Chapter 4

Non-radial DEA Models and DEA with Preference

4.1 Non-radial DEA Models

We can call the envelopment DEA models as radial efficiency measures, because these models optimize all inputs or outputs of a DMU at a certain proportion. Färe and Lovell (1978) introduce a non-radial measure which allows nonproportional reductions in positive inputs or augmentations in positive outputs. Table 4.1 summarizes the non-radial DEA models with respect to the model orientation and frontier type.

Table 4.1. Non-radial DEA Models

Frontier Type	Input-Oriented	Output-Oriented
CRS	$\begin{array}{ll} \min(\frac{1}{m}\sum_{i=1}^{m}\theta_i - \varepsilon\sum_{r=1}^{s}s_r^+) & \\ \text{subject to} & \\ \sum_{j=1}^{n}\lambda_j x_{ij} = \theta_i x_{io} & i=1,2,\ldots,m; \\ \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro} & r=1,2,\ldots,s; \\ \theta_i \le 1 & i=1,2,\ldots,m; \\ \lambda_j \ge 0 & j=1,2,\ldots,n. \end{array}$	$\begin{array}{ll} \max(\frac{1}{s}\sum_{r=1}^{s}\phi_r + \varepsilon\sum_{r=1}^{s}s_r^+) & \\ \text{subject to} & \\ \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io} & i=1,2,\ldots,m; \\ \sum_{j=1}^{n}\lambda_j y_{rj} = \phi_r y_{ro} & r=1,2,\ldots,s; \\ \phi_i \ge 1 & r=1,2,\ldots,s; \\ \lambda_j \ge 0 & j=1,2,\ldots,n. \end{array}$
VRS	Add $\sum_{j=1}^{n}\lambda_j = 1$	
NIRS	Add $\sum_{j=1}^{n}\lambda_j \le 1$	
NDRS	Add $\sum_{j=1}^{n}\lambda_j \ge 1$	
Efficient Target	$\begin{cases} \hat{x}_{io} = \theta_i^* x_{io} & i=1,2,\ldots,m \\ \hat{y}_{ro} = y_{ro} + s_r^{+*} & r=1,2,\ldots,s \end{cases}$	$\begin{cases} \hat{x}_{io} = x_{io} - s_i^{-*} & i=1,2,\ldots,m \\ \hat{y}_{ro} = \phi_r^* y_{ro} & r=1,2,\ldots,s \end{cases}$

The slacks in the non-radial DEA models are optimized in a second-stage model where θ_i^* or ϕ_r^* are fixed. For example, under CRS we have

Input Slacks for Output-oriented Non-radial DEA Model

$$\max \sum_{r=1}^{s} s_r^+$$
subject to
$$\sum_{j=1}^{n} \lambda_j x_{ij} = \theta_i^* x_{io} \qquad i = 1,2,...,m;$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1,2,...,s;$$
$$\lambda_j \geq 0 \qquad j = 1,2,...,n.$$

Output Slacks for Input-oriented Non-radial DEA Model

$$\max \sum_{i=1}^{m} s_i^-$$
subject to
$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \qquad i = 1,2,...,m;$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \phi_r^* y_{ro} \qquad r = 1,2,...,s;$$
$$\lambda_j \geq 0 \qquad j = 1,2,...,n.$$

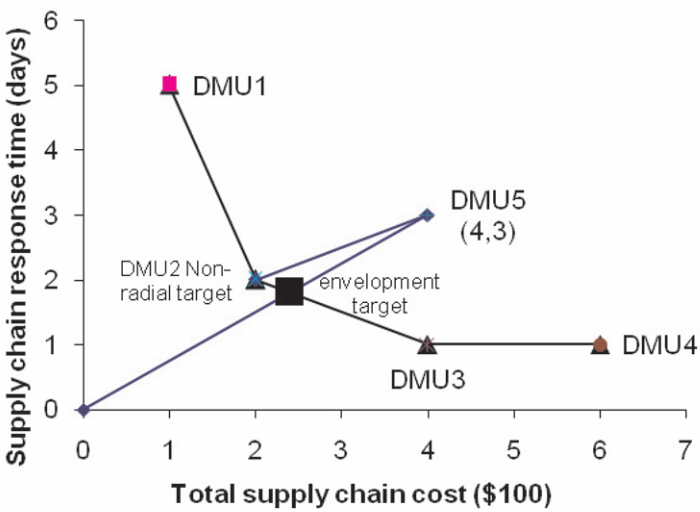

Figure 4.1. Efficient Targets

Note that input slacks do not exist in the input-oriented non-radial DEA models, and output slacks do not exist in the output-oriented non-radial DEA models.

Because $\theta_i^* \leq 1$ ($\phi_r^* \geq 1$), $\frac{1}{m}\sum_{i=1}^{m}\theta_i^* \leq 1$ and $\frac{1}{m}\sum_{i=1}^{m}\theta_i^* = 1$ if and only if $\theta_i^* = 1$ for all i ($\frac{1}{s}\sum_{r=1}^{s}\phi_r^* \geq 1$ and $\frac{1}{s}\sum_{r=1}^{s}\phi_r^* = 1$ if and only if $\phi_r^* = 1$ for all r). Thus, $\frac{1}{m}\sum_{i=1}^{m}\theta_i^*$ ($\frac{1}{s}\sum_{r=1}^{s}\phi_r^*$) can be used as an efficiency index.

Both the envelopment models and the non-radial DEA models yield the same frontier, but may yield different efficient targets (even when the envelopment models do not have non-zero slacks). For example, if we change the second input from 4 to 3 for DMU5 in Table 1.1 (Chapter 1), the input-oriented CRS envelopment model yields the efficient target of $x1$ = 2.4 and $x2$ = 1.8 (with $\lambda_2^* = 0.8$, $\lambda_3^* = 0.2$, and all zero slacks). Whereas the input-oriented CRS non-radial DEA model yields DMU2 as the efficient target for DMU5 (see Figure 4.1). Note that both models yield the same target of DMU3 for DMU4.

4.2 DEA with Preference Structure and Cost/Revenue Efficiency

Both the envelopment models and the non-radial DEA models yield efficient targets for inefficient DMUs. However, these targets may not be preferred by the management or achievable under the current management and other external conditions. Therefore, some other targets along the efficient frontier should be considered as preferred ones. This can be done by constructing preference structures over the proportions by which the corresponding current input levels (output levels) can be changed. Zhu (1996) develops a set of weighted non-radial DEA models where various efficient targets along with the frontier can be obtained.

Let A_i (i = 1, 2, …, m) and B_r (r = 1, 2,…, s) be user-specified preference weights which reflect the relative degree of desirability of the adjustments of the current input and output levels, respectively. Then we can have a set of weighted non-radial DEA models based upon Table 4.1 by changing the objective functions $\frac{1}{m}\sum_{i=1}^{m}\theta_i$ and $\frac{1}{s}\sum_{r=1}^{s}\phi_r$ to $\sum_{i=1}^{m}A_i\theta_i / \sum_{i=1}^{m}A_i$ and $\sum_{r=1}^{s}B_r\phi_r / \sum_{r=1}^{s}B_r$, respectively.

Further, if we remove the constraint $\theta_i \leq 1$ ($\phi_r \geq 1$), we obtain the DEA/preference structure (DEA/PS) models shown in Table 4.2 (Zhu, 1996a).

If some $A_i = 0$ ($B_r = 0$), then set the corresponding $\theta_i = 1$ ($\phi_r = 1$). But at least one of such weights should be positive. Note that for example, the bigger the weight A_i, the higher the priority DMU_o is allowed to adjust its ith input amount to a lower level. i.e., when inefficiency occurs, the more one wants to adjust an input or an output, the bigger the weight should be attached to θ_i or ϕ_r. If we can rank the inputs or outputs according to their relative importance, then we can obtain a set of ordinal weights. One may use Delphi-like techniques, or Analytic Hierarchy Process (AHP) to obtain the weights. However, caution should be paid when we convert the ordinal weights into preference weights. For example, if an input (output) is relatively more important and the DMU does not wish to adjust it with a higher rate, we should take the reciprocal of the corresponding ordinal weight as the preference weight. Otherwise, if the DMU does want to adjust the input (output) with a higher rate, we can take the ordinal weight as the preference weight. Also, one may use the principal component analysis to derive the information on weights (Zhu, 1998).

Note that in the DEA/PS models, some θ_i^* (ϕ_r^*) may be greater (less) than one under certain weight combinations. i.e., the DEA/PS models are *not* restricted to the case where 100% efficiency is maintained through the input decreases or output increases.

Table 4.2. DEA/Preference Structure Models

Frontier Type	Input-Oriented	Output-Oriented
CRS	$\min\left(\frac{\sum_{i=1}^{m} A_i\theta_i}{\sum_{i=1}^{m} A_i} - \varepsilon\sum_{r=1}^{s} s_r^+\right)$ subject to $\sum_{j=1}^{n}\lambda_j x_{ij} = \theta_i x_{io} \quad i=1,2,...,m;$ $\sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro} \quad r=1,2,...,s;$ $\lambda_j \geq 0 \quad j=1,2,...,n.$	$\max\left(\frac{\sum_{r=1}^{s}\phi_r}{\sum_{r=1}^{s} B_r} - \varepsilon\sum_{r=1}^{s} s_r^+\right)$ subject to $\sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io} \quad i=1,2,...,m;$ $\sum_{j=1}^{n}\lambda_j y_{rj} = \phi_r y_{ro} \quad r=1,2,...,s;$ $\lambda_j \geq 0 \quad j=1,2,...,n.$
VRS	Add $\sum_{j=1}^{n}\lambda_j = 1$	
NIRS	Add $\sum_{j=1}^{n}\lambda_j \leq 1$	
NDRS	Add $\sum_{j=1}^{n}\lambda_j \geq 1$	
Efficient Target	$\begin{cases}\hat{x}_{io} = \theta_i^* x_{io} & i=1,2,...,m\\ \hat{y}_{ro} = y_{ro} + s_r^{+*} & r=1,2,...,s\end{cases}$	$\begin{cases}\hat{x}_{io} = x_{io} - s_i^{-*} & i=1,2,...,m\\ \hat{y}_{ro} = \phi_r^* y_{ro} & r=1,2,...,s\end{cases}$

Now, in order to further investigate the property of DEA/PS models, we consider the dual program to the input-oriented CRS DEA/PS model.

$$
\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{ro} \\
&\text{subject to} \\
&\sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} \nu_i x_{ij} \le 0 \qquad j = 1,\ldots,n; \\
&\nu_i x_{io} = A_i / \sum_{i=1}^{m} A_i \qquad\qquad i = 1,\ldots,m; \\
&\mu_r, \nu_i \ge 0
\end{aligned}
\tag{4.1}
$$

We see that the normalization condition $\sum_{i=1}^{m} \nu_i x_{io} = 1$ is also satisfied in (4.1). The DEA/PS model is actually a DEA model with fixed input multipliers.

Let p_i^o denote the *i*th input price for DMU_o and $\widetilde{x}_{io}$ represents the *i*th input that minimizes the cost. Consider the following DEA model for calculating the "minimum cost".

$$
\begin{aligned}
&\min \sum_{i=1}^{m} p_i^o \widetilde{x}_{io} \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} \le \widetilde{x}_{io} \qquad i = 1,\ldots,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \qquad r = 1,\ldots,s \\
&\lambda_j, \widetilde{x}_{io} \ge 0
\end{aligned}
\tag{4.2}
$$

The dual program to (4.2) is

$$
\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{ro} \\
&\text{subject to} \\
&\sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} \nu_i x_{ij} \le 0 \qquad j = 1,\ldots,n; \\
&0 \le \nu_i \le p_i^o \qquad\qquad i = 1,\ldots,m; \\
&\mu_r, \nu_i \ge 0
\end{aligned}
\tag{4.3}
$$

By the complementary slackness condition of linear programming, we have that if $\widetilde{x}_{io}^* > 0$ then $p_i^o = \nu_i^*$. Thus, ν_i^* can be interpreted as p_i^o. Consequently, the input prices can be used to develop the preference weights.

In the DEA literature, we have a concept called "cost efficiency" which is defined as

$$
\frac{\sum_{i=1}^{m} p_i^o \widetilde{x}_{io}^*}{\sum_{i=1}^{m} p_i^o x_{io}}
$$

The following development shows that the related DEA/PS model can be used to obtain exact the cost efficiency scores. Because the actual cost – $\sum_{i=1}^{m} p_i^o x_{io}$ is a constant for a specific DMU_o, cost efficiency can be directly calculated by the following modified (4.2).

$$\min \frac{\sum_{i=1}^{m} p_i^o \tilde{x}_{io}}{\sum_{i=1}^{m} p_i^o x_{io}}$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{x}_{io} && i = 1,2,...,m; \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r = 1,2,...,s; \\
&\lambda_j \ge 0 && j = 1,2,...,n.
\end{aligned} \tag{4.4}$$

Let $\tilde{x}_{io} = \theta_i x_{io}$. Then (4.4) is equivalent to the input-oriented CRS DEA/PS model with $A_i = p_i^o x_{io}$. This indicates that if one imposes a proper set of preference weights for each DMU under consideration, then the DEA/PS model yields cost efficiency measure. (see Seiford and Zhu (2002) for an empirical investigation of DEA efficiency and cost efficiency.)

Similarly, the output-oriented DEA/PS model can be used to obtain the "revenue efficiency" which is defined as

$$\frac{\sum_{r=1}^{s} q_r^o \tilde{y}_{ro}^*}{\sum_{r=1}^{s} q_r^o y_{ro}}$$

where q_r^o indicates output price for DMU_o and $\tilde{y}_{ro}$ represents the rth output that maximizes the revenue in the following linear programming problem.

$$\max \sum_{r=1}^{s} q_r^o \tilde{y}_{ro}$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} && i = 1,2,...,m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \ge \tilde{y}_{ro} && r = 1,2,...,s \\
&\lambda_j, \tilde{y}_{ro} \ge 0
\end{aligned} \tag{4.5}$$

Let $\tilde{y}_{ro} = \phi_r y_{ro}$ and $B_r = q_r^o y_{ro}$ in the output-oriented DEA/PS model. We have

$$
\max \frac{\sum_{r=1}^{s} q_r^o \tilde{y}_{ro}}{\sum_{r=1}^{s} q_r^o y_{ro}}
$$
subject to
$$
\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} \qquad i = 1,2,...,m;
$$
$$
\sum_{j=1}^{n} \lambda_j y_{rj} \ge \tilde{y}_{ro} \qquad r = 1,2,...,s;
$$
$$
\lambda_j \ge 0 \qquad j = 1,2,...,n.
$$

which calculates the revenue efficiency.

4.3 DEA/Preference Structure Models in Spreadsheets

Figure 4.2 shows an input-oriented VRS DEA/PS spreadsheet model. Cells I2:I16 are reserved for λ_j. Cells F20:F22 are reserved for θ_i. These are the changing cells in the Solver parameters shown in Figure 4.3.

	A	B	C	D	E	F	G	H	I
1	Company	Assets	Equity	Employees		Revenue	Profit		λ
2	Mitsubishi	91920.6	10950	36000		184365.2	346.2		1
3	Mitsui	68770.9	5553.9	80000		181518.7	314.8		0
4	Itochu	65708.9	4271.1	7182		169164.6	121.2		0
5	General Motors	217123.4	23345.5	709000		168828.6	6880.7		0
6	Sumitomo	50268.9	6681	6193		167530.7	210.5		0
7	Marubeni	71439.3	5239.1	6702		161057.4	156.6		0
8	Ford Motor	243283	24547	346990		137137	4139		0
9	Toyota Motor	106004.2	49691.6	146855		111052	2662.4		0
10	Exxon	91296	40436	82000		110009	6470		0
11	Royal Dutch/Shell Group	118011.6	58986.4	104000		109833.7	6904.6		0
12	Wal-Mart	37871	14762	675000		93627	2740		0
13	Hitachi	91620.9	[illegible]	[illegible]		[illegible]	[illegible]		0
14	Nippon Life Insurance	364762.5	[illegible]	[illegible]		[illegible]	[illegible]		0
15	Nippon Telegraph & Telephone	127077.3	[illegible]	[illegible]		[illegible]	[illegible]		0
16	AT&T	88884	17274	299300		79609	139		0
17									
18		Reference		DMU under	1	Efficiency			
19	**Constraints**	set		Evaluation		3	Weights		
20	Assets	91920.6	=	91920.6		1	1		
21	Equity	10950	=	10950		1	1		
22	Employees	36000	=	36000		1	1		
23	Revenue	184365.2	≥	184365.2					
24	Profit	346.2	≥	346.2					
25	Σλ	1							

Reserved to indicate the DMU under evaluation.

Efficiency; Weighted sum of individual θi

Changing cells; Individual θi

Figure 4.2. Input-oriented VRS DEA/PS Spreadsheet Model

The target cell is cell F19 which contains the following formula

Cell F19 =SUMPRODUCT(F20:F22,G20:G22)/SUM(G20:G22) where cells G20:G22 are reserved for the input weights.

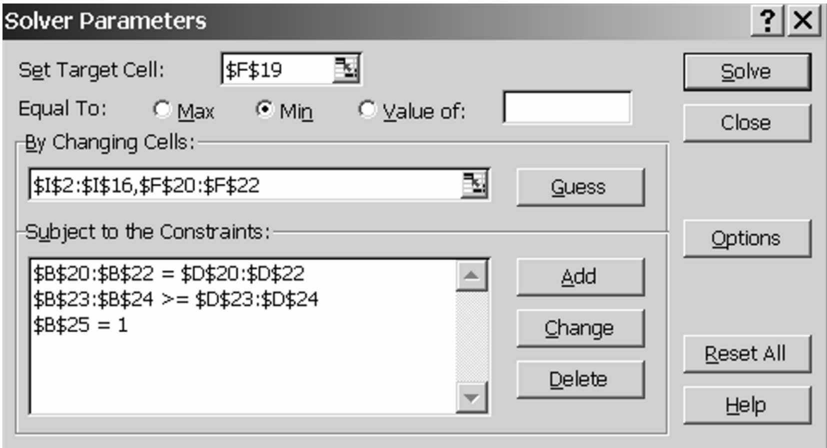

Figure 4.3. Solver Parameters for Input-oriented VRS DEA/PS Model

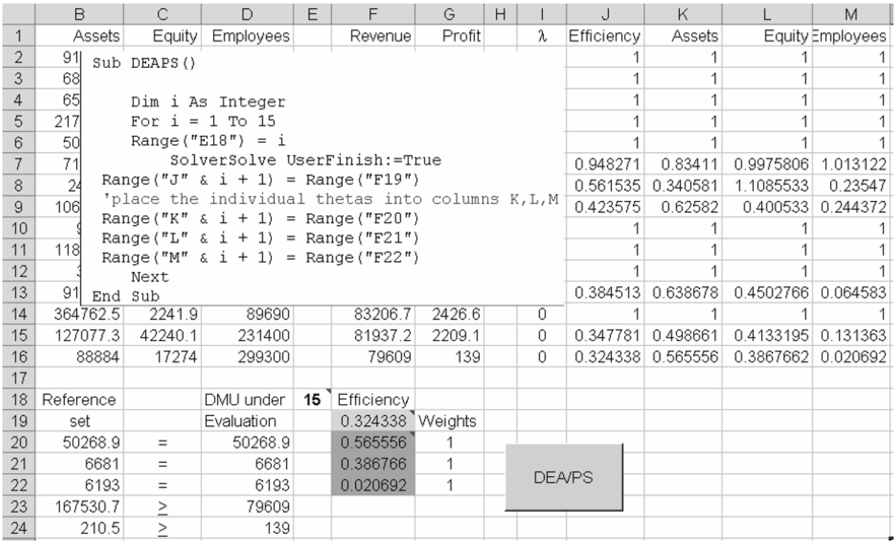

Figure 4.4. Efficiency Result for Input-oriented VRS DEA/PS Model

The formulas for cells B20:B25 are

Cell B20 =SUMPRODUCT(B2:B16,$I$2:$I$16)
Cell B21 =SUMPRODUCT(C2:C16,$I$2:$I$16)
Cell B22 =SUMPRODUCT(D2:D16,$I$2:$I$16)

Cell B23 =SUMPRODUCT(F2:F16,$I$2:$I$16)
Cell B24 =SUMPRODUCT(G2:G16,$I$2:$I$16)
Cell B25 =SUM(I2:I16)

The formulas for cells D20:D24 are

Cell D20 =F20*INDEX(B2:B16,E18,1)
Cell D21 =F21*INDEX(C2:C16,E18,1)
Cell D22 =F22*INDEX(D2:D16,E18,1)
Cell D23 =INDEX(F2:F16,E18,1)
Cell D24 =INDEX(G2:G16,E18,1)

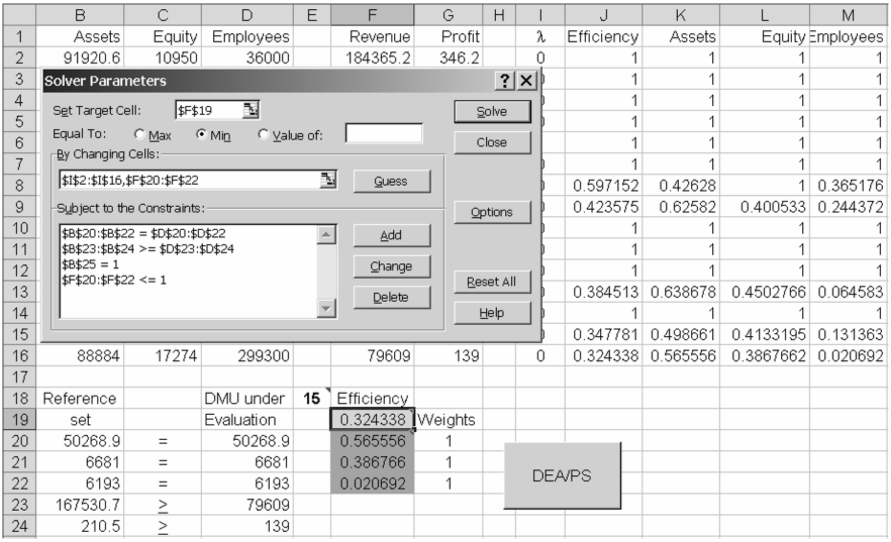

Figure 4.5. Efficiency Result for Input-oriented VRS Non-radial DEA Model

Figure 4.4 shows the results and the VBA procedure "DEAPS" which automates the calculation.

Note that the θ_i (i = 1,2,3) are not restricted in Figure 4.3. If we add $\theta_i \leq 1$ ($F$20:FF22 <=1), then we obtain the results shown in Figure 4.5.

4.4 DEA and Multiple Objective Linear Programming

Charnes, Cooper, Golany, Seiford and Stutz (1985) describe the relationship between DEA frontier and Pareto-Koopmans efficient empirical production frontier. This work points out the relation of efficiency in DEA and pareto optimality in multiple criteria decision making (MCDM) or Multiple Objective Linear Programming (MOLP). The relationship between

DEA and MOLP is again raised by Belton and Vickers (1993), Doyle and Green (1993) and Stewart (1994) in their discussion of DEA and MCDM. Joro, Korhonen and Wallenius (1998) provide a structure comparison of DEA and MOLP.

In fact, as shown in Chen (2005), the DEA/PS models have a strong relationship with MOLP. To demonstrate this, we use vector presentation of $\mathbf{x}_j = (x_{1j},...,x_{mj})$ and $\mathbf{y}_j = (y_{1j},...,y_{sj})$.

4.4.1 Output-oriented DEA

Consider the following MOLP model

$$\begin{aligned}
&\max_{\lambda_j}(\sum_{j=1}^{n}\lambda_j\mathbf{y}_j) = (\sum_{j=1}^{n}\lambda_j y_{1j},...,\sum_{j=1}^{n}\lambda_j y_{sj})\\
&\text{subject to}\\
&\sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io} \qquad i = 1,...,m;\\
&\lambda_j \ge 0, \qquad j = 1,...,n.
\end{aligned} \tag{4.6}$$

where $\mathbf{x}_o = (x_{1o},...,x_{mo})$ represents the input vector of DMU_o among others.

If all DMUs produce only one output, i.e., $\mathbf{y}_j$ is a scalar rather than a vector, then (4.6) is a single objective linear programming problem

$$\begin{aligned}
&\max_{\lambda_j}(\sum_{j=1}^{n}\lambda_j y_j)\\
&\text{subject to}\\
&\sum_{j=1}^{n}\lambda_j x_{ij} \le x_{io} \qquad i = 1,...,m;\\
&\lambda_j \ge 0, \qquad j = 1,...,n.
\end{aligned} \tag{4.7}$$

Let $\lambda_j y_j = \lambda'_j$, then (4.7) turns into

$$\begin{aligned}
&\max_{\lambda'_j}\sum_{j=1}^{n}\lambda'_j\\
&\text{subject to}\\
&\sum_{j=1}^{n}\lambda'_j x'_{ij} \le x'_{io} \qquad i = 1,...,m;\\
&\lambda'_j \ge 0, \qquad j = 1,...,n.
\end{aligned} \tag{4.8}$$

where $x'_{ij} = x_{ij} / y_j$ and $x'_{io} = x_{io} / y_o$.

As shown in Charnes, Cooper and Rhodes (1978), model (4.8) is equivalent to the output-oriented CRS envelopment model

$$\max_{\lambda_j, z_o} z_o$$
subject to
$$\sum_{j=1}^{n} \lambda_j y_j \geq z_o y_o$$
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \quad i = 1,\ldots,m;$$
$$\lambda_j \geq 0, \quad j = 1,\ldots,n.$$

Next, if $\mathbf{y}_j$ is a vector with s components, then we define

$$\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro} \tag{4.9}$$

As a result, (4.6) becomes

$$\max_{\lambda_j \sigma_r} (\sigma_1 y_{1o}, \ldots, \sigma_s y_{so})$$
subject to
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1,\ldots,s; \tag{4.10}$$
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1,\ldots,m;$$
$$\lambda_j \geq 0, \quad j = 1,\ldots,n.$$

Let $\mathbf{W} = \{w \mid w \in \mathbf{R}^s, w_r \geq 0 \text{ and } \sum_{r=1}^{s} w_r = 1\}$ be the set of nonnegative weights. The weighting problem associated with (4.10) is defined for some $w \in \mathbf{W}$ as

$$\max_{\lambda_j \sigma_r} \sum_{r=1}^{s} w_r \sigma_r y_{ro}$$
subject to
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1,\ldots,s; \tag{4.11}$$
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \quad i = 1,\ldots,m;$$
$$\lambda_j \geq 0 \quad j = 1,\ldots,n.$$

Furthermore, let $\overline{w}_r = w_r y_{ro}$ for all $r = 1, \ldots, s$, then (4.11) is equivalent to the following linear programming problem

$$\max_{\lambda_j \sigma_r} \sum_{r=1}^{s} \overline{w}_r \sigma_r$$
subject to
$$\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1,\ldots,s; \tag{4.12}$$
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} \quad i = 1,\ldots,m;$$
$$\lambda_j \geq 0 \quad j = 1,\ldots,n.$$

Model (4.12) is exactly the output-oriented CRS DEA/preference model. However, if we wish output level cannot be decreased to reach the efficient frontier, we specify (4.13) instead of (4.9).

$$\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro} \text{ such that } \sigma_r \geq 1 \text{ for all } r = 1, ..., s. \tag{4.13}$$

We see that for a specific DMU_o, $\lambda_o^* = 1$ and $\lambda_j^* = 0\,(j \neq o)$ is an optimal solution to (4.12), when $\sigma_r^* = 1$ for all $r = 1, ..., s$. Note that if some $\sigma_r^* \neq 1$, then $\lambda_o^* = 0$ is an optimal solution to (4.12). Therefore, (4.6) can be interpreted as follows: when $\mathbf{x}_o = (x_{1o},...,x_{mo})$ is regarded as resource, if the resource $\mathbf{x}_o$ can be used among other DMUs (associated with $\lambda_j^* \neq 0$), then more desirable or preferred output level $\mathbf{y}^*$ is produced and $\mathbf{y}_o$ is not a pareto solution to (4.6).

It can be seen that weighted non-radial DEA model (4.12) is equivalent to an MOLP problem. If we impose an additional on $\sum_{j=1}^{n} \lambda_j$ in (4.6), then we obtain other output-oriented DEA models.

4.4.2 Input-oriented DEA

Similar to (4.6), we write the following MOLP model.

$$\begin{aligned}
&\min_{\lambda_j} (\sum_{j=1}^{n} \lambda_j \mathbf{x}_j) = (\sum_{j=1}^{n} \lambda_j x_{1j},..., \sum_{j=1}^{n} \lambda_j x_{mj}) \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{ro} \qquad r = 1,...,s; \\
&\lambda_j \geq 0, \qquad j = 1,...,n.
\end{aligned} \tag{4.14}$$

where $\mathbf{y}_o = (y_{1o},..., y_{so})$ represents the output vector of DMU_o. If all DMUs use only one input, i.e., $\mathbf{x}_j$ is a scalar, then (4.14) is a single objective linear programming problem and is equivalent to the input-oriented CRS envelopment model with single input.

Let $\mathbf{G} = \{g \mid g \in \mathbf{R}^m, g_i \geq 0$ and $\sum_{i=1}^{m} g_i = 1\}$ be the set of nonnegative weights. Then model (4.14) can be transformed into the following linear programming problem.

$$\begin{aligned}
&\min_{\lambda_j \tau_i} \sum_{i=1}^{m} \bar{g}_i \tau_i \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = \tau_i x_{io} \qquad i = 1,...,m; \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \qquad r = 1,...,s; \\
&\lambda_j \geq 0 \qquad j = 1,...,n.
\end{aligned} \tag{4.15}$$

where $\overline{g}_i = g_i x_{io}$ for all $i = 1, \ldots, m$, and τ_i is defined in (4.16) or (4.17).

$$\sum_{j=1}^{n} \lambda_j x_{ij} = \tau_i x_{io} \tag{4.16}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} = \tau_i x_{io} \text{ such that } \tau_i \leq 1 \text{ for all } i = 1, \ldots, m. \tag{4.17}$$

Model (4.15) is a weighted non-radial DEA model incorporated with preference over the adjustment of input levels. If we use (4.16), then there is no restrictions on τ_i and model (4.15) is the input-oriented CRS DEA/PS model.

Note that for a specific DMU_o, $\lambda_o^* = 1$ and $\lambda_j^* = 0\,(j \neq o)$ is an optimal solution to (4.15), when $\tau_i^* = 1$ for all $i = 1, \ldots, m$. Note also that if some $\tau_i^* \neq 1$, then $\lambda_o^* = 0$ is an optimal solution to (4.15). If we impose an additional on $\sum_{j=1}^{n} \lambda_j$ in (4.15), then we obtain other input-oriented DEA models.

4.4.3 Non-Orientation DEA

Consider the following MOLP model.

$$\begin{aligned}
&\max_{\lambda_j} \left(\sum_{j=1}^{n} \lambda_j \mathbf{y}_j\right) = \left(\sum_{j=1}^{n} \lambda_j y_{1j}, \ldots, \sum_{j=1}^{n} \lambda_j y_{sj}\right) \\
&\min_{\lambda_j} \left(\sum_{j=1}^{n} \lambda_j \mathbf{x}_j\right) = \left(\sum_{j=1}^{n} \lambda_j x_{1j}, \ldots, \sum_{j=1}^{n} \lambda_j x_{mj}\right) \\
&\text{subject to} \\
&\lambda_j \geq 0 \qquad j = 1, \ldots, n.
\end{aligned} \tag{4.18}$$

We have the following equivalent linear programming model

$$\begin{aligned}
&\max_{\lambda_j, \sigma_r, \tau_i} \sum_{r=1}^{s} \overline{w}_r \sigma_r - \sum_{i=1}^{m} \overline{g}_i \tau_i \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j x_{ij} = \tau_i x_{io} \qquad i = 1, \ldots, m; \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = \sigma_r y_{ro}, \qquad r = 1, \ldots, s; \\
&\tau_i \leq 1, \qquad i = 1, \ldots, m; \\
&\sigma_r \geq 1, \qquad r = 1, \ldots, s; \\
&\lambda_j \geq 0, \qquad j = 1, \ldots, n.
\end{aligned} \tag{4.19}$$

Note that $\sigma_r \geq 1$ and $\tau_i \leq 1$ in (4.19). Therefore, we have $\tau_i x_{io} = x_{io} - s_i^-$ and $\sigma_r y_{ro} = y_{ro} + s_r^+$, where s_i^-, s_r^+ $\geq$ 0. Then, (4.19) becomes

$$\max_{\lambda_j, s_i^-, s_r^+} \sum_{r=1}^{s} w_r s_r^+ + \sum_{i=1}^{m} g_i s_i^- - \sum_{i=1}^{m} \overline{g}_i + \sum_{r=1}^{s} \overline{w}_r$$

subject to

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \qquad i = 1,...,m;$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \qquad r = 1,...,s;$$

$$s_i^-, s_r^+, \lambda_j \geq 0.$$

which is a weighted slack-based DEA model (see chapter 3 and Seiford and Zhu (1998)).

4.5 Solving DEA Using DEAFrontier Software

4.5.1 Non-radial Models

To run the non-radial models, select the "Non-radial Model" menu item. You will be prompted with a form as shown in Figure 4.6 for selecting the models presented in Table 4.1. The Results are reported in "Efficiency", "Slack", and "Target" sheets.

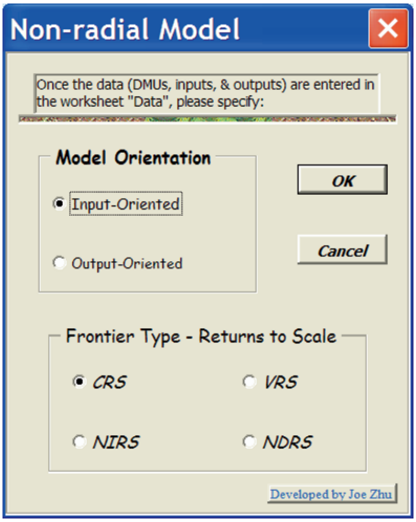

Figure 4.6. Non-radial Models

4.5.2 Preference-Structure Models

To run the preference structure models, select the "Preference Structure Model" menu item. Figure 4.6 shows the form for specifying the models.

If "Yes" is selected under "Restrict Input/Output Change?", then we have weighted non-radial models (see discussion on page 75). If "No" is selected, then we have the DEA/PS models presented in Table 4.2. The software will then ask you to specify the weights for the inputs or outputs, depending on the model orientation. The Results are reported in "Efficiency", "Slack", and "Target" sheets.

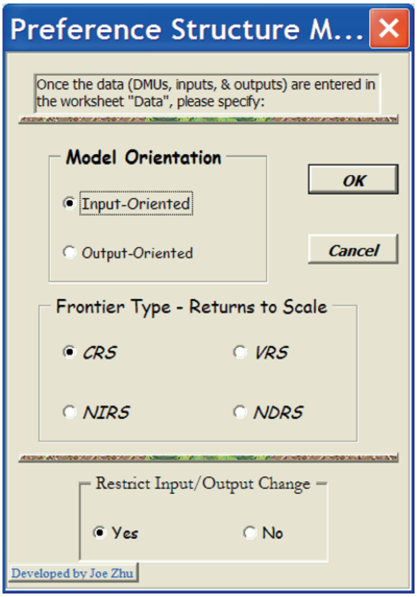

Figure 4.7. Preference Structure Models

4.5.3 Cost Efficiency, Revenue Efficiency and Profit Efficiency

These models need information on the input and output prices. Consider the Hospital example in Cooper, Tone and Seiford (2000). The input and output data are reported in the "Data" sheet (Figure 4.8), input price are reported in the "Input Price" sheet (Figure 4.9) and the output price are reported in the "Output Price" sheet (Figure 4.10).

Hospital	Doctor	Nurse		Outpat.	Inpat.
A	20	151		100	90
B	19	131		150	50
C	25	160		160	55
D	27	168		180	72
E	22	158		94	66
F	55	255		230	90
G	33	235		220	88
H	31	206		152	80
I	30	244		190	100
J	50	268		250	100
K	53	306		260	147
L	38	284		250	120

Figure 4.8. Hospital Data

Hospital	Doctor	Nurse
A	500	100
B	350	80
C	450	90
D	600	120
E	300	70
F	450	80
G	500	100
H	450	85
I	380	76
J	410	75
K	440	80
L	400	70

Figure 4.9. Input Prices

Hospital	outpat.	Inpat.
A	550	2010
B	400	1800
C	480	2200
D	600	3500
E	400	3050
F	430	3900
G	540	3300
H	420	3500
I	350	2900
J	410	2600
K	540	2450
L	295	3000

Figure 4.10. Output Price

The cost efficiency and revenue efficiency are discussed in section 4.2. Table 4.3 summarizes the related models.

Table 4.3. Cost Efficiency and Revenue Efficiency Models

Frontier Type	Cost	Revenue
CRS	$\min \sum_{i=1}^{m} p_i^o \tilde{x}_{io}$ subject to $\sum_{j=1}^{n} \lambda_j x_{ij} \le \tilde{x}_{io} \quad i = 1,2,...,m;$ $\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} \quad r = 1,2,...,s;$ $\lambda_j, \tilde{x}_{io} \ge 0$	$\max \sum_{r=1}^{s} q_r^o \tilde{y}_{ro}$ subject to $\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} \quad i = 1,2,...,m$ $\sum_{j=1}^{n} \lambda_j y_{rj} \ge \tilde{y}_{ro} \quad r = 1,2,...,s$ $\lambda_j, \tilde{y}_{ro} \ge 0$
VRS	Add $\sum_{j=1}^{n} \lambda_j = 1$	
NIRS	Add $\sum_{j=1}^{n} \lambda_j \le 1$	
NDRS	Add $\sum_{j=1}^{n} \lambda_j \ge 1$	

In Table 4.3, p_i^o and q_r^o are unit price of the input *i* and unit price of the output *r* of DMU_o, respectively. These price data may vary from one DMU to another. The cost efficiency and revenue efficiency of DMU_o is defined as

$$\frac{\sum_{i=1}^{m} p_i^o \tilde{x}_{io}^*}{\sum_{i=1}^{m} p_i^o x_{io}} \text{ and } \frac{\sum_{r=1}^{s} q_r^o y_{ro}}{\sum_{r=1}^{s} q_r^o \tilde{y}_{ro}^*}$$

Note that the revenue efficiency is defined as the reciprocal of the one defined in section 4.2. As a result, the cost and revenue efficiency scores are within the range of 0 and 1.

The efficiency scores are reported in the "Cost Efficiency" ("Revenue Efficiency") sheet. The optimal inputs (outputs) are reported in the "OptimalData Cost Efficiency" ("OptimalData Revenue Efficiency") sheet.

Table 4.4 presents the models used to calculate the profit efficiency defined as

$$\frac{\sum_{r=1}^{s} q_r^o y_{ro} - \sum_{i=1}^{m} p_i^o x_{io}}{\sum_{r=1}^{s} q_r^o \tilde{y}_{ro}^* - \sum_{i=1}^{m} p_i^o \tilde{x}_{io}^*}$$

Table 4.4. Profit Efficiency Models

Frontier Type	
CRS	$\max \sum_{r=1}^{s} q_r^o \tilde{y}_{ro} - \sum_{i=1}^{m} p_i^o \tilde{x}_{io}$ subject to $\sum_{j=1}^{n} \lambda_j x_{ij} \leq \tilde{x}_{io} \quad i = 1,2,...,m$ $\sum_{j=1}^{n} \lambda_j y_{rj} \geq \tilde{y}_{ro} \quad r = 1,2,...,s$ $\tilde{x}_{io} \leq x_{io}, \tilde{y}_{ro} \geq y_{ro}$ $\lambda_j \geq 0$
VRS	Add $\sum_{j=1}^{n} \lambda_j = 1$
NIRS	Add $\sum_{j=1}^{n} \lambda_j \leq 1$
NDRS	Add $\sum_{j=1}^{n} \lambda_j \geq 1$

The results are reported in the "Profit Efficiency" and "OptimalData Profit Efficiency" sheets.

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