, we have $0 < D_1 < D_2 < \ldots$, and $A_1,\ X_1,\ \ldots,\ A_i,\ X_i$ belong to the past $\mathcal{F}_{D_i}$. By the strong Markov property at D_i, then, the pair $(A_{i+1} - A_i,\ X_{i+1})$ is independent of $\mathcal{F}_{D_i}$ and, therefore, of $\{A_1, X_1,\ \ldots,\ A_i,\ X_i\}$, and has the same distribution as $(A_1,\ X_1)$. Noting further that A_1 and X_1 are independent, we conclude the following: (A_i) forms a Poisson random measure on $\mathbb{R}_+$ with mean c_ε Leb; (X_i) is independent of it and is an independency of variables with some distribution μ_ε on $(E, \mathcal{E})$ in common. It follows from Corollary VI.3.5 that the pairs $(A_i,\ X_i)$, $i \geq 1$, form a Poisson random measure N_ε on $\mathbb{R}_+ \times E$ whose mean measure is Leb $\times\, \nu_\varepsilon$, where $\nu_\varepsilon = c_\varepsilon\ \mu_\varepsilon$ is a finite measure on $(E, \mathcal{E})$.

Observe that, for every ω, the atoms $(A_i(\omega),\ X_i(\omega))$ are those atoms (a, x) of $N(\omega, \cdot)$ with $\zeta(x) > \varepsilon$. Thus, the Poisson random measure N_ε is the trace of N on $\mathbb{R}_+ \times E_\varepsilon$, where $E_\varepsilon = \{x \in E : \zeta(x) > \varepsilon\}$. Letting $\varepsilon \to 0$, we conclude that N is a Poisson random measure on $R_+ \times E$ whose mean measure is Leb $\times\, \nu$, where ν is the measure defined by

$$\nu f = \lim_{\varepsilon \to 0} \nu_\varepsilon f, \qquad f \in \mathcal{E}_+.$$

Since $\nu(E_\varepsilon) = \nu_\varepsilon(E) = c_\varepsilon < \infty$ for every $\varepsilon > 0$, the measure ν is σ-finite. □

Excursions determine W

We have constructed the Poisson random measure N above, ω by ω, from the Wiener process W. This can be reversed: N determines W.

Recall from 6.3 that the duration $\zeta(x)$ of an excursion x is the jump size for $S(\omega)$ at the local time corresponding to that excursion. Thus, for every ω,

6.5 $$S_a(\omega) = \int_{[0,a] \times E} N(\omega; db, dx)\, \zeta(x), \qquad a \in \mathbb{R}_+,$$

and $L(\omega)$ is the functional inverse of $S(\omega)$; and

6.6 $$W_t(\omega) = \int_{[0, L_t(\omega)] \times E} N(\omega; da, dx)\, x\,(t - S_{a-}(\omega)), \qquad t \in \mathbb{R}_+.$$

In fact, the last integral is a countable sum with at most one non-zero term, namely, the term corresponding to $a = L_t(\omega)$ if $S_{a-}(\omega) < S_a(\omega)$.

Extents of excursions

In preparation for characterizing Itô's measure ν on the excursion space $(E, \mathcal{E})$, we describe next the law it imparts on the extents of excursions.

For an excursion x in E, we define the *extent* of x to be the point touched by x that is at maximum distance from 0, that is,

6.7 $$m(x) = \begin{cases} \max_{t\in\mathbb{R}_+} x(t) & \text{if} \quad x \text{ is positive,} \\ \min_{t\in\mathbb{R}_+} x(t) & \text{if} \quad x \text{ is negative;} \end{cases}$$

recall that x is either positive or negative. The next theorem shows that the local times and extents of excursions form a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with an explicit mean measure.

6.8 THEOREM. *Let h be the mapping $(a, x) \mapsto (a,\ m(x))$ from $\mathbb{R}_+ \times E$ into $\mathbb{R}_+ \times \mathbb{R}$. Then, $\hat{N} = N \circ h^{-1}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ whose mean has the form* Leb $\times\, \hat{\nu}$*, where*

$$\hat{\nu}(db) = db\ \frac{1}{2b^2}, \qquad b \in \mathbb{R}.$$

REMARK. It is curious that $\hat{\nu}$ is the Lévy measure of a Cauchy process, namely, $(1/2\pi Y_t)$ where Y is standard Cauchy process; see Example VII.2.14.

Proof. a) Since $m:\ E \mapsto \mathbb{R}$ is continuous, the mapping h is measurable with respect to the Borel σ-algebras on $\mathbb{R}_+ \times E$ and $\mathbb{R}_+ \times \mathbb{R}$. Since N is Poisson on $\mathbb{R}_+ \times E$ with mean Leb $\times\, \nu$, it follows that $\hat{N}$ is Poisson on $\mathbb{R}_+ \times \mathbb{R}$ with mean Leb $\times\, \hat{\nu}$, where $\hat{\nu} = \nu \circ m^{-1}$. By the symmetry of W, the measure $\hat{\nu}$ on $\mathbb{R}$ must be symmetric. Thus, for every $b > 0$,

6.9 $$\hat{\nu}(b, \infty) = \hat{\nu}(-\infty, -b) = \frac{1}{2}\left[\hat{\nu}(b, \infty) + \hat{\nu}(-\infty, -b)\right] = \frac{1}{2}\nu(E_b),$$

where

$$E_b = \{x \in E : |m(x)| > b\}\,.$$

To complete the proof, we shall show that $\nu(E_b) = 1/b$.

b) Fix $b > 0$ and define

$$\tau = \inf\{t : |W_t| > b\}\,;$$

recall that $\mathbb{E}\tau = b^2$. Consider L_τ, the local time at the standard time τ. Note that it is also the local time corresponding to the first excursion belonging to E_b. Thus, for every ω,

$$L_{\tau(\omega)}(\omega) > a \Leftrightarrow N(\omega, [0, a] \times E_b) = 0.$$

Since N is Poisson with mean Leb $\times\, \nu$, then,

6.10 $$\mathbb{P}\{L_\tau > a\} = \exp\_\ a\ \nu(E_b), \qquad a \in \mathbb{R}_+.$$

c) For the same $b > 0$, define

$$\sigma = \inf\{t : M_t - W_t > b\}\,.$$

Since $(|W|,\ L)$ has the same law as $(M-W,\ M)$ by Theorem 5.15, we deduce that $(\tau,\ L_\tau)$ has the same distribution as $(\sigma,\ M_\sigma)$. Hence,

6.11 $$\mathbb{E}\,\sigma = \mathbb{E}\,\tau = b^2,\ \mathbb{E}\,M_\sigma = \mathbb{E}\,L_\tau = 1/\nu(E_b),$$

the last equality being a consequence of 6.10. Since $\sigma < \infty$ almost surely, $M_\sigma - W_\sigma = b$ by the definition of σ and the continuity of $M-W$. Hence, to complete the proof via 6.11 and 6.9, there remains to show that

6.12 $$\mathbb{E}\,W_\sigma = 0.$$

d) Consider the martingale $X = \left(W_t^2 - t\right)_{t\in\mathbb{R}_+}$. For each t, it is Doob on $[0,\ t]$ by V.5.6, and thus, $\mathbb{E}X_{\sigma\wedge t} = 0$. Hence,

$$\mathbb{E}\,W_{\sigma\wedge t}^2 = \mathbb{E}\,(\sigma\wedge t) \le \mathbb{E}\,\sigma = b^2, \qquad t\in\mathbb{R}_+,$$

which shows that the martingale $(W_{\sigma\wedge t})_{t\in\mathbb{R}_+}$ is L^2-bounded and, thus, uniformly integrable (see Remark II.3.13e). By Theorem V.5.14, this is equivalent to saying that the martingale W is Doob on $[0,\ \sigma]$. Hence, 6.12. □

Itô measure on excursions

Recall the Poisson random measure N of excursions; see Theorem 6.4. Its mean measure on $\mathbb{R}_+\times E$ is the product measure Leb $\times\,\nu$, where ν is a σ-finite measure on $(E,\mathcal{E})$. Our aim is to state a characterization for ν, the Itô measure.

Let $(A_i,\ X_i)$, $i\in\mathbb{N}$, be an enumeration of the atoms of $\mathbb{N}$, that is, the pairs $(A_i,\ X_i)$ are random variables taking values in $\mathbb{R}_+\times E$, and they form N. Then, the pairs $(A_i,\ m\circ X_i)$ are the atoms of the Poisson random measure $\hat{N}$ described in Theorem 6.8. Finally, the triplets $(A_i,\ m\circ X_i,\ X_i)$ must form a Poisson random measure $\tilde{N}$, namely, $\tilde{N} = N\circ h^{-1}$ where $h(a,\ x) = (a,\ m(x),\ x)$. The following is immediate from Theorems 6.4 and 6.8; no proof is needed.

6.13 Proposition. *The mean μ of the Poisson random measure $\tilde{N}$ is given by*

6.14 $$\mu(da,db,dx) = da\ db\ \frac{1}{2\,b^2}\ Q(b,dx), \qquad a\in\mathbb{R}_+, b\in\mathbb{R}, x\in E,$$

where Q is the transition probability kernel from $(\mathbb{R},\ \mathcal{B}_\mathbb{R})$ into $(E,\mathcal{E})$ defined by

6.15 $$Q(b,D) = \mathbb{P}\left\{X_i\in D | m\circ X_i = b\right\}, \qquad b\in\mathbb{R},\ D\in\mathcal{E}.$$

6.16 Corollary. *Itô measure ν for excursions is given by*

$$\nu(D) = \int_\mathbb{R} db\ \frac{1}{2\,b^2}\ Q(b,D), \qquad D\in\mathcal{E}.$$

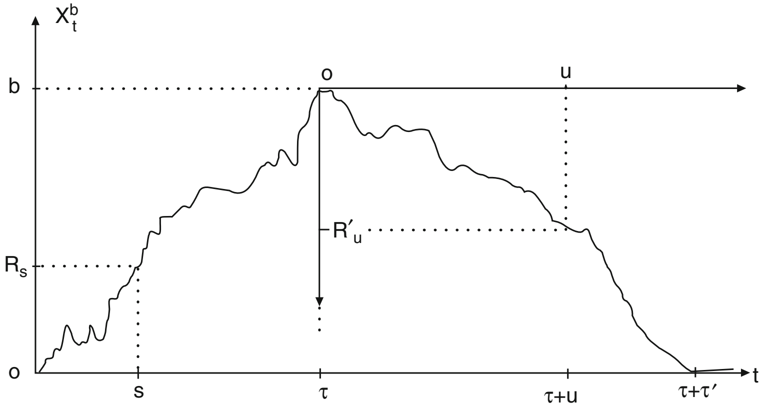

Figure 15: Excursion X^b with a given extent $b > 0$. Run a bessel process R upward until it hits b; then run a new Bessel process R' downward from b until it hits 0.

Proof is immediate from the form 6.14 for the mean of $\tilde{N}$, since $N = \tilde{N} \circ h^{-1}$ with $h(a, b, x) = (a, x)$, which implies that $\text{Leb} \times \nu = \mu \circ h^{-1}$. □

The preceding corollary reduces the task of characterizing the Itô measure ν to that of characterizing the probability measure $Q(b, \cdot)$ for each b, namely, the probability law of an excursion whose extent is given to be b.

It is obvious that $Q(b,\ D) = Q(-b,\ -D)$ for $b < 0$, with $-D = \{-x :\ x \in D\}$; this is by the symmetry of W. It is also obvious that, if $b > 0$, then $Q(b, \cdot)$ must put all its mass on the set of positive excursions. Thus, the following characterization specifies Q completely, and via the last corollary, Itô measure ν. See Figure 15 as well. This theorem of D. Williams's is put here without proof; see the notes for this chapter.

6.17 THEOREM. *Let R and R' be independent Bessel processes with index 3. Let τ_b be the hitting time of the level $b > 0$ by R, and τ'_b the same for R'. Define, for ω in Ω and t in $\mathbb{R}_+$,*

6.18
$$X_t^b(\omega) = \begin{cases} R_t(\omega) & \text{if} \quad 0 \le t \le \tau_b(\omega) \\ b - R'_{t-\tau_b(\omega)}(\omega) & \text{if} \quad \tau_b(\omega) \le t \le \tau_b(\omega) + \tau'_b(\omega) \\ 0 & \text{if} \quad t > \tau_b(\omega) + \tau'_b(\omega). \end{cases}$$

Then, $Q(b, \cdot)$ is the probability law of the process X^b.

The preceding theorem together with Corollary 6.16 characterizes the Itô measure in terms of well-understood operations. For Bessel processes see Example 1.22; recall that R here is the radial part of a three-dimensional Wiener process. See also Theorem 4.19, which shows that R has the same law as $2M - W$.

Local times of some hits

This is to expand on the observation, within the proof of Theorem 6.8, that the local time L_τ at the time τ of exit from $(-b,\ b)$ has the exponential distribution with mean b. Recall that T_a is the time W hits (a, ∞) if $a \geq 0$, and is the time of hitting $(-\infty,\ a)$ if $a \leq 0$.

6.19 PROPOSITION. *Let $a,\ b > 0$. Then, L_{T_a} and $L_{T_{-b}}$ are independent and exponentially distributed with means $2a$ and $2b$ respectively. Moreover, $L_{T_a \wedge T_{-b}}$ is equal to $L_{T_a} \wedge L_{T_{-b}}$ and has the exponential distribution with mean $2ab/(a+b)$.*

Proof. In terms of the Poisson random measure N of excursions, we have

6.20 $$\left\{L_{T_a} > u,\ L_{T_{-b}} > v\right\} = \{N([0,u] \times A) = 0\} \cap \{N([0,v] \times B) = 0\}$$

where

$$A = \{x \in E : m(x) > a\}, \qquad B = \{x \in E : m(x) < -b\}.$$

Since A and B are disjoint, the right side of 6.20 is the intersection of two independent events. Hence, by 6.4,

$$\mathbb{P}\left\{L_{T_a} > u, L_{T_{-b}} > v\right\} = e^{-u\nu(A)} e^{-v\nu(B)}, \quad u, v \in \mathbb{R}_+,$$

where, by Theorem 6.8,

$$\nu(A) = 1/2a, \qquad \nu(B) = 1/2b.$$

This proves the first statement. The second is immediate from it and the computation $\nu(A) + \nu(B) = (a+b)/2ab$. □

The arcsine law

As another illustration of the uses of excursion theory, we prove next *the* arcsine law, the most celebrated of the arcsine laws. It specifies the distribution of

6.21 $$A_t = \int_{[0,t]} ds\ 1_{\mathbb{R}_+} \circ W_s, \qquad t \in \mathbb{R}_+,$$

and is the main ingredient in computations about occupation times and Brownian quantiles; see Exercises 6.41–6.47.

6.22 THEOREM. *The distribution of A_t is the same as that of tA, where A has the arcsine distribution as in 2.15.*

6.23 REMARK. In view of the (simpler to obtain) arcsine law given in Proposition 2.18, we see that G_t and A_t have the same distribution. See Exercise 6.40 for the underlying reasons.

Proof. Consider the standard time S_a corresponding to the local time a. It is obtained via 6.5 from the Poisson random measure N of excursions. Then, (S_a) is a pure-jump Lévy process whose Lévy measure λ is given by (see 6.5 and 5.9)

6.24 $$\lambda f = \int_0^\infty ds\ \frac{1}{\sqrt{2\pi s^3}} f(s) = \int_E \nu(dx)\ f(\zeta(x)),$$

where ν is the Itô measure of excursions. We now decompose S as

$$S = S^+ + S^-,$$

where S_a^+ is the time spent on positive excursions during $[0, S_a]$, and S_a^- is that on negative excursions:

6.25 $$S_a^+ = \int_{[0,a]\times E_+} N(db, dx)\ \zeta(x), \qquad \text{where } E_+ = \{x \in E : x \geq 0\},$$

and S^- is defined similarly but with $E_- = \{x \in E : x \leq 0\}$.

Since E_+ and E_- are disjoint, and since N is Poisson, the processes S^+ and S^- are independent. Comparing 6.25 with 6.5, we see that S^+ and, by symmetry, S^- are pure-jump Lévy processes with the same Lévy measure, namely, $\frac{1}{2}\lambda$. We conclude that S_a^+ and S_a^- are independent and have the same distribution as $S_{a/2}$, that is,

$$\mathbb{P}\left\{S_a^+ \in du, S_a^- \in dv\right\} = du\ dv\ \frac{a\ e^{-a^2/8u}}{2\sqrt{2\pi u^3}} \cdot \frac{a\ e^{-a^2/8v}}{2\sqrt{2\pi v^3}}.$$

Hence, for positive Borel functions f on $\mathbb{R}_+ \times \mathbb{R}_+$, an easy computation yields

6.26 $$\mathbb{E}\int_{\mathbb{R}_+} da\ f\left(S_a^+, S_a^-\right) = \int_{\mathbb{R}_+} du \int_{\mathbb{R}_+} dv \frac{1}{\sqrt{2\pi(u+v)^3}}\ f(u,v).$$

b) It follows from the scaling property of W that A_t has the same distribution as tA_1. Thus, we concentrate on the distribution of A_1.

Fix $a > 0$; for almost every ω, the counting measure $N(\omega, \cdot)$ has exactly one atom $(a,\ x)$ such that $S_{a-}(\omega) \leq 1 < S_a(\omega) = S_{a-}(\omega) + \zeta(x)$, and, then,

$$A_1(\omega) = \begin{cases} S_{a-}^+(\omega) & \text{if} \quad x \in E_-, \\ 1 - S_{a-}^-(\omega) & \text{if} \quad x \in E_+. \end{cases}$$

In other words, for Borel $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$,

6.27 $$f(A_1) = \int_{\mathbb{R}_+\times E} N(da, dx)\ g(S_{a-}^+, S_{a-}^-, x)$$

where

$$g(u,v,x) = 1_{[0,1]}(u+v)\ 1_{(1,\infty)}(u+v+\zeta(x))[f(u)1_{E_+}(x) + f(1-v)1_{E_-}(x)].$$

Applying Theorem VI.6.2 to the Poisson integral in 6.27, recalling that the mean of N is Leb $\times\,\nu$, we see that

6.28
$$\begin{aligned}\mathbb{E}f(A_1) &= \mathbb{E}\int_{\mathbb{R}_+} da \int_E \nu(dx) \quad g\left(S_a^+, S_a^-, x\right) \\ &= \int_E \nu(dx) \int_{\mathbb{R}_+} du \int_{\mathbb{R}_+} dv \frac{1}{\sqrt{2\pi(u+v)^3}}\ g(u,v,x),\end{aligned}$$

where we used 6.26 at the last step. In view of 6.24,

$$\begin{aligned}\int_E \nu(dx) g(u,v,x) &= 1_{[0,1]}(u+v) \int_0^\infty ds \frac{1}{\sqrt{2\pi s^3}} 1_{(1,\infty)}(u+v+s) \\ &\qquad \times \left[\tfrac{1}{2} f(u) + \tfrac{1}{2} f(1-v)\right] \\ &= 1_{[0,1]}(u+v) \frac{1}{\sqrt{2\pi(1-u-v)}} \left[f(u) + f(1-v)\right].\end{aligned}$$

Putting this into 6.28 we obtain, with $s = u + v$ and $r = u/s$,

6.29
$$\begin{aligned}\mathbb{E}f(A_1) &= \int_0^1 ds\ \frac{1}{\pi\sqrt{s(1-s)}} \int_0^1 dr \left[\tfrac{1}{2} f(sr) + \tfrac{1}{2}\ f(1-s+sr)\right] \\ &= \mathbb{E}\left[\tfrac{1}{2} f(AU) + \tfrac{1}{2}\ f(1-A+AU)\right],\end{aligned}$$

where A and U are independent, A has the arcsine distribution as in 2.15, and U is uniform on (0, 1).

It is easy to show that, then, AU has the beta distribution with parameter $(\tfrac{1}{2},\ \tfrac{3}{2})$, and so does $A - AU = A(1-U)$ since $1-U$ is also uniform on (0, 1); see Exercise 6.39. Hence, 6.29 yields

$$\begin{aligned}\mathbb{E}\ f(A_1) &= \textstyle\int_0^1 dv\ \frac{2}{\pi} v^{-1/2}(1-v)^{1/2} \left[\tfrac{1}{2}\ f(v) + \tfrac{1}{2}\ f(1-v)\right] \\ &= \textstyle\int_0^1 du \frac{1}{\pi\sqrt{u(1-u)}}\ f(u).\end{aligned}$$

This proves that A_1 has the arcsine distribution as does A in 2.15, which completes the proof since A_t and tA_1 have the same distribution. □

Exercises

Notation: W, M, T, L, S, N, V retain the meanings they had within the present section and earlier. Below, for random variables X and Y (or processes X and Y), we write $X \approx Y$ to mean that X and Y have the same distribution. Throughout, A will denote a random variable having the arcsine distribution as in 2.15.

6.30 *Skew Brownian motion.* Recall the Itô measure ν regulating the excursions, and the sets E_+ and E_- of positive and negative excursions. Let ν_+ be the trace of 2ν on E_+, and ν_- the trace of 2ν on E_-. Then,

$$\nu(D) = \nu(D \cap E_+) + \nu(D \cap E_-) = \frac{1}{2}\nu_+(D) + \frac{1}{2}\nu_-(D), \qquad D \in \mathcal{E}.$$

This is a precise expression of the heuristic that each excursion is positive with probability $1/2$ and negative with $1/2$. Define, for $0 < p < 1$ and $q = 1 - p$, a new measure on $(E, \mathcal{E})$. Let

$$\nu^* = p\, \nu_+ + q\, \nu_-,$$

and let N^* be the Poisson random measure on $\mathbb{R}_+ \times E$ with mean measure $\text{Leb} \times \nu^*$. Define W^* from N^* as W is defined from N through 6.5 and 6.6. The resulting process W^* is called skew Brownian motion; it is a Markov process. It is *not* symmetric, its increments are not independent. Find the distribution of W^*_t. Compute its transition function (P_t).

6.31 *Random time changes.* Many interesting Markov processes are obtained from Wiener processes by random time changes. Here is the general setup. Let $H = (H_t)$ be a random clock; assume that $t \mapsto H_t(\omega)$ is increasing and continuous, starting from $H_0(\omega) = 0$. We think of H_t as the clock time when the standard time is t. Then,

$$\tau_u = \inf\{t : H_t > u\}$$

is the standard time when the clock reads u, and

$$X_u = W_{\tau_u}$$

is the position of the Wiener particle at that time. The simplest case is when $t \mapsto H_t$ is deterministic, strictly increasing, and continuous, then X has (possibly non-stationary) independent increments. Following are some special cases.

6.32 *Reflected Brownian motion.* In 6.31, Suppose that

$$H_t = \int_0^t ds\, 1_{\mathbb{R}_+} \circ W_s.$$

show that X is a reflected Brownian motion, that is, $X \approx |W|$.
Hint: The net effect of the time change on the picture of W is to remove the negative excursions. Modify the excursion measure N accordingly.

6.33 *Processes with two states.* Fix $b > 0$, Let

$$H_t = L_t^\circ + L_t^b,$$

where $L^0 = L$ is the local time at 0, and L^b at b. Show that $X = W_\tau$ is a process with only two states, 0 and b.

a) Show that its jump times form a Poisson process with rate 1/2b, and that the successive jump sizes are $+b,\ -b,\ +b,\ -b, \ldots$. Hint: use proposition 6.19.

b) Compute

$$p_t(x, y) = \mathbb{P}\left\{X_{s+t} = y | X_s = x\right\}$$

for $x,\ y \in \{0,\ b\}$.

6.34 *Processes with three states.* Let $a < 0 < b$ be fixed. Put

$$H_t = L_t^\circ + L_t^a + L_t^b.$$

Show that X is a Markor process whose state space is $D = \{0, a, b\}$. Of course, $X_0 = 0$. Show that the successive states visited by X is a Markov chain $(Y_n)_{n\in\mathbb{N}}$ with $Y_0 = 0$ and transition probability matrix (with states ordered as 0, a, b)

$$P = \begin{bmatrix} 0 & q & p \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where $p = \mathbb{P}\{Y_{n+1} = b | Y_n = 0\} = -a/(-a + b)$, and $q = 1 - p$. Describe X completely by specifying the distributions of

$$\mathbb{P}\left\{R_n \in dt | Y_n = x\right\}, \qquad x \in D,$$

for the time R_n between the n^{th} and $(n + 1)^{th}$ jumps (which is the sojourn time in Y_n).

6.35 *Process on the integers.* Recall that $\mathbb{Z}$ is the set of all integers, positive and negative. Define

$$H_t = \sum_{x\in\mathbb{Z}} L_t^x.$$

For fixed t and ω, show that $H_t(\omega)$ is in fact a finite sum of finite quantities; so, $H_t < \infty$ almost surely for all t in $\mathbb{R}_+$. Show that $X = W_\tau$ is a compound Poisson process, whose jump times form a Poisson process with rate 1 and whose every jump has size ± 1 with equal probabilities.

6.36 *Brownian motion with sticky* 0. Let

$$H_t = t + L_t.$$

The process X is Markov with state space $\mathbb{R}$. It goes through the same states as W does, and in the same order. Show that

$$\int_0^u ds\ 1_{\{0\}} \circ X_s = L_{\tau_u},$$

which is strictly positive for $u > 0$ and increases to $+\infty$ as $u \to \infty$. Describe a method for recovering the path $W(\omega)$ given the path $X(\omega)$.

6.37 *Distribution of* AU. Let A and U be independent, A with the arcsine distribution, and U the uniform on (0,1). Let X, Y, Z be independent gamma variables with shape indices $1/2$, $1/2$, 1 respectively, and with the same scale parameter. Show that

$$A = \frac{X}{X+Y}, \qquad U = \frac{X+Y}{X+Y+Z}$$

satisfy the assumptions on A and U. Conclude that AU has the beta distribution with the index pair $(1/2, 3/2)$.

6.38 *Joint distribution* of G *and* A_G. Write $G = G_1$. In the notation of the proof of Theorem 6.22, similar to 6.27, we can write

$$f(G, A_G) = \int_{\mathbb{R}_+ \times E} N(da, dx)\;\; h\left(S^+_{a-}, S^-_{a-}, x\right)$$

where

$$h(u, v, x) = f(u+v, u) 1_{[0,1]}(u+v) 1_{(1,\infty)}(u+v+\zeta(x)).$$

Show that

$$\mathbb{E}\, f(G, A_G) = \int_0^1 ds \frac{1}{\pi\sqrt{s(1-s)}} \int_0^1 dr \quad f(s, sr).$$

Thus, G has the arcsine distribution (as we already know from 2.18); and, given G, the variable A_G has the uniform distribution on $[0, G]$.

6.39 *Occupation times.* For t in $\mathbb{R}_+$ and r in $\mathbb{R}$, define

$$A_t(r) = \int_0^t ds\; 1_{(-\infty, r]} \circ W_s.$$

By Theorem 6.22, then, $t - A_t(0) \approx tA$, where A has the arcsine distribution as before. Show that

$$A_t(r) \approx t\; A_1\left(\frac{r}{\sqrt{t}}\right), \qquad A_1(-r) \approx 1 - A_1(r).$$

In view of these, it is enough to concentrate on $A(r) = A_1(r)$ for $r > 0$.

6.40 *Distribution* of $A(r)$. Fix $r > 0$. Show that

$$A(r, \omega) = \begin{cases} 1 & \text{if} \quad T_r(\omega) \geq 1, \\ T_r(\omega) + A_{1-T_r(\omega)}\left(\theta_{T_r(\omega)}\omega\right) & \text{if} \quad T_r(\omega) < 1. \end{cases}$$

Show that, with A independent of T_r,

$$A(r) \approx 1_{\{T_r \geq 1\}} + [T_r + (1 + T_r)A]\, 1_{\{T_r < 1\}}.$$

Conclude that, for $u \le 1$,

$$\mathbb{P}\{A(r) < u\} = \int_0^u ds \frac{re^{-r^2/2s}}{\sqrt{2\pi\ s^3}} \mathbb{P}\{s + (1-s)A < u\}.$$

6.41 *Continuation.* This is mere calculus. For $u < 1$, show that

$$\begin{aligned}\mathbb{P}\{A(r) \in du\} &= du \int_0^u ds\ \frac{re^{-r^2/2s}}{\sqrt{2\pi s^3}} \frac{1}{\pi\sqrt{(1-u)(u-s)}} \\ &= du \frac{1}{\pi\sqrt{u(1-u)}} \int_0^u ds \frac{re^{-r^2/2s}}{\sqrt{2\pi s^3(1-s/u)}} = du\ \frac{e^{-r^2/2u}}{\pi\sqrt{u(1-u)}}.\end{aligned}$$

Hint: In the last integral, replace s with $u/(1+v)$; the integral becomes

$$\int_0^\infty dv\ \frac{re^{-r^2(1+v)/2u}}{\sqrt{2\pi uv}} = e^{-r^2/2u} \int_0^\infty dv \frac{e^{-cv} c^a v^{a-1}}{r(a)} = e^{-r^2/2u},$$

where we recognize the gamma density with $a = 1/2$ and $c = r^2/2u$.

6.42 *Continuation.* To sum up, with Z standard Gaussian, show that

$$\mathbb{P}\{A(r) \in du\} = du \frac{e^{-r^2/2u}}{\pi\sqrt{u(1-u)}} 1_{(0,1)}(u) + \delta_1(du)\mathbb{P}\{|Z| \le r\}.$$

6.43 *Gamma tails and Laplace transforms.* It follows from the preceding computation that

$$\mathbb{E}\ e^{-r^2/2A} = \int_0^1 du \frac{e^{-r^2/2u}}{\pi\sqrt{u(1-u)}} = \mathbb{P}\{|Z| > r\}.$$

Taking $r = \sqrt{2p}$ and recalling that $\frac{1}{2} Z^2$ has the standard gamma distribution with shape index $\frac{1}{2}$, we obtain

$$\mathbb{E}\ e^{-p/A} = \mathbb{P}\left\{\frac{1}{2}Z^2 > p\right\} = \int_p^\infty dy \frac{e^{-y}\ y^{-1/2}}{\Gamma(1/2)}.$$

In other words, the tail of the gamma distribution with shape index $1/2$ is the Laplace transform of $1/A$, where A has the arcsine distribution.

6.44 *Brownian Quantiles.* The mapping $r \mapsto A(r)$ is the cumulative distribution function of a random probability measure on $\mathbb{R}$. We define the corresponding quantile function by

$$Q_u = \inf\{r \in \mathbb{R} : A(r) > u\}, \qquad 0 < u < 1.$$

Obviously, $\{Q_u > r\} = \{A(r) < u\}$, and the probabilities of these events can be obtained by using the results of Exercises 6.41–6.42. In particular, for $r > 0$, show that

$$\mathbb{P}\{Q_u \in dr\} = \int_0^u dv \frac{r\ e^{-r^2/2v}}{\pi\sqrt{v^3(1-v)}}.$$

6.45 *Continuation.* It is possible to give a simpler formula for the preceding expression: Since $u < 1$, and $r > 0$, in view of 6.40,

$$\begin{aligned}\mathbb{P}\{Q_u > r\} &= \mathbb{P}\{A + T_r \cdot (1-A) < u\} \\ &= \mathbb{P}\left\{A < u,\ T_r < \frac{u-A}{1-A}\right\} \\ &= \int_0^u dv \frac{1}{\pi\sqrt{v(1-v)}} \int_r^\infty dz\ \frac{2\ e^{-z^2(1-v)/2(u-v)}}{\sqrt{2\pi(u-v)/(1-v)}}\end{aligned}$$

since $T_r \approx r^2/Z^2$, with Z standard Gaussian. Some elementary operations give

$$\begin{aligned}\mathbb{P}\{Q_u \in dr\} &= \frac{2}{\sqrt{2\pi}} \int_0^u dv \frac{1}{\pi\sqrt{v(u-v)}} \exp_- \frac{r^2}{2}\frac{1-v}{u-v} \\ &= \frac{2}{\sqrt{2\pi}} \int_0^1 dx \frac{1}{\pi\sqrt{x(1-x)}} \exp_- \frac{r^2}{2}\left(1 + \left(\frac{1-u}{u}\right)\cdot\frac{1}{v}\right) \\ &= \frac{2}{\sqrt{2\pi}} e^{-r^2/2}\ \mathbb{E}\ \exp_- \frac{r^2(1-u)}{2u} \cdot \frac{1}{A}.\end{aligned}$$

The last expectation can be evaluated using 6.43 to obtain

$$\mathbb{P}\{Q_u \in dr\} = \frac{2}{\sqrt{2\pi}} e^{-r^2/2}\ \mathbb{P}\left\{|Z| > r\sqrt{\frac{1-u}{u}}\right\}.$$

7 Path Properties

This section is on the oscillatory behavior of Brownian paths. We shall see that, for almost every ω, the following are true for the Wiener path $t \mapsto W_t(\omega)$: The path is continuous, but nowhere differentiable. Over every interval, it has infinite total variation, but finite quadratic variation; thus, the path is highly oscillatory, but the oscillations are of small amplitude. In addition to clarifying these points, we shall discuss Hölder continuity of the paths, describe the exact modulus of continuity, and give the law of the iterated logarithm. These help to visualize the paths locally in terms of deterministic functions.

Throughout this section, W is a Wiener process over some probability space $(\Omega, \mathcal{H}, \mathbb{P})$. We assume that the path $W(\omega) : t \mapsto W_t(\omega)$ is continuous for every ω. By a subdivision of an interval $[a,b]$ we mean a finite collection of disjoint intervals of the form $(s,t]$ whose union is $(a,b]$; it is a partition of $(a,b]$ whose elements are intervals. If $\mathcal{A}$ is a subdivision, we write $\|\mathcal{A}\|$ for its *mesh*, defined as $\|\mathcal{A}\| = sup\{t - s : (s,t] \in \mathcal{A}\}$.

Quadratic variation

Let $f: \mathbb{R}_+ \mapsto \mathbb{R}$ be right-continuous. Fix an interval $[a, b]$ in $\mathbb{R}_+$. For $p > 0$ and $\mathcal{A}$ a subdivision of $[a, b]$, consider

7.1
$$\sum_{(s,t]\in\mathcal{A}} |f(t) - f(s)|^p \,.$$

The supremum of this over all such subdivisions $\mathcal{A}$ is called the *true p-variation* of f over $[a, b]$. For $p = 1$, the supremum is called the *total variation* of f on $[a, b]$, and for $p = 2$ the *true quadratic variation.*

These deterministic concepts prove to be too strict when applied to a typical Wiener path: for almost every ω, if $f = W(\omega)$, the total variation over $[a, b]$ is $+\infty$, and so is the true quadratic variation. However, at least for the quadratic variation, a probabilistic version proves interesting:

7.2 THEOREM. *Let the interval $[a, b]$ be fixed. Let $(\mathcal{A}_n)$ be a sequence of subdivisions of it with $\|\mathcal{A}_n\| \to 0$. Then, the sequence of random variables*

7.3
$$V_n = \sum_{(s,t]\in\mathcal{A}_n} |W_t - W_s|^2$$

converges in L^2 and in probability to the length $b - a$.

Proof. Recall that $|W_t - W_s|^2$ has the same distribution as $(t - s)\ Z^2$, where Z is standard Gaussian, and that $\mathbb{E}\ Z^2 = 1$, var $Z^2 = 2$. Since the intervals $(s, t]$ in $\mathcal{A}_n$ are disjoint, the corresponding increments $W_t - W_s$ are independent. Thus,

$$\mathbb{E}\, V_n = \sum_{(s,t]\in\mathcal{A}_n} (t - s) = b - a,$$
$$\text{Var } V_n = \sum_{(s,t]\in\mathcal{A}_n} (t - s^2) \cdot 2 \le 2 \cdot (b - a) \cdot \|\mathcal{A}_n\|.$$

Hence, $\mathbb{E}|V_n - (b-a)|^2 = \text{Var } V_n \to 0$ as $n \to \infty$. This shows the convergence in L^2 and implies the convergence in probability. □

The limit in the preceding theorem is called the *quadratic variation* of W over $[a, b]$. Heuristically, it is a sum of squares of the increments over infinitesimal subintervals. The following clarifies this picture by taking the limit for each ω separately.

7.4 PROPOSITION. *For each n in $\mathbb{N}$, let $\mathcal{A}_n$ be the subdivision of $[a, b]$ that consists of 2^n intervals of the same length. Then, (V_n) defined by 7.3 converges to $b - a$ almost surely.*

Proof. Since each $(s,t]$ in $\mathcal{A}_n$ has length $(b-a)\cdot 2^{-n}$, we have $\mathbb{E}\, V_n = b-a$ as before, but Var $V_n = 2^n \cdot 2 \cdot (b-a)^2 \cdot 2^{-2n}$. Thus, Chebyshev's inequality yields that, for $\varepsilon > 0$,

$$\mathbb{P}\{|V_n - (b-a)| > \varepsilon\} \le \frac{1}{\varepsilon^2} \cdot 2 \cdot (b-a)^2 \cdot 2^{-n}.$$

Since the right side is summable in n, Borel-Cantelli lemma I.2.6 applies, and $V_n \to b\text{-}a$ almost surely. □

7.5 Remark. a) The preceding proposition can be strengthened. The conclusion remains true when $(\mathcal{A}_n)$ is an arbitrary nested sequence with $\|\mathcal{A}_n\| \to 0$, the term *nested* meaning that each interval of the subdivision $\mathcal{A}_{n+1}$ is a subset of some interval in $\mathcal{A}_n$.

b) But, it is essential that $(\mathcal{A}_n)$ be chosen deterministically. Otherwise, there are counter-examples. For example, for almost every ω there is a nested sequence $(\mathcal{A}_n(\omega))$ with $\|\mathcal{A}_n(\omega)\| \to 0$ such that $V_n(\omega)$ defined by 7.3 goes to $+\infty$.

Total variation

The following proposition shows that each typical path is highly oscillatory over every interval. But the amplitudes must be small enough that their squares sum to the finite number called the quadratic variation.

7.6 Proposition. *For almost every ω, the path $W(\omega)$ has infinite total variation over every interval $[a,\, b]$ with $a < b$.*

Proof. In the setting of Proposition 7.4, let Ω_{ab} be the almost sure set of convergence. Pick ω in Ω_{ab}, write w for $W(\omega)$, and let $v^* \le +\infty$ be the total variation of w over $[a,b]$. We observe that, with sums and supremum over all $(s,t]$ in $\mathcal{A}_n$,

$$\sum |w_t - w_s|^2 \le (\sup|w_t - w_s|) \sum |w_t - w_s| \le (\sup|w_t - w_s|) \cdot v^*,$$

the last inequality being by the definition of v^* as the supremum over all subdivisions. Now, let $n \to \infty$. The left side goes to $b-a \ne 0$ by the way ω is picked. On the right side, the supremum goes to 0 by the uniform continuity of w on $[a,b]$. It follows that v^* cannot be finite.

Let Ω_0 be the intersection of Ω_{ab} over all rationals a and b with $0 \le a < b$. The claim of the proposition holds for $W(\omega)$ for every ω in the almost sure event Ω_0. □

Hölder continuity, nowhere differentiability

Let $\alpha \in \mathbb{R}_+$, $B \subset \mathbb{R}_+$, and $f : \mathbb{R}_+ \mapsto \mathbb{R}$. The function f is said to be *Hölder continuous* of order α on B if there is a constant k such that

7.7 $$|f(t) - f(s)| \le k \cdot |t-s|^\alpha \qquad \text{if } s,t \in B.$$

It is said to be locally Hölder continuous of order α if it is such on $[0, b]$ for every $b < \infty$. Note that if f is differentiable at some point then it is Hölder continuous of order 1 on some neighborhood of that point.

The next proposition is another consequence of the finiteness of the quadratic variation, Proposition 7.4. Its proof is similar to that of Proposition 7.6.

7.8 PROPOSITION. *For almost every ω, the Wiener path $W(\omega)$ is Hölder continuous of order α on no interval for $\alpha > 1/2$. In particular, for almost every ω, the path is nowhere differentiable.*

Proof. Pick ω, write w for $W(\omega)$, and suppose that

$$|w_t - w_s| \le k \cdot |t-s|^{\alpha}$$

for all s and t in some interval $[a, b]$, $a < b$, for some $\alpha > 1/2$ and some constant k. With $\mathcal{A}_n$ as in Proposition 7.4, with summations and supremum over $(s, t]$ in $\mathcal{A}_n$,

$$\sum |w_t - w_s|^2 \le k^2 \sum |t-s|^{2\alpha} \le k^2 \cdot (b-a) \cdot \sup |t-s|^{2\alpha - 1}.$$

As $n \to \infty$, the supremum vanishes since $2\alpha > 1$, which means that the left side vanishes as well. Thus, ω does not belong to the almost sure set Ω_{ab} of convergence in Proposition 7.4. Hence, the claims hold for every ω in the intersection of Ω_{ab} over all rationals $a < b$. □

7.9 REMARK. We shall see shortly that the claim of the preceding proposition remains true for $\alpha = 1/2$ as well; see Theorem 7.13 below.

By contrast, the following is a positive result. Its proof is based on a lemma of independent interest; the lemma is put last in this section in order to preserve the continuity of presentation; see 7.32.

7.10 PROPOSITION. *For almost every ω, the path $W(\omega)$ is locally Hölder continuous of order α for every $\alpha < 1/2$.*

Proof. For Z standard Gaussian, $c_p = \mathbb{E}\, Z^{2p} < \infty$, and

$$\mathbb{E}\, |W_t - W_s|^{2p} = c_p\, |t-s|^p, \qquad p \ge 1.$$

Thus, Lemma 7.32 below applies, and almost every path is Hölder continuous of order $\alpha = (p-1)/2p = 1/2 - 1/2p$ on [0,1]. Scaling property allows us to replace [0,1] with [0,b] for each $b < \infty$, and the proof is complete since p can be taken as large as desired. □

Modulus of continuity

Let f and g be functions from [0,1] into $\mathbb{R}$. The function g is said to be a *modulus of continuity* for f if

7.11 $$s, t \in [0,1]\,,\ [t-s] \le \delta \Rightarrow |f(t) - f(s)| \le g(\delta)$$

for every $\delta > 0$ small enough. Of course, then, so is cg for every constant $c \ge 1$. The following theorem, due to Lévy, shows that

7.12 $$g(t) = \sqrt{2t \log(1/t)}, \qquad t \in [0,1],$$

is the exact modulus of continuity for the paths of $(W_t)_{t\in[0,1]}$ in the following sense: cg is a modulus of continuity for almost every path if $c > 1$, and is a modulus of continuity for almost no path if $c < 1$. The proof will be delayed somewhat; see 7.26.

7.13 THEOREM. *Let g be as defined by 7.12. Then, for almost every ω,*

$$\limsup_{\delta\to 0} \frac{1}{g(\delta)} \sup_{\substack{0 \le s < t \le 1 \\ t - s \le \delta}} |W_t(\omega) - W_s(\omega)| = 1.$$

As a corollary, since $\sqrt{\delta}/g(\delta)$ goes to 0 as δ goes to 0, we obtain the proof of Remark 7.9. Details are left as an exercise.

Law of the iterated logarithm

This is about the oscillatory behavior of Wiener paths near the time 0 and for very large times. The name comes from its use of

7.14 $$h(t) = \sqrt{2t\ \log\log(1/t)}, \qquad t \in [0,1],$$

as the control function.

7.15 THEOREM. *With h as in 7.14, the following hold for almost every ω:*

$$\limsup_{t\to 0} \frac{1}{h(t)} W_t(\omega) = 1, \qquad \liminf_{t\to 0} \frac{1}{h(t)} W_t(\omega) = -1.$$

7.16 REMARK. By time inversion, the same results hold when $W_t(\omega)$ is replaced with $t\,W_{1/t}(\omega)$. Then, replacing $1/t$ with t, we obtain that the following hold for almost every ω:

7.17 $$\limsup_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = 1, \quad \liminf_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = -1.$$

We know from Lemma 3.3 that the running maximum increases to $+\infty$, and the running minimum decreases to $-\infty$ in the limit as $t \to \infty$. These are re-confirmed by 7.17 and are made more precise.

Proofs. We list here two approximation lemmas before giving the proofs of the last two theorems.

7.18 LEMMA. $\mathbb{P}\left\{\sup_{t\le 1}\left(W_t - \tfrac{1}{2}\, pt\right) > q\right\} \le e^{-pq}$ *for positive* p *and* q.

Proof. Let $X_t = \exp\left(pW_t - \tfrac{1}{2}\, p^2 t\right)$. The probability in question is

$$\mathbb{P}\left\{\sup_{t\le 1}\ X_t > e^{pq}\right\} \le e^{-pq}\ \mathbb{E}\ X_1 = e^{-pq},$$

where the inequality follows from the maximal inequality V.5.33 applied to the exponential martingale X. □

7.19 LEMMA. *Let* Z *be a standard Gaussian variable. Then, for* $b > 0$,

$$\frac{1}{4}\cdot\frac{b}{1+b^2}\ e^{-b^2/2} < \mathbb{P}\{Z > b\} < \frac{1}{2b}e^{-b^2/2}.$$

Proof. Observe that

$$\int_b^\infty dx\ e^{-x^2/2} < \int_b^\infty dx\ \frac{x}{b}\ e^{-x^2/2} = \frac{1}{b}\ e^{-b^2/2},$$

and

$$\int_b^\infty dx\ e^{-x^2/2} > \int_b^\infty dx\ \frac{b^2}{x^2}\ e^{-x^2/2} = b\ e^{-b^2/2} - b^2\int_b^\infty dx\ e^{-x^2/2},$$

the last equality being through integration by parts. The rest is arithmetic. □

7.20 *Proof of Theorem* 7.15. a) We show first that, for almost every ω,

7.21 $$\alpha(\omega) = \limsup_{t\to 0}\frac{1}{h(t)}W_t(\omega)$$

is at most 1.

Let $0 < a < 1 < b$. Put $p_n = b\ h(a^n)/a^n$ and $q_n = \tfrac{1}{2}\ h(a^n)$. By Lemma 7.18,

$$\mathbb{P}\left\{\sup_{t\le 1}\left(W_t - \tfrac{1}{2}\, p_n t\right) > q_n\right\} \le e^{-p_n q_n} = \left(n\log \tfrac{1}{a}\right)^{-b},$$

and the right side is summable in n. Thus, by the Borel-Cantelli lemma, there is an almost sure event Ω_{ab} such that for every ω in it there is n_ω such that

7.22 $$W_t(\omega) \le q_n + \tfrac{1}{2}\ p_n t \qquad \text{for every } t\le 1 \text{ and } n \ge n_\omega.$$

The function h is increasing on $[0, e^{-c}]$, where $c = e^{1/c}$. Choose $n \ge n_\omega$ large enough that $a^{n-1} \le e^{-c}$, and let $t \in (a^n,\ a^{n-1}]$. By 7.22,

$$W_t(\omega) \le q_n + \tfrac{1}{2}p_n\ a^{n-1} = \frac{1}{2}\cdot\left(1+\frac{b}{a}\right)h(a^n) \le \frac{1}{2}\left(1+\frac{b}{a}\right)h(t).$$

Hence, for ω in Ω_{ab},

7.23 $$\alpha(\omega) \le \frac{1}{2}\left(1+\frac{b}{a}\right).$$

write Ω_n for Ω_{ab} with $a = 1-(1/n)$ and $b = 1+(1/n)$. Now 7.23 implies that $\alpha(\omega) \le 1$ for every ω in the almost sure event $\cap_n \Omega_n$.

b) Next, we prove that $\alpha(\omega) \ge 1$ for almost every ω. Let $\varepsilon \in (0,1)$ and put $t_n = \varepsilon^{2n}$. Observe that $h(t_{n+1}) \le 2\ \varepsilon\ h(t_n)$ for all n large enough. And, by part(a) applied to the Wiener process $(-W_t)$, there is an almost sure set Ω_o such that 7.21 holds, with $-W$ replacing W, for almost every ω. Thus,

7.24 $$\omega \in \Omega_0 \Rightarrow -W_{t_{n+1}}(\omega) \le 2\ h(t_{n+1}) \le 4\ \varepsilon\ h(t_n) \quad \text{for all } n \text{ large enough.}$$

On the other hand, by Lemma 7.19 applied with $b = (1-\varepsilon)h(t_n)/\sqrt{t_n - t_{n+1}}$,

$$p_n = \mathbb{P}\left\{W_{t_n} - W_{t_{n+1}} > (1-\varepsilon)h(t_n)\right\} > \frac{1}{4}\,\frac{b}{1+b^2}\ e^{-b^2/2},$$

and $e^{-b^2/2} = (2\quad n\quad \log\quad 1/\varepsilon)^{-c}$, where $c = (1-\varepsilon)/(1+\varepsilon)$ is less than 1. It follows that $\Sigma_{p_n} = +\infty$. Since the increments $W_{t_n} - W_{t_{n+1}}$ are independent, the divergence part of the Borel-Cantelli lemma applies. There is an almost sure event Ω_ε such that

7.25 $$\omega \in \Omega_\varepsilon \Rightarrow W_{t_n}(\omega) - W_{t_{n\pm1}}(\omega) > (1-\varepsilon)\ h(t_n) \quad \text{for infinitely many } n.$$

Combining 7.24 and 7.25, we see that

$$\begin{aligned} \omega \in \Omega_0 \cap \Omega_\varepsilon &\Rightarrow W_{t_n}(\omega) > (1-5\varepsilon)\ h(t_n) \\ &\Rightarrow \limsup_{t\to 0} \frac{1}{h(t)} W_t(\omega) \ge \limsup_{n\to\infty} \frac{1}{h(t_n)} W_{t_n}(\omega) \ge 1-5\varepsilon. \end{aligned}$$

For $k \ge 1$, put $\Omega_k = \Omega_o \wedge \Omega_\varepsilon$ with $\varepsilon = 1/k$. Then, for ω in $\bigcap \Omega_k$,

$$\limsup_{t\to 0} \frac{1}{h(t)} W_t(\omega) \ge 1.$$

This completes the proof of the statement about the limit superior. The one about the limit inferior is obtained by recalling that $\lim\ \inf\ x_n = -\lim\ \sup(-x_n)$ and that $-W$ is again Wiener.

7.26 *Proof of Theorem* 7.13. a) First we show that

7.27 $$\alpha(\omega) = \limsup_{\delta\to 0} \frac{1}{g(\delta)} \sup_{\substack{0 \le s < t \le 1 \\ t-s\le\delta}} |W_t(\omega) - W_s(\omega)|$$

is equal to 1 or more for every ω in an almost sure event Ω_o.

Take a in (0,1), put $u = 2^{-n}$, and recall g from 7.12. Note that $g(u)/\sqrt{u} = b\sqrt{n}$, where $b = \sqrt{2\log 2}$, and $e^{-nb^2/2} = 2^{-n}$. For Z standard Gaussian, it follows from Lemma 7.19 that

$$p = \mathbb{P}\left\{\sqrt{u}\ |Z| > a\ g(u)\right\} > \frac{1}{2}\frac{ab\sqrt{n}}{1+a^2b^2n}\ e^{-a^2b^2n/2} > c\ 2^{-na^2}/\sqrt{n},$$

for some constant c depending on a only. Thus, since the increments $W_{ku} - W_{ku-u}$ are independent and identically distributed as $\sqrt{u}$ Z,

$$\begin{aligned}&\mathbb{P}\left\{\max_{1\le k\le 2^n} |W_{ku} - W_{ku-u}| \le a\ g(u)\right\}\\ &\quad = (1-p)^{2^n} \le e^{-p2^n} \le \exp\left(-c2^{n-na^2}/\sqrt{n}\right)\end{aligned}$$

since $1 - p \le e^{-p}$. The right-most member is summable in n. Hence, by the Borel-Cantelli lemma, there is an almost sure event Ω_a such that

$$\omega \in \Omega_a, u = 2^{-n} \Rightarrow \max_{1\le k\le 2^n} |W_{ku}(\omega) - W_{ku-u}(\omega)| > a\ g(u)$$

for all n large enough, which means that $\alpha(\omega) > a$; see 7.27 for $\alpha(\omega)$. Let Ω_0 be the intersection of Ω_a over a in $\{1/2,\ 2/3,\ 3/4,\ \ldots\}$; then, Ω_0 is almost sure, and $\alpha(\omega) \ge 1$ for ω in Ω_0.

b) We show next that $\alpha(\omega) \le 1$ for almost every ω. Choose $b > 1$. Put $a = 2/(1+b)$. Note that $a \in (0,1)$ and $ab > 1$. For u in $(0,\ 2^{-na})$, since $g(u)/\sqrt{u} \ge \sqrt{2\,na\,\log 2}$, it follows from Lemma 7.19 that

$$\mathbb{P}\left\{\sqrt{u}\,|Z| > b\,g(u)\right\} \le \mathbb{P}\left\{|Z| > b\sqrt{2\,na\,\log 2}\right\}$$

7.28
$$\le \frac{e^{-b^2\,na\,\log 2}}{b\sqrt{2\,na\,\log 2}} = c\cdot 2^{-na\,b^2}/\sqrt{n},$$

where c depends only on b.

Let B_n be the set of all pairs of numbers s and t in the set $D_n = \{k/2^n : 0 \le k \le 2^n\}$ satisfying $0 < t - s < 2^{-na}$; there are at most 2^{na} such pairs (s,t). Using 7.28 with $u = t - s$, we get

$$\mathbb{P}\left\{\max_{(s,t)\in B_n} \frac{1}{g(t-s)}\,|W_t - W_s| > b\right\} \le 2^{na}\cdot c\cdot 2^{-na\,b^2}/\sqrt{n};$$

and the right side is summable in n, since $ab^2 - a > b - a > 0$. Thus, by the Borel-Cantelli lemma, there is an almost sure event Ω_b such that for every ω in it there is n_ω such that

7.29
$$n \ge n_\omega, (s,t) \in B_n \Rightarrow |W_t(\omega) - W_s(\omega)| \le b\ g(t-s).$$

Fix ω in Ω_b; write n^* for n_ω, and w for $W(\omega)$. Let $D = \bigcup_0^\infty D_m$, the set of all dyadic numbers in [0,1]. For s and t in D, put $s_m = \inf D_m \cap [s,1]$ and

$t_m = sup\ D_m \cap [0, t]$. Then, (s_m) is decreasing, (t_m) is increasing, and $s_m = s$ and $t_m = t$ for all m large enough. Thus,

7.30 $$w_t - w_s = \sum_{m \geq n} \left(w_{t_{m+1}} - w_{t_m}\right) + w_{t_n} - w_{s_n} + \sum_{m \geq n} \left(w_{s_m} - w_{s_{m+1}}\right).$$

Suppose that $0 < t - s < 2^{-n^* a}$ and choose $n \geq n^*$ such that

7.31 $$2^{-na-a} \leq t - s < 2^{-na} < e^{-1}.$$

Then, $s \leq s_n \leq t_n \leq t$, and the times $t_{m+1},\ t_m,\ s_m,\ s_{m+1}$ belong to B_{m+1} for every $m \geq n$. It follows from 7.29 and 7.30 that

$$|w_t - w_s| \leq \sum_{m \geq n} b\ \ g(t_{m+1} - t_m) + b\ \ g(t - s) + \sum_{m \geq n} b\ \ g\left(s_m - s_{m+1}\right).$$

Moreover, g is increasing on $[0, e^{-1}]$, and $t_{m+1} - t_m \leq 2^{-m-1} \leq e^{-1}$ and $s_m - s_{m+1} \leq 2^{-m-1} \leq e^{-1}$ for $m \geq n$ by the way n is chosen. So,

$$|w_t - w_s| \leq b\ \ g(t - s) + 2b \sum_{m \geq n} g(2^{-m-1}).$$

Also,

$$\sum_{m > n} g(2^{-m}) = g(2^{-n}) \sum_{m > n} \sqrt{2^{-m+n} m/n} \leq g(2^{-n}) \sum_{p \geq 1} \sqrt{2p\, 2^{-p}}$$

and

$$g\left(2^{-n}\right) \leq g(t - s) g(2^{-n})/g(2^{-na-a}) \leq g(t - s)\sqrt{2^{-n(1-a)} \cdot 2^a/a}$$

in view of 7.31. Combining the last three expressions, we see that

$$|w_t - w_s| \leq b\ g(t - s) + 2bc\ g(t - s)\sqrt{2^{-n(1-a)}}$$

with c chosen appropriately. This was for s and t in D satisfying 7.31; by the continuity of w, the same holds for all s and t in [0,1] satisfying 7.31. Consequently, letting $n \to \infty$ and recalling that $1 - a > 0$, we see for $\alpha(\omega)$ of 7.27 that $\alpha(\omega) \leq b$ for the arbitrarily fixed ω in Ω_b. Thus, $\alpha(\omega) \leq 1$ for every ω in $\bigcap \Omega_b$, where the intersection is over b in $\{1 + 1/n : n \geq 1\}$. This completes the proof. □

Kolmogorov's moment condition

The next lemma was used to prove Hölder continuity in Proposition 7.10. It is the main part of Kolmogorov's theorem on the existence of continuous modifications. These are stated in a form that will be of use in the next section. Recall that D is the set of all dyadic numbers in [0,1]. Here, $X = (X_t)_{t \in [0,1]}$ is a process with state space $\mathbb{R}$.

7.32 LEMMA. *Suppose that there exist constants c, p, q in (0,1) such that*

7.33 $$\mathbb{E}\,|X_t - X_s|^p \le c \cdot |t-s|^{1+q}, \qquad s,t \in [0,1].$$

Then, for every α in $[0,q/p)$ there is a random variable K such that $\mathbb{E}\ K^p$ is finite and

7.34 $$|X_t - X_s| \le K \cdot |t-s|^{\alpha}, \qquad s,t \in D.$$

If X is also continuous, then 7.34 holds for all s, t in [0,1].

Proof. Fix α in $[0, q/p)$. Let

7.35 $$K = \sup_{s,t\in D,\ s\neq t} \frac{|X_t - X_s|}{|t-s|^{\alpha}}.$$

Since $D \times D$ is countable, this defines a random variable. Now, 7.34 is obvious, and it extends to s,t in [0,1] when X is continuous, since D is dense in [0,1]. Thus, the proof reduces to showing that

7.36 $$\mathbb{E} \quad K^p < \infty.$$

a) Let $M_n = \sup |X_t - X_s|$, where the supremum is over all pairs of numbers s and t in D_n with $t - s = 2^{-n}$. Since there are 2^n such pairs, the assumption 7.33 implies that

7.37 $$\mathbb{E}\, M_n^p \le 2^n \cdot c \cdot (2^{-n})^{1+q} = c \cdot 2^{-nq}.$$

b) For s and t in D, let $s_n = \inf\ D_n \cap [s,1]$ and $t_n = \sup\ D_n \cap [0,t]$. Then, (s_n) is decreasing, (t_n) is increasing, and $s_n = s$ and $t_n = t$ for all n large enough. Thus,

$$X_t - X_s = \sum_{n\ge m} \left(X_{t_{n+1}} - X_{t_n}\right) + X_{t_m} - X_{s_m} + \sum_{n \ge m} \left(X_{s_n} - X_{s_{n+1}}\right).$$

If $0 < t - s \le 2^{-m}$, then $t_m - s_m$ is either 0 or equal to 2^{-m}; hence,

7.38 $$|X_t - X_s| \le \sum_{n\ge m} M_{n+1} + M_m + \sum_{n\ge m} M_{n+1} \le 2 \sum_{n \ge m} M_n.$$

c) Consider 7.35. Take the supremum there first over s and t with $2^{-m-1} < |t-s| \le 2^{-m}$ and then over m. In view of 7.38, we get

$$K \le \sup_m \left(2^{m+1}\right)^{\alpha} \cdot 2 \sum_{n\ge m} M_n \le 2^{1+\alpha} \sum_{n\ge 0} 2^{n\alpha} M_n.$$

If $p \ge 1$, letting $\|\cdot\|$ denote the L^p-norm, we see from 7.37 that

$$\|K\| \le 2^{1+\alpha} \sum_n 2^{n\alpha}\ c^{1/p}\ 2^{-nq/p} < \infty$$

since $\alpha < q/p$. If $p < 1$, then $(x+y)^p \le x^p + y^p$ for positive x and y, and

$$\mathbb{E}K^p \le (2^{1+\alpha})^p \sum_n 2^{n\alpha p} c \cdot 2^{-nq} < \infty$$

again. Thus, 7.36 holds in either case, as needed to complete the proof. □

The following is Kolmogorov's theorem on modifications. Recall that $\tilde{X}$ is a modification of X if for every t there is an almost sure event Ω_t on which $\tilde{X}_t = X_t$.

7.39 THEOREM. *Suppose that 7.33 holds for some constants, c, p, q in $(0, \infty)$. Then, for every α in $[0, q/p]$ there is a modification $\tilde{X}$ of X such that the path $\tilde{X}(\omega)$ is Hölder continuous of order α on [0, 1] for every ω.*

Proof. Fix α as described. Let K be as in Lemma 7.32. Since $\mathbb{E}\, K^p < \infty$, the event $\Omega_0 = \{K < \infty\}$ is almost sure. For ω outside Ω_0, put $\tilde{X}(\omega) = 0$ identically. For ω in Ω_0, Lemma 7.32 ensures that $X(\omega)$ is Hölder continuous of order α on D. Thus, putting

7.40 $$\tilde{X}_t(\omega) = \lim_{\substack{s \to t \\ s \in D}} X_s(\omega), \qquad t \in [0,1], \quad \omega \in \Omega_0,$$

we obtain a path $\tilde{X}(\omega)$ that is Hölder continuous of order α on [0,1]. The same property holds trivially for $\tilde{X}(\omega)$ with $\omega \notin \Omega_0$. Finally, for each t in [0,1], we have $X_t = \tilde{X}_t$ almost surely in view of 7.40 and 7.33. □

Exercises

7.41 *p-variation.* Let $\mathcal{A}_n$ be the subdivision of [0,1] that consist of $(0, \delta]$, $(\delta, 2\delta]$, ..., $(1-\delta, 1]$ with $\delta = 1/n$. Show that, for $p > 0$,

$$\frac{1}{n}\sqrt{n^p} \sum_{(s,t] \in \mathcal{A}_n} |W_t - W_s|^p$$

converges, as $n \to \infty$, to $\mathbb{E}|Z|^p$ in probability, where Z is standard Gaussian. Hint: Use time inversion and the weak law of large numbers.

7.42 *Monotonicity.* For almost every ω, the Wiener path is monotone in no interval. Show. Hint: Compare Proposition 7.6 with Exercise I.5.24.

7.43 *Local maxima.* Let $f : [0,1] \mapsto \mathbb{R}$ be continuous. It is said to have a *local maximum* at t if there is $\varepsilon > 0$ such that $f(s) \le f(t)$ for every s in $(t-\varepsilon,\ t+\varepsilon)$. Suppose that f is monotone in no interval. Show the following:

a) f has a local maximum.

b) If f has local maxima at s and at t, then it has a local maximum at some point u in (s, t).

c) The set of all local maxima is dense in [0,1].

7.44 *Exponential inequality.* This is similar to Lemma 7.18. Let $M_t = \max_{s \le t} W_s$. Show that, for $a > 0$,

$$\mathbb{P}\{M_t > at\} \le e^{-a^2 t/2}.$$

Hint: Recall that $M_t \approx \sqrt{t}|Z|$. So, $M_t^2 \approx t\,Z^2 \le t(Z^2 + Y^2) \approx 2tX$, where Y and Z are independent standard Gaussians, and X is standard exponential.

8 Existence

This is to end the chapter by completing the circle, by showing that Brownian motions do exist. The question of existence is mathematical: Does there exist a probability space, and a process defined over it, such that the process is continuous and appropriately Gaussian.

We give two very different constructions. The first is via Kolmogorov's extension theorem and existence of continuous modifications; here, the complexities of the Wiener process are built into the probability measure in an abstract fashion. By contrast, the second, due to Lévy, uses a very simple probability space, and the intricacies of the process are built explicitly into the paths.

First construction

The basic ingredients are Theorem IV.4.18, the Kolmogorov extension theorem, and Theorem 7.39 on the existence of continuous modifications.

8.1 Theorem. *There exist a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ and a stochastic process $W = (W_t)_{t \in \mathbb{R}_+}$ such that W is a Wiener process over $(\Omega, \mathcal{H}, \mathbb{P})$.*

Proof. We follow the setup of Theorem IV.4.18. Let Ω be the set of all mappings from $\mathbb{R}_+$ into $\mathbb{R}$. For t in $\mathbb{R}_+$ and ω in Ω, put $X_t(\omega) = \omega(t)$. Let $\mathcal{H}$ be the σ-algebra on Ω generated by $\{X_t : t \in \mathbb{R}_+\}$. For each finite subset J of $\mathbb{R}_+$, if J has $n \ge 1$ elements, let π_J be the n-dimensional Gaussian distribution on $\mathbb{R}^J$ with mean 0 and covariances $s \wedge t$ for s and t in J. These finite-dimensional distributions π_J form a consistent family. Thus, by Theorem IV.4.18, there exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{H})$ such that the distribution of $(X_t)_{t \in J}$ under $\mathbb{P}$ is given by π_J for every finite subset J of $\mathbb{R}_+$. It follows that the process $X = (X_t)_{t \in \mathbb{R}_+}$ has stationary and independent increments, has $X_0 = 0$ almost surely, and every increment $X_t - X_s$ has the Gaussian distribution with mean 0 and variance $t - s$.

Consider $(X_t)_{t \in [0,1]}$. Note that the condition 7.33 holds, for instance, with $p = 4$, $q = 1$, $c = 3$. Thus, Theorem 7.39 applies: there is a modification $(\tilde{X}_t)_{t \in [0,1]}$ that is continuous. Applying 7.39 repeatedly to $(X_t)_{t \in [n, n+1]}$, $n \in \mathbb{N}$, we obtain a process $\tilde{X} = (\tilde{X}_t)_{t \in \mathbb{R}_+}$ that is continuous and has the same finite-dimensional distributions as X. Thus, $\tilde{X}$ is the Wiener process W sought. □

Lévy's construction

This starts with a probability space on which there is defined a countable independency of standard Gaussian variables Z_q, one for each q in the set D of all dyadic numbers in [0,1]. This is easy to do; see the next theorem, and also the exercises below which show that the probability space can be taken to be $((0,1), \mathcal{B}_{(0,1)}, \text{Leb})$.

The object is to construct $X = \{X(t) : t \in [0,1]\}$ such that X is a Wiener process on [0,1]. It will be obtained as the limit of a sequence of piecewise linear continuous processes X_n. The initial process is defined as

8.2 $$X_0(t) = t\ Z_1, \qquad t \in [0,1].$$

By the n^{th} step, the variables $X(t)$ will have been defined for t in $D_n = \{k/2^n : \ k = 0, 1, \ldots, 2^n\}$, and the process $X_n = \{X_n(t) = t \in [0,1]\}$ is the piecewise linear continuous process with $X_n(t) = X(t)$ for t in D_n. At the next step, $X(t)$ is specified for t in $D_{n+1} \backslash D_n$ and X_{n+1} is defined to be the piecewise linear continuous process with $X_{n+1}(t) = X(t)$ for t in D_{n+1}. See Figure 16.

To implement this plan, we need to specify $X(q)$ for q in $D_{n+1} \backslash D_n$ consistent with the values $X(t)$ for t in D_n. This problem is solved by Example 1.9 since X is to be Wiener: Given $X(p)$ and $X(r)$ for adjacent points p and r in D_n, the conditional distribution of $X(q)$ at the mid-point q of $[p, r]$ must be Gaussian with mean $\frac{1}{2}X(p) + \frac{1}{2}X(r)$ and variance 2^{-n-2}; note that the conditional mean is exactly $X_n(q)$; thus, we should put $X(q) = X_{n+1}(q) = X_n(q) + \sqrt{2^{-n-2}}\ Z_q$. Finally, piecewise linearity of X_n and X_{n+1} require that we put

8.3 $$X_{n+1}(t) = X_n(t) + \sum_{q \in D_{n+1} \backslash D_n} h_q(t)\ Z_q, \qquad n \geq 0,\ t \in [0,1],$$

where

8.4 $$h_q(t) = \sqrt{2^{-n-2}} \left(1 - |t - q| \cdot 2^{n+1}\right)^+, \qquad q \in D_{n+1} \backslash D_n.$$

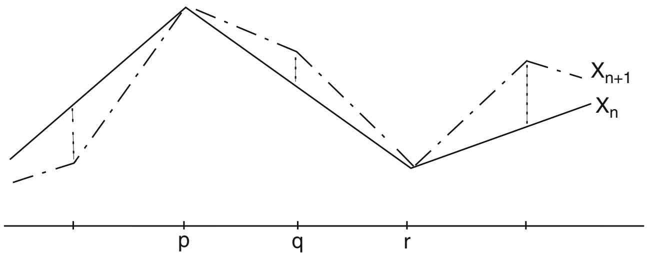

Figure 16: Approximation X_{n+1} coincides with X_n at the points p and r in D_n and differs from X_n at the midpoint q by an amount $Z_q/\sqrt{2^{n+2}}$.

8.5 REMARKS. a) For q in $D_{n+1} \backslash D_n$, the function h_q achieves its maximum $\sqrt{2^{-n-2}}$ at the point q and vanishes outside the interval of length 2^{-n} centered at q. Thus, in particular, for each t, the sum in 8.3 has at most one non-zero term.

b) Note that all the h_q are re-scaled translations of the "mother wavelet" $h(t) = (1 - |t|)^+,\ t \in [-1, 1]$.

The following is the formal construction and the proof that the sequence of process X_n converges to a Wiener process.

8.6 THEOREM. *Let μ be the standard Gaussian distribution on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. Define*

8.7 $$(\Omega, \mathcal{H}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu)^D,$$

and let Z_q, $q \in D$, be the coordinate variables. Let $X_0, X_1, \ldots$ be defined by 8.2 and 8.3. Then, there exists a process $X = \{X(t) :\ t \in [0, 1]\}$ such that, for almost every ω in Ω,

8.8 $$\lim_{n\to\infty} \sup_{t\in[0,1]} |X_n(\omega, t) - X(\omega, t)| = 0;$$

and the process X is a Wiener process with parameter set [0,1].

Proof. a) Existence and construction of the probability space of 8.7 is immediate from Theorem IV.4.7; see IV.5.1 *et seq.* as well. It is clear that $\{Z_q : q \in D\}$ is an independency of standard Gaussian variables.

b) For $f : [0, 1] \mapsto \mathbb{R}$, let $\|f\| = \sup_t |f(t)|$, the supremum norm. We shall show that (X_n) is Cauchy for almost sure convergence in the norm $\|\cdot\|$. This implies the existence of a continuous process X such that $\|X_n - X\| \to 0$ almost surely, and there remains to show that X is Gaussian with mean 0 and covariance $s \wedge t$ for $X(s)$ and $X(t)$. To that end, we observe from 8.2 and 8.3 that $\{X_n(t) :\ t \in D_n\}$ is Gaussian with mean 0 and covariance $s \wedge t$; and the same is true for $\{X_{n+k}(t) :\ t \in D_n\}$ for every k, since $X_{n+k}(t) = X_n(t)$ for $t \in D_n$. Hence, $\{X(t) : t \in D_n\}$ is Gaussian with mean 0 and variance $s \wedge t$, which means that the same is true for $\{X(t) : t \in D\}$. In view of the continuity of X, approximating $X(t)$ by $X(q)$, $q \in D$, we see that X is Gaussian as desired, thus completing the proof.

c) Fix $n \geq 8$. Put $\varepsilon = 2^{-(n+2)/4}$. In view of 8.3 and Remark 8.5a, noting that the maximum of h_q is ε^2, we see that $\|X_{n+1} - X_n\| = \varepsilon^2 M$, where M is the maximum of $|Z_q|$ as q ranges over the set $D_{n+1} \backslash D_n$ of cardinality 2^n. Since the Z_q are independent copies of the standard Gaussian Z_0,

$$\mathbb{P}\{\|X_{n+1} - X_n\| > \varepsilon\} = \mathbb{P}\{\varepsilon^2 M > \varepsilon\} \leq 2^n\ \mathbb{P}\{|Z_0| > 1/\varepsilon\} \leq 2^n \cdot \varepsilon \cdot e^{-1/2\varepsilon^2},$$

the last inequality being by Lemma 7.19. Since $(1/2\varepsilon^2) = \sqrt{2^n} \geq 2n$ for $n \geq 8$, and since $e^{-2n} \leq 2^{-2n}$, we conclude that, with $\varepsilon_n = 2^{-(n+2)/4}$,

8.9 $$\sum_n \mathbb{P}\{\|X_{n+1} - X_n\| > \varepsilon_n\} < \infty, \qquad \sum_n \varepsilon_n < \infty.$$

By the Borel-Cantelli lemma, then, there exists an almost sure event Ω_0 such that, for every ω in it there is n_ω with

$$\|X_{n+1}(\omega,\cdot) - X_n(\omega,\cdot)\| \le \varepsilon_n \qquad \text{for all} \quad n \ge n_\omega .$$

Thus, for ω in Ω_0, if $i, j \ge n \ge n_\omega$,

$$\|X_i(\omega,\cdot) - X_j \cdot (\omega,\cdot)\| \le \sum_{k=n}^{\infty} \varepsilon_k ,$$

and the right side goes to 0 as $n \to \infty$ since (ε_k) is summable. So, for ω in Ω_0, the sequence $(X_n(\omega,\cdot))$ is Cauchy for convergence in the norm and, hence, has a limit $X(\omega,\cdot)$ in the norm. We re-define $X(\omega,\cdot) = 0$ identically for ω not in Ω_0. This X is the process that was shown to be Wiener in part (b) above. □

Exercises

8.10 *Construction on* [0,1] *with its Lebesgue measure.* This is to show that, in Lévy's construction, we can take $(\Omega, \mathcal{H}, \mathbb{P})$ to be $([0,1], \mathcal{B}_{[0,1]}, \text{Leb})$. This is tedious but instructive.

Let $A = \{0,1\}$, $\mathcal{A} = 2^A$, and α the measure that puts weight ½ at the point 0, and ½ at the point 1; then $(A,\ \mathcal{A},\ \alpha)$ is a model for the toss of a fair coin once. Thus,

$$(\Omega, \mathcal{H}, \mathbb{P}) = (A, \mathcal{A}, \alpha)^{\mathbb{N}^*}$$

is a model for an infinite sequence of tosses, independently. We know that $(\Omega,\ \mathcal{H},\ \mathbb{P})$ is basically the same as ([0,1], $\mathcal{B}_{[0,1]}$, Leb).

Let $b:\ \mathbb{N}^* \times \mathbb{N}^* \mapsto \mathbb{N}^*$ be a bijection, and define

$$U_i(\omega) = \sum_{j=1}^{\infty} 2^{-j} \omega_{b(i,j)} \qquad \text{if } \omega = (\omega_1, \omega_2, \cdots) .$$

Show that $U_1,\ U_2,\ \ldots$ are independent and uniformly distributed on [0,1]. Let h be the quantile function (inverse functional) corresponding to the cumulative distribution function for the standard Gaussian. Then, $Y_1 = h \circ U_1,\ Y_2 = h \circ U_2,\ \ldots$ are independent standard Gaussian variables. Finally, let $g:\ D \mapsto \mathbb{N}^*$ be a bijection, and put $Z_q = Y_{g(q)}$ for $q \in D$. Then, $\{Z_q\ :\ q \in D\}$ is an independency of standard Gaussian variables over the probability space $(\Omega, \mathcal{H}, \mathbb{P})$. Lévy's construction yields a Wiener path $W_t(\omega),\ t \in [0,1]$, for each sequence ω of zeros and ones.

88.11 *Lévy's construction, an alternative.* Start with the probability space $(\Omega,\ \mathcal{H},\ \mathbb{P})$ and the standard Gaussians $Z_q,\ q \in D$. Put $W_0 = 0,\ W_1 = Z_1$. Having defined W_p for every p in D_n, put

$$W_q = \frac{1}{2}(W_p + W_r) + \sqrt{2^{-n-2}}\, Z_q , \qquad q \in D_{n+1} \backslash D_n ,$$

where $p = \sup D_n \cap [0,q]$ and $r = \inf D_n \cap [q,1]$.

a) Show that $\{W_t : t \in D\}$ is a Gaussian process; specify its mean and covariance function.,

b) Show that the condition 7.33 is satisfied (with $p=4$, $q=1$) for $\{W_t : t \in D\}$ with s, t in D. Then 7.34 holds. Show that this implies that, for almost every ω, the function $t \mapsto W_t(\omega)$ from D into $\mathbb{R}$ is uniformly continuous; let $t \mapsto \bar{W}_t(\omega)$ be its continuous extension onto $[0, 1]$. For the negligible set of ω remaining, put $\bar{W}_t(\omega) = 0$. Show that $\bar{W}$ is a Wiener process on [0,1].