Chapter 10

Super Efficiency

10.1 Super-efficiency DEA Models

When a DMU under evaluation is not included in the reference set of the envelopment models, the resulting DEA models are called super-efficiency DEA models. Charnes, Haag, Jaska and Semple (1992) use a super-efficiency model to study the sensitivity of the efficiency classifications. Zhu (1996) and Seiford and Zhu (1998) develop a number of new super-efficiency models to determine the efficiency stability regions (see Chapter 11). Andersen and Petersen (1993) propose using the CRS super-efficiency model in ranking the efficient DMUs. Also, the super-efficiency DEA models can be used in detecting influential observations (Wilson, 1995) and in identifying the extreme efficient DMUs (Thrall, 1996). Seiford and Zhu (1999) study the infeasibility of various super-efficiency models developed from the envelopment models in Table 1.2. Chapter 11 presents other super-efficiency models that are used in sensitivity analysis.

Table 10.1 presents the basic super-efficiency DEA models based upon the envelopment DEA models. Based upon Table 10.1, we see that the difference between the super-efficiency and the envelopment models is that the DMU_o under evaluation is excluded from the reference set in the super-efficiency models. i.e., the super-efficiency DEA models are based on a reference technology constructed from all other DMUs.

Consider the example in Table 1.1. If we measure the (CRS) super efficiency of DMU2, then DMU2 is evaluated against point A on the new facet determined by DMUs 1 and 3 (see Figure 10.1). To calculate the (CRS) super efficiency score for DMU2, we use the spreadsheet model shown in Figure 10.2.

Table 10.1. Super-efficiency DEA Models

Frontier Type	Input-Oriented		Output-Oriented	
CRS	$\min \theta^{\text{super}}$ subject to		$\max \phi^{\text{super}}$ subject to	
	$\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \le \theta^{\text{super}} x_{io}$	$i = 1,2,...,m;$	$\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \le x_{io}$	$i = 1,2,...,m;$
	$\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \ge y_{ro}$	$r = 1,2,...,s;$	$\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \ge \phi^{\text{super}} y_{ro}$	$r = 1,2,...,s;$
	$\lambda_j \ge 0$	$j \neq o.$	$\lambda_j \ge 0$	$j \neq o.$
VRS	Add $\sum_{j\neq o} \lambda_j = 1$			
NIRS	Add $\sum_{j\neq o} \lambda_j \le 1$			
NDRS	Add $\sum_{j\neq o} \lambda_j \ge 1$			

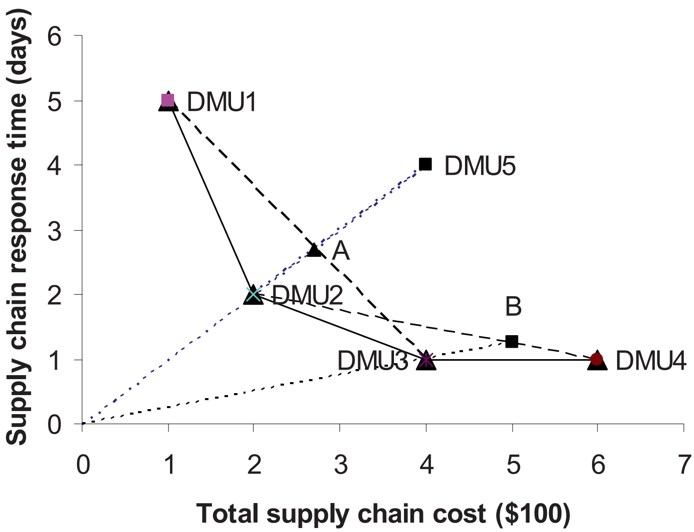

Figure 10.1. Super-efficiency

Cell E9 indicates the DMU under evaluation which is excluded from the reference set. Cells F2:F6 are reserved for λ_j $(j = 1, 2, 3, 4, 5)$, and cell F10 is reserved for the super-efficiency score (θ^{super}).

Cells B11:B13 contain the following formulas

Cell B11 =SUMPRODUCT(B2:B6,F2:F6)
Cell B12 =SUMPRODUCT(C2:C6,F2:F6)

Cell B13 =SUMPRODUCT(E2:E6,F2:F6)

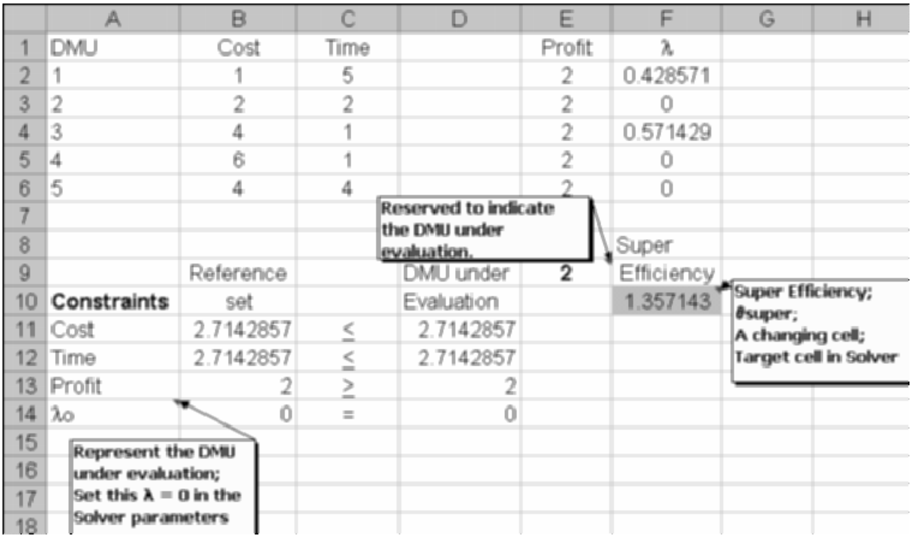

Figure 10.2. Input-oriented CRS Super-efficiency Spreadsheet Model

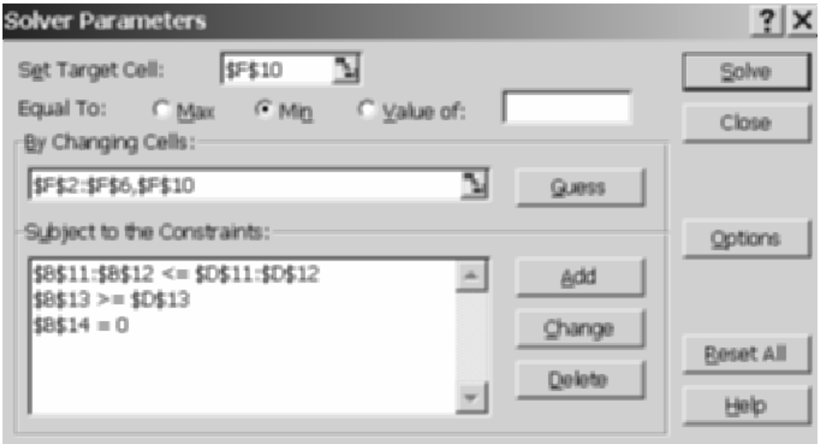

Figure 10.3. Solver Parameters for Input-oriented CRS Super-efficiency

Note that in the above formulas, the DMU under evaluation is included in the reference set. In order to exclude the DMU under evaluation from the reference set, we introduce the following formula into cell B14

Cell B14 =INDEX(F2:F6,E9,1)

which returns the λ_j for the DMU$_j$ under evaluation. In the Solver parameters shown in Figure 10.3, we set cell B14 equal to zero.

Cells D11:D13 contain the following formulas

Cell D11 =$F$10*INDEX(B2:B6,E9,1)
Cell D12 =$F$10*INDEX(C2:C6,E9,1)
Cell D13 =INDEX(E2:E6,E9,1)

Based upon Figure 10.2 and Figure 10.3, the super-efficiency score for DMU2 is 1.357, and the non-zero λ_j in cells F2 and F4 indicate that DMU1 and DMU3 form a new efficient facet.

DMU3 is evaluated against B on the new facet determined by DMUs 2 and 4. If we change the value of cell E9 to 3, we obtain the super-efficiency score for DMU3 using the Solver parameters shown in Figure 10.3. The score is 1.25 (see cell G4 in Figure 10.4).

	A	B	C	D	E	F	G
1	DMU	Cost	Time		Profit	λ	Super Efficiency
2	1	1	5		2	5.55E-17	2
3	2	2	2		2	1	1.357142857
4	3	4	1		2	0	1.25
5	4	6	1		2	0	1
6	5	4	4		2	0	0.5
7							
8						Super	
9		Reference		DMU under	5	Efficiency	
10	**Constraints**	set		Evaluation		0.5	
11	Cost	2	≤	2			
12	Time	2	≤	2		CRS Super-efficiency	
13	Profit	2	≥	2			
14	λo	0	=	0			

Figure 10.4. Super-efficiency Scores

If we remove DMU4 or DMU 5 from the reference set, the frontier remains the same. Therefore, the super-efficiency score for DMU4 (DMU5) equals to the input-oriented CRS efficiency score (see Figure 10.4).

If we measure the super-efficiency of DMU1, DMU1 is evaluated against C on the frontier extended from DMU2 (see Figure 10.5). It can be seen that C is a weakly efficient DMU in the remaining four DMUs 2, 3, 4 and 5. In fact, we may want to adjust such a super-efficiency score (see Zhu (2001b) and Chen and Sherman (2002)).

Although the super-efficiency models can differentiate the performance of the efficient DMUs, the efficient DMUs are not compared to the same "standard". Because the frontier constructed from the remaining DMUs changes for each efficient DMU under evaluation. In fact, the super-efficiency should be regarded the potential input savings or output surpluses (see Chen (2002)).

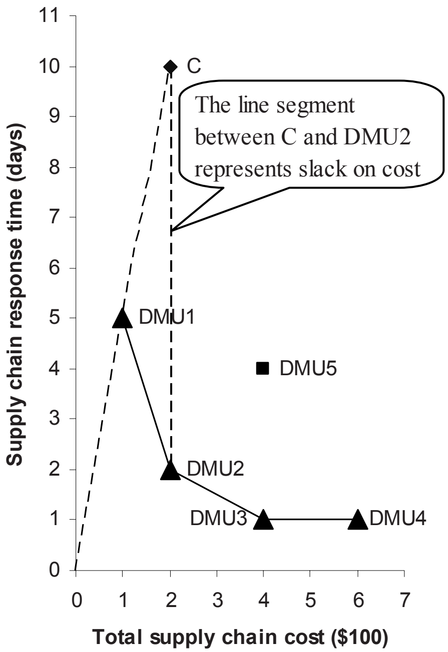

Figure 10.5. Super-efficiency and Slacks

10.2 Infeasibility of Super-efficiency DEA Models

Consider the input-oriented VRS super-efficiency model shown Figure 10.6. In fact, this is the spreadsheet model for the input-oriented VRS envelopment model except that we introduce the formula "=INDEX(I2:I16, E18,1)" into cell B26. This formula is used to exclude the DMU under evaluation from the reference set. That is, one needs to add an additional constraint of "$B$26=0" into the Solver parameters for the input-oriented VRS envelopment spreadsheet model, as shown in Figure 10.7.

Once we set up the Solver parameters, the calculation is performed by the VBA procedure "SuperEfficiency".

```
Sub SuperEfficiency()
Dim i As Integer
```

```
For i = 1 To 15
Range("E18") = i
SolverSolve UserFinish:=True
If SolverSolve(UserFinish:=True) = 5 Then
Range("J" & i + 1) = "Infeasible"
Else
Range("J" & i + 1) = Range("F19")
End If
Next
End Sub
```

	A	B	C	D	E	F	G	H	I	J
1	Company	Assets	Equity	Employees		Revenue	Profit		λ	Super efficiency
2	Mitsubishi	91920.6	10950	36000		184365.2	346.2		0	Infeasible
3	Mitsui	68770.9	5553.9	80000		181518.7	314.8		0	1.751885253
4	Itochu	65708.9	4271.1	7182		169164.6	121.2		0	1.606521649
5	General Motors	217123.4	23345.5	709000		168828.6	6880.7		0	Infeasible
6	Sumitomo	50268.9	6681	6193		167530.7	210.5		0.77	1.320957592
7	Marubeni	71439.3	5239.1	6702		161057.4	156.6		0	1.009347188
8	Ford Motor	243283	24547	346990		137137	4139		0	0.737555958
9	Toyota Motor	106004.2	49691.6	146855		111052	2662.4		0	0.603245345
10	Exxon	91296	40436	82000		110009	6470		0	1.344368672
11	Royal Dutch/Shell Group	118011.6	58986.4	104000		109833.7	6904.6		0	Infeasible
12	Wal-Mart	37871	14762	675000		93627	2740		0.23	1.765155063
13	Hitachi	91620.9	29907.2	331852		84167.1	1468.8		0	0.557595838
14	Nippon Life Insurance	364762.5	2241.9	89690		83206.7	2426.6		0	4.806917693
15	Nippon Telegraph & Telephone	127077.3	42240.1	231400		81937.2	2209.1		0	0.470610997
16	AT&T	88884	17274	299300		79609	139		0	0.533543522
17						Super				
18		Reference		DMU under	15	Efficiency				
19	**Constraints**	set		Evaluation		0.5335435				
20	Assets	47423.482	≤	47423.4824						
21	Equity	8535.6544	≤	9216.4308						
22	Employees	159689.58	≤	159689.576		VRS Super-efficiency				
23	Revenue	150569.21	≥	79609						
24	Profit	791.04056	≥	139						
25	$\Sigma\lambda$	1	=	1						
26	λo	0	=	0						

Figure 10.6. Input-oriented VRS Super-efficiency Spreadsheet Model

It can be seen that the input-oriented VRS super-efficiency model is infeasible for three VRS efficient companies (Mitsubishi, General Motors, and Royal Dutch/Shell Group). Note that in the VBA procedure "SuperEfficiency", a VBA statement on infeasibility check is added.

If we consider the output-oriented VRS super-efficiency model, we have the spreadsheet shown in Figure 10.8. Figure 10.8 is based upon the output-oriented VRS envelopment with an additional formula in cell B26 "=INDEX (I2:I16,E18,1)". To calculate the output-oriented super-efficiency scores, we need to change the "Min" to "Max" in the Solver parameters shown in Figure 10.7.

Based upon Figure 10.8, the output-oriented VRS super-efficiency model is infeasible for five output-oriented VRS efficient companies (Itochu, Sumitomo, Marubeni, Wal-Mart, and Nippon Life Insurance).

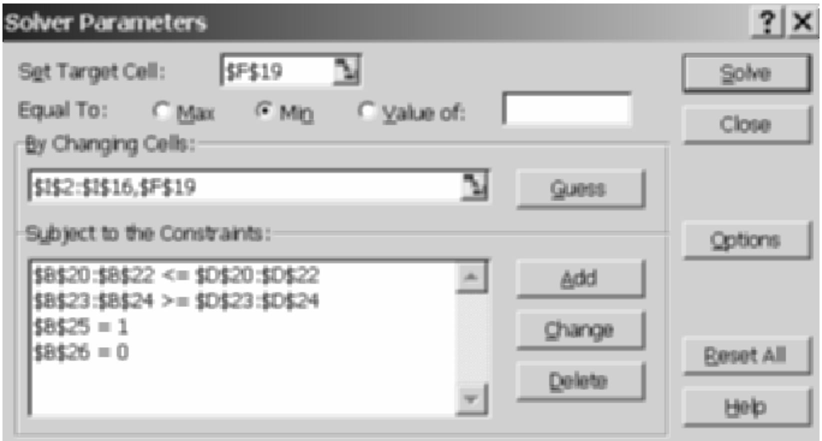

Figure 10.7. Solver Parameters for Input-oriented VRS Super-efficiency

	A	B	C	D	E	F	G	H	I	J
1	Company	Assets	Equity	Employees		Revenue	Profit		λ	Super efficiency
2	Mitsubishi	91920.6	10950	36000		184365.2	346.2		0.869	0.936120353
3	Mitsui	68770.9	5553.9	80000		181518.7	314.8		0.131	0.937264375
4	Itochu	65708.9	4271.1	7182		169164.6	121.2		0	Infeasible
5	General Motors	217123.4	23345.5	709000		168828.6	6880.7		0	0.647119111
6	Sumitomo	50268.9	6681	6193		167530.7	210.5		0	Infeasible
7	Marubeni	71439.3	5239.1	6702		161057.4	156.6		0	Infeasible
8	Ford Motor	243283	24547	346990		137137	4139		0	1.158414974
9	Toyota Motor	106004.2	49691.6	146855		111052	2662.4		0	1.371588284
10	Exxon	91296	40436	82000		110009	6470		0	0.673147631
11	Royal Dutch/Shell Group	118011.6	58986.4	104000		109833.7	6904.6		0	0.939143546
12	Wal-Mart	37871	14762	675000		93627	2740		0	Infeasible
13	Hitachi	91620.9	29907.2	331852		84167.1	1468.8		0	1.898938938
14	Nippon Life Insurance	364762.5	2241.9	89690		83206.7	2426.6		0	Infeasible
15	Nippon Telegraph & Telephone	127077.3	42240.1	231400		81937.2	2209.1		0	1.892916538
16	AT&T	88884	17274	299300		79609	139		0	2.311193684
17						Super				
18		Reference		DMU under	15	Efficiency				
19	**Constraints**	set		Evaluation		2.3111937				
20	Assets	88884	≤	88884						
21	Equity	10242.181	≤	17274		Output-oriented				
22	Employees	41771.582	≤	299300		VRS Super-efficiency				
23	Revenue	183991.82	≥	183991.818						
24	Profit	342.08119	≥	321.255922						
25	Σλ	1	=	1						
26	λo	0	=	0						

Figure 10.8. Output-oriented VRS Super-efficiency Spreadsheet Model

Thrall (1996) shows that the super-efficiency CRS model can be infeasible. However, Thrall (1996) fails to recognize that the output-oriented CRS super-efficiency model is always feasible for the trivial solution which has all variables set equal to zero. Moreover, Zhu (1996) shows that the input-oriented CRS super-efficiency model is infeasible if and only if a certain pattern of zero data occurs in the inputs and outputs.

Figure 10.9 illustrates how the VRS super-efficiency model works and the infeasibility for the case of a single output and a single input case. We have three VRS frontier DMUs, A, B and C. AB exhibits IRS and BC exhibits DRS. The VRS super-efficiency model evaluates point B by reference to B′ and B″ on section AC through output-reduction and input-increment, respectively. In an input-oriented VRS super-efficiency model, point A is evaluated against A′. However, there is no referent DMU for point C for input variations. Therefore, the input-oriented VRS super-efficiency model is infeasible at point C. Similarly, in an output-oriented VRS super-efficiency model, point C is evaluated against C′. However, there is no referent DMU for point A for output variations. Therefore, the output-oriented VRS super-efficiency model is infeasible at point A. Note that point A is the left most end point and point B is the right most end point on this frontier.

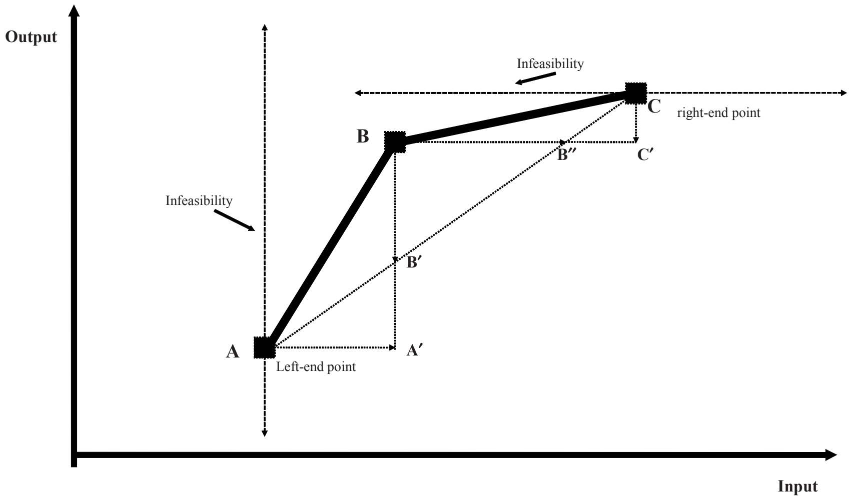

Figure 10.9. Infeasibility of Super-efficiency Model

As in Charnes, Cooper and Thrall (1991), the DMUs can be partitioned into four classes E, E', F and N described as follows. First, E is the set of extreme efficient DMUs. Second, E' is the set of efficient DMUs that are not extreme points. The DMUs in set E' can be expressed as linear combinations of the DMUs in set E. Third, F is the set of frontier points (DMUs) with non-zero slack(s). The DMUs in set F are usually called weakly efficient. Fourth, N is the set of inefficient DMUs.

For example, DMUs 1, 2, and 3 in Figure 10.1 are extreme efficient (in set E), DMU4 is in set F, and DMU5 is in set N.

Thrall (1996) shows that if the CRS super-efficiency model is infeasible, or if the super-efficiency score is greater than one for input-oriented model (less than one for output-oriented model), then $DMU_o \in$ E. This result can also be applied to other super-efficiency models. i.e., the extreme efficient DMUs can be identified by the super-efficiency models. This finding is important in empirical applications. For example, in the slack-based congestion measures discussed in Chapter 9, if we can know that the data set consists of only extreme efficient DMUs, then the congestion slacks are equal to the DEA slacks.

Note that if a specific $DMU_o \in$ E', F or N and is not included in the reference set, then the efficient frontiers (constructed by the DMUs in set E) remain unchanged. As a result, the super-efficiency DEA models are always feasible and equivalent to the original DEA models when $DMU_o \in$ E', F or N. Thus we only need to consider the infeasibility when $DMU_o \in$ E.

We next study the infeasibility of the VRS, NIRS and NDRS super-efficiency models, where we assume that all data are positive.

From the convexity constraint ($\sum_{j \neq o} \lambda_j$ = 1) on the intensity lambda variables, we immediately have

Proposition 10.1 $DMU_o \in$ E under the VRS model *if and only if* $DMU_o \in$ E under the NIRS model or NDRS model.

Thus in the discussion to follow, we limit our consideration to $DMU_o \in$E under the VRS model. We have

Proposition 10.2 Let $\theta^{\text{super*}}$ and $\phi^{\text{super*}}$ denote, respectively, optimal values to the input-oriented and output-oriented super-efficiency DEA models when evaluating an extreme efficient DMU_o, then
(i) Either $\theta^{\text{super*}}$ > 1 or the specific input-oriented super-efficiency DEA model is infeasible.
(ii) Either $\phi^{\text{super*}}$ < 1 or the specific output-oriented super-efficiency DEA model is infeasible.

Based upon Seiford and Zhu (1999), we next (i) present the necessary and sufficient conditions for the infeasibility of various super-efficiency DEA models in a multiple inputs and multiple outputs situation, and (ii) reveal the relationship between infeasibility and RTS classification. (Note that, in Figure 10.9, point A is associated with IRS and point C is associated with DRS.)

10.2.1 Output-oriented VRS Super-efficiency Model

Suppose each DMU_j (j = 1, 2, ..., n) consumes a vector of inputs, x_j, to produce a vector of outputs, y_j. We have

Theorem 10.1 For a specific extreme efficient DMU_o = (x_o, y_o), the output-oriented VRS super-efficiency model is infeasible *if and only if* (x_o, δ y_o) is efficient under the VRS envelopment model for any 0< δ ≤ 1.

[Proof]: Suppose that the output-oriented VRS super-efficiency model is infeasible and that $(x_o, \delta^o y_o)$ is inefficient, where $0 < \delta^o \leq 1$. Then

$$
\begin{aligned}
&\phi_o^{\text{super*}} = \max \phi_o^{\text{super}} \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j x_j \leq x_o \\
&\sum_{j=1}^{n} \lambda_j y_j \geq \phi_o^{\text{super}} (\delta^o y_o) \\
&\sum_{j=1}^{n} \lambda_j = 1
\end{aligned}
\tag{10.1}
$$

has a solution of $\lambda_j^*(j \neq o)$, λ_o^* = 0, $\phi_o^{\text{super*}}$ > 1. Since $\lambda_o^* = 0$, we have that model (10.1) is equivalent to an output-oriented VRS super-efficiency model and thus the output-oriented VRS super-efficiency model is feasible. A contradiction. This completes the proof of the *only if* part.

To establish the *if* part, we note that if the output-oriented VRS super-efficiency model is feasible, then $\phi^{\text{super*}}$ < 1 is the maximum radial reduction of all outputs preserving the efficiency of DMU_o. Therefore, δ cannot be less than $\phi^{\text{super*}}$. Otherwise, DMU_o will be inefficient under the output-oriented VRS envelopment model. Thus, the output-oriented VRS super-efficiency model is infeasible. ■

Theorem 10.2 The output-oriented VRS super-efficiency model is infeasible *if and only if* $\hbar^*$, where $\hbar^*$ > 1 is the optimal value to (10.2).

$$
\begin{aligned}
&\hbar^* = \min \hbar \\
&\text{subject to} \\
&\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j \leq \hbar x_o \\
&\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0, j \neq o
\end{aligned}
\tag{10.2}
$$

[Proof]: We note that for any $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$, the constraint $\sum_{j \neq o} \lambda_j y_j \geq \phi^{\text{super}} y_o$ always holds. Thus the output-oriented super-efficiency-VRS is infeasible if and only if there exists no $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$ such that $\sum_{j \neq o} \lambda_j x_j \leq x_o$ holds. This means that the optimal value to (10.2) is greater than one, i.e., $\hbar^* > 1$. ■

	A	B	C	D	E	F	G	H	I
1	Company	Assets	Equity	Employees			λ	h	Super efficiency
2	Mitsubishi	91920.6	10950	36000			0	0.57426	0.936120353
3	Mitsui	68770.9	5553.9	80000			0	0.8919	0.937264375
4	Itochu	65708.9	4271.1	7182			0	1.20984	Infeasible
5	General Motors	217123.4	23345.5	709000			0	0.25382	0.647119111
6	Sumitomo	50268.9	6681	6193			0.77	1.30639	Infeasible
7	Marubeni	71439.3	5239.1	6702			0	1.00935	Infeasible
8	Ford Motor	243283	24547	346990			0	0.23236	1.158414974
9	Toyota Motor	106004.2	49691.6	146855			0	0.4634	1.371588284
10	Exxon	91296	40436	82000			0	0.54283	0.673147631
11	Royal Dutch/Shell Group	118011.6	58986.4	104000			0	0.42008	0.939143546
12	Wal-Mart	37871	14762	675000			0.23	1.32737	Infeasible
13	Hitachi	91620.9	29907.2	331852			0	0.51532	1.898938938
14	Nippon Life Insurance	364762.5	2241.9	89690			0	1.90513	Infeasible
15	Nippon Telegraph & Telephone	127077.3	42240.1	231400			0	0.38353	1.892916538
16	AT&T	88884	17274	299300			0	0.53354	2.311193684
17									
18		Reference		DMU under	15	h			
19	**Constraints**	set		Evaluation		0.533544			
20	Assets	47423.482	≤	47423.482					
21	Equity	8535.6544	≤	9216.4308		Infeasibility			
22	Employees	159689.58	≤	159689.58					
23	Σλ	1	=	1					
24	λo	0	=	0					

Figure 10.10. Spreadsheet for Infeasibility Test (Output-oriented VRS Super-efficiency)

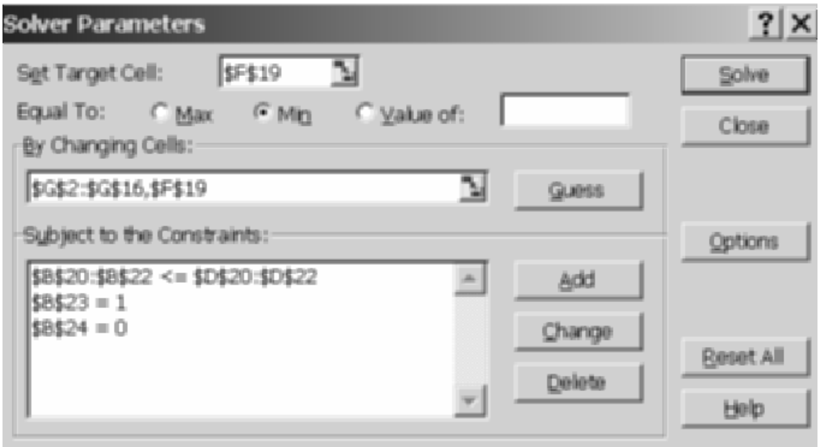

Figure 10.11. Solver Parameters for Infeasibility Test (Output-oriented)

Figure 10.10 shows the spreadsheet model for model (10.2) where the output-oriented VRS super-efficiency scores are reported in cells I2:I16.

The spreadsheet shown in Figure 10.10 is obtained by removing the output constraints from the spreadsheet shown in Figure 10.6. Figure 10.11 shows the Solver parameters. It can be seen that $\hbar^*$ > 1 if and only if model (10.2) is infeasible for a company.

Further, note that the DMU_o is also CRS efficient if and only if CRS prevail . Therefore, if IRS or DRS prevail, then DMU_o must be CRS inefficient. Thus, in this situation, the CRS super-efficiency model is identical to the CRS envelopment model. Based upon Chapter 13, IRS or DRS on DMU_o can be determined by

Lemma 10.1 The RTS for DMU_o can be identified as IRS *if and only if* $\sum_{j \neq 0} \lambda_j^* < 1$ in all optima for the CRS super-efficiency model and DRS *if and only if* $\sum_{j \neq 0} \lambda_j^*$ in all optima for the CRS super-efficiency model.

Lemma 10.2 If DMU_o exhibits DRS, then the output-oriented VRS super-efficiency model is feasible and $\phi^{\text{super}*} < 1$, where $\phi^{\text{super}*}$ is the optimal value to the output-oriented VRS super-efficiency model.

[Proof]: The output-oriented VRS super-efficiency model is as follows

$$
\begin{aligned}
&\phi^{\text{super}*} = \max \phi^{\text{super}} \\
&\text{subject to} \\
&\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j \leq x_o \\
&\sum_{\substack{j=1 \\ j \neq 0}}^{n} \lambda_j y_j \geq \phi^{\text{super}} y_o \\
&\sum_{\substack{j=1 \\ j \neq 0}}^{n} \lambda_j = 1; \\
&\phi^{\text{super}}, \lambda_j \geq 0, j \neq o
\end{aligned}
\tag{10.3}
$$

Let $\theta = 1/\phi^{\text{super}}$. Multiplying all constraints in (10.3) by θ yields

$$
\begin{aligned}
&\min \theta \\
&\text{subject to} \\
&\sum_{\substack{j=1 \\ j \neq o}}^{n} \tilde{\lambda}_j x_j \leq \theta x_o \\
&\sum_{\substack{j=1 \\ j \neq 0}}^{n} \tilde{\lambda}_j y_j \geq y_o
\end{aligned}
\tag{10.4}
$$

$$\sum_{\substack{j=1\\ j\neq 0}}^{n}\tilde{\lambda}_j = \theta = \frac{1}{\phi^{\text{super}}}$$
$$\phi^{\text{super}}, \theta, \tilde{\lambda}_j \geq 0, j \neq o$$

where $\tilde{\lambda}_j = \theta\lambda_j \ (j \neq o)$.

Since DMU_o exhibits DRS, then by Lemma 10.1, $\sum_{j\neq 0}\lambda_j^*$ > 1 in all optima to the following CRS super-efficiency model

$$\begin{aligned}&\min \theta^{\text{super}}\\ &\sum_{\substack{j=1\\ j\neq o}}^{n}\lambda_j x_j \leq \theta^{\text{super}} x_o\\ &\sum_{\substack{j=1\\ j\neq 0}}^{n}\lambda_j y_j \geq y_o\\ &\lambda_j \geq 0.\end{aligned} \tag{10.5}$$

Let $\sum_{j\neq 0}\lambda_j^*$ = θ. Obviously $\theta > \theta^{\text{super}}$ is a feasible solution to (10.5). This in turn indicates that $\lambda_j^* \ (j \neq o)$ and θ is a feasible solution to (10.4). Therefore, (10.3) is feasible. Furthermore by Proposition 10.2, we have that $\phi^{\text{super*}}$ < 1, where $\phi^{\text{super*}}$ is the optimal value to (10.3). ■

Theorem 10.3 If the output-oriented VRS super-efficiency model is infeasible, then DMU_o exhibits IRS or CRS.

[Proof]: Suppose that DMU_o exhibits DRS. By Lemma 10.2, the output-oriented VRS super-efficiency model is feasible. A contradiction. ■

Theorems 10.1 and 10.2 indicate that if the output-oriented VRS super-efficiency model is infeasible, then DMU_o is one of the *endpoints*. Moreover, if IRS prevail, then DMU_o is a *left endpoint* (see Figure 10.9).

10.2.2 Other Output-oriented Super-efficiency Models

Now, consider the output-oriented NIRS and NDRS super-efficiency models. Obviously, we have a feasible solution of $\lambda_j = 0 \ (j \neq o)$ and ϕ^{super} = 0 in the output-oriented NIRS super-efficiency model. Therefore, we have

Theorem 10.4 The output-oriented NIRS super-efficiency model is always feasible.

Lemma 10.3 The output-oriented NDRS super-efficiency model is infeasible *if and only if* the output-oriented VRS super-efficiency model is infeasible.

[Proof]: The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the output-oriented NDRS super-efficiency model is feasible. i.e., we have a feasible solution with $\sum_{j\neq o}\lambda_j \geq 1$ for the output-oriented NDRS super-efficiency model. If $\sum_{j\neq o}\lambda_j = 1$, then this solution is also feasible for the output-oriented VRS super-efficiency. If $\sum_{j\neq o}\lambda_j > 1$, let $\sum_{j\neq o}\lambda_j = d > 1$. Then $\sum_{j\neq o}\tilde{\lambda}_j x_j \leq \sum_{j\neq o}\lambda_j x \leq x_o$, where $\tilde{\lambda}_j = \lambda_j / d$ ($j \neq o$) and $\sum_{j\neq o}\tilde{\lambda}_j = 1$. Therefore $\tilde{\lambda}_j$ ($j \neq o$) is a feasible solution to the output-oriented VRS super-efficiency model. Both possible cases lead to a contradiction. Thus, the output-oriented NDRS super-efficiency model is infeasible if the output-oriented VRS super-efficiency model is infeasible. ■

On the basis of Lemma 10.3, we have

Theorem 10.5 For a specific extreme efficient $DMU_o = (x_o,\ y_o)$, we have
(i) The output-oriented NDRS super-efficiency model is infeasible *if and only if* $(x_o,\ \delta y_o)$ is efficient under the VRS envelopment model for any $0 < \delta \leq 1$.
(ii) The output-oriented NDRS super-efficiency model is infeasible *if and only if* $\hbar^* > 1$, where $\hbar^*$ is the optimal value to (10.2).

If $DMU_o \in E$ for the NDRS model, then DMU_o exhibits IRS or CRS. By Proposition 10.1, DMU_o also lies on the VRS frontier that satisfies IRS or CRS. i.e., the VRS and NDRS envelopment models are identical for DMU_o. Thus, $(x_o,\ \delta y_o)$ is also efficient under the NDRS envelopment model for any $0 < \delta \leq 1$.

10.2.3 Input-oriented VRS Super-efficiency Model

Theorem 10.6 For a specific extreme efficient $DMU_o = (x_o,\ y_o)$, the input-oriented VRS super-efficiency model is infeasible *if and only if* $(\chi x_o, y_o)$ is efficient under the VRS envelopment model for any $1 \leq \chi < +\infty$.

[Proof]: Suppose the input-oriented VRS super-efficiency model is infeasible and assume that $(\chi^o x_o, y_o)$ is inefficient, where $1 \leq \chi^o < +\infty$. Then

$$\begin{aligned}
&\theta_o^{\text{super}*} = \min \theta_o^{\text{super}} \\
&\text{subject to} \\
&\sum_{j=1}^{n} \lambda_j x_j \le \theta_o^{\text{super}} (\chi^o x_o) \\
&\sum_{j=1}^{n} \lambda_j y_j \ge y_o \\
&\sum_{j=1}^{n} \lambda_j = 1
\end{aligned} \tag{10.6}$$

has a solution of $\lambda_j^*(j \neq o)$, λ_o^* = 0, $\theta_o^{\text{super}*}$. Since λ_o^* = 0, model (10.6) is equivalent to the input-oriented VRS super-efficiency model. Thus, the input-oriented VRS super-efficiency model is feasible. This completes the proof of *only if* part.

To establish the *if* part, we note that if the input-oriented VRS super-efficiency model is feasible, then $\theta^{\text{super}*}$ > 1 is the maximum radial increase of all inputs preserving the efficiency of DMU_o. Therefore, χ cannot be bigger than $\theta^{\text{super}*}$. Otherwise, DMU_o will be inefficient under the input-oriented VRS envelopment model. Thus, the input-oriented VRS super-efficiency model is infeasible. ■

Theorem 10.7 The input-oriented super-efficiency-VRS model is infeasible *if and only if* g^* < 1, where g^* is the optimal value to (10.7).

$$\begin{aligned}
&g^* = \max g \\
&\text{subject to} \\
&\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_j \ge g y_o \\
&\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1 \\
&\lambda_j \ge 0, j \neq o
\end{aligned} \tag{10.7}$$

[Proof]: We note that for any $\lambda_j(j \neq o)$ with $\sum_{j\neq o}\lambda_j$ = 1, the constraint $\sum_{j\neq o}\lambda_j x_j$ ≤ $\theta^{\text{super}} x_o$ always holds. Thus, the input-oriented VRS super-efficiency model is infeasible if and only if $\sum_{j\neq o}\lambda_j y_j \ge y_o$ does not hold for any λ_j $(j \neq o)$ with $\sum_{j\neq o}\lambda_j$ = 1. This means that the optimal value to (10.7) is less than one, i.e., g^* < 1.■

Figure 10.12 shows the spreadsheet model for model (10.7) where the input-oriented VRS super-efficiency scores are reported in cells I2:I16. This spreadsheet is obtained from the output-oriented VRS super-efficiency model shown in Figure 10.8. Figure 10.13 shows the Solver parameters. It can be seen that g^* < 1 if and only if model (10.7) is infeasible for a company.

	A	B	C	D	E	F	G	H	I
1	Company	Revenue	Profit	λ				g	Super efficiency
2	Mitsubishi	184365.2	346.2	1				0.9843	Infeasible
3	Mitsui	181518.7	314.8	0				1.0157	1.751885253
4	Itochu	169164.6	121.2	0				1.0899	1.606521649
5	General Motors	168828.6	6880.7	0				0.7623	Infeasible
6	Sumitomo	167530.7	210.5	0				1.1005	1.320957592
7	Marubeni	161057.4	156.6	0				1.1447	1.009347188
8	Ford Motor	137137	4139	0				1.26	0.737555958
9	Toyota Motor	111052	2662.4	0				1.5777	0.603245345
10	Exxon	110009	6470	0				1.0667	1.344368672
11	Royal Dutch/Shell Group	109833.7	6904.6	0				0.9965	Infeasible
12	Wal-Mart	93627	2740	0				1.8493	1.765155063
13	Hitachi	84167.1	1468.8	0				2.1126	0.557595838
14	Nippon Life Insurance	83206.7	2426.6	0				2.0813	4.806917693
15	Nippon Telegraph & Telephone	81937.2	2209.1	0				2.124	0.470610997
16	AT&T	79609	139	0				2.3159	0.533543522
17						Super			
18		Reference		DMU under	15	Efficiency			
19	**Constraints**	set		Evaluation		2.315884			
20	Revenue	184365.2	≥	184365.2					
21	Profit	346.2	≥	321.90786		Infeasibility			
22	Σλ	1	=	1					
23	λo	0	=	0					

Figure 10.12. Spreadsheet for Infeasibility Test (Input-oriented VRS Super-efficiency)

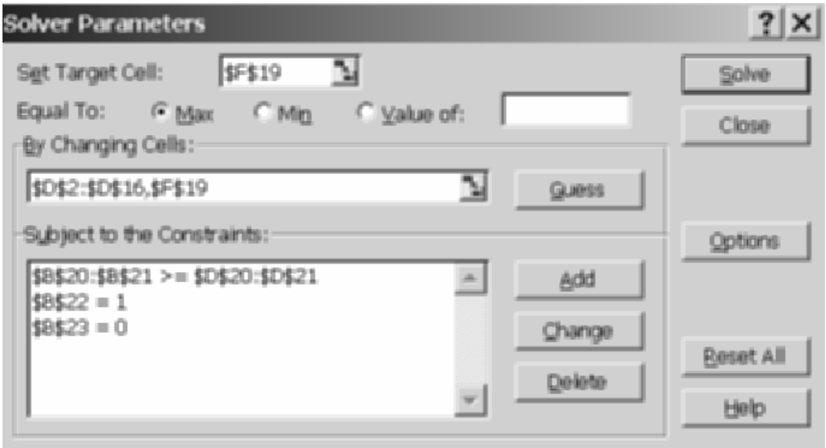

Figure 10.13. Solver Parameters for Infeasibility Test (Input-oriented)

Lemma 10.4 If DMU_o exhibits IRS, then the input-oriented VRS super-efficiency model is feasible and $\theta^{\text{super}*} > 1$, where $\theta^{\text{super}*}$ is the optimal value to the input-oriented VRS super-efficiency model.

[Proof]: Let $\vartheta = 1/\theta^{\text{super}}$, then the input-oriented VRS super-efficiency model becomes

$$
\begin{aligned}
&\max \vartheta \\
&\text{subject to} \\
&\sum_{\substack{j=1 \\ j\neq o}}^{n} \hat{\lambda}_j x_j \le x_o; \\
&\sum_{\substack{j=1 \\ j\neq 0}}^{n} \hat{\lambda}_j y_j \ge \vartheta y_o; \\
&\sum_{\substack{j=1 \\ j\neq 0}}^{n} \hat{\lambda}_j = \vartheta = \frac{1}{\theta^{\text{super}}}; \\
&\theta^{\text{super}}, \vartheta, \hat{\lambda}_j \ge 0.
\end{aligned}
\tag{10.8}
$$

where $\hat{\lambda}_j = \vartheta\lambda_j \ (j \neq o)$.

Since DMU_o exhibits IRS, then by Lemma 10.1, $\sum_{j\neq 0}\lambda_j^* < 1$ in all optima to the following output-oriented CRS super-efficiency model

$$
\begin{aligned}
&\max \phi^{\text{super}} \\
&\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_j \le x_o \\
&\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_j \ge \phi^{\text{super}} y_o \\
&\phi^{\text{super}}, \lambda_j \ge 0.
\end{aligned}
\tag{10.9}
$$

Let $\sum_{j\neq 0}\lambda_j^* = \vartheta < 1$. Since DMU_o is CRS inefficient, therefore $\phi^{\text{super}} >$ 1 and hence $\phi^{\text{super}} > \vartheta$ is a feasible solution to (10.9). This in turn indicates that ϑ and $\lambda_j^*(j \neq o)$ with $\sum_{j\neq 0}\lambda_j^* = \vartheta$ is a feasible solution to (10.8). Therefore, the input-oriented VRS super-efficiency model is feasible. Furthermore, by Proposition 10.2, we have that $\phi^{\text{super}*} > 1$, where $\phi^{\text{super}*}$ is the optimal value to the input-oriented VRS super-efficiency model. ■

Theorem 10.8 If the input-oriented VRS super-efficiency model is infeasible, then DMU_o exhibits DRS or CRS.

[Proof]: If DMU_o exhibits IRS, then by Lemma 10.4, the input-oriented VRS super-efficiency model is feasible. A contradiction.■

Theorems 10.6 and 10.7 indicate that if the input-oriented VRS super-efficiency model is infeasible, then DMU_o is one of the *endpoints*. Furthermore, if DRS prevail, then DMU_o is an *right endpoint* (see Figure 10.9).

10.2.4 Other Input-oriented Super-efficiency Models

Now, consider the input-oriented NIRS and NDRS super-efficiency models.

Theorem 10.9 The input-oriented NDRS super-efficiency model is always feasible.

[Proof]: Since $\sum_{j\neq o}\lambda_j \geq 1$ in the input-oriented DNRS super-efficiency model, there must exist some $\tilde{\lambda}_j$ with $\sum_{j\neq o}\tilde{\lambda}_j > 1$ such that $\sum_{j\neq o}\tilde{\lambda}_j y_j \geq y_o$ holds. Note that $\sum_{j\neq o}\tilde{\lambda}_j x_j \leq \theta^{\text{super}} x_o$ can always be satisfied by a proper θ^{super}. Thus, the input-oriented NDRS super-efficiency model is always feasible. ■

Lemma 10.5 The input-oriented NIRS super-efficiency model is infeasible *if and only if* the input-oriented VRS super-efficiency model is infeasible.

[Proof]: The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the input-oriented NIRS super-efficiency model is feasible. i.e., we have a feasible solution with $\sum_{j\neq o}\lambda_j \leq 1$ for the input-oriented NIRS super-efficiency model. If $\sum_{j\neq o}\lambda_j = 1$, then this solution is also feasible for the output-oriented VRS super-efficiency model. If $\sum_{j\neq o}\lambda_j < 1$, let $\sum_{j\neq o}\lambda_j = e < 1$. Then $\sum_{j\neq o}\hat{\lambda}_j y_j \geq \sum_{j\neq o}\lambda_j y_j \geq y_o$, where $\hat{\lambda}_j = \lambda_j / e$ $(j \neq o)$ and $\sum_{j\neq o}\hat{\lambda}_j = 1$. Therefore $\hat{\lambda}_j (j \neq o)$ is a feasible solution to the output-oriented VRS super-efficiency model. Both possible cases lead to a contradiction. Thus, the output-oriented NIRS super-efficiency model is infeasible if the output-oriented VRS super-efficiency model is infeasible. ■

On the basis of this Lemma 10.5, we have

Theorem 10.10 For a specific extreme efficient $DMU_o = (x_o, y_o)$, we have
(i) The input-oriented NIRS super-efficiency model is infeasible *if and only if* $(\chi x_o, y_o)$ is efficient under the VRS envelopment model for any $1 \leq \chi < +\infty$.
(ii) The input-oriented NIRS super-efficiency model is feasible *if and only if* $g^* < 1$, where g^* is the optimal value to (10.7).

If $DMU_o \in E$ under the NIRS model, then DMU_o exhibits DRS or CRS. By Proposition 10.1, the DMU_o also lies on the VRS frontier that satisfies DRS or CRS. i.e., the VRS and NIRS envelopment models are identical for DMU_o. Thus $(\chi x_o, y_o)$ is also efficient under the NIRS envelopment model for any $1 \leq \chi < +\infty$.

Furthermore, Theorems 10.3 and 10.8 demonstrate that the possible infeasibility of the output-oriented and input-oriented VRS super-efficiency models can only occur at those extreme efficient DMUs exhibiting IRS (or CRS) and DRS (or CRS), respectively. Note that IRS and DRS are not allowed in the NIRS and NDRS models, respectively. Therefore, we have the following corollary.

Corollary 10.1
(i) If $DMU_o \in E$ exhibits DRS, then all output-oriented super-efficiency DEA models are feasible.
(ii) If $DMU_o \in E$ exhibits IRS, then all input-oriented super-efficiency DEA models are feasible.

By Theorems 10.1 and 10.6, we know that infeasibility indicates that the inputs of an extreme efficient DMU_o can be proportionally increased without limit or that the outputs can be decreased in any positive proportion, while preserving the efficiency of DMU_o. This indicates that the efficiency of DMU_o is always stable under the proportional data changes.

Models (10.2) and (10.7) are useful in the determination of infeasibility while Theorems 10.1 and 10.6 are useful in the sensitivity analysis of efficiency classifications. Table 10.2 summarizes the relationship between infeasibility and the super-efficiency DEA models.

Table 10.2. Super-efficiency DEA Models and Infeasibility

Super-efficiency Models		Infeasibility	RTS
Output-oriented	VRS	Theorem 10.2 (Model (10.2))	DRS
	NIRS	always feasible	always feasible
Input-oriented	NDRS	Lemma 10.3, Theorem 10.2	Corollary 10.1 (i)
	VRS	Theorem 10.7 (Model (10.7))	IRS
	NIRS	Lemma 10.5, Theorem 7	always feasible
	NDRS	always feasible	Corollary 10.1 (ii)

Finally, we note that the super-efficiency VRS models can also be used to estimate RTS. This is a possible new usage of the super-efficiency DEA models.

10.3 Solving DEA Using DEAFrontier Software

To run the super-efficiency models presented in Table 10.1, select the “Super-efficiency” menu item. You will be prompted a form shown in Figure 10.4 for specifying the super-efficiency models. The results are reported in the “Super-efficiency” sheet.

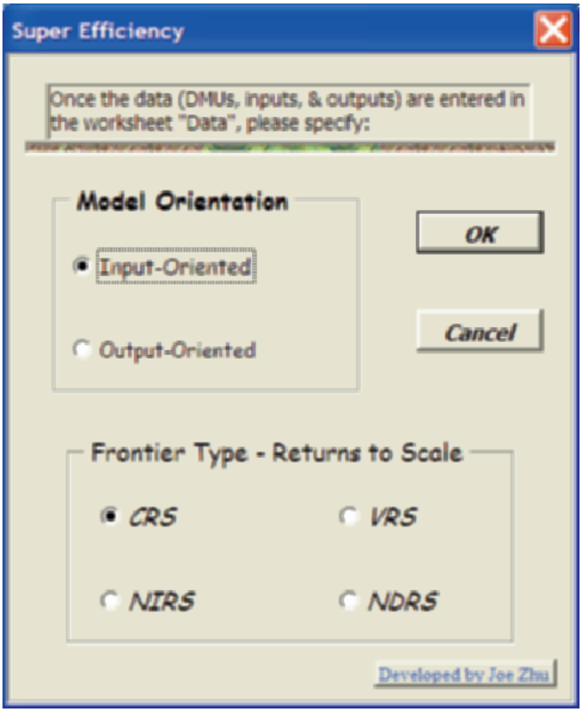

Figure 10.14. Super Efficiency Models

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