Chapter VII

Lévy Processes

This chapter is on Lévy processes with state space $\mathbb{R}^d$, their structure and general properties. Section 1 introduces them and gives a constructive survey of the range of behaviors. Section 2 illustrates those constructions in the case of stable processes, a special class.

Section 3 re-introduces Lévy processes in a modern setting, discusses the Markov and strong Markov properties for them, and shows the special nature of the filtrations they generate. Section 4 characterizes the three basic processes, Poisson, compound Poisson, and Wiener, in terms of the qualitative properties of the sample paths. Section 5 is on the famous characterization theorem of Itô and Lévy, showing that every Lévy process has the form constructed in Section 1; we follow Itô's purely stochastic treatment.

Section 6 is on the use of increasing Lévy processes in random time changes, an operation called subordination with many applications. Finally, in Section 7, we describe some basic results on increasing Lévy processes; these are aimed at applications to theories of regeneration and Markov processes.

The special case of Wiener processes is left to the next chapter for a deeper treatment.

1 Introduction

Let $(\Omega, \mathcal{H}, \mathbb{P})$be a probability space. Let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be a filtration on it. Let $X = (X_t)_{t\in\mathbb{R}_+}$ be a stochastic process with state space $\mathbb{R}^d$; here, $d \geq 1$ is the dimension, and the relevant σ-algebra on $\mathbb{R}^d$ is the Borel one.

1.1 Definition. *The process X is called a* Lévy process in $\mathbb{R}^d$ *with respect to $\mathcal{F}$ if it is adapted to $\mathcal{F}$ and*

a) for almost every ω, the path $t \mapsto X_t(\omega)$ is right-continuous and left-limited starting from $X_0(\omega) = 0$, and

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b) for every t and u in $\mathbb{R}_+$, the increment $X_{t+u} - X_t$ is independent of $\mathcal{F}_t$ and has the same distribution as X_u.

Let $\mathcal{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by X. If X is a Lévy process with respect to $\mathcal{F}$, then it is such with respect to $\mathcal{G}$ automatically. It is called a Lévy process, without mentioning a filtration, if it is such with respect to $\mathcal{G}$.

In the preceding definition, the first condition is on the regularity of paths. The second condition implies that X has *stationary and independent increments*: $X_{t+u} - X_t$ has the same distribution for all t, and the increments $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent for all choices of $n \geq 2$ and times $0 \leq t_0 < t_1 < \cdots < t_n$. Conversely, if X has stationary and independent increments, then it fulfills the condition 1.1b with $\mathcal{F} = \mathcal{G}$.

Every constant multiple of a Lévy process in $\mathbb{R}^d$ is again Lévy. The sum of a finite number of independent Lévy processes in $\mathbb{R}^d$ is again Lévy. Given a Lévy process X in $\mathbb{R}^d$ and a $d' \times d$ matrix c, the process cX is a Lévy process in $\mathbb{R}^{d'}$; in particular, every linear combination of the components of X is a Lévy process in $\mathbb{R}$; every component of X is a Lévy process in $\mathbb{R}$ - the components generally depend on each other.

1.2 Example. The simplest (and trivial) Lévy process in $\mathbb{R}^d$ is the pure-drift: it has the form $X_t = bt$ where b is a fixed vector in $\mathbb{R}^d$. Next, we recall the definitions of some Lévy processes introduced in earlier chapters.

a) According to Definition V.2.15, a Wiener process W is a Lévy process in $\mathbb{R}$ that has continuous paths and has the Gaussian distribution with mean 0 and variance u for its increments $W_{t+u} - W_t$. It is the basic continuous Lévy process: The most general continuous Lévy process in $\mathbb{R}$ has the form

$$X_t = bt + cW_t, \quad t \in \mathbb{R}_+,$$

where b and c are constants in $\mathbb{R}$. A similar result holds for processes in $\mathbb{R}^d$, in which case b is a vector in $\mathbb{R}^d$, and c is a $d \times d'$ matrix, and W is a d'-dimensional Wiener process (whose components are independent Wiener processes). See Theorem 4.3.

b) *Poisson processes.* The initial definition was given in Definition V.2.20: a Poisson process N with rate c is a Lévy process that is a counting process having the Poisson distribution with mean cu for its increments $N_{t+u} - N_t$. A list of characterizations were given in Section 5 of the preceding chapter, and also a martingale characterization in Theorem V.6.13. We shall add one more in Section 4: a Lévy process whose increments are Poisson distributed is necessarily a counting process (and, hence, is a Poisson process).

c) *Compound Poisson process.* These were introduced in Section 3 of the preceding chapter as follows. Let N be a Poisson process. Independent of it, let (Y_n) be an independency of identically distributed $\mathbb{R}^d$-valued random variables. Define

$$X_t = \sum_{n=1}^{\infty} Y_n 1_{\{n \leq N_t\}}, \quad t \in \mathbb{R}_+.$$

Then, X is a Lévy process in $\mathbb{R}^d$. Its every path is a step function; its jumps occur at the jump times of N, and the sizes of successive jumps are $Y_1, Y_2, \ldots$. We shall show in Theorem 4.6 that, conversely, every Lévy process whose paths are step functions is a compound Poisson process.

d) *Increasing Lévy processes.* According to Definition VI.4.5, these are Lévy processes in $\mathbb{R}$ whose paths are increasing. Equivalently, they are Lévy processes with state space $\mathbb{R}_+$, because the positivity of X_u and the stationarity of $X_{t+u} - X_t$ imply that every increment is positive. Every Poisson process is an increasing Lévy process. So is every compound Poisson process with positive jumps (with $\mathbb{R}_+$-valued Y_n in the preceding remark). So are gamma processes, so are stable processes with indices in $(0, 1)$; see 4.9, 4.10, 4.19, 4.20 of Chapter VI for these, and also Propositions 4.6 and 4.14 there for general constructions. It will become clear that every increasing Lévy process has the form given in Proposition VI.4.6; see Remark 5.4b to come.

Infinite divisibility, characteristic exponent

Recall that a random variable is said to be infinitely divisible if, for every integer n, it can be written as the sum of n independent and identically distributed random variables. Let X be a Lévy process in $\mathbb{R}^d$. For $t > 0$ fixed and $n \geq 1$, lettting $\delta = t/n$, we can write X_t as the sum of the increments over the intervals $(0, \delta], (\delta, 2\delta], \cdots, (n\delta - \delta, n\delta]$, and those increments are independent and identically distributed. Thus, X_t is infinitely divisible for every t, and so is every increment $X_{t+u} - X_t$. It follows that the characteristic function of X has the form

1.3 $$\mathbb{E}\, e^{ir\cdot X_t} = e^{t\psi(r)}, \qquad t \in \mathbb{R}_+,\ r \in \mathbb{R}^d\ ;$$

here, on the left, $r \cdot x = r_1 x_1 + \cdots + r_d x_d$, the inner product of r and x in $\mathbb{R}^d$. On the right side, ψ is some complex-valued function having a specific form; it is called the characteristic exponent of X. Its form is given by the Lévy-Khinchine formula; see 1.31 and 1.33 below. Its derivation is basically a corollary to Itô-Lévy decomposition of Theorem 5.2 to come.

Means and variances

Let X be a Lévy process in $\mathbb{R}^d$. It is possible that $\mathbb{E}X_t$ does not exist; this is the case, for instance, if X is a compound Poisson process as in Example 1.2c and the Y_n do not have expected values. Or, it is possible that $\mathbb{E}X_t$ is well-defined but is equal to infinity in some components. However, if the means and variances of the components of the random vector X_t are well-defined, then they must be linear in t, that is,

1.4 $$\mathbb{E}\, X_t = at, \quad \text{Var}X_t = vt, \quad t \in \mathbb{R}_+.$$

This is a consequence of the stationarity and independence of the increments; a is a fixed vector in $\mathbb{R}^d$, and $\text{Var}X_t$ is notation for the covariance matrix of

X_t, and v is a fixed symmetric $d \times d$ matrix that is positive definite, that is, $v_{ij} = v_{ji}$ for all i and j, and $r \cdot vr \geq 0$ for every r in $\mathbb{R}^d$.

Continuity in distribution

Consider 1.3 and note its continuity in t. Recall that the convergence in distribution is equivalent to the convergence of the corresponding characteristic functions; see Corollary III.5.19. Thus, (X_{t_n}) converges in distribution to X_t for every sequence (t_n) with limit t. The following is the same statement using the definition of convergence in distribution.

1.5 PROPOSITION. *Suppose that X is a Lévy process in $\mathbb{R}^d$. Then, $t \mapsto \mathbb{E}\, f \circ X_t$ is continuous for every bounded continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$.*

Probability law of X

Suppose that X is Lévy. Then, its probability law is determined by the distribution π_t of X_t for any one $t > 0$, or equivalently, by the characteristic exponent ψ appearing in 1.3. To see this, first note that the Fourier transform of π_t is $e^{t\psi}$; if it is known for one t, then it is known for all t.

Next, consider the finite-dimensional distributions of X: consider the distribution of $(X_s, X_t, \ldots, X_u, X_v)$ for finitely many times, $0 < s < t < \cdots < u < v$. That distribution is determined by the distribution of $(X_s, X_t - X_s, \ldots, X_v - X_u)$, and the latter is the product measure $\pi_s \times \pi_{t-s} \times \cdots \times \pi_{v-u}$ in view of the independence and stationarity of the increments.

Regularity of the paths and jumps

Suppose that X is a Lévy process in $\mathbb{R}^d$. Fix an outcome ω for which the regularity properties 1.1a hold. This means that the limits

1.6 $$X_{t-}(\omega) = \lim_{s \uparrow t} X_s(\omega), \quad X_{t+}(\omega) = \lim_{u \downarrow t} X_u(\omega)$$

exist for every t in $\mathbb{R}_+$ (with the convention that $X_{t-}(\omega) = 0$ for $t = 0$), the limits belong to $\mathbb{R}^d$, and $X_{t+}(\omega) = X_t(\omega)$ by right-continuity. If the two limits differ, then we say that the path $X(\omega)$ jumps from its left-limit $X_{t-}(\omega)$ to its right-hand value $X_t(\omega) = X_{t+}(\omega)$. The difference

1.7 $$\Delta X_t(\omega) = X_t(\omega) - X_{t-}(\omega),$$

if non-zero, is called the *size* of the jump at time t and its length $|\Delta X_t(\omega)|$ is called the *jump magnitude*. The path $X(\omega)$ can have no discontinuities other than the jump-type described.

Let D_ω be the discontinuity set for the path $X(\omega)$, that is,

1.8 $$D_\omega = \{t > 0 \,:\, \Delta X_t(\omega) \neq 0\}.$$

If X is continuous, then D_ω is empty for almost every ω. If X is Poisson or compound Poisson plus some continuous process, then, for almost every ω, the set D_ω is an infinite countable set, but $D_\omega \cap (s,u)$ is finite for all $0 \le s < u < \infty$. For all other processes X, for almost every ω, the set D_ω is still infinite but with the further property that $D_\omega \cap (s,u)$ is infinite for all $0 \le s < u < \infty$. This last property is apparent for gamma and stable processes of Example VI.4.9 and VI.4.10, and will follow from Itô-Lévy decomposition in general; see Theorem 5.2.

1.9 Remark. However, for every $\varepsilon > 0$, there can be at most finitely many t in $D_\omega \cap (s,u)$ for which the magnitude $|\Delta X_t(\omega)|$ exceeds ε. For, otherwise, if there were infinitely many such jump times for some $\varepsilon > 0$, then Bolzano-Weierstrass theorem would imply that there must exist a sequence (t_n) of such times that converges to some point t in $[s,u]$, and then at least one of the limits 1.6 must fail to exist.

Pure-jump processes

These are processes in $\mathbb{R}^d$ where X_t is equal to the sum of the sizes of its jumps during $[0,t]$; more precisely, for almost every ω,

1.10
$$X_t(\omega) = \sum_{s \in D_\omega \cap [0,t]} \Delta X_s(\omega), \quad t \in \mathbb{R}_+,$$

where the sum on the right side converges absolutely, that is, where

1.11
$$V_t(\omega) = \sum_{s \in D_\omega \cap [0,t]} |\Delta X_s(\omega)| < \infty.$$

Indeed, every such process has bounded variation over bounded intervals, and $V_t(\omega)$ is the total variation of the path $X(\omega)$ over $[0,t]$.

Every increasing Lévy process without drift is a pure-jump Lévy process, so is the difference of two such independent processes. The following constructs such processes in general. We shall see later that every pure-jump Lévy process in $\mathbb{R}^d$ has the form given in this theorem.

1.12 Theorem. *Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean measure Leb $\times\, \lambda$, where the measure λ on $\mathbb{R}^d$ has $\lambda\{0\} = 0$ and*

1.13
$$\int_{\mathbb{R}^d} \lambda(dx)(|x| \wedge 1) < \infty.$$

Then, for almost every ω, the integral

1.14
$$X_t(\omega) = \int_{[0,t]\times\mathbb{R}^d} M_\omega(ds,dx)x$$

converges absolutely for every t, and the path $X(\omega)$ has bounded variation over $[0,t]$ for every t in $\mathbb{R}_+$. The process X is a pure-jump Lévy process in $\mathbb{R}^d$, and its characteristic exponent is

1.15 $$\psi(r) = \int_{\mathbb{R}^d} \lambda(dx)(e^{ir\cdot x} - 1), \quad r \in \mathbb{R}^d.$$

1.16 REMARK. *Lévy measure.* The measure λ determines the probability laws of M and X. It is called the Lévy measure of X. It regulates the jumps: for every Borel subset A of $\mathbb{R}^d$ with $\lambda(A) < \infty$, the jump times of X with corresponding sizes belonging to A form the counting process $t \mapsto M((0,t] \times A)$, and the latter is a Poisson process with rate $\lambda(A)$. The condition that $\lambda\{0\} = 0$ is for reasons of convenience: to prevent linguistic faults like "jumps of size 0," and also to ensure that $X(\omega)$ and M_ω determine each other uniquely for almost every ω. The condition 1.13 is essential. It is satisfied by every finite measure. More interesting are infinite measures that satisfy it; to such measures there correspond pure-jump processes that have infinitely many jumps during every interval (s,t) with $s < t$; but, of those jumps, only finitely many may exceed ε in magnitude however small $\varepsilon > 0$ may be; see Remark 1.9.

Proof. Let $\hat{M}$ be the image of M under the mapping $(s,x) \mapsto (s,|x|)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $\mathbb{R}_+ \times \mathbb{R}_+$, and $\hat{\lambda}$ the image of λ under the mapping $x \mapsto |x|$. Then, $\hat{M}$ is Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean $Leb \times \hat{\lambda}$. Note that $\hat{\lambda}\{0\} = 0$ and 1.13 is equivalent to

1.17 $$\int_{\mathbb{R}_+} \hat{\lambda}(dv)(v \wedge 1) < \infty,$$

which, in particular, implies that $\hat{\lambda}(\varepsilon, \infty) < \infty$ for every $\varepsilon > 0$.

Thus, by Proposition VI.2.18, we can select an almost sure event Ω' such that, for every ω in it, the measure $\hat{M}_\omega$ is a counting measure, has no atoms in $\{0\} \times \mathbb{R}_+$ and no atoms in $\mathbb{R}_+ \times \{0\}$, and has at most one atom in $\{t\} \times \mathbb{R}_+$ no matter what t is.

On the other hand, for each time t,

1.18 $$V_t = \int_{[0,t]\times\mathbb{R}_+} \hat{M}(ds,dv)v = \int_{[0,t]\times\mathbb{R}^d} M(ds,dx)|x|$$

is positive and real-valued almost surely in view of 1.17 and Proposition VI.2.13. Let Ω_t be the almost sure event involved, and define Ω'' to be the intersection of Ω_t over t in $\mathbb{N}$.

Fix an outcome ω in the almost sure event $\Omega' \cap \Omega''$. The mapping $t \mapsto V_t(\omega)$ from $\mathbb{R}_+$ into $\mathbb{R}_+$ is right-continuous and increasing starting from the origin; and it has a jump of size v at time s if and only if M_ω has an atom (s, x) with $|x| = v$. It follows that the integral in 1.14 converges absolutely for all times t, and we have

$$\sum_{s \le t} |\Delta X_s(\omega)| = V_t(\omega), \quad \sum_{s \le t} \Delta X_s(\omega) = X_t(\omega).$$

Hence, X is of the pure-jump type and is right-continuous and left-limited starting from the origin, and its total variation over $[0, t]$ is equal to V_t.

It is immediate from 1.14 and the Poisson character of M that X has stationary and independent increments. The form 1.15 for the characteristic exponent follows from 1.3, 1.14, and Theorem VI.2.9. □

1.19 Remark. *Total variation.* The preceding proof has shown, in addition, that the total variation process V is defined by 1.18 as well, and that it is a pure-jump increasing Lévy process. Its Lévy measure is the image of λ under the mapping $x \mapsto |x|$. The path $X(\omega)$ has a jump of some size x at time t if and only if $V(\omega)$ has a jump of size $|x|$ at the same time t.

1.20 Remark. *Poisson and compound Poisson.* If the dimension $d = 1$, and $\lambda = c\delta_1$ (recall that δ_x is Dirac at x), then X of the last theorem becomes a Poisson process with rate c. For arbitrary d, if λ is a finite measure on $\mathbb{R}^d$, then 1.13 holds automatically and X is a compound Poisson process as in Example 1.2c: its jump times form a Poisson process N with rate $c = \lambda(\mathbb{R}^d)$, and the sizes Y_n of its jumps are independent of N and of each other and have the distribution $\mu = \frac{1}{c}\lambda$ on $\mathbb{R}^d$. Its total variation process V is an increasing compound Poisson process in $\mathbb{R}_+$; the jump times of V form the same Poisson process N, but the jump sizes are the $|Y_n|$.

1.21 Example. *Gamma, two-sided and symmetric.* Recall Example VI.4.6, the gamma process with shape rate a and scale parameter c. It is an increasing pure-jump Lévy process in $\mathbb{R}_+$. Its Lévy measure has the density ae^{-cx}/x for x in $(0, \infty)$ and puts no mass elsewhere. Its value at t has the gamma distribution with shape index at and scale c.

Let X^+ and X^- be independent gamma processes. Then,

$$X = X^+ - X^-$$

is a pure-jump Lévy process in $\mathbb{R}$; the distribution of X_t is not gamma; nevertheless, X may be called a two-sided gamma process; see Exercises 1.47 and 1.48 for some observations. In the special case where X^+ and X^- have the same law, that is, if they have the same shape rate a and the same scale parameter c, then the Lévy measure of X is given by

$$\lambda(dx) = dx\, a \frac{e^{-c|x|}}{|x|}, \quad x \in \mathbb{R}\backslash\{0\},$$

with $\lambda\{0\} = 0$; in this case, we call X a symmetric gamma process with shape rate a and scale parameter c. The distribution of X_t is not gamma and cannot be expressed explicitly; however, the characteristic function is

$$\mathbb{E}\, e^{irX_t} = \left(\frac{c}{c-ir}\right)^{at}\left(\frac{c}{c+ir}\right)^{at} = \left(\frac{c^2}{c^2+r^2}\right)^{at}, \quad r \in \mathbb{R}.$$

The total variation process $V = X^+ + X^-$ is a gamma process with shape rate $2a$ and scale parameter c. See Exercise 1.48 and also 6.26 for d-dimensional analogs of X.

Compensated sums of jumps

This is to introduce Lévy processes driven by Poisson random measures as above, but whose paths may have infinite total variation over every time interval of strictly positive length. As remarked in 1.9, there can be at most finitely many jumps of magnitude exceeding $\varepsilon > 0$ during a bounded time interval. Thus, intricacies of paths are due to the intensity of jumps of small magnitude. To concentrate on those essential issues, the next construction is for processes whose jumps are all small in magnitude, say, all less than unity. We write $\mathbb{B}$ for the unit ball in $\mathbb{R}^d$ and $\mathbb{B}_\varepsilon$ for the complement in $\mathbb{B}$ of the ball of radius ε, that is,

1.22 $$\mathbb{B} = \{\, x \in \mathbb{R}^d : |x| \le 1 \,\}, \quad \mathbb{B}_\varepsilon = \{\, x \in \mathbb{R}^d : \varepsilon < |x| \le 1 \,\}.$$

1.23 THEOREM. *Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{B}$ with mean $Leb \times \lambda$, where the measure λ on $\mathbb{B}$ satisfies $\lambda\{0\} = 0$ and*

1.24 $$\int_{\mathbb{B}} \lambda(dx)|x|^2 < \infty.$$

For ε in $(0,1)$, define

1.25 $$X_t^\varepsilon(\omega) = \int_{[0,t]\times\mathbb{B}_\varepsilon} M_\omega(ds,dx)x - t\int_{\mathbb{B}_\varepsilon}\lambda(dx)x, \quad \omega \in \Omega, \quad t \in \mathbb{R}_+.$$

Then, there exists a Lévy process X such that, for almost every ω,

$$\lim_{\varepsilon \downarrow 0} X_t^\varepsilon(\omega) = X_t(\omega),$$

the convergence being uniform in t over bounded intervals. The characteristic exponent for X is

1.26 $$\psi(r) = \int_{\mathbb{B}} \lambda(dx)(e^{ir\cdot x} - 1 - ir\cdot x), \quad r \in \mathbb{R}^d.$$

1.27 NOTATION. For future purposes, it is convenient to write

$$X_t = \int_{[0,t]\times\mathbb{B}} [M(ds,dx) - ds\lambda(dx)]\ x,$$

the exact meaning of the right side being the almost sure limit described in the preceding theorem.

1.28 REMARKS. The proof of the preceding theorem is left to the end of this section because of its length and technical nature. For the present, here are some comments on its meaning.

a) The process X^ε has the form

$$X_t^\varepsilon = Y_t^\varepsilon - a_\varepsilon t$$

where Y^ε is a compound Poisson process in $\mathbb{R}^d$ and the drift rate a_ε is a fixed vector in $\mathbb{R}^d$. To see this, we start by defining

$$b_\varepsilon = \int_{\mathbb{B}_\varepsilon} \lambda(dx)|x|, \quad c_\varepsilon = \int_{\mathbb{B}_\varepsilon} \lambda(dx)|x|^2, \qquad 0 \le \varepsilon < 1,$$

and note that $\varepsilon^2\lambda(\mathbb{B}_\varepsilon) \le \varepsilon b_\varepsilon \le c_\varepsilon \le c_0$ since $\varepsilon^2 \le \varepsilon|x| \le |x|^2$ for x in $\mathbb{B}_\varepsilon$. The condition 1.24 means that $c_0 < \infty$, which implies that $\lambda(\mathbb{B}_\varepsilon) < \infty$ and $b_\varepsilon < \infty$ for every $\varepsilon > 0$. Since $\lambda(\mathbb{B}_\varepsilon) < \infty$, the first integral on the right side of 1.25 converges absolutely, and the second defines a vector a_ε in $\mathbb{R}^d$. Hence, the claimed form for X^ε.

b) The claim of the theorem is the existence of a Lévy process X such that, for almost every ω,

$$\lim_{\varepsilon\downarrow 0} \sup_{0\le t\le u} |X_t^\varepsilon(\omega) - X_t(\omega)| = 0$$

for every u in $\mathbb{R}_+$.

c) Recall the notation introduced in Remark (a) above. If $b_0 < \infty$, then λ satisfies 1.13, and Theorem 1.12 shows that

$$Y_t = \lim_{\varepsilon\downarrow 0} Y_t^\varepsilon = \int_{[0,t]\times\mathbb{B}} M(ds,dx)x, \quad t \in \mathbb{R}_+,$$

is a pure-jump Lévy process with Lévy measure λ. In this case,

$$a = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{B}_\varepsilon} \lambda(dx)\, x = \int_{\mathbb{B}} \lambda(dx)\, x$$

is also well-defined, and we have

$$X_t = Y_t - at, \quad t \in \mathbb{R}_+.$$

d) The novelty of the theorem, therefore, occurs when $b_0 = +\infty$ and $c_0 < \infty$, that is, 1.13 fails but 1.24 holds. Then, a_ε fails to converge as $\varepsilon \to 0$, and Y_t^ε fails to converge as $\varepsilon \to 0$, but the difference $X_t^\varepsilon = Y_t^\varepsilon - a_\varepsilon t$ converges. The limit process X has infinite variation over every time interval (s, t), however small $t - s > 0$ may be.

e) Every X_t^ε is a compensated sum of jumps: the sum of the sizes of jumps during $(0, t]$ is equal to Y_t^ε, the corresponding compensator term is equal to $a_\varepsilon t$, and the resulting process X^ε is a d-dimensional martingale. For this reason, the limit X is said to be a *compensated sum of jumps.*

Construction of general Lévy processes

The next theorem introduces Lévy processes of a general nature. In Section 5, Itô-Lévy decomposition theorem will show that, conversely, every Lévy process in $\mathbb{R}^d$ has this form. In the next section, there are several concrete examples.

Recall the notation $\mathbb{B}$ for the closed unit ball in $\mathbb{R}^d$, and write $\mathbb{B}^c$ for its complement, $\mathbb{R}^d \backslash \mathbb{B}$. We shall use notation 1.27 again.

1.29 THEOREM. *Let b be a vector in $\mathbb{R}^d$, and c a $d \times d'$ matrix, and λ a measure on $\mathbb{R}^d$ satisfying $\lambda\{0\} = 0$ and*

1.30 $$\int_{\mathbb{R}^d} \lambda(dx)\ (|x|^2 \wedge 1) < \infty.$$

Let W be a d'-dimensional Wiener process and, independent of it, let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean Leb $\times \lambda$. Then,

1.31 $$X_t = bt + cW_t + \int_{[0,t]\times\mathbb{B}} [M(ds, dx) - ds\lambda(dx)]\, x + \int_{[0,t]\times\mathbb{B}^c} M(ds, dx)\, x,$$

defines a Lévy process in $\mathbb{R}^d$, and the characteristic exponent of X is, with $v = cc^T$,

1.32 $$\psi(r) = ir\cdot b - \tfrac{1}{2} r\cdot vr + \int_{\mathbb{B}} \lambda(dx)(e^{ir\cdot x} - 1 - ir\cdot x) + \int_{\mathbb{B}^c} \lambda(dx)(e^{ir\cdot x} - 1), \quad r \in \mathbb{R}^d.$$

Proof. Let X^b, X^c, X^d, X^e denote the processes defined by the four terms on the right side of 1.31 in the order they appear; then,

1.33 $$X = X^b + X^c + X^d + X^e.$$

The first term is trivially Lévy. The second, X^c, is a continuous Lévy process, since it is the product of the matrix c with the continuous Lévy process W.

The condition 1.30 is equivalent to requiring the condition 1.24 together with $\lambda(\mathbb{B}^c) < \infty$. Thus, Theorem 1.23 shows that X^d is a Lévy process. And, as remarked in 1.20, X^e is a compound Poisson process. So, all four are Lévy.

The processes X^d and X^e are independent because the traces of the Poisson random measure M on $\mathbb{R}_+ \times \mathbb{B}$ and on $\mathbb{R}_+ \times \mathbb{B}^c$ are independent. And X^c is independent of X^d and X^e by the assumed independence of W and M. Since sums of independent Lévy processes is Lévy, X is Lévy. The formula for the characteristic exponent follows from the independence of the four terms, results in Theorem 1.12 and 1.23, and the well-known formula for $\mathbb{E}\, e^{iZ}$, where $Z = r \cdot cW_t = \sum_i \sum_j r_i c_{ij} W_t^{(j)}$ is Gaussian with mean 0 and variance $(r \cdot vr)t$. □

1.34 Remarks. a) *Lévy-Khinchine formula.* This refers to the formula 1.32. If Z is an $\mathbb{R}^d$-valued infinitely divisible variable, then $\mathbb{E}\, e^{ir\cdot Z} = e^{\psi(r)}$ for some b in $\mathbb{R}^d$, some $d \times d$ symmetric positive definite matrix v, and some measure λ on $\mathbb{R}^d$ satisfying 1.30.

b) *Characteristics for X.* This refers to the triplet (b, v, λ) which determines the probability law of X.

c) *Semimartingale connection.* The decomposition 1.31–1.33 shows that X is a semimartingale (see Definition V.5.18): The drift term X^b is continuous and has locally *bounded* variation, the Gaussian term X^c is a *continuous* martingale, X^d is a *discontinuous* martingale, and X^e is a step process whose every jump *exceeds* unity in magnitude. Thus, X^c+X^d is the martingale part of X, and $X^b + X^e$ the part with locally bounded variation.

The following is immediate from Theorem 1.12 for pure-jump Lévy processes, but we state it here as a special case of the last theorem.

1.35 Corollary. *In the last theorem, suppose that λ satisfies the condition 1.13. Then, the integral*

$$a = \int_{\mathbb{B}} \lambda(dx)x$$

converges absolutely, and the process X takes the form

1.36 $$X_t = (b-a)t + cW_t + \int_{[0,t]\times\mathbb{R}^d} M(ds,dx)x, \quad t \in \mathbb{R}_+,$$

with the last term defining a pure-jump Lévy process. Accordingly, the characteristic exponent becomes

$$\psi(r) = ir\cdot(b-a) - \frac{1}{2} r\cdot vr + \int_{\mathbb{R}^d} \lambda(e^{ir\cdot x} - 1), \quad r \in \mathbb{R}^d.$$

Proof. When λ satisfies 1.13, the integral defining a converges absolutely, and λ satisfies 1.30 since $|x|^2 \le |x|$ for $x \in \mathbb{B}$. So, the conclusions of the last theorem hold. In addition, Remark 1.28c applies and $X_t^d = Y_t - at$ in the notation there. Now, writing $Y_t + X_t^e$ as one integral, we obtain 1.36 from 1.31. □

Proof of Theorem 1.23

This will be through a series of lemmas. We start with an easy extension of Kolmogorov's inequality, Lemma III.7.1, to $\mathbb{R}^d$-valued variables. This is a discrete-time result, but we state it in continuous-time format.

1.37 LEMMA. *Let $\{ Z(t) : t \in \mathbb{R}_+ \}$ be a process with state space $\mathbb{R}^d$ and $\mathbb{E}\, Z(t) = 0$ for all t. Suppose that it has independent increments. Then, for every finite set $D \subset [0,1]$ and every $\varepsilon > 0$,*

$$\mathbb{P}\{ \sup_{t\in D} |Z(t)| > \varepsilon \} \leq \frac{d}{\varepsilon^2}\, \mathbb{E}\, |Z(1)|^2.$$

Proof. Let $Z^i(t)$ denote the i-coordinate of $Z(t)$. Obviously,

$$\sup_D |Z(t)|^2 = \sup_D \sum_{i=1}^d |Z^i(t)|^2 \leq \sum_{i=1}^d \sup_D |Z^i(t)|^2,$$

and the left side exceeds ε^2 only if at least one term on the right exceeds ε^2/d. Thus,

$$\begin{aligned}\mathbb{P}\{ \sup_D |Z(t)| > \varepsilon \} &\leq \sum_{i=1}^d \mathbb{P}\{ \sup_D |Z^i(t)| > \frac{\varepsilon}{\sqrt{d}} \} \\ &\leq \sum_{i=1}^d \frac{d}{\varepsilon^2}\, \mathbb{E}\, |Z^i(1)|^2 = \frac{d}{\varepsilon^2}\, \mathbb{E}\, |Z(1)|^2,\end{aligned}$$

where Kolmogorov's inequality justifies the second inequality. □

For processes Z with right-continuous and left-limited paths, we introduce the norm

1.38 $$\|Z\| = \sup_{0\leq t\leq 1} |Z(t)|.$$

The following extends Kolmogorov's inequality to continuous-time processes; we state it for Lévy processes even though the stationarity of increments is not needed.

1.39 LEMMA. *Let Z be a Lévy process in $\mathbb{R}^d$ with mean 0. For every $\varepsilon > 0$,*

$$\mathbb{P}\{ \|Z\| > \varepsilon \} \leq \frac{d}{\varepsilon^2}\, \mathbb{E}\, |Z(1)|^2.$$

Proof. Let $q_0, q_1, \ldots$ be an enumeration of the rational numbers in $[0,1]$. Let $D_n = \{ q_0, \ldots, q_n \}$. By the right-continuity of Z, the supremum of $|Z(t)|$

over t in D_n increases to $\|Z\|$ as $n \to \infty$. Thus, by the monotone convergence theorem,

$$\mathbb{P}\{ \|Z\| > \varepsilon \} = \lim_n \mathbb{P}\{ \sup_{t \in D_n} |Z(t)| > \varepsilon \};$$

and the proof is completed via Lemma 1.37 above. □

1.40 LEMMA. *Let $Z_1, \ldots, Z_m$ be processes with state space $\mathbb{R}^d$ and paths that are right-continuous and left-limited. Suppose that $Z_1, Z_2 - Z_1, \ldots, Z_m - Z_{m-1}$ are independent. Then, for every $\varepsilon > 0$,*

1.41 $$\mathbb{P}\{ \max_{k \le m} \|Z_k\| > 3\varepsilon \} \le 3 \max_{k \le m} \mathbb{P}\{ \|Z_k\| > \varepsilon \}.$$

Proof. Let H be the event on the left side, and let 3δ denote the right side; we need to show that

1.42 $$\mathbb{P}(H) \le 3\delta.$$

Put $Z_0 = 0$ and let $H_k = \{ \max_{j \le k-1} \|Z_j\| \le 3\varepsilon < \|Z_k\| \}$ for $k = 1, \ldots, m$; these events form a partition of H. Since $\|Z_m - Z_k\| + \|Z_m\| \ge \|Z_k\|$, we have

$$H_k \cap \{ \|Z_m - Z_k\| \le 2\varepsilon \} \subset H_k \cap \{ \|Z_m\| > \varepsilon \}.$$

The two events on the left side are independent for each k by the assumed independence of the increments of $k \mapsto Z_k$. The union over k of the right side yields a subset of $\{ \|Z_m\| > \varepsilon \}$, and the latter's probability is at most δ. Thus,

1.43 $$\sum_{k=1}^{m} \mathbb{P}(H_k)\, \mathbb{P}\{ \|Z_m - Z_k\| \le 2\varepsilon \} \le \delta.$$

Since $\|Z_m - Z_k\| \le \|Z_m\| + \|Z_k\|$, on the set $\{ \|Z_m - Z_k\| > 2\varepsilon \}$ we have either $\|Z_m\| > \varepsilon$ or $\|Z_k\| > \varepsilon$. Hence,

$$1 - \mathbb{P}\{ \|Z_m - Z_k\| \le 2\varepsilon \} \le \mathbb{P}\{\|Z_m\| > \varepsilon\} + \mathbb{P}\{\|Z_k\| > \varepsilon\} \le 2\delta.$$

Putting this into 1.43 and recalling that (H_k) is a partition of H, we get

$$(1 - 2\delta)\mathbb{P}(H) \le \delta.$$

If $\delta < 1/3$, then $1 - 2\delta \ge 1/3$ and we get $\mathbb{P}(H)/3 \le \delta$ as needed to show 1.42. If $\delta \ge 1/3$, then 1.42 is true trivially. □

Proof of Theorem 1.23

Recall the setup and assumptions of the theorem. Recall the norm 1.38. Let (ε_n) be a sequence in $(0, 1)$ strictly decreasing to 0. For notational simplicity, we define

1.44 $$B_n = \mathbb{B}_{\varepsilon_n}, \quad Z_n(t) = X_t^{\varepsilon_n} = \int_{[0,t] \times B_n} M(ds, dx)x - t \int_{B_n} \lambda(dx)x.$$

We shall show that, almost surely,

1.45 $$\lim_{n\to\infty} \sup_{i,j\ge n} \|Z_i - Z_j\| = 0.$$

Assuming this, the rest of the proof is as follows: 1.45 means that (Z_n) is Cauchy for almost sure convergence in the norm $\|\cdot\|$. Hence, there is a process X such that $\|Z_n - X\| \to 0$ almost surely, and it is obvious that the limit X does not depend on the sequence (ε_n) chosen. So, in the notation of Theorem 1.23, we see that, for almost every ω, $X_t^\varepsilon(\omega) \mapsto X_t(\omega)$ uniformly in $t \le 1$ as $\varepsilon \to 0$. The uniformity of convergence implies that $X(\omega)$ is right-continuous and left limited on the interval $[0,1]$, since each X^ε is such. Since almost sure convergence implies convergence in distribution for $(X_{t_1}^\varepsilon, \dots, X_{t_k}^\varepsilon)$, and since X^ε has stationary and independent increments, the process X has stationary and independent increments, over $[0,1]$. Repeating the whole procedure for the processes $\{ X_{k+t}^\varepsilon - X_k^\varepsilon \ : \ 0 \le t \le 1 \}$ with $k = 1, 2, \dots$ completes the proof of the theorem, except for showing 1.45.

Each Z_n defined in 1.44 is a Lévy process with $\mathbb{E}\ Z_n(t) = 0$. Moreover, the processes $Z_1, Z_2 - Z_1, \dots$ are independent (and Lévy), because they are defined by the traces of M over the disjoint sets $\mathbb{R}_+ \times B_1, \mathbb{R}_+ \times (B_2 \backslash B_1), \cdots$ respectively, and M is Poisson.

Fix $\varepsilon > 0$. Applying Lemma 1.40 with processes $Z_{n+1} - Z_n, \dots, Z_{n+m} - Z_n$ and then using Lemma 1.39 with well-known formulas for the moments of Poisson integrals, we obtain

$$\begin{aligned}\mathbb{P}\{ \max_{k\le m} \|Z_{n+k} - Z_n\| > 3\varepsilon \} &\le 3 \max_{k\le m} \mathbb{P}\{ \|Z_{n+k} - Z_n\| > \varepsilon \} \\ &\le 3 \max_{k\le m} \frac{d}{\varepsilon^2}\ \mathbb{E}\ |Z_{n+k}(1) - Z_n(1)|^2 \\ &\le \frac{3d}{\varepsilon^2} \max_{k\le m} \int_{B_{n+k}\backslash B_n} \lambda(dx)|x|^2 \\ &\le \frac{3d}{\varepsilon^2} \int_{B_0\backslash B_n} \lambda(dx)|x|^2\end{aligned}$$

On the left, the random variable involved increases as m does, and the limit dominates $\frac{1}{2}\|Z_i - Z_j\|$ for all $i, j \ge n$. Thus,

$$\mathbb{P}\{ \sup_{i,j\ge n} \|Z_i - Z_j\| > 6\varepsilon \} \le \frac{3d}{\varepsilon^2} \int_{B_0\backslash B_n} \lambda(dx)|x|^2.$$

On the right side, the integrability condition 1.24 allows the use of the dominated convergence theorem as $n \to \infty$, and the limit is 0 since $B_0 \backslash B_n$ shrinks to the empty set. Hence, since the supremum over $i, j \ge n$ decreases as n increases,

$$\mathbb{P}\{ \lim_n \sup_{i,j\ge n} \|Z_i - Z_j\| > 6\varepsilon \} = \lim_n \mathbb{P}\{ \sup_{i,j\ge n} \|Z_i - Z_j\| > 6\varepsilon \} = 0.$$

Since $\varepsilon > 0$ is arbitrary, this proves that 1.45 holds almost surely.

Exercises and complements

1.46 *Simple random walk in continuous time.* Let X be a pure-jump Lévy process in $\mathbb{R}$ with Lévy measure

$$\lambda = a\delta_1 + b\delta_{-1},$$

where δ_x is Dirac at x, and a and b are positive numbers. Show that $X = X^+ - X^-$ where X^+ and X^- are independent Poisson processes with respective rates a and b. Describe the total variation process V. Show that at every time of jump for V, the process X jumps either upward or downward with respective probabilities $a/(a+b)$ and $b/(a+b)$.

1.47 *Processes with discrete jump size.* Let λ be a purely atomic finite measure in $\mathbb{R}^d$. Let X be a compound Poisson process with λ as its Lévy measure. Show that X can be decomposed as

$$X = \sum_1^\infty a_k N^{(k)}$$

where (a_k) is a sequence in $\mathbb{R}^d$, and the $N^{(k)}$ are independent Poisson processes. Identify the a_k and the rates of the $N^{(k)}$.

1.48 *Two-sided gamma processes.* As in Example 1.21, let X^+ and X^- be independent gamma processes and define $X = X^+ - X^-$. Suppose that X^+ and X^- have the same scale parameter c, and respective shape rates a and b. Let $V = X^+ + X^-$.

a) Compute the Lévy measures of X and V.
b) Show that the distribution π_t of X_t is given by

$$\pi_t f = \int_{\mathbb{R}_+} dx \frac{e^{-cx} c^{at} x^{at-1}}{\Gamma(at)} \int_{\mathbb{R}_+} dx \frac{e^{-cx} c^{bt} x^{bt-1}}{\Gamma(bt)} f(x-y), \quad f \in \mathcal{B}(\mathbb{R}).$$

c) For fixed t, show that X_t^+/V_t and V_t are independent. What are their distributions?

1.49 *Symmetric gamma distribution.* Let k_a denote the density function of the symmetric gamma distribution with shape index a and scale parameter 1, that is,

$$\int_{\mathbb{R}} dx\, k_a(x)\, e^{irx} = \left(\frac{1}{1+r^2}\right)^a, \qquad r \in \mathbb{R}.$$

a) The density for the same distribution with scale parameter c is the function $x \mapsto c k_a(cx)$. Show.
b) For $a = b$ in 1.48, show that $\pi_t(dx) = c k_{at}(cx)\, dx$.

1.50 *Alternative constructions.* Let N be a standard Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ (with mean $Leb \times Leb$). Let $j : \mathbb{R}_+ \mapsto \mathbb{R}^d$ be a Borel function satisfying

$$\int_{\mathbb{R}_+} dx\,(|j(x)| \wedge 1) < \infty.$$

a) Show that

$$X_t = \int_{[0,t]\times\mathbb{R}_+} N(ds, dx) j(x), \quad t \in \mathbb{R}_+,$$

defines a pure-jump Lévy process.

b) Compute the Lévy measure corresponding to $j(x) = e^{-cx}$, $x \in \mathbb{R}_+$.

1.51 *Continuation.* Let N be as in the preceding exercise. Let $j : \mathbb{R}_+ \mapsto \mathbb{R}^d$ be such that

$$\int_{\mathbb{R}_+} dx\,(|j(x)|^2 \wedge 1) < \infty,$$

and put $D = \{\, x \in \mathbb{R}_+ \,:\, |j(x)| \le 1 \,\}$ and $D^c = \mathbb{R}_+ \backslash D$. Let

$$X_t^d = \int_{[0,t]\times D} [\, N(ds, dx) - ds\,dx \,]\, j(x),$$

with the exact meaning to be in accord with Notation 1.27. Then, X^d is a Lévy process. So is

$$X_t^e = \int_{[0,t]\times D^c} N(ds, dx)\, j(x).$$

1.52 *Continuation.* Let j_1 and j_2 be Borel functions from $\mathbb{R}_+$ into $\mathbb{R}$, and suppose that they both satisfy the condition on j of 1.50. Define

$$X_t^{(1)} = \int_{[0,t]\times\mathbb{R}_+} N(ds, dx)\, j_1(x), \quad X_t^{(2)} = \int_{[0,t]\times\mathbb{R}_+} N(ds, dx)\, j_2(x).$$

Show that $X^{(1)}$ and $X^{(2)}$ are Lévy processes in $\mathbb{R}$, and $X = (X^{(1)}, X^{(2)})$ is a Lévy process in $\mathbb{R}^2$; all three are of the pure-jump type; $X^{(1)}$ and $X^{(2)}$ are dependent.

1.53 *Spherical coordinates.* Each point x in $\mathbb{R}^d$ can be represented as $x = vu$ by letting $v = |x|$ and $u = x/|x|$; obviously, v is the length of x, and u is its direction represented as a point on the unit sphere

$$S = \{\, x \in \mathbb{R}^d \,:\, |x| = 1 \,\}.$$

Let ρ be a σ-finite measure on $\mathbb{R}_+$ and let σ be a transition probability kernel from $\mathbb{R}_+$ into S. Define the measure λ on $\mathbb{R}^d$ by the integral formula

$$\lambda f = \int_{\mathbb{R}^d} \lambda(dx)\, f(x) \;=\; \int_{\mathbb{R}_+} \rho(dv) \int_S \sigma(v, du) f(vu)$$

for $f : \mathbb{R}^d \mapsto \mathbb{R}_+$ Borel. Then, ρ is called the radial part of λ, and σ the spherical part.

a) Show that $\int_{\mathbb{R}^d} \lambda(dx)\,(|x|^2 \wedge 1) < \infty \;\Leftrightarrow\; \int_{\mathbb{R}_+} \rho(dv)(v^2 \wedge 1) < \infty$.

b) Show that $\int_{\mathbb{R}^d} \lambda(dx)\,(|x| \wedge 1) < \infty \;\Leftrightarrow\; \int_{\mathbb{R}_+} \rho(dv)(v \wedge 1) < \infty$.

c) Let $h : \mathbb{R}^d \mapsto \mathbb{R}_+$ be the mapping $x \mapsto |x|$. Show that $\rho = \lambda \circ h^{-1}$. If λ is given somehow, one can find ρ and σ such that ρ is the radial part and σ the spherical part.

1.54 *Continuation.* Let λ, ρ, σ be as in the preceding exercise 1.53, and suppose that ρ-integral of $(v \wedge 1)$ is finite as in part (b) of 1.53. Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean $Leb \times \rho$. Let (T_i, V_i), $i \in \mathbb{N}$, be a labeling of its atoms. For each i, let U_i be a random point on the sphere S such that

$$\mathbb{P}\{\, U_i \in B \mid T_i = t, V_i = v \,\} = \sigma(v, B)$$

free of t, and assume that U_i is conditionally independent of $\{\, (T_j, V_j, U_j) \;:\; j \neq i \,\}$ given V_i. Show that

$$X_t = \sum_{i \in \mathbb{N}} V_i\, U_i\, 1_{\{T_i \le 1\}}, \quad t \in \mathbb{R}_+,$$

defines a pure-jump Lévy process X in $\mathbb{R}^d$ whose Lévy measure is λ.

2 Stable Processes

Stable processes form an important subclass of Lévy processes. This section is to introduce them and point out the explicit forms of their characteristic exponents and Lévy measures. Section 6 on subordination will have further results clarifying the relationships among them.

Let a be a number in $\mathbb{R}_+$. Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a Lévy process in $\mathbb{R}^d$. Then X is said to be a-stable, or *stable with index* a, or *self-similar with index* a if the process $\hat{X} = (s^{-1/a} X_{st})_{t \in \mathbb{R}_+}$ has the same probability law as X for every s in $(0, \infty)$. Since the law of a Lévy process X is determined by the distribution of X_1, and since $\hat{X}$ is also Lévy, the condition of a-stability is equivalent to the condition that $s^{-1/a} X_s$ have the same distribution as X_1 for every s in $(0, \infty)$, or that X_t and $t^{1/a} X_1$ have the same distribution.

If $X = 0$ almost surely, then it is a-stable for every a in $\mathbb{R}_+$; we exclude this degenerate case from now on; then $a > 0$ necessarily. Exercises 2.34 and 2.35 show that the index a cannot exceed 2. If $X = W$ or $X = cW$ with W Wiener and c a constant, then X is stable with index 2; see Exercise 2.36. All other stable processes have indices in the interval $(0, 2)$.

For stable processes in $\mathbb{R}$, we shall see the following. If the index a is in $(0, 1)$, then the process is necessarily a pure-jump Lévy process whose Lévy measure is infinite and has a specific form. If a is in $(1, 2)$, then the Lévy measure is again infinite and has a specific form, and the paths have infinite variation over every time interval and cannot be pure-jump type. If $a = 1$, there are three possibilities: the process can be pure drift and thus deterministic; or it can be a Cauchy process, the paths having the same qualitative features as in the case of indices in $(1, 2)$, but each increment having a Cauchy distribution; or it can be a Cauchy process plus some drift.

Stable processes with index in $(0, 1)$

The process introduced in Example VI.4.10 is an increasing pure-jump Lévy process which is stable with index a in $(0, 1)$. It will serve as the total variation process (see 1.12 *et seq.*) for a-stable processes in $\mathbb{R}^d$. We review the example in a form suited to our current agenda.

2.1 EXAMPLE. *Increasing stable processes.* Fix a in $(0, 1)$ and c in $(0, \infty)$. Let

$$\lambda(dv) = dv\ \frac{c}{v^{a+1}}\ 1_{(0,\infty)}(v), \quad v \in \mathbb{R}.$$

This λ satisfies the condition 1.13 of Theorem 1.12. Let V be the pure-jump Lévy process associated. Then V is strictly increasing, all its jumps are upward, and it has infinitely many jumps in every time interval of some length; the last is because $\lambda(0, \infty) = +\infty$. The process V is a-stable; conversely, every increasing stable process has this form. Recall from VI.4.10 that, for $p \geq 0$,

2.2 $$\mathbb{E}\ e^{-pV_t} = \exp_-\ t \int_{\mathbb{R}_+} dv\ \frac{c}{v^{a+1}}\ (1 - e^{-pv}) = \exp_-\ tc\ \frac{\Gamma(1-a)}{a}\ p^a.$$

We show next that the corresponding characteristic function is

$$\mathbb{E}\, e^{irV_t} = \exp\ t \int_{\mathbb{R}_+} dv\ \frac{c}{v^{a+1}} (e^{irv} - 1)$$

2.3 $$= \exp_-\ tc_a |r|^a\ [1 - i(\tan \tfrac{1}{2}\pi a)\ \mathrm{sgn}\ r],$$

$r \in \mathbb{R}$, where

2.4 $$c_a = c\ \frac{\Gamma(1-a)}{a} \cos \tfrac{1}{2}\pi a, \quad \mathrm{sgn}\ r = 1_{\mathbb{R}_+}(r) - 1_{\mathbb{R}_+}(-r).$$

We start by claiming that, for every complex number z whose real part is zero or less,

$$\int_{\mathbb{R}_+} dv \frac{c}{v^{a+1}} (e^{zv} - 1) = -c\frac{\Gamma(1-a)}{a}\ (-z)^a = -c\frac{\Gamma(1-a)}{a} |z|^a\ e^{ia\ \mathrm{Arg}(-z)},$$

2.5

where $\mathrm{Arg}\, z$ is the principal value (in the interval $(-\pi, \pi]$) of the argument of z. This claim follows from noting that 2.5 holds for all negative real z in view of 2.2, and that both sides of 2.5 are regular in the interior of the left-hand plane and are continuous on the boundary. Taking $z = ir$ in 2.5 we obtain 2.3, since $\mathrm{Arg}\ (-ir) = -\frac{1}{2}\pi\ \mathrm{sgn}\ r$.

2.6 EXAMPLE. *Stable processes in* $\mathbb{R}^d$ *with index in* $(0,1)$. Fix a in $(0,1)$ and c in $(0,\infty)$. Let $S = \{ x \in \mathbb{R}^d : |x| = 1 \}$, the unit sphere in $\mathbb{R}^d$, and let σ be a probability measure on it. Define a measure λ on $\mathbb{R}^d$ by the integral formula

$$2.7 \quad \lambda f = \int_{\mathbb{R}^d} \lambda(dx)\, f(x) = \int_{\mathbb{R}_+} dv\, \frac{c}{v^{a+1}} \int_S \sigma(du)\, f(vu), \qquad f \geq 0 \text{ Borel;}$$

see Exercise 1.53; note that the radial part of λ is the Lévy measure of the preceding example. When $f(x) = |x| \wedge 1$, we have $f(vu) = |vu| \wedge 1 = v \wedge 1$ for u in S, and it follows that $\lambda f < \infty$ since $a \in (0,1)$. Thus λ satisfies the condition 1.13.

Let X be the process constructed in Theorem 1.12 for this Lévy measure λ. Then, X is a pure-jump Lévy process in $\mathbb{R}^d$. Its total variation process V is the increasing pure-jump Lévy process of the preceding example. The processes X and V have the same jump times, infinitely many in every open interval. Moreover, 2.7 implies the following conditional structure for X given V: given that V has a jump of size v at time t, the process X has a jump of size vU at the same time t, where U is a random variable with distribution σ on the sphere S ; see Exercise 1.54 for the same description in more detail.

Consequently, the a-stability of V implies the a-stability of X: for fixed s in $(0,\infty)$, the transformation that takes the sample path $t \mapsto X_t(\omega)$ to the sample path $t \mapsto \hat{X}_t(\omega) = s^{-1/a} X_{st}(\omega)$ merely alters the times and magnitudes of the jumps, which are totally determined by $t \mapsto V_t(\omega)$. The a-stability of X can also be deduced by noting that $t^{1/a} X_1$ and X_t have the same characteristic function; for, with the notations 2.4, it follows from 2.3 and 2.7 that, with c_a as in 2.4,

$$2.8 \quad \mathbb{E}\, e^{ir\cdot X_t} = \exp t \int_S \sigma(du) \int_{\mathbb{R}_+} dv \frac{c}{v^{a+1}} (e^{ivr\cdot u} - 1)$$

$$= \exp_-\ tc_a \int_S \sigma(du) |r\cdot u|^a [1 - i(\tan \tfrac{1}{2}\pi a) \operatorname{sgn}\ \ r\cdot u],\ r \in \mathbb{R}^d.$$

2.9 EXAMPLE. *Symmetric stable processes with index in* $(0,1)$. A Lévy process X is said to be *symmetric* if $-X$ has the same law as X. This is equivalent to having the characteristic exponent ψ symmetric, that is, to having $\psi(r) = \psi(-r)$ for every r in $\mathbb{R}^d$. In the case of a pure-jump Lévy process, in view of 1.14 defining X, symmetry is equivalent to having the law of M invariant under the transformation $(t,x) \mapsto (t,-x)$ of $\mathbb{R}_+ \times \mathbb{R}^d$ onto itself. These imply, together with 2.7 and 2.8, that the following four statements are equivalent for the process X of Example 2.6:

a) The process X is symmetric.

b) The Lévy measure λ is symmetric, that is, $\lambda(B) = \lambda(-B)$ for Borel $B \subset \mathbb{R}^d$.

c) The distribution σ is symmetric, that is, $\sigma(B) = \sigma(-B)$ for Borel $B \subset S$.

d) The exponent of X is real-valued, that is, 2.8 reduces to

$$2.10 \qquad \mathbb{E}\, e^{\,ir\cdot X_t} \;=\; \exp\_\;\; tc_a \int_S \sigma(du)\;|r\cdot u|^a, \quad r\in\mathbb{R}^d.$$

2.11 EXAMPLE. *Isotropic stable processes with index in* $(0,1)$. A Lévy process X in $\mathbb{R}^d$ is said to be *isotropic*, or *rotationally invariant*, if its law is invariant under all orthogonal transformations of $\mathbb{R}^d$. This is equivalent to saying that X and gX have the same law for every orthogonal matrix g of dimension d. If $d=1$, isotropy is the same as symmetry; in higher dimensions, isotropy implies symmetry and more.

Let X be as in Example 2.6. Thinking of the jumps, it is clear that X is isotropic if and only if the law governing the jump directions is isotropic, that is, the measure σ on S is the uniform distribution on S. And then, the characteristic function 2.8 becomes even more specific than 2.10 (see Exercises 2.40 and 2.41 for the computations):

$$2.12 \qquad \mathbb{E}\, e^{\,ir\cdot X_t} \;=\; \exp\_\;\; tc_{ad}\,|r|^a, \quad r\in\mathbb{R}^d,$$

where the constant c_{ad} depends on a, c, and d; with c_a as in 2.4,

$$2.13 \qquad c_{ad} \;=\; \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{a+d}{2})\Gamma(\frac{1}{2})}\, c_a.$$

Stable processes with index 1

If a Lévy process X is 1-stable, then X_t and tX_1 have the same distribution. The meaning of stability is striking: X_5, for instance, which is the sum of 5 independent copies of X_1, has the same distribution as $5X_1$. The simplest example is the pure-drift process $X_t = bt, t\in\mathbb{R}_+$. If X is 1-stable, then so is $\hat{X} = (X_t + bt)$; but, if $\hat{X}$ is to be symmetric, X has to be symmetric and $b=0$. From now on we concentrate on processes without drift.

2.14 EXAMPLE. *Standard Cauchy process in* $\mathbb{R}$. This is a symmetric stable process with index 1. Its canonical decomposition 1.33 is $X = X^d + X^e$ and the Lévy measure defining its law is

$$2.15 \qquad \lambda(dx) \;=\; dx\,\frac{1}{\pi x^2}, \quad x\in\mathbb{R}.$$

This Lévy measure satisfies 1.30 but not 1.13. We shall show that

$$2.16 \qquad \mathbb{E}\, e^{\,irX_t} \;=\; e^{-t\,|r|}, \quad r\in\mathbb{R},$$

which makes it apparent that X is 1-stable. The corresponding distribution is

$$2.17 \qquad \mathbb{P}\{\,X_t \in dx\,\} \;=\; dx\,\frac{t}{\pi(t^2+x^2)}, \quad x\in\mathbb{R},$$

which is called the *Cauchy distribution* with scale parameter t, because it is the distribution of tX_1 and the distribution of X_1 is the standard Cauchy distribution; see II.2.27. For this reason, X is said to be a standard Cauchy process.

The symmetry of λ simplifies the construction of X. Going over Theorems 1.23 and 1.29, we observe that the λ-integral in 1.25 vanishes and thus the term X^d is a limit of pure-jump (compound Poisson) processes. Thus, $X = X^d + X^e$ can be written as

2.18 $$X_t = \int_{[0,t]\times\mathbb{R}} M(ds, dx)\ x,$$

the precise meaning of which is as follows: with $\mathbb{R}_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$, for almost every ω,

2.19 $$X_t(\omega) = \lim_{\varepsilon\downarrow 0} \int_{[0,t]\times\mathbb{R}_\varepsilon} M_\omega(ds, dx)x,$$

the convergence being uniform in t over bounded intervals. It follows from this, or from 1.32 and the mentioned vanishing of the λ-integral on the right side of 1.25, that

$$\begin{aligned}\mathbb{E}\ e^{\,irX_t} &= \exp\ t\ \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}_\varepsilon} \lambda(dx)\ (e^{irx} - 1)\\ &= \exp\_\ 2t \int_{\mathbb{R}_+} dx\ \frac{1}{\pi x^2}\ (1 - \cos rx) = e^{-t|r|},\ r \in \mathbb{R},\end{aligned}$$

as claimed in 2.16.

The Cauchy process X is not a pure-jump process despite the looks of 2.18. Indeed, since

$$\int_{(0,1)} \lambda(dx)\ x = \int_{(-1,0)} \lambda(dx)\ (-x) = +\infty,$$

it follows from Proposition VI.2.13 on the finiteness of Poisson integrals that

$$\int_{(s,t)\times(0,1)} M_\omega(du, dx)\ x = \int_{(s,t)\times(-1,0)} M_\omega(du, dx)\ (-x) = +\infty$$

for almost every ω for $s < t$. In other words, over every interval (s, t), the path $X(\omega)$ has infinitely many upward jumps whose sizes sum to $+\infty$, and infinitely many downward jumps whose sizes sum to $-\infty$. In particular, the total variation over (s, t) is equal to $+\infty$ always.

Nevertheless, the small jumps are small enough, and the positive and negative jumps balance each other well, that removing the big jumps yields a martingale. Employing a notation similar to 2.18 with precise meaning analogous to 2.19,

$$Z_t = \int_{[0,t]\times[-b,b]} M(ds, dx)\ x, \quad t \in \mathbb{R}_+,$$

defines a martingale Z for each b in $(0, \infty)$. Indeed, Z is a Lévy process with $\mathbb{E}Z_t = 0$ and $\text{Var}Z_t = 2bt/\pi$; it is not a stable process.

The process X is not a martingale for the simple reason that $\mathbb{E}X_t$ does not exist, which is because the jumps exceeding b in magnitude are very big in expectation:

$$\mathbb{E} \int_{[0,t]\times(b,\infty)} M(ds, dx)\, x \;=\; t \int_{(b,\infty)} \lambda(dx)\, x \;=\; \infty \,,$$

and similarly for the integral over $[0, t] \times (-\infty, -b)$. This fact about $\mathbb{E}X_t$ is often expressed by saying that the Cauchy distribution has fat tails; the account above is more revealing.

2.20 Example. *Half-Cauchy.* This is a Lévy process X that is not stable, and the distribution of X_t is not Cauchy. We give it here to clarify the role of symmetry in the 1-stability of Cauchy processes. Let X be a Lévy process in $\mathbb{R}$ whose canonical decomposition is $X = X^d + X^e$ and whose Lévy measure is

$$\lambda(dx) \;=\; dx\, \frac{1}{x^2}\, 1_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$

This λ is, up to a constant multiple, the one-sided version of the Lévy measure in the preceding example.

All jumps of X are upward, but X is not constrained to $\mathbb{R}_+$; for $t > 0$, the distribution of X_t puts strictly positive mass on every interval (x, y) with $-\infty < x < y < \infty$. In particular, all the jumps of X^d are upward, the jumps over (s, t) are infinitely many and their sizes sum to $+\infty$. Thus, X^d is truly a compensated sum of jumps; it is the limit of the processes X^ε with upward jumps and downward drift.

The characteristic function for X_t is, in view of 1.32, and the form of λ here,

2.21 $$\mathbb{E}\, e^{irX_t} = \exp t \left[\int_0^1 dx \frac{1}{x^2}(e^{irx} - 1 - irx) + \int_1^\infty dx \frac{1}{x^2}(e^{irx} - 1) \right]$$
$$= \exp_-\; t\; \left[\, \tfrac{1}{2}\pi|r| - ic_0 r + ir\log|r| \,\right],$$

where

$$c_0 \;=\; \int_0^1 dx\, \frac{1}{x^2}\, (\sin x - x) + \int_1^\infty dx\, \frac{1}{x^2}\, \sin x.$$

It is checked easily that it is impossible for X_t and $t^{1/a}X_1$ to have the same characteristic function for some $a > 0$. So, X is not stable at all.

2.22 Example. *Cauchy and other* 1*-stable processes in* $\mathbb{R}^d$. Let S be the unit sphere in $\mathbb{R}^d$, and let σ be a probability measure on S satisfying

2.23 $$\int_S \sigma(du)\; u \;=\; 0.$$

For example, if $d = 2$, we obtain such a measure by putting equal weights at the vertices of a regular pentagon circumscribed by the unit circle S.

Let c be a constant in $(0, \infty)$, and let λ be the measure on $\mathbb{R}^d$ given by

$$2.24 \quad \lambda f \;=\; \int_{\mathbb{R}^d} \lambda(dx)\, f(x) \;=\; \int_{\mathbb{R}_+} dv\, \frac{c}{v^2} \int_S \sigma(du)\, f(vu), \quad f \geq 0 \text{ Borel.}$$

This λ satisfies 1.30; for $f(x) = |x|^2 \wedge 1$, we get $\lambda f = 2c < \infty$. But λ fails to satisfy 1.13.

Let X be the Lévy process whose canonical decomposition is $X = X^d + X^e$ in the notations of 1.31 and 1.33 and whose Lévy measure is the current λ. Its sample path behavior is similar to that of Example 2.6, except that it is not a pure-jump process and has infinite variation over every interval. The magnitudes of its jumps are regulated by the radial part of λ, and the latter is a constant multiple of the Lévy measure in Example 2.20, the half-Cauchy process. The characteristic function of X_t can be obtained using 2.21:

$$\begin{aligned} 2.25 \qquad \mathbb{E}\, e^{\,ir\cdot X_t} &= \exp\; t\int_S \sigma(du) \int_{\mathbb{R}_+} dv\; \frac{c}{v^2}\, [\, e^{ivr\cdot u} - 1 - ivr\cdot u\; 1_{\mathbb{B}}(vu)\,] \\ &= \exp\_\;\; tc\int_S \sigma(du)\; [\, \tfrac{1}{2}\pi\; |r\cdot u| - ic_0 r\cdot u + ir\cdot u \log|r\cdot u|\,] \\ &= \exp\_\;\; tc\int_S \sigma(du)\; [\, \tfrac{1}{2}\pi |r\cdot u| + ir\cdot u \log|r\cdot u|\,]\; ; \end{aligned}$$

here, we noted that $1_{\mathbb{B}}(vu) = 1_{[0,1]}(v)$ for the unit ball $\mathbb{B}$ and the unit vector u, and then used 2.21 with r there replaced by $r \cdot u$ and finally the assumption 2.23. Replacing r by tr and using 2.23 once more, we see that the characteristic functions of X_t and tX_1 are the same. Hence, X is 1-stable.

When $d = 1$, the unit "sphere" consists of the two points $+1$ and -1, and the condition 2.23 makes σ symmetric. Thus, when $d = 1$, the process is symmetric necessarily and is a Cauchy process (equal to $(1/2)\pi c Z$ where Z is standard Cauchy). In higher dimensions, symmetry and isotropy require further conditions on σ. For example, the pentagonal σ mentioned above yields a 1-stable process X in $\mathbb{R}^2$ that is not symmetric.

When $d \geq 2$, the process X is symmetric if and only if σ is symmetric, and then 2.23 holds automatically and 2.25 becomes

$$2.26 \qquad \mathbb{E}\, e^{ir\cdot X_t} \;=\; \exp\_\;\; \tfrac{1}{2}\pi ct\; \textstyle\int_S\; \sigma(du)\; |r\cdot u|, \quad r \in \mathbb{R}^d.$$

Moreover, X is isotropic if and only if σ is the uniform distribution on S, in which case the integral over the sphere can be computed as in Exercise 2.41, and we get

$$2.27 \qquad \mathbb{E}\, e^{ir\cdot X_t} \;=\; \exp\_\;\; \hat{c}t|r|, \quad r \in \mathbb{R}^d,$$

where $\hat{c} = \frac{1}{2} c\sqrt{\pi}\,\Gamma(\frac{d}{2})/\Gamma(\frac{d+1}{2})$. This Fourier transform is invertible (see Example 6.8 for a direct computation)

$$\mathbb{P}\{X_t \in dx\} \;=\; dx\; \hat{c}t\,\Gamma\left(\tfrac{d+1}{2}\right)/[\,\pi\hat{c}^2t^2 + \pi|x|^2\,]^{(d+1)/2}, \quad x \in \mathbb{R}^d.$$

This is called the d-dimensional *Cauchy distribution* with scale factor $\hat{c}t$; thus, X_t has the same distribution as $\hat{c}tZ$, where Z has the standard d-dimensional Cauchy distribution; see Exercise 2.42 for the definition.

Stable processes with index in $(1, 2)$

These processes are similar to the stable ones with index in $(0, 1)$, except that they cannot have bounded variation over intervals.

Fix a in $(1, 2)$ and c in $(0, \infty)$. Let S be the unit sphere in $\mathbb{R}^d$, and let σ be a probability measure on it. Define a measure λ on $\mathbb{R}^d$ by

$$\lambda f \;=\; \int_{\mathbb{R}_+} dv\, \frac{c}{v^{a+1}} \int_S \sigma(du)\, f(vu), \qquad f \geq 0 \text{ Borel.}$$

This λ is the same as that in 2.7 but the shape index a is now in the interval $(1, 2)$; this λ satisfies 1.30 but not 1.13. Thus, the process we are about to introduce will have infinitely many jumps over every interval and, further, it will have infinite variation over every interval.

Theorem 1.29 shows the existence of a Lévy process $X^d + X^e$ whose Lévy measure is λ. Consider the compound Poisson process X^e whose every jump exceeds unity in magnitude; it has a well-defined mean: since $a > 1$,

$$\mathbb{E}\, X_t^e \;=\; t \int_1^\infty dv\, \frac{c}{v^{a+1}}\, v \int_S \sigma(du)\, u \;=\; t\, \frac{c}{a-1} \int_S \sigma(du)\, u \;=\; bt$$

with an apparent definition for the vector b in $\mathbb{R}^d$. We define the process X to have the canonical decomposition $X = X^b + X^d + X^e$ with $X_t^b \;=\; -bt$. In other words, in the spirit of Notation 1.27, and with M Poisson with mean $\text{Leb} \times \lambda$,

2.28 $$X_t \;=\; \int_{[0,t]\times\mathbb{R}^d} [\, M(ds, dx) - ds\; \lambda(dx)\,]\; x.$$

It is clear that X is a Lévy process in $\mathbb{R}^d$, and its every component is a martingale. It follows from 2.28 that

2.29 $$\mathbb{E}\, e^{\,ir\cdot X_t} = \exp\, t \int_{\mathbb{R}^d} \lambda(dx)\, (e^{ir\cdot x} - 1 - ir\cdot x)$$
$$= \exp\, t \int_S \sigma(du) \int_{\mathbb{R}_+} dv\, \frac{c}{v^{a+1}}\, (e^{ivr\cdot u} - 1 - ivr\cdot u).$$

It is now easy to check that X_t and $t^{1/a} X_1$ have the same characteristic function; thus, X is a-stable.

On the right side of 2.29, the integral over $\mathbb{R}_+$ can be evaluated through integration by parts using 2.3. The result is similar to 2.8: for r in $\mathbb{R}^d$,

2.30 $$\mathbb{E}\, e^{\,ir\cdot X_t} \;=\; \exp_-\; tc_a \int_S \sigma(du)\, |r\cdot u|^a\, [1 - i\, (\tan \tfrac{1}{2}\pi a)\, \text{sgn}\, r\cdot u],$$

where

$$c_a \;=\; -\,c\,\frac{\Gamma(2-a)}{a\,(a-1)}\;\cos\tfrac{1}{2}\pi a\;;$$

note that $c_a > 0$. The formula 2.30 shows that X is symmetric if and only if σ is symmetric, in which case

2.31 $$\mathbb{E}\, e^{\,ir\cdot X_t} \;=\; \exp_-\; tc_a \int_S \sigma(du)\;|r\cdot u|^a, \quad r\in\mathbb{R}^d.$$

Further, X is isotropic if and only if σ is the uniform distribution on S, in which case the result of Exercise 2.41 yields

2.32 $$\mathbb{E}\, e^{\,ir\cdot X_t} \;=\; \exp_-\; tc_{ad}\;|r|^a, \quad r\in\mathbb{R}^d,$$

with $c_{ad} \;=\; c_a\;\Gamma(\frac{a+1}{2})\Gamma(\frac{d}{2})\;/\;\Gamma(\frac{a+d}{2})\Gamma(\frac{1}{2})$ with c_a as in 2.30. Note that 2.32 has the same form as in 2.12 and 2.26.

Exercises

2.33 *Arithmetics*. Fix $a > 0$. Show the following for Lévy processes X and Y in $\mathbb{R}^d$.

a) If X is a-stable, then so is cX for every constant c in $\mathbb{R}$.

b) If X and Y are a-stable and independent, then $X+Y$ and $X-Y$ are a-stable.

c) If X is a-stable, and Y is independent of X and is b-stable for some $b > 0$ distinct from a, then $X+Y$ is not stable.

2.34 *Stability index*. Fix $a > 0$. Suppose that X is an a-stable non-degenerate Lévy process in $\mathbb{R}$ with characteristic exponent ψ. This is to show that, then, $a \in (0,2]$ necessarily.

a) Show that $t\psi(r) \;=\; \psi(t^{1/a}\;r)$ for $t > 0$ and r in $\mathbb{R}$. Show that $\psi(r) \;=\; c\;r^a$ for some complex constant c for r in $\mathbb{R}_+$.

b) Suppose that X is symmetric, that is, X and $-X$ have the same law. Then, $\psi(r) \;=\; \psi(-r)$ for all r. Show that $\psi(r) \;=\; c\;|r|^a$ for all r in $\mathbb{R}$.

c) Show that $e^{c|r|^a}$ cannot be a characteristic function when $a > 2$. Hint: See Exercise II.2.33 about the second moment of X_1. Conclude that, if X is symmetric, then $a \in (0,2]$.

d) If X is not symmetric, let Y be an independent copy of it. Then, $X-Y$ is symmetric and a-stable. So, $a \in (0,2]$ again.

2.35 *Continuation.* Let X be a Lévy process in $\mathbb{R}^d$. Suppose that it is not degenerate. If it is a-stable, then $a \in (0,2]$. Show.

2.36 *Stability with index* 2. Let X be a Lévy process in $\mathbb{R}$. Suppose that it is 2-stable. Then X_t has the Gaussian distribution with mean 0 and variance vt for some constant v. Show.

2.37 *Stable Poissons.* Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with mean $\mu = \text{Leb} \times \lambda$. Let $a > 0$ be fixed. Suppose that M and $M \circ h^{-1}$ have the same law (which means that $\mu = \mu \circ h^{-1}$) for $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}$ defined by

$$h(t, x) = (\frac{t}{s}, s^{1/a} x),$$

and that this is true for every $s > 0$. Show that, then,

$$\lambda(dx) = dx\, |x|^{-a-1} [\, b\, 1_{(0,\infty)}(x) + c\, 1_{(-\infty,0)}(x)\,], \quad x \neq 0,$$

for some constants b and c in $\mathbb{R}_+$. If λ satisfies 1.13 then $a \in (0,1)$; show. If λ satisfies 1.30, then $a \in (0,2)$; show.

2.38 *Stable processes with index in* $(0,1)$. Let X be as in Example 2.6, but the dimension is $d = 1$. Then, the "sphere" S consists of the two points $+1$ and -1.

a) Show that λ of 2.7 takes the form

$$\lambda(dx) = dx\, \frac{c}{|x|^{a+1}}\, (p\, 1_{(0,\infty)}(x) + q\, 1_{(-\infty,0)}(x)),$$

where p and q are positive numbers with $p + q = 1$.

b) Conclude that $X = X^+ - X^-$, where X^+ and X^- are independent a-stable increasing Lévy processes.

c) Show that the characteristic exponent of X is (see 2.8)

$$\psi(r) = -\, c_a\, |r|^a\, [\, 1 - i\, (p - q)(\tan\, \tfrac{1}{2}\pi a)\, \text{sgn}\, r\,], \quad r \in \mathbb{R}.$$

2.39 *Continuation: symmetric case.* When $d = 1$, symmetry and isotropy are the same concept. Show that X of the preceding exercise is symmetric if and only if $p = q$. Then, the characteristic exponent becomes $\psi(r) = -c_a\, |r|^a$. Check that 2.10 and 2.12 coincide when $d = 1$.

2.40 *Uniform distribution on* S. Let σ be the uniform distribution on the unit sphere S in $\mathbb{R}^d$. This is to show that, for every s in S and a in $\mathbb{R}_+$,

$$\int_S \sigma(du)\, |s \cdot u|^a = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{a+d}{2})\Gamma(\frac{1}{2})}.$$

The left side is $\mathbb{E}\, |s \cdot U|^a$ where U has the uniform distribution on S. The trick is to recall that, if the random vector $Z = (Z_1, \ldots, Z_d)$ has independent components each of which has the standard Gaussian distribution, then

$$Z = RU,$$

where $R = |Z|$, and U is independent of R and has the uniform distribution on S. It follows that, for every s in S,

$$\mathbb{E}\, |s \cdot Z|^a = (\mathbb{E}\, R^a)(\mathbb{E}\, |s \cdot U|^a),$$

and the problem reduces to evaluating the expectations concerning R and $s \cdot Z$.

a) Recall that R^2 has the gamma distribution with shape index $d/2$ and scale index $1/2$. Use this to show that

$$\mathbb{E}\ R^a = \int_0^\infty dx\ \frac{e^{-x}\ x^{d/2-1}}{\Gamma(d/2)}\ (2x)^{a/2}\ =\ 2^{a/2}\ \Gamma(\frac{d+a}{2})/\Gamma(d/2).$$

b) Show that $|s \cdot Z|$ has the same distribution as R but with d put equal to 1. Thus, $\mathbb{E}\ |s \cdot Z|^a\ =\ 2^{a/2}\ \Gamma(\frac{a+1}{2})/\Gamma(1/2)$.

c) Show that $\mathbb{E}\ |s \cdot U|^a$ is as claimed.

2.41 *Continuation.* For r in $\mathbb{R}^d$ and u in S, we have $|r \cdot u| = |r|\ |s \cdot u|$ with $s = r/|r|$ in S. Use this observation to show that:

$$\int_S \sigma(du)\ |r \cdot u|^a\ =\ |r|^a\ \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{a+d}{2})\Gamma(\frac{1}{2})}.$$

2.42 *Cauchy distribution on* $\mathbb{R}^d$. Let Z take values in $\mathbb{R}^d$. It is said to have the *standard Cauchy distribution* if

$$\mathbb{P}\{Z \in dx\}\ =\ dx\ \frac{d+1}{2}\ (\frac{1}{\pi(1+|x|^2)})^{(d+1)/2},\quad x \in \mathbb{R}^d.$$

Then, $\mathbb{E}\ e^{ir\cdot Z} = e^{-|r|}, r \in \mathbb{R}^d$. Show that Z has the same distribution as X/Y, where $X = (X_1, \ldots, X_d)$ is a d-dimensional standard Gaussian, and Y is a one-dimensional Gaussian independent of X. Note that each component Z_i has the standard one-dimensional Cauchy distribution, but the components are dependent. Show that, for every vector v in $\mathbb{R}^d$, the inner product $v \cdot Z$ has the one-dimensional Cauchy distribution with scale factor $|v|$, that is,

$$\mathbb{P}\{\ v \cdot Z \in dx\ \}\ =\ dx\ \frac{|v|}{\pi(|v|^2+|x|^2)},\quad x \in \mathbb{R}.$$

2.43 *Stable processes in* $\mathbb{R}$ *with index in* $(1, 2)$. Let X be defined by 2.28, but with $d = 1$. Show that the Lévy measure λ defining its law has the form

$$\lambda(dx)\ =\ dx\ \frac{c}{|x|^{a+1}}\ [\ p\ 1_{(0,\infty)}(x) + q\ 1_{(-\infty,0)}(x)\ ],\quad x \in \mathbb{R},$$

where $a \in (1, 2)$ and $c \in (0, \infty)$ as before, and p and q are positive numbers with $p+q = 1$. All the jumps are upward if $p = 1$, and all downward if $q = 1$. The process is symmetric if and only if $p = q = 1/2$. In all cases, X is a martingale. In particular, $\mathbb{E}\ X_t = 0$. Compute Var X_t. Show that

$$\mathbb{E}^{e\ irX_t}\ =\ \exp\_\ tc_a\ [\ |r|^a - i\ (p-q)(\tan\frac{1}{2}\ \pi a)\text{sgn}\ r\ ],\quad r \in \mathbb{R}.$$

2.44 *Continuation.* Recall from 2.28 that X has the form $X = X^b + X^d + X^e$, where $X_t^b = -\mathbb{E}X_t^e = -(p-q)\frac{c}{a-1}t$. Note that none of the processes $X^b, X^d, X^e, X^d + X^e, X^e + X^b, X^b + X^d$ is a-stable.

2.45 *Continuation.* Show that it is possible to decompose X as

$$X = Y - Z,$$

where Y and Z are independent a-stable processes, with Y having only upward jumps, and Z only downward jumps. Define Y and Z carefully from the same M that defines X.

3 Lévy Processes on Standard Settings

This section is to re-introduce Lévy processes in a modern setting, show the Markov and strong Markov properties for them, and reconcile the differences from the earlier definition. The motivation for modernity is two-fold: First, we prefer filtrations that are augmented and right-continuous, because of the advantages mentioned in the last section of Chapter V. Second, it is desirable to have a moving coordinate system for time and space, which would indicate what time is meant by "present" in a given argument.

Lévy processes over a stochastic base

3.1 Definition. *A* stochastic base *is a collection*

$$\mathscr{B} = (\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$$

where $(\Omega, \mathcal{H}, \mathbb{P})$ *is a complete probability space,* $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *is an augmented right-continuous filtration on it, and* $\theta = (\theta_t)_{t\in\mathbb{R}_+}$ *is a semigroup of operators* $\theta_t : \omega \mapsto \theta_t\omega$ *from* Ω *into* Ω *with*

3.2 $$\theta_0\omega = \omega, \quad \theta_u(\theta_t\omega) = \theta_{t+u}\omega, \quad t, u \in \mathbb{R}_+.$$

Operators θ_t *are called* time-shifts.

3.3 Definition. *Let* $X = (X_t)_{t\in\mathbb{R}_+}$ *be a stochastic process with state space* $\mathbb{R}^d$*. It is called a* Lévy process *over the stochastic base* $\mathscr{B}$ *if* X *is adapted to* $\mathcal{F}$ *and the following hold:*

a) Regularity*.* X *is right-continuous and left-limited, and* $X_0 = 0$.
b) Additivity*.* $X_{t+u} = X_t + X_u\circ\theta_t$ *for every* t *and* u *in* $\mathbb{R}_+$.
c) Lévy property*. For every* t *and* u *in* $\mathbb{R}_+$*, the increment* $X_u\circ\theta_t$ *is independent of* $\mathcal{F}_t$ *and has the same distribution as* X_u.

Remark. If X is a Lévy process over the base $\mathscr{B}$, then it is a Lévy process in the sense of Definition 1.1 with respect to the filtration $\mathcal{F}$. The difference between Definitions 1.1 and 3.3 is slight: we shall show in Theorem 3.20 below (see also Remark 3.21) that starting from a raw Lévy process (in the sense of Definition 1.1 and with respect to its own filtration $\mathcal{G}$), we can modify $\mathcal{H}, \mathcal{G}, \mathbb{P}$ to make them fit a stochastic base. The existence of shift operators is easy to add as well; see Exercise 3.24.

Shifts

Existence of shift operators is a condition of richness on Ω. In canonical constructions, it is usual to take Ω to be the collection of all right-continuous left-limited functions $\omega : \mathbb{R}_+ \to \mathbb{R}^d$ with $\omega(0) = 0$. Then, we may define $\theta_t\omega$ to be the function $u \mapsto \omega(t+u) - \omega(t)$, and setting $X_t(\omega) = \omega(t)$ we obtain both the semigroup property 3.2 and the additivity 3.3b.

In general, θ_t derives its meaning from what it does, which is described by the additivity condition 3.3b. We interpret it as follows: $X_u \circ \theta_t$ is the increment over the next period of length u if the present time is t. Thus, θ_t shifts the time-space origin to the point (t, X_t) of the standard coordinate system; see Figure 9 below.

In other words, θ is a moving reference frame pinned on the path X. It is an egocentric coordinate system: the present is the origin of time, the present position is the origin of space.

Our usage of shifts is in accord with the established usage in the theory of Markov processes. We illustrate this with an example and draw attention to a minor distinction in terms. Let X be a Wiener process, and put

$$Z_t \;=\; Z_0 + X_t, \quad t \in \mathbb{R}_+.$$

Then, Z is called a standard Brownian motion with initial position Z_0. The established usage would require that the Markov process Z satisfy

$$Z_u \circ \theta_t \;=\; Z_{t+u}, \quad t, u \in \mathbb{R}_+ \;;$$

This is called time-homogeneity for Z. It implies that X is additive:

$$X_u \circ \theta_t \;=\; Z_u \circ \theta_t - Z_0 \circ \theta_t \;=\; Z_{t+u} - Z_t \;=\; X_{t+u} - X_t.$$

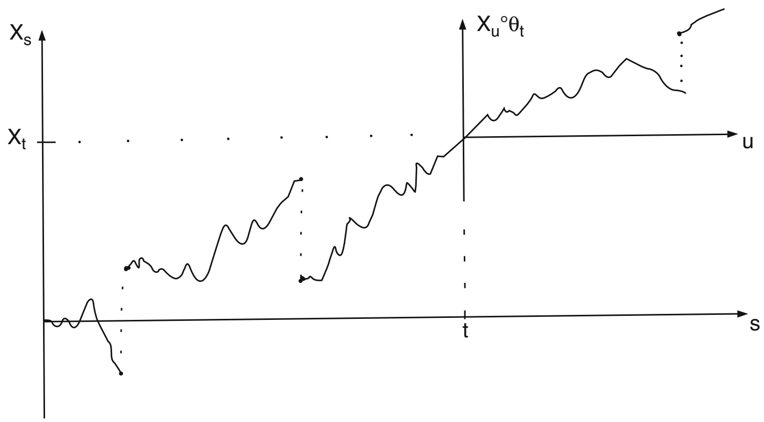

Figure 9: When the present time is t, the new coordinate system has its origin at (t, X_t) of the standard coordinate system.

Indeed, in the terminology of Markov processes, X is an additive functional of the Markov process Z.

The additivity for X implies certain measurability properties for the shifts:

3.4 LEMMA. *Let X be a Lévy process over the base $\mathscr{B}$. Let $\mathcal{G}$ be the filtration generated by it. For each t in $\mathbb{R}_+$, the mapping $\theta_t : \Omega \to \Omega$ is measurable with respect to $\mathcal{G}_{t+u}$ and $\mathcal{G}_u$ for every u in $\mathbb{R}_+$; in particular, θ_t is measurable with respect to $\mathcal{G}_\infty$ and $\mathcal{G}_\infty$.*

Proof. Fix t and u. For every s in $[0, u]$, we have $X_s \circ \theta_t = X_{t+s} - X_t$ by additivity, and $X_{t+s} - X_t$ is in $\mathcal{G}_{t+u}$. Since $\mathcal{G}_u$ is generated by X_s, with such s, this proves the first claim. The "particular" claim is obvious. □

Markov property

For a Lévy process, Markov property is the independence of future increments from the past at all times. The next theorem is the precise statement. Here, $\mathscr{B}$ is the stochastic base in 3.1, and $\mathcal{G}$ is the filtration generated by X. With the filtration $\mathcal{F}$ fixed, we write $\mathbb{E}_t$ for $\mathbb{E}(\cdot|\mathcal{F}_t)$, the conditional expectation operator given $\mathcal{F}_t$.

3.5 THEOREM. *Suppose that X is a Lévy process in $\mathbb{R}^d$ over the stochastic base $\mathscr{B}$. Then, for every time t, the process $X \circ \theta_t$ is independent of $\mathcal{F}_t$ and has the same law as X. Equivalently, for every bounded random variable V in $\mathcal{G}_\infty$,*

3.6 $$\mathbb{E}_t\, V \circ \theta_t = \mathbb{E}\, V, \quad t \in \mathbb{R}_+.$$

REMARK. The restriction to bounded V is for avoiding questions of existence for expectations. Of course, 3.6 extends to all positive V in $\mathcal{G}_\infty$ and to all integrable V in $\mathcal{G}_\infty$, and further.

Proof. a) We start by observing that the Lévy property 3.3c is equivalent to saying that, for every bounded Borel function f on $\mathbb{R}^d$,

3.7 $$\mathbb{E}_t\, f \circ X_u \circ \theta_t = \mathbb{E}\, f \circ X_u, \quad t, u \in \mathbb{R}_+.$$

b) We show next that 3.6 holds for V having the form $V = V_n$, where

3.8 $$V_n = f_1(X_{u_1})\, f_2(X_{u_2} - X_{u_1}) \cdots f_n(X_{u_n} - X_{u_{n-1}})$$

for some bounded Borel functions $f_1, \cdots, f_n$ on $\mathbb{R}^d$ and some times $0 < u_1 < u_2 < \cdots < u_n$.

The claim is true for $n = 1$ in view of 3.7. We make the induction hypothesis that the claim is true for n and consider it for $n + 1$. Observe that, with $u = u_n$ and $v = u_{n+1} - u_n$ for simplicity of notation, we have

$$V_{n+1} = V_n \cdot W \circ \theta_u, \quad \text{where} \quad W = f_{n+1} \circ X_v.$$

Thus, writing $\mathbb{E}_t = \mathbb{E}_t\ \mathbb{E}_{t+u}$ and recalling that $\theta_u \circ \theta_t = \theta_{t+u}$, we get

$$\begin{aligned}\mathbb{E}_t\ V_{n+1}\circ\theta_t &= \mathbb{E}_t\ \mathbb{E}_{t+u}(V_n\circ\theta_t)(W\circ\theta_{t+u})\\ &= \mathbb{E}_t\ V_n\circ\theta_t\ \mathbb{E}_{t+u}\ W\circ\theta_{t+u}\\ &= \mathbb{E}_t\ V_n\circ\theta_t\ \mathbb{E}\ W \ = \ \mathbb{E}\ V_n\ \mathbb{E}\ W \ = \ \mathbb{E}\ V_{n+1}\end{aligned}$$

where we used 3.7 to justify the third equality sign, induction hypothesis to justify the fourth, and 3.7 again for the fifth. So 3.6 holds for every V of the form 3.8.

c) Borel functions f having the form $f(x_1, \cdots, x_n) = f_1(x_1)\cdots f_n(x_n)$ generate the Borel σ-algebra on $(\mathbb{R}^d)^n$. Thus, by the monotone class theorem, part (b) of the proof implies that 3.6 holds for every bounded V having the form

$$V = f(X_{u_1}, X_{u_2} - X_{u_1}, \cdots, X_{u_n} - X_{u_{n-1}}) \qquad \square$$

for some bounded Borel f and some times $0 < u_1 < \cdots < u_n$. Since the increments of X generate the σ-algebra $\mathcal{G}_\infty$, the proof is completed through another application of the monotone class theorem.

Strong Markov property

This is the analog of the Markov property where the deterministic time t is replaced by a stopping time T. The setup is the same, and we write $\mathbb{E}_T$ for the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_T)$. However, in formulating 3.6 with T, we face a problem: if $T(\omega) = \infty$, then $\theta_T\omega = \theta_{T(\omega)}\omega$ makes no sense and $X_\infty(\omega)$ is not defined. The following is to handle the problem.

3.9 Convention. *Suppose that $Z(\omega)$ is well-defined for every ω for which $T(\omega) < \infty$. Then, the notation $Z\ 1_{\{T<\infty\}}$ stands for the random variable that is equal to Z on $\{T < \infty\}$ and to 0 on $\{T = \infty\}$.*

The convention is without ambiguity. If Z is already defined for all ω, then $Z\ 1_{\{T<\infty\}}$ is equal to 0 on $\{T = \infty\}$ since $x\cdot 0 = 0$ for all x in $\bar{\mathbb{R}} = [-\infty, +\infty]$. With this convention, the following is the strong Markov property. Here $\bar{\mathcal{G}}_\infty$ is the completion of $\mathcal{G}_\infty$ in $\mathcal{H}$, that is $\bar{\mathcal{G}}_\infty = \mathcal{G}_\infty \vee \mathcal{N}$ where $\mathcal{N}$ is the σ-algebra generated by the collection of negligible events in $\mathcal{H}$.

3.10 Theorem. *Suppose that X is a Lévy process over the base $\mathscr{B}$. Let T be a stopping time of $\mathcal{F}$. Then, for every bounded random variable V in $\bar{\mathcal{G}}_\infty$,*

3.11 $$\mathbb{E}_T\ V\circ\theta_T\ 1_{\{T<\infty\}} \ = \ (\mathbb{E}\ V)\ 1_{\{T<\infty\}}.$$

Remark. If $T < \infty$ almost surely, then $1_{\{T<\infty\}}$ can be deleted on both sides. In words, the preceding theorem states the following: on the event $\{T < \infty\}$, the future process $X\circ\theta_T$ is independent of the past $\mathcal{F}_T$ and has the same law as X. On the event $\{T = \infty\}$, there is no future and nothing to be said.

Proof. Let X and T be as hypothesized. In view of the defining property for $\mathbb{E}_T$, it is sufficient to show that

3.12 $$\mathbb{E}\, 1_H\, V\circ\theta_T\, 1_{\{T<\infty\}} \;=\; \mathbb{E}\, 1_{H\cap\{T<\infty\}}\, \mathbb{E}\, V$$

for every H in $\mathcal{F}_T$ and bounded positive V in $\bar{\mathcal{G}}_\infty$. Moreover, for every V in $\bar{\mathcal{G}}_\infty$ there is V_0 in $\mathcal{G}_\infty$ such that $V = V_0$ almost surely, and it is enough to show 3.12 for V_0. Hence, it is enough to prove 3.12 for H in $\mathcal{F}_T$ and bounded positive V in $\mathcal{G}_\infty$. We do this in a series of steps.

a) Assume, further, that T is countably-valued. Let D be its range intersected with $\mathbb{R}_+$. Then, $\{T < \infty\}$ is equal to the union of $\{T = t\}$ over t in D, and $H \cap \{T = t\} \in \mathcal{F}_t$ for every t. Thus, starting with the monotone convergence theorem, the left side of 3.12 becomes

$$\sum_{t\in D} \mathbb{E}\, 1_{H\cap\{T=t\}} V\circ\theta_t \;=\; \sum_{t\in D} \mathbb{E}\, 1_{H\cap\{T=t\}}\, \mathbb{E}_t\, V\circ\theta_t \;=\; \mathbb{E}\, 1_{H\cap\{T<\infty\}}\, \mathbb{E}\, V,$$

where the last equality sign is justified by the Markov property that $\mathbb{E}_t V\circ\theta_t = \mathbb{E}\, V$. This proves 3.12 for T countably-valued.

b) Now we remove the restriction on T but assume that

3.13 $$V \;=\; f\circ X_u$$

for some $u > 0$ and some bounded positive continuous function f on $\mathbb{R}^d$. Let (T_n) be the approximating sequence of stopping times discussed in Propositions V.1.20 and V.7.12: each T_n is countably-valued, $T_n < \infty$ on $\{T < \infty\}$, and the sequence decreases to T. Since $H\cap\{T < \infty\} \in \mathcal{F}_T \subset \mathcal{F}_{T_n}$, we get from 3.12 with T_n that

3.14 $$\mathbb{E}\, 1_{H\cap\{T<\infty\}}\, V\circ\theta_{T_n} \;=\; \mathbb{E}\, 1_{H\cap\{T<\infty\}}\, \mathbb{E}\, V.$$

On the event $\{T < \infty\}$, we have $T_n < \infty$ for every n, and

$$X_u\circ\theta_{T_n} \;=\; X_{T_n+u} - X_{T_n} \to X_{T+u} - X_T = X_u\circ\theta_T$$

almost surely, since (T_n) is decreasing to T and X is right-continuous. Since V has the form 3.13 with f continuous and bounded, it follows that $V\circ\theta_{T_n} \to V\circ\theta_T$ almost surely and, thus, the left side of 3.14 converges to the left side of 3.12 by the bounded convergence theorem. This proves 3.12 for V having the form 3.13 with f bounded, positive, and continuous.

c) Since continuous $f : \mathbb{R}^d \mapsto \mathbb{R}$ generate the Borel σ-algebra on $\mathbb{R}^d$, and since V satisfying 3.13 is a vector space and a monotone class, it follows that 3.12 and therefore 3.11 holds for V having the form 3.13 with f bounded, positive, Borel.

d) There remains to extend 3.11 to arbitrary bounded positive V in $\mathcal{G}_\infty$. This is done exactly as in the parts (b) and (c) of the proof of the Markov property, Theorem 3.5: put T wherever t appears and append the factor $1_{\{T<\infty\}}$ on each side of every equation having t in it.

Processes with bounded jumps

We have seen examples of Lévy processes X with $\mathbb{E}X_t$ equal to $+\infty$ (increasing stable processes) and also examples where $\mathbb{E}X_t$ does not exist (Cauchy processes). As an application of the strong Markov property, we now show that such anomalies are possible only if X has jumps of unbounded size. The converse is false: as gamma processes exemplify, X may have jumps of arbitrarily large size and still have finite moments of all orders.

3.15 PROPOSITION. *Let X be a Lévy process in $\mathbb{R}^d$ over the base $\mathscr{B}$. Suppose that all its jumps are bounded in magnitude by some fixed constant. Then, for every t, the variable $|X_t|$ has finite moments of all orders.*

Proof. a) Fix a constant b in $(0,\infty)$. Suppose that all the jumps (if any) have magnitudes bounded by b. The claim is that, then, $\mathbb{E}\,|X_t|^k < \infty$ for every integer $k \geq 1$. To prove this, it is enough to show that the distribution of $|X_t|$ has an exponential tail; indeed, we shall show that there exists a constant c in $(0,1)$ such that

3.16 $$\mathbb{P}\{\,|X_t| > (1+b)n\,\} \leq e^t\, c^n, \quad n \in \mathbb{N}.$$

b) Let R be the time of exit from the unit ball, that is,

$$R = \inf\{\,t > 0 : |X_t| > 1\,\}.$$

Note that $|X_R| \leq 1+b$ since the worst that can happen is that X exits the unit ball by a jump of magnitude b. Moreover, since $X_0 = 0$ and X is right-continuous, $R > 0$ almost surely; hence,

3.17 $$c = \mathbb{E}\, e^{-R} < 1.$$

Finally, note that $R < \infty$ almost surely. This follows from the impossibility of containing the sequence (X_m) within the unit ball, since X_m is the sum of m independent and identically distributed variables.

c) Let T be a finite stopping time of $\mathcal{F}$ and consider

$$T + R\circ\theta_T = \inf\{t > T : |X_t - X_T| > 1\}.$$

By the strong Markov property at T, by Theorem 3.10 with $V = e^{-R}$,

3.18 $$\mathbb{E}\, e^{-(T+R\circ\theta_T)} = \mathbb{E}\, e^{-T}\, \mathbb{E}_T\, e^{-R}\circ\theta_T = \mathbb{E}\, e^{-T}\, \mathbb{E}\, e^{-R} = c\,\mathbb{E}\, e^{-T}.$$

d) Put $T_0 = 0$ and define $T_n, n \geq 1$, recursively by setting $T_{n+1} = T_n + R\circ\theta_{T_n}$. Since $R < \infty$ almost surely, so is $T_1 = R$ and so is $T_2 = T_1 + R\circ\theta_{T_1}$, and so on. Thus, T_n is an almost surely finite stopping time for each n, and using 3.18 repeatedly we get

3.19 $$\mathbb{E}\, e^{-T_n} = c^n, \quad n \in \mathbb{N}.$$

e) Finally, consider the bound 3.16. Note that T_{n+1} is the first time t after T_n such that $|X_t - X_{T_n}| > 1$. Thus, for fixed t and ω,

$$|X_t(\omega)| \;>\; (1+b)n \;\Rightarrow\; T_n(\omega) \;<\; t \;\Rightarrow\; e^{-T_n(\omega)} \;>\; e^{-t}.$$

Hence, the left side of 3.16 is less than or equal to

$$\mathbb{P}\{\, e^{-T_n} \;>\; e^{-t}\,\} \;\le\; e^t \,\mathbb{E}\, e^{-T_n} \;=\; e^t\, c^n\,.$$

by Markov's inequality and 3.19. This completes the proof. □

On the definitions

Consider Definitions 1.1 and 3.3. They differ at two points: the existence of shift operators and the conditions of right-continuity and augmentedness for the filtration $\mathcal{F}$. The shifts are for reasons of convenience and clarity. For instance, replacing $X_u \circ \theta_t$ with $X_{t+u} - X_t$ would eliminate the shifts in Definition 3.3 and in Theorem 3.5 on the Markov property. The same is true more generally; we can eliminate the shifts entirely, without loss of real content, but with some loss in brevity; for example, in Theorem 3.10 on the strong Markov property, we need to replace $X_u \circ \theta_T = X_{T+u} - X_T$ with $\hat{X}_u$ and, instead of $R \circ \theta_T$, we need to introduce $\hat{R}$, which is obtained from $\hat{X}$ by the same formula that obtains R from X. We use the shifts for the clarity and economy achieved through their use; see Exercise 3.24 to see that they can always be introduced without loss of generality.

The conditions on the filtration $\mathcal{F}$ are more serious. We illustrate the issue with an example. Suppose that X is a Wiener process and T is the time of hitting some fixed level $b > 0$, that is, $T \;=\; \inf\{t > 0 \;:\; X_t > b\}$. Since X is adapted to $\mathcal{F}$, and $\mathcal{F}$ is right-continuous, T is a stopping time of $\mathcal{F}$. If $\mathcal{F}$ were *not* right-continuous, T might fail to be a stopping time of it. For instance, T is not a stopping time of $\mathcal{G}$, the filtration generated by X ; this can be inferred from the failure of Galmarino's test, Exercise V.1.28.

Nevertheless, the conditions on $\mathcal{F}$ of Definition 3.3 are natural in addition to being advantageous. We show next that, starting with the filtration $\mathcal{G}$ generated by X, we can always use the augmentation $\bar{\mathcal{G}}$ as the filtration $\mathcal{F}$.

Augmenting the raw process

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, X a stochastic process with state space $\mathbb{R}^d$, and $\mathcal{G}$ the filtration generated by X. Let $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$ be the completion of $(\Omega, \mathcal{H}, \mathbb{P})$, and let $\mathcal{N}$ be the σ-algebra generated by the collection of negligible sets in $\bar{\mathcal{H}}$. We denote by $\bar{\mathcal{G}}$ the augmentation of $\mathcal{G}$ in $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$, that is, $\bar{\mathcal{G}}_t \;=\; \mathcal{G}_t \vee \mathcal{N}$, the σ-algebra generated by the union of $\mathcal{G}_t$ and $\mathcal{N}$. See Section 7 of Chapter V for these and for the notation $\mathcal{G}_{t+} = \cap_{\varepsilon>0}\; \mathcal{G}_{t+\varepsilon}$, and recall that the right-continuity for $\bar{\mathcal{G}}$ means that $\cap_{\varepsilon>0}\; \bar{\mathcal{G}}_{t+\varepsilon} = \bar{\mathcal{G}}_t$ for every t.

3.20 Theorem. *Suppose that, over $(\Omega, \mathcal{H}, \mathbb{P})$, the process X is Lévy with respect to $\mathcal{G}$ in the sense of Definition 1.1. Then,*

a) over $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$, the process X is Lévy with respect to $\bar{\mathcal{G}}$ in the sense of Definition 1.1, and
b) the filtration $\bar{\mathcal{G}}$ is augmented and right-continuous.

Before proving this, we note two interesting corollaries; one concerning $\mathcal{G}_{t+}$ and the other the special case $\bar{\mathcal{G}}_0$. Since $\mathcal{G}_{t+\varepsilon} \subset \bar{\mathcal{G}}_{t+\varepsilon}$, we have $\mathcal{G}_{t+} \subset \cap_{\varepsilon>0} \bar{\mathcal{G}}_{t+\varepsilon}$, and the last σ-algebra is equal to $\bar{\mathcal{G}}_t$ by the preceding theorem. Since $\bar{\mathcal{G}}_t = \mathcal{G}_t \vee \mathcal{N}$, we see that $\mathcal{G}_{t_+} \subset \mathcal{G}_t \vee \mathcal{N}$; in words, the extra wisdom gained by an infinitesimal peek into the future consists of events that are either negligible or almost sure. In particular, since $X_0 = 0$ almost surely, $\mathcal{G}_0 \subset \mathcal{N}$ and we obtain the following corollary, called *Blumenthal's zero-one law.*

3.21 Corollary. *Every event in $\bar{\mathcal{G}}_0$ has probability 0 or 1.*

Going back to arbitrary t, we express the finding $\mathcal{G}_{t+} \subset \bar{\mathcal{G}}_t$ in terms of random variables; recall that $\bar{\mathcal{G}}_t = \mathcal{G}_t \vee \mathcal{N}$, which means that every random variable in $\bar{\mathcal{G}}_t$ differs from one in $\mathcal{G}_t$ over a negligible set.

3.22 Corollary. *Fix t in $\mathbb{R}_+$. For every random variable V in $\mathcal{G}_{t+}$ there is a random variable V_0 in $\mathcal{G}_t$ such that $V = V_0$ almost surely.*

Proof of Theorem 3.20

a) We prove the first claim first. Suppose X is as hypothesized. Since the restriction of $\bar{\mathbb{P}}$ to $\mathcal{H}$ is equal to $\mathbb{P}$, the events in $\mathcal{H}$ that are $\mathbb{P}$-almost sure are also events in $\bar{\mathcal{H}}$ that are $\bar{\mathbb{P}}$-almost sure. Thus, the regularity 1.1a of X over $(\Omega, \mathcal{H}, \mathbb{P})$ remains as regularity over $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$.

For the Lévy property 1.1b, we first observe that X is such over $(\Omega, \mathcal{H}, \mathbb{P})$ with respect to $\mathcal{G}$ if and only if

3.23 $$\mathbb{E}\, V\, f \circ (X_{t+u} - X_t) = \mathbb{E}\, V\, \mathbb{E}\, f \circ X_u, \quad t, u \in \mathbb{R}_+,$$

for every bounded Borel f on $\mathbb{R}^d$ and bounded variable V in $\mathcal{G}_t$. Since $\bar{\mathbb{P}}$ coincides with $\mathbb{P}$ on $\mathcal{H}$, we may replace $\mathbb{E}$ with the expectation operator $\bar{\mathbb{E}}$ with respect to $\bar{\mathbb{P}}$. Finally, if $\bar{V}$ is a bounded variable in $\bar{\mathcal{G}}_t = \mathcal{G}_t \vee \mathcal{N}$, and f as above, there is V in $\mathcal{G}_t$ such that $\bar{V} = V$ almost surely (under $\bar{\mathbb{P}}$). Thus, in 3.23, we may replace $\mathbb{E}$ with $\bar{\mathbb{E}}$ and V with $\bar{V}$; the result is the Lévy property 1.1b over $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$ with respect to $\bar{\mathcal{G}}$.

b) We are working on the complete probability space $(\Omega, \bar{\mathcal{H}}, \bar{\mathbb{P}})$ to show that the augmentation $\bar{\mathcal{G}}$ is also right-continuous. We start at $t = 0$. Let (ε_n) be a sequence decreasing strictly to 0. For $n \ge 1$, let

$$\mathcal{H}_n = \sigma\{X_t - X_s : \quad \varepsilon_n \le s < t \le \varepsilon_{n-1}\}.$$

Since X is a Lévy process (shown in part (a)), the σ-algebras $\mathcal{H}_1, \mathcal{H}_2, \cdots$ are independent. By Theorem II.5.12, Kolmogorov's zero-one law, the tail

σ-algebra defined by $(\mathcal{H}_n)$ is trivial, that is, the tail σ-algebra is contained in $\mathcal{N}$. But, since $\mathcal{H}_{n+1} \vee \mathcal{H}_{n+2} \vee \cdots = \mathcal{G}_{\varepsilon_n}$, the tail σ-algebra is equal to

$$\cap_n \mathcal{G}_{\varepsilon_n} = \cap_{\varepsilon>0} \mathcal{G}_\varepsilon = \mathcal{G}_{0+}.$$

We have shown that $\mathcal{G}_{0+} \subset \mathcal{N}$. We use this to show that $\bar{\mathcal{G}}$ is right-continuous. Fix t ; let $\hat{\mathcal{G}}$ be the filtration generated by the process $\hat{X}$, where $\hat{X}_u = X_{t+u} - X_t$ for every time u. Since $\hat{X}$ is a Lévy process, what we have just shown applies to $\hat{\mathcal{G}}$ and we have $\hat{\mathcal{G}}_{0+} \subset \mathcal{N}$. It follows that

$$\bigcap_{\varepsilon>0} \bar{\mathcal{G}}_{t+\varepsilon} = \bigcap_{\varepsilon} (\mathcal{G}_t \vee \hat{\mathcal{G}}_\varepsilon \vee \mathcal{N}) = \mathcal{G}_t \vee \mathcal{N} \vee \hat{\mathcal{G}}_{0+} = \mathcal{G}_t \vee \mathcal{N} = \bar{\mathcal{G}}_t$$

because $\bar{\mathcal{G}}_{t+\varepsilon} = \mathcal{G}_{t+\varepsilon} \vee \mathcal{N}$ and $\mathcal{G}_{t+\varepsilon} = \mathcal{G}_t \vee \hat{\mathcal{G}}_\varepsilon$. In words, $\bar{\mathcal{G}}$ is right-continuous as claimed.

Exercises

3.24 *Processes of canonical type.* The aim is to introduce the probability law of a Lévy process in a concrete fashion. As a byproduct, this will show that every Lévy process is equivalent to a Lévy process of the type in Definition 3.3. *Setup.*

i) Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, X a Lévy process with state space $\mathbb{R}^d$ in the sense of Definition 1.1, and $\mathcal{G}$ the filtration generated by X.

ii) Let W be the collection of all mappings $w : \mathbb{R}_+ \mapsto \mathbb{R}^d$ with $w(0) = 0$. Let Y_t be the coordinate mapping with $Y_t(w) = w(t)$, and let $\theta_t : W \mapsto W$ be defined by $\theta_t w(u) = w(t+u) - w(t)$. Define $\mathcal{K}_t$ to be the σ-algebra on W generated by $Y_s, s \le t$, and let $\mathcal{K} = (\mathcal{K}_t)$.

iii) Define the transformation $\varphi : \Omega \mapsto W$ by letting $\varphi\omega$ to be the path $X(\omega) : t \mapsto X_t(\omega)$.

a) Note that $Y_t \circ \varphi = X_t$ for every t. Use this to show that φ is measurable with respect to $\mathcal{G}_t$ and $\mathcal{K}_t$ for every t, and with respect to $\mathcal{G}_\infty$ and $\mathcal{K}_\infty$, and therefore with respect to $\mathcal{H}$ and $\mathcal{K}_\infty$.

b) It follows that $\mathbb{Q} = \mathbb{P} \circ \varphi^{-1}$ is a probability measure on $(W, \mathcal{K}_\infty)$. This $\mathbb{Q}$ is the distribution of X.

c) Show that, over the probability space $(W, \mathcal{K}_\infty, \mathbb{Q})$, the process $Y = (Y_t)$ is a Lévy process, in the sense of Definition 1.1, with respect to its own filtration $\mathcal{K}$.

d) Let $\bar{\mathcal{K}}$ be the augmentation of $\mathcal{K}$ in the completion $(W, \bar{\mathcal{K}}_\infty, \bar{\mathbb{Q}})$ of $(W, \mathcal{K}_\infty, \mathbb{Q})$. Show that

$$\mathscr{B} = (W, \bar{\mathcal{K}}_\infty, \bar{\mathcal{K}}, \theta, \bar{\mathbb{Q}})$$

is a stochastic base in the sense of Definition 3.1. Show that $Y = (Y_t)$ is a Lévy process over $\mathscr{B}$ in the sense of Definition 3.3.

4 Characterizations for Wiener and Poisson

Throughout this section $\mathscr{B}$ is the stochastic base introduced in Definition 3.1, and X is a Lévy process over $\mathscr{B}$ as in Definition 3.3. The aim is to characterize the three basic processes in qualitative terms: Poisson, the archetypical pure-jump process; Wiener, the continuous process *par excellence*; and compound Poisson process, whose paths are step functions.

Poisson processes

Recall that a Poisson process is a Lévy process whose increments are Poisson distributed. A number of characterizations were listed in Theorems VI.5.5 and VI.5.9. The following is a small addition.

4.1 Theorem. *The Lévy process X is Poisson if and only if it is a counting process.*

Proof. Sufficiency was shown in Theorem VI.5.9. To show the necessity, suppose that every increment of the Lévy process X has a Poisson distribution, $X_{t+u} - X_t$ with mean cu, where c is a constant in $\mathbb{R}_+$. Then, every increment takes values in $\mathbb{N}$ almost surely, which implies that almost every path is an increasing step function taking values in $\mathbb{N}$. To show that X is a counting process, there remains to show that every jump is of size 1 almost surely.

Fix t. Let H_t be the event that there is a jump of size 2 or more during $[0, t]$. Subdivide the interval $[0, t]$ into n intervals of equal length. The event H_t implies that, of the increments over those n subintervals, at least one increment is equal to 2 or more. Thus, by Boole's inequality,

$$\mathbb{P}(H_t) \le n\,(1 - e^{-ct/n} - (ct/n)\,e^{-ct/n})$$

since each increment has the Poisson distribution with mean ct/n. Letting $n \to \infty$ we see that $\mathbb{P}(H_t) = 0$. Taking the union of H_t over $t = t_1, t_2, \cdots$ for some sequence (t_n) increasing to infinity, we see that, almost surely, no jump exceeds 1 in size. □

In the preceding proof, the sufficiency was by appealing to Theorem VI.5.9. The bare-hands proof of the latter theorem can be replaced with the following: Suppose that the Lévy process X is a counting process. Then, all jumps are bounded in size by 1, and Proposition 3.15 shows that $\mathbb{E}X_t < \infty$. Thus, the stationarity of the increments implies that $\mathbb{E}X_t = ct$ for some finite constant c, and the Lévy property implies that $M_t = X_t - ct$ defines a martingale M. Thus, by Theorem V.6.13, the process X is Poisson.

Wiener and continuous Lévy processes

According to the earlier definition, the Lévy process X is a Wiener process if it is continuous and X_t has the Gaussian distribution with mean 0 and variance t. The following shows that continuity is enough.

4.2 THEOREM. *Suppose that the Lévy process X is continuous and has state space $\mathbb{R}$. Then, it has the form*

$$X_t = bt + c\,W_t, \quad t \in \mathbb{R}_+,$$

where b and c are constants in $\mathbb{R}$, and W is a Wiener process over the base $\mathscr{B}$.

Proof. Assume X is such. Proposition 3.15 shows that X_t has finite mean and variance; there exist constants b and c such that $\mathbb{E}X_t = bt$ and $\text{Var}X_t = c^2 t$. If $c = 0$, then there is nothing left to prove. Assuming that $c \neq 0$, define

$$W_t = (X_t - bt)/c, \quad t \in \mathbb{R}_+.$$

Then W is a continuous Lévy process over the base $\mathscr{B}$, and the Lévy property can be used to show that W is a continuous martingale, and so is $(W_t^2 - t)_{t\in\mathbb{R}_+}$, both with respect to the filtration $\mathcal{F}$. It follows from Proposition V.6.21 that W is Wiener. □

The preceding proof is via Proposition V.6.21, and the latter's proof is long and difficult. It is possible to give a direct proof using the Lévy property more fully: Start with W being a Lévy process with $\mathbb{E}\,W_t = 0$ and $\text{Var}\,W_t = t$. For each integer $n \geq 1$,

$$W_1 = Y_{n,1} + \cdots + Y_{n,n}$$

where $Y_{n,j}$ is the increment of W over the interval from $(j-1)/n$ to j/n. Since W is Lévy, those increments are independent and identically distributed with mean 0 and variance $1/n$. Now the proof of the classical central limit theorem (III.8.1) can be adapted to show that W_1 has the standard Gaussian distribution. Thus, W is a Wiener process.

Continuous Lévy processes in $\mathbb{R}^d$

If $W^1, \ldots, W^d$ are independent Wiener processes, then $W = (W^1, \ldots, W^d)$ is a d-dimensional Wiener process. Obviously, it is a continuous Lévy process in $\mathbb{R}^d$. We now show that, conversely, every continuous Lévy process in $\mathbb{R}^d$ is obtained from such a Wiener process by a linear transformation plus some drift.

4.3 THEOREM. *Suppose that the Lévy process X in $\mathbb{R}^d$ is continuous. Then,*

$$X_t = bt + c\,W_t, \quad t \in \mathbb{R}_+,$$

for some vector b in $\mathbb{R}^d$, some $d \times d'$ matrix c, and a d'-dimensional Wiener process W over the base $\mathscr{B}$.

Remark. Let v be the covariance matrix for X_1. Then, d' is the rank of v (of course $d' \le d$), and $v = cc^T$ with c^T denoting the transpose of c.

Proof. Suppose X continuous. For r in $\mathbb{R}^d$, consider the linear combination $r \cdot X_t$ of the coordinates of X_t. The process $r \cdot X$ is a continuous Lévy process in $\mathbb{R}$. It follows from Theorem 4.2 that $r \cdot X_1$ has a one-dimensional Gaussian distribution, and this is true for every r in $\mathbb{R}^d$. Thus, X_1 has a d-dimensional Gaussian distribution with some mean vector b in $\mathbb{R}^d$ and some $d \times d$ matrix v of covariances.

The matrix v is symmetric and positive definite (that is, $v = v^T$ and $r \cdot vr \ge 0$ for every r in $\mathbb{R}^d$). Let d' be its rank. There exists some $d \times d'$ matrix c of rank d' such that $v = cc^T$, that is,

4.4 $$v_{ij} = \sum_{k=1}^{d'} c_{ik} c_{jk}, \quad i, j = 1, 2, \cdots, d.$$

We define a matrix a as follows. If $d' = d$, put $a = c$. If $d' < d$, the matrix c has exactly d' linearly independent rows, which we may assume are the rows $1, 2, \ldots, d'$ by re-labeling the coordinates of X ; we let a be the $d' \times d'$ matrix formed by those first d' rows of c. Obviously a is invertible; let $\hat{a}$ be its inverse. Define, for $i = 1, \ldots, d'$,

4.5 $$W_t^i = \sum_{k=1}^{d'} \hat{a}_{ik} \, (X_t^k - b_k t), \quad t \in \mathbb{R}_+.$$

It is clear that $W = (W^1, \ldots, W^{d'})$ is a continuous Lévy process in $\mathbb{R}^{d'}$, and W_t has the d'-dimensional Gaussian distribution with mean vector 0 and covariances

$$\begin{aligned} \mathbb{E}\, W_t^i \, W_t^j &= \sum_{m=1}^{d'} \hat{a}_{im} \sum_{n=1}^{d'} \hat{a}_{jn} \, \mathbb{E}\, (X_t^m - b_m t)(X_t^n - b_n t) \\ &= \sum_{m=1}^{d'} \sum_{n=1}^{d'} \hat{a}_{im} \hat{a}_{jn} v_{mn} t \\ &= \sum_{m=1}^{d'} \sum_{n=1}^{d'} \hat{a}_{im} \hat{a}_{jn} \sum_{k=1}^{d'} a_{mk} a_{nk} t = \delta_{ij} t, \end{aligned}$$

where the third equality follows from 4.4 since $c_{mk} = a_{mk}$ and $c_{nk} = a_{nk}$ for $m, n \le d'$. This shows that W is d'-dimensional Wiener. Reversing 4.4 and 4.5 shows that X is as claimed. □

Compound Poisson processes

We adopt the construction in Example 1.2c as the definition for compound Poisson processes. Several other constructions were mentioned previously in this chapter and the last. The following characterization theorem summarizes the previous results and extends them onto the modern setting. This is basic.

4.6 THEOREM. *The Lévy process X over the base $\mathscr{B}$ is a compound Poisson process if and only if one (and therefore all) of the following statements holds.*

a) Almost every path of X is a step function.

b) There is a Poisson process (N_t) over $\mathscr{B}$ and, independent of it, an independency (Y_n) of identically distributed $\mathbb{R}^d$-valued variables such that

4.7 $$X_t = \sum_{n=1}^{\infty} Y_n \, 1_{\{n \le N_t\}}, \quad t \in \mathbb{R}_+.$$

c) There is a Poisson random measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ whose mean has the form $\text{Leb} \times \lambda$ *with some finite measure λ on $\mathbb{R}^d$ such that*

4.8 $$X_t = \int_{[0,t]\times\mathbb{R}^d} M(ds, dx)\, x, \quad t \in \mathbb{R}_+.$$

4.9 REMARKS. a) The proof will show that the Poisson random measure M is adapted to the filtration $\mathcal{F}$ and is homogeneous relative to the shifts θ_t, that is, $\omega \mapsto M(\omega, A)$ is in $\mathcal{F}_t$ for every Borel subset A of $[0,t] \times \mathbb{R}^d$, and

$$M(\theta_t\omega, B) = M(\omega, B_t)$$

for every Borel subset B of $\mathbb{R}_+ \times \mathbb{R}^d$, where B_t consists of the points $(t+u, x)$ with (u, x) in B.

b) The connection between 4.7 and 4.8 is as follows. Let $c = \lambda(\mathbb{R}^d) < \infty$ and put μ for the distribution $(1/c)\,\lambda$. Finiteness of c implies that the atoms of M can be labeled as points (T_n, Y_n) so that $0 < T_1 < T_2 < \cdots$ almost surely. The T_n are the successive jump times of X and form the Poisson process N, and the Y_n are the variables appearing in 4.7. The jump rate for N is c, and the distribution common to Y_n is μ.

Proof. $(a) \Rightarrow (b)$. Assume (a). Let N_t be the number of jumps of X over $(0,t]$. Since X is adapted to $\mathcal{F}$, so is N ; since X is a step process, N is a counting process; and since $N_u \circ \theta_t$ is the number of jumps of $X \circ \theta_t$ over $(0, u]$, we have the additivity of N with respect to the shifts. Moreover, $X \circ \theta_t$ is independent of $\mathcal{F}_t$ and has the same law as X (this is the Markov property, Theorem 3.5); thus, $N_u \circ \theta_t$ is independent of $\mathcal{F}_t$ and has the same distribution as N_u. In summary, N is a Lévy process over the base $\mathscr{B}$ and is a counting process. By Theorem 4.1, it must be a Poisson process with some rate c.

Let Y_n be the size of the jump by X at T_n. Then, 4.7 is obvious, and there remains to show that (Y_n) is independent of (T_n) and is an independency of variables having the same distribution, say, μ on $\mathbb{R}^d$. We start by showing that $R = T_1$ and $Z = Y_1$ are independent and a bit more. The distribution of

Z is μ, and the distribution of R is exponential with parameter c ; the latter is because N is Poisson with rate c. For t in $\mathbb{R}_+$ and Borel subset B of $\mathbb{R}^d$, we now show that

4.10 $$\mathbb{P}\{\ R > t,\ Z \in B\ \} \ = \ e^{-ct}\ \mu(B).$$

Note that, if $R(\omega) > t$, then $Z(\omega) \ = \ Z(\theta_t\omega)$. So, by the Markov property for X, Theorem 3.5, the left side of 4.10 is equal to

$$\mathbb{P}\{\ R > t,\ Z\circ\theta_t \in B\ \} = \mathbb{E}\ 1_{\{R>t\}}\ \mathbb{E}_t\ 1_B\circ Z\circ\theta_t$$
$$= \mathbb{E}\ 1_{\{R>t\}}\ \mathbb{E}\ 1_B\circ Z \ = \ e^{-ct}\ \mu(B)$$

as claimed in 4.10. Next, for $n \geq 1$, we note that

$$T_{n+1} - T_n \ = \ R\circ\theta_{T_n}, \quad Y_{n+1} \ = \ Z\circ\theta_{T_n},$$

and use the strong Markov property proved in Theorem 3.10 at the almost surely finite stopping times T_n . We get that

4.11 $$\mathbb{P}\{\ T_{n+1} - T_n > t\,,\ Y_{n+1} \in B \mid \mathcal{F}_{T_n}\ \}$$
$$= \mathbb{P}\{\ R\circ\theta_{T_n} > t\,,\ Z\circ\theta_{T_n} \in B \mid \mathcal{F}_{T_n}\ \} = \mathbb{P}\{\ R > t\,,\ Z \in B\ \}.$$

Putting 4.10 and 4.11 together shows that the sequences (T_n) and (Y_n) are independent, and the Y_n are independent and have the distribution μ. This completes the proof that $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$. Assume (b) and let (T_n) be the sequence of successive jump times of N. It follows from Corollary VI.3.5 that the pairs (T_n, Y_n) form a Poisson random measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean $c\text{Leb} \times \mu = \text{Leb} \times \lambda$, where $\lambda = c\mu$ is a finite measure on $\mathbb{R}^d$. It is obvious that, then, 4.7 and 4.8 are the same equation served up in differing notations.

$(c) \Rightarrow (a)$. Assume (c) ; then 4.8 shows that X is a Lévy process; and the paths are almost surely step functions, because the measure λ is finite. □

The best qualitative definition for compound Poisson processes is that they are Lévy processes whose paths are step functions. The following provides another equivalent condition for it. As before, X is a Lévy process in $\mathbb{R}^d$ over the base $\mathscr{B}$. We leave its proof to Exercise 4.14.

4.12 Proposition. *Almost every path of X is a step function if and only if the probability is strictly positive that*

$$R \ = \ \inf\{\ t > 0\ :\ X_t \neq 0\ \}$$

is strictly positive. Moreover, then $0 < R < \infty$ almost surely and has the exponential distribution with some parameter c in $(0, \infty)$.

4.13 REMARK. Obviously, R is a stopping time of $(\mathcal{G}_{t+})$. Thus, the event $\{R > 0\}$ belongs to $\mathcal{G}_{0+}$ and, by Blumenthal's zero-one law (Corollary 3.21), its probability is either 0 or 1. In other words, either $R = 0$ almost surely or $R > 0$ almost surely; in the former case the point 0 of $\mathbb{R}^d$ is said to be *instantaneous*, and in the latter case, *holding*. The preceding can now be restated: X is a compound Poisson process if and only if the point 0 is a holding point, in which case the holding time has an exponential distribution.

Exercise

4.14 *Proof of Proposition* 4.12. If the paths are step functions, then R is the time of first jump and is necessarily strictly positive. The following are steps leading to the sufficiency part, assuming that $R > 0$ almost surely in view of Remark 4.13.

a) Show that, if $R(\omega) > t$, then $R(\omega) = t + R(\theta_t\omega)$. Use the Markov property to show that the function $f(t) = \mathbb{P}\{R > t\}, t \in \mathbb{R}_+$, satisfies $f(t+u) = f(t)f(u)$.

b) Note that f is right-continuous and bounded; the case $f = 1$ is excluded by the standing assumption that X is not degenerate; show that the case $f = 0$ is also excluded. Thus, $f(t) = e^{-ct}$ for some constant c in $(0, \infty)$; in other words, $0 < R < \infty$ almost surely and R is exponential.

c) Show that, on the event $H = \{X_{R-} = X_R\}$, we have $X_{R-} = X_R = 0$ and $R\circ\theta_R = 0$. Use the strong Markov property to show that

$$\mathbb{P}(H) = \mathbb{P}(\, H \cap \{R\circ\theta_R = 0\}\,) = \mathbb{P}(H)\mathbb{P}\{R = 0\} = 0.$$

Hence, R is a jump time almost surely, obviously the first.

d) Define $T_1 = R$, and recursively put $T_{n+1} = T_n + R\circ\theta_{T_n}$. Show that the T_n form a Poisson process. Conclude that X is a step process.

5 Itô-Lévy Decomposition

This is to show the exact converse to Theorem 1.29: every Lévy process has the form given there. Throughout, $\mathscr{B} = (\Omega, \mathcal{H}, \mathcal{F}, \theta, \mathbb{P})$ is a stochastic base, and X is a stochastic process over it with state space $\mathbb{R}^d$. A random measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ is said to be *Poisson over* $\mathscr{B}$ with Lévy measure λ if

5.1 a) $M(A)$ is in $\mathcal{F}_t$ for every Borel subset A of $[0, t] \times \mathbb{R}^d$,

b) $M(\theta_t\omega, B) = M(\omega, B_t)$ for every ω and t and Borel subset B of $\mathbb{R}_+ \times \mathbb{R}^d$, where $B_t = \{(t+u, x) : (u, x) \in B\}$, and

c) M is Poisson with mean $\text{Leb} \times \lambda$, and λ is a Lévy measure, that is, $\lambda\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \lambda(dx)\; (|x|^2 \wedge 1) \;<\; \infty.$$

With this preparation, we list the main theorem of this section next. It is called the *Itô-Lévy decomposition theorem.* Its sufficiency part is Theorem 1.29. The necessity part will be proved in a series of propositions of interesting technical merit. Recall Notation 1.27 and its meaning.

5.2 THEOREM. *The process X is a Lévy process over $\mathscr{B}$ if and only if, for every t in $\mathbb{R}_+$,*

$$X_t \;=\; bt + cW_t + \int_{[0,t]\times\mathbb{B}} [M(ds,dx) - ds\lambda(dx)]x + \int_{[0,t]\times\mathbb{B}^c} M(ds,dx)x$$

for some vector b in $\mathbb{R}^d$, some $d \times d'$ matrix c, some d'-dimensional Wiener process W over $\mathscr{B}$, and, independent of W, a random measure M on $\mathbb{R}_+ \times \mathbb{R}^d$ that is Poisson over $\mathscr{B}$ with some Lévy measure λ.

5.3 COROLLARY. *If X is a Lévy process, then its characteristic exponent is*

$$\psi(r) \;=\; ib\cdot r - \tfrac{1}{2} r\cdot vr + \int_{\mathbb{R}^d} \lambda(dx)\ (e^{ir\cdot x} - 1 - ir\cdot x\ 1_{\mathbb{B}}(x)), \quad r \in \mathbb{R}^d$$

for some b in $\mathbb{R}^d$, some $d\times d$ matrix v that is symmetric and positive definite, and some measure λ on $\mathbb{R}^d$ that is a Lévy measure. Conversely, if (b, v, λ) is such a triplet, there is a Lévy process whose characteristic exponent is ψ above.

The preceding corollary is immediate from Theorems 5.2 and 1.29. This is basically the *Lévy-Khinchine formula* stated in stochastic terms. Obviously, b and λ are as in Theorem 5.2, and $v = cc^T$.

5.4 REMARKS. a) *Characteristic triplet.* This refers to (b, v, λ) ; it defines the law of X by defining the characteristic exponent in the canonical form given in the preceding corollary.

b) *Semimartingaleness.* It follows from the theorem above that every Lévy process is a semimartingale; see Remark 1.34c also.

c) *Special cases.* Characterizations for Lévy processes with special properties can be deduced from the theorem above and discussion in Section 1. See Theorems 4.2 and 4.3 for X continuous, Theorem 4.6 for X step process, Theorem 1.12 for X pure-jump.

d) *Increasing processes.* Suppose that the state space is $\mathbb{R}$ and X is increasing. Then, in the theorem above and its corollary, we must have $v = 0, \lambda(-\infty, 0] = 0$, and, further, λ must satisfy 1.13. It is usual to represent such X in the form

$$X_t \;=\; at + \int_{[0,t]\times(0,\infty)} M(ds,dx)\ x, \quad t \in \mathbb{R}_+.$$

The corresponding characteristic triplet is $(b, 0, \lambda)$ with λ as noted and

$$b \;=\; a + \int_{(0,1]} \lambda(dx)x.$$

Jumps exceeding ε in magnitude

Recall the notation $\Delta X_t = X_t - X_{t-}$ and also Remark 1.9 on the sparseness of jumps exceeding $\varepsilon > 0$ in magnitude.

5.5 PROPOSITION. *Suppose that X is a Lévy process over $\mathscr{B}$. For $\varepsilon > 0$, let*

$$X_t^\varepsilon = \sum_{s \le t} \Delta X_s \, 1_{\{|\Delta X_s| > \varepsilon\}}, \quad t \in \mathbb{R}_+.$$

Then, X^ε is a compound Poisson process over $\mathscr{B}$.

Proof. a) It is clear that X^ε is adapted to $\mathcal{F}$ and is additive with respect to shifts. Since X_u^ε is $\mathcal{G}_\infty$-measurable, it follows from the Markov property (Theorem 3.5) that $X_u^\varepsilon \circ \theta_t$ is independent of $\mathcal{F}_t$ and has the same distribution as X_u^ε. Thus, X^ε is a Lévy process over the base $\mathscr{B}$.

b) Since the paths of X are right-continuous and left-limited with $X_0 = 0$, the paths of X^ε are step functions; see Remark 1.9. It follows from the characterization theorem 4.6 that X^ε is a compound Poisson process. □

Some independence

This is to show that X^ε above is independent of $X - X^\varepsilon$. Generally, proofs of independence are easy consequences of assumptions made beforehand. This is a rare case where the proof requires serious work.

5.6 PROPOSITION. *Suppose that X is a Lévy process over $\mathscr{B}$. For fixed $\varepsilon > 0$, let X^ε be as defined in the preceding proposition. Then, X^ε and $X - X^\varepsilon$ are independent Lévy processes over $\mathscr{B}$.*

Proof. a) The preceding proposition has shown that X^ε is a Lévy process over $\mathscr{B}$ and more; part (a) of its proof is easily adapted to show that $X - X^\varepsilon$ is a Lévy process over $\mathscr{B}$, and that the pair $(X^\varepsilon, X - X^\varepsilon)$ is a Lévy process over $\mathscr{B}$ with state space $\mathbb{R}^d \times \mathbb{R}^d$. To show that X^ε and $X - X^\varepsilon$ are independent, then, is reduced to showing that X_t^ε and $X_t - X_t^\varepsilon$ are independent for every t. To show the latter, it is enough to show that, for every q and r in $\mathbb{R}^d$,

5.7 $$\mathbb{E} \exp\left[iq \cdot X_t^\varepsilon + ir \cdot (X_t - X_t^\varepsilon) \right] = \left(\mathbb{E} \exp iq \cdot X_t^\varepsilon \right) \left(\mathbb{E} \exp ir \cdot (X_t - X_t^\varepsilon) \right).$$

b) Fix q and r in $\mathbb{R}^d$. Recall (see 1.3) that the characteristic functions on the right side of 5.7 have the form $e^{t\varphi}$ and $e^{t\psi}$ for some complex numbers φ and ψ depending on q and r respectively. With these notations, the Lévy property for X^ε and, separately, for $X - X^\varepsilon$ shows that

$$L_t = 1 - \exp\left(iq \cdot X_t^\varepsilon - t\varphi \right), \quad M_t = 1 - \exp\left(ir \cdot (X_t - X_t^\varepsilon) - t\psi \right)$$

define complex-valued $\mathcal{F}$-martingales. We shall show that

5.8 $$\mathbb{E}\, L_t M_t = 0, \quad t \in \mathbb{R}_+,$$

thus showing 5.7 and completing the proof.

c) Fix $t > 0$. Fix $n \geq 1$. Let $\mathcal{D}$ be the subdivision of $(0, t]$ into n equi-length intervals of the type $(,]$. For each interval A in $\mathcal{D}$, if $A = (u, v]$, we put $L_A = L_v - L_u$ and $M_A = M_v - M_u$. With this notation, since $L_0 = M_0 = 0$,

$$L_t M_t \; = \; \sum_{A \in \mathcal{D}} L_A \sum_{B \in \mathcal{D}} M_B.$$

Take expectations on both sides. Note that $\mathbb{E}\, L_A M_B = 0$ if A and B are disjoint; this is by the martingale property for L and M. It follows that

5.9 $$\mathbb{E}\, L_t M_t \; = \; \mathbb{E}\, R_n, \quad \text{where} \quad R_n \; = \; \sum_{A \in \mathcal{D}} L_A M_A.$$

d) We show now that $|R_n| \leq R$ where R is an integrable random variable free of n. First, observe that

5.10 $$|M_A| \; \leq \; 2 \sup_{s \leq t} |M_s|, \quad A \in \mathcal{D},$$

and that, by Doob's norm inequality (V.3.26),

5.11 $$\mathbb{E} \sup_{s \leq t} |M_s|^2 \; \leq \; 4\, \mathbb{E}\, |M_t|^2 \; < \infty.$$

Next, by the definition of L, with $X_A^\varepsilon = X_v^\varepsilon - X_u^\varepsilon$ for $A = (u, v]$,

$$\begin{aligned} |L_A| &= |e^{\,iq \cdot X_u^\varepsilon}\, e^{\,-u\varphi} - e^{\,iq \cdot X_v^\varepsilon}\, e^{\,-v\varphi}| \\ &= |e^{\,-u\varphi} - e^{\,-v\varphi} + e^{\,-v\varphi}\,(1 - e^{\,iq \cdot X_A^\varepsilon})| \\ &\leq |e^{\,-u\varphi} - e^{\,-v\varphi}| + |e^{\,-v\varphi}| \cdot |1 - e^{\,iq \cdot X_A^\varepsilon}| \\ &\leq \int_A a e^{as}\, ds \; + \; 2\, e^{at}\, 1_{\{X_A^\varepsilon \neq 0\}} \end{aligned}$$

where we put $a = |\varphi|$ and observed that $|1 - e^{\,iq \cdot x}|$ is equal to 0 if $x = 0$ and is bounded by 2 if $x \neq 0$. Thus,

5.12 $$\sum_{A \in \mathcal{D}} |L_A| \; \leq \; e^{at} \; + \; 2\, e^{at} \sum_{A \in \mathcal{D}} 1_{\{X_A^\varepsilon \neq 0\}} \; \leq \; e^{at}\,(1 + 2\,K_t)$$

where K_t is the number of jumps X^ε has during $(0, t]$. Since K_t has the Poisson distribution with some mean ct, and therefore variance ct,

5.13 $$\mathbb{E}\,(1 + 2K_t)^2 \; < \; \infty.$$

It follows from 5.10 and 5.12 that

$$|R_n| \; \leq \; 2 \sup_{s \leq t} |M_s| \sum_{A \in \mathcal{D}} |L_A| \; \leq \; 2\, e^{at}\, (\sup_{s \leq t} |M_s|)(1 + 2K_t) \; = \; R$$

where R is integrable in view of 5.11 and 5.13.

e) Finally, we let $n \to \infty$ in 5.9. Since X^ε is a step process, the martingale L has finitely many jumps during $[0,t]$ and is smooth between the jumps. Thus,

$$\lim R_n = \sum_{s \le t} (L_s - L_{s-})(M_s - M_{s-}) = 0,$$

where the sum is over the finitely many jump times s of X^ε and the last equality is because $M_s - M_{s-} = 0$ at those times s since $X - X^\varepsilon$ has no jumps in common with X^ε. In view of part (d) above, the dominated convergence theorem applies, and we have

$$\lim \mathbb{E}\, R_n = 0.$$

This proves 5.8 via 5.9 and completes the proof. □

Jump measure

This is to show that jumps of X are governed by a Poisson random measure. We start by defining the random measure, to be called the *jump measure* of X. Recall the notation ΔX_t and also the set D_ω of discontinuities of the path $X(\omega)$. Let $\omega \in \Omega$, and A Borel subset of $\mathbb{R}_+ \times \mathbb{R}^d$; if the path $X(\omega)$ is right-continuous, left-limited, and $X_0(\omega) = 0$, we put

5.14 $$M(\omega, A) = \sum_{t \in D_\omega} 1_A(t, \Delta X_t(\omega));$$

for all other ω, put $M(\omega, A) = 0$. For each ω, this defines a counting measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

5.15 Proposition. *Suppose that X is a Lévy process over $\mathscr{B}$. Then, M is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean* Leb $\times\, \lambda$, *where λ is a Lévy measure on $\mathbb{R}^d$; that is, 5.1 holds.*

Proof. It follows from the definition of M and augmentedness of $\mathcal{F}$ that the condition 5.1a holds. Similarly, 5.1b is satisfied by the additivity of X. There remains to show 5.1c.

For $\varepsilon > 0$, let M_ε be the trace of M on $\mathbb{R}_+ \times \varepsilon\mathbb{B}^c$, where $\varepsilon\mathbb{B}^c$ is the set of all εx with x outside the unit ball $\mathbb{B}$. Comparing 5.14 with the definition of X^ε in Proposition 5.5, we see that M_ε is the jump measure of X^ε. Since X^ε is compound Poisson, it follows from Theorem 4.6 that M_ε is a Poisson random measure with mean $\mu_\varepsilon = \text{Leb} \times \lambda_\varepsilon$, where λ_ε is a finite measure on $\mathbb{R}^d$.

It is obvious that λ_ε puts all its mass outside $\varepsilon\mathbb{B}$. Define the measure λ on $\mathbb{R}^d$ by letting, for g positive Borel, λg be the increasing limit of $\lambda_\varepsilon g$ as $\varepsilon > 0$ decreases to 0. Put $\mu = \text{Leb} \times \lambda$. It is obvious that $\lambda\{0\} = 0$, and that μ_ε is the trace of μ on $\mathbb{R}_+ \times \varepsilon\mathbb{B}^c$.

Let f be a positive Borel function on $\mathbb{R}_+ \times \mathbb{R}^d$. Then, $\omega \mapsto M_\varepsilon f(\omega)$ is a random variable for each ε, and $M_\varepsilon f(\omega)$ increases to $Mf(\omega)$ as $\varepsilon \to 0$. Thus, Mf is a random variable, and

$$\mathbb{E}\, e^{-Mf} \;=\; \lim_{\varepsilon\downarrow 0}\, \mathbb{E}\, e^{-M_\varepsilon f} \;=\; \lim_{\varepsilon\downarrow 0}\, \exp_-\; \mu_\varepsilon(1-e^{-f}) \;=\; \exp_-\; \mu(1-e^{-f}).$$

Thus, M is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with mean $\mu = \text{Leb} \times \lambda$. The proof is complete via the next lemma. □

5.16 LEMMA. *The measure λ is a Lévy measure.*

Proof. We have noted that $\lambda\{0\} = 0$ by definition. Recall that λ_ε is the Lévy measure for the compound Poisson process X^ε. In particular, then, λ_ε is finite for $\varepsilon = 1$. Thus, to show that the λ-integral of $x \mapsto |x|^2 \wedge 1$ is finite, it is sufficient to show that

5.17
$$\int_{\mathbb{B}} \lambda(dx)\; |x|^2 \;<\; \infty.$$

By Proposition 5.6 above, X^ε and $X - X^\varepsilon$ are independent. Thus,

$$\begin{aligned}|\mathbb{E}\; e^{ir\cdot X_t}| &= |\mathbb{E}\; e^{ir\cdot(X_t - X_t^\varepsilon)}\; \mathbb{E}\; e^{ir\cdot X_t^\varepsilon}| \le |\mathbb{E}\; e^{ir\cdot X_t^\varepsilon}| \\ &\le |\exp\; t\int_{\varepsilon\,\mathbb{B}^c} \lambda(dx)\; (e^{ir\cdot x} - 1)| \\ &\le \exp_-\; t\int_{\varepsilon\,\mathbb{B}^c} \lambda(dx)\; (1-\cos\, r\cdot x) \;\le\; \exp_-\; \frac{t}{4}\int_{\mathbb{B}\setminus\varepsilon\mathbb{B}} \lambda(dx)\; |r\cdot x|^2 \qquad (5.18)\end{aligned}$$

for r in $\mathbb{B}$, where the last step used the observation that $\varepsilon\,\mathbb{B}^c = \mathbb{R}^d \setminus \varepsilon\,\mathbb{B} \supset \mathbb{B}\setminus\varepsilon\mathbb{B}$ and $1 - \cos\, u \;\ge\; u^2/4$ for $|u| \le 1$. Since the left-most member is free of ε, we let $\varepsilon \to 0$ in the right-most member to conclude that

$$\int_{\mathbb{B}} \lambda(dx)\; |r\cdot x|^2 \;<\; \infty$$

for every r in $\mathbb{B}$; this implies 5.17 as needed. □

Proof of the decomposition theorem 5.2

Suppose that X is a Lévy process. Recall Proposition 5.15 about the jump measure M. In terms of it, $X^e = X^\varepsilon$ with $\varepsilon = 1$ is given by

5.19
$$X_t^e \;=\; \int_{[0,t]\times\mathbb{B}^c} M(ds, dx)\; x, \quad t \in \mathbb{R}_+.$$

Since λ is a Lévy measure, Theorem 1.23 (and using Notation 1.27) yields a Lévy process X^d over $\mathscr{B}$ through

5.20
$$X_t^d \;=\; \int_{[0,t]\times\mathbb{B}} [\, M(ds, dx) - ds\; \lambda(dx)\,]\, x, \quad t \in \mathbb{R}_+.$$

Poisson nature of M implies that X^d and X^e are independent. The definition of M shows that $X - X^d - X^e$ has almost surely continuous paths, and the Markov property for X shows that the latter is a Lévy process over $\mathscr{B}$. Thus, by Theorem 4.2,

5.21 $$X_t - X_t^d - X_t^e \;=\; bt \;+\; c\,W_t, \quad t \in \mathbb{R},$$

where b, c, W are as claimed in Theorem 5.2. Putting 5.19, 5.20, and 5.21 together yields the decomposition wanted, and the main claim of the theorem is proved, except for the independence of W and M.

For $\varepsilon > 0$, the process X^ε of 5.5 is determined by M_ε, the trace of M on $\mathbb{R}_+ \times \varepsilon\mathbb{B}^c$, whereas W is determined by $X - X^\varepsilon$. Independence of X^ε and $X - X^\varepsilon$ proved in Proposition 5.6 implies that W and M_ε are independent. This is true for every $\varepsilon > 0$, and $M_\varepsilon f$ increases to Mf for every positive Borel f on $\mathbb{R}_+ \times \mathbb{R}^d$. It follows that W and M are independent.

6 Subordination

This is about time changes using increasing Lévy processes as clocks. In deterministic terms, the operation is as follows. Imagine a clock, something like the odometer of a car; suppose that, when the clock points to the number t, the standard time is s_t. Imagine, also, a particle whose position in $\mathbb{R}^d$ is z_s when the standard time is s. Then, when the clock shows t, the particle's position is z_{s_t}.

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space. Let $S = (S_t)$ be an increasing process. Let $Z = (Z_s)$ be a process with state space $\mathbb{R}^d$. Define

6.1 $$X_t(\omega) \;=\; Z_{S_t(\omega)}(\omega), \qquad \omega \in \Omega,\ t \in \mathbb{R}_+.$$

Then, $X = (X_t)$ is said to be obtained by *subordinating* Z to S, and S is called the *subordinator*. We write $X = Z_S$ to express 6.1.

The concept of subordination can be extended: all that is needed is that the subordinator's state space be contained in the parameter set of Z. For instance, the compound Poisson process X of Example 1.2c is obtained by subordinating

$$Z_n = Z_0 + Y_1 + \cdots + Y_n, \qquad n \in \mathbb{N},\ Z_0 = 0,$$

to the Poisson process $N = (N_t)$; this is immediate upon noting $X_t = Z_{N_t}$ is another way of expressing the sum defining X_t in 1.2c. Another such example is where $Z = (Z_n)_{n\in\mathbb{N}}$ is a Markov chain with some state space $(E, \mathcal{E})$, and $N = (N_t)$ is a Poisson process independent of Z; then, $X = Z_N$ is a Markov process (in continuous time) with state space $(E, \mathcal{E})$.

For the remainder of this section, Z and S will be independent Lévy processes. To keep the setting simple, and also because there are three processes and two time scales, we use Definition 1.1 for Lévy processes (with respect to their own filtrations).

Main results

6.2 THEOREM. *Let S be an increasing Lévy process. Let Z be a Lévy process in $\mathbb{R}^d$. Suppose that the two are independent. Then, $X = Z_S$ is a Lévy process in $\mathbb{R}^d$.*

Proof. Since S is increasing and the regularity condition 1.1a holds for S and for Z, the same condition holds for X. We now show that 1.1b holds for X with $\mathcal{F}$ as the filtration generated by X.

Fix times $0 \le t_0 < t_1 < \cdots < t_n < \infty$ and let $f_1, \cdots, f_n$ be positive Borel functions on $\mathbb{R}^d$. Conditioning on the σ-algebra $\mathcal{G}_\infty$ generated by S, using the independence of Z and S, and also the Lévy property for Z, we obtain

$$\mathbb{E} \prod_{i=1}^{n} f_i(\, X_{t_i} - X_{t_{i-1}}\,) \;=\; \mathbb{E} \prod_{i=1}^{n} g_i(\, S_{t_i} - S_{t_{i-1}}\,),$$

where $g_i(s) = \mathbb{E}\; f_i \circ Z_s$. Since S is a Lévy process, the right side is equal to

$$\prod_{i=1}^{n} \mathbb{E}\; g_i(\, S_{t_i} - S_{t_{i-1}}\,) \;=\; \prod_{i=1}^{n} \mathbb{E}\; g_i(S_{t_i - t_{i-1}}).$$

But, by the definition of g_i and the independence of S and Z,

$$\mathbb{E}\; g_i(S_t) \;=\; \mathbb{E}\; f_i(Z_{S_t}) \;=\; \mathbb{E}\; f_i(X_t).$$

We have shown that the increments of X over the intervals $(t_{i-1}, t_i]$, $1 \le i \le n$, are independent and stationary. □

In the remainder of this section, we present a number of examples of subordination and give a characterization of the law of X in terms of the laws of S and Z. For the present, we list the following useful result without proof; it is a corollary to Theorem 6.18 to be proved at the end of the section.

6.3 PROPOSITION. *Let Z, S, X be as in Theorem 6.2. Suppose that S is pure-jump with Lévy measure ν. Then, the Lévy measure of X is,*

6.4 $$\lambda(B) \;=\; \int_{(0,\infty)} \nu(ds)\; \mathbb{P}\{\; Z_s \in B\backslash\{0\}\;\}, \qquad \textit{Borel } B \subset \mathbb{R}^d.$$

The heuristic reasoning behind this is as follows. Since S is an increasing pure-jump process, S_t is equal to the sum of the lengths of the intervals $(S_{u-}, S_u]$, $u \le t$. This implies that X_t is equal to the sum of the increments of Z over those intervals. Now, $\nu(ds)$ is the rate (per unit of clock time) of S-jumps of size belonging to the small interval ds around the value s; and, given that an interval $(S_{u-}, S_u]$ has length s, the corresponding increment of Z has the same distribution as Z_s. See Theorem 6.18 for more.

Gamma subordinators

6.5 EXAMPLE. *Wiener subordinated to gamma.* In Theorem 6.2, take Z to be a Wiener process, and let S be a gamma process with shape rate a and scale parameter c; see Example 1.21. Then, given S_t, the conditional distribution of X_t is Gaussian with mean 0 and variance S_t; thus,

$$\mathbb{E}\ \exp\ irX_t \ = \ \mathbb{E}\exp_{-}\frac{1}{2}r^2 S_t \ = \ \left(\frac{c}{c+r^2/2}\right)^{at} \ = \ \left(\frac{2c}{2c+r^2}\right)^{at}.$$

Hence, in the terminology of Example 1.21, X is a symmetric gamma process with shape rate a and scale parameter $\sqrt{2c}$. As described there, X is the difference of two independent gamma processes, each with rate a and scale $\sqrt{2c}$, say $X = X^+ - X^-$. Indeed, by the reasoning following Proposition 6.3,

$$X_t^+ \ = \ \sum_{u\le t} (Z_{S_u} - Z_{S_{u-}})^+, \qquad X_t^- \ = \ \sum_{u\le t} (Z_{S_u} - Z_{S_{u-}})^-,$$

each sum being over the countable set of jump times u of S.

The Lévy measure of X is as given in 1.21 with c there replaced by $\sqrt{2c}$ here. We re-derive it to illustrate 6.4: for x in $\mathbb{R}$,

$$\begin{aligned}\lambda(dx) &= dx \int_0^\infty ds\ a\ \frac{e^{-cs}}{s}\cdot\frac{e^{-x^2/2s}}{\sqrt{2\pi s}}\\ &= dx\ a\ \frac{1}{|x|}\int_0^\infty ds\ e^{-cs}\ \frac{|x|\,e^{-x^2/2s}}{\sqrt{2\pi s^3}} \ = \ dx\ a\ \frac{e^{-|x|\sqrt{2c}}}{|x|},\end{aligned}$$

where we evaluated the last integral by recognizing it as the Laplace transform of a stable distribution with index $\frac{1}{2}$; see VI.4.10. See also Exercise 6.26 for the d-dimensional version of this example.

Stable subordinators

Subordination operation is especially interesting when the subordinator S is an increasing stable process with index a; the index must be in $(0, 1)$ since S is increasing. Exercise 6.29 is an example where Z is a gamma process. The following is about the case when Z is stable; it shows that the stability of Z is inherited by X.

6.6 PROPOSITION. *Let S, Z, X be as in Theorem 6.2. Suppose that S is an increasing a-stable process, $a \in (0, 1)$, and that Z is a b-stable process in $\mathbb{R}^d$, $b \in (0, 2]$. Then, X is a stable process in $\mathbb{R}^d$ with index ab.*

6.7 REMARK. In particular, taking Z to be a Wiener process in $\mathbb{R}^d$, the subordination yields an isotropic stable process in $\mathbb{R}^d$ with index $ab = 2a$. Every isotropic stable process X with index in $(0, 2)$ is obtained in this manner by taking a such that $2a$ is equal to the index of X.

Proof. Let S, Z, X be as assumed. Since X is Lévy by Theorem 6.2, we need to check only its stability; we need to show that X_t has the same distribution as $t^{1/ab}X_1$. Fix t. Since S is a-stable, S_t has the same distribution as $t^{1/a}S_1$, which implies that X_t has the same distribution as Z_{uS_1} with $u = t^{1/a}$. On the other hand, since Z is b-stable, Z_{us} has the same distribution as $u^{1/b}Z_s$. Thus, since S and Z are independent, Z_{uS_1} has the same distribution as $u^{1/b}Z_{S_1} = t^{1/ab}X_1$. □

6.8 EXAMPLE. *Cauchy in* $\mathbb{R}^d$. This is to illustrate the uses of the preceding theorem; we shall re-establish the results on Cauchy processes in $\mathbb{R}^d$. Let Z be a Wiener process in $\mathbb{R}^d$. Independent of it, let S be the increasing stable process of Example 2.1 with $a = 1/2$ and $c = 1/\sqrt{2\pi}$; then, the Lévy measure is $\nu(ds) = ds\, 1/\sqrt{2\pi s^3}$, and (see 2.1 and VI.4.10)

6.9 $$\mathbb{E}\, e^{-pS_t} \;=\; e^{-t\sqrt{2p}}, \qquad \mathbb{P}\{\, S_t \in ds \,\} \;=\; ds\, \frac{te^{-t^2/2s}}{\sqrt{2\pi s^3}},$$

for p and s positive. According to the preceding proposition, X is an isotropic stable process with index $ab = \frac{1}{2}\cdot 2 = 1$, a Cauchy process.

It follows from the well-remembered formula $\mathbb{E}\,\exp\, ir\cdot Z_t \;=\; \exp\_\; t|r|^2/2$ and the independence of S and Z that

6.10 $$\mathbb{E}\, e^{ir\cdot X_t} \;=\; \mathbb{E}\;\exp\_\;\frac{1}{2}\,S_t\,|r|^2 \;=\; e^{-t|r|}, \qquad r \in \mathbb{R}^d,$$

in view of the Laplace transform in 6.9. So, X is the standard Cauchy process in $\mathbb{R}^d$. The distribution of X_t can be obtained by inverting the Fourier transform in 6.10; we do it directly from the known distributions of S_t and Z_s:

6.11 $$\begin{aligned}\mathbb{P}\{\, X_t \in dx \,\} &= dx \int_0^\infty \mathbb{P}\{S_t \in ds\}\frac{e^{-|x|^2/2s}}{(2\pi s)^{d/2}} \\ &= dx\; \frac{t\,\Gamma\left(\frac{d+1}{2}\right)}{[\,\pi t^2 + \pi|x|^2\,]^{(d+1)/2}}\end{aligned}$$

here we used 6.9, replaced $2s$ with $1/u$, and noted that the integral is a constant times a gamma density.

Comparing 6.10 with 2.27, we see that the Lévy measure of X is the measure λ given by 2.24 with $c = 2\Gamma\left(\frac{d+1}{2}\right)\Big/\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)$, and σ the uniform distribution on the unit sphere. Here is a confirmation of it in Cartesian coordinates: using Proposition 6.3,

6.12 $$\lambda(dx) \;=\; dx \int_0^\infty ds\, \frac{1}{\sqrt{2\pi s^3}}\cdot\frac{e^{-|x|^2/2s}}{(2\pi s)^{d/2}} \;=\; dx\,\frac{\hat{c}}{|x|^{d+1}}, \qquad x \in \mathbb{R}^d,$$

where $\hat{c} = \Gamma\left(\frac{d+1}{2}\right)\Big/\pi^{(d+1)/2}$. For $d = 1$, this reduces to 2.15 as it should.

6.13 REMARK. The preceding exercise contains the distributions of S_t, X_t, and Z_t, namely, the strictly stable distributions with indices 1/2, 1, and 2. These three seem to be the only stable distributions that can be displayed explicitly in terms of common functions.

Transformation of laws under subordination

This is to characterize the probability law of X in terms of the laws of Z and S. To specify the laws of Z and X, we employ characteristic triplets. In general, for an arbitrary Lévy process X, we shall use the shorthand $X \sim (b, v, \lambda)$ to mean that X has (b, v, λ) as its characteristic triplet. We recall Corollary 5.3 and Remark 5.4a:

6.14 $$X \sim (b, v, \lambda) \Leftrightarrow \mathbb{E}\, e^{ir\cdot X_t} = \exp\, t\,[\,ib\cdot r - \frac{1}{2} r\cdot vr + \lambda f_r\,]$$

where $f_r(x) = e^{ir\cdot x} - 1 - ir\cdot x 1_{\mathbb{B}}(x)$. The following lemma is obvious and needs no proof.

6.15 LEMMA. *a) If X is a compound Poisson process with Lévy measure λ, then $X \sim (\lambda h, 0, \lambda)$, where $h(x) = x1_{\mathbb{B}}(x)$ for x in $\mathbb{R}^d$.*

b) If $X \sim (b, v, \lambda)$ and $X'_t = X_{at}$ for some fixed a in $\mathbb{R}_+$, then $X' \sim (ab, av, a\lambda)$.

c) If $X' \sim (b', v', \lambda')$ and $X'' \sim (b'', v'', \lambda'')$, and if X' and X'' are independent, then $X' + X'' \sim (b' + b'',\ v' + v'',\ \lambda' + \lambda'')$.

The next theorem gives a complete characterization of the law of $X = Z_S$. In preparation, we introduce

6.16 $$K(s, B) = \mathbb{P}\{\, Z_s \in B,\ Z_s \neq 0 \,\}, \qquad s \in \mathbb{R}_+, \quad \text{Borel } B \subset \mathbb{R}^d$$

and note that K is a sub-probability transition kernel. For S we use the representation

6.17 $$S_t = at + S^o_t, \qquad t \in \mathbb{R}_+,$$

with a in $\mathbb{R}_+$ and S^o pure-jump with Lévy measure ν. This is the general form of an increasing Lévy process (see Remark 5.4d).

6.18 THEOREM. *Let Z, S, X be as in Theorem 6.2. Suppose that $Z \sim (b, v, \lambda)$. Then, with $h(x) = x1_{\mathbb{B}}(x)$ for x in $\mathbb{R}_d$,*

$$X \sim (\, ab + \nu Kh,\ av,\ a\lambda + \nu K\,).$$

Proof. a) We write $X_t = Z(S_t)$ for ease of notation. The process X is Lévy by Theorem 6.2; thus, the claim here is about the characteristic function of X_t. It follows from 6.17 that, for fixed t,

$$X_t = Z(at) + [\,Z(at + S^o_t) - Z(at)\,] = X'_t + X''_t,$$

say, where X'_t and X''_t are independent, the former has the same distribution as $Z(at)$, and the latter as $Z(S^o_t)$. By part (b) of the last lemma, $X' \sim (ab, av, a\lambda)$; and by part (c), the triplet for $X_t = X'_t + X''_t$ is the sum of the triplets for X'_t and X''_t. Hence, the proof is reduced to showing that

6.19 $$X^o \;=\; Z(S^o) \;\sim\; (\;\nu K h,\; 0,\; \nu K\;).$$

b) Let S^ε be the pure-jump process where jumps are those of S with sizes exceeding $\varepsilon > 0$. Then, S^ε is a compound Poisson process, and its Lévy measure ν_ε is the trace of ν on (ε, ∞). Its successive jump times T_n form a Poisson process with rate $\nu(\varepsilon, \infty)$, and the corresponding sequence (U_n) of jump sizes is independent of (T_n) and is an independency with the distribution $\mu \;=\; \nu_\varepsilon \,/\, \nu(\varepsilon, \infty)$ for each U_n. It follows from this picture that

$$X^\varepsilon_t \;=\; Z(S^\varepsilon_t) \;=\; \sum_n \; Y_n \, 1_{\{T_n \le t\}},$$

where, with $U_0 = 0$,

$$Y_n \;=\; Z(U_0 + \cdots + U_n) \;-\; Z(U_0 + \cdots + U_{n-1}), \quad n \ge 1.$$

Note that (Y_n) is independent of (T_n) and is an independency with the common distribution

$$\mathbb{P}\{\; Y_1 \in B \;\} \;=\; \int_{\mathbb{R}_+} \mu(ds)\; \mathbb{P}\{\; Z_s \in B \;\}, \qquad \textit{Borel } B \subset \mathbb{R}^d.$$

Hence, X^ε is a compound Poisson process with Lévy measure $\nu_\varepsilon K$; and we were careful to exclude the mass at the origin which the distribution of Z_s might have. So, the characteristic exponent of X^ε is

6.20 $$\psi_\varepsilon(r) \;=\; \int_{(\varepsilon,\infty)} \nu(ds) \int_{\mathbb{R}^d} K(s, dx)\; (e^{ir\cdot x} - 1), \qquad r \in \mathbb{R}^d.$$

c) Let $\varepsilon \to 0$. Since S^o is pure-jump, S^ε_t increases to S^o_t, which implies that $Z(S^\varepsilon_t) \to Z(S^o_t-)$ by the left-limitedness of Z. But, for fixed s, we have $Z_s = Z_{s-}$ almost surely, and this remains true for $s = S^o_t$ by the independence of Z from S. Hence, $X^\varepsilon_t \to X^o_t$ almost surely, and the characteristic exponent of X^o is

6.21 $$\psi_o(r) \;=\; \lim_{\varepsilon \downarrow 0} \psi_\varepsilon(r).$$

d) Let φ be the characteristic exponent of Z. For $s \le 1$,

$$\left| \int_{\mathbb{R}^d} K(s, dx)\; (e^{ir\cdot x} - 1) \right| \;=\; \left| e^{s\varphi(r)} - 1 \right| \le s\; |\varphi(r)| \;;$$

and

$$\int_{(0,1]} \nu(ds)\; s < \infty$$

since ν is the Lévy measure of an increasing process. Thus, the dominated convergence theorem applies, and

$$\lim_{\varepsilon\downarrow 0}\int_{(\varepsilon,1]}\nu(ds)\int_{\mathbb{R}^d}K(s,dx)\,(e^{ir\cdot x}-1) \;=\; \int_{(0,1]}\nu(ds)\int_{\mathbb{R}^d}K(s,dx)\,(e^{ir\cdot x}-1).$$

Putting this together with 6.20 and 6.21, we get

6.22 $$\psi_o(r) \;=\; \int_{(0,\infty)}\nu(ds)\int_{\mathbb{R}^d}K(s,dx)\,(e^{ir\cdot x}-1).$$

e) We now show that, as 6.22 suggests, the Lévy measure of X^o is νK. Let M^ε be the jump measure of the process X^ε. We have shown in part (b) that it is Poisson with mean $Leb \times \nu_\varepsilon K$. For positive Borel functions f on $\mathbb{R}_+ \times \mathbb{R}^d$, it is clear that $M^\varepsilon f$ increases to some limit $M^o f$ as $\varepsilon \to 0$, and since

$$\mathbb{E}e^{-M^\varepsilon f} \;=\; \exp_-\int_{\mathbb{R}_+}dt\int_{(\varepsilon,\infty)}\nu(ds)\int_{\mathbb{R}^d}K(s,dx)\,[1-e^{-f(t,x)}],$$

we have

$$\mathbb{E}e^{-Mf} \;=\; \exp_-\int_{\mathbb{R}_+}dt\int_{(0,\infty)}\nu(ds)\int_{\mathbb{R}^d}K(s,dx)\,[1-e^{-f(t,x)}]$$

by the monotone convergence theorem. Thus M^o is Poisson with mean $Leb\times \nu K$. It now follows from part (c) of the proof that M^o is the jump measure of X^o. Hence, in particular, the Lévy measure of X^o is νK.

f) Since νK is a Lévy measure on $\mathbb{R}^d$,

$$\psi_1(r) = \int_{\mathbb{B}}\nu K(dx)\,(e^{ir\cdot x}-1-ir\cdot x),$$

6.23 $$\psi_2(r) = \int_{\mathbb{B}^c}\nu K(dx)\,(e^{ir\cdot x}-1)$$

are well-defined complex numbers for each r in $\mathbb{R}^d$. Writing

$$ir\cdot x \;=\; (e^{ir\cdot x}-1)-(e^{ir\cdot x}-1-ir\cdot x)$$

and recalling that $h(x) = x1_{\mathbb{B}}(x)$, we see from 6.22 and 6.23 that

$$\left|\int_{\mathbb{R}^d}\nu K(dx)\; r\cdot h(x)\right| \;=\; \left|\psi_o(r)-\psi_2(r)-\psi_1(r)\right| \;<\infty.$$

Taking $r = e_j$, the j^{th} unit vector, for $j = 1,\cdots,d$, we see that $\nu K h$ is a well-defined vector in $\mathbb{R}^d$, and that

6.24 $$\psi_o(r) \;=\; i\,(\nu K h)\cdot r + \psi_1(r)+\psi_2(r).$$

In view of 6.23, this implies through 6.14 that $X^o \sim (\nu K h, 0, \nu K)$; hence, 6.19 is true, and the proof is complete. □

Exercises

6.25 *Symmetric gamma.* Let k_a be as defined in Exercise 1.49 , that is, k_a is the density function for the difference of two independent gamma variables with the same shape index a and the same scale parameter 1. Let X be as in Example 6.5.

a) Show that the density function for X_t is $\sqrt{2c}\,k_{at}(\sqrt{2c}\;x)$.
b) Show that

$$k_a(x) \;=\; \int_0^\infty du\; \frac{e^{-u}\,u^{a-1}}{\Gamma(a)} \cdot \frac{e^{-x^2/4u}}{\sqrt{4\pi u}}, \qquad x \in \mathbb{R}.$$

This is more appealing than its close relative, the *modified Bessel function* K_ν. The latter is given by

$$K_\nu(x) \;=\; \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \int_0^\infty du\; e^{-u}\,u^{\nu-1} e^{-x^2/4u}, \quad \nu \in \mathbb{R}, \quad x \in \mathbb{R}_+.$$

Thus, for $a > 0$ and x in $\mathbb{R}$,

$$k_a(x) \;=\; \frac{|x/2|^{a-1/2}}{\sqrt{\pi}\,\Gamma(a)}\; K_{a-1/2}(\,|x|\,).$$

6.26 *Wiener subordinated to gamma.* In Theorem 6.2, let Z be a Wiener process in $\mathbb{R}^d$, and S a gamma process with shape rate a and scale parameter c . In view of Example 6.5 , every component of $X = Z_S$ is a symmetric gamma process with shape rate a and scale parameter $\sqrt{2c}$. The process X is isotropic.

a) Show that

$$\mathbb{E}\; e^{ir\cdot X_t} \;=\; \left(\frac{2c}{2c + |r|^2}\right)^{at}, \quad r \in \mathbb{R}^d.$$

b) Let λ be the Lévy measure of X. In spherical coordinates (see 1.52) , its spherical part σ is the uniform distribution on the unit sphere in $\mathbb{R}^d$, and its radial part ρ is given by

$$\rho(dv) \;=\; dv \int_0^\infty ds\; \frac{ae^{-cs}}{s} \cdot \frac{2v^{d-1}\,e^{-v^2/2s}}{(2s)^{d/2}\,\Gamma(d/2)}, \quad v > 0.$$

Show this. Show that

$$\rho(dv) \;=\; dv \cdot \frac{\Gamma(\frac{d+1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} \cdot \frac{4a}{v}\, k_b(\sqrt{2c}\,v)$$

with $b = \frac{d+1}{2}$; see the preceding exercise for k_b .

6.27 *Stable subordinated to gamma*. Let Z be an isotropic a-stable process in $\mathbb{R}^d$ having the characteristic exponent $\psi(r) = -|r|^a$. Let S be a gamma process with shape rate b and scale parameter c. As usual, we assume that Z and S are independent. For $X = Z_S$, show that

$$\mathbb{E}\, e^{ir\cdot X_t} = \left(\frac{c}{c+|r|^a}\right)^{bt}, \quad r \in \mathbb{R}^d.$$

The distribution of X_t is called *Linnik distribution.*

6.28 *Continuation*. Suppose, now, that Z is an increasing stable process with index a, necessarily, $a \in (0,1)$. Then, $X = Z_S$ is an increasing pure-jump process. Suppose that the scale parameter is chosen to make Z_t have the Laplace transform e^{-tp^a}, $p \in \mathbb{R}_+$.

a) Show that

$$\mathbb{E}\, e^{-pX_t} = \left(\frac{c}{c+p^a}\right)^{bt}, \quad p \in \mathbb{R}_+.$$

b) Show that, when $a = 1/2$, the Lévy measure for X is

$$\lambda(dx) = dx\; b\,\frac{e^{c^2x}}{x} \int_{c\sqrt{2x}}^{\infty} du\; \frac{e^{-u^2/2}}{\sqrt{2\pi}}, \quad x > 0.$$

6.29 *Gamma subordinated to stable*. Let Z be a gamma process with shape rate b and scale parameter 1. Let S be an increasing stable process with shape index a and scale c, that is, $\mathbb{E}\,\exp_-\, pS_t = \exp_-\, tcp^a$ for p in $\mathbb{R}_+$; here $c > 0$ and $a \in (0,1)$. Show that, then,

$$\mathbb{E}\, e^{-pX_t} = \exp_-\; tc\,[\,b\,\log(1+p)\,]^a, \quad p \in \mathbb{R}_+.$$

7 Increasing Lévy Processes

These processes play an important role in the theories of regeneration and Markov processes in continuous time. Moreover, they are useful as subordinators and interesting in themselves. In this section, we give a highly selective survey concentrating on potentials and hitting times.

Throughout, $(\Omega, \mathcal{H}, \mathbb{P})$ is a complete probability space, $\mathcal{F} = (\mathcal{F}_t)$ is an augmented right-continuous filtration, and $S = (S_t)$ is an increasing Lévy process relative to $\mathcal{F}$. The assumptions on $\mathcal{F}$ are without loss of generality in view of Theorem 3.20.

We let b denote the drift rate and λ the Lévy measure for S. Thus, b is a constant in $\mathbb{R}_+$, and the measure λ on $\mathbb{R}_+$ satisfies 1.13 and $\lambda\{0\} = 0$. More explicitly,

7.1 $$S_t = bt + \int_{(0,t]\times\mathbb{R}_+} M(ds,dx)\; x, \quad t \in \mathbb{R}_+,$$

where M is Poisson on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean $Leb \times \lambda$. We let π_t be the distribution of S_t and recall that, for p in $\mathbb{R}_+$,

7.2 $$\mathbb{E}e^{-p\,S_t} \;=\; \int_{\mathbb{R}_+} \pi_t(dx)\; e^{-px} \;=\; \exp\_\; t\,[\,bp + \int_{\mathbb{R}_+} \lambda(dx)\;(1-e^{-px})\,].$$

We exclude from further consideration the trivial case where $\lambda = 0$. When $b = 0$ and λ finite, S is a compound Poisson process, and its paths are step functions. Otherwise, S is strictly increasing.

Potential measure

For Borel subsets B of $\mathbb{R}_+$, we define

7.3 $$U(B) \;=\; \mathbb{E} \int_{\mathbb{R}_+} dt\; 1_B \circ S_t \;=\; \int_{\mathbb{R}_+} dt\; \pi_t(B),$$

the expected amount of time spent in B by S. Then, U is called the *potential measure* of S. Explicit computations are rare, but the Laplace transform

7.4 $$\hat{u}_p \;=\; \int_{\mathbb{R}_+} U(dx)\; e^{-px} \;=\; \int_{\mathbb{R}_+} dt\; \mathbb{E}\; e^{-pS_t}$$

is readily available: in view of 7.2,

7.5 $$[\,bp + \int_{\mathbb{R}_+} \lambda(dx)\;(1-e^{-px})\,]\;\hat{u}_p \;=\; 1, \qquad p \in (0,\infty).$$

7.6 EXAMPLE. a) *Poisson process.* Suppose that S is a Poisson process with rate c. Then, it spends an exponential amount with mean $1/c$ at each positive integer n. So, $U = (1/c)\,(\delta_0 + \delta_1 + \cdots)$, where δ_x is Dirac at x as usual.

b) *Stable process.* Suppose that S is increasing stable with index a; the index is necessarily in $(0,1)$. Then, the Lévy measure has the density c/x^{a+1} with respect to Lebesgue on $(0,\infty)$; see 2.1. Choosing $c = a/\Gamma(1-a)$, the Laplace transform for S_t becomes $\exp\_\; tp^a$, and hence

$$\int_{\mathbb{R}_+} U(dx)\; e^{-px} \;=\; \frac{1}{p^a} \;=\; \int_{\mathbb{R}_+} dx\; \frac{e^{-px}\,x^{a-1}}{\Gamma(a)}.$$

It follows that the potential measure is absolutely continuous, and

$$U(dx) \;=\; dx\; \frac{x^{a-1}}{\Gamma(a)}, \quad x \in \mathbb{R}_+.$$

7.7 REMARK. a) The measure U is diffuse, except when S is compound Poisson: if S is not compound Poisson, then it is strictly increasing, which implies that the amount of time spent in the singleton $\{x\}$ is equal to zero. When S is compound Poisson, the point 0 is an atom for U, because S spends an exponential amount of time at 0 with parameter $c = \lambda(\mathbb{R}_+) < \infty$; there are atoms beyond 0 only if λ has atoms.

b) The potential measure is finite on compacts: For $B \subset [0, x]$,

$$U(B) \le \int_{\mathbb{R}_+} dt\, \mathbb{P}\{\, S_t \le x \,\} = \int_{\mathbb{R}_+} dt\, \mathbb{P}\{\, e^{-S_t} \ge e^{-x} \,\}$$
$$\le \int_{\mathbb{R}_+} dt\, e^{x}\, \mathbb{E}\, e^{-S_t} \;=\; e^{x}\, \hat{u}_1,$$

where the inequality is Markov's; and $\hat{u}_1 < \infty$ since the first factor on the left side of 7.5 is strictly positive for $p = 1$.

Absolute continuity of the potential

This is a closer examination of the equation 7.5 for the Laplace transform $\hat{u}_p$. We start by introducing a measure φ on $\mathbb{R}_+$:

7.8 $$\varphi(dx) \;=\; b\,\delta_0(dx) + dx\; \lambda(x, \infty)\; 1_{(0,\infty)}(x), \qquad x \in \mathbb{R}_+.$$

Since the Lévy measure λ satisfies 1.13, its tail $x \mapsto \lambda(x, \infty)$ is a real-valued decreasing locally integrable function on $(0, \infty)$. Thus, the measure φ is finite on compacts. Note that its Laplace transform is

$$\hat{\varphi}_p \;=\; \int_{\mathbb{R}_+} \varphi(dx)\; e^{-px} \;=\; b + \frac{1}{p} \int_{\mathbb{R}_+} \lambda(dx)\; (1 - e^{-px}), \qquad p > 0.$$

Hence, we may re-write 7.5 as $\hat{\varphi}_p\, \hat{u}_p = \frac{1}{p}$; in other words, the convolution of the measures U and φ is equal to the Lebesgue measure on $\mathbb{R}_+$, that is,

$$\int_{\mathbb{R}_+} \varphi(dx) \int_{\mathbb{R}_+} U(dy)\; 1_B(x + y) \;=\; \text{Leb } B,$$

or equivalently,

7.9 $$b\,U(B) + \int_B dx \int_{[0,x]} U(dy)\; \lambda(x - y, \infty) \;=\; \text{Leb } B, \qquad B \in \mathcal{B}_{\mathbb{R}_+}.$$

7.10 Remark. Suppose that $b = 0$. Then, the preceding equation shows that

7.11 $$\int_{[0,x]} U(dy)\; \lambda(x - y, \infty) \;=\; 1$$

for Lebesgue-almost every x in $(0, \infty)$. It is known that, in fact, this is true for every x in $(0, \infty)$. We shall show this when S is compound Poisson; see 7.25ff. The proof in the remaining case, where $b = 0$ and $\lambda(\mathbb{R}_+) = +\infty$, is famously difficult; see notes and comments for this chapter.

7.12 THEOREM. *Suppose that* $b > 0$. *Then,* U *is absolutely continuous and admits a bounded continuous function* $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *as its density; and*

7.13 $$b\,u(x) + \int_0^x dy\ u(y)\ \lambda(x-y,\infty) \ = \ 1, \qquad x \in \mathbb{R}_+.$$

Proof. It follows from 7.9 that $bU \le Leb$, which implies via Radon-Nikodym theorem that $U(dx) = dx\ u(x)$, $x \in \mathbb{R}_+$,for some positive Borel function bounded by $1/b$. Then, 7.9 implies that 7.13 holds for Leb-almost every x. Since u is bounded and $x \mapsto \lambda(x,\infty)$ is right-continuous and locally integrable, the second term on the left side of 7.13 is continuous in x. Thus, we may take u continuous, and 7.13 holds for every x. □

Level crossings

Let T_x denote the time of hitting (x,∞) by S; we call it also the time of crossing the level x:

7.14 $$T_x \ = \ \inf\{\, t \ge 0 \ : \ S_t > x \,\}, \qquad x \in \mathbb{R}_+.$$

Each T_x is a stopping time of $(\mathcal{G}_{t+})$, where $\mathcal{G}$ is the filtration generated by S; it is also a stopping time of $\mathcal{F}$ since $\mathcal{F}$ is right-continuous. The processes (S_t) and (T_x) are functional inverses of each other. If S is compound Poisson, then $T_0 > 0$ almost surely, and (T_x) is a step process. Otherwise, S is strictly increasing, and $T_0 = 0$ almost surely, and (T_x) is continuous.

7.15 PROPOSITION. *For fixed* x *and* t *in* $\mathbb{R}_+$,

7.16 $$\mathbb{P}\{\ T_x \le t\ \} \ = \ \begin{cases} \mathbb{P}\{\ S_t > x\ \} & \text{if } S \text{ is compound Poisson,} \\ \mathbb{P}\{\ S_t \ge x\ \} & \text{otherwise.} \end{cases}$$

In both cases,

$$\mathbb{E}\ T_x \ = \ U\,[0,x] \ = \ \int_0^\infty dt\ \mathbb{P}\{\ S_t \le x\ \}.$$

Proof. Pick ω such that the regularity conditions hold for the corresponding path of S. If S is compound Poisson, the path is a step function, and $T_x(\omega) \le t$ if and only if $S_t(\omega) > x$; this proves 7.16 in this case. Otherwise, the path is strictly increasing; then, $S_t(\omega) \ge x$ implies that $T_x \le t$, and the latter implies that $x \le S(\omega, T_x(\omega)) \le S(\omega,t)$; thus, $T_x(\omega) \le t$ if and only if $S_t(\omega) \ge x$, and this proves 7.16 in this case.

As to expected values, it follows from 7.16 that

$$\mathbb{E}\ T_x \ = \ \int_{\mathbb{R}_+} dt\ \mathbb{P}\{\ T_x > t\ \} \ = \ \begin{cases} U\,[0,x] & \text{if S is compound Poisson,} \\ U\,[0,x) & \text{otherwise .} \end{cases}$$

But if S is not compound Poisson, then U is diffuse (see Remark 7.7a), and we have $U\,[0,x) = U\,[0,x]$. □

Jumping across

In the remainder of this section we shall consider the joint distribution of T_x and the values of S just before and after T_x. We introduce (G for *gauche*, and D for *droit*)

7.17 $$G_x = S_{T_x -}, \quad D_x = S_{T_x}, \qquad x \in \mathbb{R}_+,$$

with the convention that $S_{0-} = 0$ always; see Figure 10 below. In general, $G_x \le x \le D_x$. Crossing into (x, ∞) occur either by drifting across x, which is the case on the event $\{ G_x = x = D_x \}$, or by jumping across x, which is the case on $\{ G_x < D_x \}$. The following gives the joint distribution in the jump case.

7.18 THEOREM. *Let $x \in (0, \infty)$. Let $f : (\mathbb{R}_+)^3 \mapsto \mathbb{R}_+$ be Borel. Then,*

$$\mathbb{E} f(T_x, G_x, D_x) 1_{\{G_x \ne D_x\}} = \int_{\mathbb{R}_+} dt \int_{[0,x]} \pi_t(dy) \int_{[x-y,\infty)} \lambda(dz)\, f(t, y, y+z).$$

7.19

7.20 REMARK. Case of $x = 0$. If S is not compound Poisson, there is nothing to do, since $T_0 = 0$ and $G_0 = D_0 = 0$. If S is compound Poisson, then 7.19 remains true for $x = 0$: Then, T_0 is the time of first jump, which has the exponential distribution with parameter $c = \lambda(\mathbb{R}_+)$; and $G_0 = 0$ almost surely; and D_0 is the size of the first jump, which is independent of T_0 and has the distribution $(1/c)\,\lambda$; whereas, $\pi_t\{0\} = e^{-ct}$.

Proof. Fix $x > 0$ and f Borel. Let Z denote the random variable on the left side of 7.19. Being increasing, S can cross the level x only once. For almost every ω, therefore, there is at most one jump time t with $S_{t-}(\omega) \le x \le S_t(\omega)$; and if t is such, putting $z = S_t(\omega) - S_{t-}(\omega) > 0$, we obtain an

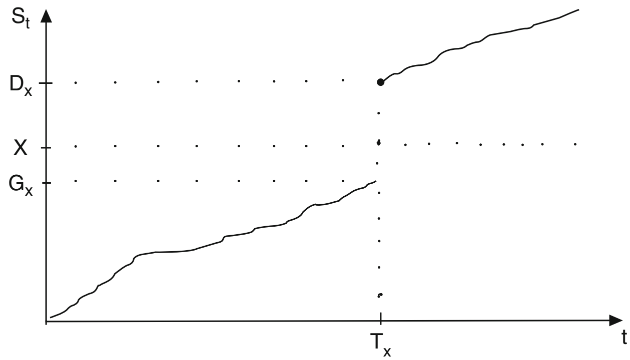

Figure 10: Level x is crossed at time T_x by a jump from the point G_x in $[0, x]$ to the point D_x in (x, ∞).

atom (t, z) of the measure $M(\omega, \cdot)$ defining $S(\omega)$; see 7.1. Thus,

$$Z = \int_{\mathbb{R}_+ \times (0,\infty)} M(dt, dz)\ f(t, S_{t-}, S_{t-} + z)\ 1_{\{S_{t-} \le x \le S_{t-}+z\}}\ ;$$

indeed, for almost every ω, the integral is a sum with at most one term, namely, the term corresponding to $t = T_x(\omega)$ if $S_t(\omega) - S_{t-}(\omega) = z > 0$. So, Z is a Poisson integral, and the integrand is predictable (see Theorem VI.6.2) since $t \mapsto S_{t-}$ is left-continuous and adapted. Hence,

$$\begin{aligned} \mathbb{E}\ Z &= \mathbb{E} \int_{\mathbb{R}_+} dt \int_{(0,\infty)} \lambda(dz)\ f(t, S_{t-}, S_{t-} + z)\ 1_{\{S_{t-} \le x \le S_{t-}+z\}} \\ &= \mathbb{E} \int_{\mathbb{R}_+} dt \int_{(0,\infty)} \lambda(dz)\ f(t, S_t, S_t + z)\ 1_{\{S_t \le x \le S_t + z\}}, \end{aligned}$$

where the last equality is justified by noting that replacing S_{t-} with S_t cannot alter the Lebesgue integral over t, since $S_{t-}(\omega)$ differs from $S_t(\omega)$ for only countably many t. We obtain 7.19 by evaluating the last expectation using the distribution π_t of S_t and recalling that $\lambda\{0\} = 0$. □

At the time S crosses x, its left-limit G_x belongs to $[0, x]$ and its right-hand value D_x belongs to $[x, \infty)$. Thus, if the crossing is by a jump, the jump is either from somewhere in $[0, x]$ into (x, ∞) or from somewhere in $[0, x)$ to the point x. The following shows that the last possibility is improbable.

7.21 Corollary. *For x in $(0, \infty)$,*

$$\mathbb{P}\{\ G_x < x = D_x\ \} = 0.$$

Proof. Fix x. This is obvious when S is compound Poisson, because $D_x > x$ then. Suppose that S is not compound Poisson, and recall that, then, the potential measure is diffuse. From the preceding theorem, taking $f(t, y, z) = 1_{[0,x)}(y)\, 1_{\{x\}}(z)$, we get

$$\mathbb{P}\{\ G_x < x = D_x\ \} = \int_{[0,x)} U(dy)\ \lambda\{\, x - y\,\}.$$

Since λ is σ-finite, it can have $x - y$ as an atom for at most countably many y; let A be the set of such y in $[0, x)$. We have $U(A) = 0$ since U is diffuse. So, the last integral is zero as claimed. □

7.22 Corollary. *For every x in $\mathbb{R}_+$,*

7.23 $$\mathbb{P}\{\ G_x = D_x\ \} = \mathbb{P}\{\ D_x = x\ \} = 1 - \int_{[0,x]} U(dy)\ \lambda\,(x - y, \infty).$$

Proof. For $x = 0$, this is by direct checking; see Remark 7.20. Suppose $x > 0$. It follows from the last corollary that, on the event $\{ D_x = x \}$, we have $G_x = x$ almost surely; hence,

$$\mathbb{P}\{ D_x = x \} = \mathbb{P}\{ G_x = x = D_x \} = \mathbb{P}\{ G_x = D_x \}.$$

This proves the first equality. The second is obtained by computing $\mathbb{P}\{D_x > x\}$ from 7.19 by taking $f(t, y, z) = 1_{(x,\infty)}(z)$. □

Consider the preceding corollary in light of Theorem 7.12. If $b > 0$, the potential measure admits a density u, and comparing 7.13 and 7.23, we see that the probability of drifting across x is

7.24 $$\mathbb{P}\{ G_x = D_x \} = \mathbb{P}\{ D_x = x \} = b\,u(x), \qquad x \in \mathbb{R}_+.$$

If $b = 0$ and λ finite, that is, if S is compound Poisson, then $D_x > x$ for every x; hence,

7.25 $$\mathbb{P}\{ G_x = D_x \} = \mathbb{P}\{ D_x = x \} = 1 - \int_{[0,x]} U(dy)\, \lambda(x - y, \infty) = 0,$$

for x in $\mathbb{R}_+$; and we see that 7.11 is true for every x as a by-product. Indeed, as remarked in 7.10, it can be shown that 7.25 is true for every x as long as $b = 0$. Here is an example.

7.26 Example. *Stable processes.* Suppose that S is the increasing stable process of Example 7.6. Recall that $\lambda(dx) = dx\, a \,/\, x^{a+1}\,\Gamma(1 - a)$, which yielded the potential measure $U(dx) = dx \,/\, x^{1-a}\,\Gamma(a)$. Then, for $0 \le y \le x < z$, we see from 7.19 that

$$\begin{aligned}\mathbb{P}\{ G_x \in dy,\ D_x \in dz \} &= U(dy)\ \lambda(dz - x)\\ &= dy\ dz\ \frac{a}{\Gamma(a)\,\Gamma(1-a) y^{1-a}(z-y)^{1+a}}\\ &= dy\ dz\ \frac{a \sin \pi a}{\pi y^{1-a}(z-y)^{1+a}}.\end{aligned}$$

Integrating over y in $[0, x]$ and z in (x, ∞), we get $\mathbb{P}\{ D_x > x \} = 1$, confirming 7.25 and 7.11 once more.

Drifting across

We concentrate here on the distribution of T_x in the event x is crossed by drifting. Define

7.27 $$\mu_x(A) = \mathbb{P}\{ T_x \in A,\ G_x = D_x \}, \qquad x \in \mathbb{R}_+,\ A \in \mathcal{B}_{\mathbb{R}_+}.$$

If $b = 0$ then this is zero. Suppose that $b > 0$. Then, S is strictly increasing, which implies that $x \mapsto T_x$ is continuous, which in turn implies that

$x \mapsto \mu_x(A)$ is Borel measurable for each A in $\mathcal{B}_{\mathbb{R}_+}$. Hence, $(x, A) \mapsto \mu_x(A)$ is a transition kernel; it is bounded, since $\mu_x(\mathbb{R}_+) = b\,u(x)$ in view of 7.24, and u is bounded by $1/b$. The following identifies it.

7.28 Theorem. *Suppose that $b > 0$. Then, μ is a transition kernel from $\mathbb{R}_+$ into $\mathbb{R}_+$ which satisfies*

7.29 $$dx\ \mu_x(dt) \ = \ dt\ \pi_t(dx)\ b, \qquad x \in \mathbb{R}_+,\ t \in \mathbb{R}_+.$$

Proof. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be Borel. With $b > 0$, the form 7.1 of S shows that $dS_t(\omega) = b\,dt$ if $S_{t-}(\omega) = S_t(\omega)$. Hence,

$$\begin{aligned}\int_{\mathbb{R}_+} dt\ b\,f(t, S_t) &= \int_{\mathbb{R}_+} f(t, S_t)\ 1_{\{S_{t-}=S_t\}} dS_t \\ &= \int_{\mathbb{R}_+} dx\,f(T_x, D_x)\ 1_{\{G_x = D_x\}} \\ &= \int_{\mathbb{R}_+} dx\,f(T_x, x)\,1_{\{G_x = D_x\}},\end{aligned}$$

where we used the time change $t = T_x$, the definitions 7.17 of G and D, and the observation that $D_x = x$ on $\{G_x = D_x\}$. Next, we take expectations on both sides; using the definition 7.27, we get

$$b \int_{\mathbb{R}_+} dt \int_{\mathbb{R}_+} \pi_t(dx)\ f(t, x) \ = \ \int_{\mathbb{R}_+} dx \int_{\mathbb{R}_+} \mu_x(dt)\ f(t, x).$$

This proves 7.29 since f is an arbitrary positive Borel function. □

7.30 Remark. Fix $t > 0$. let $\nu(A)$ be the expected amount of time that S spends in the set A during the time interval $[0, t]$. Then, ν is a measure on $\mathbb{R}_+$ whose total mass is t. According to 7.29, ν is absolutely continuous with respect to the Lebesgue measure, and

$$\mu_x\,[0, t] \ = \ b\ \frac{\nu(dx)}{dx}, \qquad x \in \mathbb{R}_+.$$

7.31 Example. Suppose that $S_t = bt + S_t^o$, where S^o is a gamma process with shape rate a and scale parameter c. Then,

$$\pi_t(dx) \ = \ dx\ \frac{e^{-c(x-bt)}\ c^{at}\ (x - bt)^{at-1}}{\Gamma(at)}, \qquad x > bt;$$

and

$$\mu_x(dt) \ = \ dt\ \frac{b\,c^{at}\ e^{-c(x-bt)}\ (x - bt)^{at-1}}{\Gamma(at)}\ 1_{(0,x)}(bt).$$

Exercises

7.32 *Compound Poisson.* Suppose that S is a compound Poisson process with an exponential jump size distribution, that is, its Lévy measure is $\lambda(dx) = ca\, e^{-ax}\, dx$ for some constants a and c in $(0,\infty)$. Show that the corresponding potential measure is

$$U(dx) \;=\; \frac{1}{c}\,\delta_0(dx) + \frac{a}{c}\,dx, \qquad x \in \mathbb{R}_+.$$

7.33 *Atoms of π_t.* Theorem 7.28 might suggest that, when $b > 0$, the distribution π_t is absolutely continuous. This is false: Suppose that $S_t = bt + N_t$ where N is Poisson with rate c. For fixed $x > 0$, then

$$\pi_t\{\,x\,\} \;=\; \mathbb{P}\{\; S_t = x \;\} \;=\; \mathbb{P}\{\; N_t \;=\; x - bt \;\},$$

which is strictly positive if $x - bt = n$ for some integer $n \ge 0$. In the positive direction, it is known that π_t is diffuse whenever the Lévy measure is infinite.

7.34 *Poisson with drift.* Suppose that $S_t \;=\; t + N_t$ where N is a Poisson process with rate 1. Fix $x > 0$. Show that

$$\{\; D_x = x \;\} \;=\; \bigcup_k \;\{\; T_x = x - k\,,\;\; N_{x-k} = k \;\} \;=\; \bigcup_k \;\{\; N_{x-k} = k \;\}$$

where the sum is over all integers k in $[0, x)$. Show that

$$u(x) \;=\; \mathbb{P}\{\; G_x = D_x = x \;\} \;=\; \sum_{k<x} \frac{e^{-(x-k)}\,(x-k)^k}{k!}.$$

Compute

$$\begin{aligned} \mu_x[0,t] &= \mathbb{P}\{\; T_x \le t,\;\; G_x = D_x = x \;\} \\ &= \mathbb{P}\{\; T_x \le t \;\} \;-\; \mathbb{P}\{\; T_x \le t,\;\; G_x \ne D_x \;\}. \end{aligned}$$

7.35 *Stable process with index $a = 1/2$.* Suppose that S is stable with index $1/2$; then, $b = 0$ and the Lévy measure is $\lambda(dx) \;=\; dx\,(c/x^{a+1})\,1_{(0,\infty)}(x)$ for some constant c. Show that the distribution of $(\,G_x, D_x\,)$ is free of c. Use Example 7.26 to show that

$$\mathbb{P}\{\; G_x \in dy,\;\; D_x \in dz \;\} \;=\; dy\,dz\; \frac{1}{2\pi\,\sqrt{y\,(z-y)^3}}, \qquad y < x < z.$$

Show that, for $y < x < z$ again,

$$\mathbb{P}\{\; G_x < y,\;\; D_x > z \;\} \;=\; \frac{2}{\pi}\;\arcsin\sqrt{\frac{y}{z}}.$$

In particular, then, for $y < x$,

$$\mathbb{P}\{\; G_x < y \;\} \;=\; \mathbb{P}\{\; D_y > x \;\} \;=\; \frac{2}{\pi}\;\arcsin\sqrt{\frac{y}{x}}.$$

The distribution involved here is called the arcsine distribution; it is the beta distribution with index pair $(\frac{1}{2}, \frac{1}{2})$.

7.36 *Drifting.* In general, if $b > 0$, show that

$$\mathbb{P}\{\, T_x > t,\ G_x = D_x \,\} \;=\; \pi_t\,[0,x] \;-\; \int_t^\infty du \int_{[0,x]} \pi_u(dy)\ \lambda\,[\,x-y,\infty\,).$$

7.37 *Laplace transforms.* Let $\psi(p) \;=\; bp + \int \lambda(dx)\,(1 - e^{-px})$, the Laplace exponent for S, for $p \geq 0$. Show that, for $p > 0$,

$$\int_{\mathbb{R}_+} dx\ e^{-px}\ \mathbb{P}\{\, T_x > t \,\} = \frac{1}{p}\ e^{-t\psi(p)},$$

$$\int_{\mathbb{R}_+} dx\ e^{-px}\ \mathbb{P}\{\, T_x > t,\, G_x = D_x \,\} = \frac{b}{\psi(p)}\ e^{-t\psi(p)},$$

$$\int_{\mathbb{R}_+} dx\ p\,e^{-px}\ \mathbb{E}\ T_x = \frac{1}{\psi(p)} \;=\; \hat{u}(p).$$

7.38 *Time changes.* Let $c\;:\;\mathbb{R}_+ \mapsto \mathbb{R}_+$ be a strictly increasing continuous function with $c(0) = 0$ and $\lim_{t\to\infty} c(t) = +\infty$. Define

$$\hat{S}_t \;=\; S_{c(t)}\,, \qquad t \in \mathbb{R}_+.$$

Then, $\hat{S}$ is a process with independent increments, but the stationarity of increments is lost unless $c(t) = c_0\,t$. Define $\hat{T}$, $\hat{G}$, $\hat{D}$ from the process $\hat{S}$ in the same manner that T, G, D are defined from S.

a) Show that $\hat{G}_x = G_x$ and $\hat{D}_x = D_x$ for all x.

b) Show that $c\,(\hat{T}_x) = T_x$; thus, $\hat{T}_x \;=\; a\,(T_x)$ where a is the functional inverse of c.

7.39 *Continuation.* Observe that the preceding results remain true when c is replaced by a stochastic clock C whose paths $t \mapsto C(\omega, t)$ satisfy the conditions on c for every ω.