

INTERNATIONAL SERIES IN OPERATIONS
RESEARCH AND MANAGEMENT SCIENCE



Quantitative Models for Performance Evaluation and Benchmarking

Data Envelopment Analysis with Spreadsheets

Joe Zhu



 Springer

Quantitative Models for
Performance Evaluation and Benchmarking

Data Envelopment Analysis with Spreadsheets
Second Edition

INT. SERIES IN OPERATIONS RESEARCH & MANAGEMENT SCIENCE

Series Editor: Frederick S. Hillier, Stanford University

Special Editorial Consultant: Camille C. Price, Stephen F. Austin State University

Titles with an asterisk (*) were recommended by Dr. Price

Axsäter/ *INVENTORY CONTROL*, 2nd Ed.

Hall/ *PATIENT FLOW: Reducing Delay in Healthcare Delivery*

Józefowska & Węglarz/ *PERSPECTIVES IN MODERN PROJECT SCHEDULING*

Tian & Zhang/ *VACATION QUEUEING MODELS: Theory and Applications*

Yan, Yin & Zhang/ *STOCHASTIC PROCESSES, OPTIMIZATION, AND CONTROL THEORY
APPLICATIONS IN FINANCIAL ENGINEERING, QUEUEING NETWORKS, AND
MANUFACTURING SYSTEMS*

Saaty & Vargas/ *DECISION MAKING WITH THE ANALYTIC NETWORK PROCESS: Economic,
Political, Social & Technological Applications w. Benefits, Opportunities, Costs & Risks*

Yu/ *TECHNOLOGY PORTFOLIO PLANNING AND MANAGEMENT: Practical Concepts and Tools*

Kandiller/ *PRINCIPLES OF MATHEMATICS IN OPERATIONS RESEARCH*

Lee & Lee/ *BUILDING SUPPLY CHAIN EXCELLENCE IN EMERGING ECONOMIES*

Weintraub/ *MANAGEMENT OF NATURAL RESOURCES: A Handbook of Operations Research Models,
Algorithms, and Implementations*

Hooker/ *INTEGRATED METHODS FOR OPTIMIZATION*

Dawande et al/ *THROUGHPUT OPTIMIZATION IN ROBOTIC CELLS*

Friesz/ *NETWORK SCIENCE, NONLINEAR SCIENCE and INFRASTRUCTURE SYSTEMS*

Cai, Sha & Wong/ *TIME-VARYING NETWORK OPTIMIZATION*

Mamon & Elliott/ *HIDDEN MARKOV MODELS IN FINANCE*

del Castillo/ *PROCESS OPTIMIZATION: A Statistical Approach*

Józefowska/ *JUST-IN-TIME SCHEDULING: Models & Algorithms for Computer & Manufacturing
Systems*

Yu, Wang & Lai/ *FOREIGN-EXCHANGE-RATE FORECASTING WITH ARTIFICIAL NEURAL
NETWORKS*

Beyer et al/ *MARKOVIAN DEMAND INVENTORY MODELS*

Shi & Olafsson/ *NESTED PARTITIONS OPTIMIZATION: Methodology and Applications*

Samaniego/ *SYSTEM SIGNATURES AND THEIR APPLICATIONS IN ENGINEERING RELIABILITY*

Kleijnen/ *DESIGN AND ANALYSIS OF SIMULATION EXPERIMENTS*

Førsund/ *HYDROPOWER ECONOMICS*

Kogan & Tapiero/ *SUPPLY CHAIN GAMES: Operations Management and Risk Valuation*

Vanderbei/ *LINEAR PROGRAMMING: Foundations & Extensions*, 3rd Edition

Chhajed & Lowe/ *BUILDING INTUITION: Insights from Basic Operations Mgmt. Models and Principles*

Luenberger & Ye/ *LINEAR AND NONLINEAR PROGRAMMING*, 3rd Edition

Drew et al/ *COMPUTATIONAL PROBABILITY: Algorithms and Applications in the Mathematical
Sciences**

Chinneck/ *FEASIBILITY AND INFEASIBILITY IN OPTIMIZATION: Algorithms and Computation
Methods*

Tang, Teo & Wei/ *SUPPLY CHAIN ANALYSIS: A Handbook on the Interaction of Information, System
and Optimization*

Ozcan/ *HEALTH CARE BENCHMARKING AND PERFORMANCE EVALUATION: An Assessment using
Data Envelopment Analysis (DEA)*

Wierenga/ *HANDBOOK OF MARKETING DECISION MODELS*

Agrawal & Smith/ *RETAIL SUPPLY CHAIN MANAGEMENT: Quantitative Models and Empirical
Studies*

Brill/ *LEVEL CROSSING METHODS IN STOCHASTIC MODELS*

Zsidisin & Ritchie/ *SUPPLY CHAIN RISK: A Handbook of Assessment, Management & Performance*

Matsui/ *MANUFACTURING AND SERVICE ENTERPRISE WITH RISKS: A Stochastic Management
Approach*

~A list of the early publications in the series is found at the end of the book~

Quantitative Models for
Performance Evaluation and Benchmarking

Data Envelopment Analysis with Spreadsheets
Second Edition

Joe Zhu
Worcester Polytechnic Institute
USA

 Springer

Joe Zhu
Worcester Polytechnic Institute
Worcester, MA, USA
jzhu@wpi.edu

ISSN: 0884-8289

ISBN-13: 978-0-387-85981-1

DOI: 10.1007/978-0-387-85981-1

e-ISBN-13: 978-0-387-85982-8

Library of Congress Control Number: 2008934031

© 2009 by Springer Science+Business Media, LLC

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

springer.com

To Yao, Alec and Andie

Contents

| | |
|--|----|
| Preface..... | xi |
| Chapter 1 Envelopment DEA Models | |
| 1.1 Performance Evaluation, Tradeoffs, and DEA..... | 1 |
| 1.2 Envelopment Model..... | 5 |
| 1.3 Envelopment Models in Spreadsheets..... | 14 |
| 1.4 Solving DEA Using DEA Frontier Software..... | 35 |
| References..... | 41 |
| Chapter 2 Multiplier and Slack-based Models | |
| 2.1 Multiplier Model with Weight Restrictions..... | 43 |
| 2.2 Multiplier Models in Spreadsheets..... | 45 |
| 2.3 Slack-based Model..... | 51 |
| 2.4 Slack-based Models in Spreadsheets..... | 53 |
| 2.5 Solving DEA Using DEA Frontier Software..... | 57 |
| References..... | 61 |
| Chapter 3 Measure-specific DEA Models | |
| 3.1 Measure-specific Models..... | 63 |
| 3.2 Measure-specific Models in Spreadsheets..... | 64 |
| 3.3 Performance Evaluation of Fortune 500 Companies..... | 66 |
| 3.4 Solving DEA Using DEA Frontier Software..... | 75 |
| References..... | 76 |

| | |
|---|-----|
| Chapter 4 Non-radial DEA Models and DEA with Preference | |
| 4.1 Non-radial DEA Models | 77 |
| 4.2 DEA with Preference Structure and Cost/Revenue Efficiency ... | 79 |
| 4.3 DEA/Preference Structure Models in Spreadsheets | 83 |
| 4.4 DEA and Multiple Objective Linear Programming | 85 |
| 4.5 Solving DEA Using DEA Frontier Software | 90 |
| References | 95 |
| Chapter 5 Modeling Undesirable Measures | |
| 5.1 Introduction..... | 97 |
| 5.2 Efficiency Invariance..... | 98 |
| 5.3 Undesirable Outputs | 99 |
| 5.4 Undesirable Inputs | 103 |
| 5.5 Solving DEA Using DEA Frontier Software..... | 104 |
| Appendix: Negative Data..... | 105 |
| References | 107 |
| Chapter 6 Context-dependent Data Envelopment Analysis | |
| 6.1 Introduction..... | 109 |
| 6.2 Stratification DEA Method | 111 |
| 6.3 Input-oriented Context-dependent DEA | 115 |
| 6.4 Output-oriented Context-dependent DEA..... | 123 |
| 6.5 Solving DEA Using DEA Frontier Software | 127 |
| References | 129 |
| Chapter 7 Benchmarking Models | |
| 7.1 Introduction..... | 131 |
| 7.2 Variable-benchmark Model | 132 |
| 7.3 Fixed-benchmark Model | 143 |
| 7.4 Fixed-benchmark Model and Efficiency Ratio | 146 |
| 7.5 Minimum Efficiency Model..... | 150 |
| 7.6 Buyer-seller Efficiency Model..... | 153 |
| 7.7 Solving DEA Using DEA Frontier Software | 158 |
| References | 160 |
| Chapter 8 Models for Evaluating Supply Chains | |
| 8.1 Supply Chain Efficiency | 161 |
| 8.2 Supply Chain Efficiency | 163 |
| 8.3 Cooperative and Non-Cooperative Approaches..... | 174 |
| References | 186 |

Chapter 9 Congestion

9.1 Congestion Measure..... 187
9.2 Congestion and Slacks 193
9.3 Slack-based Congestion Measure 195
9.4 Solving DEA Using DEA Frontier Software 201
References 203

Chapter 10 Super Efficiency

10.1 Super-efficiency DEA Models..... 205
10.2 Infeasibility of Super-efficiency DEA Models 209
10.3 Solving DEA Using DEA Frontier Software 223
References 225

Chapter 11 Sensitivity Analysis

11.1 DEA Sensitivity Analysis 227
11.2 Stability Region..... 230
11.3 Infeasibility and Stability 242
11.4 Simultaneous Data Change 246
11.5 Solving DEA Using DEA Frontier Software 264
References 266

Chapter 12 Identifying Critical Measures in DEA

12.1 Introduction..... 269
12.2 Performance Evaluation and DEA 269
12.3 Identifying Critical Output Measures..... 274
12.4 Identifying Critical Input Measures 275
12.5 Numerical Example and Extension 277
12.6 Application to Fortune E-Companies..... 278
References 287

Chapter 13 Returns-to-Scale

13.1 Introduction..... 289
13.2 RTS Regions 290
13.3 RTS Estimation 291
13.4 Scale Efficient Targets 299
13.5 Solving DEA Using DEA Frontier Software 301
References 303

Chapter 14 DEA Models for Two-Stage Processes

14.1 Introduction..... 305
14.2 VRS Two-Stage Model..... 306

| | |
|--|-----|
| 14.3 CRS Two-Stage Model | 312 |
| 14.4 Solving DEA Using DEA Frontier Software | 317 |
| References | 318 |
| Author Index | 319 |
| Topic Index | 323 |

Preface

Managers are often under great pressure to improve the performance of their organizations. To improve performance, one needs to constantly evaluate operations or processes related to producing products, providing services, and marketing and selling products. Performance evaluation and benchmarking are a widely used method to identify and adopt best practices as a means to improve performance and increase productivity, and are particularly valuable when no objective or engineered standard is available to define efficient and effective performance. For this reason, benchmarking is often used in managing service operations, because service standards (benchmarks) are more difficult to define than manufacturing standards.

Benchmarks can be established but they are somewhat limited as they work with single measurements one at a time. It is difficult to evaluate an organization's performance when there are multiple inputs and outputs to the system. The difficulties are further enhanced when the relationships between the inputs and the outputs are complex and involve unknown tradeoffs. It is critical to show benchmarks where multiple measurements exist. The current book introduces the methodology of data envelopment analysis (DEA) and its uses in performance evaluation and benchmarking under the context of multiple performance measures.

DEA uses mathematical programming techniques and models to evaluate the performance of peer units (e.g., bank branches, hospitals and schools) in terms of multiple inputs used and multiple outputs produced. DEA examines the resources available to each unit and monitors the "conversion" of these resources (inputs) into the desired outputs. Researchers in a number of fields have quickly recognized that DEA is an excellent methodology for modeling operational processes. DEA's empirical orientation and absence of *a priori* assumptions have resulted in its use in a number of studies involving efficient frontier estimation in the nonprofit sector, in the regulated sector,

and in the private sector. DEA applications involve a wide range of contexts, such as education, health care, banking, armed forces, auditing, market research, retail outlets, organization effectiveness, transportation, public housing, and manufacturing.

The motivation for this book is three-fold. First, as DEA is being applied to a variety of efficiency evaluation problems, managers may want to conduct performance evaluation and analyze decision alternatives without the help of sophisticated modeling programs. For this purpose, spreadsheet modeling is a suitable vehicle. In fact, spreadsheet modeling has been recognized as one of the most effective ways to evaluate decision alternatives. It is easy for the managers to apply various DEA models in spreadsheets. The book introduces spreadsheet modeling into DEA, and shows how various conventional and new DEA approaches can be implemented using Microsoft® Excel and Solver. With the assistance of the developed DEA spreadsheets, the user can easily develop new DEA models to deal with specific evaluation scenarios.

Second, new models for performance evaluation and benchmarking are needed to evaluate business operations and processes in a variety of contexts. After briefly presenting the basic DEA techniques, the current book introduces new DEA models and approaches. For example, a context-dependent DEA measures the relative attractiveness of competitive alternatives. Sensitivity analysis techniques can be easily applied, and used to identify critical performance measures. Two-stage DEA models deal with multi-stage efficiency evaluation problems. DEA benchmarking models incorporate benchmarks and standards into DEA evaluation.

All these new models can be useful in benchmarking and analyzing complex operational efficiency in manufacturing organizations as well as evaluating processes in banking, retail, franchising, health care, e-business, public services and many other industries.

Third, although the spreadsheet modeling approach is an excellent way to build new DEA models, an integrated easy-to-use DEA software can be helpful to managers, researchers, and practitioners. Therefore the current version includes a DEAFrontier software for both Excel 97-2003 and Excel 2007. DEAFrontier is a DEA Add-In for Microsoft Excel and offers the user the ability to perform a variety of DEA models and approaches – it provides a custom Excel menu which calculates more than 150 different DEA models.

This second edition improves a number of DEA spreadsheet models and provides a DEAFrontier software for use with Excel 2007. Several new DEA models and approaches are added. For example, a new DEA-based supply chain model (chapter 8) and DEA models for two-stage processes (Chapter 14) are new additions to the book. Models with restricted multipliers are also discussed and added into the DEAFrontier software. A detailed use of

sensitivity analysis in identifying critical measures under DEA is provided. A demonstration of how to use the DEAFrontier software is provided at the end of related chapters.

I would like to offer my sincere thanks to my mentor, friend and collaborator, Dr. Lawrence M. Seiford who helped and enabled me to contribute to dual areas of DEA methodology and applications, and to Dr. William W. Cooper who constantly supports my DEA research. I also want to thank Dr. Frederick S. Hiller – the series Editor, and Gary Folven and Carolyn Ford of Springer for their support in publishing this edition of the book.

I would also like to thank Dr. Timothy Anderson, David B. Learner and Bao-cheng Zhang for pointing some of the errors in the first edition. However, any errors in this edition are entirely my responsibility, and I would be grateful if anyone would bring any such errors to my attention.

Joe Zhu, February 2008.

Chapter 1

Envelopment DEA Models

1.1 Performance Evaluation, Tradeoffs, and DEA

All business operations/processes involve transformation – adding values and changes to materials and turning them into goods and services that customers want. The transformation involves the use of inputs made up of labor, materials, energy, machines, and other resources, and the generation of outputs of finished products, services, customer satisfaction, and other outcomes. Consider hospital operations, for example. The inputs include doctors, nurses, medical supplies, equipment, laboratories, beds and others, and outputs include number of patients treated, number of interns and residents trained, and others. Managers are often interested in evaluating how efficiently various processes operate with respect to multiple performance measures (or inputs and outputs). For example, in a buyer-seller supply chain, the buyer may be interested in comparing the performance of several sellers with respect to response time, costs, flexibility, customer service, quality, and customization. Eliminating or improving inefficient operations decreases the cost of inputs and increases productivity. Performance evaluation and benchmarking help business operations/processes to become more productive and efficient.

Performance evaluation is an important continuous improvement tool for staying competitive and plays an important role in the high-technology world of computers and telecommunications where competition is intense and grows more so each day. Performance evaluation and benchmarking positively force any business unit to constantly evolve and improve in order to survive and prosper in a business environment facing global competition. Through performance evaluation, one can (i) reveal the strengths and weaknesses of business operations, activities, and processes; (ii) better prepare the business to meet its customers' needs and requirements; and (iii)

identify opportunities to improve current operations and processes, and create new products, services and processes.

Single-measure based gap analysis is often used as a fundamental method in performance evaluation and benchmarking. However, as pointed out by Camp (1995), one of the dilemmas that we face is how to show benchmarks where multiple measurements exist. It is rare that one single measure can suffice for the purpose of performance evaluation. The single output to input financial ratios, such as, return on investment (ROI) and return on sales (ROS), may be used as indices to characterize the financial performance. However, they are unsatisfactory discriminants of “best-practice”, and are not sufficient to evaluate operating efficiency. Since a business unit’s performance is a complex phenomenon requiring more than a single criterion to characterize it. For example, as pointed out by Sherman (1984), a bank branch may be profitable when profit reflects the interest and the revenues earned on funds generated by the branch less the cost of these funds and less the costs of operating the branch. However, this profit measure does not indicate whether the resources used to provide customer services are being managed efficiently.

Further, the use of single measures ignores any interactions, substitutions or tradeoffs among various performance measures. Each business operation has specific performance measures with tradeoffs. For example, consider the tradeoff between total supply chain cost and supply chain response time, measured by the amount of time between an order and its corresponding delivery. Figure 1.1 illustrates alternate supply chain operations S1, S2, S3, and S, and the efficient frontier or tradeoff curve determined by them. A supply chain whose performance (or strategy) is on the efficient frontier is non-dominated (efficient) in the sense that no alternate supply chain’s performance is strictly better in both cost and response time. Through performance evaluation, the efficient frontier that represents the best practice is identified, and an inefficient strategy (e.g., point S) can be improved (moved to the efficient frontier) with suggested directions for improvement (to S1, S2, S3 or other points along the frontier).

Optimization techniques can be used to estimate the efficient frontier if we know the functional forms for the relationships among various performance measures. For example, stockout levels and inventory turns are two mutually dependent variables with performance tradeoffs. Technological and process innovations can shift the cost tradeoff curves by reducing the cost of achieving lower inventories at a particular stockout level or the cost of achieving lower stockouts at a particular inventory level. Unfortunately, such information is usually not completely available.

Without *a priori* information on the tradeoffs, the functional forms cannot be specified. Consequently, we cannot fully characterize the business

operations and processes. Note that the objective of performance evaluation is to evaluate the current business operation internally and to benchmark against similar business operations externally to identify the best practice. Thus, such best-practices can be empirically identified. We can empirically estimate the efficient frontier based upon observations on one business operation/process over time or similar business operations at a specific time period.

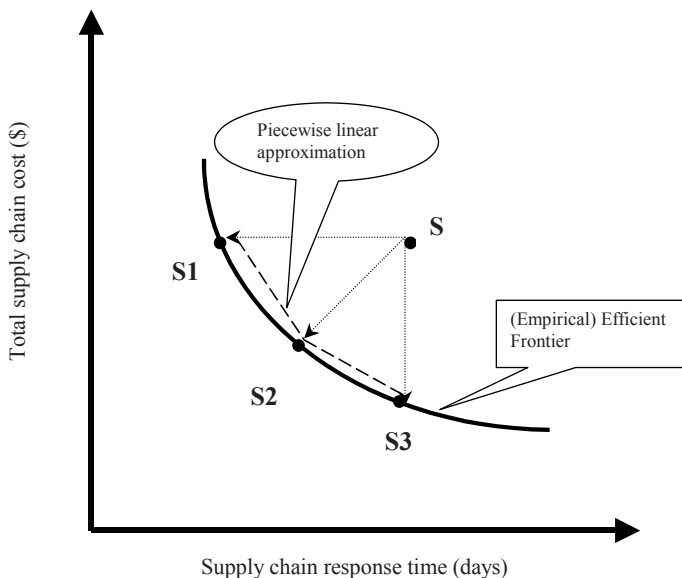


Figure 1.1. Efficient Frontier of Supply Chain Operations

Throughout the book, we use decision making units (DMUs) to represent business operations or processes. Each DMU has a set of inputs and outputs, representing multiple performance measures. Consider a set of n observations on the DMUs. Each observation, DMU_j ($j = 1, \dots, n$), uses m inputs x_{ij} ($i = 1, 2, \dots, m$) to produce s outputs y_{rj} ($r = 1, 2, \dots, s$).

The (empirical) efficient frontier or best-practice frontier is determined by these n observations. The following two properties ensure that we can develop a piecewise linear approximation to the efficient frontier and the area dominated by the frontier.

Property 1.1 Convexity. $\sum_{j=1}^n \lambda_j x_{ij}$ ($i = 1, 2, \dots, m$) and $\sum_{j=1}^n \lambda_j y_{rj}$ ($r = 1, 2, \dots, s$) are possible inputs and outputs achievable by the DMU_j , where λ_j ($j = 1, \dots, n$) are nonnegative scalars such that $\sum_{j=1}^n \lambda_j = 1$.

Property 1.2 Inefficiency. The same y_{rj} can be obtained by using \hat{x}_{ij} , where $\hat{x}_{ij} \geq x_{ij}$ (i.e., the same outputs can be produced by using more inputs); The same x_{ij} can be used to obtain \hat{y}_{rj} , where $\hat{y}_{rj} \leq y_{rj}$ (i.e., the same inputs can be used to produce less outputs).

Consider Figure 1.1 where total supply chain cost and supply chain response time represent two inputs. Applying Property 1.1 to S1, S2, and S3 yields the piecewise linear approximation to the curve shown in Figure 1.1. Applying both properties expands the line segments S1S2 and S2S3 into the area dominated by the curve.

For specific x_i ($i = 1, 2, \dots, m$) and y_i ($r = 1, 2, \dots, s$), we have

$$\begin{cases} \sum_{j=1}^n \lambda_j x_{ij} \leq x_i & i = 1, 2, \dots, m \\ \sum_{j=1}^n \lambda_j y_{rj} \geq y_i & r = 1, 2, \dots, s \\ \sum_{j=1}^n \lambda_j = 1 \end{cases} \quad (1.1)$$

The next step is to estimate the empirical (piecewise linear) efficient frontier characterized by (1.1). Data Envelopment Analysis (DEA) introduced by Charnes, Cooper and Rhodes (1978) has been proven an effective tool in identifying such empirical frontiers and in evaluating relative efficiency. DEA uses mathematical programming to implicitly estimate the tradeoffs inherent in the empirical efficient frontier.

DEA was designed to measure the relative efficiency where market prices are not available (see, e.g., Charnes, Cooper and Rhodes, 1981; Johnson and Zhu, 2002). However, by its ability to model multiple-input and multiple-output relationships without *a priori* underlying functional form assumption, DEA has also been widely applied to other areas. For example, Bank failure prediction (Barr and Siems, 1997), electric utilities evaluation (Färe, Grosskopf, Logan and Lovell, 1985), textile industry performance (Zhu, 1996), steel industry productivity (Ray, Seiford and Zhu, 1998), highway maintenance efficiency (Cook, Roll and Kazakov, 1990), health care (Chilingerian and Sherman, 2004), software development (Banker and Kemerer, 1989), spatial efficiency (Desai and Storbeck, 1990), sports (Anderson and Sharp, 1997), logistics systems (Kleinsorge, Schary and Tanner, 1989) among others. See Charnes, Cooper, Lewin and Seiford (1994) for a collection of DEA applications and Cooper, Seiford and Zhu (2004) for surveys in particular application areas. Such previous DEA studies provide useful managerial information on improving the performance. In particular, DEA is an excellent tool for improving the productivity of service businesses (Sherman and Zhu, 2006).

In the current book, we present various DEA approaches that can be used in empirical efficient frontier estimation and further in performance evaluation and benchmarking. For readers who are interested in detailed discussion on fundamental DEA, please refer to Cooper, Seiford and Tone (2000), Cooper, Seiford and Zhu (2004), Zhu and Cook (2007), and Cook and Zhu (2008).

1.2 Envelopment Model

1.2.1 Envelopment Models with Variable Returns to Scale

Two alternative approaches are available in DEA to determine the efficient frontier characterized by (1.1). One is input-oriented, and the other output-oriented.

The following DEA model is an input-oriented model where the inputs are minimized and the outputs are kept at their current levels (Banker, Charnes and Cooper, 1984)

$$\begin{aligned}
 \theta^* &= \min \theta \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i = 1, 2, \dots, m; \\
 &\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\
 &\sum_{j=1}^n \lambda_j = 1 \\
 &\lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{1.2}$$

where DMU_o represents one of the n DMUs under evaluation, and x_{io} and y_{ro} are the i th input and r th output for DMU_o , respectively.

Since $\theta = 1$ is a feasible solution to (1.2), the optimal value to (1.2), $\theta^* \leq 1$. If $\theta^* = 1$, then the current input levels cannot be reduced (proportionally), indicating that DMU_o is on the frontier. Otherwise, if $\theta^* < 1$, then DMU_o is dominated by the frontier. θ^* represents the (input-oriented) efficiency score of DMU_o .

Consider a simple numerical example shown in Table 1.1 where we have five DMUs (supply chain operations). Within a week, each DMU generates the same profit of \$2,000 with a different combination of supply chain cost and response time.

Table 1.1. Supply Chain Operations Within a Week

| DMU | Cost (\$100) | Response time (days) | Profit (\$1,000) |
|-----|--------------|----------------------|------------------|
| 1 | 1 | 5 | 2 |
| 2 | 2 | 2 | 2 |
| 3 | 4 | 1 | 2 |
| 4 | 6 | 1 | 2 |
| 5 | 4 | 4 | 2 |

Figure 1.2 presents the five DMUs and the piecewise linear frontier. DMUs 1, 2, 3, and 4 are on the frontier. If we calculate model (1.2) for DMU5,

$$\begin{aligned}
 & \text{Min } \theta \\
 & \text{Subject to} \\
 & 1 \lambda_1 + 2\lambda_2 + 4\lambda_3 + 6\lambda_4 + 4\lambda_5 \leq 4\theta \\
 & 5 \lambda_1 + 2\lambda_2 + 1\lambda_3 + 1\lambda_4 + 4\lambda_5 \leq 4\theta \\
 & 2 \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 \geq 2 \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\
 & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
 \end{aligned}$$

we obtain a set of unique optimal solutions of $\theta^* = 0.5$, $\lambda_2^* = 1$, and $\lambda_j^* = 0$ ($j \neq 2$), indicating that DMU2 is the benchmark for DMU5, and DMU5 should reduce its cost and response time to the amounts used by DMU2.

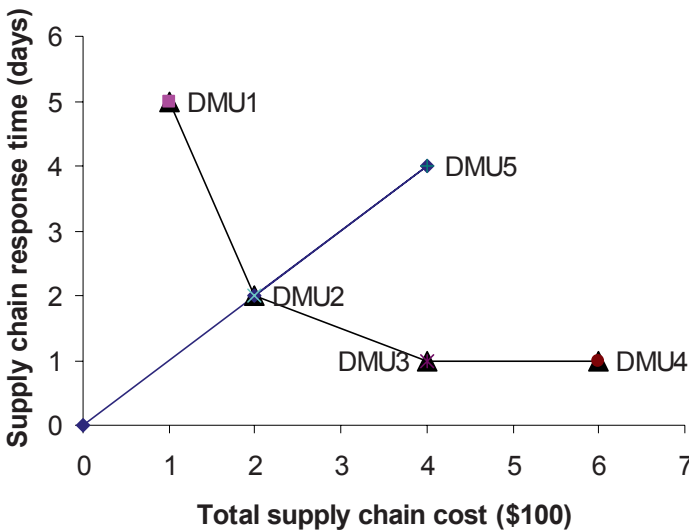


Figure 1.2. Five Supply Chain Operations

Now, if we calculate model (1.2) for DMU4, we obtain $\theta^* = 1$, $\lambda_4^* = 1$, and $\lambda_j^* = 0$ ($j \neq 4$), indicating that DMU4 is on the frontier. However, Figure 1.2 indicates that DMU4 can still reduce its response time by 2 days to reach DMU3. This individual input reduction is called input slack.

In fact, both input and output slack values may exist in model (1.2). Usually, after calculating (1.2), we have

$$\begin{cases} s_i^- = \theta^* x_{io} - \sum_{j=1}^n \lambda_j x_{ij} & i = 1, 2, \dots, m \\ s_r^+ = \sum_{j=1}^n \lambda_j y_{rj} - y_{ro} & r = 1, 2, \dots, s \end{cases} \quad (1.3)$$

where s_i^- and s_r^+ represent input and output slacks, respectively. An alternate optimal solution of $\theta^* = 1$ and $\lambda_3^* = 1$ exists when we calculate model (1.2) for DMU4. This leads to $s_1^- = 2$ for DMU4. However, if we obtain $\theta^* = 1$ and $\lambda_4^* = 1$ from model (1.2), we have all zero slack values. i.e., because of possible multiple optimal solutions, (1.3) may not yield all the non-zero slacks.

Therefore, we use the following linear programming model to determine the possible non-zero slacks after (1.2) is solved.

$$\begin{aligned} & \max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta^* x_{io} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \quad (1.4)$$

For example, applying (1.4) to DMU4 yields

$$\begin{aligned} & \text{Max } s_1^- + s_2^- + s_1^+ \\ & \text{Subject to} \\ & 1 \lambda_1 + 2\lambda_2 + 4\lambda_3 + 6\lambda_4 + 4\lambda_5 + s_1^- = 6 \theta^* = 6 \\ & 5 \lambda_1 + 2\lambda_2 + 1\lambda_3 + 1\lambda_4 + 4\lambda_5 + s_2^- = \theta^* = 1 \\ & 2 \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 - s_1^+ = 2 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, s_1^-, s_2^-, s_1^+ \geq 0 \end{aligned}$$

with optimal slacks of $s_1^{*-} = 2$, $s_2^{*-} = s_1^{*+} = 0$.

DMU_o is efficient if and only if $\theta^* = 1$ and $s_i^{-*} = s_r^{+*} = 0$ for all i and r . DMU_o is weakly efficient if $\theta^* = 1$ and $s_i^{-*} \neq 0$ and (or) $s_r^{+*} \neq 0$ for some i and r . In Figure 1.2, DMUs 1, 2, and 3 are efficient, and DMU 4 is weakly efficient. (The slacks obtained by (1.4) are called DEA slacks, see Definition 9.2 in Chapter 9.)

In fact, models (1.2) and (1.4) represent a two-stage DEA process involved in the following DEA model.

$$\begin{aligned}
 & \min \theta - \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{1.5}$$

The presence of the non-Archimedean ε in the objective function of (1.5) effectively allows the minimization over θ to preempt the optimization involving the slacks, s_i^- and s_r^+ . Thus, (1.5) is calculated in a two-stage process with maximal reduction of inputs being achieved first, via the optimal θ^* in (1.2); then, in the second stage, movement onto the efficient frontier is achieved via optimizing the slack variables in (1.4).

In fact, the presence of weakly efficient DMUs is the cause of multiple optimal solutions. Thus, if weakly efficient DMUs are not present, the second stage calculation (1.4) is not necessary, and we can obtain the slacks using (1.3). However, priori to calculation, we usually do not know whether weakly efficient DMUs are present.

Note that the frontier determined by model (1.5) exhibits variable returns to scale (VRS). Model (1.5) is called input-oriented VRS envelopment model. (see Chapter 13 for a detailed discussion on DEA and Returns-to-Scale (RTS).)

Consider Figure 1.3 where we have 5 DMUs (A, B, C, D, and E) with one input and one output. The VRS frontier consists of DMUs A, B, C, and D. AB exhibits increasing RTS (IRS), B exhibits constant RTS (CRS), and BC and CD exhibit decreasing RTS (DRS).

Based upon (1.5), DMU E is inefficient and is compared to F (a convex combination of A and B) on the VRS frontier, i.e., E should reduce its input to F, or F is the efficient target for E. If we use an output-oriented model, E is compared to C by increasing output.

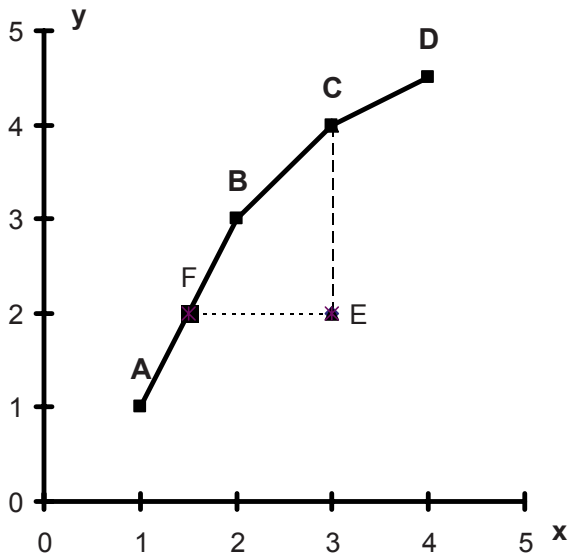


Figure 1.3. VRS Frontier

The output-oriented VRS envelopment model can be expressed as

$$\begin{aligned}
 & \max \phi + \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{1.6}$$

Model (1.6) is also calculated in a two-stage process. First, we calculate ϕ^* by ignoring the slacks, and then we optimize the slacks by fixing the ϕ^* in the following linear programming problem.

$$\begin{aligned}
 & \max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi^* y_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{1.7}$$

DMU_o is efficient if and only if $\phi^* = 1$ and $s_i^{-*} = s_r^{+*} = 0$ for all i and r . DMU_o is weakly efficient if $\phi^* = 1$ and $s_i^{-*} \neq 0$ and (or) $s_r^{+*} \neq 0$ for some i and r . If weakly efficient DMUs are not present, then we need not to calculate (1.7), and we can obtain the slacks via

$$\begin{cases} s_i^- = x_{io} - \sum_{j=1}^n \lambda_j x_{ij} & i = 1, 2, \dots, m \\ s_r^+ = \sum_{j=1}^n \lambda_j y_{rj} - \phi^* y_{ro} & r = 1, 2, \dots, s \end{cases}$$

Note that $\phi^* \geq 1$, and $\phi^* = 1$ if and only if $\theta^* = 1$. This indicates that models (1.5) and (1.6) identify the same frontier. Also, $\theta^* = 1/\phi^*$ (see Lemma 3.2 in Chapter 13).

Figure 1.2 shows an input efficient frontier when outputs are fixed at their current levels. Similarly, we can obtain an output efficient frontier when inputs are fixed at their current levels. Consider the four DMUs shown in Figure 1.4 where we have two outputs.

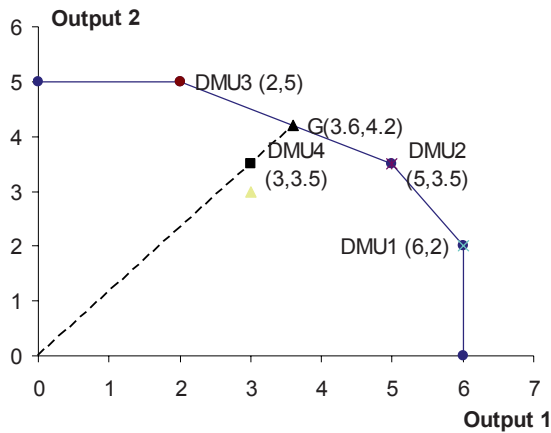


Figure 1.4. Output Efficient Frontier

In Figure 1.4, DMUs 1, 2 and 3 are efficient. If we calculate model (1.6) for DMU4, we have

$$\begin{aligned}
 & \text{Max } \phi \\
 & \text{Subject to} \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \leq 1 \\
 & 6\lambda_1 + 5\lambda_2 + 2\lambda_3 + 3\lambda_4 \geq 3\phi \\
 & 2\lambda_1 + 3.5\lambda_2 + 5\lambda_3 + 3.5\lambda_4 \geq 3.5\phi \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
 & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
 \end{aligned}$$

The optimal solution is $\phi^* = 1.2$, $\lambda_2^* = 8/15$, and $\lambda_3^* = 7/15$. i.e., DMU4 is inefficient and is compared to G in Figure 1.4, or DMU4 should increase its two output levels to G.

1.2.2 Other Envelopment Models

The constraint on $\sum_{j=1}^n \lambda_j$ actually determines the RTS type of an efficient frontier. If we remove $\sum_{j=1}^n \lambda_j = 1$ from models (1.5) and (1.6), we obtain CRS envelopment models where the frontier exhibits CRS. Figure 1.5 shows an CRS frontier – ray OB. Based upon this CRS frontier, only B is efficient.

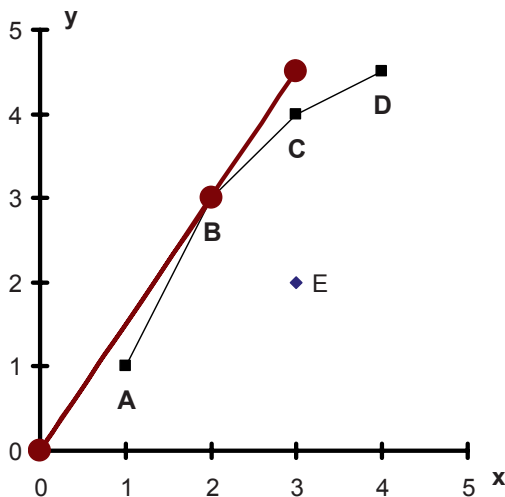


Figure 1.5. CRS Frontier

If we replace $\sum_{j=1}^n \lambda_j = 1$ with $\sum_{j=1}^n \lambda_j \leq 1$, then we obtain non-increasing RTS (NIRS) envelopment models. In Figure 1.6, the NIRS frontier consists of DMUs B, C, D and the origin.

If we replace $\sum_{j=1}^n \lambda_j = 1$ with $\sum_{j=1}^n \lambda_j \geq 1$, then we obtain non-decreasing RTS (NDRS) envelopment models. In Figure 1.7, the NDRS frontier consists of DMUs A, B, and the section starting with B on ray OB.

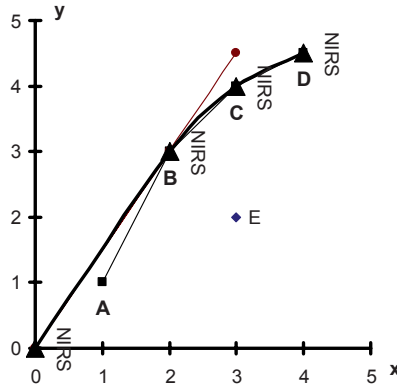


Figure 1.6. NIRS Frontier

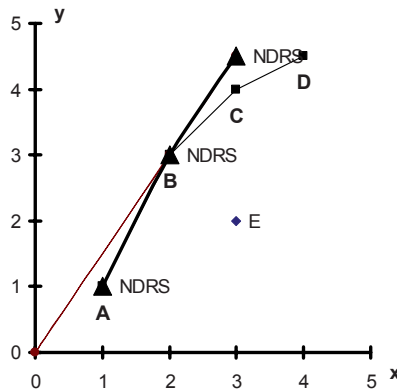


Figure 1.7. NDRS Frontier

Table 1.2 summarizes the envelopment models with respect to the orientations and frontier types. The last row presents the efficient target (DEA projection) of a specific DMU under evaluation.

Table 1.2. Envelopment Models

| Frontier Type | Input-Oriented | Output-Oriented |
|------------------|---|---|
| | $\min \theta - \varepsilon (\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+)$ | $\max \phi + \varepsilon (\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+)$ |
| | subject to | subject to |
| CRS | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad i = 1, 2, \dots, m;$ | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m;$ |
| | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s;$ | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi y_{ro} \quad r = 1, 2, \dots, s;$ |
| | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ |
| VRS | | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | | Add $\sum_{j=1}^n \lambda_j \geq 1$ |
| Efficient Target | $\begin{cases} \hat{x}_{io} = \theta^* x_{io} - s_i^{-*} & i = 1, 2, \dots, m \\ \hat{y}_{ro} = y_{ro} + s_r^{+*} & r = 1, 2, \dots, s \end{cases}$ | $\begin{cases} \hat{x}_{io} = x_{io} - s_i^{-*} & i = 1, 2, \dots, m \\ \hat{y}_{ro} = \phi^* y_{ro} + s_r^{+*} & r = 1, 2, \dots, s \end{cases}$ |

The interpretation of the envelopment model results can be summarized as

i) If $\theta^* = 1$ or $\phi^* = 1$, then the DMU under evaluation is a frontier point. i.e., there is no other DMUs that are operating more efficiently than this DMU. Otherwise, if $\theta^* < 1$ or $\phi^* > 1$, then the DMU under evaluation is inefficient. i.e., this DMU can either increase its output levels or decrease its input levels.

ii) The left-hand-side of the envelopment models is usually called the “Reference Set”, and the right-hand-side represents a specific DMU under evaluation. The non-zero optimal λ_j^* represent the benchmarks for a specific DMU under evaluation. The Reference Set provides coefficients (λ_j^*) to define the hypothetical efficient DMU. The Reference Set or the efficient target shows how inputs can be decreased and outputs increased to make the DMU under evaluation efficient.

1.3 Envelopment Models in Spreadsheets

Table 1.3 presents 15 companies from the top Fortune Global 500 list in 1995. We have three inputs: (1) number of employees, (2) assets (\$ millions), and (3) equity (\$ millions), and two outputs: (1) profit (\$ millions), and (2) revenue (\$ millions).

1.3.1 Input-oriented VRS Envelopment Spreadsheet Model

The input-oriented VRS envelopment model (model (1.5)) requires 15 calculations – one for each company. We illustrate how to formulate this efficiency evaluation problem in a spreadsheet, and then illustrate how Excel Solver can be used to calculate the efficiency scores for the 15 companies.

We begin by organizing the data in Table 1.3 in a spreadsheet (see Figure 1.8). A spreadsheet model of an envelopment model contains the following four major components: (1) cells for the decision variables (e.g., λ_j and θ); (2) cell for the objective function (efficiency) (e.g., θ); (3) cells containing formulas for computing the DEA reference set (the right-hand-side of the constraints) ($\sum_{j=1}^n \lambda_j x_{ij}$, $\sum_{j=1}^n \lambda_j y_{rj}$, and $\sum_{j=1}^n \lambda_j$); and (4) cells containing formulas for computing the DMU under evaluation (left-hand-sided of the constraints) (e.g., θx_{io} and y_{ro}).

Table 1.3. Fortune Global 500 Companies

| Company | Assets | Equity | Employees | Revenue | Profit |
|------------------------------|----------|---------|-----------|----------|--------|
| Mitsubishi | 91920.6 | 10950 | 36000 | 184365.2 | 346.2 |
| Mitsui | 68770.9 | 5553.9 | 80000 | 181518.7 | 314.8 |
| Itochu | 65708.9 | 4271.1 | 7182 | 169164.6 | 121.2 |
| General Motors | 217123.4 | 23345.5 | 709000 | 168828.6 | 6880.7 |
| Sumitomo | 50268.9 | 6681 | 6193 | 167530.7 | 210.5 |
| Marubeni | 71439.3 | 5239.1 | 6702 | 161057.4 | 156.6 |
| Ford Motor | 243283 | 24547 | 346990 | 137137 | 4139 |
| Toyota Motor | 106004.2 | 49691.6 | 146855 | 111052 | 2662.4 |
| Exxon | 91296 | 40436 | 82000 | 110009 | 6470 |
| Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | 109833.7 | 6904.6 |
| Wal-Mart | 37871 | 14762 | 675000 | 93627 | 2740 |
| Hitachi | 91620.9 | 29907.2 | 331852 | 84167.1 | 1468.8 |
| Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | 83206.7 | 2426.6 |
| Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | 81937.2 | 2209.1 |
| AT&T | 88884 | 17274 | 299300 | 79609 | 139 |

In Figure 1.8, cells I2 through I16 represent λ_j ($j = 1, 2, \dots, 15$). Cell F19 represents the efficiency score θ which is the objective function.

For the DEA reference set (left-hand-side of the envelopment model), we enter the following formulas that calculate the weighted sums of inputs and outputs across all DMUs, respectively.

- Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)
- Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)
- Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)
- Cell B23 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)
- Cell B24 =SUMPRODUCT(G2:G16,\$I\$2:\$I\$16)

For the DMU under evaluation (DMU1: Mitsubishi), we enter the following formulas into cells D20:D24.

- Cell D20 =\$F\$19*INDEX(B2:B16,E18,1)
- Cell D21 =\$F\$19*INDEX(C2:C16,E18,1)
- Cell D22 =\$F\$19*INDEX(D2:D16,E18,1)
- Cell D23 =INDEX(F2:F16,E18,1)
- Cell D24 =INDEX(G2:G16,E18,1)

| | A | B | C | D | E | F | G | H | I | J | K |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|------------|---|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency | changing cell |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | | changing cell |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | | changing cell |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | | changing cell |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | | changing cell |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0.77 | | changing cell |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | | changing cell |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | | changing cell |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 0.6 | changing cell |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | | changing cell |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | | changing cell |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.23 | | changing cell |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.5 | changing cell |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | | changing cell |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.6 | changing cell |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.5 | changing cell |
| 17 | | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | | Reserved to indicate the DMU under evaluation. |
| 19 | Constraints | set | | Evaluation | | 0.533544 | | | | | Efficiency: 0; A changing cell; Target cell in Solver |
| 20 | Assets | 47423.482 | \leq | 47423.482 | | | | | | | |
| 21 | Equity | 8535.6544 | \leq | 9216.4308 | | | | | | | |
| 22 | Employees | 159689.58 | \leq | 159689.58 | | | | | | | |
| 23 | Revenue | 150569.21 | \geq | 79609 | | | | | | | |
| 24 | Profit | 791.04056 | \geq | 139 | | | | | | | |
| 25 | $\Sigma \lambda$ | 1 | $=$ | 1 | | | | | | | |

Figure 1.8. Input-oriented VRS Envelopment Spreadsheet Model

Finally, we enter the formula for $\sum_{j=1}^n \lambda_j = 1$ into cells B25 (=SUM(I2:I16)) and D25 (=1), respectively.

Cell E18 is reserved to indicate the DMU under evaluation. The function INDEX(array,row number,column number) returns the value in the specified row and column of the given array. Because cell E18 contains the current value of 1, the INDEX function in cell D23 returns the value in first row and

first column of the Revenue array F2:F16 (or the value in cell F2, the Revenue output for DMU1). When the value in cell E18 changes from 1 to 15, the INDEX functions in cells D20:D24 return the input and output values for a specific DMU under evaluation. This feature becomes obvious and useful when we provide the Visual Basic for Applications (VBA) code to automate the DEA computation.

1.3.2 Using Solver

After the DEA model is set up in the spreadsheet, we can use Solver to find the optimal solutions. First, we need to invoke Solver in Excel by using the **Solver** command which is available in the **Data** tab as shown in Figure 1.9.

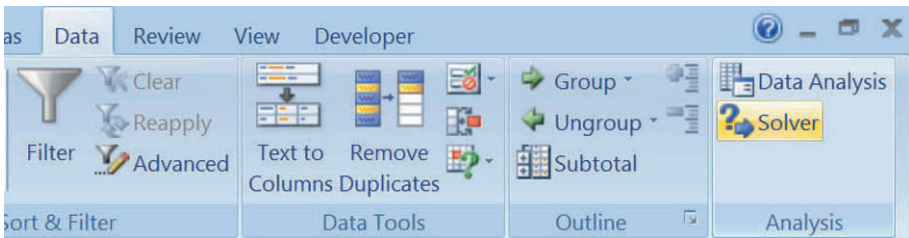


Figure 1.9. Display Solver Parameters Dialog Box

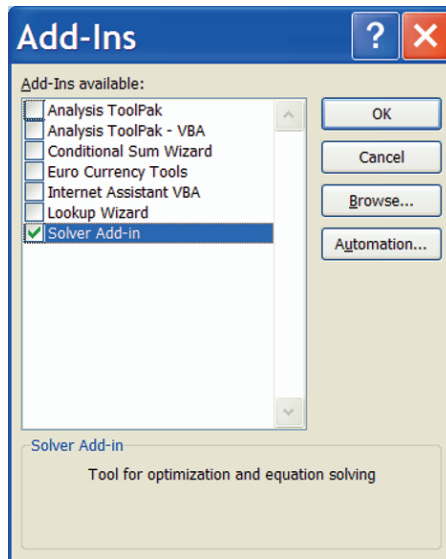



Figure 1.10. Solver Add-In

If Solver does not exist, you need to do the followings:

1. Click the **Microsoft Office Button** , and then click **Excel Options**.
2. Click **Add-Ins**, and then in the **Manage** box, select **Excel Add-ins**.
3. Click **Go**.
4. In the **Add-Ins available** box, select the **Solver Add-in** check box, and then click **OK**. (If **Solver Add-in** is not listed in the **Add-Ins available** box, click **Browse** to locate the add-in. If you get prompted that the Solver Add-in is not currently installed on your computer, click **Yes** to install it.)

Now, you should see the Solver Parameters dialog box shown in Figure 1.11.

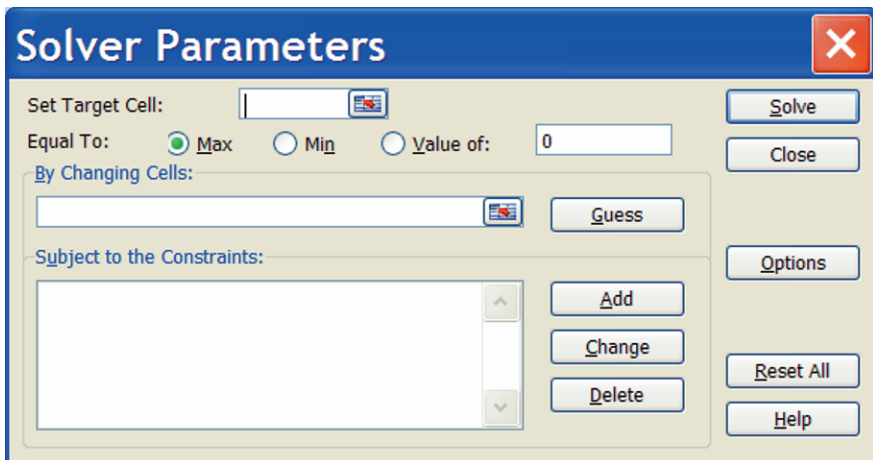


Figure 1.11. Solver Parameters Dialog Box

1.3.3 Specifying the Target Cell

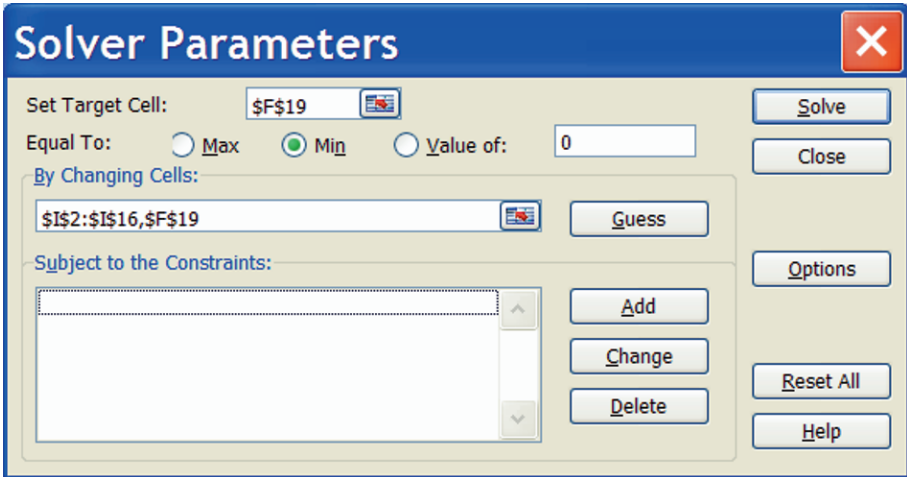


Figure 1.12. Specifying Target Cell and Changing Cells

Set Target Cell indicates the objective function cell in the spreadsheet, and whether its value should be maximized or minimized. In our case, the target cell is the DEA efficiency represented by cell F19, and its value should be minimized, because we use the input-oriented VRS envelopment model (1.5) (see Figure 1.12).

1.3.4 Specifying Changing Cells

Changing Cells represent the decision variables in the spreadsheet. In our case, they represent the λ_j ($j = 1, 2, \dots, 15$) and θ , and should be cells I2:I16 and F19, respectively (see Figure 1.12).

1.3.5 Adding Constraints

Constraints represent the constraints in the spreadsheet. In our case, they are determined by cells B20:B25 and D20:D25. For example, click the Add button shown in Figure 1.12, you will see the Add Constraint dialog box shown in Figure 1.13.

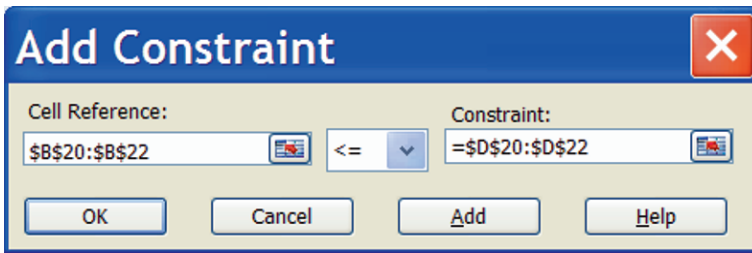


Figure 1.13. Adding Constraints

In the spreadsheet model shown in Figure 1.8, we have six constraints. The “Cell Reference” corresponds to the DEA Reference Set, and “Constraint” corresponds to the DMU under evaluation. The first three constraints are related to the three inputs (see Figure 1.13). Click the Add button to add additional constraints (output constraints and $\sum_{j=1}^n \lambda_j = 1$), and click the OK button when you have finished adding the constraints. The set of the constraints are shown in Figure 1.15.

1.3.6 Non-Negativity and Linear Model

Note that λ_j and θ are all non-negative, and the envelopment model is a linear programming problem. This can be achieved by clicking the Option button in Figure 1.12, and then checking the Assume Non-Negative and Assume Linear Model boxes, as shown in Figure 1.14. This action should be performed for each DEA model. In the rest of the book, we will not show the Solver Options dialog box.

When the Assuming Linear Model option is checked, Solver conducts a number of internal tests to see if the model is truly linear. When the data are poorly scaled, Solver may show that the conditions for linearity are not satisfied. To circumvent this, we may check the box of “Use Automatic Scaling” in the Solver Options dialog box.

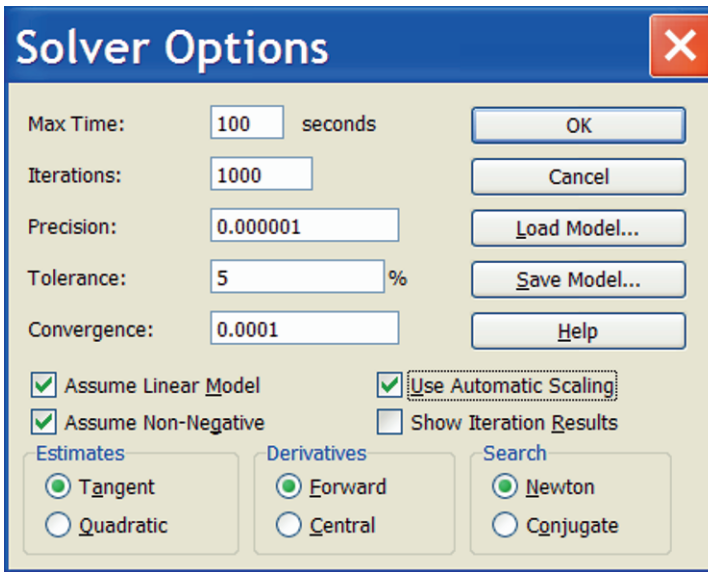


Figure 1.14. Non-Negative and Linear Model

1.3.7 Solving the Model

Now, we have successfully set up the Solver Parameters dialog box, as shown in Figure 1.15. Click the Solve button to solve the model. When Solver finds an optimal solution, it displays the Solver Results dialog box, as shown in Figure 1.16.

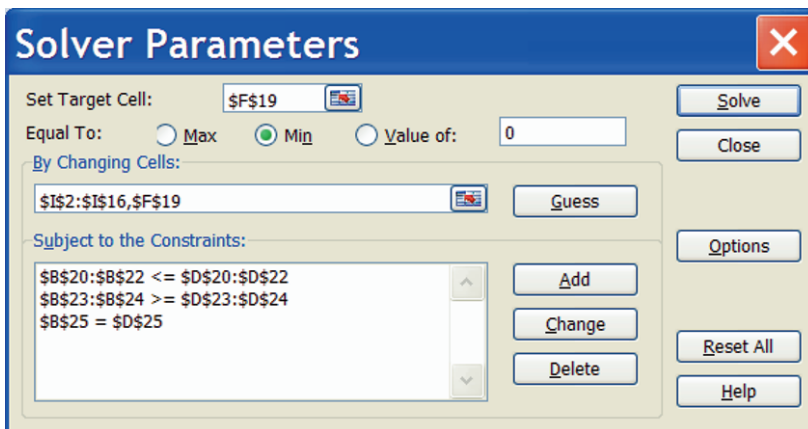


Figure 1.15. Solver Parameters for Input-oriented VRS Envelopment Model

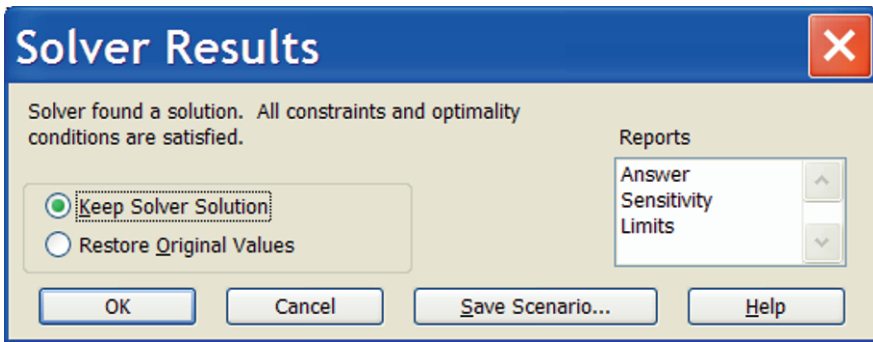


Figure 1.16. Solver Results Dialog Box

1.3.8 Automating the DEA Calculation

To complete the analysis for the remaining 14 companies, one needs to manually change the value in cell E18 to 2, 3, ..., 15 and use Solver to re-optimize the spreadsheet model for each company and record the efficiency scores (in column J, for instance). When the number of DMUs becomes large, the manual process is apparently cumbersome. Note that exactly the same Solver settings will be used to find the optimal solutions for the remaining DMUs. This allows us to write a simple VBA code to carry out the process automatically.

Before we write the VBA code, we need to set a reference to Solver Add-In in Visual Basic (VB) Editor. Otherwise, VBA will not recognize the Solver functions and you will get a “Sub or function not defined” error message.

We may follow the following procedure to set the reference. Enter the VB Editor by pressing *Alt-F11* key combination (or using the Developer Tab¹). Open the Tools/References menu in the VB Editor. This brings up a list of references. One of these should be Solver (**Solver.xlam**²) (see Figure 1.17).

To add the reference, simply check its box. If it says “Missing: Solver.xla”, then click the Browse button and search for **Solver.xlam**. If you are using Excel 2007, the **Solver.xlam** is usually located at C:\Program Files\Microsoft Office\Office12\Library\ Solver. However, this depends on where the Microsoft Office is installed.

¹ To display the Developer Tab, please check “Show Developer tab in the Ribbon” box under the Excel Options.

² In Excel 97-2003, the reference is Solver.xla.

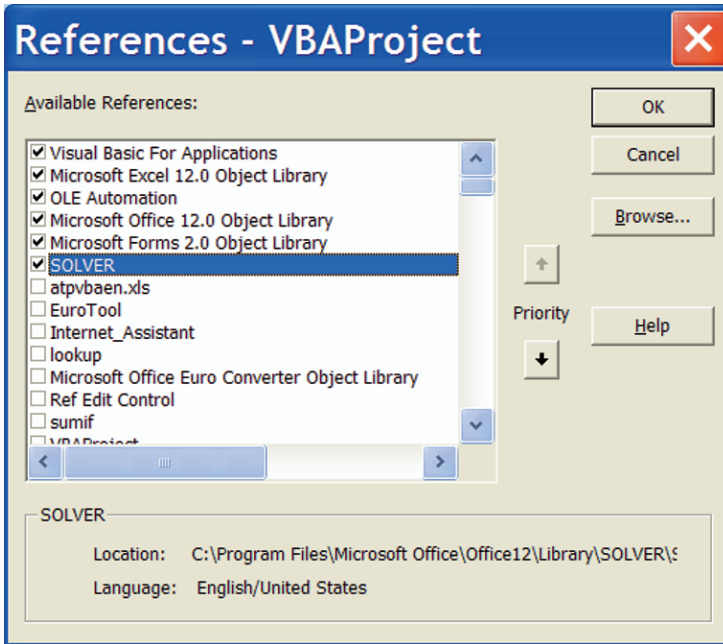


Figure 1.17. Adding Reference to Solver Add-In

After the Solver reference is added, we should see “Reference to Solver.xlam” under the “References” in the VBA Project Explorer window shown in Figure 1.18. (The file “envelopment spreadsheet.xls” in the CD contains the spreadsheet model.)

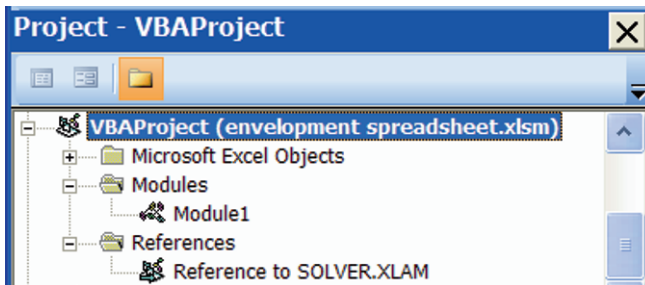


Figure 1.18. Reference to Solver Add-In in VBA Project

Next, select the Insert/Module menu item in the VB Editor (Figure 1.19). This action will add a Module (e.g., Module1) into the Excel file. (You can change the name of the inserted module in the Name property of the module.)

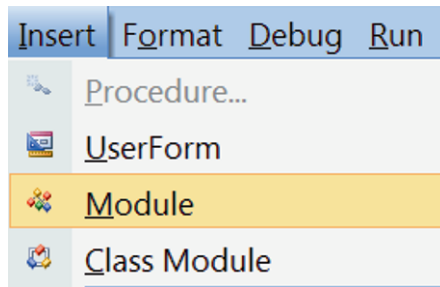


Figure 1.19. Insert a Module

Now, we can insert the VBA code into the Module1. Type “Sub DEA()” in the code window. This generates a VBA procedure called DEA which is also the Macro name (see Figure 1.21). Figure 1.20 shows the VBA code for automating the DEA calculation.

The Macro statement “SolverSolve UserFinish:=True” tells the Solver to solve the DEA problem without displaying the Solver Results dialog box. The “Offset(rowOffset, columnOffset)” property takes two arguments that correspond to the relative position from the upper-left cell of the specified Range. When we evaluate the first DMU, i.e., DMUNo = 1, Range(“J1”).Offset(1,0) refers to cell J2. The statements “With Range(“J1”) and “.Offset(DMUNo, 0)=Range(“F19”) take the optimal objective function value (efficiency score) in cell F19 and place it in cell J “DMUNo” (that is, cell J2, J3, ..., J16).

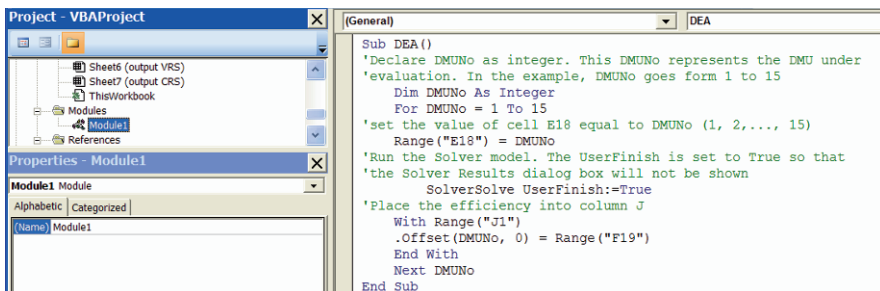


Figure 1.20. VBA Code for Input-oriented VRS Envelopment Model

Enter the Run Macro dialog box by pressing *Alt-F8* key combination (or using the Developer/Macros tab). You should see “DEA”, as shown in Figure 1.21. Select “DEA” and then click the Run button. This action will generate the efficiency scores (cells J2:J16) for the 15 companies, as shown in Figure 1.22.

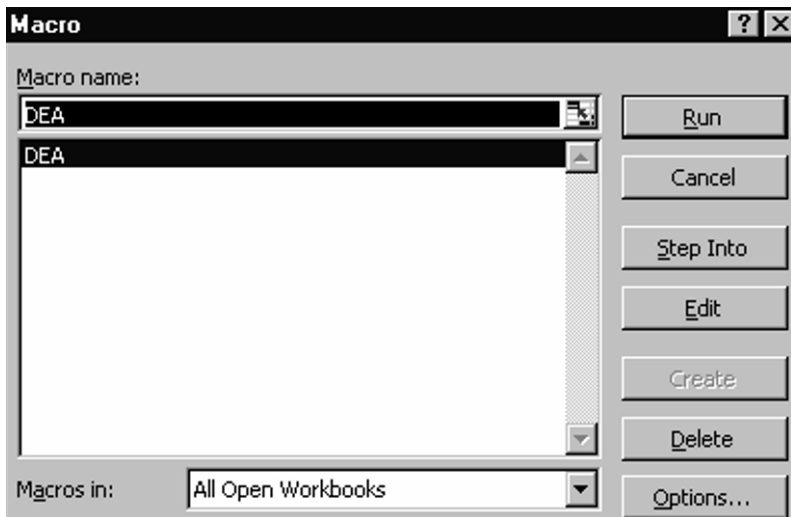


Figure 1.21. Run "DEA" Macro

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0.77 | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 1 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.737556 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 0.603245 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 1 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.23 | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.557596 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.470611 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.533544 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 0.533544 | | | | |
| 20 | Assets | 47423.482 | < | 47423.482 | | | | | | |
| 21 | Equity | 8535.6544 | < | 9216.4308 | | | | | | |
| 22 | Employees | 159689.58 | < | 159689.58 | | | | | | |
| 23 | Revenue | 150569.21 | > | 79609 | | | | | | |
| 24 | Profit | 791.04056 | > | 139 | | | | | | |
| 25 | $\Sigma \lambda$ | | 1 | = | 1 | | | | | |

Figure 1.22. Input-oriented VRS Envelopment Efficiency

Ten companies are efficient (on the VRS frontier). For the inefficient companies, the non-zero optimal λ_j indicate the benchmarks. For example, the efficiency score for AT&T is 0.53354 and the benchmarks for AT&T are Sumitoma ($\lambda_5 = 0.77$ in cell I6) and Wal-Mart ($\lambda_{11} = 0.23$ in cell I12).

The previous macro “DEA” does not record the optimal λ_j in the worksheet. This can be done by the adding a VBA procedure named “DEA1” into the existing module.

```

Sub DEA1()
'Declare DMUNo as integer. This DMUNo represents the DMU under
'evaluation. In the example, DMUNo goes form 1 to 15
  Dim DMUNo As Integer
  For DMUNo = 1 To 15
'set the value of cell E18 equal to DMUNo (1, 2, ..., 15)
    Range("E18") = DMUNo
'Run the Solver model. The UserFinish is set to True so that
'the Solver Results dialog box will not be shown
    SolverSolve UserFinish:=True
'Place the efficiency into column J
    Range("J" & DMUNo + 1) = Range("F19")
'Select the cells containing the optimal lambdas
    Range("I2:I16").Select
'copy the selected lambdas and paste them to row "DMUNo+1"
'(that is row 2, 3, ..., 16) starting with column K
    Selection.Copy
    Range("K" & DMUNo + 1).Select
    Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
  Next DMUNo
End Sub

```

In the Run Macro dialog box, select “DEA1” and then click the Run button. The procedure “DEA1” will record both the efficiency scores and the related optimal values on λ_j ($j = 1, 2, \dots, 15$) (see file “envelopment spreadsheet.xls” in the CD).

1.3.9 Calculating Slacks

Based upon the efficiency scores and the optimal values on λ_j ($j = 1, 2, \dots, 15$), we can calculate the slack values using (1.3). However, because of possible multiple optimal solutions, we need to use model (1.4) to optimize the input and output slacks.

Figure 1.23 shows the spreadsheet model for calculating the slacks after the efficiency scores are obtained. This spreadsheet model is built upon the spreadsheet model shown in Figure 1.18 with efficiency scores reported in column J.

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|--|----------------------|--------|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184385.2 | 346.2 | | 0 | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 0.7705 | | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 1 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.737556 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 0.603245 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 1 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.2295 | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.557596 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.470611 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.533544 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Reserved to indicate the DMU under evaluation. | | | | |
| 19 | Constraints | set | | Evaluation | | | | | | |
| 20 | Assets | 47423.482 | = | 47423.482 | | 0 | slack; changing cell | | | |
| 21 | Equity | 9216.4308 | = | 9216.4308 | | 680.7764 | slack; changing cell | | | |
| 22 | Employees | 159689.58 | = | 159689.58 | | 1.75E-09 | slack; changing cell | | | |
| 23 | Revenue | 79609 | = | 79609 | | 70960.21 | slack; changing cell | | | |
| 24 | Profit | 139 | = | 139 | | 652.0406 | slack; changing cell | | | |
| 25 | $\Sigma \lambda$ | 1 | = | 1 | | 72293.02 | Objective function | | | |
| 26 | | | | | | | Sum of slacks | | | |

Figure 1.23. Second-stage Slack Spreadsheet Model

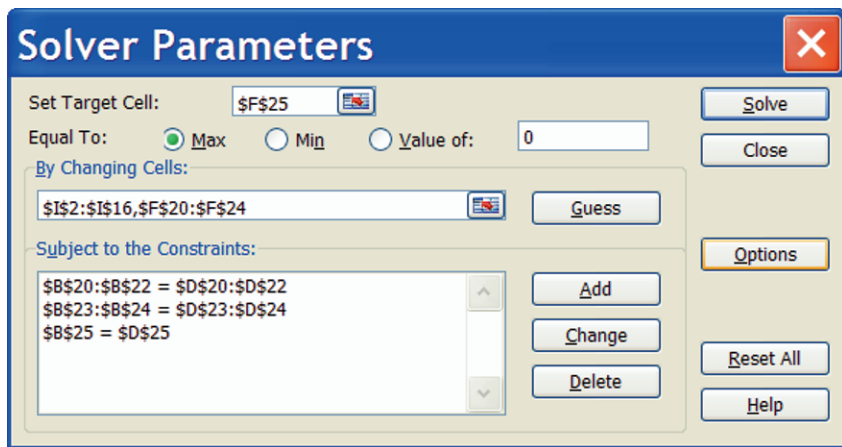


Figure 1.24. Solver Parameters for Calculating Slacks

Cells F20:F24 are reserved for input and output slacks (changing cells). The formulas for cells B25 and D25 remain unchanged. The formulas for Cells B20:B24 are changed to

$$\text{Cell B20} = \text{SUMPRODUCT}(B2:B16, \$I\$2:\$I\$16) + F20$$

$$\text{Cell B21} = \text{SUMPRODUCT}(C2:C16, \$I\$2:\$I\$16) + F21$$

$$\text{Cell B22} = \text{SUMPRODUCT}(D2:D16, \$I\$2:\$I\$16) + F22$$

Cell B23 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)-F23

Cell B24 =SUMPRODUCT(G2:G16,\$I\$2:\$I\$16)-F24

The formulas for cells D21:D24 are

Cell D20 =INDEX(J2:J16,E18,1)*INDEX(B2:B16,E18,1)

Cell D21 =INDEX(J2:J16,E18,1)*INDEX(C2:C16,E18,1)

Cell D22 =INDEX(J2:J16,E18,1)*INDEX(D2:D16,E18,1)

Cell D23 =INDEX(F2:F16,E18,1)

Cell D24 =INDEX(G2:G16,E18,1)

After the Solver parameters are set up, as shown in Figure 1.24, the VBA procedure “DEASlack” is inserted into the existing module to automate the slack calculations for the 15 companies (see file “envelopment spreadsheet.xls” in the CD for the results).

```
Sub DEASlack()
'Declare DMUNo as integer. This DMUNo represents the DMU under
'evaluation. In the example, DMUNo goes form 1 to 15
  Dim DMUNo As Integer
  For DMUNo = 1 To 15
'set the value of cell E18 equal to DMUNo (1, 2, ..., 15)
    Range("E18") = DMUNo
'Run the Slack Solver model
    SolverSolve UserFinish:=True
'Select the cells containing the slacks
    Range("F20:F24").Select
'copy the selection (slacks) and paste it to row "DMUNo+1"
'(that is, row 2,3, ...,16) starting column L
    Selection.Copy
    Range("L" & DMUNo + 1).Select
    Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
  Next DMUNo
End Sub
```

1.3.10 Other Input-oriented Envelopment Spreadsheet Models

Figures 1.8 and 1.15 represent the input-oriented VRS envelopment model. By changing the constraint on $\sum_{j=1}^n \lambda_j$, we immediately obtain other input-oriented envelopment models.

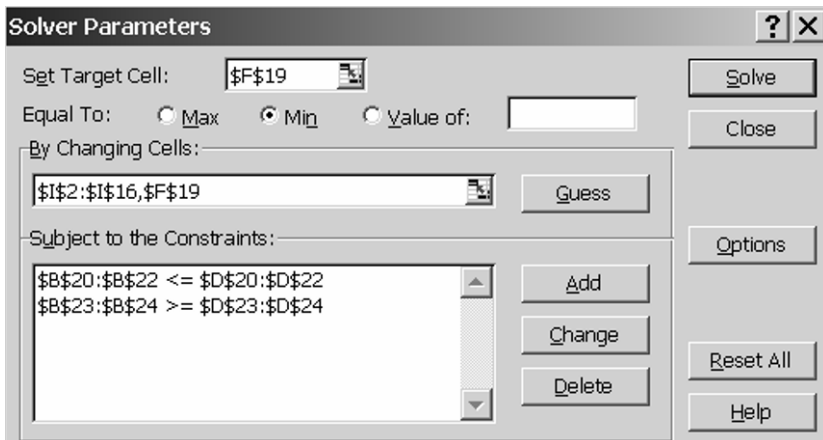


Figure 1.25. Solver Parameters for Input-oriented CRS Envelopment Model

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | 0.662832 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0.47 | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 0.971967 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.737166 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 0.524558 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 0.841424 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.01 | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.386057 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.348578 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.270382 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 0.270382 | | | | |
| 20 | Assets | 24032.613 | < | 24032.613 | | | | | | |
| 21 | Equity | 3338.6447 | < | 4670.5747 | | | | | | |
| 22 | Employees | 12922.984 | < | 80925.264 | | | | | | |
| 23 | Revenue | 79609 | > | 79609 | | | | | | |
| 24 | Profit | 139 | > | 139 | | | | | | |
| 25 | $\Sigma \lambda$ | 0.4817465 | | | | | | | | |

Figure 1.26. Input-oriented CRS Envelopment Efficiency

For example, if we select $SB\$25 = SD\25 and click the Delete button in Figure 1.15 (i.e., we remove $\sum_{j=1}^n \lambda_j = 1$), we obtain the Solver parameters for the input-oriented CRS envelopment model, as shown in Figure 1.25.

If we click the Change button, and replace $SB\$25 = SD\25 with $SB\$25 \leq SD\25 (or $SB\$25 \geq SD\25), we obtain the input-oriented NIRS (or NDRS) envelopment model.

During this process, the spreadsheet shown in Figure 1.15 and the VBA procedures remain unchanged. For example, if we run the Macro “DEA” for the input-oriented CRS envelopment model, we have the CRS efficiency scores shown in Figure 1.26. Seven DMUs are on the CRS efficient frontier.

1.3.11 Output-oriented Envelopment Spreadsheet Models

We next consider the output-oriented envelopment models. The spreadsheet model should be similar to the one in Figure 1.15, but with a different set of formulas for the DMU under evaluation. Figure 1.27 shows a spreadsheet for the output-oriented VRS envelopment model.

To make the spreadsheet more understandable, we use “range names” in the formulas. Select a range that needs to be named, and then type the desirable range name in the upper left “name box” in the Excel. This “name box” is just above the column A heading (see Figure 1.27). For example, we select cells B2:D16 containing the inputs, and then type “InputUsed” in the “name box” (see Figure 1.27). An alternative way is to use the Insert/Name/Define menu item. We can then refer to the inputs by using “InputUsed” in stead of cells B2:D16.

| InputUsed 91920.6 | | | | | | | | | | | |
|-------------------|------------------------------|----------|---------|----------------------|---|------------|--------|---|-----------|-----------------------------|--|
| 1 | A | B | C | D | E | F | G | H | I | J | K |
| 2 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Name cells I2:I16 "Lambdas" | |
| 3 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 1 | | |
| 4 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | | |
| 5 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | | |
| 6 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | | |
| 7 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0 | | |
| 8 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | | |
| 9 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | | |
| 10 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | | |
| 11 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | | |
| 12 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | | |
| 13 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 | | |
| 14 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | | |
| 15 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | | |
| 16 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | | |
| 17 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | | |
| 18 | | | | | 1 | Efficiency | | | | | |
| 19 | Constraints | set | | DMU under Evaluation | | 1 | | | | | Reserved to indicate the DMU under evaluation. |
| 20 | Assets | 91920.6 | <= | 91920.6 | | | | | | | |
| 21 | Equity | 10950 | <= | 10950 | | | | | | | |
| 22 | Employees | 36000 | <= | 36000 | | | | | | | |
| 23 | Revenue | 184365.2 | >= | 184365.2 | | | | | | | |
| 24 | Profit | 346.2 | >= | 346.2 | | | | | | | |
| 25 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | | |

Figure 1.27. Output-oriented VRS Envelopment Spreadsheet Model

We name the cells F2:G16 containing the outputs as “OutputProduced”. We also name the changing cells I2:I16 and F19 “Lambdas” and “Efficiency”, respectively. As a result, the formulas on $\sum_{j=1}^n \lambda_j$ can be

expressed as Cell B25 =SUM(Lambdas), and the formulas for the DEA reference set can be expressed as

Cell B20 = SUMPRODUCT(INDEX(InputUsed,0,1),Lambdas)
 Cell B21 = SUMPRODUCT(INDEX(InputUsed,0,2),Lambdas)
 Cell B22 = SUMPRODUCT(INDEX(InputUsed,0,3),Lambdas)
 Cell B23 = SUMPRODUCT(INDEX(OutputProduced,0,1),Lambdas)
 Cell B24 = SUMPRODUCT(INDEX(OutputProduced,0,2),Lambdas)

Note that we use “0” for the “row number” in the INDEX function. This returns the whole column in the specified array in the INDEX function. For example, INDEX(InputUsed,0,1) returns the first input across all DMUs in cells B2:B16.

We assign a range name of “DMU” to cell E18, the cell representing the DMU under evaluation. The formulas for the DMU under evaluation then can be expressed as

Cell D20 =INDEX(InputUsed,DMU,1)
 Cell D21 =INDEX(InputUsed,DMU,2)
 Cell D22 =INDEX(InputUsed,DMU,3)
 Cell D23 = Efficiency*INDEX(OutputProduced,DMU,1)
 Cell D24 = Efficiency*INDEX(OutputProduced,DMU,2)

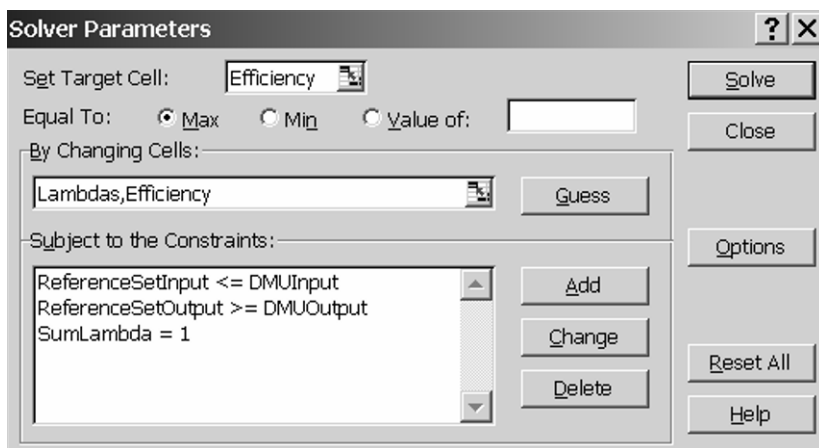


Figure 1.28. Solver Parameters for Output-oriented VRS Envelopment Model

The cells “B20:B22”, “B23:B24”, “B25”, “D20:D22”, “D23:D24” are named as “ReferenceSetInput”, “ReferenceSetOutput”, “SumLambda”, “DMUInput”, and “DMUOutput”, respectively. Based upon these range names, we obtain the Solver parameters shown in Figure 1.28. Since it is an

output-oriented envelopment model, “Max” is selected to maximize the efficiency (ϕ).

We can still apply the previous Excel macros (“DEA” or “DEA1”) to this spreadsheet model shown in Figure 1.27 with the Solver parameters shown in Figure 1.28. We next present an alternative approach to automate the DEA calculation.

First, turn on the ActiveX Controls by clicking the Developer/Insert tab. Then click the Command Button (ActiveX Controls) icon on the Insert tab, and drag it on to your worksheet (see Figure 1.29).

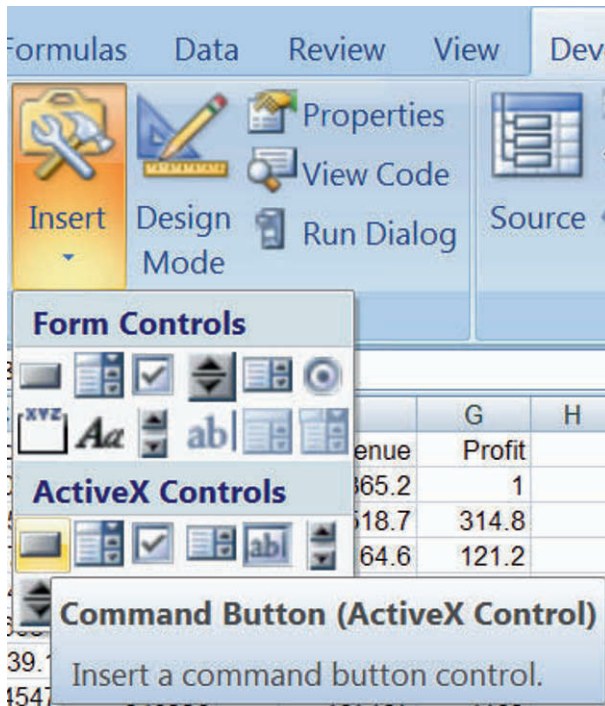


Figure 1.29. Adding Command Button

While the command button is selected, you can change its properties by clicking the Properties tab. For example, setting the TakeFocusOnClick to False leaves the worksheet selection unchanged when the button is clicked. You can change the name of this Command Button by changing the Caption property with “Output-oriented VRS” (see Figure 1.30).

Double click the command button. This should launch the VB Editor and bring up the code window for the command button’s click event. Insert the statements shown in Figure 1.31.

The word Private before the macro name “CommandButton1_Click” means that this macro will not appear in the Run Macro dialog box. The macro is only available to the worksheet “output VRS” containing the model (see Figure 1.31).

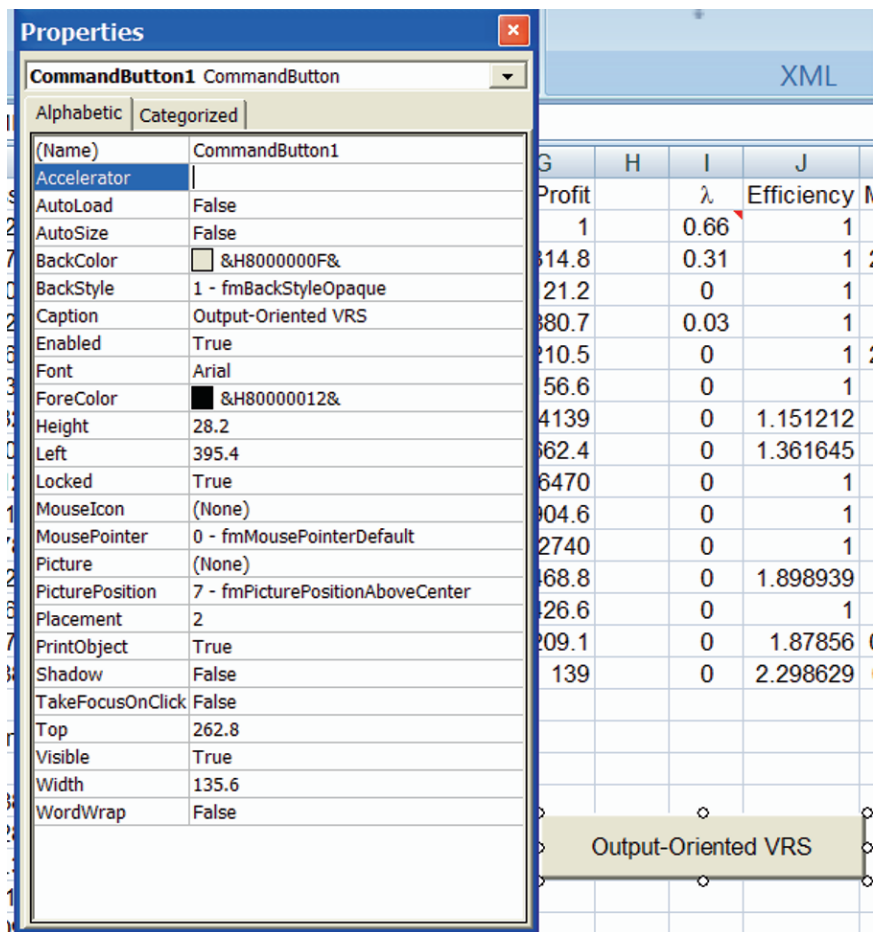


Figure 1.30. Changing Command Button Properties

```

Private Sub CommandButton1_Click()
'NDMUs, NInputs, and NInputs represent the number of DMUs, inputs,
'and outputs, respectively.

Dim NDMUs As Integer, NInputs As Integer, NOutputs As Integer

'In the example, NDMUs=15, NInputs=3, and NOutputs=2
    NDMUs = 15
    NInputs = 3
    NOutputs = 2

'This i represents the DMU under evaluation
    Dim i As Integer
    For i = 1 To NDMUs
' set the cell representing the DMU under evaluation equal to i
        Range("DMU") = i
'Run the Solver model.
        SolverSolve UserFinish:=True
'In the example, placing the efficiency into column J
        Range("A1").Offset(i, NInputs + NOutputs + 4) = Range("Efficiency")

'copy the optimal lambdas and paste the values
        Range("Lambdas").Copy
        Range("A1").Offset(i, NInputs + NOutputs + 5).Select
        Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
'calculate next DMU
        Next i
    End Sub
    
```

Figure 1.31. VBA Code for Output-oriented VRS Envelopment Model

In the macro, we introduce three variables, NDMUs, NInputs, and NOutputs, representing the number of DMUs, inputs and outputs, respectively. In the current example, NDMUs = 15, NInputs = 3, and NOutputs = 2. For a different set of DMUs, set these variables to different values, and the macro should still work.

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|------------|------------|----|------------|--------|---|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0.87 | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0.13 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0 | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 1 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 1.158415 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 1.371588 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 1 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 1.898939 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 1.892917 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 2.311194 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 2.311194 | | | | |
| 20 | Assets | 88884 | \leq | 88884 | | | | | | |
| 21 | Equity | 10242.181 | \leq | 17274 | | | | | | |
| 22 | Employees | 41771.582 | \leq | 299300 | | | | | | |
| 23 | Revenue | 183991.82 | \geq | 183991.82 | | | | | | |
| 24 | Profit | 342.08119 | \geq | 321.25592 | | | | | | |
| 25 | $\Sigma \lambda$ | | \uparrow | | | | | | | |

Figure 1.32. Output-oriented VRS Envelopment Efficiency

Close the VB Editor and click the Design Mode tab to turn off design mode. The selection handles disappear from the command button. The macro runs when you click the command button. Figure 1.32 shows the results (see file “envelopment spreadsheet.xls” in the CD).

In a similar manner, we can set up other output-oriented envelopment spreadsheet models. For example, if we remove “SumLambda=1” from Figure 1.28, we obtain the Solver parameters for output-oriented CRS envelopment model.

If one wants to use the macros established for the input-oriented envelopment spreadsheet models, one can proceed as follows using the Button (Form Controls).

First, we click the Developer tab and then select the button (the first item on the Insert tab) (see Figure 1.29). Drag the button onto your worksheet containing the output-oriented CRS envelopment spreadsheet and the Solver parameters. You will immediately be asked to assign a macro to this button. Select “DEA1”. At this point, the button is selected. You may also want to change the caption on the button to “Output-oriented CRS”, for example (see Figure 1.33). To run the selected macro, you have to deselect the button by clicking anywhere else on the worksheet. You can always assign a different macro to the button by right-clicking on the button and selecting “Assign Macro”. Figure 1.34 shows the output-oriented CRS efficiency scores.

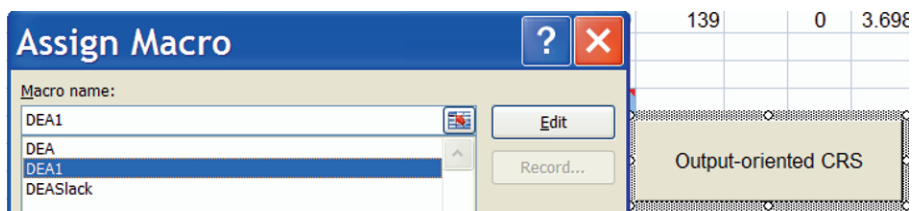


Figure 1.33. Adding a Button with Macro

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | 1.508679 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 1.73 | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 1.028842 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 1.356546 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 1.906368 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 1.188462 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.05 | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 2.590289 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 2.8688 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 3.698474 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 3.698474 | | | | |
| 20 | Assets | 88884 | \leq | 88884 | | | | | | |
| 21 | Equity | 12347.891 | \leq | 17274 | | | | | | |
| 22 | Employees | 47795.324 | \leq | 299300 | | | | | | |
| 23 | Revenue | 294431.83 | \geq | 294431.83 | | | | | | |
| 24 | Profit | 514.08791 | \geq | 514.08791 | | | | | | |
| 25 | $\Sigma\lambda$ | 1.7817269 | | | | | | | | |

Figure 1.34. Output-oriented CRS Envelopment Efficiency

1.4 Solving DEA Using DEA Frontier Software

One can solve the basic DEA models using the spreadsheets and Excel Solver. We now demonstrate how to solve the above DEA models using the student version of *DEA Frontier* software supplied with the book³. *DEA Frontier* is an Add-In for Microsoft® Excel and uses the Excel Solver. This software requires Excel 2007⁴. This student version can solve up to 100 DMUs with unlimited number of inputs and outputs.

To install the software, copy the file “DEA Frontier.xlam”⁵ to your hard drive.

Please set the Macro Security to Medium Level in the Excel. You need to do this in the Trust Center. You may check (1) enable all macros and (2) Trust access to the VBA project object model. You may also need to add the subdirectory where the *DEA Frontier* software is located as “Trusted Locations”.

If you run the *DEA Frontier* and get an error message (as shown in Figure 1.35), this error message means that the Excel Solver is not found by the *DEA Frontier* software⁶. To correct this, please use the following steps:

³ See www.deafrontier.com for other versions of the *DEA Frontier* software.

⁴ This book also provides a version for Excel 97-2003. See footnote 2.

⁵ This file is for use with Excel 2007. If you are running Excel 97-2003, please copy the file “DEA Frontier2003.xla”.

Step 1: Open Excel

Step 2: Load Excel Solver so that the Excel Solver parameters dialog box (Figure 1.12) is displaced.

Step 3: Close the Excel Solver parameters dialog box.

Step 4: Load the DEA software by opening (i) “DEAFrontier2003.xla” for Excel 97-2003, or (ii) “DEAFrontier.xlam” for Excel 2007.

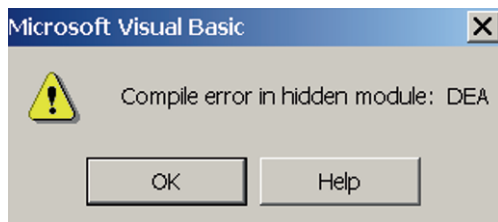
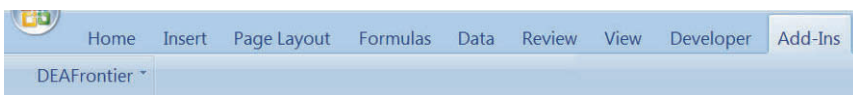


Figure 1.35. DEAFrontier Error Message

To locate the DEA Menu, Office 2007 users select the Add-Ins tab and navigate to the DEA menu option. The DEA menus will not always be visible, as they were in Office 97-2003.



You will see a new Menu item “DEAFrontier” or “DEA”⁷ in Add-Ins tab. (see Figure 1.36.) (Please see the Format for Data Sheet for proper setup of data sets.)

⁶ If the DEAFrontier software is installed in the subdirectory where the Excel Solver is installed, this error may not occur.

⁷ This depends on the version of the DEAFrontier software you are using.

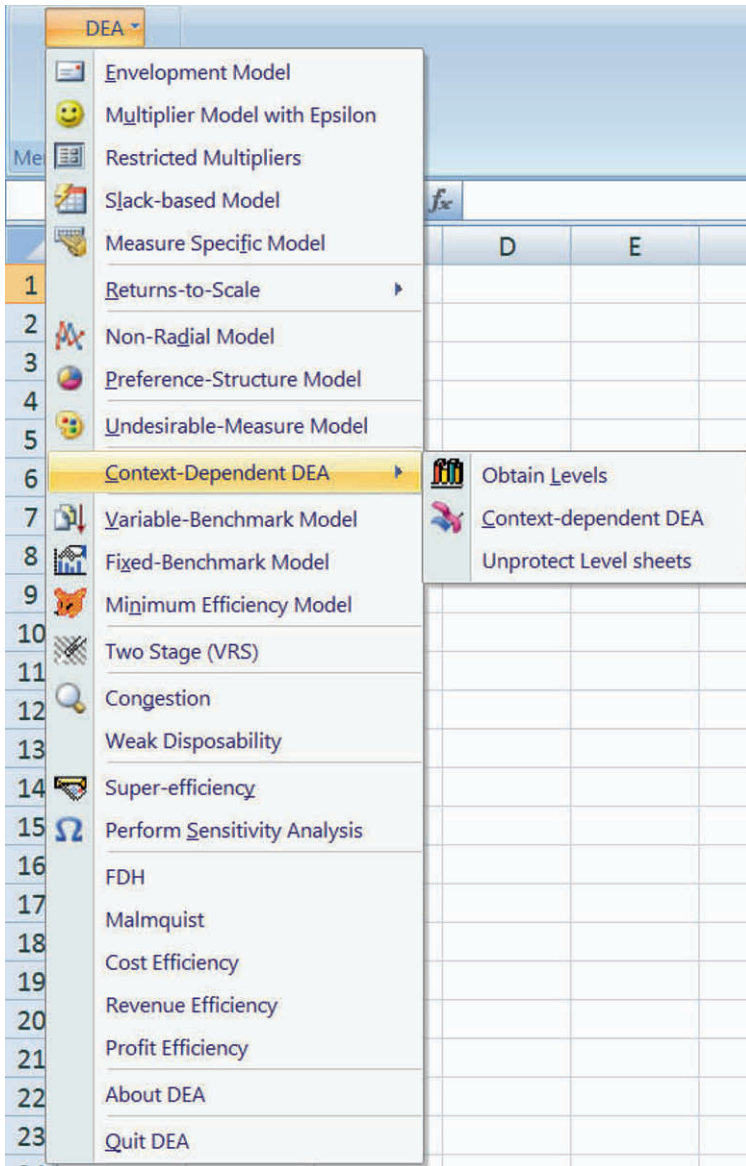


Figure 1. 36. DEA Frontier Menu

1.4.1 Data Sheet Format

In most of the cases, the data sheet containing the data for DMUs under evaluations must be named as “Data”. Other names for the data sheet will be used for the other DEA models that will be discussed in later chapters, such as, Variable-benchmark models, Fixed-benchmark models, Minimum Efficiency models, Malmquist index model, Cost Efficiency, Revenue

Efficiency and Profit Efficiency. However, all the data sheets have the same format as shown in Figures 1.37 and 1.38.

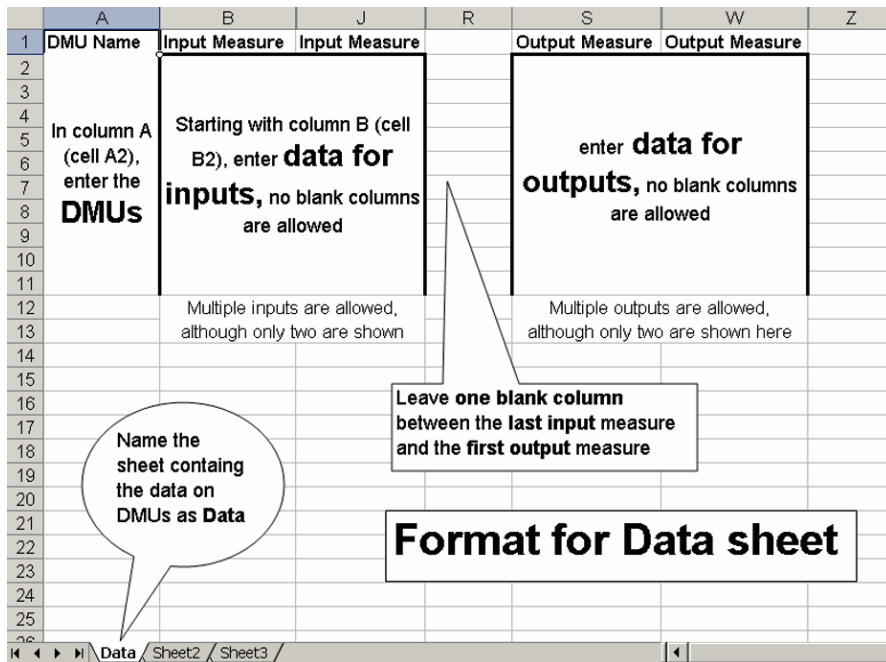


Figure 1. 37. Data Sheet Format

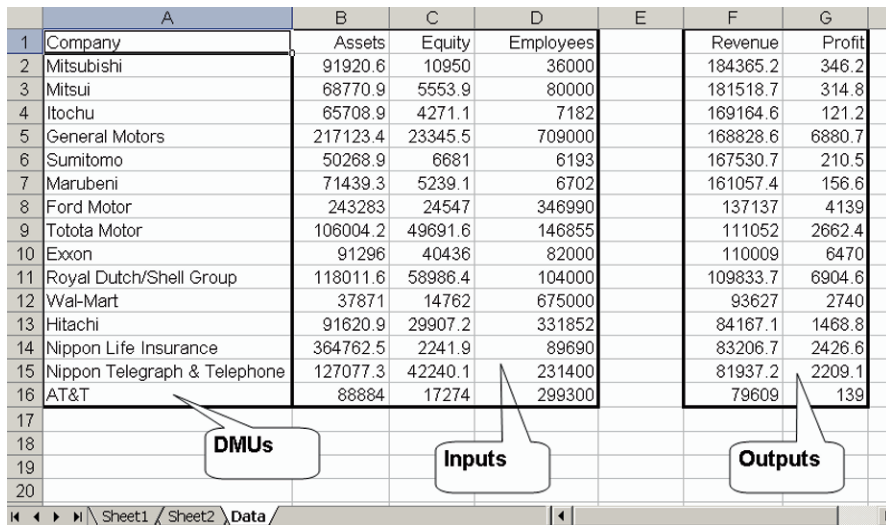


Figure 1. 38. Example Data Sheet

Leave one blank column between the input and output data. No blank columns and rows are allowed within the input and output data.

Negative or non-numerical data are deemed as invalid data. The software checks if the data are in valid form before the calculation. If the data sheet contains negative or non-numerical data, the software will quit and locate the invalid data (see Figure 1.39).

| | A | B | C | D | E | F |
|---|----------------|----------|---------|-----------|---|----------|
| 1 | Company | Assets | Equity | Employees | | Revenue |
| 2 | Mitsubishi | W91920.6 | 10950 | 36000 | | 184365.2 |
| 3 | Mitsui | 68770.9 | 5553.9 | | | |
| 4 | Itochu | 65708.9 | 4271.1 | | | |
| 5 | General Motors | 217123.4 | 23345.1 | | | |
| 6 | Sumitomo | 50268.9 | 6681.1 | | | |
| 7 | Marubeni | 71439.3 | 5239.1 | | | |
| 8 | Ford Motor | 243283 | 24547.1 | | | |
| 9 | Totota Motor | 106004.2 | 49691.1 | | | |

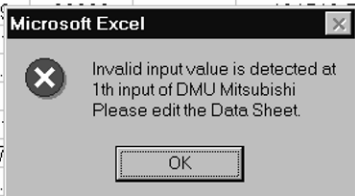


Figure 1. 39. Invalid Data

1.4.2 Envelopment Model

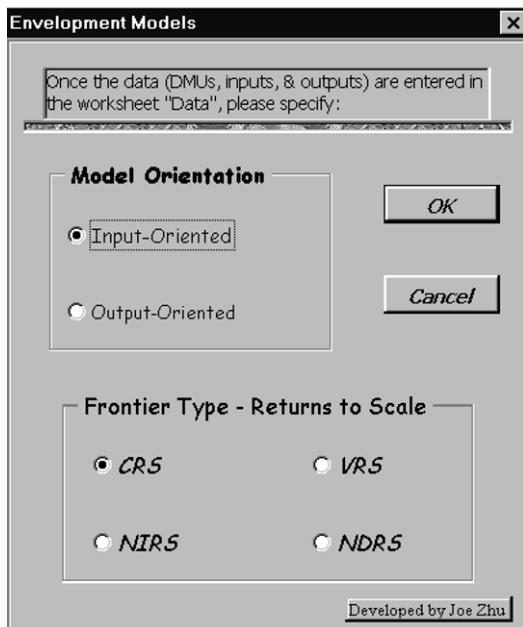


Figure 1. 40. Envelopment Models

To run the envelopment models in Table 1.2, select the “Envelopment Model” menu item. You will be prompted with a form for selecting the model orientation and frontier type, as shown in Figure 1.40.

The software performs a two-stage DEA calculation. First, the efficiency scores are calculated, and the efficiency scores and benchmarks (λ_j^*) are reported in the “Efficiency” sheet. At the same time, a “Slack” sheet and a “Target” sheet are generated based upon the efficiency scores and the λ_j^* .

Then you will be asked whether you want to perform the second-stage calculation, i.e., fixing the efficiency scores and calculating the DEA slacks (see Figure 1.41). If Yes, then the slack and target sheets will be replaced by new ones which report the DEA slacks and the efficient targets defined in Table 1.2.

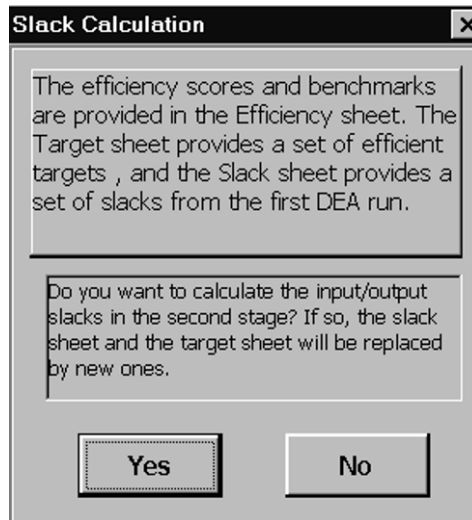


Figure 1. 41. Second Stage DEA Slack Calculation

REFERENCES

1. Anderson, T.R. and G.P. Sharp (1997), A new measure of baseball batters using DEA, *Annals of Operations Research*, 73, 141-155.
2. Banker, R.D., A. Charnes and W.W. Cooper (1984), Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, 30, 1078-1092.
3. Banker, R.D. and C.F. Kemerer (1989), Scale economies in new software development, *IEEE Transactions on Software Engineering*, 15, 1199-1205.
4. Barr, R.S. and T.F. Siems (1997), Bank failure prediction using DEA to measure management quality, In: R.S. Barr, R.V. Helgason, and J.L. Kennington., eds. *Advances in Metaheuristics, Optimization, and Stochastic Modeling Techniques*, 341-365. Kluwer Academic Publishers, Boston.
5. Camp, R.C., (1995), *Business Process Benchmarking, Finding and Implementing Best Practices*. ASQC Quality Press, Milwaukee, Wisconsin.
6. Charnes, A., W.W. Cooper and E. Rhodes (1978), Measuring efficiency of decision making units, *European Journal of Operational Research*, 2, 429-444.
7. Charnes, A., W.W. Cooper and E. Rhodes (1981), Evaluating program and managerial efficiency: an application of data envelopment analysis to program follow through, *Management Science*, 27, No. 6, 668-697.
8. Charnes, A., W.W. Cooper, A.Y. Lewin and L.M. Seiford (1994), *Data envelopment analysis: theory, methodology and applications*. Kluwer Academic Publishers, Boston.
9. Chilingerian, J.A. and H.D. Sherman (2004), Health care applications: from hospitals to physicians, from productive efficiency to quality frontiers, in *Handbook on Data Envelopment Analysis*, Chapter 17, eds W.W. Cooper, L.M. Seiford and J. Zhu, Kluwer Academic Publishers, Boston.
10. Cook, W.D., Y. Roll and A. Kazakov (1990), A DEA model for measuring the relative efficiency of highway maintenance patrols, *INFOR*, 28, No. 2, 113-124.
11. Cook, W.D. and J. Zhu (2008), *Data Envelopment Analysis: Modeling Operational Processes and Measuring Productivity*. ISBN: 1434830233. EAN-13: 9781434830234.

12. Cooper, W.W., L.M. Seiford and K. Tone (2000), *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publishers, Boston.
13. Cooper, W.W., L.M. Seiford and J. Zhu (2004), *Handbook on Data Envelopment Analysis*, Kluwer Academic Publishers, Boston.
14. Desai, A. and J.E. Storbeck (1990), A data envelopment analysis for spatial efficiency, *Computers, Environment and Urban Systems*, 14, 145-156.
15. Färe, R., Grosskopf, J. Logan and C.A.K. Lovell (1985), Measuring efficiency in production: with an application to electric utilities, In: R. Färe, S. Grosskopf and C.A.K. Lovell., eds. *The Measurement of Efficiency of Production*, 185-214. Kluwer-Nijhoff Publishing, Kluwer Academic Publishers, Boston.
16. Johnson, S.A. and J. Zhu (2002), Identifying top applicants in recruiting using data envelopment analysis, *Socio-Economic Planning Sciences*, 37, 125-139.
17. Kleinsorge, I.K., P.B. Schary and R.D. Tanner (1989), Evaluating logistics decisions, *International Journal of Physical Distribution and Materials Management*, 19, 200-219.
18. Ray, S., L.M. Seiford and J. Zhu (1998), Market entity behavior of Chinese state-owned enterprises, *OMEGA*, 26, No. 2, 263-278.
19. Sherman, H.D. (1984), Improving the productivity of service businesses, *Sloan Management Review*, 35, No. 3, 11-23.
20. Sherman, D.H. and J. Zhu (2006), *Service Productivity Management: Improving Service Performance using Data Envelopment Analysis*, Springer Science, Boston. 2006.
21. Zhu, J. (1996), DEA/AR analysis of the 1988-1989 performance of the Nanjing Textile Cooperation, *Annals of Operations Research*, 66, 311-335.
22. Zhu, J. and W.D. Cook (2007), *Modeling Data Irregularities and Structural Complexities in Data Envelopment Analysis*, Springer, Boston, 2007

Chapter 2

Multiplier and Slack-based Models

2.1 Multiplier Model with Weight Restrictions

The dual linear programming problems to the envelopment models are called multiplier models as shown in Table 2.1.

The dual variables v_i and μ_r are called multipliers. A DMU is on the frontier if and only if $\sum_{r=1}^s \mu_r y_{ro} + \mu = 1$ (or $\sum_{i=1}^m v_i x_{io} + \nu = 1$) in optimality. The ϵ in the envelopment model essentially requires that v_i and μ_r are positive in the multiplier models. The constraint $\sum_{i=1}^m v_i x_{io} = 1$ (or $\sum_{r=1}^s \mu_r y_{ro} = 1$) is known as a normalization constraint. In DEA, the weighted input and output of $\sum_{i=1}^m v_i x_{ij}$ and $\sum_{r=1}^s \mu_r y_{rj}$ are called virtual input and virtual output, respectively. See Seiford and Thrall (1990) for a detailed discussion on these models.

Table 2.1. Multiplier Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|--|--|
| | $\max \sum_{r=1}^s \mu_r y_{ro} + \mu$ | $\min \sum_{i=1}^m v_i x_{io} + \nu$ |
| | subject to | subject to |
| | $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + \mu \leq 0$ | $\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu \geq 0$ |
| | $\sum_{i=1}^m v_i x_{io} = 1$ | $\sum_{r=1}^s \mu_r y_{ro} = 1$ |
| | $\mu_r, v_i \geq 0(\epsilon)$ | $\mu_r, v_i \geq 0(\epsilon)$ |
| CRS | where $\mu = 0$ | where $\nu = 0$ |
| VRS | where μ free | where ν free |
| NIRS | where $\mu \leq 0$ | where $\nu \geq 0$ |
| NDRS | where $\mu \geq 0$ | where $\nu \leq 0$ |

Note that $\mu_r, \nu_i \geq \varepsilon$. This set of constraints ensures that a DMU with an efficiency score of one must be efficient. If a DMU's efficiency score equals one with non-zero slacks in an envelopment model, then this DMU must have a score less than one in the above related multiplier model (with ε). That is, if we impose $\mu_r, \nu_i \geq \varepsilon$ in the multiplier models, the two-stage process in the envelopment models is automatically carried out in the calculation. However, note that ε is a very small positive value and usually is set equal to 10^{-6} , and such choice does not always work. It is also possible that the multiplier model with can be infeasible because the ε is not correctly selected.

In the DEA literature, a number of approaches have been proposed to introduce additional restrictions on the values that the multipliers can assume.

Some of the techniques for enforcing these additional restrictions include imposing bounds on ratios of multipliers (Thompson et al., 1990), appending multiplier inequalities (Wong and Beasley, 1990), and requiring multipliers to belong to given closed cones (Charnes et al., 1989), among others.

We here present the assurance region (AR) approach of Thompson et al. (1990). To illustrate the AR approach, suppose we wish to incorporate additional inequality constraints of the following form into the multiplier DEA models as given in Table 2.1:

$$\begin{aligned} \alpha_i &\leq \frac{\nu_i}{\nu_{i_0}} \leq \beta_i, & i = 1, \dots, m \\ \delta_r &\leq \frac{\mu_r}{\mu_{r_0}} \leq \gamma_r, & r = 1, \dots, s \end{aligned} \quad (\text{AR})$$

Here, ν_{i_0} and μ_{r_0} represent multipliers which serve as “numeraires” in establishing the upper and lower bounds represented here by α_i , β_i , and by δ_r , γ_r for the multipliers associated with inputs $i=1, \dots, m$ and outputs $r = 1, \dots, s$ where $\alpha_{i_0} = \beta_{i_0} = \delta_{r_0} = \gamma_{r_0} = 1$. The above constraints are called Assurance Region (AR) constraints as in Thompson et al. (1990).

Uses of such bounds are not restricted to prices. For example, Zhu (1996) uses an assurance region approach to establish bounds on the weights obtained from uses of Analytic Hierarchy Processes in Chinese textile manufacturing in order to reflect how the local government in measuring the textile manufacturing performance.

For example, we can include the following AR constraints

$$1 \leq \frac{\nu_{\text{Employee}}}{\nu_{\text{Assets}}} \leq 2.5$$

$$1.5 \leq \frac{v_{Employee}}{v_{Equity}} \leq 3$$

$$3 \leq \frac{\mu_{MarketValue}}{\mu_{Revenue}} \leq 4$$

The first AR constraint indicates that Employee input should be at most 2.5 times as important as the Assets input, but at least as important as the Assets input.

It is noted that the AR constraints in the above form are non-linear, however, they can be converted into linear restrictions, namely

$$\alpha_i v_{i_o} \leq v_i \leq \beta_i v_{i_o}, \quad i = 1, \dots, m$$

$$\delta_r \mu_{r_o} \leq \mu_r \leq \gamma_r \mu_{r_o}, \quad r = 1, \dots, s$$

or

$$\alpha_i v_{i_o} \leq v_i, \quad i = 1, \dots, m$$

$$v_i \leq \beta_i v_{i_o}, \quad i = 1, \dots, m$$

$$\delta_r \mu_{r_o} \leq \mu_r, \quad r = 1, \dots, s$$

$$\mu_r \leq \gamma_r \mu_{r_o}, \quad r = 1, \dots, s$$

2.2 Multiplier Models in Spreadsheets

Figure 2.1 presents the input-oriented CRS multiplier spreadsheet model. We name the cells C2:E16 containing the inputs as “InputUsed” and the cells G2:H16 containing the outputs as “OutputProduced”. Cells C19:E19 and G19:H19 are reserved for the decision variables – input and output multipliers, and are named “InputMultiplier” and “OutputMultiplier”, respectively. Cells A2:A16 are reserved for DMU numbers which are used in the formulas in cells I2:I16.

Cell I2 contains the formula “= SUMPRODUCT(OutputMultiplier, INDEX (OutputProduced,A2,0))-SUMPRODUCT(InputMultiplier,INDEX (InputUsed,A2,0))” which represents the difference between weighted output and weighted input for DMU1. This value will be set as non-negative in the Solver parameters.

The function INDEX(array,row number,0) returns the entire row in the array. For example, the value for cell A2 is one, therefore INDEX(OutputProduced,A2,0) returns all the outputs for DMU1, i.e., cells G2:H2.

| | A | B | C | D | E | F | G | H | I | J |
|----|-----|------------------------------|----------|---------|-----------|---|----------|--------|-------------|------------|
| 1 | DMU | Company | Assets | Equity | Employees | | Revenue | Profit | Constraints | Efficiency |
| 2 | 1 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | 45840.8 | |
| 3 | 2 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | 27508.7 | |
| 4 | 3 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | 92123.8 | |
| 5 | 4 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | -773759.6 | |
| 6 | 5 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 104598.3 | |
| 7 | 6 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | 77833.6 | |
| 8 | 7 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | -473544 | |
| 9 | 8 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | -188836.4 | |
| 10 | 9 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | -97253 | |
| 11 | 10 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | -164259.7 | |
| 12 | 11 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | -631266 | |
| 13 | 12 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | -367744.2 | |
| 14 | 13 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | -371061.1 | |
| 15 | 14 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | -316571.1 | |
| 16 | 15 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | -325710 | |
| 17 | | | | | | | | | 138870.6 | |
| 18 | | | | | | | | | | |
| 19 | | Multipliers | 1 | 1 | 1 | | 1 | 1 | | |
| 20 | | DMU under evaluation | 1 | | | | | | | |
| 21 | | Efficiency | 184714.4 | | | | | | | |
| 22 | | | | | | | | | | |
| 23 | | | | | | | | | | |
| 24 | | | | | | | | | | |

Figure 2.1. Input-oriented CRS Multiplier Spreadsheet Model

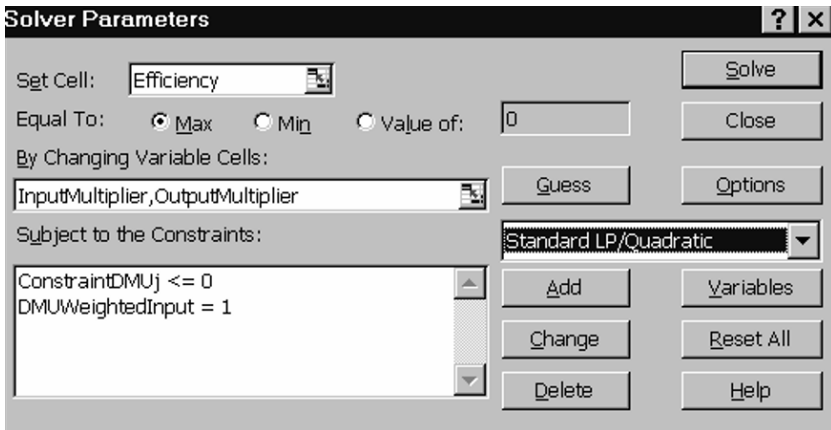


Figure 2.2. Premium Solver Parameters for Input-oriented CRS Multiplier Model

The formula in cell I2 is then copied into cells I3:I16. Cells I2:I16 are named “ConstraintDMUj”.

The formula for cell I17 is “= SUMPRODUCT (InputMultiplier, INDEX (InputUsed,DMU,0))”, where DMU is a range name for cell C20, indicating the DMU under evaluation. The value of cell I17 will be set equal to one in the Solver parameters. Cell I17 is named “DMUWeightedInput”.

The target cell is C21 which represents the efficiency – weighted output for the DMU under evaluation. The cell C21 is named “Efficiency”. Its

formula is “=SUMPRODUCT(OutputMultiplier,INDEX(OutputProduced, DMU,0))”.

Note that initial values of one are entered into the cells for the multipliers. As a result, some of the constraints are violated, and the value in cell C21 (efficiency) is greater than one. However, once the Solver solves, these values will be replaced by optimal solutions.

Figure 2.2 shows the Premium Solver parameters for the spreadsheet model in Figure 2.1. If one uses the Premium Solver, one should select “Standard LP/Quadratic” solver engine. In the Options, check the “Assume Non-Negative” box.

| Efficiency =SUMPRODUCT(OutputMultiplier, INDEX(OutputProduced,DMU,0)) | | | | | | | | | | |
|---|-----|------------------------------|-----------|----------|-----------|---|----------|---------|-------------|------------|
| | A | B | C | D | E | F | G | H | I | J |
| 1 | DMU | Company | Assets | Equity | Employees | | Revenue | Profit | Constraints | Efficiency |
| 2 | 1 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | -0.33716826 | 0.662832 |
| 3 | 2 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | -0.00211303 | 1 |
| 4 | 3 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | -3.2196E-15 | 1 |
| 5 | 4 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | -1.9984E-14 | 1 |
| 6 | 5 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 0 | 1 |
| 7 | 6 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | -0.08918393 | 0.971967 |
| 8 | 7 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | -0.95350243 | 0.737166 |
| 9 | 8 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | -1.7278984 | 0.524558 |
| 10 | 9 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | -0.16903775 | 1 |
| 11 | 10 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | -1.01666407 | 0.841424 |
| 12 | 11 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | 0 | 1 |
| 13 | 12 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | -1.24683922 | 0.386057 |
| 14 | 13 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | -1.21742056 | 1 |
| 15 | 14 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | -1.75723797 | 0.348578 |
| 16 | 15 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | -1.05154525 | 0.270382 |
| 17 | | | | | | | | | | 1 |
| 18 | | | | | | | | | | |
| 19 | | Multipliers | 5.546E-06 | 4.38E-05 | 2.827E-07 | | 3.07E-06 | 0.00028 | | |
| 20 | | DMU under evaluation | 1 | | | | | | | |
| 21 | | Efficiency | 0.6628317 | | | | | | | |
| 22 | | | | | | | | | | |
| 23 | | | | | | | | | | |
| 24 | | CRS Multiplier | | | | | | | | |

Figure 2.3. Input-oriented CRS Multiplier Efficiency

Figure 2.3 shows the optimal solutions for DMU1 with an efficiency of 0.66283. To calculate the CRS efficiencies for the remaining DMUs, we insert a VBA procedure “MultiplierCRS” to automate the computation, as shown in Figure 2.4. Note that the name of the module is changed to “MultiplierDEA”. This VBA procedure works for other sets of DMUs when setting the “NDMUs”, “NInputs”, and “NOutputs” equal to proper values. In the current example, this VBA procedure takes the efficiency in cell C21 and places it into cells J2:J16, and also takes the optimal multipliers and places them into cells K2:M16 and O2:P16 for 15 DMUs. Select and run the macro “MultiplierCRS” in the Run Macro dialog box will generate the efficiency results. You may also create a button in Forms toolbar and assign macro “MultiplierCRS” to the button (see file “multiplier spreadsheet.xls” in the CD).

```

Sub MultiplierCRS()
Dim NDMUs As Integer, NInputs As Integer, NOutputs As Integer
    NDMUs = 15
    NInputs = 3
    NOutputs = 2
    Dim i As Integer
    For i = 1 To NDMUs
        Range("DMU") = i
        SolverSolve UserFinish:=True
'record the efficiency scores
        Range("A1").Offset(i, NInputs + NOutputs + 4) = Range("Efficiency")
'record the optimal multipliers
        Range("InputMultiplier").Copy
        Range("A1").Offset(i, NInputs + NOutputs + 5).Select
        Selection.PasteSpecial Paste:=xlPasteValues
        Range("OutputMultiplier").Copy
        Range("A1").Offset(i, 2 * NInputs + NOutputs + 6).Select
        Selection.PasteSpecial Paste:=xlPasteValues

    Next i
End Sub

```

Figure 2.4. VBA Code for Input-oriented CRS Multiplier Model

Spreadsheets for other multiplier models can be set up in a similar manner. For example, Figure 2.5 shows a spreadsheet model for the input-oriented VRS multiplier model.

Because we have a decision variable that is free in sign, we need to introduce two variables in cells I19 and J19. The free variable in the VRS multiplier model is represented by cell J18 with a formula of “=I19-J19”. In the Solver parameters, cells I19 and J19 (not cell J18) along with cells C19:E19 and G19:H19 are changing cells.

The formula for cell I2 is

Cell I2 =SUMPRODUCT(G2:H2,\$G\$19:\$H\$19)-
SUMPRODUCT(C2:E2,\$C\$19:\$E\$19)+\$I\$19-\$J\$19

Cells for the multipliers and free variables are used as absolute references indicated by the dollar sign. This allows us to copy the formula in cell I2 to cells I3:I16. Figure 2.6 shows the Solver parameters for the input-oriented VRS multiplier spreadsheet model.

| | A | B | C | D | E | F | G | H | I | J |
|----|-----|------------------------------|-----------|---------|-----------|---|----------|---------|---------------|------------|
| 1 | DMU | Company | Assets | Equity | Employees | | Revenue | Profit | Constraints | Efficiency |
| 2 | 1 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | -7.9936E-15 | |
| 3 | 2 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | 0 | |
| 4 | 3 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | -0.42009487 | |
| 5 | 4 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | -4.3521E-14 | |
| 6 | 5 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | -0.2847084 | |
| 7 | 6 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | -0.7938799 | |
| 8 | 7 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | -2.19408313 | |
| 9 | 8 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | -2.22252322 | |
| 10 | 9 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | 0 | |
| 11 | 10 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | -0.098082 | |
| 12 | 11 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | -3.39289316 | |
| 13 | 12 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | -4.22139918 | |
| 14 | 13 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | -5.96194306 | |
| 15 | 14 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | -4.06208857 | |
| 16 | 15 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | -5.00032337 | |
| 17 | | | | | | | | | 1 | |
| 18 | | | | | | | | | free variable | -6.81693 |
| 19 | | Multipliers | 1.001E-05 | 0 | 2.217E-06 | | 4.14E-05 | 0.00052 | 0 | 6.816927 |
| 20 | | DMU under evaluation | 1 | | | | | | | |
| 21 | | Efficiency | 1 | | | | | | | |

Figure 2.5.. Input-oriented VRS Multiplier Spreadsheet Model

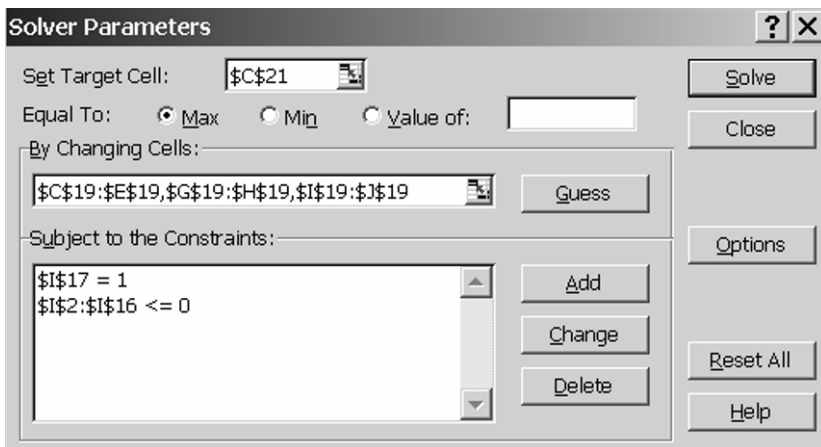


Figure 2.6. Solver Parameters for Input-oriented CRS Multiplier Model

Insert the VBA procedure “MultiplierVRS” shown in Figure 2.7 into the existing module “MultiplierDEA”. The macro records the efficiency score in cells J2:J16, optimal free variable in cells K2:K16, and optimal multipliers in cells L2:N16 and P2:Q16 for 15 DMUs (see file “multiplier spreadsheet.xls” in the CD).

```

Sub MultiplierVRS()
'Declare i as integer. This i represents the DMU under
'evaluation. In the example, i goes from 1 to 15
  Dim i As Integer
  For i = 1 To 15
'set the value of cell C20 equal to i (1, 2, ..., 15)
    Range("C20") = i
'Run the Solver model. The UserFinish is set to True so that
'the Solver Results dialog box will not be shown
    SolverSolve UserFinish:=True
'Place the efficiency in cell C20 into column J
    Range("J" & i + 1) = Range("C21")
'Select the cells containing the optimal multipliers
    Range("C19:H19").Select
'copy the selected multipliers and paste them to row "i+1"
'(that is row 2, 3, ..., 16) starting with column L
    Selection.Copy
    Range("L" & i + 1).Select
    Selection.PasteSpecial Paste:=xlPasteValues
'copy the value of free variable in cell J18 into column K
    Range("K" & i + 1) = Range("J18")
  Next
End Sub
  
```

Figure 2.7. VBA Code for the Input-oriented VRS Multiplier Model

| | A | B | C | D | E | F | G | H | I | J |
|----|-----|------------------------------|-----------|----------|-----------|-----|----------|---------|-------------|------------|
| 1 | DMU | Company | Assets | Equity | Employees | | Revenue | Profit | Constraints | Efficiency |
| 2 | 1 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | -0.21073304 | 0.604129 |
| 3 | 2 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | -0.07467892 | 0.827172 |
| 4 | 3 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | -2.3315E-15 | 1 |
| 5 | 4 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | -0.1888734 | 0.908288 |
| 6 | 5 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | -0.04003305 | 1 |
| 7 | 6 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | -0.03864332 | 0.88263 |
| 8 | 7 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | -0.4430373 | 0.697857 |
| 9 | 8 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | -1.18017405 | 0.372625 |
| 10 | 9 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | 0 | 1 |
| 11 | 10 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | -0.551944 | 0.836479 |
| 12 | 11 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | -0.62040258 | 0.523164 |
| 13 | 12 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | -1.03607407 | 0.280553 |
| 14 | 13 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | 5.5511E-15 | 1 |
| 15 | 14 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | -1.19701102 | 0.317256 |
| 16 | 15 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | -0.87923554 | 0.120764 |
| 17 | | | | | | | | | | ↑ |
| 18 | | | | | | | | | | |
| 19 | | Multipliers | 1.172E-06 | 3.16E-05 | 1.172E-06 | | 1.15E-06 | 0.00021 | | |
| 20 | | DMU under evaluation | 15 | | | | | | | |
| 21 | | Efficiency | 0.1207645 | | | AR | | | | |
| 22 | | | | | | 1st | 1.17E-06 | <= | 1.1715E-06 | |
| 23 | | | | | | 2nd | 1.17E-06 | <= | 2.9288E-06 | |
| 24 | | CRS AR | | | | | | | | |

Figure 2.8. CRS AR Multiplier Model

We next incorporate $1 \leq \frac{v_{Employee}}{v_{Assets}} \leq 2.5$ into the CRS multiplier model shown in Figure 2.3. The following two additional constraints are needed

$$1v_{Assets} \leq v_{Employee} \quad \text{and} \quad v_{Employee} \leq 2.5v_{Assets}$$

Cells G22:G23 contains the left-hand-side of the above two constraints and cells I22:I23 contains the right-hand-side of the above two constraints, as shown in Figure 2.8. In the Solver parameters, we need to add these two additional constraints, as shown in Figure 2.9.

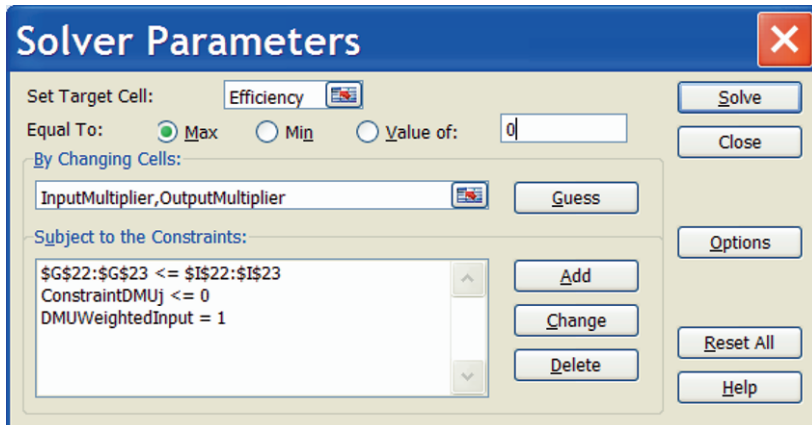


Figure 2.9. Solver Parameters for CRS AR Model

2.3 Slack-based Model

The input-oriented DEA models consider the possible (proportional) input reductions while maintaining the current levels of outputs. The output-oriented DEA models consider the possible (proportional) output augmentations while keeping the current levels of inputs. Charnes, Cooper, Golany, Seiford and Stutz (1985) develop an additive DEA model which considers possible input decreases as well as output increases simultaneously. The additive model is based upon input and output slacks. For example,

$$\begin{aligned}
 & \max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\
 & \lambda_j, s_i^-, s_r^+ \geq 0
 \end{aligned} \tag{2.1}$$

Note that model (2.1) assumes equal marginal worth for the nonzero input and output slacks. Therefore, caution should be excised in selecting the

units for different input and output measures. Some *a priori* information may be required to prevent an inappropriate summation of non-commensurable measures. Previous management experience and expert opinion, which prove important in productivity analysis, may be used (see, e.g., Seiford and Zhu (1998)).

Model (2.1) therefore is modified to a weighted CRS slack-based model as follows (Ali, Lerne and Seiford, 1995).

$$\begin{aligned}
 & \max \sum_{i=1}^m w_i^- s_i^- + \sum_{r=1}^s w_r^+ s_r^+ \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\
 & \lambda_j, s_i^-, s_r^+ \geq 0
 \end{aligned} \tag{2.2}$$

where w_i^- and w_r^+ are user-specified weights obtained through value judgment. The DMU_o under evaluation will be termed efficient *if and only if* the optimal value to (2.2) is equal to zero. Otherwise, the nonzero optimal s_i^{*-} identifies an excess utilization of the i th input, and the non-zero optimal s_r^{*+} identifies a deficit in the r th output. Thus, the solution of (2.2) yields the information on possible adjustments to individual outputs and inputs of each DMU. Obviously, model (2.2) is useful for setting targets for inefficient DMUs with *a priori* information on the adjustments of outputs and inputs.

Table 2.2. Slack-based Models

| Frontier type | Slack-based DEA Model |
|---------------|---|
| CRS | $ \begin{aligned} & \max \sum_{i=1}^m w_i^- s_i^- + \sum_{r=1}^s w_r^+ s_r^+ \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\ & \lambda_j, s_i^-, s_r^+ \geq 0 \end{aligned} $ |
| VRS | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | Add $\sum_{j=1}^n \lambda_j \geq 1$ |

One should note that model (2.2) does not necessarily yield results that are different from those obtained from the model (2.1). In particular, it will

not change the classification from efficient to inefficient (or vice versa) for any DMU.

Model (2.1) identifies a CRS frontier, and therefore is called CRS slack-based model. Table 2.2 summarizes the slack-based models in terms of the frontier types.

2.4 Slack-based Models in Spreadsheets

Figure 2.10 shows a spreadsheet model for the CRS slack-based model when DMU1 is under evaluation. Cells I2:I16 are reserved for λ_j . Cells F20:F24 are reserved for input and output slacks. The weights on slacks are entered into Cells G20:G24. Currently, the weights are all equal to one.

| | A | B | C | D | E | F | G | H | I |
|----|------------------------------|-----------|---------|------------|---|--------------|----------------|---|-----------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0.2901 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 1.4476 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0.001 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 |
| 17 | | | | | | | | | |
| 18 | | Reference | | DMU under | 1 | | | | |
| 19 | Constraints | set | | Evaluation | | Slack | Weights | | |
| 20 | Assets | 91920.6 | = | 91920.6 | | 0 | 1 | | |
| 21 | Equity | 10950 | = | 10950 | | 0 | 1 | | |
| 22 | Employees | 36000 | = | 36000 | | 24871.42 | 1 | | |
| 23 | Revenue | 184365.2 | = | 184365.2 | | 107334.9 | 1 | | |
| 24 | Profit | 346.2 | = | 346.2 | | 0 | 1 | | |
| 25 | | | | | | 132206.4 | | | |

Figure 2.10. CRS Slack-based DEA Spreadsheet Model

Cells B20:B24 contain the following formulas

- Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)+F20
- Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)+F21
- Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)+F22
- Cell B23 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)-F23
- Cell B24 =SUMPRODUCT(G2:G16,\$I\$2:\$I\$16)-F24

The input and output values of the DMU under evaluation are placed into cells D20:D24 via the following formulas

Cell D20 =INDEX(B2:B16,E18,1)

Cell D21 =INDEX(C2:C16,E18,1)

Cell D22 =INDEX(D2:D16,E18,1)

Cell D23 =INDEX(F2:F16,E18,1)

Cell D24 =INDEX(G2:G16,E18,1)

Cell F25 is the target cell which represents the weighted slack. The formula for cell F25 is

Cell F25 =SUMPRODUCT(F20:F24,G20:G24)

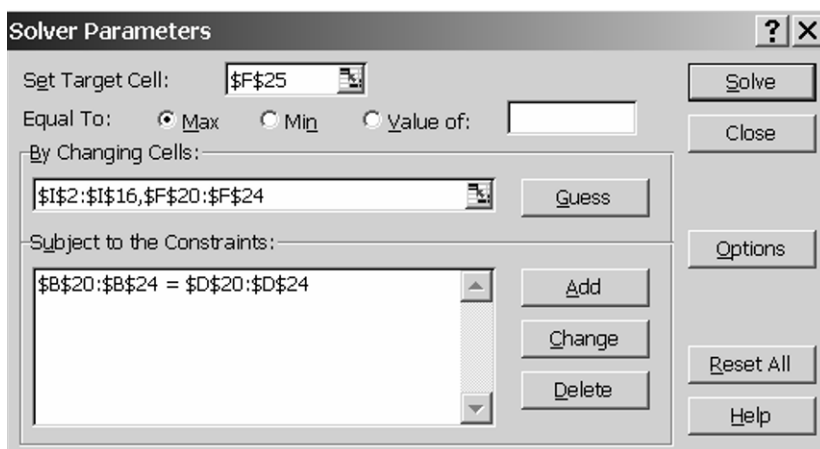


Figure 2.11. Solver Parameters for CRS Slack-based Model

Figure 2.11 shows the Solver parameters. Figure 2.12 shows the optimal slack values when DMU1 is under evaluation. Next, we insert a VBA procedure “CRSSlack” to calculate the optimal slacks for the remaining DMUs.

```
Sub CRSSlack()
Dim i As Integer
  For i = 1 To 15
    'set the value of cell E18 equal to i (=1, 2, ..., 15)
    Range("E18") = i
    'Run the Slack Solver model
    SolverSolve UserFinish:=True
    'Select the cells containing the slacks
    Range("F20:F24").Select
    'record optimal slacks in cells K2:O16
```

```

Selection.Copy
Range("K" & i + 1).Select
Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
Next
End Sub
    
```

| | J | K | L | M | N | O |
|----|------------------------------|----------|----------|------------|----------|----------|
| 1 | Company | Assets | Equity | Employees | Revenue | Profit |
| 2 | Mitsubishi | 0 | 0 | 24871.423 | 107334.9 | 0 |
| 3 | Mitsui | 0 | 0 | -3.988E-10 | 0 | 0 |
| 4 | Itochu | 3.69E-11 | 0 | 0 | -8.6E-14 | 1.75E-13 |
| 5 | General Motors | 0 | 0 | -1.649E-10 | 0 | 0 |
| 6 | Sumitomo | 0 | 2.27E-12 | 1.99E-13 | -2.1E-13 | -7.1E-14 |
| 7 | Marubeni | 12794.2 | 0 | 0 | 5857.514 | 0.812974 |
| 8 | Ford Motor | 0 | 0 | 267229.26 | 112833 | 0 |
| 9 | Toyota Motor | 0 | 25289.03 | 108011.55 | 171436.8 | 0 |
| 10 | Exxon | 0 | 8.49E-12 | 2.118E-11 | 5.85E-11 | 0 |
| 11 | Royal Dutch/Shell Group | 0 | 13499.36 | 14957.88 | 78912.18 | 0 |
| 12 | Wal-Mart | 0 | 0 | 0 | 0 | 0 |
| 13 | Hitachi | 0 | 12685.39 | 307952.81 | 186551.3 | 0 |
| 14 | Nippon Life Insurance | -1E-10 | 0 | 0 | -1.9E-10 | 0 |
| 15 | Nippon Telegraph & Telephone | 0 | 17554.49 | 196254.33 | 288061.8 | 0 |
| 16 | AT&T | 0 | 5460.851 | 288349.72 | 216613.9 | 233.1999 |
| 17 | | | | | | |
| 18 | | | | | | |
| 19 | | | | | | |
| 20 | | | | | | |
| 21 | CRS Slack | | | | | |
| 22 | | | | | | |

Figure 2.12. CRS Slacks

| | A | B | C | D | E | F | G | H | I |
|----|------------------------------|-----------|---------|------------|---|--------------|----------------|---|-----------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 |
| 17 | | | | | | | | | |
| 18 | | Reference | | DMU under | 1 | | | | |
| 19 | Constraints | set | | Evaluation | | Slack | Weights | | |
| 20 | Assets | 91920.6 | = | 91920.6 | | 0 | 1 | | |
| 21 | Equity | 10950 | = | 10950 | | 0 | 1 | | |
| 22 | Employees | 36000 | = | 36000 | | 0 | 1 | | |
| 23 | Revenue | 184365.2 | = | 184365.2 | | 0 | 1 | | |
| 24 | Profit | 346.2 | = | 346.2 | | 0 | 1 | | |
| 25 | $\Sigma \lambda$ | 1 | = | 1 | | 0 | | | |

Figure 2.13. VRS Slack-based Spreadsheet Model

By adding an additional constraint on $\sum_{j=1}^n \lambda_j$, we can obtain spreadsheet models for other slack-based models (see Excel file slack-based spreadsheet.xls in the CD). For example, Figure 2.13 shows a spreadsheet model for the VRS slack-based DEA model.

Range names are used in Figure 2.13. Cells B2:D16 are named “InputUsed” and cells F2:G16 are named “OutputProduced”. We also name cells I2:I16 “Lambdas”, cells F20:F24 “Slacks”, G20:G24 “Weights”, and cell E18 “DMU”. Accordingly, we have formulas

Cell B20 = SUMPRODUCT(INDEX(InputUsed,0,1),Lambdas)+Slacks

Cell B21 = SUMPRODUCT(INDEX(InputUsed,0,2),Lambdas)+Slacks

Cell B22 = SUMPRODUCT(INDEX(InputUsed,0,3),Lambdas)+Slacks

Cell B23 = SUMPRODUCT(INDEX(OutputProduced,0,1),Lambdas)-Slacks

Cell B24 = SUMPRODUCT(INDEX(OutputProduced,0,2),Lambdas)-Slacks

Cell B25 = SUM(Lambdas)

Cell F25 = SUMPRODUCT(Slacks,Weights)

We then name cells B20:B24 “ReferenceSet”, cells D20:D24 “DMUEvaluation”, B25 “SumLambda”, and cell F25 “SumSlack”. Figure 2.14 shows the Solver parameters for the VRS slack-based model.

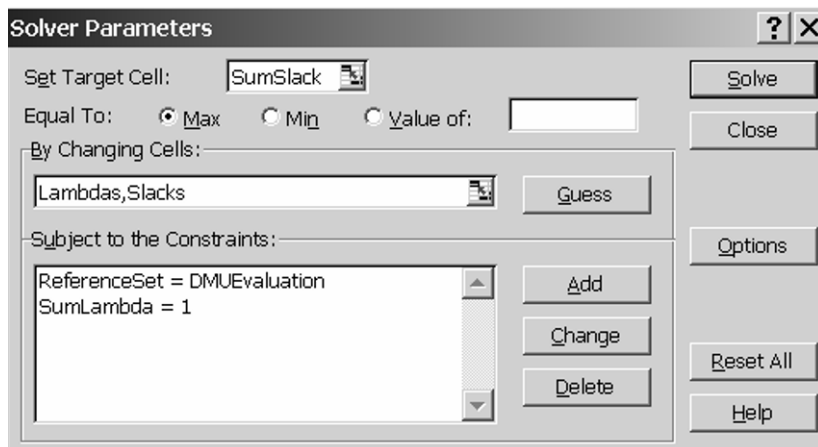


Figure 2.14. Solver Parameters for VRS Slack-based Model

Since range names are used in the Solver model, we can modify “CRSSlack” into a VBA procedure that can be applied to other data sets. The modified VBA procedure is called “Slack”.

```

Sub Slack()
Dim NDMUs As Integer, NInputs As Integer, NOutputs As Integer
    NDMUs = 15
    NInputs = 3
    NOutputs = 2
Dim i As Integer
For i = 1 To NDMUs
Range("DMU") = i
SolverSolve UserFinish:=True
Range("Slacks").Copy
Range("A1").Offset(i, NInputs + NOutputs + 5).Select
Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
Next
End Sub

```

2.5 Solving DEA Using DEAFrontier Software

2.5.1 Multiplier Model

To run the multiplier models, select the “Multiplier Model with Epsilon” menu item. You will be prompted with a form for selecting the models presented in Table 2.1. As shown in Figure 2.15, the default ε value = 0. The user can specify its own non-zero ε . The results are reported in a sheet named “Efficiency Report”.

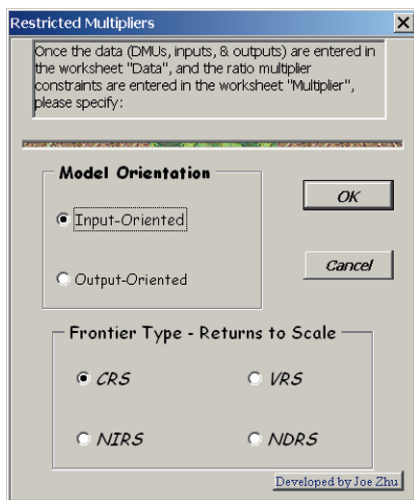


Figure 2.15. Multiplier Model

2.5.2 Restricted Multiplier Model

We need to set up the sheet “Multiplier” which contains the ARs. For example, if we want to include the following ARs

$$1 \leq \frac{v_{Employee}}{v_{Assets}} \leq 2.5$$

$$1.5 \leq \frac{v_{Employee}}{v_{Equity}} \leq 3$$

$$3 \leq \frac{\mu_{MarketValue}}{\mu_{Revenue}} \leq 4$$

then the data in the “Multiplier” sheet should be entered as shown in the following Figure 2.16.

| | A | B | C | D | E |
|---|-----|--------------|---------|-----|---|
| 1 | 1 | Employee | Assets | 2.5 | |
| 2 | 1.5 | Employee | Equity | 3 | |
| 3 | 3 | Market Value | Revenue | 4 | |
| 4 | | | | | |

Figure 2.16. Restrictions (AR) on Multipliers

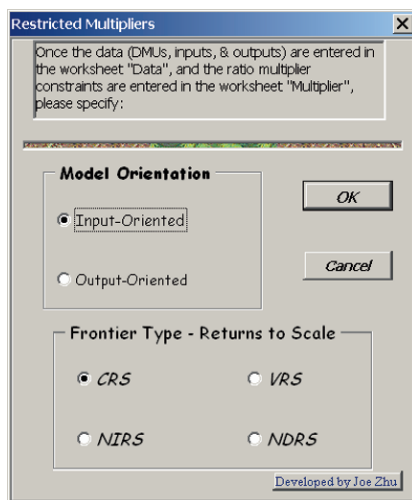


Figure 2.17. Restricted Multiplier Model

To avoid any errors, we suggest copying and pasting the input and output names from the “data” sheet when you enter the information into the “Multiplier” sheet. If the input (output) names in the two sheets do not match, the program will stop.

Once the “Multiplier” sheet is set up, select the “Restricted Multipliers” menu item and you will be prompted to choose a DEA model, as shown in Figure 2.17. Figure 2.18 shows the results of the input-oriented CRS multiplier model with the above ARs.

Note that you can also add ARs that link the input and output multipliers for the “Restricted Multipliers”. Note also that if the ARs are not properly specified, then the related DEA model may be infeasible. If that happens, the program will return a value “-9999” for the efficiency score.

| DMU No. | DMU Name | Efficiency | Employee | Assets | Equity | Market Value | Revenue |
|---------|------------------------------|------------|----------|---------|---------|--------------|---------|
| 1 | Citicorp | 1.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00002 | 0.00001 |
| 2 | BankAmerica Corp. | 0.80538 | 0.00000 | 0.00000 | 0.00000 | 0.00002 | 0.00001 |
| 3 | NationsBank Corp. | 0.84015 | 0.00001 | 0.00000 | 0.00000 | 0.00003 | 0.00001 |
| 4 | Chemical Banking Corp. | 0.84197 | 0.00001 | 0.00000 | 0.00000 | 0.00004 | 0.00001 |
| 5 | J.P. Morgan & Co. | 0.93846 | 0.00001 | 0.00000 | 0.00000 | 0.00005 | 0.00001 |
| 6 | Chase Manhattan Corp. | 0.86080 | 0.00001 | 0.00000 | 0.00000 | 0.00006 | 0.00001 |
| 7 | First Chicago NBD Corp. | 0.81792 | 0.00001 | 0.00000 | 0.00001 | 0.00005 | 0.00001 |
| 8 | First Union Corp. | 0.92079 | 0.00001 | 0.00000 | 0.00000 | 0.00005 | 0.00001 |
| 9 | Banc One Corp. | 1.00000 | 0.00001 | 0.00001 | 0.00000 | 0.00006 | 0.00001 |
| 10 | Bankers Trust New York Corp. | 0.63273 | 0.00002 | 0.00001 | 0.00001 | 0.00008 | 0.00003 |

Figure 2.18. Restricted Multiplier Model Results

2.5.3 Slack-based Model

To run the slack-based models, select the “Slack-based Model” menu item. You will be prompted with a form for selecting the models presented in Table 2.2, as shown in Figure 2.19.

If you select “Yes” under the “Weights on Slacks”, you will be asked to provide the weights, as shown in Figure 2.10. If you select “No”, then all the weights are set equal to one.

The results are reported in a sheet named “Slack Report” along with a sheet named “Efficient Target”.

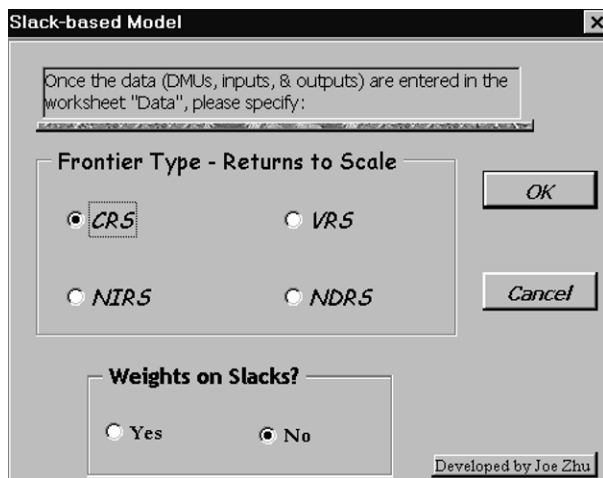


Figure 2.19. Slack-based Models

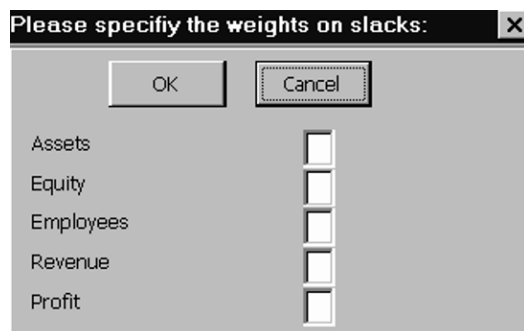


Figure 2.20. Weights on Slacks

REFERENCES

1. Ali, A. I., C.S. Lerne and L.M. Seiford (1995), Components of efficiency evaluation in data envelopment analysis, *European Journal of Operational Research*, 80, 462-473.
2. Charnes, A., W.W. Cooper, B. Golany, L.M. Seiford and J. Stutz (1985), Foundations of data envelopment analysis for Pareto-Koopman's efficient empirical production functions, *J. of Econometrics*, 30, 1-17.
3. Seiford, L.M. and R.M. Thrall (1990), Recent developments in DEA: the mathematical programming approach to frontier analysis, *Journal of Econometrics*, 46, 7-38.
4. Seiford, L.M. and J. Zhu (1998), Identifying excesses and deficits in Chinese industrial productivity (1953-1990): a weighted data envelopment analysis approach, *OMEGA*, 26, No. 2, 279-269.
5. Thompson, R. G., L. N. Langemeier, C. T. Lee, E. Lee, R. M. Thrall (1990), The role of multiplier bounds in efficiency analysis with application to Kansas farming. *J. Econometrics* 46, 93-108.
6. Wong, Y-H. B., and J. E. Beasley (1990), Restricting weight flexibility in data envelopment analysis, *Journal of Operational Research Society* 41/9, 829-835.
7. Zhu, J. (1996), Data envelopment analysis with preference structure, *Journal of Operational Research Society*, 47, 136-150.

Chapter 3

Measure-specific DEA Models

3.1 Measure-specific Models

Although DEA does not need *a priori* information on the underlying functional forms and weights among various input and output measures, it assumes proportional improvements of inputs or outputs. This assumption becomes invalid when a preference structure over the improvement of different inputs (outputs) is present in evaluating (inefficient) DMUs (see Chapter 4). We need models where a particular set of performance measures is given pre-emptive priority to improve.

Let $I \subseteq \{1, 2, \dots, m\}$ and $O \subseteq \{1, 2, \dots, s\}$ represent the sets of specific inputs and outputs of interest, respectively. Based upon the envelopment models, we can obtain a set of measure-specific models where only the inputs associated with I or the outputs associated with O are optimized (see Table 3.1).

The measure-specific models can be used to model uncontrollable inputs and outputs (see Banker and Morey (1986)). The controllable measures are related to set I or set O .

A DMU is efficient under envelopment models if and only if it is efficient under measure-specific models. i.e., both the measure-specific models and the envelopment models yield the same frontier. However, for inefficient DMUs, envelopment and measure-specific models yield different efficient targets.

Consider Figure 1.1. If the response time input is of interest, then the measure-specific model will yield the efficient target of S1 for inefficient S. If the cost input is of interest, S3 will be the target for S. The envelopment model projects S to S2 by reducing the two inputs proportionally.

Table 3.1. Measure-specific Models

| Frontier | | |
|------------------|---|---|
| Type | Input-Oriented | Output-Oriented |
| | $\min \theta - \varepsilon (\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+)$ | $\max \phi - \varepsilon (\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+)$ |
| | subject to | subject to |
| | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad i \in I;$ | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m;$ |
| CRS | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i \notin I;$ | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = \phi y_{ro} \quad r \in O;$ |
| | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s;$ | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r \notin O;$ |
| | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ |
| VRS | Add $\sum_{j=1}^n \lambda_j = 1$ | |
| NIRS | Add $\sum_{j=1}^n \lambda_j \leq 1$ | |
| NDRS | Add $\sum_{j=1}^n \lambda_j \geq 1$ | |
| Efficient Target | $\begin{cases} \hat{x}_{io} = \theta^* x_{io} - s_i^{*-} & i \in I \\ \hat{x}_{io} = x_{io} - s_i^{*-} & i \notin I \\ \hat{y}_{ro} = y_{ro} + s_r^{*+} & r = 1, 2, \dots, s \end{cases}$ | $\begin{cases} \hat{x}_{io} = x_{io} - s_i^{*-} & i = 1, 2, \dots, m \\ \hat{y}_{ro} = \phi^* y_{ro} + s_r^{*+} & r \in O \\ \hat{y}_{ro} = y_{ro} + s_r^{*+} & r \notin O \end{cases}$ |

3.2 Measure-specific Models in Spreadsheets

Since the measure-specific models are closely related to the envelopment models, the spreadsheet models can be modified from the envelopment spreadsheet models.

Figure 3.1 shows an input-oriented VRS measure-specific spreadsheet model where the Assets input is of interest. We only need to change the formulas in cells D21:D22 (representing Equity and Employee for the DMU under evaluation) in the input-oriented VRS envelopment spreadsheet model shown in Figure 1.8 to

Cell D21 =INDEX(C2:C16,E18,1)

Cell D22 =INDEX(C2:C16,E18,1)

The Solver parameters remain the same, as shown in Figure 1.15. All the VBA procedures developed for the envelopment models can be used. In Figure 3.1, the VBA procedure “DEA1” is assigned to the button “Measure-Specific”.

If we apply the same formula changes in the Second-stage Slack Spreadsheet Model shown in Figure 1.23, with the same Solver parameters shown in Figure 1.24 and with the macro “DEASlack”, we can optimize the

slacks for the spreadsheet model shown in Figure 3.1 after we obtain the efficiency scores. Figure 3.2 shows the results (see Excel file measure-specific spreadsheet.xls in the CD).

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|------|-----------|------------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 0.56 | | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | 1 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.377606 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 0.578288 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 1 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | 0.44 | | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.484837 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.42684 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.504427 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 0.504427 | | | | |
| 20 | Assets | 44835.477 | < | 44835.477 | | | | | | |
| 21 | Equity | 10222.526 | < | 17274 | | | | | | |
| 22 | Employees | 299300 | < | 299300 | | | | | | Measure-Specific |
| 23 | Revenue | 135142.15 | > | 79609 | | | | | | |
| 24 | Profit | 1319.0622 | > | 139 | | | | | | |
| 25 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | |

Figure 3.1. Input-oriented VRS Measure-specific Spreadsheet Model

| | E | F | G | H | I | J | K | L | M | N | O | P |
|----|----|----------|--------|-----------|------------|---|----------|----------|----------|-----------|----------|--------|
| 1 | | Revenue | Profit | λ | Efficiency | | | Assets | Equity | Employees | Revenue | Profit |
| 2 | | 184365.2 | 346.2 | | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 3 | | 181518.7 | 314.8 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 4 | | 169164.6 | 121.2 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 5 | | 168828.6 | 6880.7 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 6 | | 167530.7 | 210.5 | 0.7705 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 7 | | 161057.4 | 156.6 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 8 | | 137137 | 4139 | 0 | 0.737556 | | 75728.26 | 0 | 220277.5 | 0 | 0 | 0 |
| 9 | | 111052 | 2662.4 | 0 | 0.603245 | | 0 | 30247.6 | 58265.4 | 29763.39 | 0 | 0 |
| 10 | | 110009 | 6470 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 11 | | 109833.7 | 6904.6 | 0 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 12 | | 93627 | 2740 | 0.2295 | 1 | | | 0 | 0 | 0 | 0 | 0 |
| 13 | | 84167.1 | 1468.8 | 0 | 0.557596 | | 0 | 17865.99 | 146812.7 | 58813.12 | 0 | 0 |
| 14 | | 83206.7 | 2426.6 | 0 | 1 | | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | | 81937.2 | 2209.1 | 0 | 0.470611 | | 0 | 25465.33 | 122500.6 | 60995.79 | 0 | 0 |
| 16 | | 79609 | 139 | 0 | 0.533544 | | 0 | 8738.346 | 139610.4 | 70960.21 | 652.0406 | 0 |
| 17 | | | | | | | | | | | | |
| 18 | 15 | | | | | | | | | | | |
| 19 | | Slack | | | | | | | | | | |
| 20 | | 0 | | | | | | | | | | |
| 21 | | 8738.346 | | | | | | | | | | |
| 22 | | 139610.4 | | | | | | | | | | |
| 23 | | 70960.21 | | | | | | | | | | |
| 24 | | 652.0406 | | | | | | | | | | |
| 25 | | 219961 | | | | | | | | | | |

Figure 3.2. Second-stage Slacks for Input-oriented VRS Measure-specific Model

3.3 Performance Evaluation of Fortune 500 Companies

Fortune magazine analyzes the financial performance of companies by eight measures: revenue, profit, assets, number of employees (employees), stockholders' equity (equity), market value (MV), earnings per share (EPS) and total return to investors (TRI).

In order to obtain an overall performance index, Zhu (2000) employs DEA to reconcile these eight measures via a two-stage transformation process described in Figure 3.3. Each stage is defined by a group of "inputs (x)" and "outputs (y)".

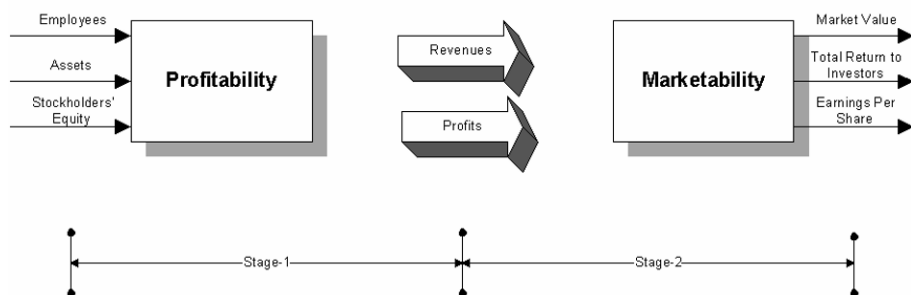


Figure 3.3. Input-output System for Fortune 500 Companies

The performance in the first stage (stage-1) may be viewed as profitability, i.e., a company's ability to generate the revenue and profit in terms of its current labor, assets and capital stock. The performance in the second stage (stage-2) may be viewed as (stock) marketability, i.e., a company's performance in stock market by its revenue and profit generated.

The data of 1995 is used. The DMU numbers correspond to the ranks by the magnitude of revenues. Because some data on MV, profit and equity are not available for some companies, we exclude these companies, and analyze the performance of the 364 companies.

2.3.1 Identification of Best Practice Frontier

Because the Fortune 500 list consists of a variety of companies representing different industries, we assume that the best-practice frontier exhibits VRS. We use the input-oriented VRS envelopment model to identify the best-practice.

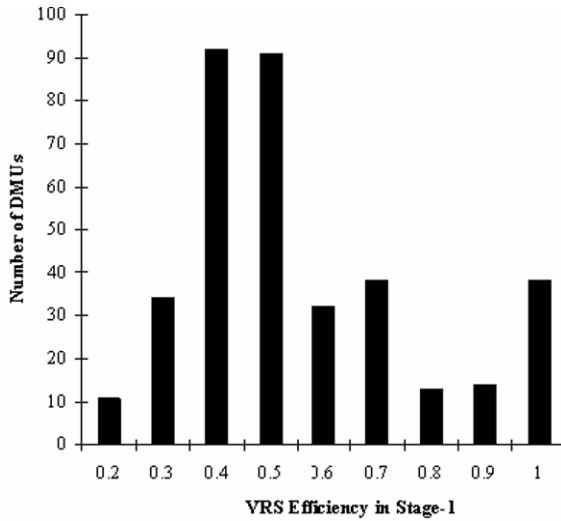


Figure 3.4. Profitability VRS Efficiency Distribution

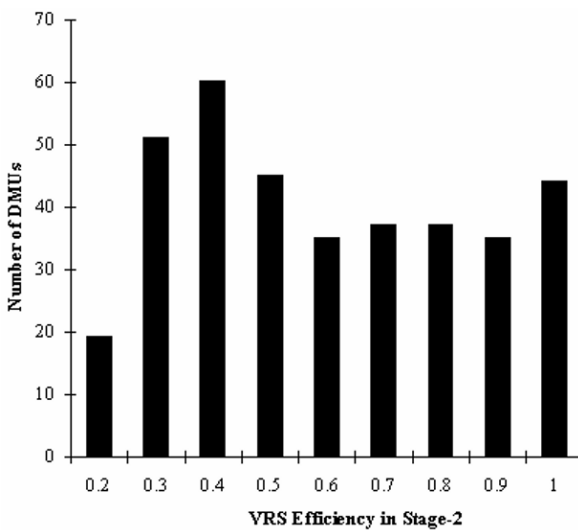


Figure 3.5. Marketability VRS Efficiency Distribution

Figures 3.4 and 3.5 report the distributions of VRS efficiency scores. 30 and 16 DMUs are VRS-efficient in profitability (stage-1) and marketability (stage-2), respectively. In stage-1, most VRS scores are distributed over [0.27, 0.51]. In stage-2, the VRS scores are almost evenly distributed over [0.16, 1]. Only four companies, namely, General Electric (DMU7), Coca-

Cola (DMU48), Nash Finch (DMU437), and CompUSA (DMU451) are on the best-practice frontiers of stage-1 and stage-2.

2.3.2 Measure-specific Performance¹

Proportional reductions of all inputs are used to determine the best practice frontier for the Fortune 500 companies. However, in an evaluation of inefficient DMUs, non-proportional input (output) improvement may be more appropriate. Therefore, we seek an alternative way to further characterize the performance of inefficient companies by measure-specific models.

Because we have already obtained the VRS best-practice frontier and the measure-specific models yield the same frontier, we modify the VRS measure-specific models for a particular inefficient DMU_d ,

$$\begin{aligned}
 \theta_d^{k*} &= \min \theta_d^k \quad d \in \mathbf{N} \\
 \text{subject to} & \\
 \sum_{j \in \mathbf{E}} \lambda_j^d x_{kj} &= \theta_d^k x_{kd} \quad k \in \{1, \dots, m\} \\
 \sum_{j \in \mathbf{E}} \lambda_j^d x_{ij} &\leq x_{id} \quad i \neq k \\
 \sum_{j \in \mathbf{E}} \lambda_j^d y_{rj} &\geq y_{rd} \quad r = 1, \dots, s \\
 \sum_{j \in \mathbf{E}} \lambda_j^d &= 1 \\
 \lambda_j^d &\geq 0, \quad j \in \mathbf{E}.
 \end{aligned} \tag{3.1}$$

$$\begin{aligned}
 \phi_d^{q*} &= \max \phi_d^q \quad d \in \mathbf{N} \\
 \text{subject to} & \\
 \sum_{j \in \mathbf{E}} \lambda_j^d y_{qj} &= \phi_d^q y_{qd} \quad q \in \{1, \dots, s\} \\
 \sum_{j \in \mathbf{E}} \lambda_j^d y_{rj} &\geq y_{rd} \quad r \neq q \\
 \sum_{j \in \mathbf{E}} \lambda_j^d x_{ij} &\leq x_{id} \quad i = 1, \dots, m \\
 \sum_{j \in \mathbf{E}} \lambda_j^d &= 1 \\
 \lambda_j^d &\geq 0, \quad j \in \mathbf{E}.
 \end{aligned} \tag{3.2}$$

where \mathbf{E} and \mathbf{N} represent the index sets for the efficient and inefficient companies, respectively, identified by the VRS envelopment DEA model.

¹ The material in this section is adapted from European Journal of Operational Research, Vol 123, Zhu, J., Multi-factor Performance Measure Model with An Application to Fortune 500 Companies, 105-124, 2000, with permission from Elsevier Science.

Table 3.2. Profitability Measure-specific Efficiency

| DMU No. | Company Name | Profitability | | | | |
|------------|--------------------|---------------|--------|--------|---------|--------|
| | | employees | assets | equity | revenue | profit |
| 1 | General Motors | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2 | Ford Motors | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | Exxon | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4 | Wal-Mart Stores | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 5 | AT&T | 0.479 | 0.503 | 0.653 | 1.172 | 41.670 |
| 6 | IBM | 0.304 | 0.598 | 0.573 | 1.307 | 1.397 |
| 7 | General Electric | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | Mobil | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 9 | Chrysler | 0.805 | 0.735 | 0.906 | 1.060 | 1.381 |
| 10 | Philip Morris | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 13 | Du Pont De Nemours | 0.933 | 0.950 | 0.976 | 1.015 | 1.039 |
| 14 | Texaco | 0.933 | 0.862 | 0.936 | 1.046 | 2.475 |
| 15 | Sears Roebuck | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 17 | Procter & Gamble | 0.325 | 0.743 | 0.654 | 1.291 | 1.413 |
| 18 | Chevron | 0.493 | 0.469 | 0.444 | 1.716 | 4.100 |
| 19 | Citicorp | 0.285 | 0.096 | 0.385 | 2.237 | 1.415 |
| 20 | Hewlett-Packard | 0.286 | 0.772 | 0.535 | 1.287 | 1.443 |
| Average | | 0.755 | 0.808 | 0.827 | 1.184 | 3.784 |

Table 3.3. Marketability Measure-specific Efficiency

| DMU No. | Company Name | Marketability | | | | |
|------------|--------------------|---------------|--------|-------|--------|---------|
| | | Revenue | Profit | MV | TRI | EPS |
| 1 | General Motors | 0.025 | 0.010 | 3.207 | 9.258 | 84.743 |
| 2 | Ford Motors | 0.028 | 0.013 | 3.314 | 33.754 | 170.670 |
| 3 | Exxon | 0.155 | 0.088 | 1.284 | 3.022 | 41.071 |
| 4 | Wal-Mart Stores | 0.058 | 0.029 | 2.011 | 43.768 | 447.912 |
| 5 | AT&T | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 6 | IBM | 0.114 | 0.032 | 1.690 | 7.235 | 59.740 |
| 7 | General Electric | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 8 | Mobil | 0.068 | 0.052 | 2.401 | 6.494 | 103.697 |
| 9 | Chrysler | 0.057 | 0.020 | 4.388 | 17.559 | 115.283 |
| 10 | Philip Morris | 0.239 | 0.095 | 1.408 | 2.465 | 47.335 |
| 13 | Du Pont De Nemours | 0.122 | 0.023 | 2.342 | 8.748 | 107.288 |
| 14 | Texaco | 0.080 | 0.059 | 3.202 | 7.863 | 243.004 |
| 15 | Sears Roebuck | 0.079 | 0.020 | 4.737 | 9.926 | 135.778 |
| 17 | Procter & Gamble | 0.168 | 0.127 | 1.789 | 5.644 | 138.595 |
| 18 | Chevron | 0.122 | 0.059 | 2.035 | 11.386 | 427.273 |
| 19 | Citicorp | 0.118 | 0.017 | 3.000 | 4.154 | 82.197 |
| 20 | Hewlett-Packard | 0.172 | 0.111 | 1.839 | 3.268 | 111.866 |
| Average | | 0.212 | 0.162 | 2.391 | 10.385 | 136.379 |

Models (3.1) and (3.2) determine the maximum potential decrease of an input and increase of an output while keeping other inputs and outputs at current levels.

Tables 3.2 and 3.3 report the results for the top-20 companies. Recall that revenue and profit are two factors served as the two outputs in stage-1 and the two inputs in stage-2. Therefore, we have two measure-specific efficiency scores for each revenue and each profit.

Table 3.4. Profitability Measure-specific Industry Efficiency

| Industries | Employees | Assets | Equity | Revenue | Profit |
|--|-------------|-------------|-------------|-------------|--------------|
| Aerospace | 0.12 (0.11) | 0.30 (0.35) | 0.24 (0.23) | 1.98 (2.26) | 3.85 (4.16) |
| Airlines | 0.12 (0.12) | 0.24 (0.29) | 0.24 (0.27) | 1.95 (2.37) | 4.54 (5.36) |
| Beverages | 0.34 (0.33) | 0.56 (0.46) | 0.56 (0.39) | 1.71 (2.17) | 1.72 (4.96) |
| Chemicals | 0.46 (0.32) | 0.56 (0.46) | 0.46 (0.31) | 1.91 (2.54) | 1.83 (2.46) |
| Commercial Banks | 0.13 (0.13) | 0.06 (0.05) | 0.31 (0.24) | 3.66 (4.26) | 2.62 (3.24) |
| Computer and Data Services | 0.20 (0.30) | 0.57 (0.54) | 0.36 (0.31) | 2.93 (3.12) | 1.69 (2.39) |
| Computers, Office Equipment | 0.25 (0.26) | 0.60 (0.59) | 0.48 (0.38) | 1.51 (2.00) | 1.77 (3.33) |
| Diversified Financials | 0.15 (0.38) | 0.62 (0.39) | 0.39 (0.43) | 2.46 (3.09) | 2.08 (2.40) |
| Electric & Gas Utilities | 0.30 (0.32) | 0.18 (0.18) | 0.16 (0.15) | 3.73 (3.85) | 2.69 (4.03) |
| Electronics, Electrical Equipment | 0.41 (0.30) | 0.86 (0.53) | 0.69 (0.38) | 1.68 (2.35) | 1.57 (3.12) |
| Entertainment | 0.12 (0.16) | 0.20 (0.24) | 0.15 (0.25) | 3.24 (3.06) | 4.33 (7.86) |
| Food | 0.29 (0.35) | 0.45 (0.55) | 0.32 (0.43) | 1.84 (2.01) | 2.49 (6.26) |
| Food & Drug Stores | 0.35 (0.23) | 0.70 (0.64) | 0.44 (0.41) | 1.49 (1.83) | 2.39 (3.34) |
| Forest & Paper Products | 0.16 (0.17) | 0.30 (0.36) | 0.21 (0.18) | 2.47 (2.83) | 3.44 (7.88) |
| General Merchandisers | 0.65 (0.32) | 0.85 (0.65) | 0.65 (0.43) | 1.35 (2.16) | 1.95 (3.52) |
| Health Care | 0.07 (0.30) | 0.34 (0.47) | 0.23 (0.32) | 2.69 (2.75) | 3.61 (4.94) |
| Industrial & Farm Equipment | 0.14 (0.13) | 0.33 (0.36) | 0.24 (0.18) | 2.43 (2.89) | 2.78 (3.59) |
| Insurance: Life & Health (stock) | 0.15 (0.25) | 0.06 (0.07) | 0.19 (0.22) | 2.57 (2.97) | 4.72 (5.23) |
| Insurance: Property & Casualty (stock) | 0.29 (0.37) | 0.29 (0.27) | 0.47 (0.35) | 2.26 (2.76) | 1.84 (2.64) |
| Metal Products | 0.12 (0.11) | 0.43 (0.42) | 0.19 (0.17) | 3.04 (3.28) | 2.82 (4.83) |
| Motor Vehicles & Parts | 0.77 (0.32) | 0.92 (0.51) | 0.84 (0.39) | 1.19 (2.11) | 1.36 (2.67) |
| Petroleum Refining | 0.64 (0.51) | 0.71 (0.50) | 0.73 (0.44) | 1.29 (1.72) | 1.85 (6.28) |
| Pharmaceuticals | 0.38 (0.41) | 0.63 (0.64) | 0.52 (0.54) | 2.02 (2.14) | 1.44 (1.59) |
| Pipelines | 0.63 (0.57) | 0.51 (0.44) | 0.59 (0.50) | 1.76 (1.91) | 1.40 (2.64) |
| Publishing, Printing | 0.13 (0.20) | 0.39 (0.43) | 0.16 (0.21) | 3.35 (3.44) | 2.77 (2.84) |
| Soaps, Cosmetics | 0.40 (0.47) | 0.64 (0.64) | 0.58 (0.58) | 1.40 (1.45) | 1.67 (3.00) |
| Special Retailers | 0.24 (0.27) | 0.69 (0.66) | 0.35 (0.40) | 1.72 (2.11) | 2.39 (4.28) |
| Telecommunications | 0.36 (0.25) | 0.41 (0.34) | 0.42 (0.33) | 1.70 (2.45) | 3.72 (10.19) |
| Temporary Help | 0.50 (0.56) | 0.84 (0.87) | 0.68 (0.68) | 1.30 (1.31) | 1.39 (1.31) |
| Wholesalers | 0.54 (0.58) | 0.69 (0.74) | 0.38 (0.55) | 1.34 (1.46) | 2.37 (2.08) |

* The number in parenthesis represents the arithmetic average.

We may use the average measure-specific efficiency scores (optimal values to (3.1) or (3.2)) within each industry to characterize the measure-specific industry efficiency. However, different companies with different sizes may exist in each industry. Therefore arithmetic averages may not be a

good way to characterize the industry efficiency. Usually, one expects large input and output levels, e.g., assets and revenue, from relatively big companies.

Table 3.5. Marketability Measure-specific Industry Efficiency

| Industries | Revenue | Profit | MV | TRI | EPS |
|--|-------------|-------------|--------------|---------------|-----------------|
| Aerospace | 0.21 (0.29) | 0.06 (0.06) | 3.44 (4.51) | 5.88 (6.39) | 102.60 (131.24) |
| Airlines | 0.35 (0.55) | 0.34 (0.38) | 4.73 (4.10) | 2.66 (5.80) | 35.65 (54.23) |
| Beverages | 0.64 (0.61) | 0.80 (0.37) | 1.37 (2.67) | 4.57 (4.94) | 102.55 (95.84) |
| Chemicals | 0.32 (0.52) | 0.06 (0.11) | 3.55 (4.16) | 9.63 (13.37) | 84.31 (100.13) |
| Commercial Banks | 0.34 (0.51) | 0.07 (0.13) | 3.94 (4.05) | 6.40 (5.98) | 85.39 (104.16) |
| Computer and Data Services | 0.82 (0.85) | 0.83 (0.66) | 1.20 (1.52) | 4.01 (5.68) | 38.83 (38.77) |
| Computers, Office Equipment | 0.24 (0.50) | 0.09 (0.02) | 2.37 (3.90) | 4.28 (8.59) | 84.83 (86.03) |
| Diversified Financials | 0.33 (0.47) | 0.13 (0.17) | 2.82 (3.79) | 5.24 (7.31) | 6.88 (117.12) |
| Electric & Gas Utilities | 0.56 (0.65) | 0.08 (0.14) | 4.99 (5.00) | 10.34 (11.68) | 128.75 (126.03) |
| Electronics, Electrical Equipment | 0.63 (0.54) | 0.47 (0.19) | 2.14 (3.54) | 6.34 (14.22) | 86.56 (87.05) |
| Entertainment | 0.40 (0.50) | 0.41 (0.39) | 1.49 (1.56) | 6.96 (10.07) | 248.91 (285.42) |
| Food | 0.30 (0.42) | 0.09 (0.20) | 3.42 (3.28) | 8.83 (16.35) | 145.47 (143.31) |
| Food & Drug Stores | 0.28 (0.44) | 0.10 (0.21) | 5.52 (5.31) | 6.24 (25.35) | 83.66 (86.30) |
| Forest & Paper Products | 0.49 (0.59) | 0.06 (0.16) | 4.24 (5.14) | 11.63 (26.57) | 67.17 (66.03) |
| General Merchandisers | 0.12 (0.32) | 0.04 (0.08) | 3.48 (4.33) | 17.41 (24.60) | 129.62 (152.9) |
| Health Care | 0.50 (0.66) | 0.16 (0.28) | 2.68 (2.91) | 8.95 (13.56) | 98.43 (95.67) |
| Industrial & Farm Equipment | 0.39 (0.52) | 0.08 (0.15) | 4.19 (4.20) | 7.56 (13.39) | 98.35 (103.26) |
| Insurance: Life & Health (stock) | 0.37 (0.53) | 0.10 (0.14) | 4.86 (5.16) | 6.25 (6.58) | 57.71 (65.72) |
| Insurance: Property & Casualty (stock) | 0.28 (0.47) | 0.05 (0.08) | 4.09 (5.55) | 6.42 (8.49) | 71.85 (80.33) |
| Metal Products | 0.63 (0.70) | 0.30 (0.36) | 2.22 (2.12) | 7.47 (8.89) | 103.54 (96.76) |
| Motor Vehicles & Parts | 0.07 (0.32) | 0.02 (0.08) | 41.19 (5.72) | 13.26 (30.38) | 77.53 (77.95) |
| Petroleum Refining | 0.17 (0.36) | 0.07 (0.16) | 2.41 (4.21) | 9.94 (18.53) | 102.81 (122.06) |
| Pharmaceuticals | 0.44 (0.44) | 0.33 (0.29) | 1.75 (2.00) | 4.77 (5.65) | 163.58 (193.30) |
| Pipelines | 0.51 (0.65) | 0.07 (0.13) | 4.38 (4.22) | 5.22 (6.47) | 46.10 (103.94) |
| Publishing, Printing | 0.67 (0.73) | 0.21 (0.22) | 3.23 (3.19) | 14.35 (19.16) | 69.09 (73.99) |
| Soaps, Cosmetics | 0.25 (0.37) | 0.13 (0.12) | 1.89 (2.23) | 10.04 (11.98) | 101.42 (109.16) |
| Special Retailers | 0.40 (0.60) | 0.15 (0.30) | 3.19 (3.90) | 6.79 (20.26) | 103.34 (98.36) |
| Telecommunications | 0.68 (0.44) | 0.12 (0.25) | 1.85 (2.53) | 7.19 (61.16) | 188.16 (214.68) |
| Temporary Help | 0.69 (0.78) | 0.32 (0.36) | 3.68 (3.52) | 18.87 (93.62) | 30.83 (30.48) |
| Wholesalers | 0.37 (0.47) | 0.18 (0.30) | 3.95 (4.61) | 6.64 (8.83) | 49.55 (49.33) |

* The number in parenthesis represents the arithmetic average.

Thus, we define weighted measure-specific scores within each industry by considering the sizes of the companies.

(size-adjusted) k th input-specific industry efficiency measure for industry \mathbf{F}

$$I_k^{\mathbf{F}} = \sum_{d \in \mathbf{F}} \theta_d^{k*} \cdot \frac{x_{kd}}{\sum_{d \in \mathbf{F}} x_{kd}} = \frac{\sum_{d \in \mathbf{F}} \hat{x}_{kd}}{\sum_{d \in \mathbf{F}} x_{kd}} \quad (3.3)$$

(size-adjusted) q th output-specific industry efficiency measure for industry \mathbf{F}

$$O_q^{\mathbf{F}} = \sum_{d \in \mathbf{F}} \phi_d^{q*} \cdot \frac{y_{qd}}{\sum_{d \in \mathbf{F}} y_{qd}} = \frac{\sum_{d \in \mathbf{F}} \hat{y}_{qd}}{\sum_{d \in \mathbf{F}} y_{qd}} \quad (3.4)$$

where \hat{x}_{kd} ($= \theta_d^{k*} x_{kd}$) and \hat{y}_{qd} ($= \phi_d^{q*} y_{qd}$) are, respectively, the projected (potentially efficient) levels for k th input and q th output of DMU_d , $d \in \mathbf{F}$.

The weights in (3.3) ($\frac{x_{kd}}{\sum_{d \in \mathbf{F}} x_{kd}}$, $d \in \mathbf{F}$) and (2.4) ($\frac{y_{qd}}{\sum_{d \in \mathbf{F}} y_{qd}}$, $d \in \mathbf{F}$) are normalized, therefore a specific industry \mathbf{F} achieves 100% efficiency, i.e., $I_k^{\mathbf{F}} = 1$ and $O_q^{\mathbf{F}} = 1$, if and only if, all of its companies are located on the best-practice frontier.

Tables 3.4 and 3.5 report the industry efficiency scores for the 30 selected industries where the number in parenthesis represents the corresponding arithmetic mean of measure-specific efficiency scores.

A relatively large discrepancy between weighted and arithmetic average scores is detected for six industries – General Merchandiser, Health Care, Motor Vehicles & Parts, Petroleum Refining, Pipelines, and Telecommunications. Since (3.3) and (3.4) determine the industry efficiency by considering the size of each company, this may imply that efficiency may highly correlate with size in these industries.

2.3.3 Benchmark Share

Non-zero λ_j^* indicates that DMU_j is used as a benchmark. As an efficient company, the role it plays in evaluating inefficiency companies is to be of interest. One wants to know the importance of each efficient DMU in measuring the inefficiencies of inefficient $DMUs$. Based upon the non-zero λ_j^* , we develop benchmark-share measures for each efficient company via (3.1) and (3.2).

We define the k th input-specific benchmark-share for each efficient DMU_j , $j \in \mathbf{E}$,

$$\Delta_j^k = \frac{\sum_{d \in \mathbf{N}} \lambda_j^{d*} (1 - \theta_d^{k*}) x_{kd}}{\sum_{d \in \mathbf{N}} (1 - \theta_d^{k*}) x_{kd}} \quad (3.5)$$

where λ_j^{d*} and θ_d^{k*} are optimal values in (3.1).

We define the q th output-specific benchmark-share for each efficient $DMU_j, j \in \mathbf{E}$,

$$\Pi_j^q = \frac{\sum_{d \in \mathbf{N}} \lambda_j^{d*} (\phi_d^{q*} - 1) y_{qd}}{\sum_{d \in \mathbf{N}} (\phi_d^{q*} - 1) y_{qd}} \quad (3.6)$$

where λ_j^{d*} and ϕ_d^{q*} are optimal values in (3.2).

The benchmark-share Δ_j^k (or Π_j^q) depends on the values of λ_j^{d*} and θ_d^{k*} (or λ_j^{d*} and ϕ_d^{q*}). Note that $(1 - \theta_d^{k*}) x_{kd}$ and $(\phi_d^{q*} - 1) y_{qd}$ characterize the potential decrease on k th input and increase on q th output, respectively.

Δ_j^k and Π_j^q are weighted λ_j^{d*} across all inefficient DMUs. The weights,

$$\frac{(1 - \theta_d^{k*}) x_{kd}}{\sum_{d \in \mathbf{N}} (1 - \theta_d^{k*}) x_{kd}} \text{ in (3.5) and } \frac{(\phi_d^{q*} - 1) y_{qd}}{\sum_{d \in \mathbf{N}} (\phi_d^{q*} - 1) y_{qd}} \text{ in (3.6) are normalized.}$$

Therefore, we have $\sum_{j \in \mathbf{E}} \Delta_j^k = 1$ and $\sum_{j \in \mathbf{E}} \Pi_j^q = 1$. (Note that $\sum_{j \in \mathbf{E}} \lambda_j^{d*} = 1$ in (3.1) and (3.2).)

It is very clear from (3.5) and (3.6) that an efficient company which does not act as a referent DMU for any inefficient DMU will have zero benchmark-share. The bigger the benchmark-share, the more important an efficient company is in benchmarking.

Table 3.6 reports the benchmark-shares for 12 selected VRS-efficient companies. The benchmark-shares for the remaining VRS-efficient companies are less than 0.01%. Of the total 60 benchmark-shares, 12 are greater than 10%. Particularly, DMU48 (Coca-Cola), DMU156 (General Mills) and DMU281 (Bindley Western) have the biggest benchmark-share with respect to employees, equity and profit, respectively. This means that, e.g., General Mills plays a leading role in setting a benchmark with respect to equity input given the current levels of employees and assets. Note that General Mills had the highest returns on equity in 1995.

In Table 2.7, DMU226 (Continental Airlines) and DMU292 (Berkshire Hathaway) are two important companies in TRI and EPS benchmarking, respectively. (Note that Continental Airlines and Berkshire Hathaway had the highest TRI and EPS in 1995.) Although Berkshire Hathaway was ranked 18 in terms of MV levels by the Fortune magazine, the benchmark-share of 39.99% indicates that it had an outstanding performance in terms of

MV given other measures at their current levels. This indicates that single financial performance alone is not sufficient to characterize a company's performance.

Finally, note that, e.g., DMU292 and DMU474 both acted as a referent DMU in 63% of the inefficient DMUs when measuring the revenue-specific efficiency. However, the benchmark-share indicates that DMU474 is more important.

Table 3.6. Benchmark-share for Profitability

| DMU No. | Company Name | Employees | Assets | Equity | Revenue | Profit |
|--------------|---------------------|---------------|---------------|---------------|---------------|---------------|
| 8 | Mobil | 3.07% | 1.51% | 0.76% | 16.00% | 0.15% |
| 32 | Fed. Natl. Mortgage | 2.78% | 0 | 2.76% | 0.89% | 0.10% |
| 44 | Loews | 7.17% | 0.14% | 0 | 0.95% | 1.41% |
| 48 | Coca-Cola | 2.58% | 12.54% | 10.65% | 2.88% | 40.65% |
| 94 | IBP | 0 | 22.51% | 0.07% | 13.16% | 0.80% |
| 153 | Bergen Brunswig | 0.60% | 0 | 0.16% | 5.91% | 0.17% |
| 156 | General Mills | 1.86% | 0.01% | 60.91% | 17.19% | 7.85% |
| 168 | Cardinal Health | 3.12% | 2.82% | 0.01% | 10.89% | 0 |
| 281 | Bindley Western | 52.91% | 4.79% | 2.93% | 5.97% | 2.86% |
| 419 | Micron Technology | 0.17% | 28.37% | 0.24% | 0.29% | 11.04% |
| 437 | Nash Finch | 0 | 10.16% | 0.02% | 0.24% | 0.27% |
| 447 | Williams | 8.68% | 0 | 0 | 0.02% | 8.62% |
| Total | | 82.94% | 82.85% | 78.51% | 74.39% | 73.92% |

Table 3.7. Benchmark-share for Marketability

| DMU No. | Company Name | Revenue | Profit | MV | TRI | EPS |
|--------------|-------------------------------|---------------|---------------|---------------|---------------|---------------|
| 5 | AT&T | 0 | 12.33% | 6.95% | 2.22% | 0 |
| 7 | IBM | 0 | 0.20% | 3.83% | 6.39% | 0.79% |
| 48 | Coca-Cola | 5.44% | 0.80% | 11.37% | 0.13% | 0.11% |
| 78 | Kimberly-Clark | 0.04% | 36.66% | 6.96% | 0 | 0.10% |
| 210 | Burlington Northern Santa FE | 0.05% | 4.29% | 6.39% | 0 | 0 |
| 219 | Microsoft | 8.46% | 0 | 9.97% | 0 | 0 |
| 226 | Continental Airlines | 0.44% | 0.69% | 1.30% | 81.91% | 0.87% |
| 292 | Berkshire Hathaway | 23.56% | 8.37% | 39.99% | 0.18% | 73.96% |
| 312 | Chiquita Brands International | 0.00% | 15.49% | 0.07% | 0.17% | 11.41% |
| 376 | Consolidated Natural Gas | 0.99% | 11.29% | 4.89% | 0.05% | 0.00% |
| 417 | Oracle | 0.00% | 0.00% | 1.37% | 0 | 0 |
| 437 | Nash Finch | 0.09% | 0.51% | 0 | 0 | 3.56% |
| 451 | CompUSA | 1.43% | 7.22% | 0.88% | 8.85% | 4.11% |
| 474 | Computer Associates | 29.90% | 0.04% | 1.69% | 0.00% | 0.00% |
| 494 | Foundation Health | 5.21% | 2.07% | 4.25% | 0.07% | 2.58% |
| 495 | State Street Boston Corp. | 24.39% | 0.04% | 0.09% | 0.03% | 2.51% |
| Total | | 100% | 100% | 100% | 100% | 100% |

We here explore the multidimensional financial performance of the Fortune 500 companies. Revenue-top-ranked companies do not necessarily have top-ranked performance in terms of profitability and (stock) marketability. Most companies exhibited serious inefficiencies. The measure-specific models enable us to study the performance based upon a specific measure while keeping the current levels of other measures. See Zhu (2000) for more discussion on measuring the performance of Fortune 500 companies.

3.4 Solving DEA Using DEA Frontier Software

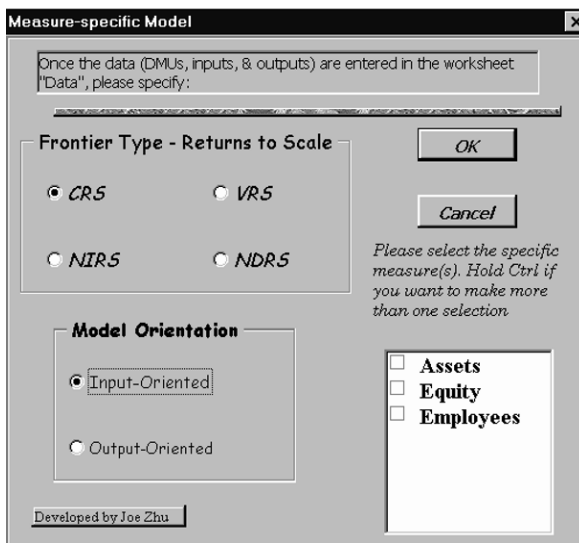


Figure 3.6. Measure-specific Models

To run the measure-specific models, select the “Measure Specific Model” menu item. You will be prompted with a form for selecting the models presented in Table 3.1, as shown in Figure 3.6.

Select the measures that are of interest. If you select all the input or all the output measures, then you have the envelopment models.

The results are reported in the “Efficiency”, “Slack” and “Target” sheets.

REFERENCES

1. Banker, R.D. and R.C. Morey (1986), Efficiency analysis for exogenously fixed inputs and outputs, *Operations Research*, 34, 513-521.
2. Zhu, J. (2000), Multi-factor performance measure model with an application to Fortune 500 companies, *European Journal of Operational Research*, 123, No. 1, 105-124.

Chapter 4

Non-radial DEA Models and DEA with Preference

4.1 Non-radial DEA Models

We can call the envelopment DEA models as radial efficiency measures, because these models optimize all inputs or outputs of a DMU at a certain proportion. Färe and Lovell (1978) introduce a non-radial measure which allows nonproportional reductions in positive inputs or augmentations in positive outputs. Table 4.1 summarizes the non-radial DEA models with respect to the model orientation and frontier type.

Table 4.1. Non-radial DEA Models

| Frontier Type | Input-Oriented | Output-Oriented |
|------------------|--|--|
| | $\min \left(\frac{1}{m} \sum_{i=1}^m \theta_i - \varepsilon \sum_{r=1}^s s_r^+ \right)$ | $\max \left(\frac{1}{s} \sum_{r=1}^s \phi_r + \varepsilon \sum_{i=1}^m s_i^- \right)$ |
| | subject to | subject to |
| | $\sum_{j=1}^n \lambda_j x_{ij} = \theta_i x_{i0} \quad i = 1, 2, \dots, m;$ | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{i0} \quad i = 1, 2, \dots, m;$ |
| CRS | $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{r0} \quad r = 1, 2, \dots, s;$ | $\sum_{j=1}^n \lambda_j y_{rj} = \phi_r y_{r0} \quad r = 1, 2, \dots, s;$ |
| | $\theta_i \leq 1 \quad i = 1, 2, \dots, m;$ | $\phi_r \geq 1 \quad r = 1, 2, \dots, s;$ |
| | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ | $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ |
| VRS | Add $\sum_{j=1}^n \lambda_j = 1$ | |
| NIRS | Add $\sum_{j=1}^n \lambda_j \leq 1$ | |
| NDRS | Add $\sum_{j=1}^n \lambda_j \geq 1$ | |
| Efficient Target | $\begin{cases} \hat{x}_{i0} = \theta_i^* x_{i0} & i = 1, 2, \dots, m \\ \hat{y}_{r0} = y_{r0} + s_r^{+*} & r = 1, 2, \dots, s \end{cases}$ | $\begin{cases} \hat{x}_{i0} = x_{i0} - s_i^{-*} & i = 1, 2, \dots, m \\ \hat{y}_{r0} = \phi_r^* y_{r0} & r = 1, 2, \dots, s \end{cases}$ |

The slacks in the non-radial DEA models are optimized in a second-stage model where θ_i^* or ϕ_r^* are fixed. For example, under CRS we have

Input Slacks for Output-oriented Non-radial DEA Model

$$\begin{aligned} & \max \sum_{r=1}^s s_r^+ \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} = \theta_i^* x_{i0} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{r0} \quad r = 1, 2, \dots, s; \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned}$$

Output Slacks for Input-oriented Non-radial DEA Model

$$\begin{aligned} & \max \sum_{i=1}^m s_i^- \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{i0} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} = \phi_r^* y_{r0} \quad r = 1, 2, \dots, s; \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned}$$

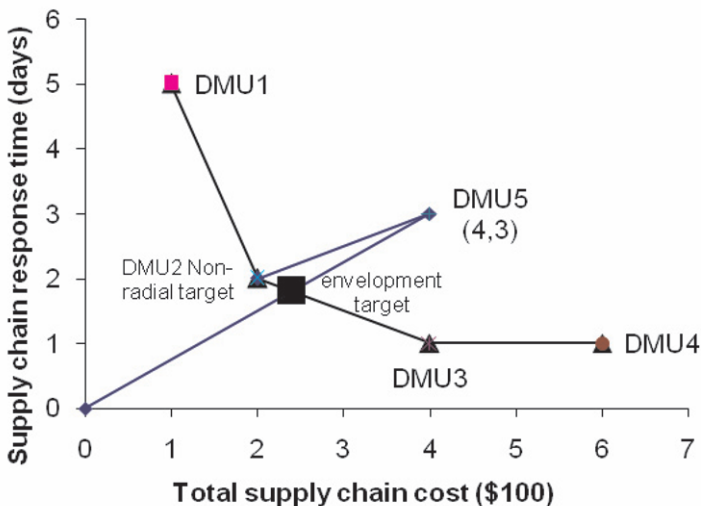


Figure 4.1. Efficient Targets

Note that input slacks do not exist in the input-oriented non-radial DEA models, and output slacks do not exist in the output-oriented non-radial DEA models.

Because $\theta_i^* \leq 1$ ($\phi_r^* \geq 1$), $\frac{1}{m} \sum_{i=1}^m \theta_i^* \leq 1$ and $\frac{1}{m} \sum_{i=1}^m \theta_i^* = 1$ if and only if $\theta_i^* = 1$ for all i ($\frac{1}{s} \sum_{r=1}^s \phi_r^* \geq 1$ and $\frac{1}{s} \sum_{r=1}^s \phi_r^* = 1$ if and only if $\phi_r^* = 1$ for all r). Thus, $\frac{1}{m} \sum_{i=1}^m \theta_i^*$ ($\frac{1}{s} \sum_{r=1}^s \phi_r^*$) can be used as an efficiency index.

Both the envelopment models and the non-radial DEA models yield the same frontier, but may yield different efficient targets (even when the envelopment models do not have non-zero slacks). For example, if we change the second input from 4 to 3 for DMU5 in Table 1.1 (Chapter 1), the input-oriented CRS envelopment model yields the efficient target of $x_1 = 2.4$ and $x_2 = 1.8$ (with $\lambda_2^* = 0.8$, $\lambda_3^* = 0.2$, and all zero slacks). Whereas the input-oriented CRS non-radial DEA model yields DMU2 as the efficient target for DMU5 (see Figure 4.1). Note that both models yield the same target of DMU3 for DMU4.

4.2 DEA with Preference Structure and Cost/Revenue Efficiency

Both the envelopment models and the non-radial DEA models yield efficient targets for inefficient DMUs. However, these targets may not be preferred by the management or achievable under the current management and other external conditions. Therefore, some other targets along the efficient frontier should be considered as preferred ones. This can be done by constructing preference structures over the proportions by which the corresponding current input levels (output levels) can be changed. Zhu (1996) develops a set of weighted non-radial DEA models where various efficient targets along with the frontier can be obtained.

Let A_i ($i = 1, 2, \dots, m$) and B_r ($r = 1, 2, \dots, s$) be user-specified preference weights which reflect the relative degree of desirability of the adjustments of the current input and output levels, respectively. Then we can have a set of weighted non-radial DEA models based upon Table 4.1 by changing the objective functions $\frac{1}{m} \sum_{i=1}^m \theta_i$ and $\frac{1}{s} \sum_{r=1}^s \phi_r$ to $\frac{\sum_{i=1}^m A_i \theta_i}{\sum_{i=1}^m A_i}$ and $\frac{\sum_{r=1}^s B_r \phi_r}{\sum_{r=1}^s B_r}$, respectively.

Further, if we remove the constraint $\theta_i \leq 1$ ($\phi_r \geq 1$), we obtain the DEA/preference structure (DEA/PS) models shown in Table 4.2 (Zhu, 1996a).

If some $A_i = 0$ ($B_r = 0$), then set the corresponding $\theta_i = 1$ ($\phi_r = 1$). But at least one of such weights should be positive. Note that for example, the bigger the weight A_i , the higher the priority DMU_o is allowed to adjust its i th input amount to a lower level. i.e., when inefficiency occurs, the more one wants to adjust an input or an output, the bigger the weight should be

attached to θ_i or ϕ_r . If we can rank the inputs or outputs according to their relative importance, then we can obtain a set of ordinal weights. One may use Delphi-like techniques, or Analytic Hierarchy Process (AHP) to obtain the weights. However, caution should be paid when we convert the ordinal weights into preference weights. For example, if an input (output) is relatively more important and the DMU does not wish to adjust it with a higher rate, we should take the reciprocal of the corresponding ordinal weight as the preference weight. Otherwise, if the DMU does want to adjust the input (output) with a higher rate, we can take the ordinal weight as the preference weight. Also, one may use the principal component analysis to derive the information on weights (Zhu, 1998).

Note that in the DEA/PS models, some θ_i^* (ϕ_r^*) may be greater (less) than one under certain weight combinations. i.e., the DEA/PS models are *not* restricted to the case where 100% efficiency is maintained through the input decreases or output increases.

Table 4.2. DEA/Preference Structure Models

| Frontier Type | Input-Oriented | Output-Oriented |
|------------------|---|---|
| | $\min \left(\frac{\sum_{i=1}^m A_i \theta_i}{\sum_{i=1}^m A_i} - \varepsilon \sum_{r=1}^s s_r^+ \right)$ | $\max \left(\frac{\sum_{r=1}^s \phi_r}{\sum_{r=1}^s B_r} - \varepsilon \sum_{r=1}^s s_r^+ \right)$ |
| | subject to | subject to |
| CRS | $\sum_{j=1}^n \lambda_j x_{ij} = \theta_i x_{io} \quad i = 1, 2, \dots, m;$ $\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s;$ $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ | $\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, 2, \dots, m;$ $\sum_{j=1}^n \lambda_j y_{rj} = \phi_r y_{ro} \quad r = 1, 2, \dots, s;$ $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ |
| VRS | | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | | Add $\sum_{j=1}^n \lambda_j \geq 1$ |
| Efficient Target | $\begin{cases} \hat{x}_{io} = \theta_i^* x_{io} & i = 1, 2, \dots, m \\ \hat{y}_{ro} = y_{ro} + s_r^{+*} & r = 1, 2, \dots, s \end{cases}$ | $\begin{cases} \hat{x}_{io} = x_{io} - s_i^{-*} & i = 1, 2, \dots, m \\ \hat{y}_{ro} = \phi_r^* y_{ro} & r = 1, 2, \dots, s \end{cases}$ |

Now, in order to further investigate the property of DEA/PS models, we consider the dual program to the input-oriented CRS DEA/PS model.

$$\begin{aligned}
& \max \sum_{r=1}^s u_r y_{ro} \\
& \text{subject to} \\
& \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad j = 1, \dots, n; \\
& v_i x_{io} = A_i / \sum_{i=1}^m A_i \quad i = 1, \dots, m; \\
& \mu_r, v_i \geq 0
\end{aligned} \tag{4.1}$$

We see that the normalization condition $\sum_{i=1}^m v_i x_{io} = 1$ is also satisfied in (4.1). The DEA/PS model is actually a DEA model with fixed input multipliers.

Let p_i^o denote the i th input price for DMU_o and \tilde{x}_{io} represents the i th input that minimizes the cost. Consider the following DEA model for calculating the “minimum cost”.

$$\begin{aligned}
& \min \sum_{i=1}^m p_i^o \tilde{x}_{io} \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \tilde{x}_{io} \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \\
& \lambda_j, \tilde{x}_{io} \geq 0
\end{aligned} \tag{4.2}$$

The dual program to (4.2) is

$$\begin{aligned}
& \max \sum_{r=1}^s u_r y_{ro} \\
& \text{subject to} \\
& \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad j = 1, \dots, n; \\
& 0 \leq v_i \leq p_i^o \quad i = 1, \dots, m; \\
& \mu_r, v_i \geq 0
\end{aligned} \tag{4.3}$$

By the complementary slackness condition of linear programming, we have that if $\tilde{x}_{io}^* > 0$ then $p_i^o = v_i^*$. Thus, v_i^* can be interpreted as p_i^o . Consequently, the input prices can be used to develop the preference weights.

In the DEA literature, we have a concept called “cost efficiency” which is defined as

$$\frac{\sum_{i=1}^m p_i^o \tilde{x}_{io}^*}{\sum_{i=1}^m p_i^o x_{io}}$$

The following development shows that the related DEA/PS model can be used to obtain exact the cost efficiency scores. Because the actual cost – $\sum_{i=1}^m p_i^o x_{io}$ is a constant for a specific DMU_o , cost efficiency can be directly calculated by the following modified (4.2).

$$\begin{aligned} & \min \frac{\sum_{i=1}^m p_i^o \tilde{x}_{io}}{\sum_{i=1}^m p_i^o x_{io}} \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq \tilde{x}_{io} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned} \tag{4.4}$$

Let $\tilde{x}_{io} = \theta_i x_{io}$. Then (4.4) is equivalent to the input-oriented CRS DEA/PS model with $A_i = p_i^o x_{io}$. This indicates that if one imposes a proper set of preference weights for each DMU under consideration, then the DEA/PS model yields cost efficiency measure. (see Seiford and Zhu (2002) for an empirical investigation of DEA efficiency and cost efficiency.)

Similarly, the output-oriented DEA/PS model can be used to obtain the “revenue efficiency” which is defined as

$$\frac{\sum_{r=1}^s q_r^o \tilde{y}_{ro}^*}{\sum_{r=1}^s q_r^o y_{ro}}$$

where q_r^o indicates output price for DMU_o and \tilde{y}_{ro} represents the r th output that maximizes the revenue in the following linear programming problem.

$$\begin{aligned} & \max \sum_{r=1}^s q_r^o \tilde{y}_{ro} \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \tilde{y}_{ro} \quad r = 1, 2, \dots, s \\ & \lambda_j, \tilde{y}_{ro} \geq 0 \end{aligned} \tag{4.5}$$

Let $\tilde{y}_{ro} = \phi_r y_{ro}$ and $B_r = q_r^o y_{ro}$ in the output-oriented DEA/PS model. We have

$$\begin{aligned} & \max \frac{\sum_{r=1}^s q_r^o \tilde{y}_{ro}}{\sum_{r=1}^s q_r^o y_{ro}} \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq \tilde{y}_{ro} \quad r = 1, 2, \dots, s; \\ & \lambda_j \geq 0 \quad j = 1, 2, \dots, n. \end{aligned}$$

which calculates the revenue efficiency.

4.3 DEA/Preference Structure Models in Spreadsheets

Figure 4.2 shows an input-oriented VRS DEA/PS spreadsheet model. Cells I2:I16 are reserved for λ_j . Cells F20:F22 are reserved for θ_i . These are the changing cells in the Solver parameters shown in Figure 4.3.

| | A | B | C | D | E | F | G | H | I |
|----|------------------------------|-----------|---------|------------|---|------------|--------|---|-----------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 1 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 |
| 13 | Hitachi | 91620.9 | 29907.2 | 231852 | | 84187.1 | 1198.8 | | 0 |
| 14 | Nippon Life Insurance | 364762.5 | | | | 83 | | | 0 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | | | | 81 | | | 0 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 |
| 17 | | | | | | | | | |
| 18 | | Reference | | DMU under | 1 | Efficiency | | | |
| 19 | Constraints | set | | Evaluation | 3 | Weights | | | |
| 20 | Assets | 91920.6 | = | 91920.6 | | 1 | 1 | | |
| 21 | Equity | 10950 | = | 10950 | | 1 | 1 | | |
| 22 | Employees | 36000 | = | 36000 | | 1 | 1 | | |
| 23 | Revenue | 184365.2 | ≥ | 184365.2 | | | | | |
| 24 | Profit | 346.2 | ≥ | 346.2 | | | | | |
| 25 | $\Sigma \lambda$ | 1 | | | | | | | |

Figure 4.2. Input-oriented VRS DEA/PS Spreadsheet Model

The target cell is cell F19 which contains the following formula

Cell F19 =SUMPRODUCT(F20:F22,G20:G22)/SUM(G20:G22)

where cells G20:G22 are reserved for the input weights.

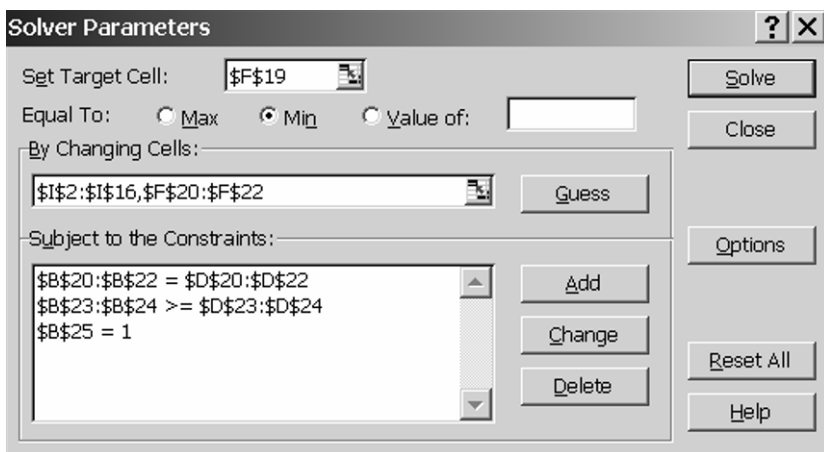


Figure 4.3. Solver Parameters for Input-oriented VRS DEA/PS Model

| | B | C | D | E | F | G | H | I | J | K | L | M |
|----|-----------|---|------------|----|------------|---------|---|---|------------|----------|-----------|-----------|
| 1 | Assets | Equity | Employees | | Revenue | Profit | | λ | Efficiency | Assets | Equity | Employees |
| 2 | 91 | Sub DEAPS () | | | | | | | 1 | 1 | 1 | 1 |
| 3 | 68 | | | | | | | 1 | 1 | 1 | 1 | |
| 4 | 65 | Dim i As Integer | | | | | | | 1 | 1 | 1 | 1 |
| 5 | 217 | For i = 1 To 15 | | | | | | | 1 | 1 | 1 | 1 |
| 6 | 50 | Range("E18") = i | | | | | | | 1 | 1 | 1 | 1 |
| 7 | 71 | SolverSolve UserFinish:=True | | | | | | | 0.948271 | 0.83411 | 0.9975806 | 1.013122 |
| 8 | 24 | Range("J" & i + 1) = Range("F19") | | | | | | | 0.561535 | 0.340581 | 1.1085533 | 0.23547 |
| 9 | 106 | 'place the individual thetas into columns K,L,M | | | | | | | 0.423575 | 0.62582 | 0.400533 | 0.244372 |
| 10 | | Range("K" & i + 1) = Range("F20") | | | | | | | 1 | 1 | 1 | 1 |
| 11 | 118 | Range("L" & i + 1) = Range("F21") | | | | | | | 1 | 1 | 1 | 1 |
| 12 | | Range("M" & i + 1) = Range("F22") | | | | | | | 1 | 1 | 1 | 1 |
| 13 | | Next | | | | | | | 1 | 1 | 1 | 1 |
| 14 | 91 | End Sub | | | | | | | 0.384513 | 0.638678 | 0.4502766 | 0.064583 |
| 15 | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 1 | 1 | 1 | 1 |
| 16 | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.347781 | 0.498661 | 0.4133195 | 0.131363 |
| 17 | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.324338 | 0.565556 | 0.3867662 | 0.020692 |
| 18 | Reference | | DMU under | 15 | Efficiency | | | | | | | |
| 19 | set | | Evaluation | | 0.324338 | Weights | | | | | | |
| 20 | 50268.9 | = | 50268.9 | | 0.565556 | 1 | | | | | | |
| 21 | 6681 | = | 6681 | | 0.386766 | 1 | | | | | | |
| 22 | 6193 | = | 6193 | | 0.020692 | 1 | | | | | | |
| 23 | 167530.7 | >= | 79609 | | | | | | | | | |
| 24 | 210.5 | >= | 139 | | | | | | | | | |

Figure 4.4. Efficiency Result for Input-oriented VRS DEA/PS Model

The formulas for cells B20:B25 are

Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)

Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)

Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)

Cell B23 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)
 Cell B24 =SUMPRODUCT(G2:G16,\$I\$2:\$I\$16)
 Cell B25 =SUM(I2:I16)

The formulas for cells D20:D24 are

Cell D20 =F20*INDEX(B2:B16,E18,1)
 Cell D21 =F21*INDEX(C2:C16,E18,1)
 Cell D22 =F22*INDEX(D2:D16,E18,1)
 Cell D23 =INDEX(F2:F16,E18,1)
 Cell D24 =INDEX(G2:G16,E18,1)

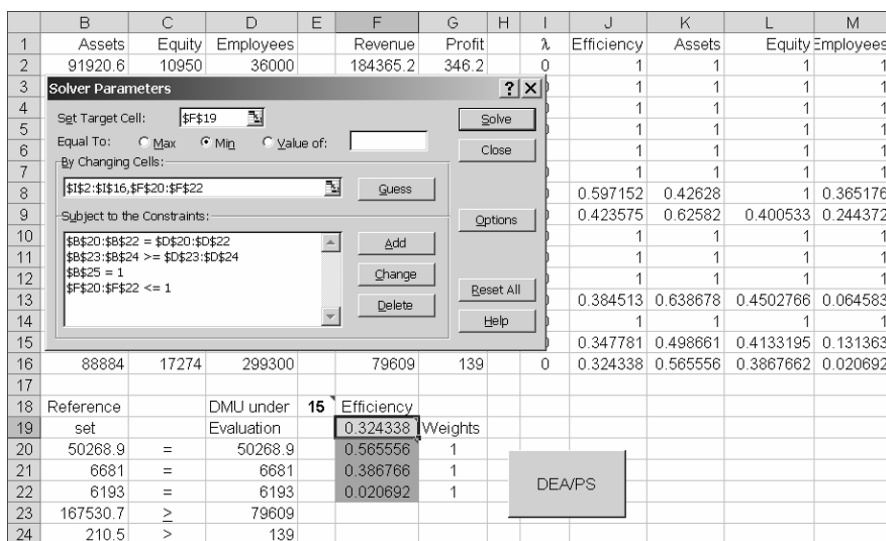


Figure 4.5. Efficiency Result for Input-oriented VRS Non-radial DEA Model

Figure 4.4 shows the results and the VBA procedure “DEAPS” which automates the calculation.

Note that the θ_i ($i = 1,2,3$) are not restricted in Figure 4.3. If we add $\theta_i \leq 1$ ($\$F\$20:\$F\$22 \leq 1$), then we obtain the results shown in Figure 4.5.

4.4 DEA and Multiple Objective Linear Programming

Charnes, Cooper, Golany, Seiford and Stutz (1985) describe the relationship between DEA frontier and Pareto-Koopmans efficient empirical production frontier. This work points out the relation of efficiency in DEA and pareto optimality in multiple criteria decision making (MCDM) or Multiple Objective Linear Programming (MOLP). The relationship between

DEA and MOLP is again raised by Belton and Vickers (1993), Doyle and Green (1993) and Stewart (1994) in their discussion of DEA and MCDM. Joro, Korhonen and Wallenius (1998) provide a structure comparison of DEA and MOLP.

In fact, as shown in Chen (2005), the DEA/PS models have a strong relationship with MOLP. To demonstrate this, we use vector presentation of $\mathbf{x}_j = (x_{1j}, \dots, x_{mj})$ and $\mathbf{y}_j = (y_{1j}, \dots, y_{sj})$.

4.4.1 Output-oriented DEA

Consider the following MOLP model

$$\begin{aligned} \max_{\lambda_j} & \left(\sum_{j=1}^n \lambda_j \mathbf{y}_j \right) = \left(\sum_{j=1}^n \lambda_j y_{1j}, \dots, \sum_{j=1}^n \lambda_j y_{sj} \right) \\ \text{subject to} & \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m; \\ & \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{4.6}$$

where $\mathbf{x}_o = (x_{1o}, \dots, x_{mo})$ represents the input vector of DMU_o among others.

If all DMUs produce only one output, i.e., \mathbf{y}_j is a scalar rather than a vector, then (4.6) is a single objective linear programming problem

$$\begin{aligned} \max_{\lambda_j} & \left(\sum_{j=1}^n \lambda_j y_j \right) \\ \text{subject to} & \\ & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m; \\ & \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{4.7}$$

Let $\lambda_j y_j = \lambda'_j$, then (4.7) turns into

$$\begin{aligned} \max_{\lambda'_j} & \sum_{j=1}^n \lambda'_j \\ \text{subject to} & \\ & \sum_{j=1}^n \lambda'_j x'_{ij} \leq x'_{io} \quad i = 1, \dots, m; \\ & \lambda'_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{4.8}$$

where $x'_{ij} = x_{ij} / y_j$ and $x'_{io} = x_{io} / y_o$.

As shown in Charnes, Cooper and Rhodes (1978), model (4.8) is equivalent to the output-oriented CRS envelopment model

$$\begin{aligned}
& \max_{\lambda_j, z_o} z_o \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j y_j \geq z_o y_o \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m; \\
& \lambda_j \geq 0, \quad j = 1, \dots, n.
\end{aligned}$$

Next, if \mathbf{y}_j is a vector with s components, then we define

$$\sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro} \quad (4.9)$$

As a result, (4.6) becomes

$$\begin{aligned}
& \max_{\lambda_j, \sigma_r} (\sigma_1 y_{1o}, \dots, \sigma_s y_{so}) \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1, \dots, s; \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io}, \quad i = 1, \dots, m; \\
& \lambda_j \geq 0, \quad j = 1, \dots, n.
\end{aligned} \quad (4.10)$$

Let $\mathbf{W} = \{w \mid w \in \mathbf{R}^s, w_r \geq 0 \text{ and } \sum_{r=1}^s w_r = 1\}$ be the set of nonnegative weights. The weighting problem associated with (4.10) is defined for some $w \in \mathbf{W}$ as

$$\begin{aligned}
& \max_{\lambda_j, \sigma_r} \sum_{r=1}^s w_r \sigma_r y_{ro} \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1, \dots, s; \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m; \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \quad (4.11)$$

Furthermore, let $\bar{w}_r = w_r y_{ro}$ for all $r = 1, \dots, s$, then (4.11) is equivalent to the following linear programming problem

$$\begin{aligned}
& \max_{\lambda_j, \sigma_r} \sum_{r=1}^s \bar{w}_r \sigma_r \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro} \quad r = 1, \dots, s; \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m; \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \quad (4.12)$$

Model (4.12) is exactly the output-oriented CRS DEA/preference model. However, if we wish output level cannot be decreased to reach the efficient frontier, we specify (4.13) instead of (4.9).

$$\sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro} \text{ such that } \sigma_r \geq 1 \text{ for all } r = 1, \dots, s. \quad (4.13)$$

We see that for a specific DMU_o , $\lambda_o^* = 1$ and $\lambda_j^* = 0 (j \neq o)$ is an optimal solution to (4.12), when $\sigma_r^* = 1$ for all $r = 1, \dots, s$. Note that if some $\sigma_r^* \neq 1$, then $\lambda_o^* = 0$ is an optimal solution to (4.12). Therefore, (4.6) can be interpreted as follows: when $\mathbf{x}_o = (x_{1o}, \dots, x_{mo})$ is regarded as resource, if the resource \mathbf{x}_o can be used among other DMUs (associated with $\lambda_j^* \neq 0$), then more desirable or preferred output level \mathbf{y}^* is produced and \mathbf{y}_o is not a pareto solution to (4.6).

It can be seen that weighted non-radial DEA model (4.12) is equivalent to an MOLP problem. If we impose an additional on $\sum_{j=1}^n \lambda_j$ in (4.6), then we obtain other output-oriented DEA models.

4.4.2 Input-oriented DEA

Similar to (4.6), we write the following MOLP model.

$$\begin{aligned} \min_{\lambda_j} (\sum_{j=1}^n \lambda_j \mathbf{x}_j) &= (\sum_{j=1}^n \lambda_j x_{1j}, \dots, \sum_{j=1}^n \lambda_j x_{mj}) \\ \text{subject to} & \\ \sum_{j=1}^n \lambda_j y_{rj} &\leq y_{ro} \quad r = 1, \dots, s; \\ \lambda_j &\geq 0, \quad j = 1, \dots, n. \end{aligned} \quad (4.14)$$

where $\mathbf{y}_o = (y_{1o}, \dots, y_{so})$ represents the output vector of DMU_o . If all DMUs use only one input, i.e., \mathbf{x}_j is a scalar, then (4.14) is a single objective linear programming problem and is equivalent to the input-oriented CRS envelopment model with single input.

Let $\mathbf{G} = \{ \mathbf{g} \mid \mathbf{g} \in \mathbf{R}^m, g_i \geq 0 \text{ and } \sum_{i=1}^m g_i = 1 \}$ be the set of nonnegative weights. Then model (4.14) can be transformed into the following linear programming problem.

$$\begin{aligned} \min_{\lambda_j \tau_i} \sum_{i=1}^m \bar{g}_i \tau_i & \\ \text{subject to} & \\ \sum_{j=1}^n \lambda_j x_{ij} &= \tau_i x_{io} \quad i = 1, \dots, m; \\ \sum_{j=1}^n \lambda_j y_{rj} &\geq y_{ro} \quad r = 1, \dots, s; \\ \lambda_j &\geq 0 \quad j = 1, \dots, n. \end{aligned} \quad (4.15)$$

where $\bar{g}_i = g_i x_{io}$ for all $i = 1, \dots, m$, and τ_i is defined in (4.16) or (4.17).

$$\sum_{j=1}^n \lambda_j x_{ij} = \tau_i x_{io} \quad (4.16)$$

$$\sum_{j=1}^n \lambda_j x_{ij} = \tau_i x_{io} \text{ such that } \tau_i \leq 1 \text{ for all } i = 1, \dots, m. \quad (4.17)$$

Model (4.15) is a weighted non-radial DEA model incorporated with preference over the adjustment of input levels. If we use (4.16), then there is no restrictions on τ_i and model (4.15) is the input-oriented CRS DEA/PS model.

Note that for a specific DMU_o , $\lambda_o^* = 1$ and $\lambda_j^* = 0 (j \neq o)$ is an optimal solution to (4.15), when $\tau_i^* = 1$ for all $i = 1, \dots, m$. Note also that if some $\tau_i^* \neq 1$, then $\lambda_o^* = 0$ is an optimal solution to (4.15). If we impose an additional on $\sum_{j=1}^n \lambda_j$ in (4.15), then we obtain other input-oriented DEA models.

4.4.3 Non-Orientation DEA

Consider the following MOLP model.

$$\begin{aligned} \max_{\lambda_j} (\sum_{j=1}^n \lambda_j \mathbf{y}_j) &= (\sum_{j=1}^n \lambda_j y_{1j}, \dots, \sum_{j=1}^n \lambda_j y_{sj}) \\ \min_{\lambda_j} (\sum_{j=1}^n \lambda_j \mathbf{x}_j) &= (\sum_{j=1}^n \lambda_j x_{1j}, \dots, \sum_{j=1}^n \lambda_j x_{mj}) \\ \text{subject to} & \\ \lambda_j \geq 0 & \quad j = 1, \dots, n. \end{aligned} \quad (4.18)$$

We have the following equivalent linear programming model

$$\begin{aligned} \max_{\lambda_j, \sigma_r, \tau_i} \sum_{r=1}^s \bar{w}_r \sigma_r - \sum_{i=1}^m \bar{g}_i \tau_i \\ \text{subject to} \\ \sum_{j=1}^n \lambda_j x_{ij} = \tau_i x_{io} \quad i = 1, \dots, m; \\ \sum_{j=1}^n \lambda_j y_{rj} = \sigma_r y_{ro}, \quad r = 1, \dots, s; \\ \tau_i \leq 1, \quad i = 1, \dots, m; \\ \sigma_r \geq 1, \quad r = 1, \dots, s; \\ \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \quad (4.19)$$

Note that $\sigma_r \geq 1$ and $\tau_i \leq 1$ in (4.19). Therefore, we have $\tau_i x_{io} = x_{io} - s_i^-$ and $\sigma_r y_{ro} = y_{ro} + s_r^+$, where $s_i^-, s_r^+ \geq 0$. Then, (4.19) becomes

$$\begin{aligned} & \max_{\lambda_j, s_i^-, s_r^+} \sum_{r=1}^s w_r s_r^+ + \sum_{i=1}^m g_i s_i^- - \sum_{i=1}^m \bar{g}_i + \sum_{r=1}^s \bar{w}_r \\ & \text{subject to} \\ & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, \dots, m; \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s; \\ & s_i^-, s_r^+, \lambda_j \geq 0. \end{aligned}$$

which is a weighted slack-based DEA model (see chapter 3 and Seiford and Zhu (1998)).

4.5 Solving DEA Using DEA Frontier Software

4.5.1 Non-radial Models

To run the non-radial models, select the “Non-radial Model” menu item. You will be prompted with a form as shown in Figure 4.6 for selecting the models presented in Table 4.1. The Results are reported in “Efficiency”, “Slack”, and “Target” sheets.

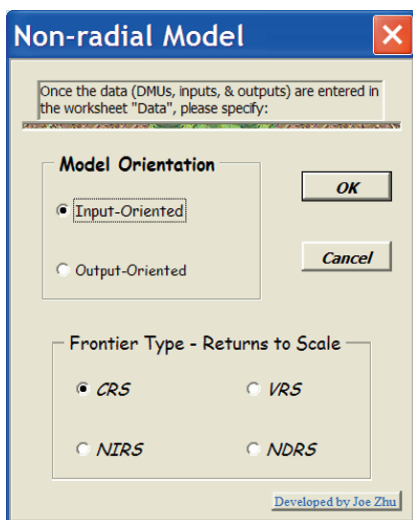


Figure 4.6. Non-radial Models

4.5.2 Preference-Structure Models

To run the preference structure models, select the “Preference Structure Model” menu item. Figure 4.6 shows the form for specifying the models.

If “Yes” is selected under “Restrict Input/Output Change?”, then we have weighted non-radial models (see discussion on page 75). If “No” is selected, then we have the DEA/PS models presented in Table 4.2. The software will then ask you to specify the weights for the inputs or outputs, depending on the model orientation. The Results are reported in “Efficiency”, “Slack”, and “Target” sheets.

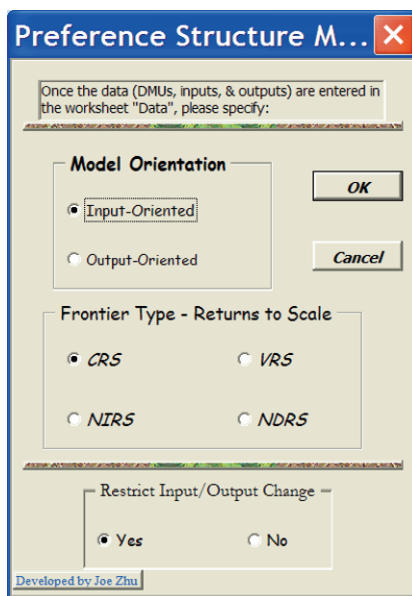


Figure 4.7. Preference Structure Models

4.5.3 Cost Efficiency, Revenue Efficiency and Profit Efficiency

These models need information on the input and output prices. Consider the Hospital example in Cooper, Tone and Seiford (2000). The input and output data are reported in the “Data” sheet (Figure 4.8), input price are reported in the “Input Price” sheet (Figure 4.9) and the output price are reported in the “Output Price” sheet (Figure 4.10).

| | A | B | C | D | E | F |
|----|----------|--------|-------|---|---------|--------|
| 1 | Hospital | Doctor | Nurse | | Outpat. | Inpat. |
| 2 | A | 20 | 151 | | 100 | 90 |
| 3 | B | 19 | 131 | | 150 | 50 |
| 4 | C | 25 | 160 | | 160 | 55 |
| 5 | D | 27 | 168 | | 180 | 72 |
| 6 | E | 22 | 158 | | 94 | 66 |
| 7 | F | 55 | 255 | | 230 | 90 |
| 8 | G | 33 | 235 | | 220 | 88 |
| 9 | H | 31 | 206 | | 152 | 80 |
| 10 | I | 30 | 244 | | 190 | 100 |
| 11 | J | 50 | 268 | | 250 | 100 |
| 12 | K | 53 | 306 | | 260 | 147 |
| 13 | L | 38 | 284 | | 250 | 120 |

Figure 4.8. Hospital Data

| | A | B | C | D | E |
|----|----------|--------|-------|---|---|
| 1 | Hospital | Doctor | Nurse | | |
| 2 | A | 500 | 100 | | |
| 3 | B | 350 | 80 | | |
| 4 | C | 450 | 90 | | |
| 5 | D | 600 | 120 | | |
| 6 | E | 300 | 70 | | |
| 7 | F | 450 | 80 | | |
| 8 | G | 500 | 100 | | |
| 9 | H | 450 | 85 | | |
| 10 | I | 380 | 76 | | |
| 11 | J | 410 | 75 | | |
| 12 | K | 440 | 80 | | |
| 13 | L | 400 | 70 | | |

Figure 4.9. Input Prices

| | A | B | C | D | E |
|----|----------|---------|--------|---|---|
| 1 | Hospital | outpat. | Inpat. | | |
| 2 | A | 550 | 2010 | | |
| 3 | B | 400 | 1800 | | |
| 4 | C | 480 | 2200 | | |
| 5 | D | 600 | 3500 | | |
| 6 | E | 400 | 3050 | | |
| 7 | F | 430 | 3900 | | |
| 8 | G | 540 | 3300 | | |
| 9 | H | 420 | 3500 | | |
| 10 | I | 350 | 2900 | | |
| 11 | J | 410 | 2600 | | |
| 12 | K | 540 | 2450 | | |
| 13 | L | 295 | 3000 | | |

Figure 4.10. Output Price

The cost efficiency and revenue efficiency are discussed in section 4.2. Table 4.3 summarizes the related models.

Table 4.3. Cost Efficiency and Revenue Efficiency Models

| Frontier Type | Cost | Revenue |
|---------------|--|--|
| | $\min \sum_{i=1}^m p_i^o \tilde{x}_{io}$ subject to | $\max \sum_{r=1}^s q_r^o \tilde{y}_{ro}$ subject to |
| CRS | $\sum_{j=1}^n \lambda_j x_{ij} \leq \tilde{x}_{io} \quad i = 1, 2, \dots, m;$ $\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s;$ $\lambda_j, \tilde{x}_{io} \geq 0$ | $\sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m$ $\sum_{j=1}^n \lambda_j y_{rj} \geq \tilde{y}_{ro} \quad r = 1, 2, \dots, s$ $\lambda_j, \tilde{y}_{ro} \geq 0$ |
| VRS | | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | | Add $\sum_{j=1}^n \lambda_j \geq 1$ |

In Table 4.3, p_i^o and q_r^o are unit price of the input i and unit price of the output r of DMU_o , respectively. These price data may vary from one DMU to another. The cost efficiency and revenue efficiency of DMU_o is defined as

$$\frac{\sum_{i=1}^m p_i^o \tilde{x}_{io}^*}{\sum_{i=1}^m p_i^o x_{io}} \quad \text{and} \quad \frac{\sum_{r=1}^s q_r^o y_{ro}}{\sum_{r=1}^s q_r^o \tilde{y}_{ro}^*}$$

Note that the revenue efficiency is defined as the reciprocal of the one defined in section 4.2. As a result, the cost and revenue efficiency scores are within the range of 0 and 1.

The efficiency scores are reported in the ‘‘Cost Efficiency’’ (‘‘Revenue Efficiency’’) sheet. The optimal inputs (outputs) are reported in the ‘‘OptimalData Cost Efficiency’’ (‘‘OptimalData Revenue Efficiency’’) sheet.

Table 4.4 presents the models used to calculate the profit efficiency defined as

$$\frac{\sum_{r=1}^s q_r^o y_{ro} - \sum_{i=1}^m p_i^o x_{io}}{\sum_{r=1}^s q_r^o \tilde{y}_{ro}^* - \sum_{i=1}^m p_i^o \tilde{x}_{io}^*}$$

Table 4.4. Profit Efficiency Models

| Frontier | |
|----------|--|
| Type | |
| | $\max \sum_{r=1}^s q_r^o \tilde{y}_{ro} - \sum_{i=1}^m p_i^o \tilde{x}_{io}$ |
| | subject to |
| CRS | $\sum_{j=1}^n \lambda_j x_{ij} \leq \tilde{x}_{io} \quad i = 1, 2, \dots, m$ |
| | $\sum_{j=1}^n \lambda_j y_{rj} \geq \tilde{y}_{ro} \quad r = 1, 2, \dots, s$ |
| | $\tilde{x}_{io} \leq x_{io}, \tilde{y}_{ro} \geq y_{ro}$ |
| | $\lambda_j \geq 0$ |
| VRS | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | Add $\sum_{j=1}^n \lambda_j \geq 1$ |

The results are reported in the “Profit Efficiency” and “OptimalData Profit Efficiency” sheets.

REFERENCES

1. Belton, V. and S.P. Vickers (1993), Demystifying DEA – a visual interactive approach based on multiple criteria analysis, *Journal of Operational Research Society*, 44, 883-896.
2. Charnes, A., W.W. Cooper and E. Rhodes (1978), Measuring efficiency of decision making units, *European Journal of Operational Research*, 2, 429-444.
3. Charnes, A., W.W. Cooper, B. Golany, L.M. Seiford and J. Stutz (1985), Foundations of data envelopment analysis for Pareto-Koopman's efficient empirical production functions, *J. of Econometrics*, 30, 1-17.
4. Chen, Y. (2005), On preference structure in Data Envelopment Analysis, *International Journal of Information Technology and Decision Making*, Vol. 4, No. 3, 411-431.
5. Cooper, W.W., L.M. Seiford and K. Tone (2000), *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publishers, Boston.
6. Doyle, J. and R. Green (1993), Data envelopment analysis and multiple criteria decision making, *OMEGA*, 21, 713-715.
7. Färe, R. and C.A.K. Lovell (1978), Measuring the technical efficiency, *Journal of Economic Theory*, 19, 150-162.
8. Joro, T., P. Korhonen and J. Wallenius (1998), Structural comparison of data envelopment analysis and multiple objective linear programming, *Management Science*, 44, 962-970.
9. Seiford, L.M. and J. Zhu (1998), Identifying excesses and deficits in Chinese industrial productivity (1953-1990): a weighted data envelopment analysis approach, *OMEGA*, 26, No. 2, 279-269.
10. Seiford, L.M. and J. Zhu (2002), Value judgment versus allocative efficiency: a case of Tennessee county jails, *Studies in Regional & Urban Planning*, Issue 4, July, 89-98.
11. Zhu, J. (1996), Data envelopment analysis with preference structure, *Journal of Operational Research Society*, 47, 136-150.
12. Zhu, J. (1998), Data envelopment analysis vs. principal component analysis: An illustrative study of economic performance of Chinese cities, *European Journal of Operational Research*, 111, No. 1, 50-61.

Chapter 5

Modeling Undesirable Measures

5.1 Introduction

Both desirable (good) and undesirable (bad) outputs and inputs may be present. For example, the number of defective products is an undesirable output. One wants to reduce the number of defects to improve the performance. If inefficiency exists in production processes where final products are manufactured with a production of wastes and pollutants, the outputs of wastes and pollutants are undesirable and should be reduced to improve the performance.

Note that in the conventional DEA models, e.g., the VRS envelopment models, it is assumed that outputs should be increased and the inputs should be decreased to improve the performance or to reach the best-practice frontier. If one treats the undesirable outputs as inputs so that the bad outputs can be reduced, the resulting DEA model does not reflect the true production process.

Situations when some inputs need to be increased to improve the performance are also likely to occur. For example, in order to improve the performance of a waste treatment process, the amount of waste (undesirable input) to be treated should be increased rather than decreased as assumed in the conventional DEA models.

Seiford and Zhu (2002) develop an approach to treat undesirable input/outputs in the VRS envelopment models. The key to their approach is the use of DEA classification invariance under which classifications of efficiencies and inefficiencies are invariant to the data transformation.

5.2 Efficiency Invariance

Suppose that the inputs and outputs are transformed to $\bar{x}_{ij} = x_{ij} + u_i$ and $\bar{y}_{rj} = y_{rj} + v_r$, where u_i and v_r are nonnegative. Then the input-oriented and the output-oriented VRS envelopment models become

$$\begin{aligned}
 & \min \theta - \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j \bar{x}_{ij} + s_i^- = \theta \bar{x}_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j \bar{y}_{rj} - s_r^+ = \bar{y}_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{5.1}$$

$$\begin{aligned}
 & \max \phi - \varepsilon \left(\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right) \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j \bar{x}_{ij} + s_i^- = \bar{x}_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j \bar{y}_{rj} - s_r^+ = \phi \bar{y}_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{5.2}$$

Ali and Seiford (1990) show that DMU_o is efficient under (1.5) or (1.6) if and only if DMU_o is efficient under (5.1) or (5.2). This conclusion is due to the presence of the convexity constraint $\sum_{j=1}^n \lambda_j = 1$. This property also enables us to treat possible negative inputs and outputs before applying the VRS model (see Appendix of this chapter.)

In general, there are three cases of invariance under data transformation in DEA. The first case is restricted to the “classification invariance” where the classifications of efficiencies and inefficiencies are invariant to the data transformation. The second case is the “ordering invariance” of the inefficient DMUs. The last case is the “solution invariance” in which the new DEA model (after data translation) must be equivalent to the old one, i.e., both mathematical programming problems must have exactly the same solution. The method of Seiford and Zhu (2002) is concerned only with the first level of invariance – classification invariance. See Pastor (1996) and Lovell and Pastor (1995) for discussions in invariance property in DEA.

5.3 Undesirable Outputs

Let y_{rj}^g and y_{rj}^b denote the desirable (good) and undesirable (bad) outputs, respectively. Obviously, we wish to increase y_{rj}^g and to decrease y_{rj}^b to improve the performance. However, in the output-oriented VRS envelopment model, both y_{rj}^g and y_{rj}^b are supposed to increase to improve the performance. In order to increase the desirable outputs and to decrease the undesirable outputs, we proceed as follows.

First, we multiply each undesirable output by “-1” and then find a proper value v_r to let all negative undesirable outputs be positive. That is, $\bar{y}_{rj}^b = -y_{rj}^b + v_r > 0$. This can be achieved by $v_r = \max_j \{y_{rj}^b\} + 1$, for example.

Based upon (5.2), we have

$$\begin{aligned}
 & \max h \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j y_{rj}^g \geq h y_{ro}^g \\
 & \sum_{j=1}^n \lambda_j \bar{y}_{rj}^b \geq h \bar{y}_{ro}^b \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{5.3}$$

Note that (5.3) increases desirable outputs and decreases undesirable outputs. The following theorem ensures that the optimized undesirable output of $y_{ro}^b (=v_r - h^* \bar{y}_{ro}^b)$ cannot be negative.

Theorem 5.1 Given a translation vector v , suppose h^* is the optimal value to (5.3), we have $h^* \bar{y}_{ro}^b \leq v_r$.

[Proof] Note that all outputs now are non-negative. Let λ_j^* be an optimal solution associated with h^* . Since $\sum_{j=1}^n \lambda_j^* = 1$, $h^* \bar{y}_{ro}^b \leq \bar{y}_r^*$, where \bar{y}_r^* is composed from (translated) maximum values among all bad outputs. Note that $\bar{y}_r^* = -y_r^* + v_r$, where y_r^* is composed from (original) minimum values among all bad outputs. Thus, $h^* \bar{y}_o^b \leq v_r$. ■

We may treat the undesirable outputs as inputs. However, this does not reflect the true production process. We may also apply a monotone decreasing transformation (e.g., $1/y_{rj}^b$) to the undesirable outputs and then to use the adapted variables as outputs. The current method, in fact, applies a linear monotone decreasing transformation. Since the use of linear transformation preserves the convexity, it is a good choice for a DEA model.

Figure 5.1 illustrates the method. The five DMUs A, B, C, D and E use an equal input to produce one desirable output (g) and one undesirable output (b). GCDEF is the (output) frontier. If we treat the undesirable output as an input, then ABCD becomes the VRS frontier. Model (5.2) rotates the output frontier at EF and obtains the symmetrical frontier. In this case, DMUs A', B' and C', which are the adapted points of A, B and C, respectively, are efficient.

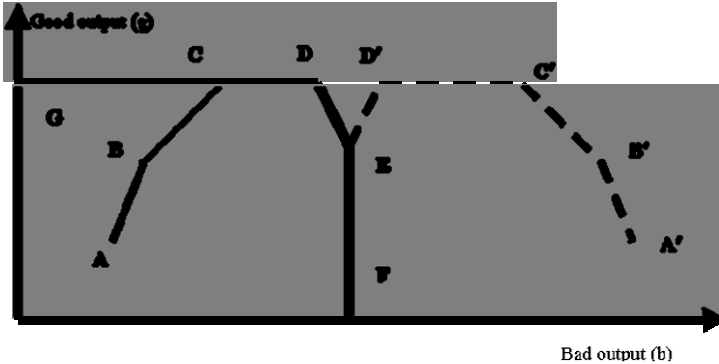


Figure 5.1. Treatment of Bad Output

The efficient target for DMU_o is

$$\begin{cases} \hat{x}_{io} = x_{io} - s_i^{-*} \\ \hat{y}_{ro}^g = h^* y_{ro}^g + s_r^{+*} \\ \hat{y}_{ro}^b = v_r - (h^* y_{ro}^b + s_r^{+*}) \end{cases}$$

Table 5.1. Vendors

| Vendors | Price (\$/unit) | %Rejects | % Late deliveries |
|---------|-----------------|----------|-------------------|
| 1 | 0.1958 | 1.2 | 5 |
| 2 | 0.1881 | 0.8 | 7 |
| 3 | 0.2204 | 0 | 0 |
| 4 | 0.2081 | 2.1 | 0 |
| 5 | 0.2118 | 2.3 | 3 |
| 6 | 0.2096 | 1.2 | 4 |

Source: Weber and Desai. (1996).

We conclude this section by applying the method to the six vendors studied in Weber and Desai (1996). Table 5.1 presents the data. The input is price per unit, and the outputs are percentage of late deliveries and

percentage of rejected units. (See Weber and Desai (1996) for detailed discussion on the input and the two outputs.)

Obviously, the two outputs are bad outputs. We use an translation vector of (3.3%, 8%). (Or one could use (100%, 100%) as in Chapter 7.) Figure 5.2 shows the translated data and the spreadsheet model. This is actually a spreadsheet model for the output-oriented VRS envelopment model. Figure 5.3 shows the Solver parameters. Column G in Figure 5.2 reports the efficiency scores.

| | A | B | C | D | E | F | G |
|----|------------------|-----------|--------|------------|-------------------|------------|------------|
| 1 | | Price | | %Rejects | % Late deliveries | λ | Efficiency |
| 2 | Vendor 1 | 0.1958 | | 2.1 | 3 | 0 | 1.08921 |
| 3 | Vendor 2 | 0.1881 | | 2.5 | 1 | 0.327059 | 1 |
| 4 | Vendor 3 | 0.2204 | | 3.3 | 8 | 0.653754 | 1 |
| 5 | Vendor 4 | 0.2081 | | 1.2 | 8 | 0.019187 | 1 |
| 6 | Vendor 5 | 0.2118 | | 1 | 5 | 0 | 1.6 |
| 7 | Vendor 6 | 0.2096 | | 2.1 | 4 | 0 | 1.427647 |
| 8 | | | | | | | |
| 9 | | Reference | | DMU under | 6 | Efficiency | |
| 10 | Constrain | set | | Evaluation | | 1.427647 | |
| 11 | price | 0.2096 | \leq | 0.2096 | | | |
| 12 | Rejects | 2.998059 | \geq | 2.998059 | | | |
| 13 | Late delive | 5.710589 | \geq | 5.710589 | | | |
| 14 | $\Sigma\lambda$ | 1 | $=$ | 1 | | | |

Figure 5.2. Bad Outputs Spreadsheet Model

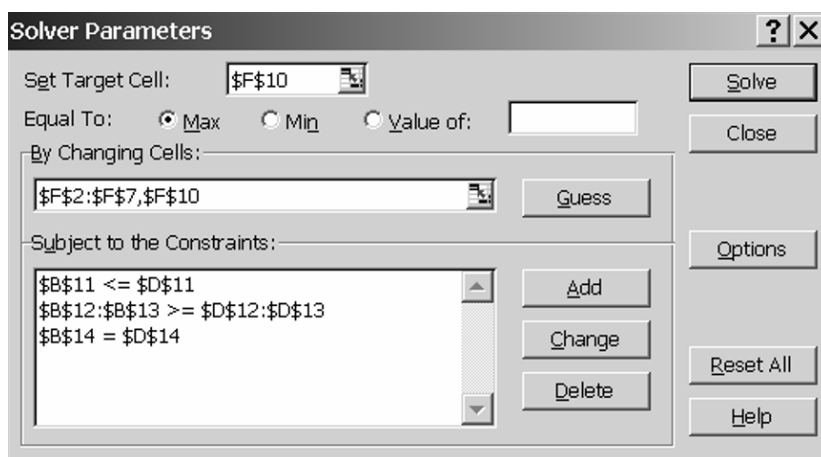


Figure 5.3. Solver Parameters for Bad Outputs Spreadsheet Model

If we do not translate the bad outputs and calculate the regular output-oriented VRS envelopment model, vendor 5 is classified as efficient, and vendor 3 is classified as inefficient. (see Figure 5.4 where 0.0001 is used to replace 0.) The same Solver parameters shown in Figure 5.3 are used.

| | A | B | C | D | E | F | G |
|----|------------------|-----------|--------|------------|-------------------|------------|------------|
| 1 | | Price | | %Rejects | % Late deliveries | λ | Efficiency |
| 2 | Vendor 1 | 0.1958 | | 1.2 | 5 | 0 | 1.075329 |
| 3 | Vendor 2 | 0.1881 | | 0.8 | 7 | 0.518519 | 1 |
| 4 | Vendor 3 | 0.2204 | | 0.0001 | 0.0001 | 0 | 23000 |
| 5 | Vendor 4 | 0.2081 | | 2.1 | 0 | 0 | 1 |
| 6 | Vendor 5 | 0.2118 | | 2.3 | 3 | 0.481481 | 1 |
| 7 | Vendor 6 | 0.2096 | | 1.2 | 4 | 0 | 1.268519 |
| 8 | | | | | | | |
| 9 | | Reference | | DMU unde | 6 | Efficiency | |
| 10 | Constrain | set | | Evaluation | | 1.268519 | |
| 11 | price | 0.199511 | \leq | 0.2096 | | | |
| 12 | Rejects | 1.522222 | \geq | 1.522222 | | | |
| 13 | Late delive | 5.074074 | \geq | 5.074074 | | | |
| 14 | $\Sigma\lambda$ | 1 | = | 1 | | | |

Figure 5.4. Efficiency Scores When Bad Outputs Are Not Translated

If we treat the two bad outputs as inputs and use the input-oriented VRS envelopment model, we obtain the efficiency scores shown in Figure 5.5 (Figure 5.6 shows the Solver parameters). In this case, we do not have outputs.

| | A | B | C | D | E | F | G |
|----|------------------|-----------|--------|------------|-------------------|------------|------------|
| 1 | | Price | | %Rejects | % Late deliveries | λ | Efficiency |
| 2 | Vendor 1 | 0.1958 | | 1.2 | 5 | 0 | 0.990548 |
| 3 | Vendor 2 | 0.1881 | | 0.8 | 7 | 0.541904 | 1 |
| 4 | Vendor 3 | 0.2204 | | 0 | 0 | 0.122632 | 1 |
| 5 | Vendor 4 | 0.2081 | | 2.1 | 0 | 0.335464 | 1 |
| 6 | Vendor 5 | 0.2118 | | 2.3 | 3 | 0 | 0.944315 |
| 7 | Vendor 6 | 0.2096 | | 1.2 | 4 | 0 | 0.948332 |
| 8 | | | | | | | |
| 9 | | Reference | | DMU under | 6 | Efficiency | |
| 10 | Constrain | set | | Evaluation | | 0.948332 | |
| 11 | price | 0.19877 | \leq | 0.19877 | | | |
| 12 | Rejects | 1.137998 | \leq | 1.137998 | | | |
| 13 | Late delive | 3.793326 | \leq | 3.793326 | | | |
| 14 | $\Sigma\lambda$ | 1 | $=$ | 1 | | | |

Figure 5.5. Efficiency Scores When Bad Outputs Are Treated As Inputs

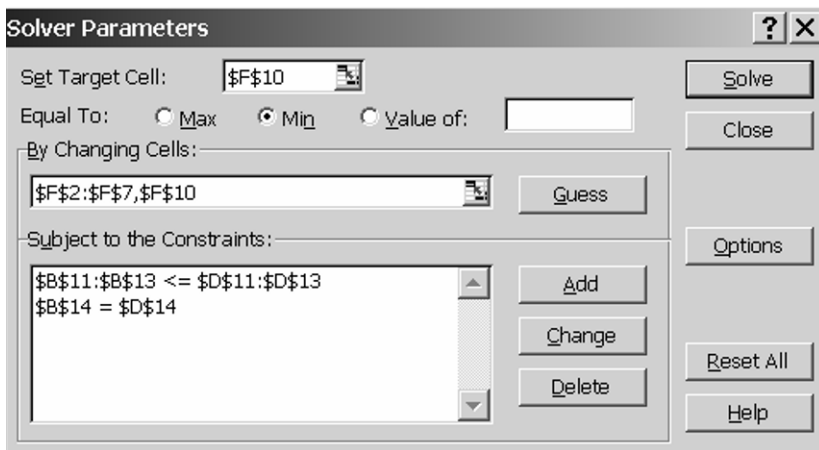


Figure 5.6. Solver Parameters When Bad Outputs Are Treated As Inputs

5.4 Undesirable Inputs

The above discussion can also be applied to situations when some inputs need to be increased rather than decreased to improve the performance. In this case, we denote x_{ij}^I and x_{ij}^D the inputs that need to be increased and decreased, respectively.

We next multiply x_{ij}^I by “-1” and then find a proper u_i to let $\bar{x}_{ij}^I = -x_{ij}^I + u_i > 0$. Based upon model (5.1), we have

$$\begin{aligned}
 & \min \tau \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij}^D \leq \bar{\alpha}_{io}^D \\
 & \sum_{j=1}^n \lambda_j \bar{x}_{ij}^I \leq \bar{\alpha}_{oi}^I \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0 \quad j=1, \dots, n.
 \end{aligned}
 \tag{5.4}$$

where x_{ij}^I is increased and x_{ij}^D is decreased for a DMU to improve the performance. The efficient target for DMU_o is

$$\begin{cases}
 \hat{x}_{io}^D = \tau^* x_{io}^I - s_i^{-*} \\
 \hat{x}_{io}^I = u_i - (\tau^* x_{io}^I - s_i^{-*}) \\
 \hat{y}_{ro} = y_{ro} + s_r^{+*}
 \end{cases}$$

5.5 Solving DEA Using DEA Frontier Software

To run the models for treating undesirable measures, select the Undesirable-Measure Model menu item. Figure 5.7 shows the form for specifying the models. The results are reported in “Efficiency”, “Slack”, and “Target” sheets.

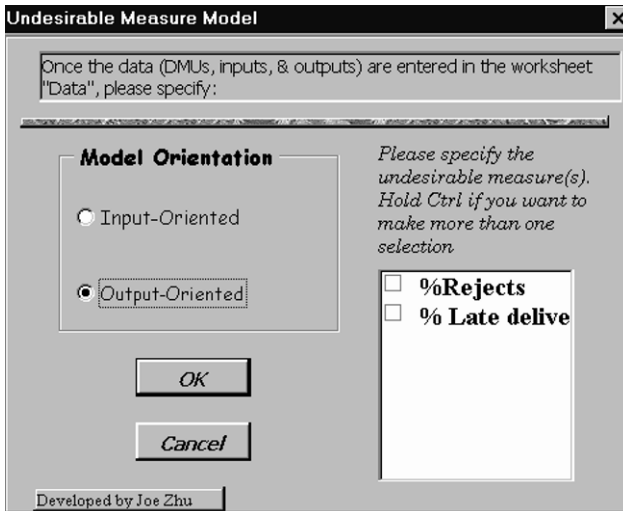


Figure 5.7. Undesirable Measure Models

APPENDIX: NEGATIVE DATA

So far, we have assumed that all inputs and outputs are either positive or zero. However, we have cases where some inputs and (or) outputs are negative. For example, when a company experiences a loss, its profit is negative. Similarly, returns on some stocks can be negative. This can be easily solved by way of the *translation invariance property* of the VRS models (Ali and Seiford, 1990). Specifically, the VRS frontier remains the same if x_{ij} and y_{rj} is replaced to \bar{x}_{ij} and \bar{y}_{rj} , respectively.

Consider the example given in the following Table where we have 10 DMUs¹. We have two inputs $x_1 =$ Standard Deviation and $x_2 =$ PropNeg (proportion of negative monthly returns during the year), and three outputs $y_1 =$ Return (average monthly return), $y_2 =$ Skewness and $y_3 =$ Min (minimum return).

Note that some values for return, skewness and Min are negative. In the table the average monthly return, skewness and minimum return are displaced by 3.7%, 2, and 26%, respectively so that all the output values are positive across all the DMUs. The translation values can be chosen randomly as long as the negative values become positive.

Table A.1 Negative Data Example

| DMU | Original Data | | | | | Transformed Data | | |
|-----|-------------------------------|--------------------------------|--------------------------------|---------------------|---------------------------|---------------------------------|-----------------------------|--------------------------------|
| | $x_1 =$ Standard Deviation | $x_2 =$ Proportion Negative | $y_1 =$ Ave. Monthly Return | $y_2 =$ Skewness | $y_3 =$ Minimum Return | $\hat{y}_1;$ $\pi_1 = 3.7\%$ | $\hat{y}_2;$ $\pi_2 = 2$ | $\hat{y}_3;$ $\pi_3 = 26\%$ |
| 1 | 6.80% | 58.30% | 0.10% | 1.13 | -8.10% | 3.80% | 3.13 | 17.90% |
| 2 | 4.00% | 41.70% | 0.70% | 0.61 | -7.90% | 4.40% | 2.61 | 18.10% |
| 3 | 3.40% | 37.50% | 0.90% | 0.58 | -4.00% | 4.60% | 2.58 | 22.00% |
| 4 | 5.00% | 50.00% | 0.60% | 1.7 | -5.60% | 4.30% | 3.7 | 20.40% |
| 5 | 4.70% | 37.50% | 1.10% | 0.28 | -8.20% | 4.80% | 2.28 | 17.80% |
| 6 | 3.80% | 50.00% | -0.10% | 0.08 | -6.30% | 3.60% | 2.08 | 19.70% |
| 7 | 11.20% | 45.80% | 3.20% | 0.39 | -17.10% | 6.90% | 2.39 | 8.90% |
| 8 | 12.80% | 58.30% | -1.00% | 0.46 | -25.70% | 2.70% | 2.46 | 0.30% |
| 9 | 8.40% | 52.20% | -1.20% | -0.26 | -17.10% | 2.50% | 1.74 | 8.90% |
| 9 | 5.00% | 54.50% | 0.40% | 1.1 | -6.70% | 4.10% | 3.1 | 19.30% |
| 10 | 8.60% | 25.00% | -3.60% | -1.98 | -16.50% | 0.10% | 0.02 | 9.50% |

Since negative data are present only in the outputs, we thus use the input-oriented VRS model. When DMU1 is under evaluation, we have $\theta_0^* = 0.75$,

¹ These DMUs are called commodity trading advisors (CTAs) in Wilkens and Zhu (2001).

indicating this DMU is inefficient, and $\lambda_3^* = 0.51$ and $\lambda_4^* = 0.49$, indicating DMU3 and DMU4 are the benchmarks.

REFERENCES

1. Ali, A.I. and L.M. Seiford (1990), Translation invariance in data envelopment analysis, *Operations Research Letters*, 9, 403-405.
2. Lovell, C.A.K. and J.T. Pastor (1995), Units invariant and translation invariant DEA models, *Operations Research Letters*, 18, 147-151.
3. Pastor, J.T. (1996), Translation invariance in data envelopment analysis: a generalization, *Annals of Operations Research*, 66, 93-102.
4. Seiford, L.M. and J. Zhu (2002), Modeling undesirable factors in efficiency evaluation, *European Journal of Operational Research*, Vol. 142, Issue 1, 16-20.
5. Weber, C.A. and A. Desai (1996), Determination of paths to vendor market efficiency using parallel coordinates representation: a negotiation tool for buyers, *European Journal of Operational Research*, 90, 142-155.
6. Wilkens, K. and J. Zhu (2001), Portfolio evaluation and benchmark selection: A mathematical programming approach, *Journal of Alternative Investments*, Vol. 4, No. 1, 9-19

Chapter 6

Context-dependent Data Envelopment Analysis

6.1 Introduction

Adding or deleting an inefficient DMU or a set of inefficient DMUs does not alter the efficiencies of the existing DMUs and the best-practice frontier. The inefficiency scores change only if the best-practice frontier is altered. i.e., the performance of DMUs depends only on the identified best-practice frontier. In contrast, researchers of the consumer choice theory point out that consumer choice is often influenced by the context. e.g., a circle appears large when surrounded by small circles and small when surrounded by larger ones. Similarly a product may appear attractive against a background of less attractive alternatives and unattractive when compared to more attractive alternatives (Simonson and Tversky, 1992).

Considering this influence within the framework of DEA, one could ask “what is the relative attractiveness of a particular DMU when compared to others?” As in Tversky and Simonson (1993), one agrees that the relative attractiveness of DMU_x compared to DMU_y depends on the presence or absence of a third option, say DMU_z (or a group of DMUs). Relative attractiveness depends on the evaluation context constructed from alternative options (or DMUs). In the original DEA methodology, each DMU is evaluated against a set of frontier DMUs. That is, the original DEA methodology can rank the performance of inefficient DMUs with respect to the best-practice frontier. However, when DMU_x and DMU_y are members of best-practice frontier, DEA cannot identify which of DMU_x and DMU_y is a better option with respect to DMU_z (or a set of inefficient DMUs). Because both DMU_x and DMU_y have an efficiency score of one. Although one may use the super-efficiency DEA models (see Chapter 10) to rank the

performance of efficient DMUs, the evaluation context or third option (the reference set) changes in each evaluation. i.e., DMU_x and DMU_y are not evaluated against the same third option by the super-efficiency concept.

From the above discussion, we see that DEA provides performance measures that are absolute in the sense that all DMUs are evaluated against the best-practice frontier. In order to obtain the relative attractiveness within DEA, Seiford and Zhu (2003) modify the original DEA methodology to a situation where the relative performance is defined with respect to a particular best-practice context (evaluation context).

In order to obtain the evaluation contexts, an algorithm is developed to remove the (original) best-practice frontier to allow the remaining (inefficient) DMUs to form a new second-level best-practice frontier. If we remove this new second-level best-practice frontier, a third-level best-practice frontier is formed, and so on, until no DMU is left. In this manner, we partition the set of DMUs into several levels of best-practice frontiers.

Note that each best-practice frontier provides an evaluation context for measuring the relative attractiveness. e.g., the second-level best-practice frontier serves as the evaluation context for measuring the relative attractiveness of the DMUs located on the first-level (original) best-practice frontier. It can be seen that the presence or absence (or the shape) of the second-level best-practice frontier affects the relative attractiveness of DMUs on the first-level best-practice frontier. A relative attractiveness measure is obtained when DMUs having worse performance are chosen as the evaluation context. When DMUs in a specific level are viewed as having equal performance, the attractiveness measure allows us to differentiate the “equal performance” based upon the same specific evaluation context (or third option).

On the other hand, we can measure the performance of DMUs on the third-level best-practice frontier with respect to the first or second level best-practice frontier. We define this type of measure as a progress measure where DMUs having better performance are chosen as the evaluation context. i.e., we measure the progress of DMUs with respect to best-practice frontiers at advanced levels. Note that the original DEA method provides a projection function to improve the performance of inefficient DMUs. However, it is likely that a particular inefficient DMU is unable to immediately improve its performance onto the first-level best-practice frontier because of such restrictions as management expertise, available resources, etc. Therefore intermediate (and more easily achievable) targets may be desirable for an inefficient DMU. By focusing on different levels of best-practice frontiers, the progress measure provides incremental improvements for a DMU's performance. i.e., we move the DMU step by step onto an attainable best-practice frontier. The resulting intermediate

targets are *local targets*, whereas the targets on the first-level (original) best-practice frontier are *global targets*.

6.2 Stratification DEA Method

Define $\mathbf{J}^1 = \{DMU_j, j = 1, \dots, n\}$ (the set of all n DMUs) and interactively define $\mathbf{J}^{l+1} = \mathbf{J}^l - \mathbf{E}^l$ where $\mathbf{E}^l = \{DMU_k \in \mathbf{J}^l \mid \theta^*(l, k) = 1\}$, and $\theta^*(l, k)$ is the optimal value to the following input-oriented CRS envelopment model when DMU_k is under evaluation

$$\begin{aligned}
 &\theta^*(l, k) = \min_{\lambda_j, \theta(l, k)} \theta(l, k) \\
 &\text{subject to} \\
 &\sum_{j \in F(\mathbf{J}^l)} \lambda_j x_{ij} \leq \theta(l, k) x_{ik} \\
 &\sum_{j \in F(\mathbf{J}^l)} \lambda_j y_{rj} \geq y_{rk} \\
 &\lambda_j \geq 0 \quad j \in F(\mathbf{J}^l).
 \end{aligned} \tag{6.1}$$

where $j \in F(\mathbf{J}^l)$ means $DMU_j \in \mathbf{J}^l$, i.e., $F(\cdot)$ represents the correspondence from a DMU set to the corresponding subscript index set.

When $l = 1$, model (6.1) becomes the original input-oriented CRS envelopment model, and \mathbf{E}^1 consists of all the frontier DMUs. These DMUs in set \mathbf{E}^1 define the first-level best-practice frontier. When $l = 2$, model (6.1) gives the second-level best-practice frontier after the exclusion of the first-level frontier DMUs. And so on. In this manner, we identify several levels of best-practice frontiers. We call \mathbf{E}^l the l th-level best practice frontier. The following algorithm accomplishes the identification of these best-practice frontiers by model (6.1).

Step 1: Set $l = 1$. Evaluate the entire set of DMUs, \mathbf{J}^1 , by model (6.1) to obtain the first-level frontier DMUs, set \mathbf{E}^1 (the first-level best-practice frontier).

Step 2: Exclude the frontier DMUs from future DEA runs. $\mathbf{J}^{l+1} = \mathbf{J}^l - \mathbf{E}^l$. (If $\mathbf{J}^{l+1} = \emptyset$ then stop.)

Step 3: Evaluate the new subset of "inefficient" DMUs, \mathbf{J}^{l+1} , by model (6.1) to obtain a new set of efficient DMUs \mathbf{E}^{l+1} (the new best-practice frontier).

Step 4: Let $l = l + 1$. Go to step 2.

Stopping rule: $\mathbf{J}^{l+1} = \emptyset$, the algorithm stops.

Thus, model (6.1) yields a stratification of the whole set of DMUs. From the algorithm, we know that l goes from 1 to L , where L is determined by the stopping rule. Consider Figure 1.2 in Chapter 1. DMUs 1, 2, 3 and 4 form

the first-level CRS frontier and DMU5 forms the second-level CRS frontier ($L = 2$).

It is easy to show that these sets of DMUs have the following properties

- (i) $\mathbf{J}^l = \bigcup_{l'=1}^l \mathbf{E}^{l'}$ and $\mathbf{E}^l \cap \mathbf{E}^{l'} = \emptyset$ for $l \neq l'$;
- (ii) The DMUs in $\mathbf{E}^{l'}$ are dominated by the DMUs in \mathbf{E}^l if $l' > l$;
- (iii) Each DMU in set \mathbf{E}^l is efficient with respect to the DMUs in set $\mathbf{E}^{l+l'}$ for all $0 < l' \leq l - L$.

Table 6.1. Data for the Flexible Manufacturing Systems

| FMS | Inputs | | | | | Outputs | | |
|-----|--------|-----|-------|-----|------|---------|----|----|
| | TC | WIP | TT | EMP | SR | VF | PF | RF |
| 1 | 1.19 | 98 | 12.33 | 5 | 5.30 | 619 | 88 | 2 |
| 2 | 4.91 | 297 | 34.84 | 14 | 1.10 | 841 | 14 | 4 |
| 3 | 4.60 | 418 | 16.68 | 12 | 6.30 | 555 | 39 | 1 |
| 4 | 3.69 | 147 | 40.83 | 10 | 3.80 | 778 | 31 | 2 |
| 5 | 1.31 | 377 | 20.82 | 3 | 9.80 | 628 | 51 | 6 |
| 6 | 3.04 | 173 | 38.87 | 4 | 1.60 | 266 | 13 | 5 |
| 7 | 1.83 | 202 | 49.67 | 13 | 4.30 | 46 | 60 | 4 |
| 8 | 2.07 | 533 | 30.07 | 14 | 8.80 | 226 | 21 | 4 |
| 9 | 3.06 | 898 | 27.67 | 2 | 3.90 | 354 | 86 | 5 |
| 10 | 1.44 | 423 | 6.02 | 10 | 5.40 | 694 | 20 | 3 |
| 11 | 2.47 | 470 | 4.00 | 13 | 5.30 | 513 | 40 | 5 |
| 12 | 2.85 | 87 | 43.09 | 8 | 2.40 | 884 | 17 | 7 |
| 13 | 4.85 | 915 | 54.79 | 5 | 2.40 | 439 | 58 | 4 |
| 14 | 1.31 | 852 | 86.87 | 3 | 0.50 | 401 | 18 | 4 |
| 15 | 4.18 | 924 | 54.46 | 4 | 6.00 | 491 | 27 | 4 |
| 16 | 1.99 | 273 | 91.08 | 3 | 2.50 | 937 | 6 | 3 |
| 17 | 1.60 | 983 | 37.93 | 13 | 8.80 | 709 | 39 | 2 |
| 18 | 4.04 | 106 | 23.39 | 11 | 2.90 | 615 | 91 | 3 |
| 19 | 3.76 | 955 | 54.98 | 1 | 9.40 | 499 | 46 | 3 |
| 20 | 4.76 | 416 | 1.55 | 9 | 1.50 | 58 | 2 | 6 |
| 21 | 3.60 | 660 | 3.98 | 6 | 3.90 | 592 | 29 | 4 |
| 22 | 3.24 | 771 | 52.26 | 8 | 1.60 | 535 | 61 | 1 |
| 23 | 3.05 | 318 | 35.09 | 4 | 9.20 | 124 | 25 | 2 |
| 24 | 1.60 | 849 | 62.83 | 15 | 7.30 | 923 | 60 | 3 |

We next use a data set from the DEA Dataset Repository at <http://java.emp.pdx.edu/etm/dea/dataset/> to illustrate the algorithm. Table 6.1 presents the data. The data set contains 24 flexible manufacturing systems (FMS). Each FMS has five inputs (1) total cost (TC) (\$millions), (2) work in process (WIP) (units), (3) throughput (TT) (hours/unit), (4) employees

(EMP) (persons), and (5) space requirements (SR) (thousands of square feet), and three outputs (1) volume flexibility (VF) (average range of production capacity per product type), (2) production mix flexibility (PF) (product types), and (3) routing flexibility (RF) (average number of operations per machining center).

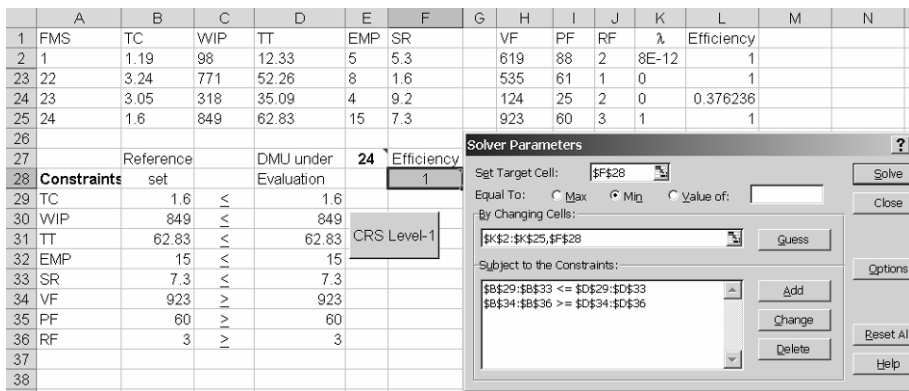


Figure 6.1. First Level CRS Frontier

Figure 6.1 shows the spreadsheet model for identifying the first level of CRS frontier. The target cell is F28, and the changing cells are K2:K25 and F28. The formulas for cells B29:B36 are

- Cell B29=SUMPRODUCT(B2:B25,K2:K25)
- Cell B30=SUMPRODUCT(C2:C25,K2:K25)
- Cell B31=SUMPRODUCT(D2:D25,K2:K25)
- Cell B32=SUMPRODUCT(E2:E25,K2:K25)
- Cell B33=SUMPRODUCT(F2:F25,K2:K25)
- Cell B34=SUMPRODUCT(H2:H25,K2:K25)
- Cell B35=SUMPRODUCT(I2:I25,K2:K25)
- Cell B36=SUMPRODUCT(J2:J25,K2:K25)

The formulas for cell D29:D36 are

- Cell D29=F28*INDEX(B2:B25,E27,1)
- Cell D30=F28*INDEX(C2:C25,E27,1)
- Cell D31=F28*INDEX(D2:D25,E27,1)
- Cell D32=F28*INDEX(E2:E25,E27,1)
- Cell D33=F28*INDEX(F2:F25,E27,1)
- Cell D34=INDEX(H2:H25,E27,1)
- Cell D35=INDEX(I2:I25,E27,1)
- Cell D36=INDEX(J2:J25,E27,1)

The button “CRS Level-1” is linked to the VBA procedure “Level1” which records the CRS efficiency scores in column L.

```

Sub Level1()
    Dim i As Integer
    For i = 1 To 24
'set the value of cell E27 equal to i (1, 2,..., 24)
        Range("E27") = i
        SolverSolve UserFinish:=True
'Place the efficiency into column L
        Range("L" & i + 1) = Range("F28")
    Next i
End Sub
    
```

Sixteen FMSs are on the first level CRS frontier. They are DMUs 1, 2, 5, 6, 9, 10, 11, 12, 14, 16, 18, 19, 20, 21, 22, and 24.

| | A | B | C | D | E | F | G | H | I | J | K | L |
|----|-------------|-----------|-----|------------|-----|------------|---|-----|----|----|-----------|------------|
| 1 | FMS | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ | Efficiency |
| 2 | 3 | 4.6 | 418 | 16.68 | 12 | 6.3 | | 555 | 39 | 1 | 0 | 1 |
| 3 | 4 | 3.69 | 147 | 40.83 | 10 | 3.8 | | 778 | 31 | 2 | 0 | 1 |
| 4 | 7 | 1.83 | 202 | 49.67 | 13 | 4.3 | | 46 | 60 | 4 | 0.201 | 1 |
| 5 | 8 | 2.07 | 533 | 30.07 | 14 | 8.8 | | 226 | 21 | 4 | 0 | 1 |
| 6 | 13 | 4.85 | 915 | 54.79 | 5 | 2.4 | | 439 | 58 | 4 | 0.157 | 1 |
| 7 | 15 | 4.18 | 924 | 54.46 | 4 | 6 | | 491 | 27 | 4 | 0.142 | 1 |
| 8 | 17 | 1.6 | 983 | 37.93 | 13 | 8.8 | | 709 | 39 | 2 | 0 | 1 |
| 9 | 23 | 3.05 | 318 | 35.09 | 4 | 9.2 | | 124 | 25 | 2 | 0 | 0.991748 |
| 10 | | | | | | | | | | | | |
| 11 | | Reference | | DMU under | 8 | Efficiency | | | | | | |
| 12 | Constraints | set | | Evaluation | | 0.991748 | | | | | | |
| 13 | TC | 1.722457 | < | 3.024832 | | | | | | | | |
| 14 | WIP | 315.3759 | < | 315.3759 | | | | | | | | |
| 15 | TT | 26.31838 | < | 34.80044 | | | | | | | | |
| 16 | EMP | 3.966992 | < | 3.966992 | | | | | | | | |
| 17 | SR | 2.093367 | < | 9.124083 | | | | | | | | |
| 18 | VF | 147.8423 | > | 124 | | | | | | | | |
| 19 | PF | 25 | > | 25 | | | | | | | | |
| 20 | RF | 2 | > | 2 | | | | | | | | |

Figure 6.2. Second Level CRS Frontier

Next, we remove those FMSs with efficiency score of one. Because no absolute references are used in the formulas in the spreadsheet shown in Figure 6.1, the Solver automatically adjusts the parameters as we remove the rows related to the FMSs with efficiency score of one. Figure 6.2 shows the new spreadsheet.

Seven FMSs are on the second level CRS frontier. Because only one FMS, namely DMU 23, is left, this DMU forms the third level CRS frontier (L = 3).

6.3 Input-oriented Context-dependent DEA

The DEA stratification model (6.1) partitions the set of DMUs into different frontier levels characterized by \mathbf{E}^l ($l = 1, \dots, L$). We present the input-oriented context-dependent DEA based upon the evaluation context \mathbf{E}^l . The context-dependent DEA is characterized by an attractiveness measure and a progress measure.

6.3.1 Attractiveness

Consider a specific $DMU_q = (x_q, y_q)$ from a specific level \mathbf{E}^{l_o} , $l_o \in \{1, \dots, L-1\}$. We have the following model to characterize the attractiveness.

$$\begin{aligned}
 H_q^*(d) &= \min H_q(d) \quad d = 1, \dots, L - l_o \\
 \text{subject to} \\
 \sum_{j \in F(\mathbf{E}^{l_o+d})} \lambda_j x_j &\leq H_q(d) x_q \\
 \sum_{j \in F(\mathbf{E}^{l_o+d})} \lambda_j y_j &\geq y_q \\
 \lambda_j &\geq 0 \quad j \in F(\mathbf{E}^{l_o+d})
 \end{aligned} \tag{6.2}$$

Lemma 6.1 For a specific $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{1, \dots, L-1\}$, model (6.2) is equivalent to the following linear programming problem

$$\begin{aligned}
 \tilde{H}_q^*(d) &= \min_{\lambda_j, \tilde{H}_q(d)} \tilde{H}_q(d) \quad d = 1, \dots, L - l_o \\
 \text{subject to} \\
 \sum_{j \in F(\mathbf{J}^{l_o+d})} \lambda_j x_j &\leq \tilde{H}_q(d) x_q \\
 \sum_{j \in F(\mathbf{J}^{l_o+d})} \lambda_j y_j &\geq y_q \\
 \lambda_j &\geq 0 \quad j \in F(\mathbf{J}^{l_o+d})
 \end{aligned} \tag{6.3}$$

[Proof]: Note that $\mathbf{J}^{l_o+d} = \bigcup_{l=\alpha}^{L-l_o} \mathbf{E}^{l_o+l}$. Therefore, (6.3) can be rewritten as

$$\begin{aligned}
 \tilde{H}_q^*(d) &= \min_{\mu_j, \lambda_j, \tilde{H}_q(d)} \tilde{H}_q(d) \quad l=\alpha \quad d = 1, \dots, L - l_o \\
 \text{subject to} \\
 \sum_{j \in F(\mathbf{E}^{l_o+d})} \mu_j x_j + \sum_{j \in F(\mathbf{J}^{l_o+d} - \mathbf{E}^{l_o+d})} \lambda_j x_j &\leq \tilde{H}_q(d) x_q \\
 \sum_{j \in F(\mathbf{E}^{l_o+d})} \mu_j y_j + \sum_{j \in F(\mathbf{J}^{l_o+d} - \mathbf{E}^{l_o+d})} \lambda_j y_j &\geq y_q \\
 \mu_j, \lambda_j &\geq 0
 \end{aligned} \tag{6.4}$$

Obviously, $\lambda_j = 0$ for all $j \in F(\mathbf{J}^{l_o+d} - \mathbf{E}^{l_o+d}) = F(\bigcup_{l=d+1}^{L-l_o} \mathbf{E}^{l_o+l})$ in any optimal solutions to (6.4). Otherwise, if some $\lambda_j \neq 0$, then $\mathbf{E}^{l_o+d} \cap$

$(\bigcup_{l=d+1}^{l_0} \mathbf{E}^{l_0+l}) \neq \emptyset$ (A contradiction). Therefore, $\tilde{\Omega}_q^*(d) = \Omega_q^*(d)$ and (6.2) is equivalent to (6.3). ■

Theorem 6.1 For a specific $DMU_q \in \mathbf{E}^{l_0}$, $l_0 \in \{1, \dots, L-1\}$, we have

- (i) $G_q^*(g) < 1$ for each $g = 1, \dots, l_0 - 1$.
- (ii) $H_q^*(d+1) > H_q^*(d)$.

[Proof]:

(i) Suppose $H_q^*(d) \leq 1$.

If $H_q^*(d) = 1$, then $DMU_q \in \mathbf{E}^{l_0+d}$. This means that $\mathbf{E}^{l_0+d} \cap \mathbf{E}^{l_0} \neq \emptyset$. A contradiction.

If $H_q^*(d) < 1$, then DMU_q is dominated by \mathbf{E}^{l_0+d} . However, \mathbf{E}^{l_0+d} is dominated by \mathbf{E}^{l_0} . Thus, DMU_q is dominated by \mathbf{E}^{l_0} . This means that $DMU_q \notin \mathbf{E}^{l_0}$. A contradiction. Therefore, $H_q^*(d) > 1$.

(ii) $H_q^*(d+1)$ is obtained by solving the following problem

$$\begin{aligned}
 H_q^*(d+1) &= \min_{\lambda_j, H_q(d+1)} H_q(d+1) \\
 \text{subject to} & \\
 \sum_{j \in F(\mathbf{E}^{l_0+d+1})} \lambda_j x_j &\leq H_q(d+1)x_q & (6.5) \\
 \sum_{j \in F(\mathbf{E}^{l_0+d+1})} \lambda_j y_j &\geq y_q \\
 \lambda_j &\geq 0 \quad j \in F(\mathbf{E}^{l_0+d+1})
 \end{aligned}$$

$H_q^*(d)$ is obtained by solving the following problem

$$\begin{aligned}
 H_q^*(d) &= \min_{\mu_j, H_q(d)} H_q(d) \\
 \text{subject to} & \\
 \sum_{j \in F(\mathbf{E}^{l_0+d})} \mu_j x_j &\leq H_q(d)x_q \\
 \sum_{j \in F(\mathbf{E}^{l_0+d})} \mu_j y_j &\geq y_q; \\
 \mu_j &\geq 0 \quad j \in F(\mathbf{E}^{l_0+d})
 \end{aligned}$$

which is, by Lemma 6.1, equivalent to the following linear programming problem

$$\begin{aligned}
 H_q^*(d) &= \min_{\mu_j, \lambda_j, H_q(d)} H_q(d) \\
 \text{subject to} & \\
 \sum_{j \in F(\mathbf{E}^{l_0+d})} \mu_j x_j + \sum_{j \in F(\mathbf{E}^{l_0+d+1})} \lambda_j x_j &\leq H_q(d)x_q & (6.6) \\
 \sum_{j \in F(\mathbf{E}^{l_0+d})} \mu_j y_j + \sum_{j \in F(\mathbf{E}^{l_0+d+1})} \lambda_j y_j &\geq y_q; \\
 \mu_j, \lambda_j &\geq 0
 \end{aligned}$$

It can be seen that any optimal solution to (6.5) is a feasible solution to (6.6). Therefore $H_q^*(d+1) \geq H_q^*(d)$. However, if $H_q^*(d+1) = H_q^*(d)$, then $E^{l_0+d} \cap E^{l_0+d+1} \neq \emptyset$. Thus, $H_q^*(d+1) > H_q^*(d)$. ■

Definition 6.1 $H_q^*(d)$ is called (input-oriented) d -degree attractiveness of DMU_q from a specific level E^{l_0} .

Suppose, e.g., each DMU in the first-level best practice frontier represents an option, or product. Customers may compare a specific DMU in E^{l_0} with other alternatives that are currently in the same level as well as with relevant alternatives that serve as evaluation contexts. The relevant alternatives are those DMUs, say, in the second or third level best-practice frontier, etc. Given the alternatives (evaluation contexts), model (6.2) enables us to select the best option – the most attractive one.

In model (6.2), each best-practice frontier of E^{l_0+d} represents an evaluation context for measuring the relative attractiveness of DMUs in E^{l_0} . The larger the value of $H_q^*(d)$, the more attractive the DMU_q is. Because this DMU_q makes itself more distinctive from the evaluation context E^{l_0+d} . We are able to rank the DMUs in E^{l_0} based upon their attractiveness scores and identify the best one.

| | A | B | C | D | E | F | G | H | I | J | K |
|----|-----------------------|--------|--------|--------|-------|-------|-------|-----|----|------|----------------|
| 1 | FMS (Level1) | TC | WIP | TT | EMP | SR | | VF | PF | RF | Attractiveness |
| 2 | 1 | 1.19 | 98 | 12.33 | 5 | 5.3 | 16 | 619 | 88 | 2 | 5.257398 |
| 3 | 2 | 4.91 | 297 | 34.84 | 14 | 1.1 | | 841 | 14 | 4 | 3.883129 |
| 4 | 5 | 1.31 | 377 | 20.82 | 3 | 9.8 | Score | 628 | 51 | 6 | 3.942397 |
| 5 | 6 | 3.04 | 173 | 38.87 | 4 | 1.6 | 1.594 | 266 | 13 | 5 | 3.222047 |
| 6 | 9 | 3.06 | 898 | 27.67 | 2 | 3.9 | | 354 | 86 | 5 | 3.706897 |
| 7 | 10 | 1.44 | 423 | 6.02 | 10 | 5.4 | | 694 | 20 | 3 | 5.347655 |
| 8 | 11 | 2.47 | 470 | 4 | 13 | 5.3 | | 513 | 40 | 5 | 10.45603 |
| 9 | 12 | 2.85 | 87 | 43.09 | 8 | 2.4 | | 884 | 17 | 7 | 4.625919 |
| 10 | 14 | 1.31 | 852 | 86.87 | 3 | 0.5 | | 401 | 18 | 4 | 4.8 |
| 11 | 16 | 1.99 | 273 | 91.08 | 3 | 2.5 | | 937 | 6 | 3 | 3.334894 |
| 12 | 18 | 4.04 | 106 | 23.39 | 11 | 2.9 | | 615 | 91 | 3 | 3.528959 |
| 13 | 19 | 3.76 | 955 | 54.98 | 1 | 9.4 | | 499 | 46 | 3 | 4.845534 |
| 14 | 20 | 4.76 | 416 | 1.55 | 9 | 1.5 | | 58 | 2 | 6 | 29.1 |
| 15 | 21 | 3.6 | 660 | 3.98 | 6 | 3.9 | | 592 | 29 | 4 | 9.24551 |
| 16 | 22 | 3.24 | 771 | 52.26 | 8 | 1.6 | | 535 | 61 | 1 | 1.789802 |
| 17 | 24 | 1.6 | 849 | 62.83 | 15 | 7.3 | | 923 | 60 | 3 | 1.593583 |
| 18 | DMU under evaluation | 2.5497 | 1353 | 100.12 | 23.9 | 11.63 | | 923 | 60 | 3 | |
| 19 | | IV | IV | IV | IV | IV | | AI | AI | AI | |
| 20 | Evaluation background | 2.5497 | 1227.1 | 57.697 | 18.73 | 11.63 | | 923 | 60 | 3.27 | |
| 21 | FMS(Level2) | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ |
| 22 | 3 | 4.8 | 418 | 16.88 | 12 | 6.3 | | 555 | 39 | 1 | 0 |
| 23 | 4 | 3.69 | 147 | 40.83 | 10 | 3.8 | | 778 | 31 | 2 | 0.082849 |
| 24 | 7 | 1.83 | 202 | 49.67 | 13 | 4.3 | | 46 | 60 | 4 | 0.177585 |
| 25 | 8 | 2.07 | 533 | 30.07 | 14 | 8.8 | | 226 | 21 | 4 | 0 |
| 26 | 13 | 4.85 | 915 | 54.79 | 5 | 2.4 | | 439 | 58 | 4 | 0 |
| 27 | 15 | 4.18 | 924 | 54.46 | 4 | 6 | | 491 | 27 | 4 | 0 |
| 28 | 17 | 1.6 | 983 | 37.93 | 13 | 8.8 | | 709 | 39 | 2 | 1.1994 |

Figure 6.3. First Degree Attractiveness Spreadsheet Model

Figure 6.3 shows a spreadsheet model for the attractiveness measure – model (6.2). Cells A1:J17 and A21:J28 store the DMUs in the first and second levels, respectively. This spreadsheet model measures the first-degree attractiveness. Cell G2 is reserved to indicate the DMU under evaluation. Cell G5 represents the attractiveness score, and is the target cell and a changing cell. Cells K22:K28 are reserved for changing cells of λ_j .

Cells B20:F20, and cells H20:J20 contain formulas for the reference set (DMUs in the second level CRS frontier). The formula for cell B20 is “=SUMPRODUCT(B22:B28,\$K\$22:\$K\$28)” and is copied into cells C20:F20 and cells H20:J20.

Cells B18:F18, and cells H18:J18 contain formulas for the DMU under evaluation (DMUs in the first level CRS frontier). The formula for cell B18 is “=\$G\$5*INDEX(B2:B17,\$G\$2,1)” and is copied into cells C18:F18. The formula for cell H18 is “=INDEX(H2:H17,\$G\$2,1)” and is copied into cells I18:J18.

Figure 6.4 shows the Solver parameters. After solve the model for the first DMU, we use the VBA procedure “Attractiveness” to obtain the attractiveness scores for the remaining DMUs.

```
Sub Attractiveness()
    Dim i As Integer
    For i = 1 To 16
        'set the value of cell G2 equal to i (1, 2,..., 16)
        Range("G2") = i
        SolverSolve UserFinish:=True
        'Place the attractiveness score in cell G5 into column K
        Range("K" & i + 1) = Range("G5")
    Next
End Sub
```

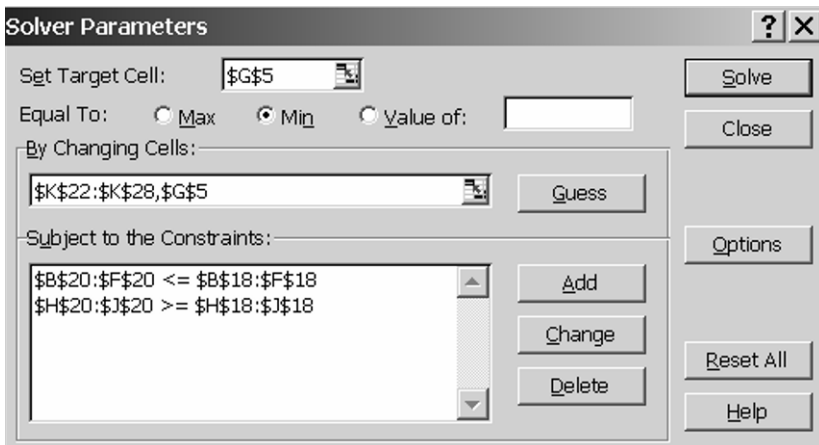


Figure 6.4. Solver Parameters for First Degree Attractiveness

| | A | B | C | D | E | F | G | H | I | J | K |
|----|-----------------------|--------|-------|--------|-------|-------|-------|-----|-------|------|----------------|
| 1 | FMS (Level1) | TC | WIP | TT | EMP | SR | | VF | PF | RF | Attractiveness |
| 2 | 1 | 1.19 | 98 | 12.33 | 5 | 5.3 | 16 | 619 | 88 | 2 | 16.19832 |
| 3 | 2 | 4.91 | 297 | 34.84 | 14 | 1.1 | | 841 | 14 | 4 | 56.72434 |
| 4 | 5 | 1.31 | 377 | 20.82 | 3 | 9.8 | Score | 628 | 51 | 6 | 11.79143 |
| 5 | 6 | 3.04 | 173 | 38.87 | 4 | 1.6 | 14.19 | 266 | 13 | 5 | 14.375 |
| 6 | 9 | 3.06 | 898 | 27.67 | 2 | 3.9 | | 354 | 86 | 5 | 8.114872 |
| 7 | 10 | 1.44 | 423 | 6.02 | 10 | 5.4 | | 694 | 20 | 3 | 32.62306 |
| 8 | 11 | 2.47 | 470 | 4 | 13 | 5.3 | | 513 | 40 | 5 | 36.29268 |
| 9 | 12 | 2.85 | 87 | 43.09 | 8 | 2.4 | | 884 | 17 | 7 | 27.32796 |
| 10 | 14 | 1.31 | 852 | 86.87 | 3 | 0.5 | | 401 | 18 | 4 | 59.50323 |
| 11 | 16 | 1.99 | 273 | 91.08 | 3 | 2.5 | | 937 | 6 | 3 | 27.80774 |
| 12 | 18 | 4.04 | 106 | 23.39 | 11 | 2.9 | | 615 | 91 | 3 | 15.73415 |
| 13 | 19 | 3.76 | 955 | 54.98 | 1 | 9.4 | | 499 | 46 | 3 | 16.09677 |
| 14 | 20 | 4.76 | 416 | 1.55 | 9 | 1.5 | | 58 | 2 | 6 | 67.91613 |
| 15 | 21 | 3.6 | 660 | 3.98 | 6 | 3.9 | | 592 | 29 | 4 | 42.09207 |
| 16 | 22 | 3.24 | 771 | 52.26 | 8 | 1.6 | | 535 | 61 | 1 | 24.80847 |
| 17 | 24 | 1.6 | 849 | 62.83 | 15 | 7.3 | | 923 | 60 | 3 | 14.18926 |
| 18 | DMU under evaluation | 22.703 | 12047 | 891.51 | 212.8 | 103.6 | | 923 | 60 | 3 | |
| 19 | | IV | IV | IV | IV | IV | | ΔJ | ΔJ | ΔJ | |
| 20 | Evaluation background | 22.703 | 2367 | 261.19 | 29.77 | 68.48 | | 923 | 186.1 | 14.9 | |
| 21 | FMS(Level3) | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ |
| 22 | 23 | 3.05 | 318 | 35.09 | 4 | 9.2 | | 124 | 25 | 2 | 7.443548 |

Figure 6.5. Second Degree Attractiveness Spreadsheet Model

If we change the evaluation background to the third level CRS frontier (DMU23), we obtain the spreadsheet model for measuring the second degree attractiveness (see Figure 6.5). This spreadsheet can be obtained via replacing the second level CRS frontier by the DMU23 in Figure 6.3.

Based upon the attractiveness scores shown in Figure 6.3, DMU20 and DMU11 are ranked as the top two most attractive systems. However, if we change the evaluation context to the third level CRS frontier, DMU11 is ranked fifth, and DMU14 becomes the second most attractiveness system. This example illustrates that under a different evaluation context, the attractiveness of DMUs on the same level may be different. Therefore, the context-dependent DEA can differentiate the performance of efficient DMUs, or DMUs on the same performance level.

6.3.2 Progress

Consider the following linear programming problem for determining the progress measure for $DMU_q \in E^{l_o}$, $l_o \in \{2, \dots, L\}$.

$$\begin{aligned}
 G_q^*(g) &= \min_{\lambda_j, G_q(g)} G_q(g) \quad g = 1, \dots, l_o - 1 \\
 \text{subject to} & \\
 \sum_{j \in F(E^{l_o-g})} \lambda_j x_j &\leq G_q(\beta) x_q & (6.7) \\
 \sum_{j \in F(E^{l_o-g})} \lambda_j y_j &\geq y_q \\
 \lambda_j &\geq 0 \quad j \in F(E^{l_o-g})
 \end{aligned}$$

Lemma 6.2 For a specific $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{2, \dots, L\}$, model (6.7) is equivalent to the following linear programming problem

$$\begin{aligned} \tilde{G}_q^*(g) &= \min_{\lambda_j, \tilde{G}_q(\beta)} \tilde{G}_q(g) \\ \text{subject to} & \\ \sum_{j \in F(\mathbf{J}^{l_o-g})} \lambda_j x_j &\leq \tilde{G}_q(g) x_q & (6.8) \\ \sum_{j \in F(\mathbf{J}^{l_o-g})} \lambda_j y_j &\geq y_q \\ \lambda_j &\geq 0 \quad j \in F(\mathbf{J}^{l_o-g}) \end{aligned}$$

[Proof]: Note that $\mathbf{J}^{l_o-g} = \mathbf{E}^{l_o-g} \cup \mathbf{J}^{l_o-g+1}$, since all DMUs in \mathbf{J}^{l_o-g} are dominated by the frontiers constructed by the DMUs in \mathbf{E}^{l_o-g} . Therefore, by the nature of DEA method, we know that $\lambda_j = 0 (j \notin F(\mathbf{E}^{l_o-g}))$ in any optimal solutions to (6.8). Thus, $\tilde{G}_q^*(g) = G_q^*(g)$ and (6.7) is equivalent to (6.8). ■

Theorem 6.2 For a specific $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{2, \dots, L\}$, we have

- (i) $G_q^*(g) < 1$ for each $g = 1, \dots, l_o - 1$.
- (ii) $G_q^*(g+1) < G_q^*(g)$.

[Proof]: The proof is similar to that of Theorem 6.1 by using Lemma 6.2 and therefore is omitted. ■

Definition 6.2: $M_q^*(g) \equiv 1/G_q^*(g)$ is called (input-oriented) g -degree progress of DMU_q from a specific level \mathbf{E}^{l_o} .

Obviously $M_q^*(g) > 1$. For a larger $M_q^*(g)$, more progress is expected. Each best-practice frontier, \mathbf{E}^{l_o-g} , contains a possible target for a specific DMU in \mathbf{E}^{l_o} to improve its performance. The progress here is a level-by-level improvement.

Now consider the following linear programming problem

$$\begin{aligned} \max & \|S^+(g)\|_1 + \|S^-(g)\|_1 \quad g = 1, \dots, l_o - 1 \\ \text{subject to} & \\ S^-(g) &= G_q^*(g)x_q - \sum_{j \in F(\mathbf{E}^{l_o-g})} \lambda_j x_j \\ S^+(g) &= \sum_{j \in F(\mathbf{E}^{l_o-g})} \lambda_j y_j - y_q & (6.9) \\ S^+(g), S^-(g) &\geq 0 \\ \lambda_j &\geq 0 \quad j \in F(\mathbf{E}^{l_o-g}) \end{aligned}$$

where $\|S^+(g)\|_1$ and $\|S^-(g)\|_1$ represent L_1 -norms for $S^+(g) = (s_1^+(g), \dots, s_s^+(g))$ and $S^-(g) = (s_1^-(g), \dots, s_m^-(g))$, respectively, i.e., $\|S^+(g)\|_1 + \|S^-(g)\|_1 = \sum_{r=1}^s s_r^+(g) + \sum_{i=1}^m s_i^-(g)$.

Definition 6.3 (Global Efficient Target and Local Efficient Target).

The following point

$$\begin{cases} \hat{y}_q = P_q^*(g)y_q + S^{+*}(g) \\ \hat{x}_q = x_q - S^{-*}(g) \end{cases}$$

is the *global efficient target* for $DMU_q \in E^{l_o}$, $l_o \in \{2, \dots, L\}$ if $g = l_o - 1$; Otherwise, if $g < l_o - 1$, it represents a *local efficient target*, where $G_q^*(g)$ is the optimal value to (6.7), and $S^{+*}(g)$ and $S^{-*}(g)$ represent the optimal values in (6.9).

It can be seen that if the first-level best-practice frontier is chosen as the evaluation context ($g = l_o - 1$), then we obtain the global efficient target, i.e., the original DEA efficient target. The local efficient targets are obtained when other best-practice frontiers are selected. Although the local efficient targets are not non-dominated points compared to the DMUs in E^1 , they may represent a better alternative and an attainable target for a specific inefficient DMU. That is, in the presence of possible external or internal restrictions, a DMU may be unable to move itself onto the first-level best-practice frontier (global efficient target). Thus, our progress measure extends the original DEA projection function and enables an inefficient DMU to improve its performance at a reasonable and desirable scale.

| | A | B | C | D | E | F | G | H | I | J | K |
|----|-----------------------|--------|--------|--------|------|-------|---|-------|----|----|----------|
| 1 | FMS(Level3) | TC | WIP | TT | EMP | SR | | VF | PF | RF | Progress |
| 2 | 23 | 3.05 | 318 | 35.09 | 4 | 9.2 | 1 | 124 | 25 | 2 | 0.991748 |
| 3 | | | | | | | | | | | |
| 4 | DMU under evaluation | 3.0248 | 315.38 | 34.8 | 3.97 | 9.124 | | 124 | 25 | 2 | |
| 5 | | IV | IV | IV | IV | IV | | Δ1 | Δ1 | Δ1 | |
| 6 | Evaluation background | 1.7225 | 315.38 | 26.318 | 3.97 | 2.093 | | 147.8 | 25 | 2 | |
| 7 | FMS(Level2) | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ |
| 8 | 3 | 4.6 | 418 | 16.68 | 12 | 6.3 | | 555 | 39 | 1 | 0 |
| 9 | 4 | 3.69 | 147 | 40.83 | 10 | 3.8 | | 778 | 31 | 2 | 0 |
| 10 | 7 | 1.83 | 202 | 49.67 | 13 | 4.3 | | 46 | 60 | 4 | 0.201125 |
| 11 | 8 | 2.07 | 533 | 30.07 | 14 | 8.8 | | 226 | 21 | 4 | 0 |
| 12 | 13 | 4.85 | 915 | 54.79 | 5 | 2.4 | | 439 | 58 | 4 | 0.156867 |
| 13 | 15 | 4.18 | 924 | 54.46 | 4 | 6 | | 491 | 27 | 4 | 0.142008 |
| 14 | 17 | 1.6 | 983 | 37.93 | 13 | 8.8 | | 709 | 39 | 2 | 0 |

Figure 6.6. First Degree Progress Spreadsheet Model

Figure 6.6 shows a spreadsheet for the progress measure where the evaluation background is the second level DMUs and the DMU under evaluation is the DMU23.

The formula for cell B4 is “=\$K\$2*INDEX(B2,\$G\$2,1), where cell K2 is the target cell (its reciprocal represents the first-degree progress). This formula is copied into cells C4:F4.

The formula for H4 is “=INDEX(H2,\$G\$2,1)” and is copied into cells I4:J4

The formula for cell B6 is “=SUMPRODUCT(B8:B14,\$K\$8:\$K\$14)” and is copied into cells C6:F6, and cells H6:J6.

Figure 6.7 shows the Solver parameters for the spreadsheet model shown in Figure 6.6. In Figure 6.6, the first degree progress score for DMU23 is $1/0.99175 = 1.0083$. The optimal values in cells B6:F6 and cells H6:J6 represent the local target for DMU23.

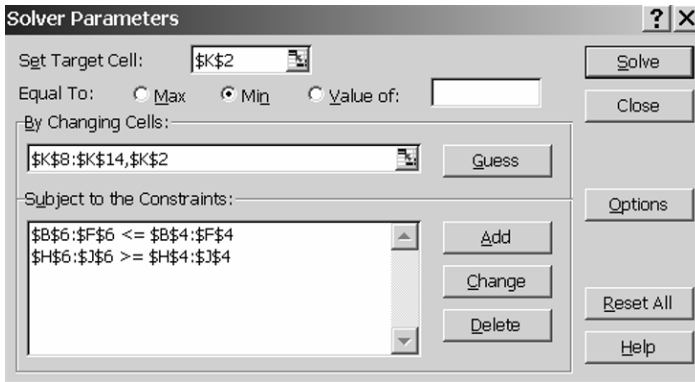


Figure 6.7. Solver Parameters for First Degree Progress

| | A | B | C | D | E | F | G | H | I | J | K |
|----|-----------------------|--------|--------|--------|-------|-------|-----|-------|----|----|----------|
| 1 | FMS(Level3) | TC | WIP | TT | EMP | SR | | VF | PF | RF | Progress |
| 2 | 23 | 3.05 | 318 | 35.09 | 4 | 9.2 | 1 | 124 | 25 | 2 | 0.376236 |
| 3 | | | | | | | | | | | |
| 4 | DMU under evaluation | 1.1475 | 119.64 | 13.202 | 1.5 | 3.461 | | 124 | 25 | 2 | |
| 5 | | IV | IV | IV | IV | IV | | ΔI | ΔI | ΔI | |
| 6 | Evaluation background | 0.5355 | 119.64 | 7.7206 | 1.5 | 3.434 | | 258.6 | 25 | 2 | |
| 7 | FMS(Level1) | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ |
| 8 | | 1 | 1.19 | 98 | 12.33 | 5 | 5.3 | 619 | 88 | 2 | 0.117385 |
| 9 | | 2 | 4.91 | 297 | 34.84 | 14 | 1.1 | 841 | 14 | 4 | 0 |
| 10 | | 5 | 1.31 | 377 | 20.82 | 3 | 9.8 | 628 | 51 | 6 | 0.285026 |
| 11 | | 6 | 3.04 | 173 | 38.87 | 4 | 1.6 | 266 | 13 | 5 | 0 |
| 12 | | 9 | 3.06 | 898 | 27.67 | 2 | 3.9 | 354 | 86 | 5 | 0 |
| 13 | | 10 | 1.44 | 423 | 6.02 | 10 | 5.4 | 694 | 20 | 3 | 0 |
| 14 | | 11 | 2.47 | 470 | 4 | 13 | 5.3 | 513 | 40 | 5 | 0 |
| 15 | | 12 | 2.85 | 87 | 43.09 | 8 | 2.4 | 884 | 17 | 7 | 0.007867 |
| 16 | | 14 | 1.31 | 852 | 86.87 | 3 | 0.5 | 401 | 18 | 4 | 0 |
| 17 | | 16 | 1.99 | 273 | 91.08 | 3 | 2.5 | 937 | 6 | 3 | 0 |
| 18 | | 18 | 4.04 | 106 | 23.39 | 11 | 2.9 | 615 | 91 | 3 | 0 |
| 19 | | 19 | 3.76 | 955 | 54.98 | 1 | 9.4 | 499 | 46 | 3 | 0 |
| 20 | | 20 | 4.76 | 416 | 1.55 | 9 | 1.5 | 58 | 2 | 6 | 0 |
| 21 | | 21 | 3.6 | 660 | 3.98 | 6 | 3.9 | 592 | 29 | 4 | 0 |
| 22 | | 22 | 3.24 | 771 | 52.26 | 8 | 1.6 | 535 | 61 | 1 | 0 |
| 23 | | 24 | 1.6 | 849 | 62.83 | 15 | 7.3 | 923 | 60 | 3 | 0 |

Figure 6.8. Second Degree Progress Spreadsheet Model

If we replace the second-level DMUs with the first-level DMUs, we obtain the second-degree progress measure for DMU23. Figure 6.8 shows the spreadsheet.

This spreadsheet model is actually the input-oriented CRS envelopment model when DMU23 is under evaluation. Figure 6.9 shows the Solver parameters for the model shown in Figure 6.8. The second degree progress score for DMU23 is $1/0.37624 = 2.6579$, and the optimal values in cells B6:F6 and H6:J6 represent the global target for DMU23

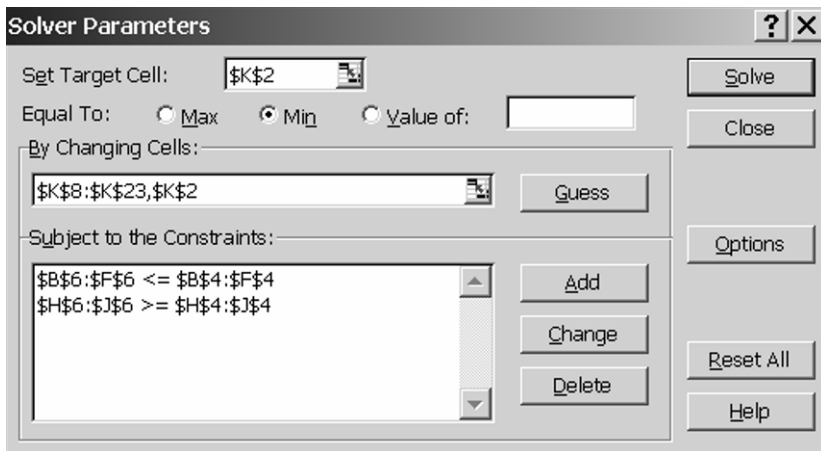


Figure 6.9. Solver Parameters for Second Degree Progress

6.4 Output-oriented Context-dependent DEA

Similar to the discussion on the input-oriented context-dependent DEA, for a specific $DMU_q = (x_q, y_q)$ from a specific level E^{l_o} , $l_o \in \{1, \dots, L-1\}$, we have the following model to characterize the output-oriented attractiveness

$$\begin{aligned}
 \Omega_q^*(d) &= \max_{\lambda_j, \Omega_q(d)} \Omega_q(d) \quad d = 1, \dots, L - l_o \\
 \text{subject to} & \\
 \sum_{j \in F(E^{l_o+d})} \lambda_j y_j &\geq \Omega_q(d) y_q \\
 \sum_{j \in F(E^{l_o+d})} \lambda_j x_j &\leq x_q \\
 \lambda_j &\geq 0 \quad j \in F(E^{l_o+d})
 \end{aligned} \tag{6.10}$$

Similar to Theorem 6.1, we have

Theorem 6.3 For a specific $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{1, \dots, L-1\}$, we have

- (i) $\Omega_q^*(d) < 1$ for each $d = 1, \dots, L - l_o$.
- (ii) $\Omega_q^*(d + 1) < \Omega_q^*(d)$.

Definition 6.4 $A_q^*(d) \equiv 1/\Omega_q^*(d)$ is called the (output-oriented) d -degree attractiveness of DMU_q from a specific level \mathbf{E}^{l_o} .

Note that $A_q^*(d)$ is the reciprocal of the optimal value to (6.10), therefore $A_q^*(d) > 1$. The larger the value of $A_q^*(d)$, the more attractive the DMU_q is. Because this DMU_q makes itself more distinctive from the evaluation context \mathbf{E}^{l_o+d} . We are able to rank the DMUs in \mathbf{E}^{l_o} based upon their attractiveness scores and identify the best one.

Next, consider the following linear programming problem for determining the progress measure for $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{2, \dots, L\}$.

$$\begin{aligned}
 P_q^*(g) &= \max_{\lambda_j, P_q(g)} P_q(g) \quad g = 1, \dots, l_o - 1 \\
 \text{subject to} & \\
 \sum_{j \in F(E^{l_o-g})} \lambda_j y_j &\geq P_q(g) y_q & (6.11) \\
 \sum_{j \in F(E^{l_o-g})} \lambda_j x_j &\leq x_q \\
 \lambda_j &\geq 0 \quad j \in F(E^{l_o-g})
 \end{aligned}$$

Similar to Theorem 6.2, we have

Theorem 6.4 For a specific $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{2, \dots, L\}$, we have

- (i) $P_q^*(g) > 1$ for each $g = 1, \dots, l_o - 1$.
- (ii) $P_q^*(g + 1) > P_q^*(g)$.

Definition 6.5 The optimal value to (6.11), i.e., $P_q^*(g)$, is called the (output-oriented) g -degree progress of DMU_q from a specific level \mathbf{E}^{l_o} .

For a larger $P_q^*(g)$, more progress is expected for DMU_q . Thus, a smaller value of $P_q^*(g)$ is preferred. To obtain the efficient target, we consider

$$\begin{aligned}
 \max & \|S^+(g)\|_1 + \|S^-(g)\|_1 \quad g = 1, \dots, l_o - 1 \\
 \text{subject to} & \\
 S^+(g) &= \sum_{j \in F(E^{l_o-g})} \lambda_j y_j - P_q^*(g) y_q \\
 S^-(g) &= x_q - \sum_{j \in F(E^{l_o-g})} \lambda_j x_j & (6.12) \\
 S^+(g), S^-(g) &\geq 0 \\
 \lambda_j &\geq 0 \quad j \in F(E^{l_o-g})
 \end{aligned}$$

Definition 6.6 (Global Efficient Target and Local Efficient Target)

The following point

$$\begin{cases} \hat{y}_q = P_q^*(g)y_q + S^{+*}(g) \\ \hat{x}_q = x_q - S^{-*}(g) \end{cases}$$

is the *global efficient target* for $DMU_q \in \mathbf{E}^{l_o}$, $l_o \in \{2, \dots, L\}$ if $g = l_o - 1$; otherwise, if $g < l_o - 1$, it represents a *local efficient target*, where $P_q^*(g)$ is the optimal value to (6.11), and $S^{+*}(g)$ and $S^{-*}(g)$ represent the optimal values in (6.12).

The relationship between the input-oriented and output-oriented context-dependent DEA can be summarized in the following Theorem.

Theorem 6.5 $H_q^*(d) = 1/\Omega_q^*(d)$, and $G_q^*(g) = 1/P_q^*(g)$.

Theorem 6.5 indicates that the output-oriented attractiveness and progress measures can be obtained from the input-oriented context-dependent DEA. However, Theorem 6.5 is not necessarily true when the frontiers do not exhibit CRS.

| | A | B | C | D | E | F | G | H | I | J | K | L | |
|----|-----------------------|-----|------|--------|-------|-----|-----|-------|-------|------|----------------|----------------|-------------|
| 1 | FMS (Level1) | TC | WIP | TT | EMP | SR | | VF | PF | RF | Attractiveness | input-oriented | |
| 2 | | 1 | 1.19 | 98 | 12.33 | 5 | 5.3 | 16 | 619 | 88 | 2 | 0.190208157 | 5.257398084 |
| 3 | | 2 | 4.91 | 297 | 34.84 | 14 | 1.1 | | 841 | 14 | 4 | 0.257524313 | 3.88312851 |
| 4 | | 5 | 1.31 | 377 | 20.82 | 3 | 9.8 | | 628 | 51 | 6 | 0.253652805 | 3.942396767 |
| 5 | | 6 | 3.04 | 173 | 38.87 | 4 | 1.6 | | 266 | 13 | 5 | 0.310361728 | 3.222046761 |
| 6 | | 9 | 3.06 | 898 | 27.67 | 2 | 3.9 | | 354 | 86 | 5 | 0.269767442 | 3.706896552 |
| 7 | | 10 | 1.44 | 423 | 6.02 | 10 | 5.4 | | 694 | 20 | 3 | 0.186997849 | 5.347655092 |
| 8 | | 11 | 2.47 | 470 | 4 | 13 | 5.3 | | 513 | 40 | 5 | 0.095638588 | 10.45603059 |
| 9 | | 12 | 2.85 | 87 | 43.09 | 8 | 2.4 | | 884 | 17 | 7 | 0.216173255 | 4.62591916 |
| 10 | | 14 | 1.31 | 852 | 86.87 | 3 | 0.5 | | 401 | 18 | 4 | 0.208333333 | 4.8 |
| 11 | | 16 | 1.99 | 273 | 91.08 | 3 | 2.5 | | 937 | 6 | 3 | 0.299859601 | 3.334894047 |
| 12 | | 18 | 4.04 | 106 | 23.39 | 11 | 2.9 | | 615 | 91 | 3 | 0.283369716 | 3.528958618 |
| 13 | | 19 | 3.76 | 955 | 54.98 | 1 | 9.4 | | 499 | 46 | 3 | 0.20637561 | 4.845533835 |
| 14 | | 20 | 4.76 | 416 | 1.55 | 8 | 1.5 | | 58 | 2 | 6 | 0.034364261 | 29.1 |
| 15 | | 21 | 3.6 | 660 | 3.98 | 6 | 3.9 | | 592 | 29 | 4 | 0.108160605 | 9.245510401 |
| 16 | | 22 | 3.24 | 771 | 52.26 | 8 | 1.6 | | 535 | 61 | 1 | 0.558720872 | 1.789802475 |
| 17 | | 24 | 1.6 | 849 | 62.83 | 15 | 7.3 | | 923 | 60 | 3 | 0.627516581 | 1.593583388 |
| 18 | DMU under evaluation | 1.6 | 849 | 62.83 | 15 | 7.3 | | 579.2 | 37.65 | 1.88 | | | |
| 19 | | iv | iv | iv | iv | iv | | Δ1 | Δ1 | Δ1 | Attractiveness | | |
| 20 | Evaluation background | 1.6 | 770 | 36.206 | 11.75 | 7.3 | | 579.2 | 37.65 | 2.06 | | | |
| 21 | FMS(Level2) | TC | WIP | TT | EMP | SR | | VF | PF | RF | λ | | |
| 22 | | 3 | 4.6 | 418 | 16.88 | 12 | 6.3 | | 555 | 39 | 1 | 0 | |
| 23 | | 4 | 3.89 | 147 | 40.83 | 10 | 3.8 | | 778 | 31 | 2 | 0.05198893 | |
| 24 | | 7 | 1.83 | 202 | 49.67 | 13 | 4.3 | | 46 | 60 | 4 | 0.111437381 | |
| 25 | | 8 | 2.07 | 533 | 30.07 | 14 | 8.8 | | 226 | 21 | 4 | 0 | |
| 26 | | 13 | 4.85 | 915 | 54.79 | 5 | 2.4 | | 439 | 58 | 4 | 0 | |
| 27 | | 15 | 4.18 | 924 | 54.46 | 4 | 6 | | 491 | 27 | 4 | 0 | |
| 28 | | 17 | 1.6 | 983 | 37.93 | 13 | 8.8 | | 709 | 39 | 2 | 0.752643173 | |

Figure 6.10. Output-oriented First Degree Attractiveness Spreadsheet Model

Figure 6.10 shows the output-oriented spreadsheet for the first-degree attractiveness for DMUs in the first-level CRS frontier. This spreadsheet is

similar to the one shown in Figure 6.3. The formula for cell B18 is changed to

Cell B18 =INDEX(B2:B17,\$G\$2,1)

and is copied into cells C18:F18.

The formula for cell H18 is changed to

Cell H18 =\$G\$5*INDEX(H2:H17,\$G\$2,1)

where cell G5 represents the output-oriented attractiveness measure. This formula is then copied into cells I18:J18.

To obtain the Solver parameters for the model shown in Figure 6.10, we change the “Min” to “Max” in Figure 6.4, as shown in Figure 6.11. The results are reported in cells K2:K17. Cells L2:L17 report the input-oriented attractiveness scores. It can be seen that Theorem 6.5 is true.

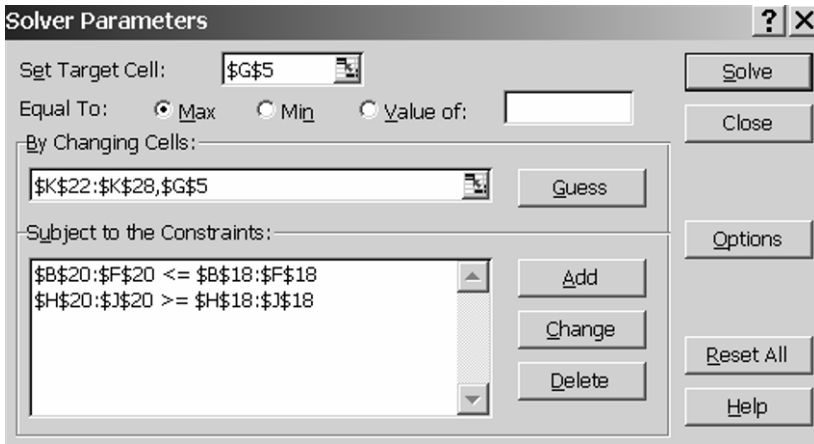


Figure 6.11. Solver Parameters for Output-oriented First Degree Attractiveness

Finally, the discussion in this chapter is based upon CRS frontier. Similar discussion can be obtained for other RTS frontiers. However, the related context-dependent DEA may be infeasible. See Chapters 7 and 10 for the discussion on infeasibility of DEA-type models. The DEAFrontier software allows you to calculate the context-dependent DEA under non-CRS assumptions.

6.5 Solving DEA Using DEA Frontier Software

The context-dependent DEA consists of three functions: (i) Obtain levels, (ii) Calculate context-depend DEA models, and (iii) Unprotect the sheets containing the levels (see Figure 6.12).



Figure 6.12. Context-dependent DEA Menu

The first function is the stratification model (6.1). It generates all the efficient frontiers – levels (Figure 6.13). This function will first delete any sheet with a name starting with “Level” and then generate a set of new sheets named as “Level i (Frontier)” where i indicates the level and *Frontier* represents the frontier type. For example, Level1(CRS) means the first level CRS frontier. The “level” sheets are protected for use in the context-dependent DEA. However, they can be unprotected by using the “Unprotect the sheets” menu item. *The format of these level sheets must not be modified. Otherwise, the context-dependent DEA will not run properly and accurately.*

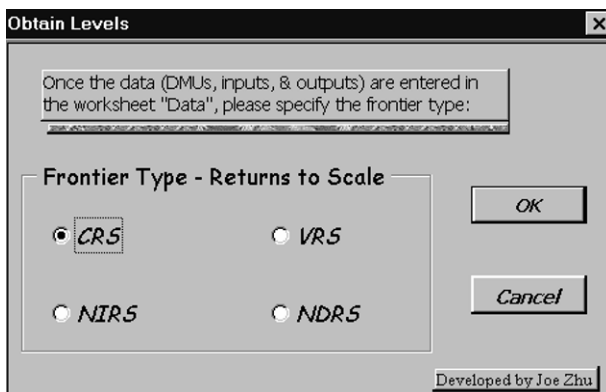


Figure 6.13. Obtain Levels

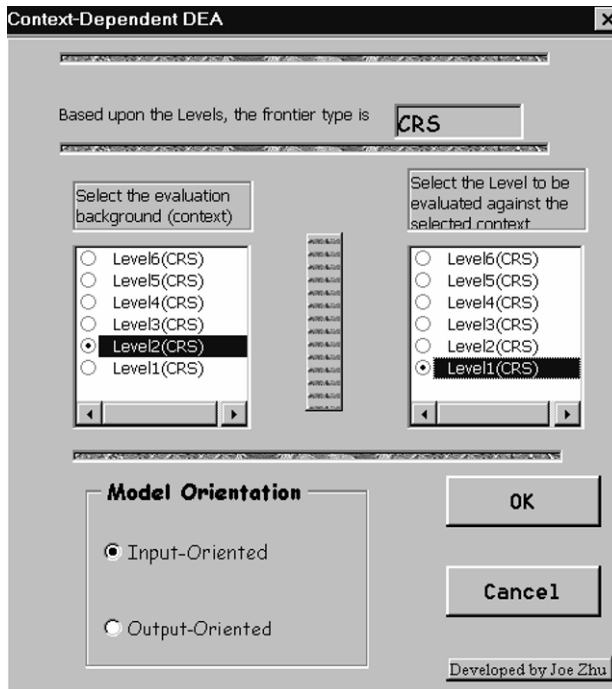


Figure 6.14. Context-dependent DEA

Once the efficient frontiers are obtained, the context-dependent DEA can be calculated using the “Context-dependent DEA” submenu item (Figure 6.14).

The results are reported in the “Context Dependent Result” sheet. In this sheet, the context-dependent scores are the optimal values to model (6.2) (or model (6.7), model (6.10), model (6.11)). To obtain the attractiveness or progress scores, one has to adjust the context-dependent scores based upon Definitions 6.1, 6.2, 6.4, and 6.5.

REFERENCES

1. Seiford, L.M. and J. Zhu (2003), Context-dependent data envelopment analysis: measuring attractiveness and progress. *OMEGA*, Vol. 31, Issue 5, 397-480.
2. Simonson, I. and A. Tversky (1992), Choice in context: tradeoff contrast and extremeness aversion, *Journal of Marketing Research*, 29 281-295.
3. Tversky, A. and I. Simonson (1993), Context-dependent preferences. *Management Sciences*, 39 1179-1189.

Chapter 7

Benchmarking Models

7.1 Introduction

Gap analysis is often used as a fundamental method in performance evaluation and benchmarking. However, gap analysis only deals one measure at a time. It is rare that one single measure can suffice for the purpose of performance evaluation (Camp, 1995). As a result, some multi-factor based gap analysis methods have been developed. e.g., Spider charts, AHP maturity index, and Z charts. Although gaps can be identified with respect to individual performance measures, it remains a challenging task to combine the multiple measures in the final stage. Therefore, benchmarking models that can deal with multiple performance measures and provide an integrated benchmarking measure are needed.

Benchmarking is a process of defining valid measures of performance comparison among peer DMUs, using them to determine the relative positions of the peer DMUs and, ultimately, establishing a standard of excellence. In that sense, DEA can be regarded as a benchmarking tool, because the frontier identified can be regarded as an empirical standard of excellence.

Once the frontier is established, we may compare a set of new DMUs to the frontier. However, when a new DMU outperforms the identified frontier, a new frontier is generated by DEA. As a result, we do not have the same benchmark (frontier) for other (new) DMUs.

In the current chapter, we present a number of DEA-based benchmarking models where each (new) DMU is evaluated against a set of given benchmarks (standards).

7.2 Variable-benchmark Model

Cook, Seiford and Zhu (2004) develop a set of variable-benchmark model. Let E^* represent the set of benchmarks or the best-practice identified by the DEA. Based upon the input-oriented CRS envelopment model, we have

$$\begin{aligned}
 & \min \delta^{CRS} \\
 & \text{subject to} \\
 & \sum_{j \in E^*} \lambda_j x_{ij} \leq \delta^{CRS} x_i^{new} \\
 & \sum_{j \in E^*} \lambda_j y_{rj} \geq y_r^{new} \\
 & \lambda_j \geq 0, j \in E^*
 \end{aligned} \tag{7.1}$$

where a new observation is represented by DMU^{new} with inputs x_i^{new} ($i = 1, \dots, m$) and outputs y_r^{new} ($r = 1, \dots, s$). The superscript of CRS indicates that the benchmark frontier composed by benchmark DMUs in set E^* exhibits CRS.

Model (7.1) measures the performance of DMU^{new} with respect to benchmark DMUs in set E^* when outputs are fixed at their current levels. Similarly, based upon the output-oriented CRS envelopment model, we can have a model that measures the performance of DMU^{new} in terms of outputs when inputs are fixed at their current levels.

$$\begin{aligned}
 & \max \tau^{CRS} \\
 & \text{subject to} \\
 & \sum_{j \in E^*} \lambda_j x_{ij} \leq x_i^{new} \\
 & \sum_{j \in E^*} \lambda_j y_{rj} \geq \tau^{CRS} y_r^{new} \\
 & \lambda_j \geq 0, j \in E^*
 \end{aligned} \tag{7.2}$$

Theorem 7.1 $\delta^{CRS*} = 1/\tau^{CRS*}$, where δ^{CRS*} is the optimal value to model (7.1) and τ_o^{CRS*} is the optimal value to model (7.2).

[Proof]: Suppose λ_j^* ($j \in E^*$) is an optimal solution associated with δ^{CRS*} in model (7.1). Now, let $\tau^{CRS*} = 1/\delta^{CRS*}$, and $\lambda_j' = \lambda_j^* \delta_o^{CRS*}$. Then τ^{CRS*} and λ_j' are optimal in model (7.2). Thus, $\delta^{CRS*} = 1/\tau^{CRS*}$. ■

Model (7.1) or (7.2) yields a benchmark for DMU^{new} . The i th input and the r th output for the benchmark can be expressed as

$$\begin{cases} \sum_{j \in E^*} \lambda_j^* x_{ij} & (ith \text{ input}) \\ \sum_{j \in E^*} \lambda_j^* y_{rj} & (rth \text{ output}) \end{cases} \quad (7.3)$$

Note also that although the DMUs associated with set E^* are given, the resulting benchmark may be different for each new DMU under evaluation. Because for each new DMU under evaluation, (7.3) may represent a different combination of DMUs associated with set E^* . Thus, models (7.1) and (7.2) represent a variable-benchmark scenario.

Theorem 7.2

- (i) $\delta^{CRS^*} < 1$ or $\tau^{CRS^*} > 1$ indicates that the performance of DMU_o^{new} is dominated by the benchmark in (7.3).
- (ii) $\delta^{CRS^*} = 1$ or $\tau^{CRS^*} = 1$ indicates that DMU_o^{new} achieve the same performance level of the benchmark in (7.3).
- (iii) $\delta^{CRS^*} > 1$ or $\tau^{CRS^*} < 1$ indicates that input savings or output surpluses exist in DMU_o^{new} when compared to the benchmark in (7.3).

[Proof]: (i) and (ii) are obvious results in terms of DEA efficiency concept.

Now, $\delta^{CRS^*} > 1$ indicates that DMU_o^{new} can increase its inputs to reach the benchmark. This in turn indicates that $\delta^{CRS^*} - 1$ measures the input saving achieved by DMU_o^{new} . Similarly, $\tau^{CRS^*} < 1$ indicates that DMU_o^{new} can decrease its outputs to reach the benchmark. This in turn indicates that $1 - \tau^{CRS^*}$ measures the output surplus achieved by DMU_o^{new} . ■

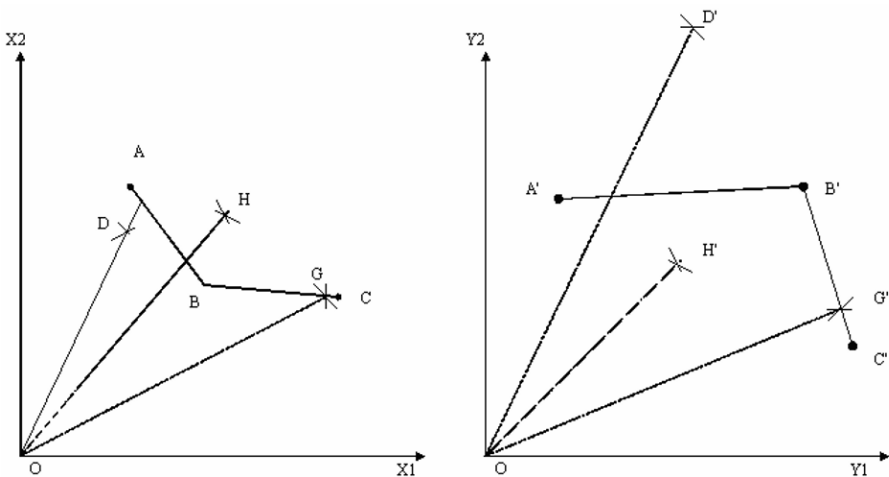


Figure 7.1. Variable-benchmark Model

Figure 7.1 illustrates the three cases described in Theorem 7.2. ABC (A'B'C') represents the input (output) benchmark frontier. D, H and G (or D', H', and G') represent the new DMUs to be benchmarked against ABC (or A'B'C'). We have $\delta_D^{CRS*} > 1$ for DMU D ($\tau_{D'}^{CRS*} < 1$ for DMU D') indicating that DMU D can increase its input values by δ_D^{CRS*} while producing the same amount of outputs generated by the benchmark (DMU D' can decrease its output levels while using the same amount of input levels consumed by the benchmark). Thus, $\delta_D^{CRS*} > 1$ is a measure of input savings achieved by DMU D and $\tau_{D'}^{CRS*} < 1$ is a measure of output surpluses achieved by DMU D'.

For DMU G and DMU G', we have $\delta_G^{CRS*} = 1$ and $\tau_{G'}^{CRS*} = 1$ indicating that they achieve the same performance level of the benchmark and no input savings or output surpluses exist. For DMU H and DMU H', we have $\delta_H^{CRS*} < 1$ and $\tau_{H'}^{CRS*} > 1$ indicating that inefficiency exists in the performance of these two DMUs.

Note that for example, in Figure 7.1, a convex combination of DMU A and DMU B is used as the benchmark for DMU D while a convex combination of DMU B and DMU C is used as the benchmark for DMU G. Thus, models (7.1) and (7.2) are called variable-benchmark models.

From Theorem 7.2, we can define $\delta^{CRS*} - 1$ or $1 - \tau^{CRS*}$ as the performance gap between DMU^{new} and the benchmark. Based upon δ^{CRS*} or τ^{CRS*} , a ranking of the benchmarking performance can be obtained.

It is likely that scale inefficiency may be allowed in the benchmarking. We therefore modify models (7.1) and (7.2) to incorporate scale inefficiency by assuming VRS.

$$\begin{aligned}
 & \min \delta^{VRS} \\
 & \text{subject to} \\
 & \sum_{j \in E^*} \lambda_j x_{ij} \leq \delta^{VRS} x_i^{new} \\
 & \sum_{j \in E^*} \lambda_j y_{rj} \geq y_r^{new} \\
 & \sum_{j \in E^*} \lambda_j = 1 \\
 & \lambda_j \geq 0, j \in E^*
 \end{aligned} \tag{7.4}$$

$$\begin{aligned}
 & \max \tau^{VRS} \\
 & \text{subject to} \\
 & \sum_{j \in E^*} \lambda_j x_{ij} \leq x_i^{new} \\
 & \sum_{j \in E^*} \lambda_j y_{rj} \geq \tau^{VRS} y_r^{new} \\
 & \sum_{j \in E^*} \lambda_j = 1 \\
 & \lambda_j \geq 0, j \in E^*
 \end{aligned} \tag{7.5}$$

Similar to Theorem 7.2, we have

Theorem 7.3

- (i) $\delta^{VRS^*} < 1$ or $\tau^{VRS^*} > 1$ indicates that the performance of DMU^{new} is dominated by the benchmark in (7.3).
- (ii) $\delta^{VRS^*} = 1$ or $\tau^{VRS^*} = 1$ indicates that DMU^{new} achieve the same performance level of the benchmark in (7.3).
- (iii) $\delta^{VRS^*} > 1$ or $\tau^{VRS^*} < 1$ indicates that input savings or output surpluses exist in DMU^{new} when compared to the benchmark in (7.3).

Note that model (7.2) is always feasible, and model (7.1) is infeasible only if certain patterns of zero data are present (Zhu 1996b). Thus, if we assume that all the data are positive, (7.1) is always feasible. However, unlike models (7.1) and (7.2), models (7.4) and (7.5) may be infeasible.

Theorem 7.4

- (i) If model (7.4) is infeasible, then the output vector of DMU^{new} dominates the output vector of the benchmark in (7.3).
- (ii) If model (7.5) is infeasible, then the input vector of DMU^{new} dominates the input vector of the benchmark in (7.3).

[Proof]: The proof follows directly from the necessary and sufficient conditions for infeasibility in super-efficiency DEA model provided in Seiford and Zhu (1999). ■

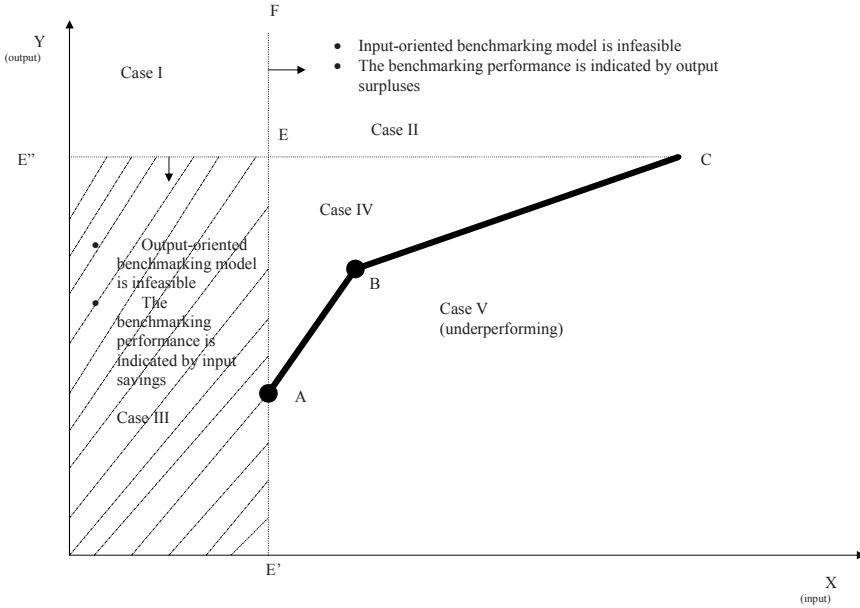


Figure 7.2. Infeasibility of VRS Variable-benchmark Model

The implication of the infeasibility associated with models (7.4) and (7.5) needs to be carefully examined. Consider Figure 7.2 where ABC represents the benchmark frontier. Models (7.4) and (7.5) yield finite optimal values for any DMU^{new} located below EC and to the right of EA. Model (7.4) is infeasible for DMU^{new} located above ray E"C and model (7.5) is infeasible for DMU^{new} located to the left of ray E'E.

Both models (7.4) and (7.5) are infeasible for DMU^{new} located above E"E and to the left of ray EF. Note that if DMU^{new} is located above E"C, its output value is greater than the output value of any convex combinations of A, B and C.

Note also that if DMU^{new} is located to the left of E'F, its input value is less than the input value of any convex combinations of A, B and C.

Based upon Theorem 7.4 and Figure 7.2, we have four cases:

Case I: When both models (7.4) and (7.5) are infeasible, this indicates that DMU^{new} has the smallest input level and the largest output level compared to the benchmark. Thus, both input savings and output surpluses exist in DMU^{new} .

Case II: When model (7.4) is infeasible and model (7.5) is feasible, the infeasibility of model (7.4) is caused by the fact that DMU^{new} has the largest output level compared to the benchmark. Thus, we use model (7.5) to characterize the output surpluses.

Case III: When model (7.5) is infeasible and model (7.4) is feasible, the infeasibility of model (7.5) is caused by the fact that DMU^{new} has the smallest input level compared to the benchmark. Thus, we use model (7.4) to characterize the input savings.

Case IV: When both models (7.4) and (7.5) are feasible, we use both of them to determine whether input savings and output surpluses exist.

If we change the constraint $\sum \lambda_j = 1$ to $\sum \lambda_j \leq 1$ and $\sum \lambda_j \geq 1$, then we obtain the NIRS and NDRS variable-benchmark models, respectively. Infeasibility may be associated with these two types of RTS frontiers, and we should apply the four cases discussed above. Table 7.1 summarizes the variable-benchmark models.

We next use 22 internet companies to illustrate the variable-benchmark models. Table 7.2 presents the data. We have four inputs: (1) number of website visitors (thousand), (2) number of employees (person), (3) marketing expenditure (\$ million), and (4) development expenditure (\$ million), and two outputs: (1) number of customers, and (2) revenue (\$ million).

Table 7.1. Variable-benchmark Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|--|--|
| | $\min \delta^{Frontier}$ | $\max \tau^{Frontier}$ |
| | subject to | subject to |
| CRS | $\sum_{j \in E^*} \lambda_j x_{ij} \leq \delta^{Frontier} x_i^{new}$ | $\sum_{j \in E^*} \lambda_j x_{ij} \leq x_i^{new}$ |
| | $\sum_{j \in E^*} \lambda_j y_{rj} \geq y_r^{new}$ | $\sum_{j \in E^*} \lambda_j y_{rj} \geq \tau^{Frontier} y_r^{new}$ |
| | $\lambda_j \geq 0, j \in E^*$ | $\lambda_j \geq 0, j \in E^*$ |
| VRS | | Add $\sum \lambda_j = 1$ |
| NIRS | | Add $\sum \lambda_j \leq 1$ |
| NDRS | | Add $\sum \lambda_j \geq 1$ |

Table 7.2. Data for the Internet Companies

| Company | Visitors | Employee | Marketing | Development | Customers | Revenue |
|----------------|----------|----------|-----------|-------------|-----------|---------|
| Barnes&Noble | 64812 | 1237 | 111.55 | 21.01 | 4700000 | 202.57 |
| Amazon.com | 177744 | 7600 | 413.2 | 159.7 | 16900000 | 1640 |
| CDnow | 79848 | 502 | 89.73 | 23.42 | 3260000 | 147.19 |
| eBay | 168384 | 300 | 95.96 | 23.79 | 10010000 | 224.7 |
| 1-800-Flowers | 11940 | 2100 | 92.15 | 8.07 | 7800000 | 52.89 |
| Buy.com | 27372 | 255 | 71.3 | 7.84 | 1950000 | 596.9 |
| FTD.com | 11856 | 75 | 29.93 | 5.29 | 1800000 | 62.6 |
| Autobytel.com | 12000 | 225 | 44.18 | 14.26 | 2065000 | 40.3 |
| Beyond.com | 17076 | 250 | 81.35 | 10.39 | 2000000 | 117.28 |
| eToys | 13896 | 940 | 120.46 | 43.43 | 1900000 | 151.04 |
| E*Trade | 29532 | 2400 | 301.7 | 78.5 | 1551000 | 621.4 |
| Garden.com | 16344 | 290 | 16 | 4.8 | 1070000 | 8.2 |
| Drugstore.com | 19092 | 408 | 61.5 | 14.9 | 695000 | 34.8 |
| Outpost.com | 7716 | 164 | 41.67 | 7 | 627000 | 188.6 |
| iPrint | 42132 | 225 | 8.13 | 3.54 | 380000 | 3.26 |
| Furniture.com | 10668 | 213 | 33.949 | 6.685 | 260000 | 10.904 |
| PlanetRX.com | 17124 | 390 | 55.18 | 12.95 | 254000 | 8.99 |
| NextCard | 46836 | 365 | 24.65 | 22.05 | 220000 | 26.56 |
| PetsMart.com | 18564 | 72 | 33.47 | 2.43 | 180000 | 10.45 |
| Peapod | 2076 | 1020 | 7.17 | 3.54 | 111900 | 73.13 |
| Webvan | 1680 | 1000 | 11.75 | 15.24 | 47000 | 13.31 |
| CarsDirect.com | 15612 | 702 | 33.43 | 2.14 | 12885 | 98.56 |

Suppose we select the first seven companies (Barnes & Noble, Amazon.com, CDnow, eBay, 1-800-Flowers, Buy.com, and FTD.com) as the benchmarks. If we apply the output-oriented CRS envelopment model to the seven companies, the top three companies (Barnes & Noble, Amazon.com, and CDnow) are not on the best-practice frontier, and therefore can be excluded. However, if we include them in the benchmark set, the benchmarking results will not be affected. Because λ_j^* related to the three companies must be equal to zero.

The spreadsheet model of the variable-benchmark models is very similar to the context-dependent DEA spreadsheet model. In fact, the evaluation background now is the selected benchmarks. Figure 7.3 shows the spreadsheet model for the output-oriented CRS variable-benchmark model where the benchmarks (evaluation background) are entered in rows 2-8.

Cell F2 is reserved to indicate the DMU under benchmarking. Cell F4 is the target cell which represent the τ_o^{CRS} in model (7.2). Cells I2:I8 represent

the λ_j for the benchmarks. Cell B9 contains the formula “=SUMPRODUCT(B2:B8,\$I\$2:\$I\$8)”. This formula is then copied into cells C9:E9. Cell G9 contains the formula “=SUMPRODUCT(G2:G8,\$I\$2:\$I\$8)”. This formula is then copied into cell H9.

| | A | B | C | D | E | F | G | H | I |
|----|-----------------------------|--------------------|-------------------|-----------------|-------------|--------------------|------------------|-------------------|--------------|
| 1 | Company | Visitors | Employee | Marketing | Development | | Customers | Revenue | λ |
| 2 | Barnes&Noble | 64812 | 1237 | 111.55 | 21.01 | 15 | 4700000 | 202.57 | 0 |
| 3 | Amazon.com | 177744 | 7800 | 413.2 | 159.7 | Score | 16900000 | 1640 | 0 |
| 4 | Cdnaw | 79848 | 502 | 89.73 | 23.42 | 1.653097978 | 3260000 | 147.19 | 0 |
| 5 | eBay | 168384 | 300 | 95.96 | 23.79 | | 10010000 | 224.7 | 0 |
| 6 | 1-800-Flowers | 11940 | 2100 | 92.15 | 8.07 | | 7800000 | 52.89 | 0 |
| 7 | Buy.com | 27372 | 255 | 71.3 | 7.84 | | 1950000 | 596.9 | 0.272959184 |
| 8 | FTD.com | 11856 | 75 | 29.93 | 5.29 | | 1800000 | 62.6 | 0 |
| 9 | Benchmarks | 7471.438776 | 69.6045918 | 19.46199 | 2.14 | | 532270.41 | 162.929337 | |
| 10 | | λ^A | λ^A | λ^A | λ^A | | v^I | v^I | Benchmarking |
| 11 | DMU under evaluation | 15612 | 702 | 33.43 | 2.14 | | 21300.167 | 162.929337 | Score |
| 12 | Autobytel.com | 12000 | 225 | 44.18 | 14.26 | | 2065000 | 40.3 | 1.095779422 |
| 13 | Beyond.com | 17076 | 250 | 81.35 | 10.39 | | 2000000 | 117.28 | 1.327240986 |
| 14 | eToys | 13896 | 940 | 120.46 | 43.43 | | 1900000 | 151.04 | 1.600761668 |
| 15 | E*Trade | 29532 | 2400 | 301.7 | 78.5 | Variable Benchmark | 1551000 | 621.4 | 1.036374356 |
| 16 | Garden.com | 16344 | 290 | 16 | 4.8 | | 1070000 | 8.2 | 1.42713759 |
| 17 | Drugstore.com | 18092 | 408 | 61.5 | 14.9 | | 895000 | 34.8 | 4.852307242 |
| 18 | Outpost.com | 7716 | 164 | 41.67 | 7 | | 627000 | 188.8 | 0.880769639 |
| 19 | iPrint | 42132 | 225 | 8.13 | 3.54 | | 380000 | 3.26 | 2.231776947 |
| 20 | Furniture.com | 10668 | 213 | 33.949 | 6.885 | | 260000 | 10.904 | 7.369719683 |
| 21 | PlanetRX.com | 17124 | 390 | 55.18 | 12.95 | | 2540000 | 8.99 | 12.93451824 |
| 22 | NextCard | 46836 | 365 | 24.65 | 22.05 | | 220000 | 26.56 | 5.917828607 |
| 23 | PetsMart.com | 18564 | 72 | 33.47 | 2.43 | | 180000 | 10.45 | 5.549892175 |
| 24 | Peapod | 2076 | 1020 | 7.17 | 3.54 | | 111900 | 73.13 | 0.619051561 |
| 25 | Webvan | 1680 | 1000 | 11.75 | 15.24 | | 47000 | 13.31 | 2.732844609 |
| 26 | CarsDirect.com | 15612 | 702 | 33.43 | 2.14 | | 12885 | 98.56 | 1.653097978 |

Figure 7.3. Output-oriented CRS Variable-benchmark Spreadsheet Model

Cells B11:E11, and Cells G11:H11 contain the formulas for the DMU under benchmarking – the right-hand-side of model (7.2). The formula for B11 is “=INDEX(B12:B26,\$F\$2,1)”, and is copied into cells C11:E11. The formula for cell G11 is “=\$F\$4*INDEX(G12:G26,\$F\$2,1)”, and is copied into cell H11.

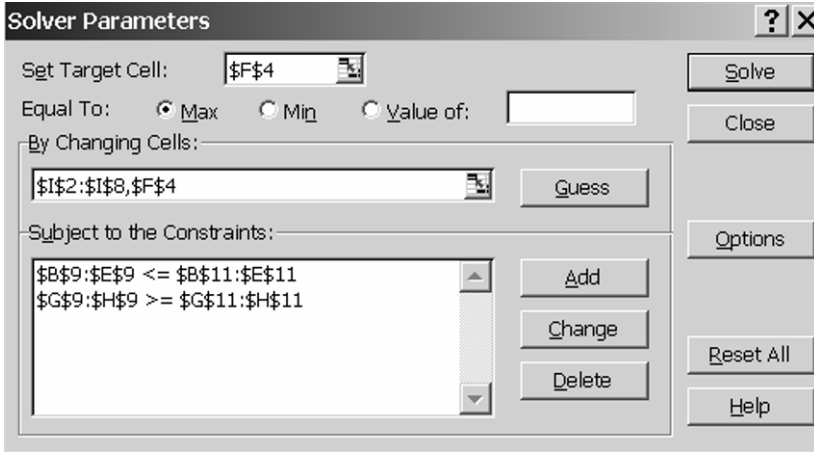


Figure 7.4. Solver Parameters for Output-oriented CRS Variable-benchmark Model

Figure 7.4 shows the Solver parameters for the spreadsheet model shown in Figure 7.3. A VBA procedure “VariableBenchmark” is used to record the benchmarking scores into cells I12:I26.

```
Sub VariableBenchmark()
Dim i As Integer
For i = 1 To 15
Range("F2") = i
SolverSolve UserFinish:=True
Range("I" & i + 11) = Range("F4")
Next
End Sub
```

Because the model in Figure 7.3 is an output-oriented model, a smaller score (τ^{CRS^*}) indicates a better performance. Thus, Peapod is the best company with respect to the specified benchmarks. The non-zero optimal λ_j^* indicates the actual benchmark for a company under benchmarking. For example, Buy.com is used as the actual benchmark for CarsDirect.com (see cell I7 in Figure 7.3).

If we use the input-oriented CRS variable-benchmark model, we need change the formula for cell B11 in Figure 7.3 to “=SF\$4*INDEX(B12:B26,SF\$2,1)”. This formula is then copied into cells C11:E11. The formula for cell G11 is changed to “=INDEX(G12:G26,SF\$2,1)” and is copied into cell H11. All the other formulas in Figure 7.3 remain unchanged.

We also need to change the Solver parameters shown in Figure 7.4 by selecting “Min”, as shown in Figure 7.5. Figure 7.6 shows the spreadsheet

model for the input-oriented CRS variable-benchmark model and the benchmarking scores. It can be seen that Theorem 7.1 is true.

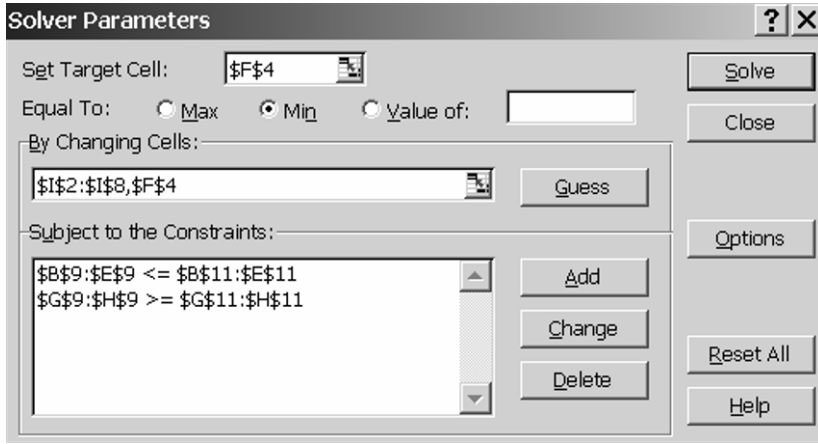


Figure 7.5. Solver Parameters for Input-oriented CRS Variable-benchmark Model

| | A | B | C | D | E | F | G | H | I |
|----|-----------------------------|-------------|------------|-----------|-------------|--------------------|-----------|---------|----------------|
| 1 | Company | Visitors | Employee | Marketing | Development | | Customers | Revenue | λ |
| 2 | Barnes&Noble | 64812 | 1237 | 111.55 | 21.01 | 15 | 4700000 | 202.57 | 0 |
| 3 | Amazon.com | 177744 | 7800 | 413.2 | 158.7 | Score | 16900000 | 1640 | 0 |
| 4 | Cdnw | 79848 | 502 | 89.73 | 23.42 | 0.604924822 | 3260000 | 147.19 | 0 |
| 5 | eBay | 168384 | 300 | 95.96 | 23.79 | | 10010000 | 224.7 | 0 |
| 6 | 1-800-Flowers | 11940 | 2100 | 92.15 | 8.07 | | 7800000 | 52.89 | 0 |
| 7 | Buy.com | 27372 | 255 | 71.3 | 7.84 | | 19500000 | 596.9 | 0.165119786 |
| 8 | FTD.com | 11856 | 75 | 29.93 | 5.29 | | 1800000 | 62.6 | 0 |
| 9 | Benchmarks | 4519.85877 | 42.1055453 | 11.77304 | 1.294539119 | | 321983.58 | 98.56 | input-oriented |
| 10 | | λ | λ | λ | λ | | v_i | v_j | Benchmarking |
| 11 | DMU under evaluation | 9444.086319 | 424.657225 | 20.22264 | 1.294539119 | | 12885 | 98.56 | Score |
| 12 | Autobytel.com | 12000 | 225 | 44.18 | 14.26 | | 2065000 | 40.3 | 0.912592425 |
| 13 | Beyond.com | 17076 | 250 | 81.35 | 10.39 | | 2000000 | 117.28 | 0.753442676 |
| 14 | eToys | 13896 | 940 | 120.46 | 43.43 | | 1900000 | 151.04 | 0.624702615 |
| 15 | E*Trade | 29532 | 2400 | 301.7 | 78.5 | Variable Benchmark | 1551000 | 621.4 | 0.9649023 |
| 16 | Garden.com | 16344 | 290 | 16 | 4.8 | | 1070000 | 8.2 | 0.700703287 |
| 17 | Drugstore.com | 19092 | 408 | 61.5 | 14.9 | | 695000 | 34.8 | 0.206087527 |
| 18 | Outpost.com | 7716 | 184 | 41.67 | 7 | | 627000 | 188.6 | 1.122624702 |
| 19 | iPrint | 42132 | 225 | 8.13 | 3.54 | | 380000 | 3.26 | 0.448073452 |
| 20 | Furniture.com | 10668 | 213 | 33.949 | 6.885 | | 260000 | 10.904 | 0.135690371 |
| 21 | PlanetFX.com | 17124 | 390 | 55.18 | 12.95 | | 254000 | 8.99 | 0.077312505 |
| 22 | NextCard | 46836 | 365 | 24.65 | 22.05 | | 220000 | 26.56 | 0.168980899 |
| 23 | PetsMart.com | 18564 | 72 | 33.47 | 2.43 | | 180000 | 10.45 | 0.180183681 |
| 24 | Peapod | 2076 | 1020 | 7.17 | 3.54 | | 111900 | 73.13 | 1.615374328 |
| 25 | Webvan | 1680 | 1000 | 11.75 | 15.24 | | 47000 | 13.31 | 0.365919085 |
| 26 | CarsDirect.com | 15612 | 702 | 33.43 | 2.14 | | 12885 | 98.56 | 0.604924822 |

Figure 7.6. Input-oriented CRS Variable-benchmark Spreadsheet Model

We now consider the input-oriented VRS variable-benchmark model. We need to add a cell representing $\sum \lambda_j$ in the spreadsheet shown in Figure 7.6. We select cell I9, and enter the formula “=SUM(I2:I8)”. We also need to add an additional constraint on $\sum \lambda_j = 1$ in the Solver parameters shown in Figure 7.5. This constraint is “\$I\$9 = 1”, as shown in Figure 7.7.

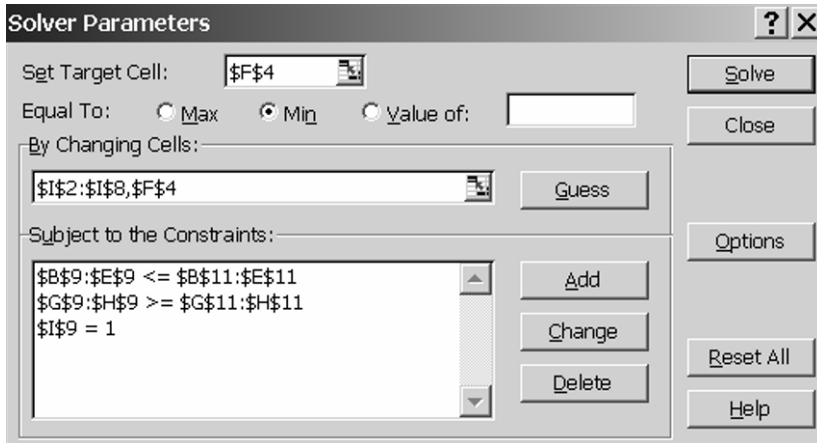


Figure 7.7. Solver Parameters for Input-oriented VRS Variable-benchmark Model

| | A | B | C | D | E | F | G | H | I |
|----|-----------------------------|--------------------|-------------------|-----------------|--------------------|------------------------|------------------|--------------|--------------|
| 1 | Company | Visitors | Employee | Marketing | Development | | Customers | Revenue | λ |
| 2 | Barnes&Noble | 64812 | 1237 | 111.55 | 21.01 | 15 | 4700000 | 202.57 | 0 |
| 3 | Amazon.com | 177744 | 7600 | 413.2 | 159.7 | Score | 16900000 | 1640 | 0 |
| 4 | Cdnw | 79848 | 502 | 89.73 | 23.42 | 2.552160133 | 3260000 | 147.19 | 0 |
| 5 | eBay | 168394 | 300 | 95.96 | 23.79 | | 10010000 | 224.7 | 0 |
| 6 | 1-800-Flowers | 11940 | 2100 | 92.15 | 8.07 | | 7800000 | 52.89 | 0 |
| 7 | Buy.com | 27372 | 255 | 71.3 | 7.84 | | 1950000 | 596.9 | 0.067303013 |
| 8 | FTD.com | 11856 | 75 | 29.93 | 5.29 | | 1800000 | 62.6 | 0.932696987 |
| 9 | Benchmarks | 12900.27355 | 87.1145424 | 32.71433 | 5.461622684 | | 1810095.5 | 98.56 | 1 |
| 10 | | IA | IA | IA | IA | | VI | VI | |
| 11 | DMU under evaluation | 39844.32399 | 1791.61841 | 85.31871 | 5.461622684 | | 12885 | 98.56 | Score |
| 12 | Autobytel.com | 12000 | 225 | 44.18 | 14.26 | | 2065000 | 40.3 | 0.988309167 |
| 13 | Beyond.com | 17076 | 250 | 81.35 | 10.39 | | 2000000 | 117.28 | 0.787957227 |
| 14 | eToys | 13896 | 940 | 120.46 | 43.43 | | 1900000 | 151.04 | 1.038346917 |
| 15 | E*Trade | 29532 | 2400 | 301.7 | 78.5 | VRS Variable Benchmark | 1551000 | 621.4 | 0.95011279 |
| 16 | Garden.com | 16344 | 290 | 16 | 4.8 | | 1070000 | 8.2 | Infeasible |
| 17 | Drugstore.com | 19092 | 408 | 61.5 | 14.9 | | 695000 | 34.8 | Infeasible |
| 18 | Outpost.com | 7716 | 164 | 41.67 | 7 | | 627000 | 188.6 | Infeasible |
| 19 | iPrint | 42132 | 225 | 8.13 | 3.54 | | 380000 | 3.26 | Infeasible |
| 20 | Furniture.com | 10668 | 213 | 33.949 | 6.685 | | 280000 | 10.904 | Infeasible |
| 21 | PlanetRX.com | 17124 | 390 | 55.18 | 12.95 | | 254000 | 8.99 | Infeasible |
| 22 | NextCard | 46836 | 365 | 24.65 | 22.05 | | 220000 | 26.56 | Infeasible |
| 23 | PetsMart.com | 18564 | 72 | 33.47 | 2.43 | | 180000 | 10.45 | Infeasible |
| 24 | Peapod | 2076 | 1020 | 7.17 | 3.54 | | 111900 | 73.13 | 5.858280241 |
| 25 | Webvan | 1680 | 1000 | 11.75 | 15.24 | | 47000 | 13.31 | Infeasible |
| 26 | CarsDirect.com | 15612 | 702 | 33.43 | 2.14 | | 12885 | 98.56 | 2.552160133 |

Figure 7.8. Input-oriented VRS Variable-benchmark Spreadsheet Model

Figure 7.8 shows the spreadsheet for the input-oriented VRS variable-benchmark model and the benchmarking scores in cells I12:I26. The button “VRS Variable Benchmark” is linked to the VBA procedure “VRSVariableBenchmark”.

```
Sub VRSVariableBenchmark ()
Dim i As Integer
For i = 1 To 15
```

```

Range("F2") = i
SolverSolve UserFinish:=True
If SolverSolve(UserFinish:=True) = 5 Then
Range("I" & i + 11) = "Infeasible"
Else
Range("I" & i + 11) = Range("F4")
End If
Next
End Sub

```

Because of the VRS frontier, the model may be infeasible. The SolverSolve function returns an integer value that indicates Solver's "success". If this value is 5, it means that there are no feasible solutions. This is represented by the statement "SolverSolve(UserFinish:=True) = 5". In the procedure, if the Solver returns a value of 5, then the procedure records "infeasible". Otherwise, the procedure records the optimal value in cell F4 of Figure 7.8.

7.3 Fixed-benchmark Model

Although the benchmark frontier is given in the variable-benchmark models, a DMU^{new} under benchmarking has the freedom to choose a subset of benchmarks so that the performance of DMU^{new} can be characterized in the most favorable light. Situations when the same benchmark should be fixed are likely to occur. For example, the management may indicate that DMUs A and B in Figure 7.1 should be used as the fixed benchmark. i.e., DMU C in Figure 7.1 may not be used in constructing the benchmark.

To couple with this situation, Cook, Seiford and Zhu (2004) turn to the multiplier models. For example, the input-oriented CRS multiplier model determines a set of referent best-practice DMUs represented by a set of binding constraints in optimality. Let set $B = \{DMU_j : j \in I_B\}$ be the selected subset of benchmark set E^* . i.e., $I_B \subset E^*$. Based upon the input-oriented CRS multiplier model, we have

$$\begin{aligned}
\tilde{\sigma}^{CRS*} &= \max \sum_{r=1}^s \mu_r y_r^{new} \\
\text{subject to} & \\
\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^s \nu_i x_{ij} &= 0 \quad j \in I_B \\
\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^s \nu_i x_{ij} &\leq 0 \quad j \notin I_B \\
\sum_{i=1}^m \nu_i x_i^{new} &= 1 \\
\mu_r, \nu_i &\geq 0.
\end{aligned} \tag{7.6}$$

By applying equalities in the constraints associated with benchmark DMUs, model (7.6) measures DMU^{new} 's performance against the benchmark constructed by set \mathbf{B} . At optimality, some $DMU_j, j \notin \mathbf{I}_B$, may join the fixed-benchmark set if the associated constraints are binding.

Note that model (7.6) may be infeasible. For example, the DMUs in set \mathbf{B} may not be fit into the same facet when they number greater than $m+s-1$, where m is the number of inputs and s is the number of outputs. In this case, we need to adjust the set \mathbf{B} .

Three possible cases are associated with model (7.6). $\tilde{\sigma}^{CRS*} > 1$ indicating that DMU^{new} outperforms the benchmark. $\tilde{\sigma}^{CRS*} = 1$ indicating that DMU^{new} achieves the same performance level of the benchmark. $\tilde{\sigma}^{CRS*} < 1$ indicating that the benchmark outperforms DMU^{new} .

By applying RTS frontier type and model orientation, we obtain the fixed-benchmark models in Table 7.3

Table 7.3. Fixed-benchmark Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|---|---|
| | $\max \sum_{r=1}^s \mu_r y_r^{new} + \mu$ subject to $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + \mu = 0 \quad j \in \mathbf{I}_B$ $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + \mu \leq 0 \quad j \notin \mathbf{I}_B$ $\sum_{i=1}^m v_i x_i^{new} = 1$ $\mu_r, v_i \geq 0$ | $\min \sum_{i=1}^m v_i x_i^{new} + \nu$ subject to $\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu = 0 \quad j \in \mathbf{I}_B$ $\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu \geq 0 \quad j \notin \mathbf{I}_B$ $\sum_{r=1}^s \mu_r y_r^{new} = 1$ $\mu_r, v_i \geq 0$ |
| CRS | where $\mu = 0$ | where $\nu = 0$ |
| VRS | where μ free | where ν free |
| NIRS | where $\mu \leq 0$ | where $\nu \geq 0$ |
| NDRS | where $\mu \geq 0$ | where $\nu \leq 0$ |

DMU^{new} is not included in the constraints of $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + \mu \leq 0$ ($j \notin \mathbf{I}_B$) ($\sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu \geq 0$ ($j \notin \mathbf{I}_B$)). However, other peer DMUs ($j \notin \mathbf{I}_B$) are included.

Figure 7.9 shows the output-oriented CRS fixed-benchmark spreadsheet model where 1-800-Flowers and Buy.com are two fixed benchmarks. Cells B5:E5 and G5:H5 are reserved for input and output multipliers, respectively. They are the changing cells in the Solver parameters.

Cell C7 is the target cell and contains the formula “=SUMPRODUCT (B5:E5,INDEX(B10:E24,C6,0))”, where cell C6 indicates the DMU under evaluation – Autobotel.com.

Cell C8 contains the formula representing $\sum_{r=1}^s \mu_r y_r^{new}$

Cell C8=SUMPRODUCT(G5:H5,INDEX (G10:H24,C6,0))

The formula for cell I2 is “=SUMPRODUCT(B2:E2,\$B\$5:\$E\$5)-SUMPRODUCT(G2:H2,\$G\$5:\$H\$5)”, and is copied into cells I3 and I10:I24.

| | A | B | C | D | E | F | G | H | I |
|----|----------------------|----------|----------|-----------|-------------|---|-----------|---------|-------------|
| 1 | Company | Visitors | Employee | Marketing | Development | | Customers | Revenue | Constraints |
| 2 | 1-800-Flowers | 11940 | 2100 | 92.15 | 8.07 | | 7800000 | 52.89 | 4.885E-15 |
| 3 | Buy.com | 27372 | 255 | 71.3 | 7.84 | | 1950000 | 596.9 | 4.5519E-15 |
| 4 | | | | | | | | | |
| 5 | Multipliers | 3.4E-06 | 0.001489 | 0.006619 | 0 | | 4.843E-07 | 0 | |
| 6 | DMU under evaluation | | 1 | | | | | | |
| 7 | Score | | 0.668082 | | | | | | |
| 8 | Weighted output | | 1 | | | | | | |
| 9 | | | | | | | | | |
| 10 | Autobytel.com | 12000 | 225 | 44.18 | 14.26 | | 2065000 | 40.3 | -0.3319178 |
| 11 | Beyond.com | 17076 | 250 | 81.35 | 10.39 | | 2000000 | 117.90 | -0.5479E-15 |
| 12 | eToys | 13896 | 940 | 120.46 | 43.43 | | 1900000 | | 0.32392744 |
| 13 | E*Trade | 29532 | 2400 | 301.7 | 78.5 | | 1551000 | | 0.91942618 |
| 14 | Garden.com | 16344 | 290 | 16 | 4.8 | | 1070000 | 8.2 | 0.07489194 |
| 15 | Drugstore.com | 19092 | 408 | 61.5 | 14.9 | | 695000 | 34.8 | 0.74265995 |
| 16 | Outpost.com | 7716 | 164 | 41.67 | 7 | | 627000 | 188.6 | 0.24250252 |
| 17 | iPrint | 42132 | 225 | 8.13 | 3.54 | | 380000 | 3.26 | 0.34747683 |
| 18 | Furniture.com | 10668 | 213 | 33.949 | 6.685 | | 260000 | 10.904 | 0.45207709 |
| 19 | PlanetRX.com | 17124 | 390 | 55.18 | 12.95 | | 254000 | 8.99 | 0.88092163 |
| 20 | NextCard | 46836 | 365 | 24.65 | 22.05 | | 220000 | 26.56 | 0.75869065 |
| 21 | PetsMart.com | 18564 | 72 | 33.47 | 2.43 | | 180000 | 10.45 | 0.30443706 |
| 22 | Peapod | 2076 | 1020 | 7.17 | 3.54 | | 111900 | 73.13 | 1.51906364 |
| 23 | Webvan | 1680 | 1000 | 11.75 | 15.24 | | 47000 | 13.31 | 1.54968667 |
| 24 | CarsDirect.com | 15612 | 702 | 33.43 | 2.14 | | 12885 | 98.56 | 1.31316454 |

Figure 7.9. Output-oriented CRS Fixed-benchmark Spreadsheet Model

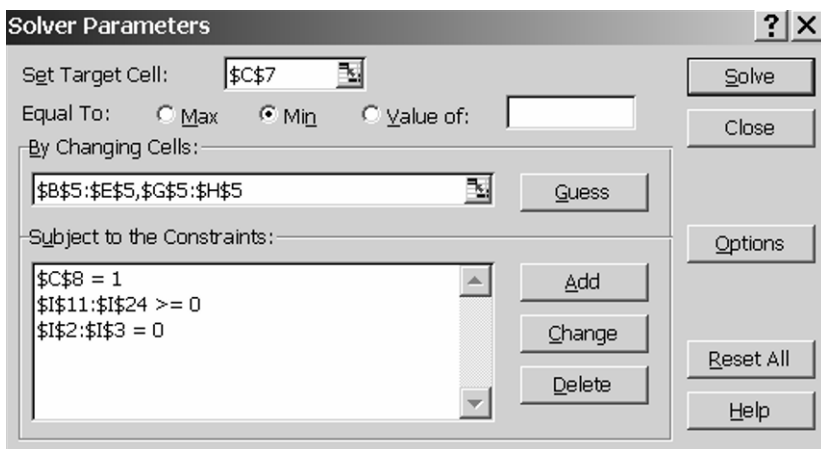


Figure 7.10. Solver Parameters for Output-oriented CRS Fixed-benchmark Model

Figure 7.10 shows the Solver parameters for Autobytel.com. Note that we have “\$I\$2:\$I\$3 = 0” for the two benchmarks. Note also that “\$I\$11:\$I\$24 >=0” does not include the DMU under evaluation, Autobytel.com.

To solve the remaining DMUs, we need to set up different Solver parameters. Because the constraints change for each DMU under evaluation. For example, if we change the value of cell C6 to 15, i.e., we benchmark CarsDirect.com, we obtain a set of new Solver parameters by removing “\$I\$24>=0” from the Solver parameters shown in Figure 7.10 and then adding “\$I\$10>=0”, as shown in Figure 7.11.

Because different Solver parameters are used for different DMUs under benchmarking, a set of sophisticated VBA codes is required to automate the calculation. We here do not discuss it, and suggest using the “DEA Excel Solver” – a DEA Add-In for Microsoft Excel described in Chapter 12 to obtain the scores (see cells J10:J24 in Figure 7.11).

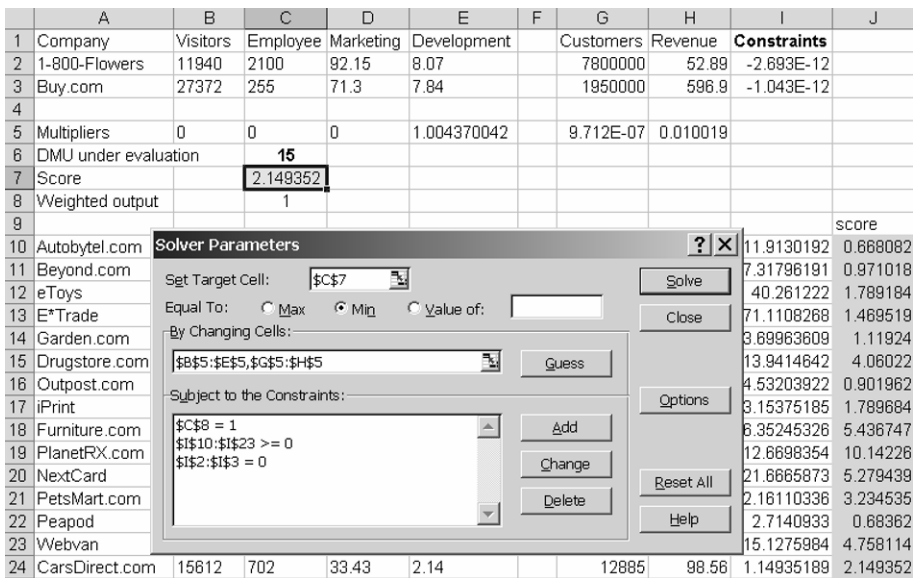


Figure 7.11. Output-oriented CRS Fixed-benchmark Scores for Internet Companies

7.4 Fixed-benchmark Model and Efficiency Ratio

A commonly used measure of efficiency is the ratio of output to input. For example, profit per employee measures the labor productivity. When multiple inputs and outputs are present, we may define the following efficiency ratio

$$\frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}}$$

where v_i and u_r represent the input and output weights, respectively.

DEA calculate the ratio efficiency without the information on the weights. In fact, the multiplier DEA models can be transformed into linear fractional programming problems. For example, if we define $v_i = t v_i$ and $\mu_r = t u_r$, where $t = 1/\sum v_i x_{io}$, the input-oriented CRS multiplier model can be transformed into

$$\begin{aligned} & \max \frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}} \\ & \text{subject to} \\ & \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j = 1, 2, \dots, n \\ & u_r, v_i \geq 0 \quad \forall r, i \end{aligned} \tag{7.7}$$

The objective function in (7.7) represents the efficiency ratio of a DMU under evaluation. Because of the constraints in (7.7), the (maximum) efficiency cannot exceed one. Consequently, a DMU with an efficiency score of one is on the frontier. It can be seen that no additional information on the weights or tradeoffs are incorporated into the model (7.7).

If we apply the input-oriented CRS fixed-benchmark model to (7.7), we obtain

$$\begin{aligned} & \max \frac{\sum_{r=1}^s u_r y_r^{new}}{\sum_{i=1}^m v_i x_i^{new}} \\ & \text{subject to} \\ & \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} = 1 \quad j \in \mathbf{I}_B \\ & \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j \notin \mathbf{I}_B \\ & u_r, v_i \geq 0 \quad \forall r, i \end{aligned} \tag{7.8}$$

It can be seen from (7.8) that the fixed benchmarks incorporate implicit tradeoff information into the efficiency evaluation. i.e., the constraints associated with I_B can be viewed as incorporation of tradeoffs or weight restrictions in DEA. Model (7.8) yields the (maximum) efficiency under the implicit tradeoff information represented by the benchmarks.

As more DMUs are selected as fixed benchmarks, more complete information on the weights becomes available. For example, if we add FTD.com to the fixed-benchmark set, the benchmarking score for Autobytel.com becomes 1.1395, as shown in Figure 7.12. As expected, the performance of those internet companies becomes worse when the set of fixed benchmarks expands.

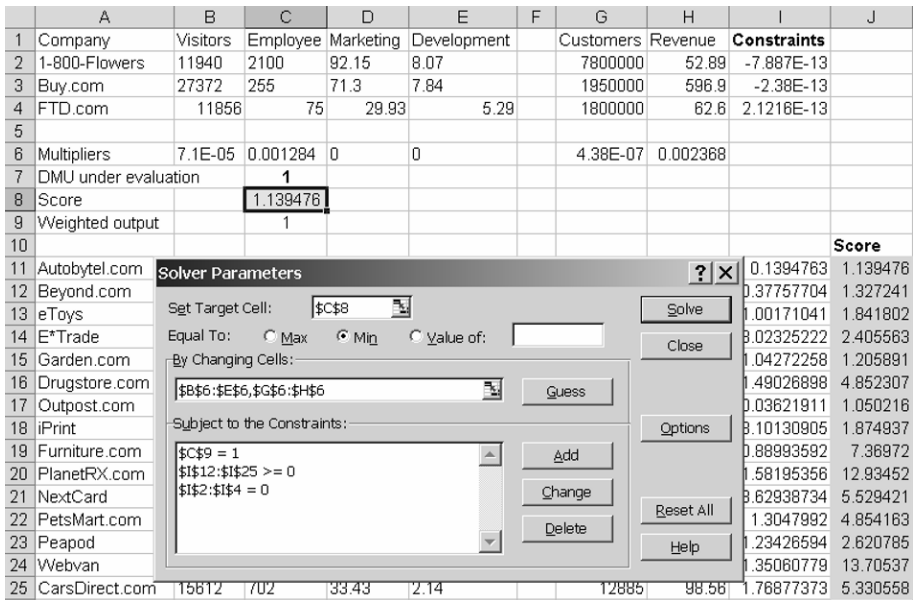


Figure 7.12. Spreadsheet Model and Solver Parameters for Fixed-benchmark Model

Similarly, the output-oriented CRS fixed-benchmark model is equivalent to

$$\min \frac{\sum_{i=1}^m v_i x_i^{new}}{\sum_{r=1}^s u_r y_r^{new}}$$

subject to

$$\frac{\sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} = 1 \quad j \in I_B$$

$$\frac{\sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} \geq 1 \quad j \notin \mathbf{I}_B$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

Note that we may define an ideal benchmark whose r th output y_r^{ideal} is the maximum output value across all DMUs, and i th input x_i^{ideal} the minimum input value across all DMUs. If we replace the fixed-benchmark set by the ideal benchmark, we have

$$\max \frac{\sum_{r=1}^s u_r y_r^{new}}{\sum_{i=1}^m v_i x_i^{new}}$$

subject to

$$\frac{\sum_{r=1}^s u_r y_r^{ideal}}{\sum_{i=1}^m v_i x_i^{ideal}} = 1 \tag{7.9}$$

$$\frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

Because the ideal benchmark dominates all DMUs (unless DMU_j is one of the ideal benchmark), the optimal value to (7.9) must not be greater than one. Further, $\sum u_r y_{rj} / \sum v_i x_{ij} \leq 1$ are redundant, and model (7.9) can be simplified as

$$\max \frac{\sum_{r=1}^s u_r y_r^{new}}{\sum_{i=1}^m v_i x_i^{new}}$$

subject to

$$\frac{\sum_{r=1}^s u_r y_r^{ideal}}{\sum_{i=1}^m v_i x_i^{ideal}} = 1 \tag{7.10}$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

Model (7.10) is equivalent to the following linear programming problem

$$\begin{aligned}
& \max \sum_{r=1}^s \mu_r y_r^{new} \\
& \text{subject to} \\
& \sum_{r=1}^s \mu_r y_r^{ideal} - \sum_{i=1}^s \nu_i x_i^{ideal} = 0 \\
& \sum_{i=1}^m \nu_i x_i^{new} = 1 \\
& \mu_r, \nu_i \geq 0.
\end{aligned} \tag{7.11}$$

Model (7.10) or (7.11) calculate the maximum efficiency of a specific DMU under evaluation given that the efficiency of the ideal benchmark is set equal to one. If we introduce RTS frontier type and model orientation into (7.10), we obtain other ideal-benchmark models, as shown in Table 7.4.

Table 7.4. Ideal-benchmark Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|---|--|
| | $ \begin{aligned} & \max \sum_{r=1}^s \mu_r y_r^{new} + \mu \\ & \text{subject to} \\ & \sum_{r=1}^s \mu_r y_r^{ideal} - \sum_{i=1}^s \nu_i x_i^{ideal} + \mu = 0 \\ & \sum_{i=1}^m \nu_i x_i^{new} = 1 \\ & \mu_r, \nu_i \geq 0. \end{aligned} $ | $ \begin{aligned} & \min \sum_{i=1}^m \nu_i x_i^{new} + \nu \\ & \text{subject to} \\ & \sum_{i=1}^s \nu_i x_i^{ideal} - \sum_{r=1}^s \mu_r y_r^{ideal} + \nu = 0 \\ & \sum_{r=1}^s \mu_r y_r^{new} = 1 \\ & \mu_r, \nu_i \geq 0 \end{aligned} $ |
| CRS | where $\mu = 0$ | where $\nu = 0$ |
| VRS | where μ free | where ν free |
| NIRS | where $\mu \leq 0$ | where $\nu \geq 0$ |
| NDRS | where $\mu \geq 0$ | where $\nu \leq 0$ |

7.5 Minimum Efficiency Model

Note that the fixed-benchmark models yield the maximum efficiency scores when the tradeoffs are implicitly defined by the benchmarks. If we change the objective function of model (7.8) into minimization, we have

$$\begin{aligned}
& \min \frac{\sum_{r=1}^s u_r y_r^{new}}{\sum_{i=1}^m \nu_i x_i^{new}} \\
& \text{subject to} \\
& \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m \nu_i x_{ij}} = 1 \quad j \in \mathbf{I}_B
\end{aligned} \tag{7.12}$$

$$\frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j \notin \mathbf{I}_B$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

We refer to (7.12) as the input-oriented CRS minimum efficiency model. Although the benchmarks implicitly define the tradeoffs amongst inputs and outputs, the exact tradeoffs are still unavailable to us. Thus, the optimal value to (7.12) gives the lower efficiency bound for DMU^{new} . The optimal value to (7.8) yields the upper efficiency bound. The true efficiency of DMU^{new} lies in-between the bounds.

In fact, model (7.12) describes the worst efficiency scenario whereas model (7.8) describe the best efficiency scenario. The minimum efficiency for the original input-oriented DEA models (e.g., model (7.7)) is zero, and for the original output-oriented DEA models is infinite.

Similarly, we can obtain the output-oriented CRS minimum efficiency model,

$$\max \frac{\sum_{i=1}^m v_i x_i^{new}}{\sum_{r=1}^s u_r y_r^{new}}$$

subject to

$$\frac{\sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} = 1 \quad j \in \mathbf{I}_B$$

$$u_r, v_i \geq 0 \quad \forall r, i$$
(7.13)

Recall that a smaller score indicates a better performance in the output-oriented DEA models. Therefore, the output-oriented CRS minimum efficiency score (optimal value to model (7.13)) is greater than or equal to the efficiency score obtained from the output-oriented CRS fixed-benchmark model.

The linear program equivalents to (7.12) and (7.13) are presented in Table 7.5 which summarizes the minimum efficiency models.

The spreadsheet models for the minimum efficiency models are similar to the fixed-benchmark spreadsheet models. We only need to change the “Max” to “Min” in the Solver parameters for the input-oriented models, and change the “Min” to “Max” for the output-oriented models. For example, consider the output-oriented CRS fixed-benchmark model shown in Figure 7.9. Figure 7.13 shows the corresponding minimum efficiency spreadsheet model.

Table 7.5. Minimum Efficiency Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|---|---|
| | $\min \sum_{r=1}^s \mu_r y_r^{new} + \mu$ subject to $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^s v_i x_{ij} + \mu = 0 \quad j \in \mathbf{I}_B$ $\sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^s v_i x_{ij} + \mu \leq 0 \quad j \notin \mathbf{I}_B$ $\sum_{i=1}^m v_i x_i^{new} = 1$ $\mu_r, v_i \geq 0$ | $\max \sum_{i=1}^m v_i x_i^{new} + \nu$ subject to $\sum_{i=1}^s v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu = 0 \quad j \in \mathbf{I}_B$ $\sum_{i=1}^s v_i x_{ij} - \sum_{r=1}^s \mu_r y_{rj} + \nu \geq 0 \quad j \notin \mathbf{I}_B$ $\sum_{r=1}^s \mu_r y_r^{new} = 1$ $\mu_r, v_i \geq 0$ |
| CRS | where $\mu = 0$ | where $\nu = 0$ |
| VRS | where μ free | where ν free |
| NIRS | where $\mu \leq 0$ | where $\nu \geq 0$ |
| NDRS | where $\mu \geq 0$ | where $\nu \leq 0$ |

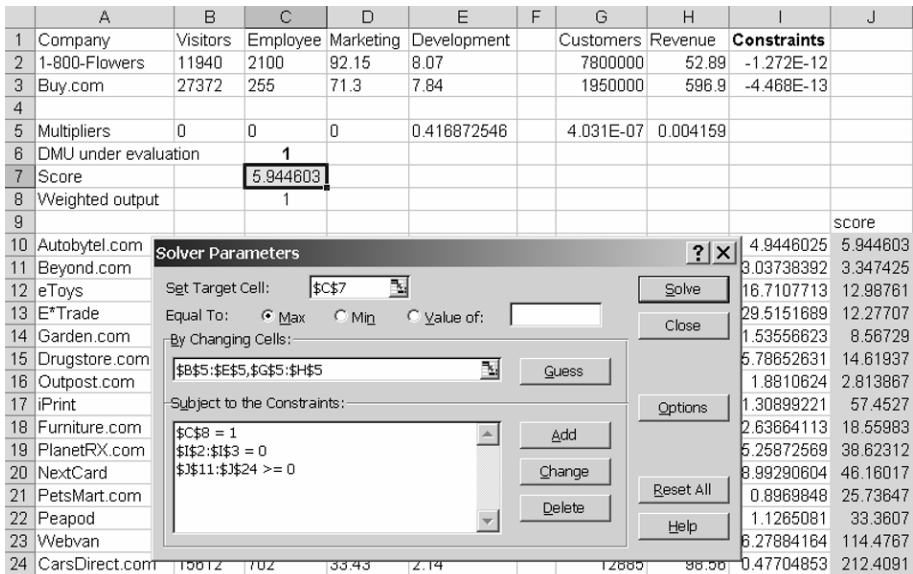


Figure 7.13. Output-oriented CRS Minimum Efficiency Spreadsheet Model

Under the tradeoffs characterized by the two benchmarks, the true efficiency of Autobytel.com lies in [0.6681, 5.9446]. Cells J10:J24 report the “minimum efficiency” for the 15 internet companies. The scores are calculated by the DEA Excel Solver discussed in Chapter 12.

If we introduce the ideal benchmark into the minimum efficiency models, we obtain, for example, the input-oriented VRS ideal-benchmark minimum efficiency model

$$\begin{aligned}
 & \min \sum_{r=1}^s \mu_r y_r^{new} + \mu \\
 & \text{subject to} \\
 & \sum_{r=1}^s \mu_r y_r^{ideal} - \sum_{i=1}^s v_i x_i^{ideal} + \mu = 0 \\
 & \sum_{i=1}^m v_i x_i^{new} = 1 \\
 & \mu_r, v_i \geq 0 \text{ and } \mu \text{ free in sign}
 \end{aligned} \tag{7.14}$$

Table 7.6 presents the ideal-benchmark minimum efficiency models.

Table 7.6. Ideal-benchmark Minimum Efficiency Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|---|--|
| | $ \begin{aligned} & \min \sum_{r=1}^s \mu_r y_r^{new} + \mu \\ & \text{subject to} \\ & \sum_{r=1}^s \mu_r y_r^{ideal} - \sum_{i=1}^s v_i x_i^{ideal} + \mu = 0 \\ & \sum_{i=1}^m v_i x_i^{new} = 1 \\ & \mu_r, v_i \geq 0. \end{aligned} $ | $ \begin{aligned} & \max \sum_{i=1}^m v_i x_i^{new} + \nu \\ & \text{subject to} \\ & \sum_{i=1}^s v_i x_i^{ideal} - \sum_{r=1}^s \mu_r y_r^{ideal} + \nu = 0 \\ & \sum_{r=1}^s \mu_r y_r^{new} = 1 \\ & \mu_r, v_i \geq 0 \end{aligned} $ |
| CRS | where $\mu = 0$ | where $\nu = 0$ |
| VRS | where μ free | where ν free |
| NIRS | where $\mu \leq 0$ | where $\nu \geq 0$ |
| NDRS | where $\mu \geq 0$ | where $\nu \leq 0$ |

7.6 Buyer-seller Efficiency Model

As pointed out by Wise and Morrison (2000), one of the major flaws in the current business-to-business (B2B) model is that it focuses on price-driven transactions between buyers and sellers, and fails to recognize other important vendor attributes such as response time, quality and customization. In fact, a number of efficiency-based negotiation models have been developed to deal with multiple attributes – inputs and outputs. For example, DEA is used by Weber and Desai (1996) to develop models for vendor evaluation and negotiation. The fixed-benchmark models and the minimum efficiency models can better help the vendor in evaluating and selecting the vendors.

Talluri (2002) proposes a buyer-seller game model that evaluates the efficiency of alternative bids with respect to the ideal target set by the buyer. Zhu (2004) shows that this buyer-seller game model is closely related to DEA and can be simplified as the models presented in Tables 7.4 and 7.6.

We next use the data in Table 5.1 to demonstrate the use of DEA benchmarking models. A Fortune 500 pharmaceutical company was involved in the implementation of a Just-in-Time manufacturing system. Therefore, price, delivery performance, and quality were considered to be the three most important criteria in evaluating and selecting vendors. In Weber and Desai (1996), the price criterion is measured by the total purchase price based on a per unit contract delivered price, the delivery criterion is measured by the percentage of late deliveries, and the quality criterion is measured by the percentage of units rejected. Obviously, the measures for delivery and quality are bad outputs. Therefore, we re-define the delivery and quality by percentage on-time deliveries and percentage of accepted units, respectively. (Otherwise, we should use the method described in Chapter 5.)

Table 7.7. Data for the Six Vendors

| Vendor | Price (\$/unit) | % accepted units | % on-time deliveries |
|--------|-----------------|------------------|----------------------|
| 1 | 0.1958 | 98.8 | 95 |
| 2 | 0.1881 | 99.2 | 93 |
| 3 | 0.2204 | 100 | 100 |
| 4 | 0.2081 | 97.9 | 100 |
| 5 | 0.2118 | 97.7 | 97 |
| 6 | 0.2096 | 98.8 | 96 |

Table 7.8. Input-oriented CRS Efficiency and Efficient Target for Vendors

| Vendor | Efficiency | Price (\$/units) | % acceptance | % on-time deliveries |
|--------|------------|------------------|-----------------|----------------------|
| 1 | 0.981 | 0.192145 | 101.3333 | 95 |
| 2 | 1 | 0.1881 | 99.2 | 93 |
| 3 | 0.918 | 0.202258 | 106.6667 | 100 |
| 4 | 0.972 | 0.202258 | 106.6667 | 100 |
| 5 | 0.926 | 0.19619 | 103.4667 | 97 |
| 6 | 0.926 | 0.194168 | 102.4 | 96 |

The results are based upon the input-oriented CRS envelopment model.

Table 7.7 presents the data for six vendors that are obtained from the data presented in Table 5.1. The second column reports the input, and the third and fourth columns report the two outputs. We next need to determine the frontier type. Because the outputs are measured in percentages, we assume the vendors form a VRS frontier. Otherwise, unreasonable results may be

obtained if we assume CRS frontier. For example, Table 7.8 reports the input-oriented CRS efficiency scores (second column) with the efficient targets. It can be seen that the efficient targets on percentage of accepted units are impossible to achieve.

If we use the input-oriented VRS envelopment model, vendors 2, 3, and 4 are efficient, and can be selected. However, if we specify an ideal benchmark by the minimum input value and the maximum output values, as shown in Figure 7.14, we can further characterize the six vendors.

| | A | B | C | D | E | F | G |
|----|----------------------|------------------|----------|------------------|----------------------|---------------|-------------------------|
| 1 | | Price (\$/units) | | % accepted units | % on-time deliveries | constraint | free variable= F4-G4 |
| 2 | Ideal target | 0.1881 | | 100 | 100 | 0 | |
| 3 | | | | | | free variable | 0.897424 |
| 4 | multipliers | 4.770992 | | 0 | 0 | 0.8974237 | 0 |
| 5 | DMU under evaluation | | 6 | | | | |
| 6 | Score | | 0.8974 | | | | |
| 7 | Weighted input | | 1 | | | Maximum | |
| 8 | | | | | | Efficiency | |
| 9 | Vendor 1 | 0.1958 | | 98.8 | 95 | 0.9606742 | |
| 10 | Vendor 2 | 0.1881 | | 99.2 | 93 | 1 | |
| 11 | Vendor 3 | 0.2204 | | 100 | 100 | 0.8534483 | Max |
| 12 | Vendor 4 | 0.2081 | | 97.9 | 100 | 0.9038924 | |
| 13 | Vendor 5 | 0.2118 | | 97.7 | 97 | 0.888102 | |
| 14 | Vendor 6 | 0.2096 | | 98.8 | 96 | 0.8974237 | |

Figure 7.14. Input-oriented VRS Ideal-benchmark Spreadsheet Model

Figure 7.14 shows the spreadsheet for the input-oriented VRS ideal-benchmark model. Cell C4 and cells D4:E4 are reserved for the input and output multipliers. The free variable is represented by cell G3 which contains the formula “=F4-G4”. Cells F4:G4 are specified as changing cells in the Solver parameters (see Figure 7.15).

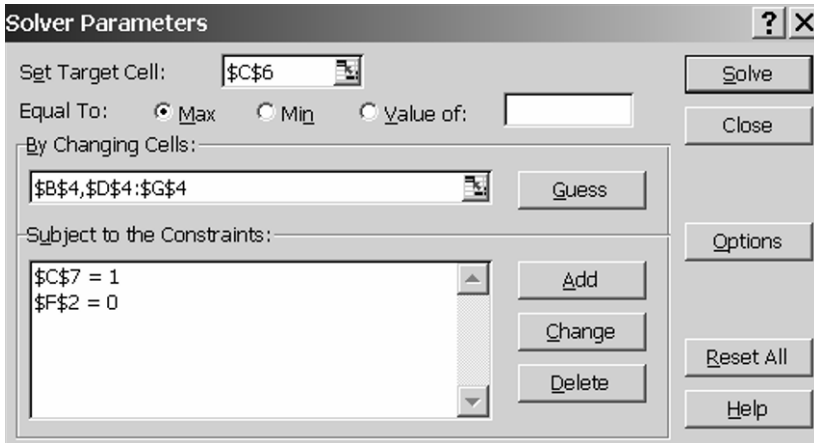


Figure 7.15. Solver Parameters for Input-oriented VRS Ideal-benchmark Model

Cell F2 contains the formula for the ideal benchmark, that is

$$\text{Cell F2} = \text{SUMPRODUCT}(D2:E2, D4:E4) - B2 * B4 + G3$$

Cell C5 is reserved to indicate the vendor under evaluation. The (maximum) efficiency is presented in cell C6 which contains the formula

$$\text{Cell C6} = \text{SUMPRODUCT}(D4:E4, \text{INDEX}(D9:E14, C5, 0)) + G3$$

Cell C7 is the weighted input and contains the formula

$$\text{Cell C7} = B4 * \text{INDEX}(B9:B14, C5, 1)$$

The Solver parameters shown in Figure 7.15 remain the same for all the vendors, and the calculation is performed by the VBA procedure “IdealBenchmark”.

```
Sub IdealBenchmark()
Dim i As Integer
For i = 1 To 6
Range("C5") = i
SolverSolve UserFinish:=True
Range("F" & i + 8) = Range("C6")
Next
End Sub
```

Based upon the scores in cells F9:F14 in Figure 7.14, vendor 2 has the best performance.

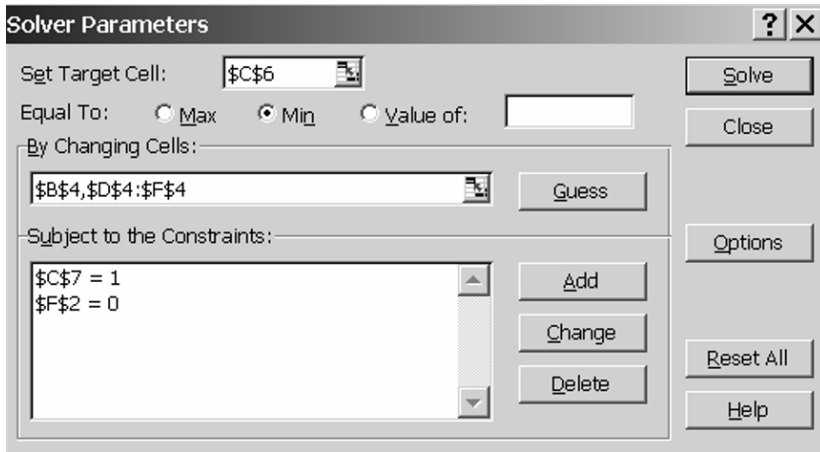


Figure 7.16. Solver Parameters for VRS Ideal-benchmark Minimum Efficiency Model

Next, we turn to the ideal-benchmark minimum efficiency model (7.14). The spreadsheet is the same as the one shown in Figure 7.14. However, we need to change “Max” to “Min” in the Solver parameters shown in Figure 7.15. Figure 7.16 shows the result. Figure 7.17 shows the minimum efficiency scores in cells F9:F14. The minimum efficiency model also indicates that vendor 2 is the best one.

| | A | B | C | D | E | F | G |
|----|----------------------|------------------|----------|------------------|----------------------|---------------|-----|
| | | Price (\$/units) | | % accepted units | % on-time deliveries | constraint | |
| 1 | | | | 100 | 100 | -1.11E-16 | |
| 2 | Ideal target | 0.1881 | | | | free variable | 0 |
| 3 | | | | | | 0 | 0 |
| 4 | multipliers | 4.770992 | | 0 | 0.008974 | | |
| 5 | DMU under evaluation | | 6 | | | | |
| 6 | Score | | 0.8615 | | | | |
| 7 | Weighted input | | 1 | | | Minimum | |
| 8 | | | | | | Efficiency | |
| 9 | Vendor 1 | 0.1958 | | 98.8 | 95 | 0.9126404 | |
| 10 | Vendor 2 | 0.1881 | | 99.2 | 93 | 0.93 | Min |
| 11 | Vendor 3 | 0.2204 | | 100 | 100 | 0.8534483 | |
| 12 | Vendor 4 | 0.2081 | | 97.9 | 100 | 0.8849106 | |
| 13 | Vendor 5 | 0.2118 | | 97.7 | 97 | 0.8614589 | |
| 14 | Vendor 6 | 0.2096 | | 98.8 | 96 | 0.8615267 | |

Figure 7.17. Minimum Efficiency Scores for the Six Vendors

7.7 Solving DEA Using DEA Frontier Software

7.7.1 Variable-benchmark Models

To run the variable-benchmark models presented in Table 7.1, we need set up the data sheets. *Store the benchmarks in a sheet named “Benchmarks” and the DMUs under evaluation in a sheet named “DMUs”*. The format for these two sheets is the same as that shown in Figure 12.3. Then select the Variable Benchmark Model menu item. You will be prompted a form for selecting the model orientation and the frontier type as shown in Figure 7.18. Note that if you select a frontier type other than CRS, the results may be infeasible. The benchmarking results are reported in the sheet “Benchmarking Results”.

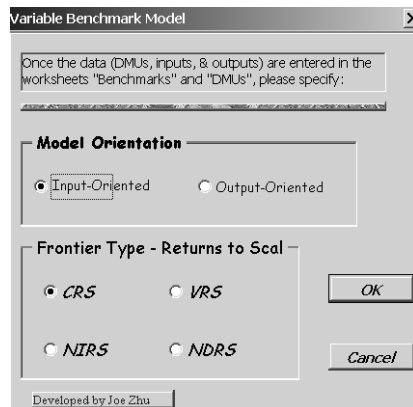


Figure 7.18. Variable Benchmark Models

7.7.2 Fixed-benchmark Models

To run the fixed-benchmark models presented in Table 7.3, we *store the benchmarks in a sheet named “Benchmarks” and the DMUs under evaluation in a sheet named “DMUs”*. Then select the Fixed-Benchmark Model menu item. You will be prompted a form for selecting the model orientation and the frontier type. The results are reported in the “Efficiency Report” sheet. If the benchmarks are not properly selected, you will have infeasible results and need to adjust the benchmarks.

The Ideal-benchmark Models in Table 7.4 should be calculated using the Fixed-Benchmark Model menu item. The data for the ideal benchmark is stored in the “Benchmarks” sheet.

7.7.3 Minimum Efficiency Models

To run the minimum efficiency models presented in Table 7.5, we *store the benchmarks in a sheet named “Benchmarks” and the DMUs under evaluation in a sheet named “DMUs”*. Then select the Minimum Efficiency Model menu item. You will be prompted a form for selecting the model orientation and the frontier type. The results are reported in the “Minimum Efficiency” sheet.

The Ideal-benchmark Minimum Efficiency Models in Table 7.6 should be calculated using the Minimum Efficiency menu item. The data for the ideal benchmark is stored in the “Benchmarks” sheet.

REFERENCES

1. Camp, R.C., (1995), *Business Process Benchmarking, Finding and Implementing Best Practices*. ASQC Quality Press, Milwaukee, Wisconsin.
2. Cook, W.D., L.M. Seiford and J. Zhu (2004), Models for performance benchmarking: Measuring the effect of e-commerce activities on banking performance, *OMEGA*, Vol. 32, Issue 4, 313-322
3. Seiford, L.M. and J. Zhu (1999), Infeasibility of super efficiency data envelopment analysis models, *INFOR*, 37, No. 2, 174-187.
4. Talluri, S. (2002), A buyer-seller game model for selection and negotiation of purchasing bids. *European Journal of Operational Research* 143 (1), 171-180.
5. Weber, C.A. and A. Desai (1996), Determination of paths to vendor market efficiency using parallel coordinates representation: a negotiation tool for buyers, *European Journal of Operational Research*, 90, 142-155.
6. Zhu, J. (2004), A buyer-seller game model for selection and negotiation of purchasing bids: Extensions and New Models, *European Journal of Operational Research*, Vol. 154, Issue 1, 150-156.

Chapter 8

Models for Evaluating Supply Chains

8.1 Supply Chain Efficiency

So far, the value-added processes or systems have been treated as a “black-box”. We examine the resources available to the processes or systems and monitor the “conversions” of these resources (inputs) into the desired outputs. However, each process or system can include many subprocesses. For example, if the process is to make a car, then important subprocesses include assembling and painting. If we evaluate the efficiency of a supply chain system, then we need to measure the performance of each individual supply chain components, including suppliers, manufacturers, retailers, and customers.

While there are studies on supply chain performance using the methodology of data envelopment analysis (DEA), the research has been focused on a single member of the supply chain.

Within the context of DEA, there are a number of methods that have the potential to be used in supply chain efficiency evaluation. Seiford and Zhu (1999a) and Chen and Zhu (2004) provide two approaches in modeling efficiency as a two-stage process. Färe and Grosskopf (2000) develop the network DEA approach to model general multi-stage processes with intermediate inputs and outputs. Golany, Hackman and Passy (2006) provide an efficiency measurement framework for systems composed of two subsystems arranged in series that simultaneously compute the efficiency of the aggregate system and each subsystem.

Note that an effective management of the supply chain requires knowing the performance of the overall chain rather than simply the performance of the individual supply chain members. Each supply chain member has its own

strategy to achieve efficiency, however, what is best for one member may not work in favor of another member. Sometimes, because of the possible conflicts between supply chain members, one member's inefficiency may be caused by another's efficient operations. For example, the supplier may increase its raw material price to enhance its revenue and to achieve an efficient performance. This increased revenue means increased cost to the manufacturer. Consequently, the manufacturer may become inefficient unless it adjusts its current operating policy. Measuring supply chain performance becomes a difficult and challenging task because of the need to deal with the multiple performance measures related to the supply chain members, and to integrate and coordinate the performance of those members.

Two hurdles are present in measuring the performance of value chains. One is the existence of multiple measures that characterize the performance of each member in a supply chain. The other is the existence of conflicts between supply chain members with respect to specific measures.

Consider three supplier-manufacturer supply chains presented in Table 8.1 where the supplier has two inputs (shipping cost and labor) and one output (revenue from selling the raw materials to the manufacturer), and the manufacturer has one input (raw material cost which is the supplier's revenue) and one output (profit).

Applying the input-oriented VRS envelopment model to the suppliers and the manufacturers indicate that the suppliers in supply chains A and B, and the manufacturer in supply chain C are efficient. Now, if we ignore the intermediate measure of revenue (raw material cost) and apply the input-oriented VRS envelopment model, the last column of Table 8.1 indicates that all supply chains are efficient.

Table 8.1. Simple Supplier-Manufacturer Example

| Supply Chain | Supplier | | Manufacturer | | Supplier efficiency | Manufacturer efficiency | Overall efficiency |
|--------------|----------------|-------|--------------|--------|---------------------|-------------------------|--------------------|
| | Shipping Costs | Labor | Revenue | Profit | | | |
| A | 7 | 9 | 4 | 16 | 1 | 0.75 | 1 |
| B | 9 | 4 | 6 | 14 | 1 | 0.5 | 1 |
| C | 11 | 6 | 3 | 23 | 0.791 | 1 | 1 |

This simple numerical example indicates that the conventional DEA fails to correctly characterize the performance of supply chain. Since an overall DEA efficient performance does not necessarily indicate efficient performance in individual components in the supply chain. Consequently, improvement to the best-practice can be distorted. i.e., the performance

improvement of one member affects the efficiency status of the other, because of the presence of intermediate measures. Seiford and Zhu (1999a) develop a procedure for value chain performance improvement by using returns to scale (RTS) sensitivity analysis (Seiford and Zhu, 1999b). In this chapter, we present models that can directly evaluate the performance of supply chains or value chains that have more than one members/components.

8.2 Supply Chain Efficiency

Supply chain management has been proven a very effective tool to provide prompt and reliable delivery of high-quality products and services at the least cost. To achieve this, performance evaluation of entire supply chain is extremely important, since it means utilizing the combined resources of the entire supply chain in the most efficient way possible to provide market-winning and cost-effective products and services. However, a lack of appropriate performance measurement systems has been a major obstacle to an effective supply chain management (Lee and Billington, 1992).

This is due to the fact that the concept of supply chain management requires the performance of overall supply chain rather than only the performance of the individual supply chain members. Each supply chain member has its own strategy to achieve 100% efficiency. One supply chain member's 100% efficiency does not necessarily mean another's 100% efficiency. Sometimes, because of the possible conflicts between supply chain members, one member's inefficiency may be caused by another's efficient operations. For example, the supplier may increase its raw material price to increase its revenue and to achieve an efficient performance. This increased revenue means increased cost to the manufacturer. Consequently, the manufacturer may become inefficient unless the manufacturer adjusts its current operating policy.

As demonstrated in Table 8.1, some measures linked to related supply chain members cannot be simply classified as "outputs" or "inputs" of the supply chain. For example, the supplier's revenue is not only an output of the supplier (the supplier wishes to maximize it), but also an input to the manufacturer (the manufacturer wishes to minimize it). Simply minimizing the total supply chain cost or maximizing the total supply chain revenue (profit) does not model and solve the conflicts. Therefore, the meaning of supply chain efficiency needs to be carefully defined and studied, and we need models that can both define and measure the efficiency of supply chain as well as supply chain members.

Methods have been developed to estimate the exact performance of supply chain members based upon single performance measures (e.g.,

Cheung and Hausman, 2000). However, no attempts have been made to identify the best practice of the supply chain. No solid mathematical models have been developed to simultaneously (i) define and measure the whole supply chain performance with possible conflicts on specific measures, (ii) evaluate the performance of supply chain members, and (iii) identify the best practice and provide directions to achieve the supply chain best practice.

The new DEA model measures the efficiency of supply chain system as a whole as well as each supply chain member, and provides directions for supply chain improvement to reach the best practice. This eliminates the needs for unrealistic assumptions in typical supply chain optimization models and probabilistic models, e.g., a typical EOQ model assumes constant and known demand rate and lead-time for delivery.

8.2.1 Supply Chain as an Input-Output System

A typical supply chain can be presented in Figure 8.1 with four echelons – suppliers, manufacturers, distributors, and retailers. The traditional objective of supply chain management is to minimize the total supply chain cost to meet the customer needs through coordination efforts among supply chain members. To achieve this objective, timely and accurate assessment of the supply chain system and individual member performance is of extreme importance. Because an effective performance evaluation system (i) provides the basis to understand the supply chain operations, (ii) monitors and manages supply chain performance through identifying the best-practice supply chain operations, and (iii) provides directions for further supply chain improvement.

Supply chain systems can be viewed as an integrated input-output system where each supply chain member uses inputs to produce. Consequently, we may classify supply chain member's performance measures into inputs and outputs. Caution should be paid when we classify the performance measures into inputs and outputs based upon specific supply chain members, since incorrect classification may lead to false conclusion on the efficiency of supply chain members as well as supply chain. The classification can be based upon the material and information flows in a supply chain system.

Let I^Δ and R^Δ represent the input and output subscript sets for a supply chain member Δ , respectively. We denote x_i^Δ ($i \in I^\Delta$) and y_r^Δ ($r \in R^\Delta$) the inputs and outputs associated with each supply chain member, respectively. Now, let \bar{x}_Δ and \bar{y}_Δ be the vectors consisting of x_i^Δ ($i \in I^\Delta$) and y_r^Δ ($r \in R^\Delta$), respectively. The following Pareto-Koopmans efficiency is used to define an efficient supply chain member.

Definition 8.1 (Efficient Supply Chain Member) A supply chain member Δ is efficient if $(\bar{x}_\Delta, \bar{y}_\Delta)$ is not dominated.

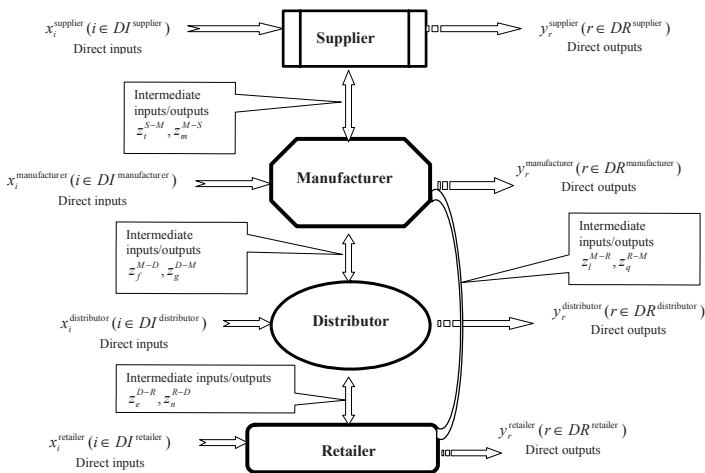


Figure 8.1. Supply Chain

Some measures are associated with a specific supply chain member only. We call these measures the “direct” inputs and outputs. For example, labor and manufacturing lead time are two direct inputs to the manufacturer. These direct inputs and outputs of supply chain members can be viewed as the inputs and outputs of the supply chain.

We also have “intermediate” inputs/outputs associated with two supply chain members. For example, the number of finished products shipped from the manufacturer to the retailer and the distributor represent outputs of the manufacturer. These outputs then become inputs to the distributor and the retailer (see Figure 8.1). These intermediate measures cannot be simply treated as inputs or outputs of the supply chain, although they are inputs/outputs of specific supply chain members.

In supply chain management, it is believed that values of intermediate measures should be determined through coordination among related supply chain members (Parlar and Weng, 1997; Thomas and Griffin, 1996). Because such intermediate measures are usually cost to one supply chain member and benefit to the other. Simply minimizing the total supply chain cost or maximizing the supply chain revenue does not model situations with intermediate measures. This poses a challenge to defining and measuring the supply chain efficiency.

To facilitate our discussion, let DI^Δ and DR^Δ represent the direct input and direct output subscript sets for a supply chain member Δ , respectively. We then use the following notions to represent intermediate inputs and outputs, x_i^Δ ($i \notin DI^\Delta \subseteq I^\Delta$) and y_r^Δ ($r \notin DR^\Delta \subseteq R^\Delta$),

z_t^{S-M} = t th intermediate output from the supplier to manufacturer, $t = 1, \dots, T$;
 z_m^{M-S} = m th intermediate output from the manufacturer to the supplier, $m = 1, \dots, M$;
 z_f^{M-D} = f th intermediate output from the manufacturer to the distributor, $f = 1, \dots, F$;
 z_g^{D-M} = g th intermediate output from the distributor to the manufacturer, $g = 1, \dots, G$;
 z_l^{M-R} = l th intermediate output from the manufacturer to the retailer, $l = 1, \dots, L$;
 z_q^{R-M} = q th intermediate output from the retailer to the manufacturer, $q = 1, \dots, Q$;
 z_e^{D-R} = e th intermediate output from the distributor to the retailer, $e = 1, \dots, E$;
 z_n^{R-D} = n th intermediate output from the retailer to the distributor, $n = 1, \dots, N$.

Note that only intermediate outputs are defined, since each such output also represents an input to an associated supply chain member. For example, z_t^{S-M} (output of the supplier) also represents an input to the manufacturer.

8.2.2 Supply Chain Efficiency Model

Suppose we have J observations associated with each supply chain member. i.e., we have observed input and output values of x_{ij}^Δ ($i \in I^\Delta$) and y_{rj}^Δ ($r \in R^\Delta$), where $j = 1, \dots, J$. The efficiency of supply chain member Δ can be measured by the following DEA model – input-oriented CRS envelopment model

$$\begin{aligned} \theta^{\Delta*} &= \min_{\phi_j^\Delta, \theta^\Delta} \theta^\Delta \\ \text{subject to} \\ \sum_{j=1}^J \phi_j^\Delta x_{ij}^\Delta &\leq \theta^\Delta x_{i_0}^\Delta \quad i \in I^\Delta \\ \sum_{j=1}^J \phi_j^\Delta y_{rj}^\Delta &\geq y_{r_0}^\Delta \quad r \in R^\Delta \\ \phi_j^\Delta &\geq 0, \quad j = 1, \dots, J \end{aligned} \quad (8.1)$$

If $\theta^{\Delta^*} = 1$, then a supply chain member Δ is efficient (including weakly efficient). Also, for inefficient performance, model (8.1) provides projection paths onto the efficient frontier via the optimal values of $\sum_{j=1}^J \phi_j^{\Delta^*} x_{ij}^{\Delta}$ and $\sum_{j=1}^J \phi_j^{\Delta^*} y_{rj}^{\Delta}$.

Because of the possible conflicts represented by the intermediate measures between associated supply chain members, the supply chain's performance cannot be simply defined and characterized by non-dominancy through using model (8.1). Let w_i be the user-specified weights reflecting the preference over supply chain member's performance (operation). We establish the following liner programming problem for the supply chain

$$\Omega^* = \min_{\Omega_i, \lambda_j, \beta_j, \delta_j, \gamma_j, \tilde{z}} \frac{\sum_{i=1}^4 w_i \Omega_i}{\sum_{i=1}^4 w_i}$$

subject to

(supplier)

$$\sum_{j=1}^J \lambda_j x_{ij}^{\text{supplier}} \leq \Omega_1 x_{ij_o}^{\text{supplier}} \quad i \in DI^{\text{supplier}}$$

$$\sum_{j=1}^J \lambda_j y_{rj}^{\text{supplier}} \geq y_{rj_o}^{\text{supplier}} \quad r \in DR^{\text{supplier}}$$

$$\sum_{j=1}^J \lambda_j z_{tj}^{\text{S-M}} \geq \tilde{z}_{tj_o}^{\text{S-M}} \quad t = 1, \dots, T$$

$$\sum_{j=1}^J \lambda_j z_{mj}^{\text{M-S}} \leq \tilde{z}_{mj_o}^{\text{M-S}} \quad m = 1, \dots, M$$

$$\lambda_j \geq 0, \quad j = 1, \dots, J$$

(manufacturer)

$$\sum_{j=1}^J \beta_j x_{ij}^{\text{manufacturer}} \leq \Omega_2 x_{ij_o}^{\text{manufacturer}} \quad i \in DI^{\text{manufacturer}}$$

$$\sum_{j=1}^J \beta_j y_{rj}^{\text{manufacturer}} \geq y_{rj_o}^{\text{manufacturer}} \quad r \in DR^{\text{manufacturer}}$$

$$\sum_{j=1}^J \beta_j z_{tj}^{\text{S-M}} \leq \tilde{z}_{tj_o}^{\text{S-M}} \quad t = 1, \dots, T$$

$$\sum_{j=1}^J \beta_j z_{mj}^{\text{M-S}} \geq \tilde{z}_{mj_o}^{\text{M-S}} \quad m = 1, \dots, M$$

$$\sum_{j=1}^J \beta_j z_{fj}^{\text{M-D}} \geq \tilde{z}_{fj_o}^{\text{M-D}} \quad f = 1, \dots, F$$

$$\sum_{j=1}^J \beta_j z_{gj}^{\text{D-M}} \leq \tilde{z}_{gj_o}^{\text{D-M}} \quad g = 1, \dots, G$$

$$\sum_{j=1}^J \beta_j z_{lj}^{\text{M-R}} \geq \tilde{z}_{lj_o}^{\text{M-R}} \quad l = 1, \dots, L$$

$$\sum_{j=1}^J \beta_j z_{qj}^{\text{R-M}} \leq \tilde{z}_{qj_o}^{\text{R-M}} \quad q = 1, \dots, Q$$

$$\beta_j \geq 0, \quad j = 1, \dots, J$$

(8.2)

(distributor)

$$\begin{aligned} \sum_{j=1}^J \delta_j x_{ij}^{\text{distributor}} &\leq \Omega_3 x_{i_0}^{\text{distributor}} & i \in DI^{\text{distributor}} \\ \sum_{j=1}^J \delta_j y_{rj}^{\text{distributor}} &\geq y_{r_0}^{\text{distributor}} & r \in DR^{\text{distributor}} \\ \sum_{j=1}^J \delta_j z_{fj}^{M-D} &\leq \tilde{z}_{f_0}^{M-D} & f = 1, \dots, F \\ \sum_{j=1}^J \delta_j z_{gj}^{D-M} &\geq \tilde{z}_{g_0}^{D-M} & g = 1, \dots, G \\ \sum_{j=1}^J \delta_j z_{ej}^{D-R} &\geq \tilde{z}_{e_0}^{D-R} & e = 1, \dots, E \\ \sum_{j=1}^J \delta_j z_{nj}^{D-R} &\leq \tilde{z}_{n_0}^{D-R} & n = 1, \dots, N \\ \delta_j &\geq 0, \quad j = 1, \dots, J \end{aligned}$$

(retailer)

$$\begin{aligned} \sum_{j=1}^J \gamma_j x_{ij}^{\text{retailer}} &\leq \Omega_4 x_{i_0}^{\text{retailer}} & i \in DI^{\text{retailer}} \\ \sum_{j=1}^J \gamma_j y_{rj}^{\text{retailer}} &\geq y_{r_0}^{\text{retailer}} & r \in DR^{\text{retailer}} \\ \sum_{j=1}^J \gamma_j z_{lj}^{M-R} &\leq \tilde{z}_{l_0}^{M-R} & l = 1, \dots, L \\ \sum_{j=1}^J \gamma_j z_{qj}^{R-M} &\geq \tilde{z}_{q_0}^{R-M} & q = 1, \dots, Q \\ \sum_{j=1}^J \gamma_j z_{ej}^{D-R} &\leq \tilde{z}_{e_0}^{D-R} & e = 1, \dots, E \\ \sum_{j=1}^J \gamma_j z_{nj}^{R-D} &\geq \tilde{z}_{n_0}^{R-D} & n = 1, \dots, N \\ \gamma_j &\geq 0, \quad j = 1, \dots, J \end{aligned}$$

Additional constraints can be added into model (8.2). For example, if z_f^{M-D} represents the number of product f shipped from the manufacturer to the distributor, and if the capacity of this manufacturer in producing product f is C_f , then we may add $\tilde{z}_f^{M-D} \leq C_f$.

Obviously, if $\Omega^* = 1$, then there must exist an optimal solution such that $\lambda_{j_0}^* = \beta_{j_0}^* = \delta_{j_0}^* = \gamma_{j_0}^* = 1$, where (*) represents optimal value in model (8.2). Further, if $\Omega^* = 1$, then $\theta^* = 1$, where θ^* is the optimal value to model (8.1). i.e., when $\Omega^* = 1$, all supply chain members are efficient.

If $\Omega^* \neq 1$, then we immediately have the following result

All supply chain members are efficient with respect to $\Omega_1 x_{i_0}^{\text{supplier}}$ ($i \in DI^{\text{supplier}}$), $\Omega_2 x_{i_0}^{\text{manufacturer}}$ ($i \in DI^{\text{manufacturer}}$), $\Omega_3 x_{i_0}^{\text{distributor}}$ ($i \in DI^{\text{distributor}}$), $\Omega_4 x_{i_0}^{\text{retailer}}$ ($i \in DI^{\text{retailer}}$), $y_{r_0}^{\text{supplier}}$ ($r \in DR^{\text{supplier}}$), $y_{r_0}^{\text{manufacturer}}$ ($r \in DR^{\text{manufacturer}}$), $y_{r_0}^{\text{distributor}}$ ($r \in DR^{\text{distributor}}$), $y_{r_0}^{\text{retailer}}$ ($r \in DR^{\text{retailer}}$), $\tilde{z}_{t_0}^{S-M^}$ ($t = 1, \dots, T$), $\tilde{z}_{m_0}^{M-S^*}$ ($m = 1, \dots, M$), $\tilde{z}_{f_0}^{M-D^*}$ ($f = 1, \dots, F$), $\tilde{z}_{g_0}^{D-M^*}$ ($g = 1, \dots, G$), $\tilde{z}_{l_0}^{M-R^*}$ ($l = 1, \dots,$*

L), \tilde{z}_{qjo}^{R-M*} ($q = 1, \dots, Q$), \tilde{z}_{ejo}^{D-R*} ($e = 1, \dots, E$), where (*) represents optimal value in model (8.2).

Definition 8.2 (Efficient Supply Chain) A supply chain is efficient if $\Omega^* = 1$, where Ω^* is the optimal value to model (8.2).

$\theta^{\Delta*}$ measures the efficiency of supply chain member Δ under the context of supply chain member best practice. Ω_i^* can actually be used as a new efficiency measure for a specific supply chain member under the context of supply chain best practice. We have

Definition 8.3 Ω_i^* is called supply-chain-best-practice-dependent efficiency score for a specific supply chain member.

Note that Ω^* can be viewed as an index for input or cost savings for (inefficient) supply chains. The smaller the Ω^* , more savings could be achieved to reach the best practice. The same observation can also be applied to $\theta^{\Delta*}$ in the context of supply chain member best practice. Let $(w_1\theta^{\text{supplier}^*} + w_2\theta^{\text{manufacturer}^*} + w_3\theta^{\text{distributor}^*} + w_4\theta^{\text{retailer}^*}) / \sum_{i=1}^4 w_i$ represent the index for input savings achievable by all supply chain members combined. The following Theorem indicates that supply chain as a whole has potential to achieve more input savings and a better performance

Theorem 8.1 $\Omega^* \leq (w_1\theta^{\text{supplier}^*} + w_2\theta^{\text{manufacturer}^*} + w_3\theta^{\text{distributor}^*} + w_4\theta^{\text{retailer}^*}) / \sum_{i=1}^4 w_i$.

8.2.3 An example

We establish a spreadsheet model for a numerical example constructed as follows. For the supplier, we use labor and operating cost as two direct inputs, and revenue as the intermediate output. This revenue becomes an intermediate input of the manufacturer.

For the manufacturer, we use manufacturing cost and manufacturing lead time as two direct inputs, in addition to the intermediate input – supplier's revenue. We also have three intermediate manufacturer outputs: number of products shipped to the distributor, number of products shipped to the retailer, and distributor's fill rate. These outputs then become inputs to the distributor and the retailer. Note that the distributor's fill rate is actually a cost measure to the distributor, since the fill rate is associated with inventory holding cost and the amount of products required from the manufacturer. The distributor's fill rate implies benefit to the manufacturer, since more products are needed from the manufacturer (meaning more revenue to the

manufacturer) if the distributor wishes to maintain a higher fill rate. Thus, the distributor's fill rate is treated as an output from the manufacturer and an input to the distributor. From a distributor's point of view, the distributor always tries to meet the needs of its customer while maintaining a fill rate as low as possible, because unnecessary high fill rate incurs additional cost to the distributor.

For the distributor, we use inventory cost and distribution cost as two direct inputs in addition to the above intermediate inputs linked with the manufacturer. Two intermediate outputs from the distributor are the number of products shipped from the distributor to the retailer, and the percentage of on-time delivery.

For the retailer, in addition to the intermediate inputs from the manufacturer and the distributor, we have one direct input of number of backorders, and one direct output of profit. Figure 8.2 presents the data with ten observations, i.e., $J = 10$.

In Figure 8.8, cells D18:M21 represents λ_j , β_j , δ_j , and γ_j . Cell B25 indicates the observation under evaluation. Cells D23:D26 represents Ω_i ($i = 1, 2, 3, 4$). Cell C25 is the objective function of model (8.2), and contains the formula “=(D23+D24+D25+D26)/4”.

Cells B27:B40 record the performance measures for a specific observation under evaluation. Cell B27 contains the formula “=INDEX(D2:M2,1,\$B\$25)” which is copied into cell B28:B40.

Cells D27:D39 are used to represent the decision variables. The formulas used in the rest of the spreadsheet model shown in Figure 8.2 are

Cell F27=\$D\$23*B27

Cell F28=\$D\$23*B28

Cell F30=\$D\$24*B30

Cell F31 =\$D\$24*B31

Cell F32 =\$D\$25*B32

Cell F33 =\$D\$25*B33

Cell F39 =\$D\$26*B39

| | A | B | C | D | E | F | G | H | I | J | K | L | M |
|----|----------------------|------|----------------------|----------|----------|------------|--------------|----------|----------|----------|----------|----------|-----------|
| 1 | | | Observation | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | | | Supplier-labor | 150 | 140 | 130 | 165 | 170 | 145 | 155 | 175 | 160 | 125 |
| 3 | | | Supplier-cost | 130 | 150 | 165 | 170 | 200 | 185 | 135 | 190 | 185 | 190 |
| 4 | | | Supplier-revenue | 20 | 21 | 23 | 24 | 27 | 25 | 24 | 30 | 28 | 25 |
| 5 | | | mfg cost | 125 | 120 | 110 | 150 | 146 | 115 | 105 | 100 | 135 | 120 |
| 6 | | | mfg time | 3 | 2 | 3 | 4 | 2 | 3 | 2 | 2 | 4 | 3 |
| 7 | | | DC cost | 90 | 100 | 80 | 70 | 85 | 77 | 78 | 90 | 78 | 68 |
| 8 | | | customer res time | 3 | 3 | 2 | 4 | 2 | 1 | 3 | 2 | 1 | 1 |
| 9 | | | fill rate | 70% | 90% | 78% | 88% | 73% | 95% | 89% | 87% | 95% | 90% |
| 10 | | | product DC-retailer | 1800 | 2000 | 2400 | 2300 | 2500 | 2500 | 2000 | 2000 | 2500 | 2300 |
| 11 | | | on-time | 96% | 95% | 97% | 89% | 99% | 89% | 93% | 88% | 99% | 83% |
| 12 | | | product mfg-retailer | 2000 | 2100 | 2500 | 1900 | 2600 | 2300 | 2200 | 2300 | 2500 | 2500 |
| 13 | | | product mfg-DC | 500 | 300 | 450 | 200 | 300 | 250 | 350 | 450 | 300 | 400 |
| 14 | | | Retailer cost | 100 | 110 | 130 | 125 | 140 | 135 | 125 | 155 | 135 | 130 |
| 15 | | | Retailer revenue | 310 | 220 | 300 | 230 | 320 | 240 | 350 | 370 | 325 | 355 |
| 16 | | | | | | | | | | | | | |
| 17 | | | | | | | | | | | | | |
| 18 | | | λ | 0 | 0 | 0 | 0 | 0 | 0 | 0.884 | 0 | 0 | 0 |
| 19 | | | β | 0 | 1.011 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | | | δ | 0 | 0 | 0.07784 | 0 | 0 | 0 | 0 | 0 | 0.893 | 0 |
| 21 | | | γ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3E-12 |
| 22 | | | | | | | | | | | | | |
| 23 | | | | 0.92 | | | | | | | | | |
| 24 | | | | 0.97 | | | Supply Chain | | | | | | |
| 25 | | | | 0.84 | | | | | | | | | |
| 26 | | 1 | 0.932955109 | 1 | | Efficiency | supplier | mfg | | DC | | | Retailer |
| 27 | Supplier-labor | 150 | S-labor | 138 | | 137.733 | 137.05 | | | | | | |
| 28 | Supplier-cost | 130 | S-cost | 119 | | 119.368 | 119.37 | | | | | | |
| 29 | Supplier-revenue | 20 | S-revenue | 21.2 | | | 21.221 | | 21.2 | | | | |
| 30 | mfg cost | 125 | mfg cost | 121 | | 121.263 | | | 121 | | | | |
| 31 | mfg time | 3 | mfg time | 2.91 | | 2.91032 | | | 2.02 | | | | |
| 32 | DC cost | 90 | DC cost | 75.9 | | 75.9147 | | | | 75.91 | | | |
| 33 | customer res time | 3 | customer res time | 2.53 | | 2.53049 | | | | 1.943 | | | |
| 34 | fill rate | 0.7 | fill rate | 0.91 | | | | | 0.91 | | 0.909 | | |
| 35 | product DC-retailer | 1800 | product DC-retailer | 2420 | | | | | | 2420 | | 1800 | |
| 36 | on-time | 0.96 | on-time | 0.96 | | | | | 0.96 | | 0.96 | | 0.96 |
| 37 | product mfg-retailer | 2000 | product mfg-retailer | 2122 | | | | | 2122 | | | 2000 | |
| 38 | product mfg-DC | 500 | product mfg-DC | 303 | | | | | 303 | | 303.1 | | |
| 39 | Retailer cost | 100 | Retailer cost | 100 | | 100 | | | | | | | 100 |
| 40 | Retailer revenue | 310 | Retailer revenue | | | | | | | | | | 310 |

Figure 8.2. Supply Chain Efficiency Spreadsheet Model

Cell G27=SUMPRODUCT(\$D\$18:\$M\$18,D2:M2)
 Cell G28 =SUMPRODUCT(\$D\$18:\$M\$18,D3:M3)
 Cell G29 =SUMPRODUCT(\$D\$18:\$M\$18,D4:M4)

Cell I29=SUMPRODUCT(\$D\$19:\$M\$19,D4:M4)
 Cell I30=SUMPRODUCT(\$D\$19:\$M\$19,D5:M5)
 Cell I31 =SUMPRODUCT(\$D\$19:\$M\$19,D6:M6)
 Cell I34 =SUMPRODUCT(\$D\$19:\$M\$19,D9:M9)
 Cell I36 =SUMPRODUCT(\$D\$19:\$M\$19,D11:M11)
 Cell I37 =SUMPRODUCT(\$D\$19:\$M\$19,D12:M12)
 Cell I38 =SUMPRODUCT(\$D\$19:\$M\$19,D13:M13)

Cell K32 =SUMPRODUCT(\$D\$20:\$M\$20,D7:M7)
 Cell K33=SUMPRODUCT(\$D\$20:\$M\$20,D8:M8)
 Cell K34 =SUMPRODUCT(\$D\$20:\$M\$20,D9:M9)
 Cell K35 =SUMPRODUCT(\$D\$20:\$M\$20,D10:M10)

Cell K36 =SUMPRODUCT(\$D\$20:\$M\$20,D11:M11)

Cell K38 =SUMPRODUCT(\$D\$20:\$M\$20,D13:M13)

Cell M35 =SUMPRODUCT(\$D\$21:\$M\$21,D10:M10)

Cell M36 =SUMPRODUCT(\$D\$21:\$M\$21,D11:M11)

Cell M37 =SUMPRODUCT(\$D\$21:\$M\$21,D12:M12)

Cell M39 =SUMPRODUCT(\$D\$21:\$M\$21,D14:M14)

Cell M40 =SUMPRODUCT(\$D\$21:\$M\$21,D15:M15)

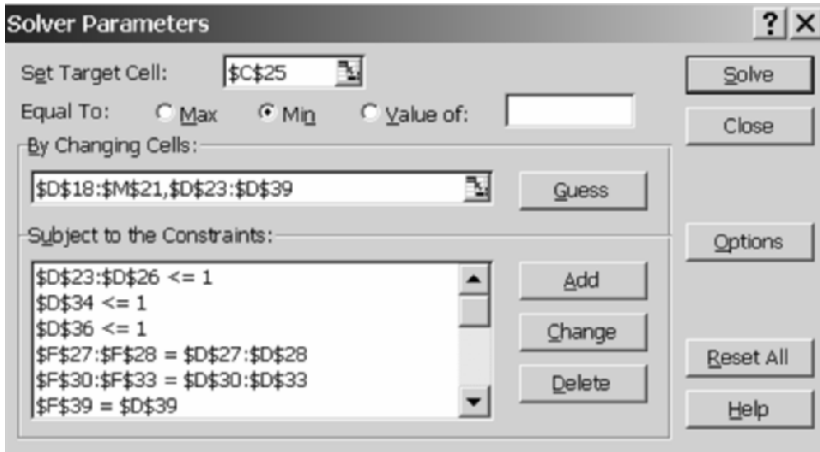


Figure 8.3. Solver Parameters for Supply Chain Efficiency

Figure 8.3 shows the Solver parameters for the spreadsheet shown in Figure 8.2 where cells D18:M21 and cells D23:D39 are changing cells. In this case, two additional constraints are added into model (8.2). One is “fill rate $\leq 100\%$ ”, and the other “percentage of on-time delivery $\leq 100\%$ ”. The constraints include

```

$D$23:$D$26 <= 1
$D$34 <= 1
$D$36 <= 1
$F$27:$F$28 = $D$27:$D$28
$F$30:$F$33 = $D$30:$D$33
$F$39 = $D$39
$G$27:$G$28 <= $D$27:$D$28
$G$29 >= $D$29
$I$29:$I$31 <= $D$29:$D$31
$I$34 >= $D$34
$I$36:$I$38 >= $D$36:$D$38
$K$32:$K$34 <= $D$32:$D$34

```

$\theta_{35} \geq \theta_{36}$
 $\theta_{38} \leq \theta_{38}$
 $\theta_{35} \leq \theta_{37}$
 $\theta_{39} \leq \theta_{39}$
 $\theta_{40} \geq \theta_{40}$

Table 8.2. Supply Chain Efficiency

| Observation | Member Efficiency | | | | Supply Chain Efficiency | | | | | |
|-------------|-----------------------|---------------------------|--------------------------|-----------------------|-------------------------|--------------|--------------|--------------|--------------|--------------|
| | Supplier | Manufacturer | Distributor | Retailer | Average | Supply Chain | Supplier | Manufacturer | Distributor | Retailer |
| | $\theta_{supplier}^*$ | $\theta_{manufacturer}^*$ | $\theta_{distributor}^*$ | $\theta_{retailer}^*$ | | Ω^* | Ω_1^* | Ω_2^* | Ω_3^* | Ω_4^* |
| 1 | 0.865 | 1 | 1 | 1 | 0.966 | 0.933 | 0.918 | 0.970 | 0.843 | 1 |
| 2 | 0.881 | 1 | 0.880 | 0.673 | 0.859 | 0.669 | 0.599 | 0.624 | 0.465 | 0.714 |
| 3 | 0.964 | 1 | 1 | 0.810 | 0.944 | 0.791 | 0.720 | 0.747 | 0.747 | 0.948 |
| 4 | 0.870 | 0.856 | 1 | 0.754 | 0.870 | 0.576 | 0.518 | 0.460 | 0.658 | 0.667 |
| 5 | 0.895 | 1 | 1 | 0.820 | 0.929 | 0.795 | 0.688 | 0.768 | 0.768 | 0.954 |
| 6 | 0.937 | 0.999 | 1 | 0.673 | 0.902 | 0.613 | 0.529 | 0.579 | 0.625 | 0.717 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 0.811 | 1 | 0.953 | 0.943 | 1 | 1 | 0.770 | 1 |
| 9 | 0.994 | 0.904 | 1 | 0.856 | 0.938 | 0.783 | 0.696 | 0.694 | 0.874 | 0.868 |
| 10 | 1 | 0.986 | 1 | 1 | 0.997 | 0.992 | 1 | 0.968 | 1 | 1 |

Table 8.2 reports the efficiency scores, optimal values to models (8.1) and (8.2) with $w_i = 1$ ($i = 1, \dots, 4$).

Columns 2-5 characterize the performance of supply chain members based upon model (8.1). The sixth column reports the average efficiency score of the supply chain members. The supply chain performance is reported in the seventh column with Ω_i^* reported in the last four columns.

Although a number of observations on supply chain members are efficient, only one supply chain performance (observation 7) is efficient. i.e., the observation 7 represents the best practice of the supply chain system. Note that in this case, all supply chain members are efficient.

We observe that the average supply chain member efficiency score (column 6) is greater than the supply chain efficiency score (Ω^*). For example, consider observation 5 where two supply chain members (manufacturer and distributor) are efficiently operating. The average supply chain member efficiency score is 0.79456 and the supply chain efficiency score is 0.79456, indicating that the supply chain system could achieve more input savings.

Model (8.2) yields optimal values on the performance measures for (an inefficient) supply chain to reach the best practice. Consider observation 1 in Figure 8.8 where a set of optimal solutions is shown in cells D27:D39. Since $\Omega_4^* = 1$ indicating the retailer is efficient, no adjustments for measures

related to the retailer are required. However, in order to reach the best practice, the supplier, the manufacturer and the distributor should reduce their “direct inputs” based upon Ω_i^* ($i = 1, 2, 3$). In addition, the supplier and the manufacturer should reach an agreement on the selling price of raw materials to increase the supplier’s revenue by 6%. The distributor’s fill rate should be increased to 90.95% (from the current rate of 70%). The products shipped from the manufacturer to the distributor should be reduced by 39%. This solution indicates that based upon the best practice, the distributor should be able to maintain the fill rate of 90.95% while the manufacturer reduces its shipment to the distributor.

Additional managerial information is available from the optimal values of λ_j^* , β_j^* , δ_j^* , and γ_j^* , since they provide information on which observations of supply chain members are used as benchmarks. For example, when the observation 1 is under evaluation by model (8.2), we have (i) $\lambda_7^* = 0.884$, indicating that the supplier in observation 7 is used as the benchmark; (ii) $\beta_2^* = 1.011$, indicating that the manufacturer in observation 2 is used as the benchmark; (iii) $\delta_3^* = 0.08$ and $\delta_9^* = 0.893$, indicating that the distributor in observations 3 and 9 is used as the benchmark; and (iv) $\gamma_1^* = 1$, indicating that the retailer in observation 1 is efficient and itself is used as the benchmark.

Some supply chains may choose to operate with high cost and high availability while others are lean with lower levels of service. The notion of DEA efficiency (i) provides an approach for characterizing and measuring the efficiency of supply chain as well as supply chain members, and (ii) makes it clear that two supply chains may have different input-output mix yet both may be efficient. Model (8.2) enables supply chain members to collectively improve the supply chain performance. Through the use of model (8.2), any supply chains can find ways to achieve best-practice performance and to gain a competitive edge. The approach also provides information on which supply chain members are used as a benchmark when a specific supply chain observation is under evaluation.

8.3 Cooperative and Non-Cooperative Approaches

In this section, we present several models due to Liang, Feng, Cook and Zhu (2006) that directly evaluate the performance of the supply chain as well as its members, while considering the relationship between the buyer and the seller. The modeling processes are based upon the concept of non-cooperative and cooperative games (see, e.g., Simaan and Cruz, 1973; Li, Huang and Ashley, 1995; Huang 2000).

Suppose there are N similar supply chains or N observations on one supply chain. Consider a buyer-seller supply chain as described in Figure 8.4, where for $j = 1, \dots, N$, $X_A = (x_{ij}^A, i = 1, \dots, I)$ is the input vector of the seller, and $Y_A = (y_{rj}^A, r=1, \dots, R)$ is the seller's output vector. Y_A is also an input vector of the buyer. The buyer also has an input vector $X_B = (x_{sj}^B, s = 1, \dots, S)$ and the output vector for the buyer is $Y_B = (y_{tj}^B, t=1, \dots, T)$.

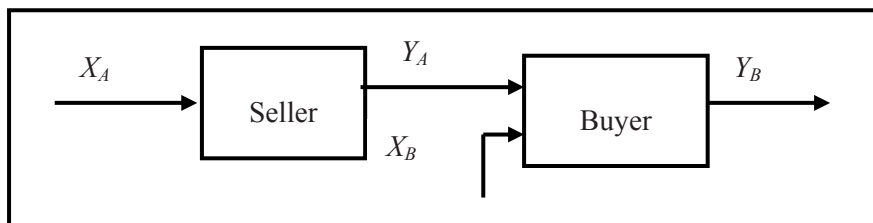


Figure 8.4. Seller-Buyer Supply Chain

8.3.1 The Non-cooperative Model

We propose the seller-buyer interaction be viewed as a two-stage non-cooperative game with the seller as the leader and the buyer as the follower. First, we use the CRS (ratio) model to evaluate the efficiency of the seller as the leader:

$$\begin{aligned} \text{Maximize } E_{AA} &= \frac{\sum_{r=1}^R u_r^A y_{r0}^A}{\sum_{i=1}^I v_i^A x_{i0}^A} \\ \text{subject to} & \\ \frac{\sum_{r=1}^R u_r^A y_{rj}^A}{\sum_{i=1}^I v_i^A x_{ij}^A} &\leq 1 \quad j = 1, \dots, N \\ u_r^A, v_i^A &\geq 0 \quad r=1, \dots, R, i = 1, \dots, I \end{aligned} \tag{8.3}$$

This model is equivalent to the following standard CRS multiplier model:

$$\begin{aligned} \text{Maximize } E_{AA} &= \sum_{r=1}^R \mu_r^A y_{r0}^A \\ \text{subject to} & \\ \sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R \mu_r^A y_{rj}^A &\geq 0 \quad j = 1, \dots, N \\ \sum_{i=1}^I \omega_i^A x_{i0}^A &= 1 \\ \mu_r^A, \omega_i^A &\geq 0 \quad r=1, \dots, R, i = 1, \dots, I \end{aligned} \tag{8.4}$$

Suppose we have an optimal solution of model (8.4) $\mu_r^{A*}, \omega_i^{A*}, E_{AA}^*$ ($r=1, \dots, R, i=1, \dots, I$), and denote the seller's efficiency as E_{AA}^* . We then use the following model to evaluate the buyer's efficiency:

$$\begin{aligned} \text{Maximize } E_{AB} &= \frac{\sum_{t=1}^T u_t^B y_{t0}^B}{D \times \sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S v_s^B x_{s0}^B} \\ \text{subject to} \\ \frac{\sum_{t=1}^T u_t^B y_{tj}^B}{D \times \sum_{r=1}^R \mu_r^A y_{rj}^A + \sum_{s=1}^S v_s^B x_{sj}^B} &\leq 1 \quad j = 1, \dots, N \\ \sum_{r=1}^R \mu_r^A y_{r0}^A &= E_{AA}^* \\ \sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R \mu_r^A y_{rj}^A &\geq 0 \quad j = 1, \dots, N \\ \sum_{i=1}^I \omega_i^A x_{i0}^A &= 1 \\ \mu_r^A, \omega_i^A, u_t^B, v_s^B, D &\geq 0 \\ r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S \end{aligned} \tag{8.5}$$

Note that in model (8.5), we try to determine the buyer's efficiency given that the seller's efficiency remains at E_{AA}^* . Model (8.5) is equivalent to the following non-linear model:

$$\begin{aligned} \text{Maximize } E_{AB} &= \sum_{t=1}^T \mu_t^B y_{t0}^B \\ \text{subject to} \\ d \times \sum_{r=1}^R \mu_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B &\geq 0 \quad j = 1, \dots, N \\ d \times \sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B &= 1 \\ \sum_{r=1}^R \mu_r^A y_{r0}^A &= E_{AA}^* \\ \sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R \mu_r^A y_{rj}^A &\geq 0 \quad j = 1, \dots, N \\ \sum_{i=1}^I \omega_i^A x_{i0}^A &= 1 \\ \mu_r^A, \omega_i^A, \mu_t^B, \omega_s^B, d &\geq 0 \\ r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S \end{aligned} \tag{8.6}$$

Note that $d \times \sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1$ and $\sum_{r=1}^R \mu_r^A y_{r0}^A = E_{AA}^*$.

Thus, we have $0 \leq d < 1/\sum_{r=1}^R \mu_r^A y_{r0}^A = 1/E_{AA}^*$. i.e, we have the upper and lower bounds on d . Therefore, d can be treated as a parameter, and model (8.6) can be solved as a parametric linear program.

In computation, we set the initial d value as the upper bound, namely, $d_0 = 1/E_{AA}^*$, and solve the resulting linear program. We then start to decrease d according to $d_t = 1/E_{AA}^* - \epsilon \times t$ for each step t , where ϵ is a small positive number¹. We solve each linear program of model (8.6) corresponding to d_t and denote the optimal objective value as $E_{BA}^*(d_t)$.

Let $E_{BA}^* = \text{Max}_t E_{BA}^*(d_t)$. Then we obtain a best heuristic search solution E_{BA}^* to model (8.6)². This E_{BA}^* represents the buyer's efficiency when the seller is given the pre-emptive priority to achieve its best performance. The efficiency of the supply chain can then be defined as

$$e_{AB} = \frac{1}{2}(E_{AA}^* + E_{AB}^*)$$

Similarly, one can develop a procedure for the situation when the buyer is the leader and the seller the follower. For example, in the October 6, 2003 issue of the Business Week, its cover story reports that Walmart dominates its suppliers and not only dictates delivery schedules and inventory levels, but also heavily influences product specifications.

We first evaluate the efficiency of the buyer using the standard CRS ratio model:

$$\begin{aligned} \text{Maximize } E_{BB} &= \frac{\sum_{t=1}^T u_t^B y_{t0}^B}{\sum_{r=1}^R u_r^A y_{r0}^A + \sum_{s=1}^S v_s^B x_{s0}^B} \\ \text{subject to} & \\ \frac{\sum_{t=1}^T u_t^B y_{tj}^B}{\sum_{r=1}^R u_r^A y_{rj}^A + \sum_{s=1}^S v_s^B x_{sj}^B} &\leq 1 \quad j = 1, \dots, N \quad (8.7) \\ u_r^A, u_t^B, v_s^B &\geq 0 \end{aligned}$$

¹ In the current study, we set $\epsilon = 0.01$. If we use a smaller ϵ , the difference only shows in the fourth decimal place in the current study.

² The obtained solution can be regarded as the global solution using a heuristic technique, as it searches through the entire feasible region of d when d is decreased from its upper bound to lower bound of zero. It is likely that estimation error exists. The smaller the decreased step, the better the heuristic search solution will be.

$$r=1, \dots, R, t=1, \dots, T, s=1, \dots, S$$

Model (8.7) is equivalent to the following standard CRS multiplier model:

$$\begin{aligned} &\text{Maximize } E_{BB} = \sum_{t=1}^T \mu_t^B y_{t0}^B \\ &\text{subject to} \\ &\sum_{r=1}^R \mu_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B \geq 0 \quad j = 1, \dots, N \quad (8.8) \\ &\sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1 \\ &\mu_r^A, \mu_t^B, \omega_s^B \geq 0 \\ &r=1, \dots, R, t=1, \dots, T, s=1, \dots, S \end{aligned}$$

Let $\mu_r^{A*}, \mu_t^{B*}, \omega_s^{B*}, E_{BB}^*$ ($r=1, \dots, R, t=1, \dots, T, s=1, \dots, S$) be an optimal solution from model (8.8), where E_{BB}^* represents the buyer's efficiency score. To obtain the seller's efficiency given that the buyer's efficiency is equal to E_{BB}^* , we solve the following model:

$$\begin{aligned} &\text{Maximize } E_{BA} = \frac{U \times \sum_{r=1}^R \mu_r^A y_{r0}^A}{\sum_{i=1}^I v_i^A x_{i0}^A} \\ &\text{subject to} \\ &\frac{U \times \sum_{r=1}^R \mu_r^A y_{rj}^A}{\sum_{i=1}^I v_i^A x_{ij}^A} \leq 1 \quad j = 1, \dots, N \quad (8.9) \\ &\sum_{t=1}^T \mu_t^B y_{t0}^B = E_{BB}^* \\ &\sum_{r=1}^R \mu_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B \geq 0 \quad j = 1, \dots, N \\ &\sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1 \\ &v_i^A, \mu_r^A, \mu_t^B, \omega_s^B, U \geq 0 \\ &r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S \end{aligned}$$

Model (8.9) is equivalent to the following non-linear program:

$$\begin{aligned} &\text{Maximize } E_{BA} = u \times \sum_{r=1}^R \mu_r^A y_{r0}^A \\ &\text{subject to} \end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^I \omega_i^A x_{ij}^A - u \times \sum_{r=1}^R \mu_r^A y_{rj}^A &\geq 0 & j = 1, \dots, N \\
\sum_{i=1}^I \omega_i^A x_{i0}^A &= 1 & (8.10) \\
\sum_{t=1}^T \mu_t^B y_{t0}^B &= E_{BB}^* \\
\sum_{r=1}^R \mu_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B &\geq 0 & j = 1, \dots, N \\
\sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B &= 1 \\
\omega_i^A, \mu_r^A, \mu_t^B, \omega_s^B, u &\geq 0 \\
r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S
\end{aligned}$$

This model (8.10) is similar to model (8.6) and can be treated as a linear program with u as the parameter. We next show how to select the initial value of this parameter.

We first solve the following model:

$$\begin{aligned}
\text{Maximize } EF_{BA} &= \frac{U \times \sum_{r=1}^R \mu_r^{A*} y_{r0}^A}{\sum_{i=1}^I v_i^A x_{i0}^A} \\
\text{subject to} & \\
\frac{U \times \sum_{r=1}^R \mu_r^{A*} y_{rj}^A}{\sum_{i=1}^I v_i^A x_{ij}^A} &\leq 1 & j = 1, \dots, N & (8.11) \\
v_i^A, U &\geq 0 \quad i = 1, \dots, I
\end{aligned}$$

where μ_r^{A*} ($r=1, \dots, R$) is an optimal solution from model (8.8).

Model (8.11) is equivalent to the following linear program:

$$\begin{aligned}
\text{Maximize } EF_{BA} &= u \times \sum_{r=1}^R \mu_r^{A*} y_{r0}^A \\
\text{subject to} & \\
\sum_{i=1}^I \omega_i^A x_{ij}^A - u \times \sum_{r=1}^R \mu_r^{A*} y_{rj}^A &\geq 0 & j = 1, \dots, N & (8.12) \\
\sum_{i=1}^I \omega_i^A x_{i0}^A &= 1 \\
\omega_i^A, u &\geq 0 \quad i = 1, \dots, I
\end{aligned}$$

Let $\omega_i^{A*}, u^*, EF_{BA}^*$ ($i = 1, \dots, I$) be an optimal solution from model (8.12). Note that the optimal value to model (8.12), EF_{BA}^* , may not be the maximum value for the seller because of possible multiple optima in model

(8.8). We have $u \times \sum_{r=1}^R \mu_r^A y_{r0}^A \geq EF_{BA}^*$. Further, based upon $\sum_{r=1}^R \mu_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1$, we have $\sum_{r=1}^R \mu_r^A y_{r0}^A \leq 1$. Therefore, $u \geq EF_{BA}^*$. We then utilize EF_{BA}^* as the lower bound for the parameter u when solving for seller's efficiency using model (8.10). However, this lower bound can be converted into an upper bound as follows.

Let $u \times \mu_r^A = c_r^A, g = 1/u$, then model (8.10) is equivalent to the following model:

$$\begin{aligned}
 &\text{Maximize } E_{BA} = \sum_{r=1}^R c_r^A y_{r0}^A \\
 &\text{subject to} \\
 &\sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R c_r^A y_{rj}^A \geq 0 \quad j = 1, \dots, N \\
 &\sum_{i=1}^I \omega_i^A x_{i0}^A = 1 \\
 &\sum_{t=1}^T \mu_t^B y_{t0}^B = E_{BB}^* \tag{8.13} \\
 &\frac{1}{g} \sum_{r=1}^R c_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B \geq 0 \quad j = 1, \dots, N \\
 &\frac{1}{g} \sum_{r=1}^R c_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1 \\
 &\omega_i^A, c_r^A, \mu_t^B, \omega_s^B, g \geq 0, \\
 &r=1, \dots, R, t=1, \dots, T, i = 1, \dots, I, s = 1, \dots, S
 \end{aligned}$$

where $0 \leq g \leq 1/EF_{BA}^*$ can be treated as a parameter.

We solve model (8.13) for the seller's efficiency. The computational procedure is similar to the one used in model (8.6). Denote the heuristic search solution to (8.13) as E_{BA}^* . Then the efficiency of the supply chain can be defined as

$$e_{BA} = \frac{1}{2}(E_{BA}^* + E_{BB}^*)$$

Table 8.3. Numerical Example

| DMU | x_1^A | x_2^A | y_1^A | y_2^A | x^B | y^B |
|-----|---------|---------|---------|---------|-------|-------|
| 1 | 8 | 50 | 20% | 10 | 8 | 100 |
| 2 | 10 | 18 | 10% | 15 | 10 | 70 |
| 3 | 15 | 30 | 10% | 20 | 8 | 95 |
| 4 | 8 | 25 | 20% | 25 | 10 | 80 |
| 5 | 10 | 40 | 15% | 20 | 15 | 85 |

Table 8.4. Leader-follower Results

| DMU | Model (8.4) Seller | Model (8.6) Buyer | e_{AB} | Model (8.8) Buyer | Model (8.13) Seller | e_{BA} |
|-----|--------------------|-------------------|----------|-------------------|---------------------|----------|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0.833 | 0.56 | 0.697 | 0.875 | 0.766 | 0.821 |
| 3 | 0.667 | 0.95 | 0.808 | 1 | 0.546 | 0.773 |
| 4 | 1 | 0.653 | 0.827 | 0.653 | 1 | 0.827 |
| 5 | 0.64 | 0.453 | 0.547 | 0.756 | 0.621 | 0.688 |

We now illustrate the above DEA procedures with five supply chain operations (DMUs) given in Table 8.3. The seller has two inputs, x_1^A (labor) and x_2^A (cost) and two outputs, y_1^A (buyer’s fill rate) and y_2^A (number of product shipped). The buyer has another input x^B (labor) and one output: y^B .

Table 8.4 reports the efficiency scores obtained from the supply chain efficiency models.

When the seller is treated as the leader, two seller operations in DMUs 1 and 4 are efficient with only one efficient buyer operation in DMU1. This indicates that only DMU1 is the efficient supply chain.

When the buyer is treated as the leader, model (8.8) shows that three buyer operations are inefficient and model (8.13) shows that only two seller operations are efficient. This also implies that only DMU1 is efficient.

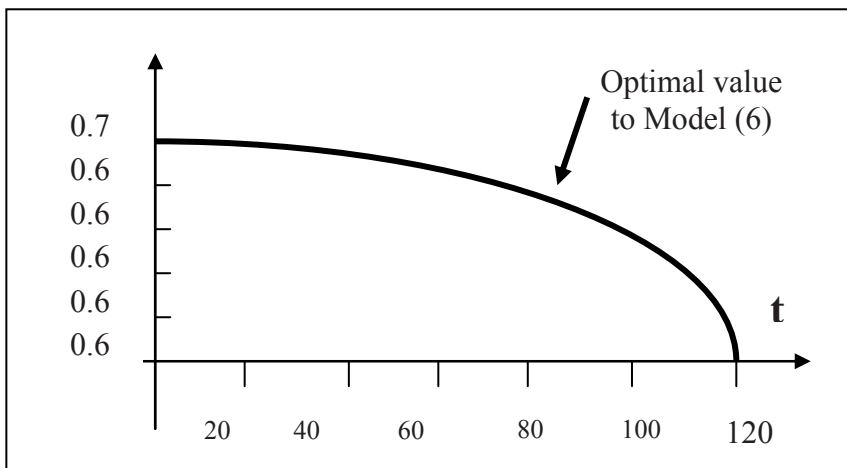


Figure 8.5. Solving Non-Cooperative Model for DMU2

Figure 8.5 shows how the best heuristic search is obtained when solving model (8.6) for DMU2. We set $d_t = 1/E_{AA}^* - 0.01 \times t$, where $E_{AA}^* = 0.833$ and $t = 0, \dots, 120$. Note that when $t = 120$, the parameter $d = 0$, the lower

bound, and the optimal value to model (8.6) is 0.650. Therefore, we have completed the search over the entire feasible region of d and the best solution is obtained at $t = 0$, that is $E_{AB}^* = 0.697$.

8.3.2 The Cooperative Model

In game theory, when the buyer-seller relation was treated as leader-follower, the buyer does not have control over the seller, and the seller determines the optimal strategy (optimal weights for the intermediate measures). Recent studies however have demonstrated that many retailers (buyers) have increased their bargaining power relative to the manufactures' (sellers) bargaining power (Porter, 1974; Li, Huang and Ashley, 1996). The shift of power from manufacturers to retailers is one of the most significant phenomena in manufacturing and retailing. Walmart is an extreme case where the manufacturer becomes a "follower". Therefore, it is in the best interest of the supply chain to encourage cooperation. This section considers the case where both the seller and buyer have the same degree of power to influence the supply chain system. Our new DEA model seeks to maximize both the seller's and buyer's efficiency, subject to a condition that the weights on the intermediate measures must be equal:

$$\text{Maximize } \frac{1}{2} \left[\frac{\sum_{r=1}^R c_r y_{r0}^A}{\sum_{i=1}^I v_i^A x_{i0}^A} + \frac{\sum_{t=1}^T u_t^B y_{t0}^B}{\sum_{r=1}^R c_r y_{r0}^A + \sum_{s=1}^S v_s^B x_{s0}^B} \right]$$

subject to

$$\frac{\sum_{r=1}^R c_r y_{rj}^A}{\sum_{i=1}^I v_i^A x_{ij}^A} \leq 1 \quad j = 1, \dots, N \tag{8.14}$$

$$\frac{\sum_{t=1}^T u_t^B y_{tj}^B}{\sum_{r=1}^R c_r y_{rj}^A + \sum_{s=1}^S v_s^B x_{sj}^B} \leq 1 \quad j = 1, \dots, N$$

$$c_r, u_t^B, v_i^A, v_s^B \geq 0$$

$$r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S$$

We call model (8.14) the cooperative efficiency evaluation model, because it maximizes the joint efficiency of the buyer and seller, and forces the two players to agree on a common set of weights on the intermediate measures³.

We apply the following Charnes-Cooper transformation to model (8.14):

³ In cooperative game theory, the joint profit of seller and buyer is maximized.

$$t_1 = \frac{1}{\sum_{i=1}^I v_i^A x_{i0}^A} \quad t_2 = \frac{1}{\sum_{r=1}^R c_r^A y_{r0}^A + \sum_{s=1}^S v_s^B x_{s0}^B}$$

$$\omega_i^A = t_1 v_i^A \quad c_r^A = t_1 c_r \quad \mu_t^B = t_2 u_t^B$$

$$\omega_s^B = t_2 v_s^B \quad c_r^B = t_2 c_r$$

$r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S$

Note that in the above transformation, $c_r^A = t_1 c_r$ and $c_r^B = t_2 c_r$ imply a linear relationship between c_r^A and c_r^B . Therefore, we can assume $c_r^B = k \times c_r^A$, $k \geq 0$. Then model (8.14) can be changed into:

$$\text{Maximize } \frac{1}{2} \left[\sum_{r=1}^R c_r^A y_{r0}^A + \sum_{t=1}^T \mu_t^B y_{t0}^B \right]$$

subject to

$$\sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R c_r^A y_{rj}^A \geq 0 \quad j = 1, \dots, N$$

$$\sum_{r=1}^R c_r^B y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B \geq 0 \quad j = 1, \dots, N$$

$$\sum_{i=1}^I \omega_i^A x_{i0}^A = 1 \quad (15)$$

$$\sum_{r=1}^R c_r^B y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1$$

$$c_r^B = k \times c_r^A$$

$$c_r^A, c_r^B, \mu_t^B, \omega_i^A, \omega_s^B, k \geq 0$$

$r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S$

Model (8.15) is a non-linear programming problem, and can be converted into the following model:

$$\text{Maximize } \frac{1}{2} \left[\sum_{r=1}^R c_r^A y_{r0}^A + \sum_{t=1}^T \mu_t^B y_{t0}^B \right]$$

subject to

$$\sum_{i=1}^I \omega_i^A x_{ij}^A - \sum_{r=1}^R c_r^A y_{rj}^A \geq 0 \quad j = 1, \dots, N$$

$$\sum_{r=1}^R k \times c_r^A y_{rj}^A + \sum_{s=1}^S \omega_s^B x_{sj}^B - \sum_{t=1}^T \mu_t^B y_{tj}^B \geq 0 \quad j = 1, \dots, N \quad (8.16)$$

$$\sum_{i=1}^I \omega_i^A x_{i0}^A = 1$$

$$\sum_{r=1}^R k \times c_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1$$

$$c_r^A, \mu_t^B, \omega_i^A, \omega_s^B, k \geq 0$$

$$r=1, \dots, R, t=1, \dots, T, i=1, \dots, I, s=1, \dots, S$$

Note that

$$\sum_{r=1}^R k \times c_r^A y_{r0}^A + \sum_{s=1}^S \omega_s^B x_{s0}^B = 1, \quad \sum_{r=1}^R c_r^A y_{r0}^A \leq 1, \quad \sum_{s=1}^S \omega_s^B x_{s0}^B > 0$$

in model (8.15). We have $k = (1 - \sum_{s=1}^S \omega_s^B x_{s0}^B) / \sum_{r=1}^R c_r^A y_{r0}^A < 1 / \sum_{r=1}^R c_r^A y_{r0}^A$.

Note also that the optimal $\sum_{r=1}^R c_r^{A*} y_{r0}^A$ in model (8.15) will not be less than E_{BA}^* in model (8.10). Thus, we have $0 \leq k < 1 / E_{BA}^*$. That is, model (8.15) can be treated as a parametric linear program, and we can obtain a heuristic search solution using the procedure developed for models (8.6) and (8.13).

At the optima, let $\theta_A^* = \sum_{r=1}^R c_r^{A*} y_{r0}^A$ and $\theta_B^* = \sum_{t=1}^T \mu_t^{B*} y_{t0}^B$ represent the efficiency scores for the seller and buyer respectively. The following two remarks show that in general, the supply chain efficiency under the assumption of cooperation will not be less than the efficiency under the assumption of non-cooperation.

Remark 8.1: If we set $\sum_{r=1}^R c_r^A y_{r0}^A = E_{AA}^*$ as a constraint in model (8.15), then the feasible region of model (8.15) is the same as that of model (8.6). Therefore, $V_P^* = e_{AB}$.

Remark 8.2: If we set $\sum_{t=1}^T \mu_t^B y_{t0}^B = E_{BB}^*$ as a constraint in model (8.15), then the feasible region of model (8.15) is the same as that of model (8.13). Therefore, $V_P^* = e_{BA}$.

We consider again the numerical example in Table 8.3. Table 8.6 reports the results from model (8.15), where columns 2 and 3 report the efficiency scores for the seller and buyer respectively, and the last column reports the optimal value to model (8.15), the supply chain efficiency.

Table 8.5. Cooperative structure results

| DMU | θ_A^* | θ_B^* | Supply chain |
|-----|--------------|--------------|--------------|
| 1 | 1 | 1 | 1 |
| 2 | 0.766 | 0.875 | 0.821 |
| 3 | 0.667 | 0.95 | 0.808 |
| 4 | 1 | 0.653 | 0.827 |
| 5 | 0.621 | 0.756 | 0.688 |

Table 8.6. Comparison of Non-cooperative and Cooperative results

| DMU | e_{AB} | e_{BA} | Model (15) |
|-----|----------|----------|------------|
| 1 | 1 | 1 | 1 |
| 2 | 0.697 | 0.821 | 0.821 |
| 3 | 0.808 | 0.773 | 0.808 |
| 4 | 0.827 | 0.827 | 0.827 |
| 5 | 0.547 | 0.688 | 0.688 |

Table 8.7 compares the efficiency scores for the cooperative and non-cooperative assumptions. In this numerical example, for all DMUs, one of the two leader-follower models achieves efficiency under the cooperative assumption. This indicates that no better solution can be found to yield a higher efficiency in the cooperative assumption. However, in other examples, the supply chain is likely to show a better performance when assuming cooperative operation.

REFERENCES

1. Chen, Y. and J. Zhu, (2004), Measuring information technology's indirect impact on firm performance, *Information Technology & Management Journal*, Vol. 5, Issue 1-2, 9-22.
2. Cheung, K.L. and W.H. Hansman (2000), An exact performance evaluation for the supplier in a two-echelon inventory system, *Operations Research* 48, 646-653.
3. Färe, R. and S. Grosskopf (2000), Network DEA, *Socio-Economic Planning Sciences*, 34, 35-49.
4. Golany, B., S.T. Hackman and U. Passy (2003), An efficiency measurement framework for multi-stage production systems," Working paper.
5. Huang, Z. (2000), Franchising cooperation through chance cross-constrained games, *Naval Research Logistics*, 47 669-685.
6. Lee, H.L. and C. Billington (1992), Managing supply chain inventory: pitfalls and opportunities, *Sloan Management Review* 33, No. 3, 65-73.
7. Li, S.X., Z.Huang and A. Ashley (1995), Seller-buyer system cooperation in a monopolistic market, *Journal of the Operational Research Society*, 46, 1456-1470.
8. Li, S.X., Z.Huang and A. Ashley (1996), Improving buyer-seller system cooperation through inventory control, *International Journal of Production Economics*, 43, 37-46.
9. Liang, L., F. Yang, W.D. Cook and J. Zhu (2006), DEA models for supply chain efficiency evaluation, *Annals of Operations Research*, Vol. 145, No. 1, 35-49.
10. Porter, M.E., (1974), Consumer behavior, retailer power and market performance in consumer goods industries, *Review of Econom. Statist.* LVI, 419-436.
11. Seiford, L.M. and J. Zhu (1999a), Profitability and marketability of the top 55 US commercial banks, *Management Science*, 45, No. 9, 1270-1288.
12. Seiford, L.M. and J. Zhu (1999b), Sensitivity and stability of the classification of returns to scale in data envelopment analysis, *Journal of Productivity Analysis*, 12, No. 1, 55-75.
13. Simaan, M. and J.B. Cruz (1973), On the Stackelberg strategy in nonzero-sum games, *Journal of Optimization Theory and Applications* 11, 533-555.

Chapter 9

Congestion

9.1 Congestion Measure

Congestion, as used in economics, refers to situations where reductions in one or more inputs generate an increase in one or more outputs. Examples can be found in underground mining and agriculture. For example, too much fertilizer applied to a given plot could reduce the overall output. We here adopt the following definition of congestion from Cooper, Thompson and Thrall (1996).

Definition 9.1 (Congestion) Evidence of congestion is present when reductions in one or more inputs can be associated with increases in one or more outputs – or, proceeding in reverse, when increases in one or more inputs can be associated with decreases in one or more outputs – without worsening any other input or output.

Färe and Grosskopf (1983) apply this concept to DEA using strong and weak input disposabilities. The envelopment DEA models discussed in Chapter 1 are strong input/output disposability models. We re-write the VRS envelopment models as

$$\begin{aligned} \theta^* &= \min \theta \\ \text{subject to} \\ \sum_{j=1}^n \lambda_j x_{ij} &\leq \theta x_{io} & i = 1, 2, \dots, m; \\ \sum_{j=1}^n \lambda_j y_{rj} &\geq y_{ro} & r = 1, 2, \dots, s; \end{aligned} \tag{9.1}$$

$$\begin{aligned}
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned}$$

$$\begin{aligned}
\phi^* &= \max \phi \\
&\text{subject to} \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \phi y_{ro} \quad r = 1, 2, \dots, s; \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \tag{9.2}$$

If we assume weak disposability of inputs and outputs in models (9.1) and (9.2), respectively, we obtain

$$\begin{aligned}
\tilde{\theta}^* &= \min \tilde{\theta} \\
&\text{subject to} \\
& \sum_{j=1}^n \lambda_j x_{ij} = \tilde{\theta} x_{io} \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \tag{9.3}$$

$$\begin{aligned}
\tilde{\phi}^* &= \max \tilde{\phi} \\
&\text{subject to} \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n \lambda_j y_{rj} = \tilde{\phi} y_{ro} \quad r = 1, 2, \dots, s; \\
& \sum_{j=1}^n \lambda_j = 1 \\
& \lambda_j \geq 0 \quad j = 1, \dots, n.
\end{aligned} \tag{9.4}$$

Note that, for example, the difference between models (9.1) and (9.3) is that input inequalities are changed into input equalities. If we apply weak disposability to other envelopment models, we obtain the weak disposability DEA models shown in Table 9.1.

The input and output congestion measures are then defined as $C(\theta^*, \tilde{\theta}^*) = \theta^* / \tilde{\theta}^*$, and $C(\phi^*, \tilde{\phi}^*) = \phi^* / \tilde{\phi}^*$, respectively. Note that we must have $\theta^* \leq \tilde{\theta}^*$ because the latter is associated with equalities. As shown by Färe, Grosskopf and Lovell (1994), we can use $C(\theta^*, \tilde{\theta}^*)$ (or $C(\phi^*, \tilde{\phi}^*)$) as a

measure of congestion with the following properties. If $C(\theta^*, \tilde{\theta}^*) = 1$ ($C(\phi^*, \tilde{\phi}^*) = 1$), then input (output) is not congested; alternatively, if $C(\theta^*, \tilde{\theta}^*) < 1$ ($C(\phi^*, \tilde{\phi}^*) > 1$), then input (output) congestion is present.

Table 9.1. Weak Disposability DEA Models

| Frontier Type | Weak Input Disposability | Weak Output Disposability |
|---------------|---|---|
| | $\min \tilde{\theta}$ subject to $\sum_{j=1}^n \lambda_j x_{ij} = \tilde{\theta} x_{io} \quad i = 1, 2, \dots, m;$ $\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s;$ $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ | $\max \tilde{\phi}$ subject to $\sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m;$ $\sum_{j=1}^n \lambda_j y_{rj} = \tilde{\phi} y_{ro} \quad r = 1, 2, \dots, s;$ $\lambda_j \geq 0 \quad j = 1, 2, \dots, n.$ |
| CRS | | |
| VRS | | Add $\sum_{j=1}^n \lambda_j = 1$ |
| NIRS | | Add $\sum_{j=1}^n \lambda_j \leq 1$ |
| NDRS | | Add $\sum_{j=1}^n \lambda_j \geq 1$ |

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------|-----------|------|------------|-----|------------|---|--------|-----------|------|
| 1 | Mines | Labor | K1 | K2 | T1 | 1/D1 | | Output | λ | Weak |
| 2 | 1 | 98.5 | 142 | 245 | 6 | 0.016 | | 3264 | 0 | 1 |
| 3 | 2 | 96.5 | 30 | 215 | 6 | 0.016 | | 3065 | 0 | 1 |
| 4 | 3 | 57.6 | 18 | 105 | 5.6 | 0.026 | | 2275 | 0 | 1 |
| 5 | 4 | 59.2 | 180 | 0 | 5.9 | 0.025 | | 1978 | 0 | 1 |
| 6 | 5 | 57.6 | 200 | 0 | 8 | 0.022 | | 1833 | 0 | 1 |
| 7 | 6 | 49.9 | 27 | 85 | 4.5 | 0.019 | | 1218 | 0 | 1 |
| 8 | 7 | 53.5 | 143 | 65 | 6 | 0.01 | | 928 | 0 | 1 |
| 9 | 8 | 34 | 70 | 65 | 6 | 0.02 | | 919 | 0 | 1 |
| 10 | 9 | 39.6 | 67.5 | 40 | 6.5 | 0.013 | | 777 | 0 | 1 |
| 11 | 10 | 51.3 | 0 | 145 | 3.2 | 0.019 | | 745 | 0 | 1 |
| 12 | 11 | 74.2 | 110 | 65 | 2.1 | 0.014 | | 742 | 0 | 1 |
| 13 | 12 | 24 | 25 | 65 | 4.4 | 0.012 | | 488 | 0 | 1 |
| 14 | 13 | 26.5 | 58 | 0 | 3 | 0.014 | | 407 | 0 | 1 |
| 15 | 14 | 43.1 | 70 | 0 | 6.5 | 0.012 | | 402 | 0 | 1 |
| 16 | 15 | 20.7 | 236 | 0 | 5.7 | 0.01 | | 396 | 1 | 1 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 1 | | | | |
| 20 | Labor | 20.7 | = | 20.7 | | | | | | |
| 21 | K1 | 236 | = | 236 | | | | | | |
| 22 | K2 | 0 | = | 0 | | | | | | Weak |
| 23 | T1 | 5.7 | = | 5.7 | | | | | | |
| 24 | 1/D1 | 0.01 | = | 0.01 | | | | | | |
| 25 | output | 396 | ≥ | 396 | | | | | | |
| 26 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | |

Figure 9.1. VRS Weak Input Disposability Spreadsheet Model

Byrnes, Färe and Grosskopf (1984) study the congestion of 15 Illinois coal mines. Figure 9.1 presents the 15 mines with one output (thousands tons) and five inputs, namely, labor (thousand miner-days), dragline capacity (K1) (cubic yards), power-shovel capacity (K2) (cubic yards), thickness of first-seam mined (T1) (feet), and reciprocal of depth to first-seam mined (1/D1) (D1 in feet).

In Figure 9.1, cells B20:B26 contain the formulas

Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)

Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)

Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)

Cell B23 =SUMPRODUCT(E2:E16,\$I\$2:\$I\$16)

Cell B24 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)

Cell B25 =SUMPRODUCT(H2:H16,\$I\$2:\$I\$16)

Cell B26 =SUM(I2:I16)

where cells I2:I16 represent the changing cells, λ_j ($j = 1, \dots, 15$).

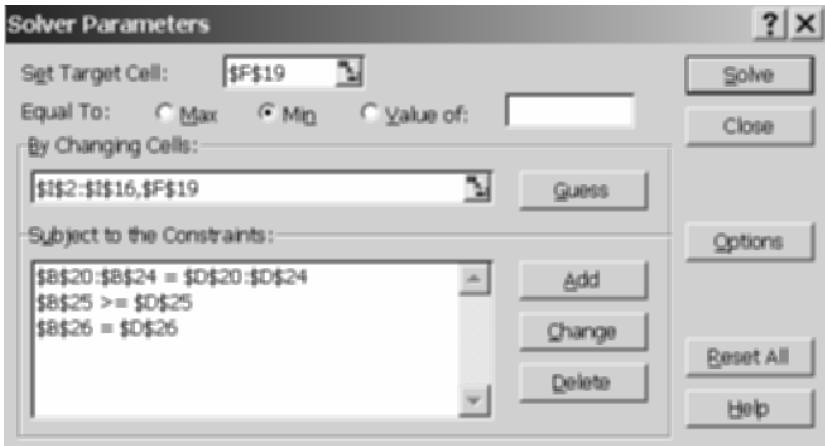


Figure 9.2. Solver Parameters for VRS Weak Input Disposability Model

Cells D20:D25 contain the formulas

D20 = \$F\$19*INDEX(\$B\$2:\$F\$16,\$E\$18,1)

D21 = \$F\$19*INDEX(\$B\$2:\$F\$16,\$E\$18,2)

D22 = \$F\$19*INDEX(\$B\$2:\$F\$16,\$E\$18,3)

D23 = \$F\$19*INDEX(\$B\$2:\$F\$16,\$E\$18,4)

D24 = \$F\$19*INDEX(\$B\$2:\$F\$16,\$E\$18,5)

D25 = INDEX(H2:H16,\$E\$18,1)

where F19 is the target cell ($\tilde{\theta}$), and cell E18 indicates the DMU under evaluation.

Figure 9.2 shows the Solver parameters for model (9.3) shown in Figure 9.1.

The optimal value to model (9.3) in this case is equal to one across all DMUs. i.e., each mine is on the frontier. To obtain the congestion measure, we also need to calculate model (9.1). Figure 9.3 shows the results. Figure 9.3 is the input-oriented VRS envelopment model where the inputs are strongly disposable. The related Solver parameters can be obtained by changing the equalities to inequalities in Figure 9.2, as shown in Figure 9.4. The efficiency scores are reported in cells J2:J16. The efficiency scores for weak input disposability are reported in cells K2:K16. It can be seen that congestion is present at DMUs 6 and 8.

| | A | B | C | D | E | F | G | H | I | J | K |
|----|------------------|-----------|------|------------|-----|------------|---|--------|-----------|---------|------|
| 1 | Mines | Labor | K1 | K2 | T1 | 1/D1 | | Output | λ | Strong | Weak |
| 2 | 1 | 98.5 | 142 | 245 | 6 | 0.016 | | 3264 | 0 | 1 | 1 |
| 3 | 2 | 96.5 | 30 | 215 | 6 | 0.016 | | 3065 | 0 | 1 | 1 |
| 4 | 3 | 57.6 | 18 | 105 | 5.6 | 0.026 | | 2275 | 0 | 1 | 1 |
| 5 | 4 | 59.2 | 160 | 0 | 5.9 | 0.025 | | 1978 | 0 | 1 | 1 |
| 6 | 5 | 57.6 | 200 | 0 | 8 | 0.022 | | 1833 | 0 | 1 | 1 |
| 7 | 6 | 49.9 | 27 | 85 | 4.5 | 0.019 | | 1218 | 0 | 0.97129 | 1 |
| 8 | 7 | 53.5 | 143 | 65 | 6 | 0.01 | | 928 | 0 | 1 | 1 |
| 9 | 8 | 34 | 70 | 65 | 6 | 0.02 | | 919 | 0 | 0.93747 | 1 |
| 10 | 9 | 39.6 | 67.5 | 40 | 6.5 | 0.013 | | 777 | 0 | 1 | 1 |
| 11 | 10 | 51.3 | 0 | 145 | 3.2 | 0.019 | | 745 | 0 | 1 | 1 |
| 12 | 11 | 74.2 | 110 | 65 | 2.1 | 0.014 | | 742 | 0 | 1 | 1 |
| 13 | 12 | 24 | 25 | 65 | 4.4 | 0.012 | | 488 | 0 | 1 | 1 |
| 14 | 13 | 26.5 | 58 | 0 | 3 | 0.014 | | 407 | 0 | 1 | 1 |
| 15 | 14 | 43.1 | 70 | 0 | 8.5 | 0.012 | | 402 | 0 | 1 | 1 |
| 16 | 15 | 20.7 | 238 | 0 | 5.7 | 0.01 | | 396 | 1 | 1 | 1 |
| 17 | | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | | |
| 19 | Constr | set | | Evaluation | | 1 | | | | | |
| 20 | Labor | 20.7 | <= | 20.7 | | | | | | | |
| 21 | K1 | 236 | <= | 236 | | | | | | | |
| 22 | K2 | 0 | <= | 0 | | | | | | | |
| 23 | T1 | 5.7 | <= | 5.7 | | | | | | | |
| 24 | 1/D1 | 0.01 | <= | 0.01 | | | | | | | |
| 25 | output | 396 | >= | 396 | | | | | | | |
| 26 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | | |

Figure 9.3. Congestion Measure For 15 Mines

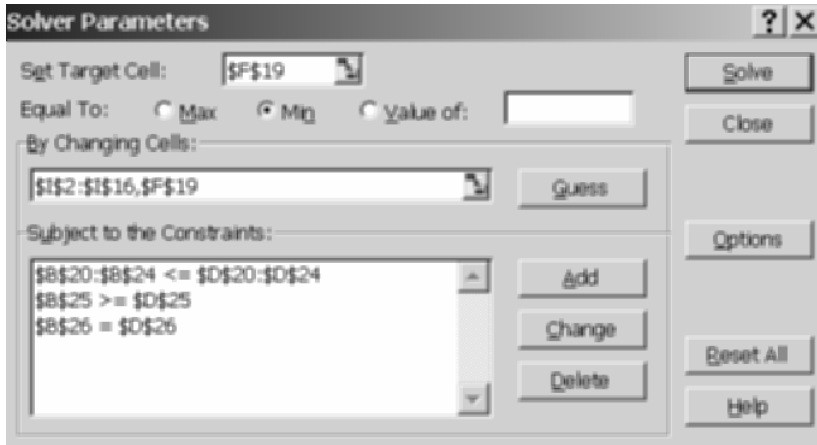


Figure 9.4. Solver Parameters for Input-oriented VRS Strong Input Disposability Model

When input congestion is present, we need to identify sources and amounts of congestion. Färe, Grosskopf and Lovell (1994) suggest a procedure for identifying input measure responsible for the input congestion.

$$\begin{aligned}
 &\alpha^* = \min \alpha \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_{ij} = \alpha x_{io} \quad i \in A; \\
 &\sum_{j=1}^n \lambda_j x_{ij} \leq \alpha x_{io} \quad i \in \bar{A}; \\
 &\sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s; \\
 &\sum_{j=1}^n \lambda_j = 1 \\
 &\lambda_j \geq 0 \quad j = 1, \dots, n.
 \end{aligned} \tag{9.5}$$

where $A \subseteq \{1, 2, \dots, m\}$ and \bar{A} is the complement. Using $\tilde{\theta}^*$ and α^* for each $A \subseteq \{1, 2, \dots, m\}$, if $C(\theta^*, \tilde{\theta}^*) < 1$, and $\theta^* = \alpha^*$, as obtained from (9.1) and (9.5), the components of the subvectors associated with \bar{A} ($= \{i \mid i \notin A\}$) then identify sources and amounts of congestion. Similar models can be established for different RTS frontier and orientation assumptions. For example, if we remove $\sum \lambda_j = 1$, we obtain the model under VRS.

The suggested route requires additional computation which can be onerous because it involves obtaining solutions over all possible partitions of A . In fact, the route followed by Färe, Grosskopf and Lovell (1994) emphasizes efficiency measurements with identification of sources and amounts of inefficiencies to be undertaken as an additional job.

9.2 Congestion and Slacks

We first provide the following definition.

Definition 9.2 (DEA Slacks) An optimal value of s_i^- and s_r^+ in (1.4) (or (1.7)), which we represent by s_i^{-*} and s_r^{+*} , are respectively called DEA input and output slack values. i.e., we refer to the slacks obtained in the second stage of DEA calculation as DEA slacks.

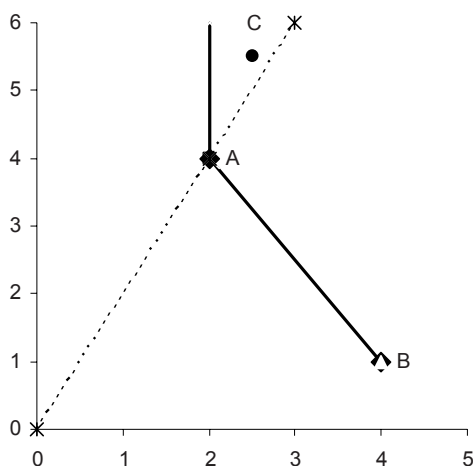


Figure 9.5. Congestion at point C

To illustrate the nature of congestion. Figures 9.5 and 9.6 plot an input isoquant. The input isoquant bends at point A in Figure 9.5 because of the weak input disposability. As a result, AC is part the frontier, and for C, the optimal value to model (9.3) is one. However, for C, the optimal value to model (9.1) is less than one. Thus, input congestion is presented at C in Figure 9.5.

In Figure 9.6, the isoquant bends at point D. Because of the existence of D, the optimal values to models (9.1) and (9.3) are equal. Thus, input congestion is absent at C in Figure 9.6.

Furthermore, note that if the efficient reference set consists of A, point C will have a positive DEA slack value for the second input x_2 . Because of the presence of the weakly efficient point D (a frontier point with non-zero DEA

slacks), if the efficient reference set consists of points A and D, point C will not have slack values. (The (input) slacks do not necessarily represent DEA slack values.)

However, if all frontier DMUs are extreme efficient, e.g., A and B, in Figure 9.5, then the input slacks are the same as the DEA slack values. In Figure 9.6, because C can be compared to a convex combination of D (weakly efficient) and A, no input slack is detected.

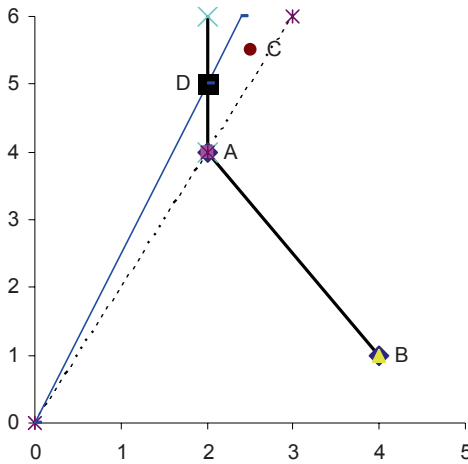


Figure 9.6. No Congestion at Point C

Theorem 9.1 Input congestion as defined by $C(\theta^*, \tilde{\theta}^*)$ is not present in the performance of DMU_o if and only if an optimal solution is associated with referent frontier DMUs such that non-zero input slack values are not detected in model (9.1).

[Proof]: Recall that the only difference between (9.1) and (9.3) is that the input inequalities are changed to equalities. The referent frontier DMUs are those in the basis when calculating the strong disposability model (9.1). If we have some referent DMUs such that no non-zero input slack values are detected for DMU_o , then we have, at optimality,

$$\sum_{j \in B} \lambda_j^* x_{ij} = \theta^* x_{io} \text{ for } i = 1, \dots, m$$

where B represents the set of referent DMUs, $B = \{j \mid \lambda_j^* > 0\}$. Obviously, λ_j^* and θ^* are also optimal for (9.3). Therefore $\theta^* = \beta^*$. Thus, no input congestion occurs. This completes the *if* part.

To establish the *only if* part, we note that if no input congestion is identified when an optimum is associated with a basis B' such that $\sum_{j \in B'} \lambda_j^* x_{ij} = \beta^* x_{io} = \theta^* x_{io}$, then this same optimum provides referent DMUs such that the input constraints are binding in (9.1). Therefore no non-zero input slack values are detected by reference to those DMUs in B' . ■

It is well known that in the single input and the single output situation, no input or output slack will occur for CRS envelopment models, whereas non-zero slack values may occur for VRS models. That is to say, in the single input and the single output situation, congestion will never occur with CRS but can possibly happen with VRS.

Based upon Theorem 9.1, we have

Corollary 9.1 If the observed values on the efficient frontier are composed only of extreme efficient DMUs, then congestion can occur *if and only if* non-zero DEA slack values are detected. Furthermore, the sources of congestion can then only be found in these non-zero DEA slack values.

Corollary 9.1 can be important in real world applications, since the frontiers in most real world data sets contain only the extreme efficient DMUs. Consequently, the congestion and its amount can simply be represented by the DEA slacks (see Ray, Seiford and Zhu, 1998).

The discussion here is based upon the VRS envelopment model and input congestion measure. The discussion for output congestion measures is the same.

9.3 Slack-based Congestion Measure

The previous section indicates that there is a strong relationship between (input) slacks and the measure of (input) congestion. In fact, Brockett, Cooper, Shin and Wang (1998) develop a new slack-based approach to capture input congestion and identify its sources and amounts. Cooper, Seiford and Zhu (2000) study the relationship between these two DEA congestion approaches, and show that the work of Brockett, Cooper, Shin and Wang (1998) improves upon the work of Färe, Grosskopf and Lovell (1994) in that it not only (i) detects congestion but also (ii) determines the amount of congestion, and simultaneously, (iii) identifies factors responsible for congestion and distinguishes congestion amounts from other components of inefficiency.

The following model is employed by Brockett, Cooper, Shin and Wang (1998) after solving the input-oriented VRS envelopment model

$$\begin{aligned}
 & \max \sum_{i=1}^m \delta_i^+ \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} - \delta_i^+ = \theta^* x_{io} - s_i^{-*} = \hat{x}_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} = y_{ro} + s_r^{+*} = \hat{y}_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, s_i^{-*} \geq \delta_i^+
 \end{aligned} \tag{9.6}$$

where θ^* is obtained from (9.1) while s_i^{-*} and s_r^{+*} are obtained from (1.4). The amount of congestion in each input can then be determined by the difference between each pair of s_i^{-*} and δ_i^{+*} , where δ_i^{+*} are optimal values in (9.6). That is,

$$s_i^c = s_i^{-*} - \delta_i^{+*}, i = 1, 2, \dots, m \tag{9.7}$$

Definition 9.3 (Congestion Slacks) s_i^c defined in (9.7) are called input congestion slacks.

Similarly, we can calculate the output congestion slacks by

$$\begin{aligned}
 & \max \sum_{r=1}^s \delta_r^- \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} = x_{io} - s_i^{-*} = \hat{x}_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} + \delta_r^- = \phi^* y_{ro} + s_r^{+*} = \hat{y}_{ro} \quad r = 1, 2, \dots, s; \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, s_r^{+*} \geq \delta_r^-
 \end{aligned}$$

where ϕ^* is obtained from (9.2) while s_i^{-*} and s_r^{+*} are obtained from (1.7).

To establish the relationship between model (9.6) and $C(\theta^*, \tilde{\theta}^*)$, we proceed as follows. Let $x(s_i^c)$ be an input subvector in which its i th component corresponds to $s_i^c \neq 0$, i.e., $x(s_i^c)$ is a congesting subvector. Next, let \mathbf{X}^C be the set of all congesting subvectors obtained via (9.5). We have

Theorem 9.2 $x(s_i^c) \in \mathbf{X}^C$. Furthermore, if (9.6) yields a unique optimal solution, then $\mathbf{X}^C = \{x(s_i^c)\}$.

[Proof]: Let $A = \{i \mid s_i^c = 0\}$ and $\bar{A} = \{i \mid s_i^c \neq 0\}$. Then the constraints of (9.6) become

$$\begin{aligned} \sum_{j=1}^n \lambda_j x_{ij} &= \theta^* x_{i0} \quad i \in A; \\ \sum_{j=1}^n \lambda_j x_{ij} &\leq \theta^* x_{i0} \quad i \in \bar{A}; \\ \sum_{j=1}^n \lambda_j y_{rj} &= y_{r0} + s_r^{+*} \quad r=1,2,\dots,s; \\ \sum_{j=1}^n \lambda_j &= 1 \\ \lambda_j &\geq 0, s_i^{-*} \geq \delta_i. \end{aligned} \tag{9.8}$$

where θ^* is the optimal value to (9.1). This implies that θ^* is a feasible solution to (9.5). Thus, $\alpha^* \leq \theta^*$, where α^* is the optimal value to (9.5) associated with A and \bar{A} . On the other hand, any optimal solution to (9.5) is a feasible solution (1), therefore $\alpha^* \geq \theta^*$. Thus, $\theta^* = \alpha^*$ indicating that the input subvector associated with \bar{A} , $x(s_i^c)$, is a source of congestion. Therefore, $x(s_i^c) \in \mathbf{X}^C$.

Moreover, if (9.6) yields a unique optimal solution, then the solution in (9.8) is also unique. This means that $\theta^* = \alpha^*$ does not hold for other input subvectors. Thus, $\mathbf{X}^C = \{x(s_i^c)\}$. ■

Theorem 9.2 indicates that under the condition of uniqueness, congestion will occur in the Brockett, Cooper, Shin and Wang (1998) approach if and only if it appears in Färe, Grosskopf and Lovell (1994) approach. However, the Brockett, Cooper, Shin and Wang (1998) approach identifies technical or mix inefficiencies and distinguishes these from congestion components via (9.7).

We observe that the use of (9.5) may result in different congestion factors because of possible multiple optimal solutions. Theorem 9.2 indicates that the results from (9.6) then yield one of the congesting subvectors obtained from (9.5). As a result, the procedure by Färe, Grosskopf and Lovell (1994) for detecting the factors responsible for the congestion may be replaced by model (9.6) and one can more easily find and identify congestion and its sources without having to conduct a series of solutions as required for (9.5).

Consider the mine example again. Before we solve model (9.6), we need to determine the DEA slacks for the spreadsheet shown in Figure 9.3. i.e., we need to perform the second stage calculation for the input-oriented VRS envelopment model.

Figure 9.7 shows the spreadsheet for calculating the DEA slacks. Cells F20:F25 and F26 represent the input slacks and output slack, respectively. Cell F19 represent the sum of slacks and is the target cell in the Solver parameters shown in Figure 9.8.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q |
|----|------------------|-----------|------|------------|-----|---------------|--------|-----------|---------|-------|-------|-------|-------|-------|--------|-------|-------|
| 1 | Mines | Labor | K1 | K2 | T1 | 1/D1 | Output | λ | Strong | Labor | K1 | K2 | T1 | 1/D1 | Output | | |
| 2 | 1 | 98.5 | 142 | 245 | 6 | 0.016 | 3264 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3 | 2 | 96.5 | 30 | 215 | 6 | 0.016 | 3065 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 3 | 57.6 | 18 | 105 | 5.6 | 0.026 | 2275 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 5 | 4 | 59.2 | 160 | 0 | 5.9 | 0.025 | 1979 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 6 | 5 | 57.6 | 200 | 0 | 8 | 0.022 | 1833 | 1E-15 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 6 | 49.9 | 27 | 85 | 4.5 | 0.019 | 1218 | 0 | 0.97129 | 5.384 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 7 | 53.5 | 143 | 65 | 6 | 0.01 | 928 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 9 | 8 | 34 | 70 | 65 | 6 | 0.02 | 919 | 0 | 0.93747 | 0.000 | 0.000 | 0.000 | 0.000 | 0.693 | 0.004 | 0.000 | 0.000 |
| 10 | 9 | 39.6 | 67.5 | 40 | 6.5 | 0.013 | 777 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 10 | 51.3 | 0 | 145 | 3.2 | 0.019 | 745 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | 11 | 74.2 | 110 | 65 | 2.1 | 0.014 | 742 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 13 | 12 | 24 | 25 | 65 | 4.4 | 0.012 | 488 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | 13 | 26.5 | 58 | 0 | 3 | 0.014 | 407 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | 14 | 43.1 | 70 | 0 | 6.5 | 0.012 | 402 | 0 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 16 | 15 | 20.7 | 236 | 0 | 5.7 | 0.01 | 396 | 1 | 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 17 | | | | | | | | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Sum of Slacks | | | | | | | | | | | |
| 19 | Constraints | set | | Evaluation | | 0.0000 | | DEA Slack | | | | | | | | | |
| 20 | Labor | 20.7 | = | 20.7 | | 0.0000 | | | | | | | | | | | |
| 21 | K1 | 236 | = | 236 | | 0.0000 | | | | | | | | | | | |
| 22 | K2 | 0 | = | 0 | | 0.0000 | | | | | | | | | | | |
| 23 | T1 | 5.7 | = | 5.7 | | 0.0000 | | | | | | | | | | | |
| 24 | 1/D1 | 0.01 | = | 0.01 | | 0.0000 | | | | | | | | | | | |
| 25 | output | 396 | = | 396 | | 0.0000 | | | | | | | | | | | |
| 26 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | | | | | | | | |

Figure 9.7. DEA Slacks for 15 Mines

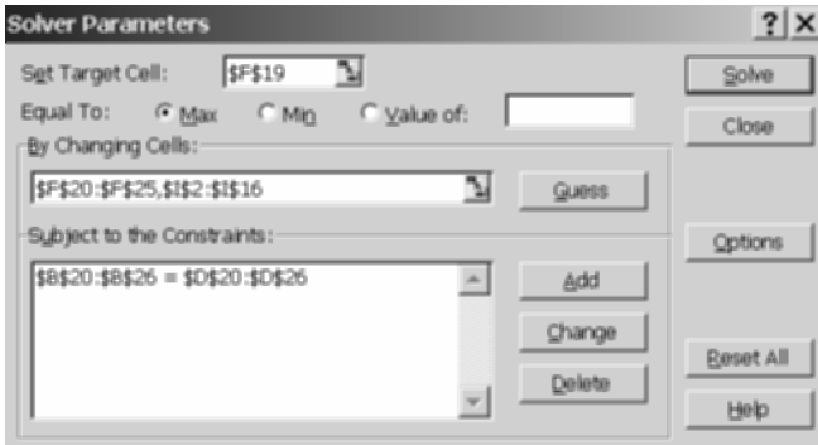


Figure 9.8. Solver Parameters for Calculating DEA Slacks for 15 Mines

The formulas for cells B20:B25 are

Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)+F20

Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)+F21
 Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)+F22
 Cell B23 =SUMPRODUCT(E2:E16,\$I\$2:\$I\$16)+F23
 Cell B24 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)+F24
 Cell B25 =SUMPRODUCT(H2:H16,\$I\$2:\$I\$16)-F25

Cell F26 represents the sum of λ_j (=SUM(I2:I16)). Cells D20:D25 contains

D20 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,1)
 D21 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,2)
 D22 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,3)
 D23 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,4)
 D24 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,5)
 D25 =INDEX(H2:H16,\$E\$18,1)

The DEA slack calculation is performed by the VBA procedure “DEASlack”

```
Sub DEASlack()
Dim i As Integer
For i = 1 To 15
Range("E18") = i
SolverSolve UserFinish:=True
Range("F20:F25").Copy
Range("L" & i + 1).Select
Selection.PasteSpecial Paste:=xlPasteValues, Transpose:=True
Next i
End Sub
```

We next calculate model (9.6) for DMUs 6 and 8. Based upon the DEA slacks in cells L2:Q16, Figure 9.9 shows the spreadsheet for calculating the congestion slacks.

Cells F20:F24 now represent δ_i^+ . Cell F19 contains the formula “=SUM(F20:F24)”, and is the target cell. We change the formulas for cells B20:B25 and D20:D25 to

Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)-F20
 Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)-F21
 Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)-F22
 Cell B23 =SUMPRODUCT(E2:E16,\$I\$2:\$I\$16)-F23
 Cell B24 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)-F24

Cell B25 =SUMPRODUCT(H2:H16,\$I\$2:\$I\$16)
 D20 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,1)-
 INDEX(L2:L16,E18,1)
 D21 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,2)-
 INDEX(M2:M16,E18,1)
 D22 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,3)-
 INDEX(N2:N16,E18,1)
 D23 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,4)-
 INDEX(O2:O16,E18,1)
 D24 =INDEX(J2:J16,E18,1)*INDEX(\$B\$2:\$F\$16,\$E\$18,5)-
 INDEX(P2:P16,E18,1)
 D25 =INDEX(H2:H16,\$E\$18,1)+INDEX(Q2:Q16,E18,1)

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q |
|----|-------------|-----------|------|------------|-----|-----------------------|--------|------------|---------|---|-------|-------|-------|-------|-------|--------|-------|
| 1 | Mines | Labor | K1 | K2 | T1 | 1/D1 | Output | λ | Strong | | Labor | K1 | K2 | T1 | 1/D1 | Output | |
| 2 | 1 | 98.5 | 142 | 245 | 8 | 0.016 | 3264 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 3 | 2 | 96.5 | 30 | 215 | 8 | 0.016 | 3065 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 4 | 3 | 57.8 | 18 | 105 | 5.8 | 0.026 | 2275 | 0.2522751 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 5 | 4 | 59.2 | 160 | 0 | 5.9 | 0.025 | 1978 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 6 | 5 | 57.8 | 200 | 0 | 8 | 0.022 | 1833 | 7.9E-17 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 6 | 49.9 | 27 | 85 | 4.5 | 0.019 | 1218 | 0 | 0.97129 | | 5.384 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 8 | 7 | 53.5 | 143 | 85 | 6 | 0.01 | 828 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 9 | 8 | 34 | 70 | 85 | 6 | 0.02 | 919 | 0 | 0.93747 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.893 | 0.004 | 0.000 |
| 10 | 9 | 39.8 | 67.5 | 40 | 6.5 | 0.013 | 777 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 11 | 10 | 51.3 | 0 | 145 | 3.2 | 0.019 | 745 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 12 | 11 | 74.2 | 110 | 85 | 2.1 | 0.014 | 742 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 13 | 12 | 24 | 25 | 85 | 4.4 | 0.012 | 498 | 0.5299453 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 14 | 13 | 26.5 | 58 | 0 | 3 | 0.014 | 407 | 0.0200183 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 15 | 14 | 43.1 | 70 | 0 | 6.5 | 0.012 | 402 | 0 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 16 | 15 | 20.7 | 236 | 0 | 5.7 | 0.01 | 396 | 0.1977633 | 1 | | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 17 | | | | | | | | | | | | | | | | | |
| 18 | | Reference | | DMU under | 8 | Sum of s _i | DEA | Congestion | | | | | | | | | |
| 19 | Constraints | set | | Evaluation | | 0.0000 | Slacks | Slacks | | | | | | | | | |
| 20 | Labor | 31.8738 | = | 31.873864 | | 0.0000 | 0 | 0.0000 | | | | | | | | | |
| 21 | K1 | 65.6227 | = | 65.622662 | | 0.0000 | 0 | 0.0000 | | | | | | | | | |
| 22 | K2 | 60.9353 | = | 60.935329 | | 0.0000 | 0 | 0.0000 | | | | | | | | | |
| 23 | T1 | 4.9318 | = | 4.9317994 | | 0.0000 | 0.893 | 0.6930 | | | | | | | | | |
| 24 | 1/D1 | 0.0152 | = | 0.0151764 | | 0.0000 | 0.004 | 0.0036 | | | | | | | | | |
| 25 | output | 919 | = | 919.000 | | | | | | | | | | | | | |
| 26 | Σλ | 1 | = | 1 | | | | | | | | | | | | | |

Figure 9.9. Congestion Slack Spreadsheet Model

Cells H20:H24 represent the DEA slacks for a DMU under evaluation and return the DEA slacks reported in cells L2:Q16. The formulas are

Cells H20 =INDEX(L2:L16,E18,1)
 Cells H21 =INDEX(M2:M16,E18,1)
 Cells H22 =INDEX(N2:N16,E18,1)
 Cells H23 =INDEX(O2:O16,E18,1)
 Cells H24 =INDEX(P2:P16,E18,1)

Figure 9.10 shows the Solver parameters for calculating the congestion slacks. The congestion slacks are reported in cells I20:I24. In this example, the congestion slacks are equal to the DEA slacks for DMUs 6 and 8,

because $\delta_i^+ = 0$ in optimality. For example, for DMU 6, the congestion factor is labor with a congestion slack of 5.384.

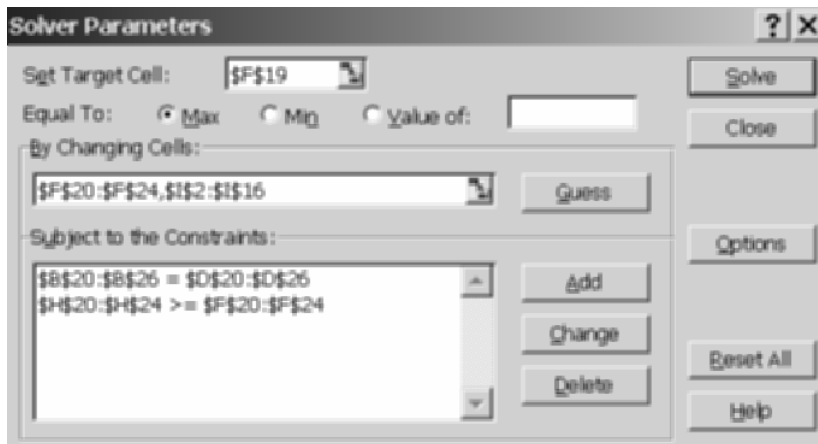


Figure 9.10. Solver Parameters for Calculating Congestion Slacks

9.4 Solving DEA Using DEAFrontier Software

To run the **weak disposability models** presented in Table 9.1, select the “Weak Disposability” menu item. The results are reported in the “Efficiency” sheet.

To calculate the **congestion slacks**, select the “Congestion” menu item. The Congestion will use the Slack and Target sheets. If there exist a slack sheet and a target sheet that are generated by the same envelopment model, you will be prompted a form shown in Figure 9.11.

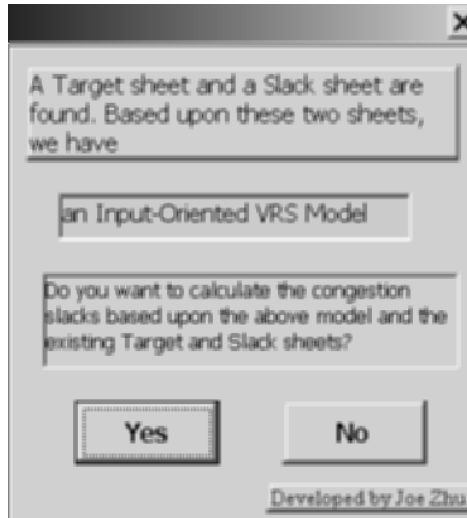


Figure 9.11. Congestion

If you choose Yes, then the software will calculate the congestion slacks based upon the information stored in the “Slack”, “Target” and “Data” sheets.

If you choose No, then the software will ask you to select an envelopment model. Then, the software will calculate the specified envelopment model, generate the “Slack” and “Target” sheets, and report the congestion slacks in “Congestion Slacks” sheet. The same procedure will be applied if there do not exist the “Slack” sheet and the “Target” sheet, or the “Slack” and “Target” sheets are generated by different envelopment models.

REFERENCES

1. Brockett, P.L., W.W. Cooper, H.C. Shin and Y. Wang (1998), Inefficiency and congestion in Chinese production before and after the 1978 economic reforms, *Socio-Economic Planning Sciences*, 32, 1-20.
2. Byrnes, P., R. Färe and S. Grosskopf (1984), Measuring productive efficiency: an application to Illinois strip mines, *Management Science*, 30, 671-681.
3. Cooper, W.W., R.G. Thompson and R.M. Thrall (1996), Introduction: extensions and new developments in DEA, *Annals of Operations Research*, 66, 3-45.
4. Cooper, W.W., L.M. Seiford and J. Zhu (2000), A unified additive model approach for evaluating inefficiency and congestion with associated measures in DEA. *Socio-Economic Planning Sciences*, 34, No. 1, 1-25.
5. Färe, R. and S. Grosskopf (1983), Measuring congestion in production, *J. of Economics*, 43, 257-271.
6. Färe, R., S. Grosskopf and C.A.K. Lovell (1994), *Production Frontiers*. Cambridge University Press.
7. Ray, S., L.M. Seiford and J. Zhu (1998), Market entity behavior of Chinese state-owned enterprises, *OMEGA*, 26, No. 2, 263-278.

Chapter 10

Super Efficiency

10.1 Super-efficiency DEA Models

When a DMU under evaluation is not included in the reference set of the envelopment models, the resulting DEA models are called super-efficiency DEA models. Charnes, Haag, Jaska and Semple (1992) use a super-efficiency model to study the sensitivity of the efficiency classifications. Zhu (1996) and Seiford and Zhu (1998) develop a number of new super-efficiency models to determine the efficiency stability regions (see Chapter 11). Andersen and Petersen (1993) propose using the CRS super-efficiency model in ranking the efficient DMUs. Also, the super-efficiency DEA models can be used in detecting influential observations (Wilson, 1995) and in identifying the extreme efficient DMUs (Thrall, 1996). Seiford and Zhu (1999) study the infeasibility of various super-efficiency models developed from the envelopment models in Table 1.2. Chapter 11 presents other super-efficiency models that are used in sensitivity analysis.

Table 10.1 presents the basic super-efficiency DEA models based upon the envelopment DEA models. Based upon Table 10.1, we see that the difference between the super-efficiency and the envelopment models is that the DMU_o under evaluation is excluded from the reference set in the super-efficiency models. i.e., the super-efficiency DEA models are based on a reference technology constructed from all other DMUs.

Consider the example in Table 1.1. If we measure the (CRS) super efficiency of DMU2, then DMU2 is evaluated against point A on the new facet determined by DMUs 1 and 3 (see Figure 10.1). To calculate the (CRS) super efficiency score for DMU2, we use the spreadsheet model shown in Figure 10.2.

Table 10.1. Super-efficiency DEA Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|--|--|
| | $\min \theta^{\text{super}}$ subject to | $\max \phi^{\text{super}}$ subject to |
| | $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta^{\text{super}} x_{io} \quad i = 1, 2, \dots, m;$ | $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, 2, \dots, m;$ |
| CRS | $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s;$ | $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq \phi^{\text{super}} y_{ro} \quad r = 1, 2, \dots, s;$ |
| | $\lambda_j \geq 0 \quad j \neq o.$ | $\lambda_j \geq 0 \quad j \neq o.$ |
| VRS | Add $\sum_{j \neq o} \lambda_j = 1$ | |
| NIRS | Add $\sum_{j \neq o} \lambda_j \leq 1$ | |
| NDRS | Add $\sum_{j \neq o} \lambda_j \geq 1$ | |

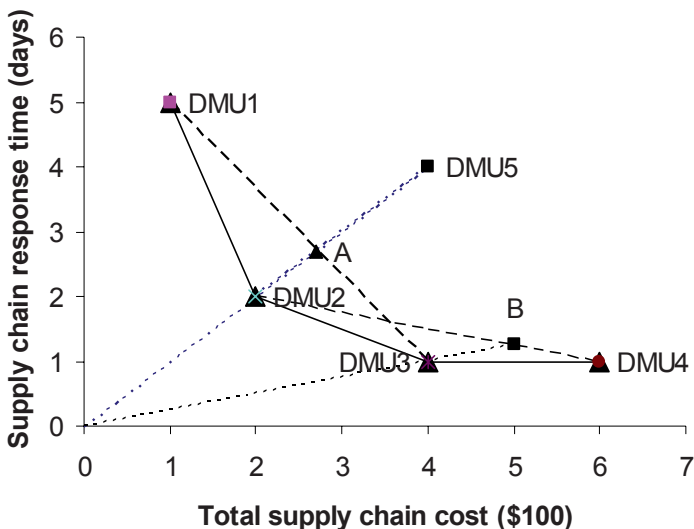


Figure 10.1. Super-efficiency

Cell E9 indicates the DMU under evaluation which is excluded from the reference set. Cells F2:F6 are reserved for λ_j ($j = 1, 2, 3, 4, 5$), and cell F10 is reserved for the super-efficiency score (θ^{super}).

Cells B11:B13 contain the following formulas

Cell B11 =SUMPRODUCT(B2:B6,F2:F6)

Cell B12 =SUMPRODUCT(C2:C6,F2:F6)

Cell B13 =SUMPRODUCT(E2:E6,F2:F6)

| | A | B | C | D | E | F | G | H |
|----|--------------------|-----------|------|------------|--------|------------------|---|---|
| 1 | DMU | Cost | Time | | Profit | λ | | |
| 2 | 1 | 1 | 5 | | 2 | 0.428571 | | |
| 3 | 2 | 2 | 2 | | 2 | 0 | | |
| 4 | 3 | 4 | 1 | | 2 | 0.571429 | | |
| 5 | 4 | 6 | 1 | | 2 | 0 | | |
| 6 | 5 | 4 | 4 | | 2 | 0 | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | Reference | | DMU under | 2 | Super Efficiency | | |
| 10 | Constraints | set | | Evaluation | | 1.357143 | | |
| 11 | Cost | 2.7142857 | <= | 2.7142857 | | | | |
| 12 | Time | 2.7142857 | <= | 2.7142857 | | | | |
| 13 | Profit | 2 | >= | 2 | | | | |
| 14 | λ_0 | 0 | = | 0 | | | | |
| 15 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 18 | | | | | | | | |

Figure 10.2. Input-oriented CRS Super-efficiency Spreadsheet Model

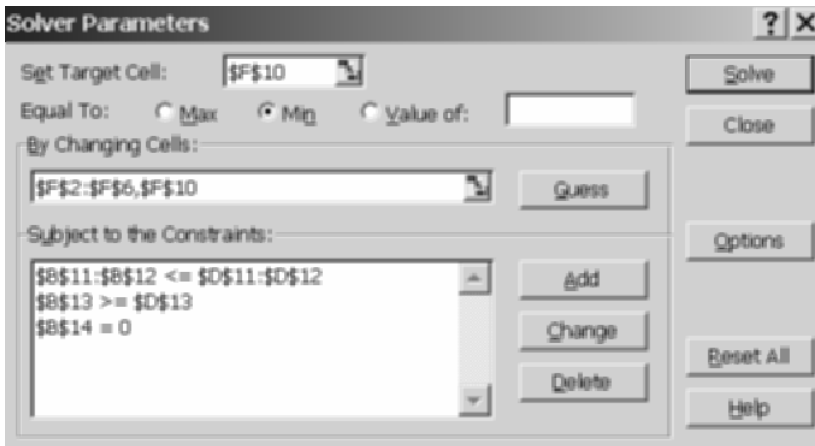


Figure 10.3. Solver Parameters for Input-oriented CRS Super-efficiency

Note that in the above formulas, the DMU under evaluation is included in the reference set. In order to exclude the DMU under evaluation from the reference set, we introduce the following formula into cell B14

Cell B14 =INDEX(F2:F6,E9,1)

which returns the λ_j for the DMU_j under evaluation. In the Solver parameters shown in Figure 10.3, we set cell B14 equal to zero.

Cells D11:D13 contain the following formulas

Cell D11 = $\$F\$10*\text{INDEX}(B2:B6,E9,1)$

Cell D12 = $\$F\$10*\text{INDEX}(C2:C6,E9,1)$

Cell D13 = $\text{INDEX}(E2:E6,E9,1)$

Based upon Figure 10.2 and Figure 10.3, the super-efficiency score for DMU2 is 1.357, and the non-zero λ_j in cells F2 and F4 indicate that DMU1 and DMU3 form a new efficient facet.

DMU3 is evaluated against B on the new facet determined by DMUs 2 and 4. If we change the value of cell E9 to 3, we obtain the super-efficiency score for DMU3 using the Solver parameters shown in Figure 10.3. The score is 1.25 (see cell G4 in Figure 10.4).

| | A | B | C | D | E | F | G |
|----|--------------------|-----------|------|------------|--------|------------|------------------|
| 1 | DMU | Cost | Time | | Profit | λ | Super Efficiency |
| 2 | 1 | 1 | 5 | | 2 | 5.55E-17 | 2 |
| 3 | 2 | 2 | 2 | | 2 | 1 | 1.357142857 |
| 4 | 3 | 4 | 1 | | 2 | 0 | 1.25 |
| 5 | 4 | 6 | 1 | | 2 | 0 | 1 |
| 6 | 5 | 4 | 4 | | 2 | 0 | 0.5 |
| 7 | | | | | | | |
| 8 | | | | | | | |
| 9 | | Reference | | DMU under | 5 | Super | |
| 10 | Constraints | set | | Evaluation | | Efficiency | |
| 11 | Cost | 2 | < | 2 | | 0.5 | |
| 12 | Time | 2 | < | 2 | | | |
| 13 | Profit | 2 | > | 2 | | | |
| 14 | λ_0 | 0 | = | 0 | | | |

Figure 10.4. Super-efficiency Scores

If we remove DMU4 or DMU 5 from the reference set, the frontier remains the same. Therefore, the super-efficiency score for DMU4 (DMU5) equals to the input-oriented CRS efficiency score (see Figure 10.4).

If we measure the super-efficiency of DMU1, DMU1 is evaluated against C on the frontier extended from DMU2 (see Figure 10.5). It can be seen that C is a weakly efficient DMU in the remaining four DMUs 2, 3, 4 and 5. In fact, we may want to adjust such a super-efficiency score (see Zhu (2001b) and Chen and Sherman (2002)).

Although the super-efficiency models can differentiate the performance of the efficient DMUs, the efficient DMUs are not compared to the same “standard”. Because the frontier constructed from the remaining DMUs

changes for each efficient DMU under evaluation. In fact, the super-efficiency should be regarded the potential input savings or output surpluses (see Chen (2002)).

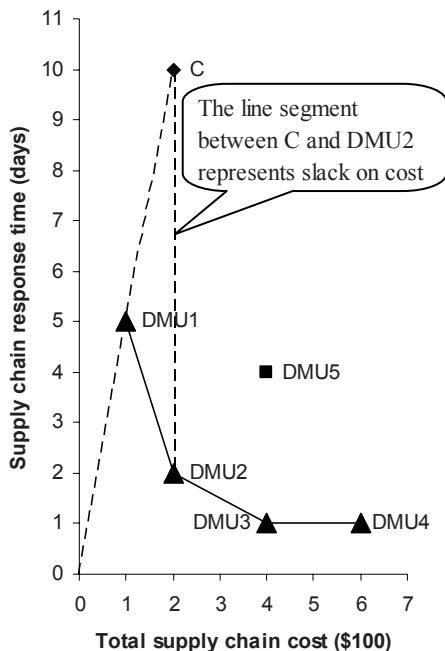


Figure 10.5. Super-efficiency and Slacks

10.2 Infeasibility of Super-efficiency DEA Models

Consider the input-oriented VRS super-efficiency model shown Figure 10.6. In fact, this is the spreadsheet model for the input-oriented VRS envelopment model except that we introduce the formula “=INDEX(I2:I16, E18,1)” into cell B26. This formula is used to exclude the DMU under evaluation from the reference set. That is, one needs to add an additional constraint of “\$B\$26=0” into the Solver parameters for the input-oriented VRS envelopment spreadsheet model, as shown in Figure 10.7.

Once we set up the Solver parameters, the calculation is performed by the VBA procedure “SuperEfficiency”.

```
Sub SuperEfficiency()
Dim i As Integer
```

```

For i = 1 To 15
Range("E18") = i
SolverSolve UserFinish:=True
If SolverSolve(UserFinish:=True) = 5 Then
Range("J" & i + 1) = "Infeasible"
Else
Range("J" & i + 1) = Range("F19")
End If
Next
End Sub
    
```

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|----------------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Super efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | Infeasible |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1.751985253 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | 1.606521649 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 8880.7 | | 0 | Infeasible |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0.77 | 1.320957592 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 158.6 | | 0 | 1.009347198 |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.737555958 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 148855 | | 111052 | 2662.4 | | 0 | 0.803245345 |
| 10 | Exon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 1.344368672 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 8904.6 | | 0 | Infeasible |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.23 | 1.765155063 |
| 13 | Hitachi | 91620.9 | 29807.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.557595838 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | 4.806917693 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 0.470610997 |
| 16 | AT&T | 89994 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.533543522 |
| 17 | | | | | | Super | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 0.5335435 | | | | |
| 20 | Assets | 47423.482 | \leq | 47423.4824 | | | | | | |
| 21 | Equity | 8535.6544 | \leq | 9216.4308 | | | | | | |
| 22 | Employees | 159689.58 | \leq | 159689.576 | | | | | | VRS Super-efficiency |
| 23 | Revenue | 150569.21 | \geq | 79609 | | | | | | |
| 24 | Profit | 791.04056 | \geq | 139 | | | | | | |
| 25 | $\Sigma \lambda$ | 1 | $=$ | 1 | | | | | | |
| 26 | λ_0 | 0 | $=$ | 0 | | | | | | |

Figure 10.6. Input-oriented VRS Super-efficiency Spreadsheet Model

It can be seen that the input-oriented VRS super-efficiency model is infeasible for three VRS efficient companies (Mitsubishi, General Motors, and Royal Dutch/Shell Group). Note that in the VBA procedure “SuperEfficiency”, a VBA statement on infeasibility check is added.

If we consider the output-oriented VRS super-efficiency model, we have the spreadsheet shown in Figure 10.8. Figure 10.8 is based upon the output-oriented VRS envelopment with an additional formula in cell B26 “=INDEX(I2:I16,E18,1)”. To calculate the output-oriented super-efficiency scores, we need to change the “Min” to “Max” in the Solver parameters shown in Figure 10.7.

Based upon Figure 10.8, the output-oriented VRS super-efficiency model is infeasible for five output-oriented VRS efficient companies (Itochu, Sumitomo, Marubeni, Wal-Mart, and Nippon Life Insurance).

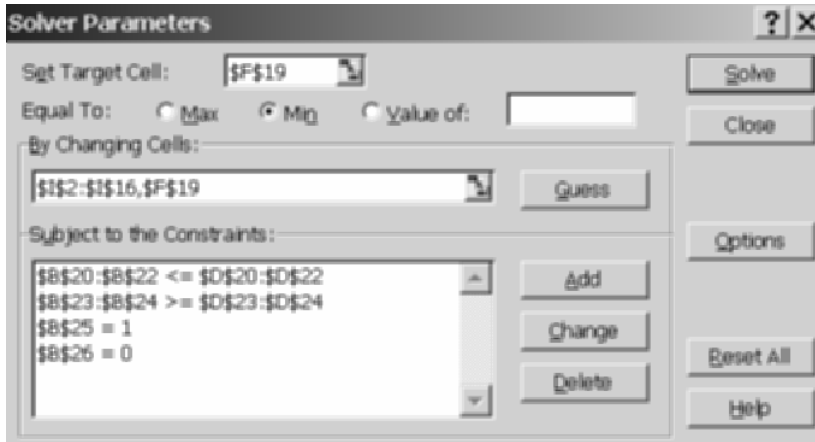


Figure 10.7. Solver Parameters for Input-oriented VRS Super-efficiency

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|------------|--------|---|-----------|---|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Super efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0.869 | 0.936120353 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0.131 | 0.937264375 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | Infeasible |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | 0.847119111 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0 | Infeasible |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | Infeasible |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 1.158414974 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | | 0 | 1.371588284 |
| 10 | Ecoon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | 0.873147631 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | 0.939143546 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0 | Infeasible |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 1.898938938 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | Infeasible |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | 1.892916538 |
| 16 | AT&T | 88894 | 17274 | 299300 | | 79609 | 139 | | 0 | 2.311193684 |
| 17 | | | | | | Super | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | |
| 19 | Constraints | set | | Evaluation | | 2.3111937 | | | | |
| 20 | Assets | 88894 | \leq | 88894 | | | | | | |
| 21 | Equity | 10242.181 | \leq | 17274 | | | | | | Output-oriented VRS Super-efficiency |
| 22 | Employees | 41771.592 | \leq | 299300 | | | | | | |
| 23 | Revenue | 183981.82 | \geq | 183981.818 | | | | | | |
| 24 | Profit | 342.08119 | \geq | 321.255922 | | | | | | |
| 25 | $\Sigma \lambda$ | | $=$ | 1 | | | | | | |
| 26 | λ_0 | | $=$ | 0 | | | | | | |

Figure 10.8. Output-oriented VRS Super-efficiency Spreadsheet Model

Thrall (1996) shows that the super-efficiency CRS model can be infeasible. However, Thrall (1996) fails to recognize that the output-oriented CRS super-efficiency model is always feasible for the trivial solution which

has all variables set equal to zero. Moreover, Zhu (1996) shows that the input-oriented CRS super-efficiency model is infeasible if and only if a certain pattern of zero data occurs in the inputs and outputs.

Figure 10.9 illustrates how the VRS super-efficiency model works and the infeasibility for the case of a single output and a single input case. We have three VRS frontier DMUs, A, B and C. AB exhibits IRS and BC exhibits DRS. The VRS super-efficiency model evaluates point B by reference to B' and B'' on section AC through output-reduction and input-increment, respectively. In an input-oriented VRS super-efficiency model, point A is evaluated against A' . However, there is no referent DMU for point C for input variations. Therefore, the input-oriented VRS super-efficiency model is infeasible at point C. Similarly, in an output-oriented VRS super-efficiency model, point C is evaluated against C' . However, there is no referent DMU for point A for output variations. Therefore, the output-oriented VRS super-efficiency model is infeasible at point A. Note that point A is the left most end point and point B is the right most end point on this frontier.

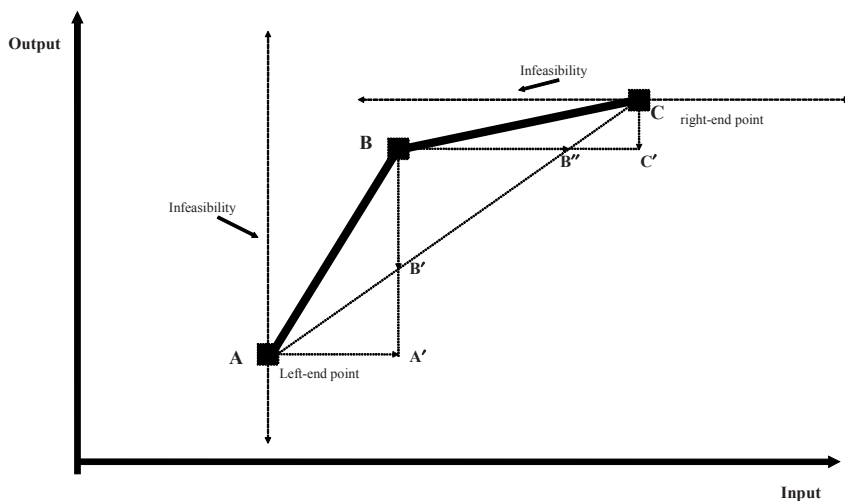


Figure 10.9. Infeasibility of Super-efficiency Model

As in Charnes, Cooper and Thrall (1991), the DMUs can be partitioned into four classes E, E', F and N described as follows. First, E is the set of extreme efficient DMUs. Second, E' is the set of efficient DMUs that are not extreme points. The DMUs in set E' can be expressed as linear combinations of the DMUs in set E. Third, F is the set of frontier points (DMUs) with non-

zero slack(s). The DMUs in set F are usually called weakly efficient. Fourth, N is the set of inefficient DMUs.

For example, DMUs 1, 2, and 3 in Figure 10.1 are extreme efficient (in set E), DMU4 is in set F, and DMU5 is in set N.

Thrall (1996) shows that if the CRS super-efficiency model is infeasible, or if the super-efficiency score is greater than one for input-oriented model (less than one for output-oriented model), then $DMU_o \in E$. This result can also be applied to other super-efficiency models. i.e., the extreme efficient DMUs can be identified by the super-efficiency models. This finding is important in empirical applications. For example, in the slack-based congestion measures discussed in Chapter 9, if we can know that the data set consists of only extreme efficient DMUs, then the congestion slacks are equal to the DEA slacks.

Note that if a specific $DMU_o \in E', F$ or N and is not included in the reference set, then the efficient frontiers (constructed by the DMUs in set E) remain unchanged. As a result, the super-efficiency DEA models are always feasible and equivalent to the original DEA models when $DMU_o \in E', F$ or N . Thus we only need to consider the infeasibility when $DMU_o \in E$.

We next study the infeasibility of the VRS, NIRS and NDRS super-efficiency models, where we assume that all data are positive.

From the convexity constraint ($\sum_{j \neq o} \lambda_j = 1$) on the intensity lambda variables, we immediately have

Proposition 10.1 $DMU_o \in E$ under the VRS model *if and only if* $DMU_o \in E$ under the NIRS model or NDRS model.

Thus in the discussion to follow, we limit our consideration to $DMU_o \in E$ under the VRS model. We have

Proposition 10.2 Let θ^{super*} and ϕ^{super*} denote, respectively, optimal values to the input-oriented and output-oriented super-efficiency DEA models when evaluating an extreme efficient DMU_o , then

- (i) Either $\theta^{super*} > 1$ or the specific input-oriented super-efficiency DEA model is infeasible.
- (ii) Either $\phi^{super*} < 1$ or the specific output-oriented super-efficiency DEA model is infeasible.

Based upon Seiford and Zhu (1999), we next (i) present the necessary and sufficient conditions for the infeasibility of various super-efficiency DEA models in a multiple inputs and multiple outputs situation, and (ii) reveal the relationship between infeasibility and RTS classification. (Note

that, in Figure 10.9, point A is associated with IRS and point C is associated with DRS.)

10.2.1 Output-oriented VRS Super-efficiency Model

Suppose each DMU_j ($j = 1, 2, \dots, n$) consumes a vector of inputs, x_j , to produce a vector of outputs, y_j . We have

Theorem 10.1 For a specific extreme efficient $DMU_o = (x_o, y_o)$, the output-oriented VRS super-efficiency model is infeasible *if and only if* $(x_o, \delta y_o)$ is efficient under the VRS envelopment model for any $0 < \delta \leq 1$.

[Proof]: Suppose that the output-oriented VRS super-efficiency model is infeasible and that $(x_o, \delta^o y_o)$ is inefficient, where $0 < \delta^o \leq 1$. Then

$$\begin{aligned}
 \phi_o^{\text{super}*} &= \max \phi_o^{\text{super}} \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_j \leq x_o \\
 &\sum_{j=1}^n \lambda_j y_j \geq \phi_o^{\text{super}} (\delta^o y_o) \\
 &\sum_{j=1}^n \lambda_j = 1
 \end{aligned} \tag{10.1}$$

has a solution of $\lambda_j^* (j \neq o)$, $\lambda_o^* = 0$, $\phi_o^{\text{super}*} > 1$. Since $\lambda_o^* = 0$, we have that model (10.1) is equivalent to an output-oriented VRS super-efficiency model and thus the output-oriented VRS super-efficiency model is feasible. A contradiction. This completes the proof of the *only if* part.

To establish the *if* part, we note that if the output-oriented VRS super-efficiency model is feasible, then $\phi_o^{\text{super}*} < 1$ is the maximum radial reduction of all outputs preserving the efficiency of DMU_o . Therefore, δ cannot be less than $\phi_o^{\text{super}*}$. Otherwise, DMU_o will be inefficient under the output-oriented VRS envelopment model. Thus, the output-oriented VRS super-efficiency model is infeasible. ■

Theorem 10.2 The output-oriented VRS super-efficiency model is infeasible *if and only if* \hat{h}^* , where $\hat{h}^* > 1$ is the optimal value to (10.2).

$$\begin{aligned}
 \hat{h}^* &= \min \hat{h} \\
 &\text{subject to} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_j \leq \hat{h} x_o \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1 \\
 &\lambda_j \geq 0, j \neq o
 \end{aligned} \tag{10.2}$$

[Proof]: We note that for any $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$, the constraint $\sum_{j \neq o} \lambda_j y_j \geq \phi^{\text{super}} y_o$ always holds. Thus the output-oriented super-efficiency-VRS is infeasible if and only if there exists no $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$ such that $\sum_{j \neq o} \lambda_j x_j \leq x_o$ holds. This means that the optimal value to (10.2) is greater than one, i.e., $\hat{h}^* > 1$. ■

| | A | B | C | D | E | F | G | H | I |
|----|------------------------------|-----------|---------|------------|----|----------|-----------|---------|------------------|
| 1 | Company | Assets | Equity | Employees | | | λ | h | Super efficiency |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | | 0 | 0.57426 | 0.936120353 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | | 0 | 0.8919 | 0.937264375 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | | 0 | 1.20984 | Infeasible |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | | 0 | 0.25382 | 0.647119111 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | | 0.77 | 1.30639 | Infeasible |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | | 0 | 1.00935 | Infeasible |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | | 0 | 0.23236 | 1.158414974 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | | 0 | 0.4634 | 1.371588284 |
| 10 | Exxon | 91296 | 40436 | 82000 | | | 0 | 0.54283 | 0.673147631 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 56966.4 | 104000 | | | 0 | 0.42008 | 0.939143546 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | | 0.23 | 1.32737 | Infeasible |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | | 0 | 0.51532 | 1.898938938 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | | 0 | 1.90513 | Infeasible |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | | 0 | 0.38353 | 1.892916538 |
| 16 | AT&T | 88884 | 17274 | 299300 | | | 0 | 0.53354 | 2.311193684 |
| 17 | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | | h | | |
| 19 | Constraints | set | | Evaluation | | 0.533544 | | | |
| 20 | Assets | 47423.482 | < | 47423.482 | | | | | |
| 21 | Equity | 8535.6544 | < | 9216.4308 | | | | | |
| 22 | Employees | 159689.58 | < | 159689.58 | | | | | |
| 23 | $\Sigma \lambda$ | 1 | = | 1 | | | | | |
| 24 | λ_o | 0 | = | 0 | | | | | |

Figure 10.10. Spreadsheet for Infeasibility Test (Output-oriented VRS Super-efficiency)

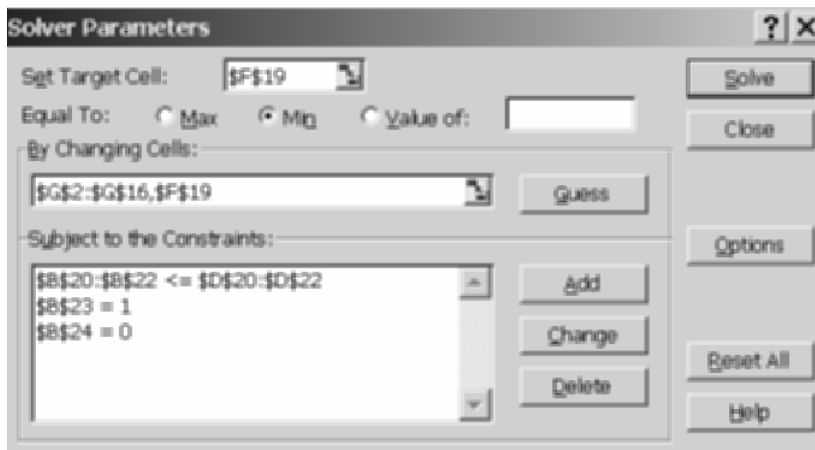


Figure 10.11. Solver Parameters for Infeasibility Test (Output-oriented)

Figure 10.10 shows the spreadsheet model for model (10.2) where the output-oriented VRS super-efficiency scores are reported in cells I2:I16.

The spreadsheet shown in Figure 10.10 is obtained by removing the output constraints from the spreadsheet shown in Figure 10.6. Figure 10.11 shows the Solver parameters. It can be seen that $\hat{h}^* > 1$ if and only if model (10.2) is infeasible for a company.

Further, note that the DMU_o is also CRS efficient if and only if CRS prevail. Therefore, if IRS or DRS prevail, then DMU_o must be CRS inefficient. Thus, in this situation, the CRS super-efficiency model is identical to the CRS envelopment model. Based upon Chapter 13, IRS or DRS on DMU_o can be determined by

Lemma 10.1 The RTS for DMU_o can be identified as IRS if and only if $\sum_{j \neq 0} \lambda_j^* < 1$ in all optima for the CRS super-efficiency model and DRS if and only if $\sum_{j \neq 0} \lambda_j^*$ in all optima for the CRS super-efficiency model.

Lemma 10.2 If DMU_o exhibits DRS, then the output-oriented VRS super-efficiency model is feasible and $\phi^{\text{super}^*} < 1$, where ϕ^{super^*} is the optimal value to the output-oriented VRS super-efficiency model.

[Proof]: The output-oriented VRS super-efficiency model is as follows

$$\begin{aligned}
 \phi^{\text{super}^*} &= \max \phi^{\text{super}} \\
 \text{subject to} \\
 \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_j &\leq x_o \\
 \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_j &\geq \phi^{\text{super}} y_o \\
 \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j &= 1; \\
 \phi^{\text{super}}, \lambda_j &\geq 0, j \neq 0
 \end{aligned} \tag{10.3}$$

Let $\theta = 1/\phi^{\text{super}^*}$. Multiplying all constraints in (10.3) by θ yields

$$\begin{aligned}
 \min \theta \\
 \text{subject to} \\
 \sum_{\substack{j=1 \\ j \neq 0}}^n \tilde{\lambda}_j x_j &\leq \theta x_o \\
 \sum_{\substack{j=1 \\ j \neq 0}}^n \tilde{\lambda}_j y_j &\geq y_o
 \end{aligned} \tag{10.4}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \tilde{\lambda}_j = \theta = \frac{1}{\phi^{\text{super}}}$$

$$\phi^{\text{super}}, \theta, \tilde{\lambda}_j \geq 0, j \neq o$$

where $\tilde{\lambda}_j = \theta \lambda_j (j \neq o)$.

Since DMU_o exhibits DRS, then by Lemma 10.1, $\sum_{j \neq o} \lambda_j^* > 1$ in all optima to the following CRS super-efficiency model

$$\begin{aligned} \min & \theta^{\text{super}} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_j \leq \theta^{\text{super}} x_o \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_j \geq y_o \\ & \lambda_j \geq 0. \end{aligned} \tag{10.5}$$

Let $\sum_{j \neq o} \lambda_j^* = \theta$. Obviously $\theta > \theta^{\text{super}}$ is a feasible solution to (10.5). This in turn indicates that $\lambda_j^* (j \neq o)$ and θ is a feasible solution to (10.4). Therefore, (10.3) is feasible. Furthermore by Proposition 10.2, we have that $\phi^{\text{super}^*} < 1$, where ϕ^{super^*} is the optimal value to (10.3). ■

Theorem 10.3 If the output-oriented VRS super-efficiency model is infeasible, then DMU_o exhibits IRS or CRS.

[Proof]: Suppose that DMU_o exhibits DRS. By Lemma 10.2, the output-oriented VRS super-efficiency model is feasible. A contradiction. ■

Theorems 10.1 and 10.2 indicate that if the output-oriented VRS super-efficiency model is infeasible, then DMU_o is one of the *endpoints*. Moreover, if IRS prevail, then DMU_o is a *left endpoint* (see Figure 10.9).

10.2.2 Other Output-oriented Super-efficiency Models

Now, consider the output-oriented NIRS and NDRS super-efficiency models. Obviously, we have a feasible solution of $\lambda_j = 0 (j \neq o)$ and $\phi^{\text{super}} = 0$ in the output-oriented NIRS super-efficiency model. Therefore, we have

Theorem 10.4 The output-oriented NIRS super-efficiency model is always feasible.

Lemma 10.3 The output-oriented NDRS super-efficiency model is infeasible *if and only if* the output-oriented VRS super-efficiency model is infeasible.

[Proof]: The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the output-oriented NDRS super-efficiency model is feasible. i.e., we have a feasible solution with $\sum_{j \neq o} \lambda_j \geq 1$ for the output-oriented NDRS super-efficiency model. If $\sum_{j \neq o} \lambda_j = 1$, then this solution is also feasible for the output-oriented VRS super-efficiency. If $\sum_{j \neq o} \lambda_j > 1$, let $\sum_{j \neq o} \lambda_j = d > 1$. Then $\sum_{j \neq o} \tilde{\lambda}_j x_j \leq \sum_{j \neq o} \lambda_j x \leq x_o$, where $\tilde{\lambda}_j = \lambda_j / d$ ($j \neq o$) and $\sum_{j \neq o} \tilde{\lambda}_j = 1$. Therefore $\tilde{\lambda}_j$ ($j \neq o$) is a feasible solution to the output-oriented VRS super-efficiency model. Both possible cases lead to a contradiction. Thus, the output-oriented NDRS super-efficiency model is infeasible if the output-oriented VRS super-efficiency model is infeasible. ■

On the basis of Lemma 10.3, we have

Theorem 10.5 For a specific extreme efficient $DMU_o = (x_o, y_o)$, we have
 (i) The output-oriented NDRS super-efficiency model is infeasible *if and only if* $(x_o, \delta y_o)$ is efficient under the VRS envelopment model for any $0 < \delta \leq 1$.
 (ii) The output-oriented NDRS super-efficiency model is infeasible *if and only if* $h^* > 1$, where h^* is the optimal value to (10.2).

If $DMU_o \in E$ for the NDRS model, then DMU_o exhibits IRS or CRS. By Proposition 10.1, DMU_o also lies on the VRS frontier that satisfies IRS or CRS. i.e., the VRS and NDRS envelopment models are identical for DMU_o . Thus, $(x_o, \delta y_o)$ is also efficient under the NDRS envelopment model for any $0 < \delta \leq 1$.

10.2.3 Input-oriented VRS Super-efficiency Model

Theorem 10.6 For a specific extreme efficient $DMU_o = (x_o, y_o)$, the input-oriented VRS super-efficiency model is infeasible *if and only if* $(\chi x_o, y_o)$ is efficient under the VRS envelopment model for any $1 \leq \chi < +\infty$.

[Proof]: Suppose the input-oriented VRS super-efficiency model is infeasible and assume that $(\chi^o x_o, y_o)$ is inefficient, where $1 \leq \chi^o < +\infty$. Then

$$\begin{aligned}
 &\theta_o^{\text{super}^*} = \min \theta_o^{\text{super}} \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_j \leq \theta_o^{\text{super}} (\chi^o x_o) \\
 &\sum_{j=1}^n \lambda_j y_j \geq y_o \\
 &\sum_{j=1}^n \lambda_j = 1
 \end{aligned} \tag{10.6}$$

has a solution of $\lambda_j^* (j \neq o)$, $\lambda_o^* = 0$, $\theta_o^{\text{super}^*}$. Since $\lambda_o^* = 0$, model (10.6) is equivalent to the input-oriented VRS super-efficiency model. Thus, the input-oriented VRS super-efficiency model is feasible. This completes the proof of *only if* part.

To establish the *if* part, we note that if the input-oriented VRS super-efficiency model is feasible, then $\theta_o^{\text{super}^*} > 1$ is the maximum radial increase of all inputs preserving the efficiency of DMU_o . Therefore, χ cannot be bigger than $\theta_o^{\text{super}^*}$. Otherwise, DMU_o will be inefficient under the input-oriented VRS envelopment model. Thus, the input-oriented VRS super-efficiency model is infeasible. ■

Theorem 10.7 The input-oriented super-efficiency-VRS model is infeasible if and only if $g^* < 1$, where g^* is the optimal value to (10.7).

$$\begin{aligned}
 &g^* = \max g \\
 &\text{subject to} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_j \geq g y_o \\
 &\sum_{j=1}^n \lambda_j = 1 \\
 &\lambda_j \geq 0, j \neq o
 \end{aligned} \tag{10.7}$$

[Proof]: We note that for any $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$, the constraint $\sum_{j \neq o} \lambda_j x_j \leq \theta_o^{\text{super}} x_o$ always holds. Thus, the input-oriented VRS super-efficiency model is infeasible if and only if $\sum_{j \neq o} \lambda_j y_j \geq y_o$ does not hold for any $\lambda_j (j \neq o)$ with $\sum_{j \neq o} \lambda_j = 1$. This means that the optimal value to (10.7) is less than one, i.e., $g^* < 1$. ■

Figure 10.12 shows the spreadsheet model for model (10.7) where the input-oriented VRS super-efficiency scores are reported in cells I2:I16. This spreadsheet is obtained from the output-oriented VRS super-efficiency model shown in Figure 10.8. Figure 10.13 shows the Solver parameters. It can be seen that $g^* < 1$ if and only if model (10.7) is infeasible for a company.

| | A | B | C | D | E | F | G | H | I |
|----|------------------------------|-----------|--------|------------|----|------------|---|--------|------------------|
| 1 | Company | Revenue | Profit | λ | | | | g | Super efficiency |
| 2 | Mitsubishi | 184365.2 | 346.2 | 1 | | | | 0.9843 | Infeasible |
| 3 | Mitsui | 181518.7 | 314.8 | 0 | | | | 1.0157 | 1.751885253 |
| 4 | Itochu | 169164.6 | 121.2 | 0 | | | | 1.0899 | 1.606521649 |
| 5 | General Motors | 168828.6 | 6880.7 | 0 | | | | 0.7623 | Infeasible |
| 6 | Sumitomo | 167530.7 | 210.5 | 0 | | | | 1.1005 | 1.320957592 |
| 7 | Marubeni | 161057.4 | 156.6 | 0 | | | | 1.1447 | 1.009347188 |
| 8 | Ford Motor | 137137 | 4139 | 0 | | | | 1.26 | 0.737555958 |
| 9 | Toyota Motor | 111052 | 2662.4 | 0 | | | | 1.5777 | 0.603245345 |
| 10 | Exxon | 110009 | 6470 | 0 | | | | 1.0667 | 1.344368672 |
| 11 | Royal Dutch/Shell Group | 109833.7 | 6904.6 | 0 | | | | 0.9965 | Infeasible |
| 12 | Wal-Mart | 93627 | 2740 | 0 | | | | 1.8493 | 1.765155063 |
| 13 | Hitachi | 84167.1 | 1468.8 | 0 | | | | 2.1126 | 0.557595838 |
| 14 | Nippon Life Insurance | 83206.7 | 2426.6 | 0 | | | | 2.0813 | 4.806917693 |
| 15 | Nippon Telegraph & Telephone | 81937.2 | 2209.1 | 0 | | | | 2.124 | 0.470610997 |
| 16 | AT&T | 79609 | 139 | 0 | | | | 2.3159 | 0.533543522 |
| 17 | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Super | | | |
| 19 | Constraints | set | | Evaluation | | Efficiency | | | |
| 20 | Revenue | 184365.2 | >= | 184365.2 | | 2.315884 | | | |
| 21 | Profit | 346.2 | >= | 321.90786 | | | | | Infeasibility |
| 22 | $\sum \lambda$ | 1 | = | 1 | | | | | |
| 23 | λ_0 | 0 | = | 0 | | | | | |

Figure 10.12. Spreadsheet for Infeasibility Test (Input-oriented VRS Super-efficiency)

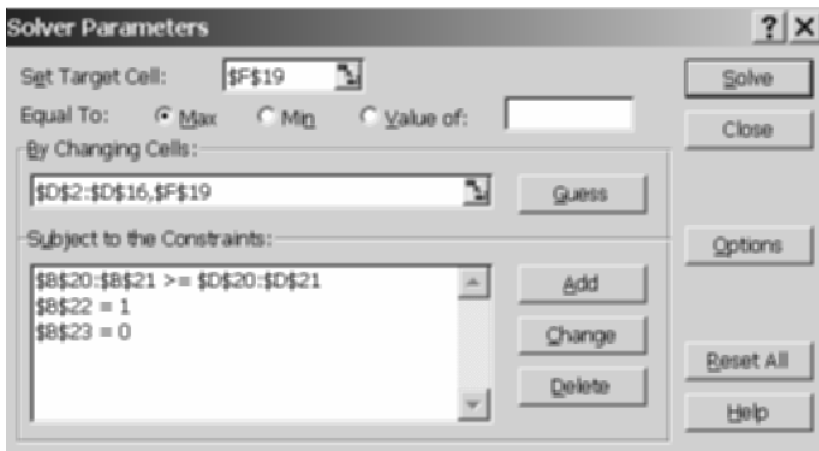


Figure 10.13. Solver Parameters for Infeasibility Test (Input-oriented)

Lemma 10.4 If DMU_o exhibits IRS, then the input-oriented VRS super-efficiency model is feasible and $\theta^{super*} > 1$, where θ^{super*} is the optimal value to the input-oriented VRS super-efficiency model.

[Proof]: Let $\vartheta = 1/\theta^{super}$, then the input-oriented VRS super-efficiency model becomes

$$\begin{aligned}
 & \max \vartheta \\
 & \text{subject to} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \hat{\lambda}_j x_j \leq x_o; \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \hat{\lambda}_j y_j \geq \vartheta y_o; \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \hat{\lambda}_j = \vartheta = \frac{1}{\theta^{\text{super}}}; \\
 & \theta^{\text{super}}, \vartheta, \hat{\lambda}_j \geq 0.
 \end{aligned} \tag{10.8}$$

where $\hat{\lambda}_j = \vartheta \lambda_j$ ($j \neq o$).

Since DMU_o exhibits IRS, then by Lemma 10.1, $\sum_{j \neq o} \lambda_j^* < 1$ in all optima to the following output-oriented CRS super-efficiency model

$$\begin{aligned}
 & \max \phi^{\text{super}} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_j \leq x_o \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_j \geq \phi^{\text{super}} y_o \\
 & \phi^{\text{super}}, \lambda_j \geq 0.
 \end{aligned} \tag{10.9}$$

Let $\sum_{j \neq o} \lambda_j^* = \vartheta < 1$. Since DMU_o is CRS inefficient, therefore $\phi^{\text{super}} > 1$ and hence $\phi^{\text{super}} > \vartheta$ is a feasible solution to (10.9). This in turn indicates that ϑ and λ_j^* ($j \neq o$) with $\sum_{j \neq o} \lambda_j^* = \vartheta$ is a feasible solution to (10.8). Therefore, the input-oriented VRS super-efficiency model is feasible. Furthermore, by Proposition 10.2, we have that $\phi^{\text{super}*} > 1$, where $\phi^{\text{super}*}$ is the optimal value to the input-oriented VRS super-efficiency model. ■

Theorem 10.8 If the input-oriented VRS super-efficiency model is infeasible, then DMU_o exhibits DRS or CRS.

[Proof]: If DMU_o exhibits IRS, then by Lemma 10.4, the input-oriented VRS super-efficiency model is feasible. A contradiction. ■

Theorems 10.6 and 10.7 indicate that if the input-oriented VRS super-efficiency model is infeasible, then DMU_o is one of the endpoints. Furthermore, if DRS prevail, then DMU_o is an right endpoint (see Figure 10.9).

10.2.4 Other Input-oriented Super-efficiency Models

Now, consider the input-oriented NIRS and NDRS super-efficiency models.

Theorem 10.9 The input-oriented NDRS super-efficiency model is always feasible.

[Proof]: Since $\sum_{j \neq o} \lambda_j \geq 1$ in the input-oriented DNRS super-efficiency model, there must exist some $\tilde{\lambda}_j$ with $\sum_{j \neq o} \tilde{\lambda}_j > 1$ such that $\sum_{j \neq o} \tilde{\lambda}_j y_j \geq y_o$ holds. Note that $\sum_{j \neq o} \tilde{\lambda}_j x_j \leq \theta^{\text{super}} x_o$ can always be satisfied by a proper θ^{super} . Thus, the input-oriented NDRS super-efficiency model is always feasible. ■

Lemma 10.5 The input-oriented NIRS super-efficiency model is infeasible if and only if the input-oriented VRS super-efficiency model is infeasible.

[Proof]: The *only if* part is obvious and hence is omitted. To establish the *if* part, we suppose that the input-oriented NIRS super-efficiency model is feasible. i.e., we have a feasible solution with $\sum_{j \neq o} \lambda_j \leq 1$ for the input-oriented NIRS super-efficiency model. If $\sum_{j \neq o} \lambda_j = 1$, then this solution is also feasible for the output-oriented VRS super-efficiency model. If $\sum_{j \neq o} \lambda_j < 1$, let $\sum_{j \neq o} \lambda_j = e < 1$. Then $\sum_{j \neq o} \hat{\lambda}_j y_j \geq \sum_{j \neq o} \lambda_j y_j \geq y_o$, where $\hat{\lambda}_j = \lambda_j / e$ ($j \neq o$) and $\sum_{j \neq o} \hat{\lambda}_j = 1$. Therefore $\hat{\lambda}_j$ ($j \neq o$) is a feasible solution to the output-oriented VRS super-efficiency model. Both possible cases lead to a contradiction. Thus, the output-oriented NIRS super-efficiency model is infeasible if the output-oriented VRS super-efficiency model is infeasible. ■

On the basis of this Lemma 10.5, we have

Theorem 10.10 For a specific extreme efficient $DMU_o = (x_o, y_o)$, we have
 (i) The input-oriented NIRS super-efficiency model is infeasible if and only if $(\chi x_o, y_o)$ is efficient under the VRS envelopment model for any $1 \leq \chi < +\infty$.
 (ii) The input-oriented NIRS super-efficiency model is feasible if and only if $g^* < 1$, where g^* is the optimal value to (10.7).

If $DMU_o \in E$ under the NIRS model, then DMU_o exhibits DRS or CRS. By Proposition 10.1, the DMU_o also lies on the VRS frontier that satisfies DRS or CRS. i.e., the VRS and NIRS envelopment models are identical for DMU_o . Thus $(\chi x_o, y_o)$ is also efficient under the NIRS envelopment model for any $1 \leq \chi < +\infty$.

Furthermore, Theorems 10.3 and 10.8 demonstrate that the possible infeasibility of the output-oriented and input-oriented VRS super-efficiency models can only occur at those extreme efficient DMUs exhibiting IRS (or CRS) and DRS (or CRS), respectively. Note that IRS and DRS are not allowed in the NIRS and NDRS models, respectively. Therefore, we have the following corollary.

Corollary 10.1

- (i) If $DMU_o \in E$ exhibits DRS, then all output-oriented super-efficiency DEA models are feasible.
- (ii) If $DMU_o \in E$ exhibits IRS, then all input-oriented super-efficiency DEA models are feasible.

By Theorems 10.1 and 10.6, we know that infeasibility indicates that the inputs of an extreme efficient DMU_o can be proportionally increased without limit or that the outputs can be decreased in any positive proportion, while preserving the efficiency of DMU_o . This indicates that the efficiency of DMU_o is always stable under the proportional data changes.

Models (10.2) and (10.7) are useful in the determination of infeasibility while Theorems 10.1 and 10.6 are useful in the sensitivity analysis of efficiency classifications. Table 10.2 summarizes the relationship between infeasibility and the super-efficiency DEA models.

Table 10.2. Super-efficiency DEA Models and Infeasibility

| Super-efficiency Models | | Infeasibility | RTS |
|-------------------------|------|-----------------------------|---------------------|
| Output-oriented | VRS | Theorem 10.2 (Model (10.2)) | DRS |
| | NIRS | always feasible | always feasible |
| Input-oriented | NDRS | Lemma 10.3, Theorem 10.2 | Corollary 10.1 (i) |
| | VRS | Theorem 10.7 (Model (10.7)) | IRS |
| | NIRS | Lemma 10.5, Theorem 7 | always feasible |
| | NDRS | always feasible | Corollary 10.1 (ii) |

Finally, we note that the super-efficiency VRS models can also be used to estimate RTS. This is a possible new usage of the super-efficiency DEA models.

10.3 Solving DEA Using DEA Frontier Software

To run the super-efficiency models presented in Table 10.1, select the “Super-efficiency” menu item. You will be prompted a form shown in Figure 10.4 for specifying the super-efficiency models. The results are reported in the “Super-efficiency” sheet.

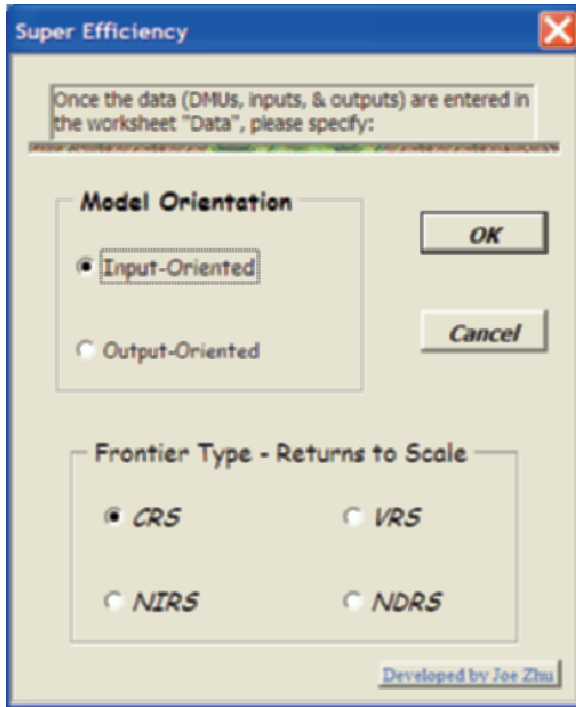


Figure 10.14. Super Efficiency Models

REFERENCES

1. Andersen, P. and N.C. Petersen (1993), A procedure for ranking efficient units in data envelopment analysis, *Management Science* 39, 1261-1264.
2. Charnes, A., W.W. Cooper and R.M. Thrall (1991), A structure for classifying and characterizing efficiencies and inefficiencies in DEA, *Journal of Productivity Analysis*, 2, 197-237.
3. Charnes, A., S. Haag, P. Jaska and J. Semple (1992), Sensitivity of efficiency classifications in the additive model of data envelopment analysis, *Int. J. Systems Sci.* 23, 789-798.
4. Seiford, L.M. and J. Zhu (1998), Stability regions for maintaining efficiency in data envelopment analysis, *European Journal of Operational Research*, 108, No. 1, 127-139.
5. Seiford, L.M. and J. Zhu (1999), Infeasibility of super efficiency data envelopment analysis models, *INFOR*, 37, No. 2, 174-187.
6. Thrall, R.M. (1996), Duality, classification and slacks in DEA, *Annals of Operations Research*, 66, 109-138.
7. Wilson, P.W. (1993), Detecting outliers in deterministic nonparametric frontier models with multiple outputs, *Journal of Business and Economic Statistics*, 11, 319-323.
8. Zhu, J. (1996), Robustness of the efficient DMUs in data envelopment analysis, *European Journal of Operational Research*, 90, No. 3, 451-460.

Chapter 11

Sensitivity Analysis

11.1 DEA Sensitivity Analysis

One important issue in DEA which has been studied by many DEA researchers is the efficiency sensitivity to perturbations in the data. Some DEA sensitivity studies focus on the sensitivity of DEA results to the variable and model selection, e.g., Ahn and Seiford (1993). Most of the DEA sensitivity analysis studies focus on the misspecification of efficiency classification of a test DMU. However, note that DEA is an extremal method in the sense that all extreme points are characterized as efficient. If data entry errors occur for various DMUs, the resulting isoquant may vary substantially. We say that the calculated frontiers of DEA models are stable if the frontier DMUs that determine the DEA frontier remain on the frontier after particular data perturbations are made.

By updating the inverse of the basis matrix associated with a specific efficient DMU in a DEA linear programming problem, Charnes, Cooper, Lewin, Morey and Rousseau (1985) study the sensitivity of DEA model to a single output change. This is followed by a series of sensitivity analysis articles by Charnes and Neralic in which sufficient conditions preserving efficiency are determined (see, e.g. Charnes and Neralic (1990)).

Another type of DEA sensitivity analysis is based on super-efficiency DEA models. Charnes, Haag, Jaska and Semple (1992), Rousseau and Semple (1995) and Charnes, Rousseau and Semple (1996) develop a super-efficiency DEA sensitivity analysis technique for the situation where simultaneous proportional change is assumed in all inputs and outputs for a specific DMU under consideration. This data variation condition is relaxed in Zhu (1996) and Seiford and Zhu (1998a) to a situation where inputs or

outputs can be changed individually and the entire (largest) stability region which encompasses that of Charnes, Haag, Jaska and Semple (1992) is obtained. As a result, the condition for preserving efficiency of a test DMU is necessary and sufficient.

The DEA sensitivity analysis methods we have just reviewed are all developed for the situation where data variations are only applied to the test efficient DMU and the data for the remaining DMUs are assumed fixed. Obviously, this assumption may not be realistic, since possible data errors may occur in each DMU. Seiford and Zhu (1998b) generalize the technique in Zhu (1996) and Seiford and Zhu (1998a) to the worst-case scenario where the efficiency of the test DMU is deteriorating while the efficiencies of the other DMUs are improving. In their method, same maximum percentage data change of a test DMU and the remaining DMUs is assumed and sufficient conditions for preserving an extreme efficient DMU's efficiency are determined. Note that Thompson, Dharmapala and Thrall (1994) use the SCSC (strong complementary slackness condition) multipliers to analyze the stability of the CRS model when the data for all efficient and all inefficient DMUs are simultaneously changed in opposite directions and in same percentages. Although the data variation condition is more restrictive in Seiford and Zhu (1998b) than that in Thompson, Dharmapala and Thrall, (1994), the super-efficiency based approach generates a larger stability region than the SCSC method. Also, the SCSC method is dependent upon a particular SCSC solution, among others, and therefore the resulting analysis may vary (see Cooper, Li, Seiford, Thrall and Zhu, 2001).

Seiford and Zhu (1999) (Chapter 10) develop the necessary and sufficient conditions for infeasibility of various super-efficiency DEA models. Although the super-efficiency DEA models employed in Charnes, Haag, Jaska and Semple (1992) and Charnes, Rousseau and Semple (1996) do not encounter the infeasibility problem, the models used in Seiford and Zhu (1998a) do. Seiford and Zhu (1998a) discover the relationship between infeasibility and stability of efficiency classification. That is, infeasibility means that the efficiency of the test DMU remains stable to data changes in the test DMU. Furthermore, Seiford and Zhu (1998b) show that this relationship is also true for the simultaneous data change case and other DEA models, such as the VRS model and the additive model of Charnes, Cooper, Golany, Seiford and Stutz (1985). This finding is critical since super-efficiency DEA models in Seiford and Zhu (1998b) are frequently infeasible for real-world data sets, indicating efficiency stability with respect to data variations in inputs/outputs associated with infeasibility.

Zhu (2001) extends the results in Seiford and Zhu (1998a; b) to a situation when different data variations are applied to the test DMU and the remaining DMUs, respectively.

In this chapter, we focus on the DEA sensitivity analysis methods based upon super-efficiency DEA models that are developed by Zhu (1996; 2001) and Seiford and Zhu (1998a;b). For the DEA sensitivity analysis based upon the inverse of basis matrix, the reader is referred to Neralic (1994).

Since an increase of any output or a decrease of any input cannot worsen the efficiency of DMU_o , we restrict our attention to decreases in outputs and increases in inputs for DMU_o . We consider proportional increases of inputs or proportional decreases of outputs of the form

$$\hat{x}_{i_o} = \beta_i x_{i_o} \quad \beta_i \geq 1, i = 1, \dots, m \tag{11.1}$$

$$\hat{y}_{r_o} = \alpha_r y_{r_o} \quad 0 < \alpha_r \leq 1, r = 1, \dots, s \tag{11.2}$$

where x_{i_o} ($i = 1, 2, \dots, m$) and y_{r_o} ($r = 1, \dots, s$) are respectively, the inputs and outputs for a specific extreme efficient $DMU_o = DMU_{j_o}$ among n DMUs.

Zhu (1996) provides a super-efficiency model to compute a stability region in which DMU_o remains efficient. Specifically, for an increase in inputs of form (11.1), this model is given by

$$\begin{aligned} &\min \beta_k^o \quad \text{for each } k=1, \dots, m \\ &\text{subject to} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{kj} \leq \beta_k^o x_{ko} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{i_o} \quad i \neq k \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{r_o} \quad r = 1, \dots, s \\ &\lambda_j, \beta_k^o \geq 0 \end{aligned} \tag{11.3}$$

where x_{ij} and y_{rj} are the i th input and r th output of DMU_j ($j = 1, \dots, n$), respectively. It can be seen that model (11.3) is developed from the input-oriented measure-specific model – CRS k th input-specific model.

Zhu's (1996) approach requires two assumptions: (i) the hyperplane constructed by the m hypothetical observations obtained from model (11.3) is not dominated by other DMUs and (ii) model (11.3) is feasible. However, in real word situations, these two assumptions may not be satisfied.

Note that any increase of input or any decrease of output will cause the DMUs in set E' (efficient but not extreme efficient) to become inefficient. For those DMUs in set F (weakly efficient with non-zero slacks), the amount of inputs (or outputs) which have non-zero slacks can be increased (or decreased) without limit, and these DMUs will remain in the set F. However, for inputs and outputs which have no slack, any input increase of (11.1) or

any output decrease of (11.2) will cause these DMUs to become inefficient. Therefore, the sensitivity issue of DMUs in set E' or F is straightforward if not trivial. Thus, we first focus on the efficiency of the DMUs in set E, i.e., the extreme efficient DMUs.

11.2 Stability Region^a

11.2.1 Input Stability Region

For $DMU_o \in E$, we first suppose that (11.3) is feasible for each input and consider input changes of form (11.1). As shown in Zhu (1996), the optimal value to (11.3), β_k^{o*} , gives the maximum possible increase for each individual input which allows DMU_o to remain efficient with the other inputs and all outputs held constant. Also, (11.3) provides m hypothetical frontier points (efficient DMUs) when DMU_o is excluded from the reference set. The k th point is generated by increasing the k th input from x_{ko} to $\beta_k^{o*} x_{ko}$ and holding all other inputs and outputs constant. We denote these k hypothetical observations by

$$DMU(\beta_k^{o*}) = (x_{1o}, \dots, \beta_k^{o*} x_{ko}, \dots, x_{mo}, y_{1o}, \dots, y_{so}) \tag{11.4}$$

Consider the following linear programming problem

$$\begin{aligned} & \min \sum_{i=1}^m \rho_i^o \\ & \text{subject to} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \rho_i^o x_{io} \quad i = 1, \dots, m \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \\ & \rho_i^o \geq 1, \lambda_j \geq 0. \end{aligned} \tag{11.5}$$

This model determines the smallest summation of the proportions to move DMU_o to the boundary of the convex hull of the other DMUs.

Lemma 11.1 Denote the optimal solution for (11.5) by ρ_i^{o*} ($i = 1, 2, \dots, m$). For $i = 1, 2, \dots, m$, we have $\beta_i^{o*} \geq \rho_i^{o*} \geq 1$.

[Proof]: Suppose for some i_o , $\beta_{i_o}^{o*} < \rho_{i_o}^{o*}$, then $\sum_{j \neq o} \lambda_j x_{ij} \leq \beta_{i_o}^{o*} x_{i_o o} < \rho_{i_o}^{o*} x_{i_o o}$. Therefore, any optimal solution to (11.3) is a feasible solution to (11.5). Thus, $m-1 + \beta_{i_o}^{o*} \geq \sum_{i=1}^m \rho_i^{o*} > m-1 + \beta_{i_o}^{o*}$. A contradiction. ■

Associated with ρ_i^{o*} ($i = 1, 2, \dots, m$), m additional points (or DMUs) can be generated as

$$DMU(\rho_k^{o*}) = (x_{1o}, \dots, \rho_k^{o*} x_{ko}, \dots, x_{mo}, y_{1o}, \dots, y_{so}) \tag{11.6}$$

Theorem 11.1 For $DMU_o = (x_{1o}, \dots, x_{mo}, y_{1o}, \dots, y_{so})$, denote an increase of inputs of form (11.1) by $DMU_o(\beta_1, \dots, \beta_m) = (\beta_1 x_{1o}, \dots, \beta_m x_{mo}, y_{1o}, \dots, y_{so})$ and define $\Omega^o = \{(\beta_1, \dots, \beta_m) \mid 1 \leq \beta_i \leq \rho_i^{o*}, i = 1, \dots, m\}$. If $(\beta_1, \dots, \beta_m) \in \Omega^o$, then $DMU_o(\beta_1, \dots, \beta_m)$ remains efficient.

[Proof]: Suppose $(\tilde{\beta}_1, \dots, \tilde{\beta}_m) \in \Omega^o$ and DMU_o with inputs of $\tilde{\beta}_i x_{io}$ ($i = 1, \dots, m$) is inefficient. In fact, (11.5) is equivalent to the following linear programming problem where $\rho_i^o x_{io} = x_{io} + \delta_i^o$ (and $\tilde{\beta}_i x_{io} = x_{io} + \tilde{\delta}_i$ in which $0 \leq \tilde{\delta}_i \leq \delta_i^{o*}$)

$$\begin{aligned} & \min \sum_{i=1}^m \delta_i^o \\ & \text{subject to} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} - \delta_i^o \leq x_{io} \quad i = 1, \dots, m \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, \dots, s \\ & \delta_i^o, \lambda_j \geq 0. \end{aligned}$$

$\sum_{i=1}^m \delta_i^o$ is $\sum_{i=1}^m \delta_i^{o*}$ at optimality when $\sum_{i=1}^m \rho_i^o = \sum_{i=1}^m \rho_i^{o*}$, and there exist λ_j ($j \neq 0$), $s_i^-, s_r^+ \geq 0$ that satisfy

$$\begin{aligned} & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} + s_i^- = x_{io} + \tilde{\delta}_i \quad i = 1, \dots, m \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\ & \lambda_j, s_i^-, s_r^+ \geq 0 \quad j = 1, \dots, n \end{aligned}$$

violating the optimality of $\sum_{i=1}^m \delta_i^{o*}$. Thus, $DMU_o(\beta_1, \dots, \beta_m)$ with inputs of $\tilde{\beta}_i x_{io}$ ($i = 1, \dots, m$) is efficient. ■

Definition 11.1 A region of allowable input increases is called an **Input Stability Region** if and only if DMU_o remains efficient after such increases occur.

The input stability region (ISR) determines by how much all of DMU_o 's inputs can be increased before DMU_o is within the convex hull of the other DMUs. From Lemma 11.1 and Theorem 11.1 we know that (i) Ω^o is only a

subset of ISR and (ii) the sets $\{\beta_i \mid 1 \leq \beta_i \leq \beta_i^{o*}\}$ $i = 1, \dots, m$, form part of the boundary of ISR.

If the input hyperplane constructed by the m points, $DMU(\beta_k^{o*})$, associated with the optimal values to (11.3), is not dominated by other DMUs except DMU_o , i.e., that input hyperplane is a new efficient facet when excluding DMU_o , then the following set Γ^o is precisely the ISR (Zhu, 1996b)

$$\Gamma^o = \{(\beta_1, \dots, \beta_m) \mid 1 \leq \beta_i \leq \beta_i^{o*}, i = 1, \dots, m \text{ and } B_1^o \beta_1 + \dots + B_m^o \beta_m \leq 1\}$$

where B_1^o, \dots, B_m^o are parameters determined by the following system of equations

$$\begin{cases} \beta_1^{o*} B_1^o + B_2^o + \dots + B_m^o = 1 \\ B_1^o + \beta_2^{o*} B_2^o + \dots + B_m^o = 1 \\ \dots \dots \dots \dots \dots \\ B_1^o + B_2^o + \dots + \beta_m^{o*} B_m^o = 1 \end{cases}$$

Zhu (1996) shows the following result

Theorem 11.2 In the case of input increases of form (11.1), for any extreme efficient DMU_o , if the m points, $DMU(\beta_k^{o*})$, which are associated with the optimal β values to (11.3), determine an efficient input hyperplane, then DMU_o remains efficient if and only if $(\beta_1, \dots, \beta_m) \in \Gamma^o$.

Next, suppose that the hyperplane constructed by the m points in (11.4) is dominated by some other DMUs which are inefficient when including DMU_o . In this case, the ISR is no longer the set of Γ^o . Thus, we develop the following procedure.

Initiation ($t = 0$). Solve model (11.3) for each k , $k = 1, \dots, m$. If the input hyperplane, which is determined by the m points of $DMU(\beta_k^{o*})$ in (11.4), is not dominated by other DMUs, then we obtain the ISR defined by Γ^o . Otherwise solve model (11.5). Associated with the optimal solutions to (11.5), ρ_i^{o*} , we obtain m new points, $DMU(\rho_i^{o*})$ ($i = 1, 2, \dots, m$) as given in (11.6) and Ω^o .

Iteration $t = 1, 2, \dots, T$. At iteration t , for each point of iteration $t-1$, say point p , which is associated with the optimal ρ values to (11.5), we solve model (11.3) at each new k th input, $k = 1, 2, \dots, m$ and apply the *Stopping Rule*. (a) If the rule is satisfied for a particular point p , then we have a similar set Γ_p^t determined by the optimal β values, say β_k^{p*} , $k = 1, \dots, m$. We

continue for the remaining points. (b) Otherwise solve model (11.5) for point p to obtain m new points and a similar set Ω'_p determined by the optimal ρ values, say ρ_k^* . Apply iteration $t+1$ to each of these m new points.

Stopping Rule. If the input hyperplane determined by the m points that are associated with the m optimal β values is not dominated by other DMUs, then iteration stops.

From the above procedure, we see that if a Γ -like set is obtained, then the iteration stops at a specific point. i.e., the Γ -like set indicates the termination of the iteration.

Theorem 11.3 The input stability region is a union of Ω^o and some Ω'_p and some Γ'_p .

[Proof]: Obviously, DMU_o remains efficient when its input increases $(\beta_1, \dots, \beta_m)$ belong to Ω^o or any of the Ω'_p or Γ'_p . Conversely, from the iterations we know that the ISR is connected. Since the input increases occurred in Ω -like sets, DMU_o is first moved to a particular point p which is used to construct a Γ -like set. By Theorem 11.2, we know that the sets of Γ'_p are the boundary sets of the ISR. This means that if further input increases are not in this kind of set, then DMU_o will become inefficient. Therefore, if DMU_o remains efficient, then the input increase of form (11.1) must be in Ω^o or any of the Ω'_p or Γ'_p . ■

11.2.2 Output Stability Region

Similarly, Seiford and Zhu (1998a) develop a sensitivity analysis procedure for output decreases of (11.2). For a specific extreme efficient DMU_o , we consider the following linear program (Zhu, 1996)

$$\begin{aligned}
 & \min \alpha_k^o \quad \text{for each } k=1, \dots, s \\
 & \text{subject to} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{kj} \geq \alpha_k^o y_{ro} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \neq k \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \\
 & \lambda_j, \alpha_k^o \geq 0
 \end{aligned} \tag{11.7}$$

Model (11.7) is a super-efficiency model based upon the CRS k th output specific model. The optimal values to (11.7), α_k^{o*} , $k = 1, \dots, s$, give s hypothetical frontier points (or DMUs) and a set Λ^o defined as follows

$$DMU(\alpha_k^{o*}) = (x_{10}, \dots, x_{m0}, y_{10}, \dots, \alpha_k^{o*} y_{k0}, \dots, y_{s0})$$

and

$$\Lambda^o = \{(\alpha_1, \dots, \alpha_m) \mid \alpha_r^{o*} \leq \alpha_r \leq 1, r = 1, \dots, s \text{ and } A_1^o \alpha_1 + \dots + A_s^o \alpha_s \geq 1\}$$

in which the parameters of A_r^o are determined by the following system of equations

$$\begin{cases} \alpha_1^{o*} A_1^o + A_2^o + \dots + A_s^o = 1 \\ A_1^o + \alpha_2^{o*} A_2^o + \dots + A_s^o = 1 \\ \dots \dots \dots \dots \dots \dots \\ A_1^o + A_2^o + \dots + \alpha_s^{o*} A_s^o = 1 \end{cases}$$

Here, we rewrite the result of Zhu (1996) as the following theorem

Theorem 11.4 In the case of output decreases of form (11.2), for any extreme efficient DMU_o , if the s points, $DMU(\alpha_k^{o*})$, which are associated with the optimal α values to (11.7), determine an efficient output hyperplane, then DMU_o remains efficient *if and only if* $(\alpha_1, \dots, \alpha_s) \in \Lambda^o$.

Definition 11.2 A region of allowable output decreases is called an **Output Stability Region** *if and only if* DMU_o remains efficient after such decreases occur.

Now, suppose that the output hyperplane constructed by the s points, $DMU(\alpha_k^{o*})$, is dominated by some other DMUs which are originally inefficient, then the output stability region (OSR) is not the set Λ^o . We consider the following linear programming problem

$$\begin{aligned} & \max \sum_{r=1}^s \varphi_r^o \\ & \text{subject to} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq \varphi_r^o y_{ro} \quad r=1, \dots, s \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i=1, \dots, m \\ & \lambda_j \geq 0, \varphi_r^o \leq 1. \end{aligned} \tag{11.8}$$

Similar to Theorem 11.1, we have

Theorem 11.5 For a decrease in outputs of form (11.2), if

$$(\alpha_1, \dots, \alpha_s) \in \Psi^o = \{(\alpha_1, \dots, \alpha_m) \mid \varphi_r^{o*} \leq \alpha_r \leq 1, r = 1, \dots, s\}$$

then DMU_o remains efficient.

We also have the following s new points that associated with the optimal solutions, φ_r^{o*} , of (11.8)

$$DMU(\varphi_k^{o*}) = (x_{10}, \dots, x_{m0}, y_{10}, \dots, \varphi_k^{o*} y_{k0}, \dots, y_{s0})$$

To obtain the output stability region, we apply model (11.7) and model (11.8) at each iteration in the procedure for input stability region until no Λ -like sets can be obtained. Similarly, we have

Theorem 11.6 The output stability region is a union of Λ -like sets and Ψ -like sets.

11.2.3 Geometrical Presentation of Input Stability Region

We now illustrate the sensitivity analysis procedure geometrically for the following five DMUs with a single output and two inputs. For convenience, we suppose the five DMUs produce an equal amount of output and thus omit the output quantities in the following discussion. With the help of Figure 11.1, we will see how to keep track of newly generated points (DMUs) by the procedure.

Table 11.1. DMUs for Illustration of Input Stability Region

| DMU | 1 (X_1) | 2 (X_2) | 3 (X_3) | 4 (X_4) | 5 (X_5) |
|---------------|-------------|-------------|-------------|-------------|-------------|
| input 1 x_1 | 5 | 2 | 1 | 5/2 | 9/4 |
| input 2 x_2 | 1 | 2 | 5 | 5/2 | 11/4 |

It is obvious that DMUs 1, 2, and 3 are extreme efficient, and DMUs 4 and 5 are inefficient. Let $DMU_o = DMU_2$ ($x_{10} = 2, x_{20} = 2$), i.e., we consider the robustness of the efficiency of DMU2 when the two inputs increase.

Initiation (t=0). First we solve model (11.3) for DMU_o (point X_2), that is

$$\beta_1^{o*} = \min \beta_1^o$$

subject to

$$5\lambda_1 + \lambda_3 + \frac{5}{2}\lambda_4 + \frac{9}{4}\lambda_5 \leq 2\beta_1^o$$

$$\lambda_1 + 5\lambda_3 + \frac{5}{2}\lambda_4 + \frac{11}{4}\lambda_5 \leq 2$$

$$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geq 1$$

$$\lambda_1, \lambda_3, \lambda_4, \lambda_5, \beta_1^o \geq 0$$

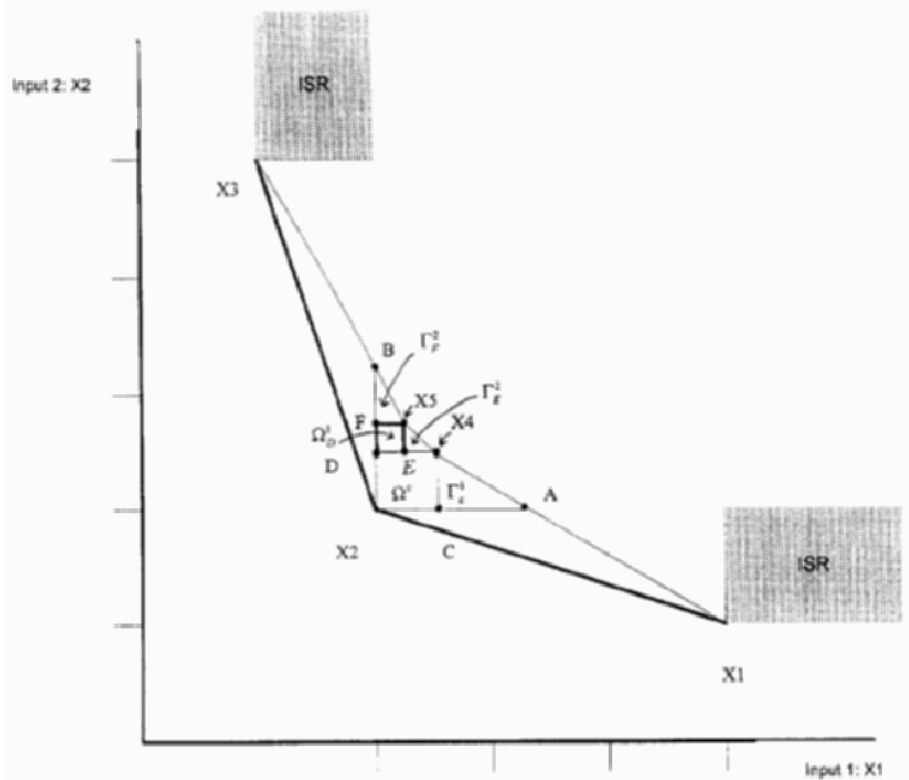


Figure 11.1. Geometrical Presentation of Input Stability Region

Figure 11.2 shows the spreadsheet model. Cell D8 is reserved to indicate the DMU under consideration, and is equal to 2 now, indicating DMU2. Cells F2:F6 are reserved to indicate λ_j . Cell E9 represents β_j . This spreadsheet model is developed from the spreadsheet model for measure-specific models. The formulas for the spreadsheet shown in Figure 11.2 are

- Cell B10=SUMPRODUCT(B2:B6,F2:F6)
- Cell B11=SUMPRODUCT(C2:C6,F2:F6)
- Cell B12=SUMPRODUCT(E2:E6,F2:F6)
- Cell B13=INDEX(F2:F6,D8,1)

Cell D10=E9*INDEX(B2:B6,D8,1)

Cell D11=INDEX(C2:C6,D8,1)

Cell D12=INDEX(E2:E6,D8,1)

Cell D13=0

| | A | B | C | D | E | F |
|----|------------------|-------------------------|---------|-------|---------|-----------|
| 1 | DMUs | input 1 | input 2 | | output | λ |
| 2 | 1 | 5 | 1 | | 1 | 0.333333 |
| 3 | 2 | 2 | 2 | | 1 | 0 |
| 4 | 3 | 1 | 5 | | 1 | 0 |
| 5 | 4 | 2 1/2 | 2 1/2 | | 1 | 0.666667 |
| 6 | 5 | 2 1/4 | 2 3/4 | | 1 | 0 |
| 7 | | | | | | |
| 8 | | DMU under consideration | | 2 | β | |
| 9 | Constraints | | | | 1 2/3 | |
| 10 | input1 | 3.333333 | < | 3 1/3 | | point A |
| 11 | input2 | 2 | < | 2 | | |
| 12 | output | 1 | > | 1 | | |
| 13 | DMU ₀ | 0 | = | 0 | | |

Figure 11.2. Spreadsheet for Input Stability Region (Input 1)

Figure 11.3 shows the Solver parameters for the spreadsheet shown in Figure 11.2. We have $\beta_1^{o*} = 5/3$ (see cell E9 in Figure 11.2). For $k = 2$, we have the spreadsheet model shown in Figure 11.4. The formulas for cells B10:B13 and cells D12:D13 remain the same. We need to change the formulas in cells D10:D11 to

Cell D10=INDEX(B2:B6,D8,1)

Cell D11=E9*INDEX(C2:C6,D8,1)

Using the Solver parameters shown in Figure 11.3, we obtain $\beta_2^{o*} = 8/5$ (see cell F9 in Figure 11.4). Furthermore, we have the following two newly generated points associated with the optimal β values (cells D10:D11 in Figures 11.2 and 11.4)

$$\begin{cases} A = (\beta_1^{o*} x_{10}, x_{20}) = (\frac{10}{3}, 2) \\ B = (x_{10}, \beta_2^{o*} x_{20}) = (2, \frac{16}{5}) \end{cases}$$

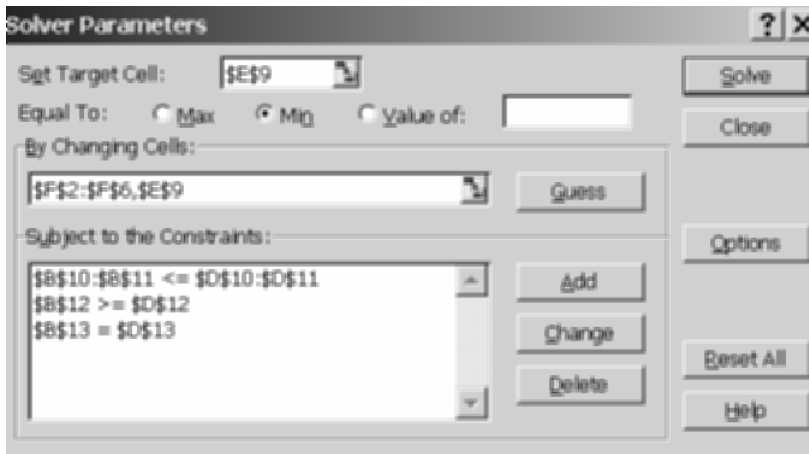


Figure 11.3. Solver Parameters for Input Stability Region

| | A | B | C | D | E | F |
|----|-------------|-------------------------|---------|-------|---------|-----------|
| 1 | DMUs | input 1 | input 2 | | output | λ |
| 2 | 1 | 5 | 1 | | 1 | 0 |
| 3 | 2 | 2 | 2 | | 1 | 0 |
| 4 | 3 | 1 | 5 | | 1 | 0.2 |
| 5 | 4 | 2 1/2 | 2 1/2 | | 1 | 0 |
| 6 | 5 | 2 1/4 | 2 3/4 | | 1 | 0.8 |
| 7 | | | | | | |
| 8 | | DMU under consideration | | 2 | β | |
| 9 | Constraints | | | | 1 3/5 | |
| 10 | input1 | 2 | <= | 2 | | |
| 11 | input2 | 3.2 | <= | 3 1/5 | | Point B |
| 12 | output | 1 | >= | 1 | | |
| 13 | DMUo | 0 | = | 0 | | |

Figure 11.4. Spreadsheet for Input Stability Region (Input 2)

Obviously, the input hyperplane (line segment AB) constructed by A and B is dominated by DMU4 and DMU5. Thus, we solve model (11.5) for DMU_o , that is

$$\begin{aligned} & \min \rho_1^o + \rho_2^o \\ & \text{subject to} \\ & 5\lambda_1 + \lambda_3 + \frac{5}{2}\lambda_4 + \frac{9}{4}\lambda_5 \leq 2\rho_1^o \end{aligned}$$

$$\lambda_1 + 5\lambda_3 + \frac{5}{2}\lambda_4 + \frac{11}{4}\lambda_5 \leq 2\rho_2^o$$

$$\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geq 1$$

$$\lambda_1, \lambda_3, \lambda_4, \lambda_5 \geq 0, \rho_1^o, \rho_2^o \geq 1$$

Figure 11.5 shows the spreadsheet model for model (11.5). Cell E9 and Cell F9 represent ρ_1 and ρ_2 , respectively. The target cell G9 (=E9+F9) represents the objective function of model (11.5). We change the formulas of cells D10:D11 to

Cell D10=E9*INDEX(B2:B6,D8,1)

Cell D11=F9*INDEX(C2:C6,D8,1)

| | A | B | C | D | E | F | G |
|----|-------------|-------------------------|---------|-------|----------|-----------|-------|
| 1 | DMUs | input 1 | input 2 | | output | λ | |
| 2 | 1 | 5 | 1 | | 1 | 0 | |
| 3 | 2 | 2 | 2 | | 1 | 0 | |
| 4 | 3 | 1 | 5 | | 1 | 0 | |
| 5 | 4 | 2 1/2 | 2 1/2 | | 1 | 1 | |
| 6 | 5 | 2 1/4 | 2 3/4 | | 1 | 0 | |
| 7 | | | | | | | |
| 8 | | DMU under consideration | | 2 | ρ_1 | ρ_2 | |
| 9 | Constraints | | | | 1 1/4 | 1 1/4 | 2 1/2 |
| 10 | input1 | 2.5 | < | 2 1/2 | | | |
| 11 | input2 | 2.5 | < | 2 1/2 | | | |
| 12 | output | 1 | > | 1 | | | |
| 13 | DMUo | 0 | = | 0 | | | |

Figure 11.5. Spreadsheet for Input Stability Region (Model (11.5))

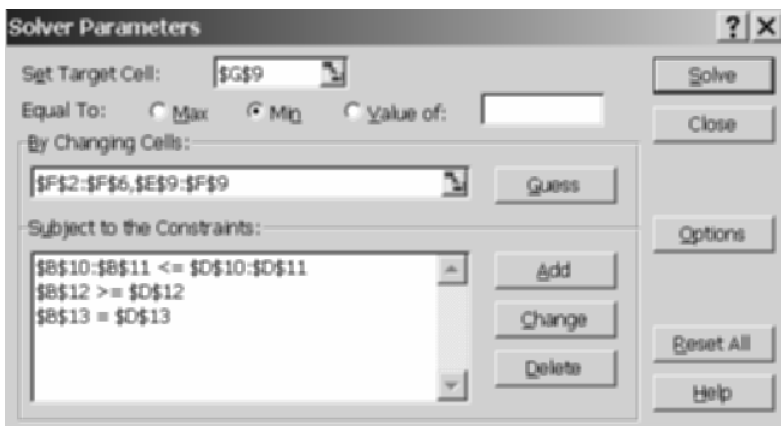


Figure 11.6. Solver Parameters for Model (11.5)

Figure 11.6 shows the Solver parameters for model (11.5). We obtain $\rho_1^{o*} = \rho_2^{o*} = 5/4$. Moreover, we have $\Omega^o = \{(\beta_1, \beta_2) | 1 \leq \beta_1 \leq 5/4, 1 \leq \beta_2 \leq 5/4\}$ as shown in Figure 11.1 and obtain the following two additional points associated with optimal ρ values

$$\begin{cases} C = (\rho_1^{o*} x_{10}, x_{20}) = (x_{10}^C, x_{20}^C) = (5/2, 2) \\ D = (x_{10}, \rho_2^{o*} x_{20}) = (x_{10}^D, x_{20}^D) = (2, 5/2) \end{cases}$$

Iteration: t=1. For the first point C, we solve model (11.3)

$$\begin{aligned} & \min \beta_1^C \\ & \text{subject to} \\ & 5\lambda_1 + \lambda_3 + \frac{5}{2}\lambda_4 + \frac{9}{4}\lambda_5 \leq \beta_1^C x_{10}^C = \frac{5}{2}\beta_1^C \\ & \lambda_1 + 5\lambda_3 + \frac{5}{2}\lambda_4 + \frac{11}{4}\lambda_5 \leq x_{20}^C = 2 \\ & \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geq 1 \\ & \lambda_1, \lambda_3, \lambda_4, \lambda_5, \beta_1^C \geq 0 \end{aligned}$$

We have $\beta_1^{C*} = 4/3$. Similarly, $\beta_2^{C*} = 5/4$. The two corresponding new points are as follows

$$\begin{cases} A = (\beta_1^{C*} x_{10}^C, x_{20}^C) = (\frac{10}{3}, 2) \\ X_4 = (x_{10}^C, \beta_2^{C*} x_{20}^C) = (\frac{5}{2}, \frac{5}{2}) \end{cases}$$

The input hyperplane constructed by these two points (line segment $A X_4$) is not dominated by other DMUs, therefore the iteration for point C stops and we have the following results.

Let $\hat{x}_{10} = c_1 x_{10}^C = c_1 \rho_1^{o*} x_{10}$ and $\hat{x}_{20} = c_2 x_{20}^C = c_2 x_{20}$. By Zhu (1996b), we have

$$\Gamma_c^1 = \{(c_1, c_2) | 1 \leq c_k \leq \beta_k^{C*}, k = 1, 2 \text{ and } B_1^C c_1 + B_2^C c_2 \leq 1\}$$

where B_1^C and B_2^C are determined as follows

$$\begin{cases} B_1^C \beta_1^{C*} + B_2^C = 1 \\ B_1^C + B_2^C \beta_2^{C*} = 1 \end{cases} \Rightarrow \begin{cases} \frac{4}{3} B_1^C + B_2^C = 1 \\ B_1^C + \frac{5}{4} B_2^C = 1 \end{cases} \Rightarrow \begin{cases} B_1^C = \frac{3}{8} \\ B_2^C = \frac{1}{2} \end{cases}$$

Hence,

$$\Gamma_c^1 = \{(c_1 \rho_1^{o*}, c_2) | \rho_1^{o*} \leq c_1 \rho_1^{o*} \leq \rho_1^{o*} \beta_1^{C*} = \beta_1^{o*}, 1 \leq c_2 \leq \beta_2^{C*}, \text{ and } \frac{3}{8} \rho_1^{o*} c_1 + \frac{1}{2} \rho_1^{o*} c_2 \leq \rho_1^{o*}\}$$

Let $\beta_1 = c_1 \rho_1^{o*}$ and $\beta_2 = c_2$. Then $\Gamma_c^1 = \{(\beta_1, \beta_2) | 5/4 \leq \beta_1 \leq 5/3, 1 \leq \beta_2 \leq 5/4, 3\beta_1 + 5\beta_2 \leq 10\}$.

Next, for the second point D, solving model (11.3) when $k = 1$ and $k = 2$ yields $\beta_1^{D*} = 5/4$ and $\beta_2^{D*} = 32/25$, respectively. Associated with these two optimal β values, we have two new points

$$\begin{cases} X_4 = (\beta_1^{D*} x_{10}^D, x_{20}^D) = (\frac{5}{2}, \frac{5}{2}) \\ B = (x_{10}^D, \beta_2^{D*} x_{20}^D) = (2, \frac{16}{5}) \end{cases}$$

The input hyperplane determined by these two points (X_4 and B) is dominated by DMU5, therefore we compute model (11.5) for point D,

$$\begin{aligned} & \min \rho_1^D + \rho_2^D \\ & \text{subject to} \\ & 5\lambda_1 + \lambda_3 + \frac{5}{2}\lambda_4 + \frac{9}{4}\lambda_5 \leq \rho_1^D x_{10}^D = 2\rho_1^D \\ & \lambda_1 + 5\lambda_3 + \frac{5}{2}\lambda_4 + \frac{11}{4}\lambda_5 \leq \rho_2^D x_{20}^D = \frac{5}{2}\rho_2^D \\ & \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 \geq 1 \\ & \lambda_1, \lambda_3, \lambda_4, \lambda_5 \geq 0, \rho_1^D, \rho_2^D \geq 1 \end{aligned}$$

We have $\rho_1^{D*} = 9/8$ and $\rho_2^{D*} = 11/10$. Next we compute Ω_D^1 .

First, let $\hat{x}_{10} = d_1 x_{10}^D = d_1 x_{10}$ and $\hat{x}_{20} = d_2 x_{20}^D = d_2 \rho_2^{D*} x_{20}$. By Theorem 11.1, we have

$$\Omega_D^1 = \{(d_1, d_2) \mid 1 \leq d_k \leq \rho_k^{D*}, k = 1, 2\}$$

and

$$\Omega_D^1 = \{(d_1, \rho_2^{D*} d_2) \mid 1 \leq d_1 \leq \rho_1^{D*}, \rho_2^{D*} \leq \rho_2^{D*} d_2 \leq \rho_2^{D*} \rho_1^{D*}\}$$

and further $\Omega_D^1 = \{(\beta_1, \beta_2) \mid 1 \leq \beta_1 \leq 9/8, 5/4 \leq \beta_2 \leq 11/8\}$.

Associated with the two optimal values of ρ_1^{D*} and ρ_2^{D*} , we now have the following two points

$$\begin{cases} E = (\rho_1^{D*} x_{10}^D, x_{20}^D) = (x_{10}^E, x_{20}^E) = (\frac{9}{4}, \frac{5}{2}) \\ F = (x_{10}^D, \rho_2^{D*} x_{20}^D) = (x_{10}^F, x_{20}^F) = (2, \frac{11}{4}) \end{cases}$$

Iteration t = 2. For the point E generated from the point D in the first iteration, we obtain, by solving model (11.3), $\beta_1^{E*} = 10/9$, $\beta_2^{E*} = 11/10$, and two corresponding points

$$\begin{cases} X_4 = (\beta_1^{E*} x_{10}^E, x_{20}^E) = (\frac{5}{2}, \frac{5}{2}) \\ X_5 = (x_{10}^E, \beta_2^{E*} x_{20}^E) = (\frac{9}{4}, \frac{11}{4}) \end{cases}$$

The input hyperplane constructed by the two points of X_4 and X_5 is not dominated by other DMUs, therefore the iteration stops.

Let $\hat{x}_{10} = e_1 x_{10}^E = e_1 \rho_1^{D^*} x_{10}^D = e_1 \rho_1^{D^*} x_{10}$ and $\hat{x}_{20} = e_2 x_{20}^E = e_2 x_{20}^D = e_2 \rho_2^{\sigma^*} x_{20}$. Similar to Γ^o , we have $\Gamma_E^2 = \{(e_1, e_2) | 1 \leq e_k \leq \beta_k^{E^*}, k = 1, 2 \text{ and } B_1^E e_1 + B_2^E e_2 \leq 1\}$ in which B_1^E and B_2^E are determined by

$$\begin{cases} B_1^E \beta_1^{E^*} + B_2^E = 1 \\ B_1^E + B_2^E \beta_2^{E^*} = 1 \end{cases} \Rightarrow \begin{cases} \frac{10}{9} B_1^E + B_2^E = 1 \\ B_1^E + \frac{11}{10} B_2^E = 1 \end{cases} \Rightarrow \begin{cases} B_1^E = \frac{9}{20} \\ B_2^E = \frac{1}{2} \end{cases}$$

Thus, $\Gamma_E^2 = \{(e_1 \rho_1^{D^*}, e_2 \rho_2^{\sigma^*}) | \rho_1^{D^*} \leq e_1 \rho_1^{D^*} \leq \rho_1^{D^*} \beta_1^{E^*} = \rho_1^{\sigma^*}, \rho_2^{\sigma^*} \leq e_2 \rho_2^{\sigma^*} \leq \beta_2^{E^*} \rho_2^{\sigma^*}, \text{ and } (9/20)\rho_1^{D^*} \rho_2^{\sigma^*} e_1 + (1/2)\rho_1^{D^*} \rho_2^{\sigma^*} e_2 \leq \rho_1^{D^*} \rho_2^{\sigma^*}\} = \{(\beta_1, \beta_2) | 9/8 \leq \beta_1 \leq 5/4, 5/4 \leq \beta_2 \leq 11/8, 18\beta_1 + 2\beta_2 \leq 45\}$, where $\beta_1 = e_1 \rho_1^{D^*}$ and $\beta_2 = e_2 \rho_2^{\sigma^*}$.

For the point F, we have $\beta_1^{F^*} = 9/8$ and $\beta_2^{F^*} = 64/55$, and two corresponding points

$$\begin{cases} X_5 = (\beta_1^{F^*} x_{10}^F, x_{20}^F) = (\frac{9}{4}, \frac{11}{4}) \\ B = (x_{10}^F, \beta_2^{F^*} x_{20}^F) = (2, \frac{16}{5}) \end{cases}$$

The input hyperplane constructed by these two points of X_5 and B is not dominated by other DMUs, therefore the iteration stops.

Let $\hat{x}_{10} = f_1 x_{10}^F = f_1 x_{10}$ and $\hat{x}_{20} = f_2 x_{20}^F = f_2 \rho_2^{D^*} x_{20}^D = f_2 \rho_2^{D^*} \rho_2^{\sigma^*} x_{20}$. Similar to Γ^o , we have $\Gamma_F^2 = \{(f_1, f_2) | 1 \leq f_k \leq \beta_k^{F^*}, k = 1, 2 \text{ and } B_1^F f_1 + B_2^F f_2 \leq 1\}$ in which B_1^F and B_2^F are the solutions to the following system of equations

$$\begin{cases} B_1^F \beta_1^{F^*} + B_2^F = 1 \\ B_1^F + B_2^F \beta_2^{F^*} = 1 \end{cases} \Rightarrow \begin{cases} \frac{9}{8} B_1^F + B_2^F = 1 \\ B_1^F + \frac{64}{55} B_2^F = 1 \end{cases} \Rightarrow \begin{cases} B_1^F = \frac{9}{17} \\ B_2^F = \frac{55}{136} \end{cases}$$

Thus, $\Gamma_F^2 = \{(\beta_1, \beta_2) | 1 \leq \beta_1 \leq 9/8, 11/8 \leq \beta_2 \leq 8/5, 9\beta_1 + 5\beta_2 \leq 17\}$. Finally, we obtain the following input stability region for DMU2 (point X_5) as shown in Figure 11.1.

$$ISR = \Omega^o \cup \Omega_D^1 \cup \Gamma_C^1 \cup \Gamma_E^2 \cup \Gamma_F^2$$

11.3 Infeasibility and Stability

The previous sensitivity analysis procedure is developed under the assumption that model (11.3) (or model (11.7)) is feasible. However, this may not be always the case. For example, if we calculate (11.3) for DMU3 in Table 11.1, then we have $\beta_1^* = 2$ for the first input but infeasibility for the second input. If we calculate (11.3) for DMU1, then we have infeasibility for the first input. Figure 11.7 presents the results for the three efficient DMUs 1, 2, and 3. The calculation is performed by a VBA procedure "InputStabilityRegion".

```

Sub InputStabilityRegion()
Dim i As Integer
For i = 1 To 3
Range("D8") = i
SolverSolve UserFinish:=True
If SolverSolve(UserFinish:=True) = 5 Then
Range("G" & i + 1) = "Infeasible"
Else
Range("G" & i + 1) = Range("E9")
End If
Next
End Sub
    
```

Note that, in fact, we can increase infinitely the amount of DMU3's second input (DMU1's first input) while maintaining the efficiency of DMU3.

| | A | B | C | D | E | F | G |
|----|-------------|---------|---------|---|---------|-----------|------------|
| 1 | DMUs | input 1 | input 2 | | output | λ | ISR |
| 2 | 1 | 5 | 1 | | 1 | 0 | Infeasible |
| 3 | 2 | 2 | 2 | | 1 | 1 | 1 2/3 |
| 4 | 3 | 1 | 5 | | 1 | 0 | 2 |
| 5 | 4 | 2 1/2 | 2 1/2 | | 1 | 0 | |
| 6 | 5 | 2 1/4 | 2 3/4 | | 1 | 0 | |
| 7 | | | | | | | |
| 8 | | | | 3 | β | | |
| 9 | Constraints | | | | 2 | | |
| 10 | input1 | 2 | \leq | 2 | | | |
| 11 | input2 | 2 | \leq | 5 | | | |
| 12 | output | 1 | \geq | 1 | Input1 | | |
| 13 | DMUo | 0 | = | 0 | | | |

| | A | B | C | D | E | F | G |
|----|-------------|---------|---------|---|---------|-----------|------------|
| 1 | | input 1 | input 2 | | output | λ | ISR |
| 2 | 1 | 5 | 1 | | 1 | 0 | 2 |
| 3 | 2 | 2 | 2 | | 1 | 0.5 | 1 3/5 |
| 4 | 3 | 1 | 5 | | 1 | 0 | Infeasible |
| 5 | 4 | 2 1/2 | 2 1/2 | | 1 | 0 | |
| 6 | 5 | 2 1/4 | 2 3/4 | | 1 | 0 | |
| 7 | | | | | | | |
| 8 | | | | 3 | β | | |
| 9 | Constraints | | | | 1/5 | | |
| 10 | input1 | 1 | \leq | 1 | | | |
| 11 | input2 | 1 | \leq | 1 | | | |
| 12 | output | 0.5 | \geq | 1 | Input2 | | |
| 13 | DMUo | 0 | = | 0 | | | |

Figure 11.7. Optimal β

Theorem 11.7 For an efficient DMU_o , an increase of the k th input only, model (11.3) is infeasible, *if and only if*, the amount of k th input of DMU_o can be increased without limitation while maintaining the efficiency of DMU_o .

[Proof]: The *if* part is obvious from the fact that if (11.3) is feasible, then the optimal value to (11.3) gives the maximum increase proportion of the k th input. Therefore, the amount of k th input cannot be infinitely increased.

To establish the *only if* part we suppose that the k th input is increased by $M \geq 1$ and DMU_o is inefficient. By substituting DMU_o into CRS envelopment model, we obtain an optimal solution $\theta^* \leq 1$, $\lambda_o^* = 0, \lambda_j^*(j \neq o)$, in which $\theta^* \leq 1$ implies $DMU_o \in F$. Therefore,

$$\begin{cases} \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j^* x_{kj} \leq \theta^* M x_{ko} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j^* x_{ij} \leq \theta^* x_{io} \leq x_{io} & i \neq k \end{cases}$$

This means that $\lambda_j^*(j \neq o), \beta_k = \theta^* M$ is a feasible solution to (11.3) and leads to a contradiction. Since M is arbitrary, the amount of the k th input can be infinitely increased while maintaining DMU_o 's efficiency. ■

As the Theorem 11.7 indicates, if (11.3) is infeasible, then $\beta_k^{o*} = +\infty$. Thus, in this situation, we must modify the sensitivity analysis procedure. Because we are unable to express the new frontier point associated with $\beta_k^{o*} = +\infty$, and further, to apply the stopping rule. Note that if we here assume that all data are positive, then model (11.5) is always feasible. But in the case of infeasibility, (11.5) does not perform well. For example, if we apply (11.5) to DMU3, we obtain $\rho_1^* = 2$ and $\rho_2^* = 1$. i.e., $\rho_i^{o*} = 1$ relative to the unbounded input i . Consequently, we are unable to determine the stability region. Thus, from a computational point of view, in this situation, we apply model (11.5) with $\rho_i^o = \theta_o$ ($i = 1, \dots, m$). That is,

$$\begin{aligned} & \min \theta_o \\ & \text{subject to} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta_o x_{io} & i = 1, \dots, m \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} & r = 1, \dots, s \\ & \lambda_j \geq 0 & j = 1, \dots, n. \end{aligned} \tag{11.9}$$

At each point, in each iteration, we first apply (11.9) when (11.3) is infeasible, and then, for the newly generated points, we apply (11.3). If (11.3) is feasible, we use the procedure suggested previously. If (11.3) is still infeasible, then apply (11.9) again (go to next iteration). We can, in fact, regard infeasibility as the rejection of the stopping rule, and then we calculate model (11.9) instead of (11.5) to generate new frontier DMUs for the next iteration. In this situation, the set Ω^o obtained from (11.9) corresponds to the ∞ -norm in Charnes, Haag, Jaska and Semple (1992).

This general procedure for the infeasibility case is stated below

Step 1: Solve model (11.9).

Step 2: Solve (11.3) for the newly generated points by (11.9):

$$DMU(\theta_o^*) = (x_{1o}, \dots, \theta_o^* x_{ko}, \dots, x_{mo}, y_{1o}, \dots, y_{so}), k=1, \dots, m.$$

- (a) If (11.3) is feasible, then go to the procedure given in section 11.2.1;
- (b) If (11.3) is infeasible, then go to step 1.

Note that infeasibility often occurs in real world situations. In theory, one can always use this general procedure to determine the ISR. However, in practice one may use this procedure to approximate the ISR due to the fact that some inputs' amount can be infinitely increased. For example, it is obvious that the IRS for DMU3 in Table 11.1 is

$$ISR = \{(\beta_1, \beta_2) | 1 \leq \beta_1 < 2, 1 \leq \beta_2 < +\infty\} \tag{11.10}$$

which is the shaded region shown in Figure 11.1. Furthermore, we have

Theorem 11.8 For the two-input case, one of the two optimal β values in (11.3) is equal to the corresponding optimal value to (11.9), *if and only if*, $ISR = \{(\beta_1, \beta_2) | 1 \leq \beta_i < \beta_i^{o*}, i = 1, 2\}$, where one of the β_i^{o*} is finite and the other is $+\infty$.

[Proof]: Without loss of generality, assume that β_1^{o*} is finite and $\beta_2^{o*} = +\infty$.

Suppose that $ISR = \{(\beta_1, \beta_2) | 1 \leq \beta_1 < \beta_1^{o*}, 1 \leq \beta_2 < +\infty\}$. Let $\beta_1 = \beta_1^{o*}$, then DMU_o with $(\beta_1^{o*} x_{1o}, \beta_1^{o*} x_{2o})$ is a frontier point. Therefore $\theta_o^* = \beta_1^{o*}$. This completes the proof of the *if* part.

Suppose $\theta_o^* = \beta_1^{o*}$. Obviously, if DMU_o with inputs of $(\beta_1 x_{1o}, \beta_2 x_{2o})$ is in set E, then $(\beta_1, \beta_2) \in ISR = \{(\beta_1, \beta_2) | 1 \leq \beta_1 < \beta_1^{o*} = \theta_o^*, 1 \leq \beta_2 < +\infty\}$. Next, note that the original DMU_o belongs to set E, therefore $\theta_o^* > 1$. By θ_o^* and β_1^{o*} , we obtain two frontier points $A = (\theta_o^* x_{1o}, \theta_o^* x_{2o})$ and $B = (\beta_1^{o*} x_{1o}, x_{2o}) = (\theta_o^* x_{1o}, x_{2o})$. Thus,

$$\theta_o^* = \beta_1^{o*} \Rightarrow A \in F \text{ with nonzero slack on the second input} \Rightarrow (\beta_1^{o*} x_{1o}, \beta_2 x_{2o}) \in F, \text{ where } 1 \leq \beta_2 < +\infty \Rightarrow (\beta_1 x_{1o}, \beta_2 x_{2o}) \in E, \text{ where } 1 \leq \beta_1 < \beta_1^{o*}, 1 \leq \beta_2 < +\infty.$$

Therefore, if $(\beta_1, \beta_2) \in ISR$, then DMU_o preserves its efficiency. This completes the *only if* part. ■

By the proof of Theorem 11.8 and the result of Theorem 11.7, we can easily obtain

Corollary 11.1 For the two-input case, if one of the two optimal β values in (11.3) is equal to the corresponding optimal value to (11.9), then (11.3) is infeasible for the other input.

Note that equality is not held in the right hand side of the inequalities in (11.10) of β_i . Otherwise, DMU_o will be in set F. For instance, if $\beta_1 = 2$ in (11.10), then DMU3 (X3) is moved into set F. However, if we only consider weak efficiency, then the equality can be imposed. Because the efficiency ratings are equal to one for the DMUs in set F.

Finally, the above discussion and development holds for the output case when (11.7) is infeasible. That is,

Theorem 11.9 For an efficient DMU_o , an increase of the k th output only, model (11.7) is infeasible, *if and only if*, the amount of k th output of DMU_o can be increased without limitation while maintaining the efficiency of DMU_o .

Theorems 11.7 and 11.9 indicate that if model (11.3) or (11.7) is infeasible, then the test DMU remains efficient when data variations are applied to the specific input or output. This conclusion is also true when the data variations are applied to both the test DMU and the remaining DMUs.

11.4 Simultaneous Data Change^b

Zhu (2001) shows that a particular super-efficiency score can be decomposed into two data perturbation components of a particular test DMU and the remaining DMUs. Also, necessary and sufficient conditions for preserving a DMU’s efficiency classification are developed when various data changes are applied to all DMUs. As a result, DEA sensitivity analysis can be easily applied if we employ various super-efficiency DEA models.

We rewrite the input-oriented CRS envelopment model and its dual as

$$\begin{aligned}
 &\theta_{CRS}^{o*} = \min \theta_{CRS}^o \\
 &\text{subject to} \\
 &\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta_{CRS}^o x_{io} \quad i = 1, 2, \dots, m; \\
 &\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, 2, \dots, s; \\
 &\theta_{CRS}^o, \lambda_j, s_i^-, s_r^+ \geq 0.
 \end{aligned}
 \tag{11.10}$$

$$\begin{aligned}
 & \max \sum_{r=1}^s u_r y_{ro} \\
 & \text{subject to} \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n \\
 & \sum_{i=1}^m v_i x_{io} = 1 \\
 & u_r, v_i \geq 0.
 \end{aligned} \tag{11.11}$$

We also present the input-oriented and output-oriented CRS super-efficiency models

$$\begin{aligned}
 & \theta_o^{\text{super}*} = \min \theta_o^{\text{super}} \\
 & \text{subject to} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta_o^{\text{super}} x_{io} \quad i = 1, 2, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s \\
 & \theta_o^{\text{super}}, \lambda_j (j \neq o) \geq 0.
 \end{aligned} \tag{11.12}$$

$$\begin{aligned}
 & \phi_o^{\text{super}*} = \max \phi_o^{\text{super}} \\
 & \text{subject to} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq \phi_o^{\text{super}} y_{ro} \quad r = 1, \dots, s \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \\
 & \phi_o^{\text{super}}, \lambda_j (j \neq o) \geq 0.
 \end{aligned} \tag{11.13}$$

In order to simultaneously consider the data changes for other DMUs, we suppose increased output and decreased input for all other DMUs. i.e., our discussion is based on a worst-case scenario in which efficiency of DMU_o declines and the efficiencies of all other DMU_j ($j \neq o$) improve.

Let **I** and **O** denote respectively the input and output subsets in which we are interested. i.e., we consider the data changes in set **I** and set **O**. Then the simultaneous data perturbations in input/output of all DMU_j ($j \neq o$) and DMU_o can be written as

Percentage data perturbation (variation)

For DMU_o

$$\begin{cases} \hat{x}_{io} = \delta_i x_{io} & \delta_i \geq 1, i \in \mathbf{I} \\ \hat{x}_{io} = x_{io} & i \notin \mathbf{I} \end{cases} \text{ and } \begin{cases} \hat{y}_{ro} = \tau_r y_{ro} & 0 < \tau_r \leq 1, r_r \in \mathbf{O} \\ \hat{y}_{ro} = y_{ro} & r \notin \mathbf{O} \end{cases}$$

For DMU_j ($j \neq o$)

$$\begin{cases} \hat{x}_{ij} = x_{ij} / \tilde{\delta}_i & \tilde{\delta}_i \geq 1, i \in \mathbf{I} \\ \hat{x}_{ij} = x_{ij} & i \notin \mathbf{I} \end{cases} \text{ and } \begin{cases} \hat{y}_{rj} = y_{rj} / \tilde{\tau}_r & 0 < \tilde{\tau}_r \leq 1, r \in \mathbf{O} \\ \hat{y}_{rj} = y_{rj} & r \notin \mathbf{O} \end{cases}$$

where $(\hat{})$ represents adjusted data. Note that the data perturbations represented by δ_i and $\tilde{\delta}_i$ (or τ_r and $\tilde{\tau}_r$) can be different for each $i \in \mathbf{I}$ (or $r \in \mathbf{O}$).

Lemma 11.2 Suppose $DMU_o \in$ set F with non-zero input/output slack values associated with set \mathbf{I} /set \mathbf{O} . Then DMU_o with inputs of \hat{x}_{io} and outputs of \hat{y}_{ro} as defined above still belongs to set F when other DMUs are fixed.

[Proof]: Applying the complementary slackness theorem for models (11.10) and (11.11), we have $s_i^* v_i^* = s_r^* u_r^* = 0$. Since $s_i^- \neq 0$ for $i \in \mathbf{I}$ and $s_r^+ \neq 0$ for $r \in \mathbf{O}$, we have $v_i^* = 0$ for $i \in \mathbf{I}$ and $u_r^* = 0$ for $r \in \mathbf{O}$. Therefore, v_i^* and u_r^* is a feasible solution to (11.11) for DMU_o with inputs of \hat{x}_{io} and outputs of \hat{y}_{ro} . Note that $\sum_{r=1}^s u_r^* \hat{y}_{ro} = \sum_{r \in \mathbf{O}} u_r^* \hat{y}_{ro} = \sum_{r \in \mathbf{O}} u_r^* y_{ro} = 1$ indicating that the maximum value of 1 is achieved. Therefore, DMU_o still belongs to set F. ■

11.4.1 Sensitivity Analysis Under CRS

We first modify models (11.12) and (11.13) to the following two super-efficiency DEA models that are based upon the measure-specific models

$$\begin{aligned} &\theta_1^{o*} = \min \theta_1^o \\ &\text{subject to} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta_1^o x_{io} & i \in \mathbf{I} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} & i \notin \mathbf{I} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} & r = 1, 2, \dots, s \\ &\theta_1^o, \lambda_j (j \neq o) \geq 0. \end{aligned} \tag{11.14}$$

and

$$\begin{aligned}
 &\phi_0^{o*} = \max \phi_0^o \\
 &\text{subject to} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq \phi_0^o y_{ro} \quad r \in \mathbf{O} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin \mathbf{O} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \\
 &\phi_0^o, \lambda_j (j \neq o) \geq 0.
 \end{aligned} \tag{11.15}$$

If $\mathbf{I} = \{k\}$, $k \in \{1, \dots, m\}$ and $\mathbf{O} = \{l\}$, $l \in \{1, \dots, s\}$, then optimal values of $\theta_1^{o*} = (\theta_k^{o*})$ ($k = 1, \dots, m$) and $\phi_0^{o*} = (\phi_l^{o*})$ ($l = 1, \dots, s$) are the optimal values to models (11.3) and (11.7), respectively.

Table 11.2. Sample DMUs

| DMU | y | x_1 | x_2 | $\theta_o^{\text{super}^*} = \theta_{l=(1,2)}^{o*}$ | $\theta_{l=(1)}^{o*}$ | $\theta_{l=(2)}^{o*}$ |
|-----|-----|-------|-------|---|-----------------------|-----------------------|
| A | 1 | 2 | 5 | 15/13 | 5/4 | 7/5 |
| B | 1 | 3 | 3 | 26/21 | 14/9 | 17/12 |
| C | 1 | 6 | 2 | 3/2 | infeasible | 3/2 |
| D | 1 | 2 | 7 | 1 | 1 | 5/7 |

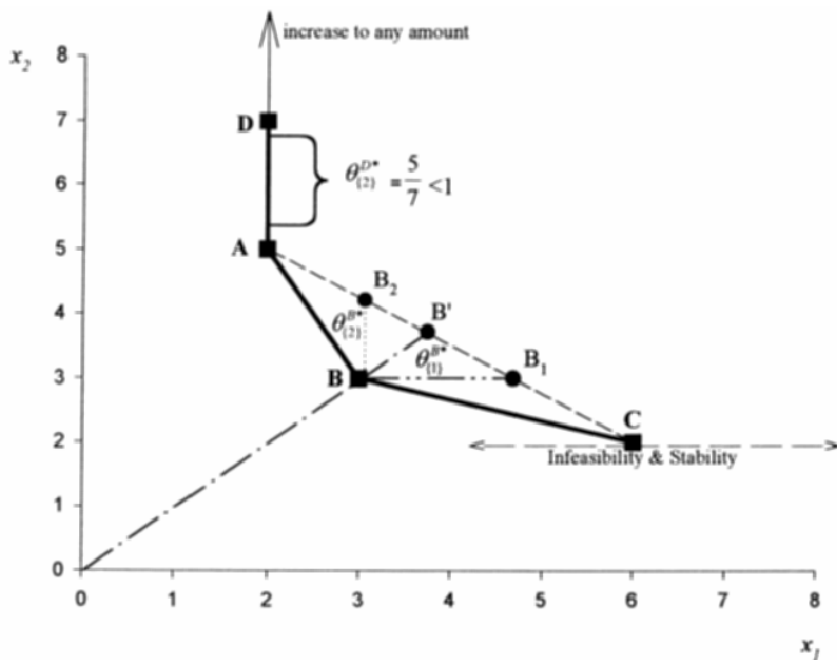


Figure 11.8. Super-efficiency and Sensitivity Analysis

Models (11.14) and (11.15) measure the maximum increase rate of inputs associated with \mathbf{I} and the maximum decrease rate of outputs associated with \mathbf{O} , respectively, required for DMU_o to reach the frontier of DMU_j ($j \neq o$) when other inputs and outputs are kept at their current levels. For example, consider B in Table 11.2 (Figure 11.8) and model (11.14). If $\mathbf{I} = \{1\}$, $\theta_{\{1\}}^{B*} = 14/9$ indicates that B reaches B_1 by changing its x_1 to $(14/9) \times 3 = 14/3$. If $\mathbf{I} = \{2\}$, $\theta_{\{2\}}^{B*} = 17/12$ indicates that B reaches B_2 by changing its x_2 to $(17/12) \times 3 = 17/4$. If $\mathbf{I} = \{1, 2\}$, $\theta_{\{1,2\}}^{B*} = \theta_B^{\text{super}*} = 26/21$ gives the input increase rate for B in order to reach B' .

Associated with the optimal values in models (11.12), (11.13), (11.14) and (11.15), we have

Lemma 11.3

- (i) If $\theta_o^{\text{super}} = 1$, then $\theta_1^{o*} \leq 1$.
- (ii) If $\phi_o^{\text{super}*} = 1$, then $\phi_o^{o*} \geq 1$.

[Proof]: The proof is obvious from the fact that $\theta_o^{\text{super}} = 1$ is a feasible solution to (11.4) and $\phi_o^{\text{super}*} = 1$ is a feasible solution to (11.5). ■

Lemma 11.4

- (i) If $\theta_o^{\text{super}} = 1$ and $\theta_1^{o*} < 1$, then $DMU_o \in F$.
- (ii) If $\phi_o^{\text{super}*} = 1$ and $\phi_o^{o*} > 1$, then $DMU_o \in F$.

[Proof]: (i) $\theta_o^{\text{super}} = 1$ indicates that $DMU_o \in E' \cup F$. $\theta_1^{o*} < 1$ further indicates that there are non-zero slack values in x_{i_o} for $i \in \mathbf{I}$. Thus, $DMU_o \in F$.

(ii) The proof is similar to that of (i). ■

Theorem 11.10

- (i) If $\theta_o^{\text{super}} = 1$ and $\theta_1^{o*} < 1$, then for any $\delta_i \geq 1$ and $\tilde{\delta}_i \geq 1$ ($i \in \mathbf{I}$), DMU_o remains in set F.
- (ii) If $\phi_o^{\text{super}*} = 1$ and $\phi_o^{o*} > 1$, then for any $0 < \tau_r \leq 1$ and $0 < \tilde{\tau}_r \leq 1$ ($r \in \mathbf{O}$), DMU_o remains in set F.

[Proof]: (i) From Lemma 11.4, we know that $DMU_o \in F$ with non-zero slack values in x_{i_o} for $i \in \mathbf{I}$. Based upon Lemma 11.2 and the proof of Lemma 11.2, we know that for any $\delta_i \geq 1$ and $\tilde{\delta}_i \geq 1$, with an objective function value of 1, v_i^* and u_r^* is a feasible solution to (11.11) in which inputs are replaced by \hat{x}_{ij} for $i \in \mathbf{I}$ and x_{ij} for $i \notin \mathbf{I}$. Thus, DMU_o remains in set F after input data changes set \mathbf{I} in all DMUs.

(ii) The proof is similar to that of (i). ■

In fact, Lemma 11.2 and Theorem 11.10 indicate that the classification of DMUs in set F is stable under any data perturbations in all DMUs occurred in inputs (outputs) which have non-zero slack values in DMU_o . For example, if $\mathbf{I} = \{2\}$, then model (11.14) yields $\theta_{\{2\}}^{D*} = 5/7 < 1$ for D indicating that D has non-zero slack value in its second input. From Figure 11.8, it is clear that D can increase its x_2 to any amount and still belongs to set F while other DMUs, A, B and C decrease their amount of x_2 . This finding is very useful for the sensitivity analysis of the DMUs in set F.

Theorem 11.10 gives the sufficient condition for $DMU_o \in$ set F to preserve its efficiency classification. By Lemma 11.3, we immediately have

Corollary 11.2

- (i) If for any $\delta_i \geq 1$ and $\tilde{\delta}_i \geq 1$ ($i \in \mathbf{I}$), DMU_o remains in set F, then (a) $\theta_o^{super} = 1$ and $\theta_1^{o*} < 1$, or (b) $\theta_o^{super} = 1$ and $\theta_1^{o*} = 1$.
- (ii) If for any $0 < \tau_r \leq 1$ and $0 < \tilde{\tau}_r \leq 1$ ($r \in \mathbf{O}$), DMU_o remains in set F, then (a) $\phi_o^{super*} = 1$ and $\phi_o^{o*} > 1$, or (b) $\phi_o^{super*} = 1$ and $\phi_o^{o*} = 1$.

Corollary 11.2 implies that for $DMU_o \in$ set F, some inputs without slack values may also be increased while preserving the efficiency of DMU_o . For example, consider two DMUs: $DMU_1 = (y, x_1, x_2, x_3)$ and $DMU_2 = (y, x_1, x_2, \pi x_3)$, where $\pi > 1$, a constant. Obviously, $DMU_1 \in$ set E and $DMU_2 \in$ set F with non-zero slack value on the third input. Now, let $\mathbf{I} = \{2, 3\}$. We have that DMU_2 with $(y, x_1, \delta x_2, \delta \pi x_3)$ ($\delta > 1$) remains in set F while DMU_1 is changed to $(y, x_1, x_2 / \tilde{\delta}, x_3 / \tilde{\delta}) \in$ set E ($\tilde{\delta} > 1$). In this situation, $\theta_{\{2,3\}}^{2*} = 1$ in (11.14).

From Lemma 11.3, we know that θ_1^{o*} or ϕ_o^{o*} may also be equal to one. Obviously, in this situation, $DMU_o \in$ set E' or set F and our approach indicates that no data variations are allowed in DMU_o and other DMUs. In fact, any data perturbation defined above will change the efficiency classification of DMUs in set E'. Note also that $\theta_{\{1\}}^{D*} = 1$ for D in Table 11.2. Thus, any data variation in the first input will let D become non-frontier point (see Figure 11.8).

Furthermore, from Lemma 11.3, we have

Corollary 11.3 Infeasibility of model (11.14) or model (11.15) can only be associated with extreme efficient DMUs in set E.

[Proof]: Lemma 11.3 implies that models (11.14) and (11.15) are always feasible for DMUs in set E' or set F. Also, models (11.14) and (11.15) are always feasible for non-frontier DMUs. Therefore, infeasibility of models (11.14) and (11.15) may only occur for extreme efficient DMUs in set E. ■

Seiford and Zhu (1998b) show that infeasibility of a super-efficiency DEA model means stability of the efficiency classification of DMU_o with respect to the changes of corresponding inputs and (or) outputs in all DMUs. We summarize Seiford and Zhu's (1998b) finding as the following theorem.

Theorem 11.11

- (i) If a specific super-efficiency DEA model associated with set **I** is infeasible, *if and only if* for any $\delta_i \geq 1$ and $\tilde{\delta}_i \geq 1$ ($i \in \mathbf{I}$), DMU_o remains extreme efficient.
- (ii) If a specific super-efficiency DEA model associated with set **O** is infeasible, *if and only if* for any $0 < \tau_r \leq 1$ and $0 < \tilde{\tau}_r \leq 1$ ($r \in \mathbf{O}$), DMU_o remains extreme efficient.

Theorem 11.11 indicates that, for example, if mode (11.14) is infeasible, then DMU_o will still be extreme efficient no matter how much its inputs associated with set **I** are increased while the corresponding inputs of other DMUs are decreased. Consider C in Table 11.2. If $\mathbf{I} = \{1\}$, then model (11.14) is infeasible. (Note that model (11.12) is feasible for C.) From Figure 11.8, it is clear that C will remain extreme efficient if its first input is increased to any amount while DMUs A, C, and D decrease their amount of x_1 .

In the discussion to follow, we assume that super-efficiency DEA models (11.14) and (11.15) are feasible. Otherwise, the efficiency classification of DMU_o is stable to data perturbations in all DMUs by Theorem 11.11.

Lemma 11.5

- (i) If model (11.14) is feasible and $\theta_o^{super} > 1$ then $\theta_1^{o*} > 1$.
- (ii) If model (11.15) is feasible and $\phi_o^{super*} < 1$ then $\phi_o^{o*} < 1$.

[Proof]: (i) Suppose $\theta_1^{o*} \leq 1$. Then the input constraints of (11.14) turn into

$$\begin{cases} \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta_1^{o*} x_{io} \leq x_{io} & i \in \mathbf{I} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} & i \notin \mathbf{I} \end{cases}$$

which indicates that $\theta_o^{super} = 1$ is a feasible solution to (11.12). Therefore, $\theta_o^{super*} \leq 1$. A contradiction. Thus, $\theta_1^{o*} > 1$.

(ii) The proof is similar to that of (i). ■

Lemma 11.5 indicates that if $DMU_o \in$ set E and model (11.14) (or model (11.15)) is feasible, then θ_1^{o*} must be greater than one (or ϕ_o^{o*} must be

less than one). We next study the efficiency stability of extreme efficient DMUs and we relax the assumption that same percentage change holds for data variation of DMU_o and DMU_j ($j \neq o$) and generalize the results in Seiford and Zhu (1998b).

Theorem 11.12 Suppose $\theta_o^{super*} > 1$ and $\phi_o^{super*} < 1$, then

- (i) If $1 \leq \delta_i \tilde{\delta}_i < \theta_1^{o*}$ for $i \in \mathbf{I}$, then DMU_o remains extreme efficient. Furthermore, if equality holds for $\delta_i \tilde{\delta}_i = \theta_1^{o*}$, i.e., $1 \leq \delta_i \tilde{\delta}_i \leq \theta_1^{o*}$, then DMU_o remains on the frontier, where θ_1^{o*} is the optimal value to (11.14).
- (ii) If $\phi_o^{o*} < \tau_r \tilde{\tau}_r \leq 1$ for $r \in \mathbf{O}$, then DMU_o remains extreme efficient. Furthermore, if equality holds for $\tau_r \tilde{\tau}_r = \phi_o^{o*}$, i.e., $\phi_o^{o*} \leq \tau_r \tilde{\tau}_r \leq 1$, then DMU_o remains on the frontier, where ϕ_o^{o*} is the optimal value to (11.15).

[Proof]: (i) Note that from Lemma 11.5, $\theta_1^{o*} > 1$. Now suppose $1 \leq \delta_i^o \tilde{\delta}_i^o < \theta_1^{o*}$, and DMU_o is not extreme efficient when $\hat{x}_{io} = \delta_i^o x_{io}$ and $\hat{x}_{ij} = x_{ij} / \tilde{\delta}_i^o$, $i \in \mathbf{I}$. Then, there exist λ_j ($j \neq o$) ≥ 0 and $\theta_o^{super*} \leq 1$ in (11.12) such that

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \frac{x_{ij}}{\tilde{\delta}_i^o} &\leq \theta_o^{super*} \delta_i^o x_{io} & i \in \mathbf{I} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} &\leq \theta_o^{super*} x_{io} \leq x_{io} & i \notin \mathbf{I} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} &\geq y_{ro} & r = 1, 2, \dots, s. \end{aligned}$$

This means that λ_j ($j \neq o$) ≥ 0 and $\theta_o^{super*} \delta_i^o \tilde{\delta}_i^o$ is a feasible solution to (11.14). But $\theta_o^{super*} \delta_i^o \tilde{\delta}_i^o < \theta_o^{super*} \theta_1^{o*} \leq \theta_1^{o*}$ violating the optimality of θ_1^{o*} . Thus, if $1 \leq \delta_i^o \tilde{\delta}_i^o < \theta_1^{o*}$, then DMU_o remains extreme efficient.

Next, if $\delta_i^o \tilde{\delta}_i^o = \theta_1^{o*}$, then we assume DMU_o is not a frontier when $\hat{x}_{io} = \delta_i^o x_{io}$ and $\hat{x}_{ij} = x_{ij} / \tilde{\delta}_i^o$, $i \in \mathbf{I}$. Thus, we have $\theta_o^{super*} < 1$ in (11.12). Now we have $\theta_o^{super*} \delta_i^o \tilde{\delta}_i^o \leq \theta_o^{super*} \theta_1^{o*} < \theta_1^{o*}$ violating the optimality of θ_1^{o*} . Thus, if $1 \leq \delta_i^o \tilde{\delta}_i^o \leq \theta_1^{o*}$, then DMU_o remains on the frontier.

(ii) The proof is similar to (i), but is based upon (11.13) and (11.15). ■

Theorem 11.12 indicates that the optimal value to a super-efficiency DEA model can actually be decomposed into a data perturbation component (e) for DMU_o and a data perturbation component (\tilde{e}) for the remaining DMUs, DMU_j ($j \neq o$). Define

$$\Omega^o = \begin{cases} \theta_1^{o*} & \text{if } e = \delta \text{ and } \tilde{e} = \tilde{\delta} \\ \phi_o^{o*} & \text{if } e = \tau \text{ and } \tilde{e} = \tilde{\tau} \end{cases}$$

Then, the data perturbation can be expressed in a quadratic function,

$$e \tilde{e} = \Omega^o \tag{11.16}$$

Function (11.16) gives an upper boundary for input changes and a lower boundary for output changes. Figures 11.9 and 11.10 illustrate the admissible regions for e and \tilde{e} . For example, in Figure 11.9, since $\delta \geq 1$ and $\tilde{\delta} \geq 1$, only part of the function $\delta \tilde{\delta} = \theta_1^{o*}$ forms the upper boundary of an admissible region for δ and $\tilde{\delta}$. Any data variations fall below MN and above lines $\delta = 1$ and $\tilde{\delta} = 1$ will preserve the frontier status of DMU_o . The bigger the θ_1^{o*} (or the smaller the ϕ_o^{o*}), the larger the input (output) variation regions will be. In fact, the function given by (11.16) defines the maximum percentage change rates for DMU_o and DMU_j ($j \neq o$).

Theorem 11.12 gives sufficient conditions for preserving efficiency. The following theorem implies necessary conditions for preserving efficiency of an extreme efficient DMU_o .

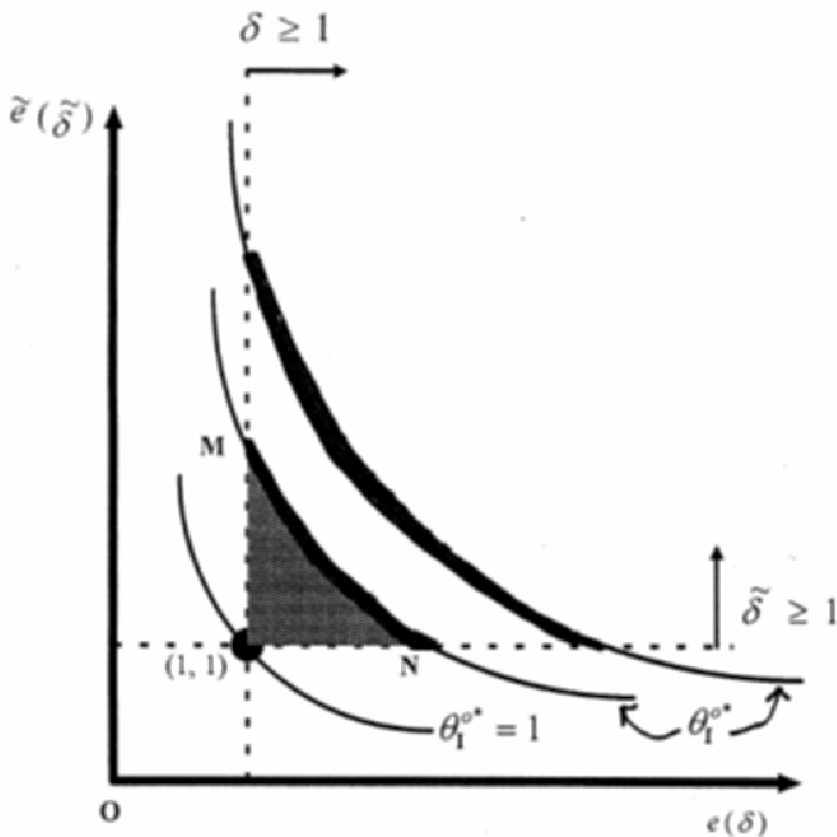


Figure 11.9. Input Variations

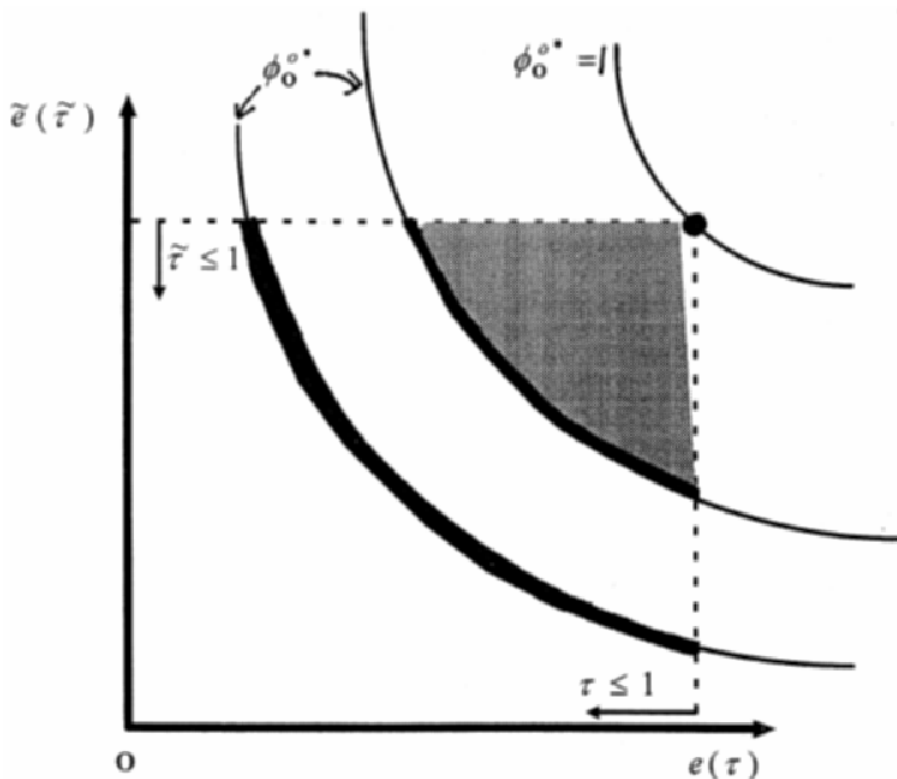


Figure 11.10. Output Variations

Theorem 11.13 Suppose $\theta_o^{super*} > 1$ and $\phi_o^{super*} < 1$, then

- (i) If $\delta_i \tilde{\delta}_i > \theta_1^{**}$ for $i \in \mathbf{I}$, then DMU_o will not be extreme efficient, where θ_1^{**} is the optimal value to (11.14).
- (ii) If $\tau_r \tilde{\tau}_r < \phi_o^{**}$ for $r \in \mathbf{O}$, then DMU_o will not be extreme efficient, where ϕ_o^{**} is the optimal value to (11.15).

[Proof]: (i) We assume that DMU_o remains extreme efficient after the data changes in all DMUs with $\delta_i \tilde{\delta}_i > \theta_1^{**}$. Consider the input constraints associated with set \mathbf{I} in (11.14),

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \frac{x_{ij}}{\tilde{\delta}_i} \leq \hat{\theta}_1^o \delta_i x_{io}, \quad i \in \mathbf{I} \tag{11.17}$$

where $\hat{\theta}_1^o$ is the objective function in (11.14).

Equation (11.17) is equivalent to

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \hat{\theta}_1^o \delta_i \tilde{\delta}_i x_{io}, \quad i \in \mathbf{I}$$

Let $\hat{\theta}_1^{o*}$ be the optimal value. Obviously, $\hat{\theta}_1^{o*} = \theta_1^{o*} / \delta_i \tilde{\delta}_i < 1$ where θ_1^{o*} is the optimal value to (11.14). On the basis of Lemma 11.5 (i), $\hat{\theta}_1^{o*}$ must be greater than one in (11.14) with input constraints of (11.17). A contradiction.
 (ii) The proof is similar to (i), but is based upon (11.17). ■

Theorem 11.13 indicates that input (output) data perturbations in all DMUs beyond the variation regions prescribed by function (11.16) will change the efficiency classification of extreme efficient DMUs.

Note that $\delta_i \tilde{\delta}_i = \theta_1^{o*}$ (or $\tau_r \tilde{\tau}_r = \phi_0^{o*}$) may or may not keep the efficiency classification of an extreme efficient DMU_o . For example, in Figure 11.8, A remains extreme efficient if $\delta_2 \tilde{\delta}_2 = \theta_{\{2\}}^{A*} = 7/5$. (In this situation, A coincides D and both become extreme efficient.) However, if we consider C and if C's second input is increased to $\theta_{\{2\}}^{C*} x_{2C} = (3/2) \times 2 = 3$, then C becomes a member of set F along the ray BB_1 in Figure 11.8. (In this situation, we assume $\delta_2 = \theta_{\{2\}}^{C*} = 3/2$ for C and $\tilde{\delta}_2 = 1$ for the remaining DMUs of A, B and D.)

Turning to point A again. If we are only interested in whether a DMU remains on the frontier, rather than in its original efficiency classification, then we may still increase A's second input after A coincides D. We can find this "extra" data perturbation by applying a very small data perturbation to the changed DMU_o and then applying model (11.14) or (11.15). For example, we apply a data perturbation of ϵ to \hat{x}_{2A} which is the new input value when $\delta_2 \tilde{\delta}_2 = \theta_{\{2\}}^{A*} = 7/5$. If we use models (11.12) and (11.14), then we know that this changed DMU A with it second input equal to $\hat{x}_{2A} + \epsilon$ is now in set F, and therefore A can still increase it x_2 to any amount larger than 7 and remains on the frontier. Note that, in this case, A may no longer be extreme efficient. In fact, $\delta_2 \tilde{\delta}_2 = \theta_{\{2\}}^{A*} = 7/5$ prescribes a point on line segment AB including A and B. If $\delta_2 \tilde{\delta}_2 > 7/5$, then A and D switch their positions. Namely, A becomes a weakly efficient DMU and D becomes an extreme efficient DMU.

Above developments consider the input changes or output changes in all DMUs. Next we consider the following modified DEA model for simultaneous variations of inputs and outputs

$$\begin{aligned}
 &\Gamma^* = \min \Gamma \\
 &\text{subject to} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq (1 + \Gamma)x_{io} \quad i \in \mathbf{I} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i \notin \mathbf{I} \\
 &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq (1 - \Gamma)y_{ro} \quad r \in \mathbf{O}
 \end{aligned} \tag{11.18}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin \mathbf{O}$$

$$\lambda_j (j \neq o) \geq 0, \Gamma \text{ unrestricted}$$

If $\mathbf{I} = \{1, 2, \dots, m\}$ and $\mathbf{O} = \{1, 2, \dots, s\}$, then (11.18) is identical to the model of Charnes, Rousseau and Semple (1996) when variations in the data are only applied to DMU_o . Note that if DMU_o is a frontier point, then $\Gamma \geq 0$.

Theorem 11.14 Suppose DMU_o is a frontier point. If $1 \leq \delta_i \tilde{\delta}_i \leq 1 + \Gamma^*$ and $1 - \Gamma^* \leq \tau_r \tilde{\tau}_r \leq 1$, then DMU_o remains as a frontier point, where Γ^* is the optimal value to (11.18).

[Proof]: Equivalently we prove that if $\delta_i \tilde{\delta}_i = 1 + \Gamma^*$ and $\tau_r \tilde{\tau}_r = 1 - \Gamma^*$, then DMU_o still remains on the frontier. We assume that after the data changes, DMU_o is a nonfrontier point, and therefore can be enveloped by the adjusted $DMU_o (j \neq o)$. Thus,

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \frac{x_{ij}}{\tilde{\delta}_i} \leq \delta_i x_{io} \quad i \in \mathbf{I}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i \notin \mathbf{I}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \frac{y_{rj}}{\tilde{\tau}_r} \geq \tau_r y_{ro} \quad r \in \mathbf{O}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin \mathbf{O}$$

That is

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq (1 + \Gamma^*) x_{io} \quad i \in \mathbf{I}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i \notin \mathbf{I}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq (1 - \Gamma^*) y_{ro} \quad r \in \mathbf{O}$$

$$\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin \mathbf{O}$$

This means that the adjusted DMU_o with $(1 + \Gamma^*)x_{io} (i \in \mathbf{I})$, $x_{io} (i \notin \mathbf{I})$, $(1 - \Gamma^*)y_{ro} (r \in \mathbf{O})$ and $y_{ro} (r \notin \mathbf{O})$ can be enveloped by the original $DMU_j (j \neq o)$. However, by Charnes, Rousseau and Semple (1996), we know that proportional changes to inputs and outputs respectively within the

computed values of $(1 + \Gamma^*)$ and $(1 - \Gamma^*)$ cannot change the efficiency of DMU_o when the remaining DMU_j ($j \neq o$) are fixed. Therefore, this leads to a contradiction and completes the proof. ■

The result in Theorem 11.14 generalizes the finding of Charnes, Rousseau and Semple (1996) to the situation where variations in the data are applied to all DMUs. Similar to Theorem 11.13, for an extreme efficient DMU_o , if $\delta_i \tilde{\delta}_i > 1 + \Gamma^*$ and $\tau_r \tilde{\tau}_r < 1 - \Gamma^*$, then DMU_o will not remain extreme efficient.

11.4.2 Sensitivity Analysis under VRS

It is obvious that the results in the previous section hold for the VRS frontier DMUs if we add the additional constraint of $\sum_{j \neq o} \lambda_j = 1$ into models (11.12), (11.13), (11.14) and (11.15), respectively.

Because of the translation invariance property resulted from the convex constraint of $\sum_{j \neq o} \lambda_j = 1$ in the VRS models, we are able to discuss the simultaneous absolute data changes in all DMUs. That is,

Absolute Data Perturbations (Variations)

For DMU_o

$$\begin{cases} \hat{x}_{io} = x_{io} + \alpha_i & \alpha_i \geq 0, i \in \mathbf{I} \\ \hat{x}_{io} = x_{io} & i \notin \mathbf{I} \end{cases} \text{ and } \begin{cases} \hat{y}_{ro} = y_{ro} - \beta_r & \beta_r \geq 0, r \in \mathbf{O} \\ \hat{y}_{ro} = y_{ro} & r \notin \mathbf{O} \end{cases}$$

For DMU_j ($j \neq o$)

$$\begin{cases} \hat{x}_{ij} = x_{ij} - \tilde{\alpha}_i & \tilde{\alpha}_i \geq 0, i \in \mathbf{I} \\ \hat{x}_{ij} = x_{ij} & i \notin \mathbf{I} \end{cases} \text{ and } \begin{cases} \hat{y}_{rj} = y_{rj} + \tilde{\beta}_r & \tilde{\beta}_r \geq 0, r \in \mathbf{O} \\ \hat{y}_{rj} = y_{rj} & r \notin \mathbf{O} \end{cases}$$

where (^) represents adjusted data. Note that the data changes defined above are not only applied to all DMUs, but also different in various inputs and outputs. In this case the sensitivity analysis results are also suitable to the slack-based models.

We modify model (11.18) to the following linear programming problem

$$\begin{aligned} &\gamma^* = \min \gamma \\ &\text{subject to} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} + \gamma & i \in \mathbf{I} \\ &\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} & i \notin \mathbf{I} \end{aligned} \tag{11.19}$$

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} &\geq y_{ro} - \gamma & r \in \mathbf{O} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} &\geq y_{ro} & r \notin \mathbf{O} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j &= 1 \\ \gamma, \lambda_j (j \neq o) &\geq 0. \end{aligned}$$

If $\mathbf{I} = \{1, 2, \dots, m\}$ and $\mathbf{O} = \{1, 2, \dots, s\}$, then model (11.19) is used by Charnes, Haag, Jaska and Semple (1992) to study the sensitivity of efficiency classifications in the additive model via L_∞ norm when variations in the data are only applied to DMU_o .

Theorem 11.5 Suppose DMU_o is a frontier point. If $0 \leq \alpha_i + \tilde{\alpha}_i \leq \gamma^*$ ($i \in \mathbf{I}$), $0 \leq \beta_r + \tilde{\beta}_r \leq \gamma^*$ ($r \in \mathbf{O}$), then DMU_o remains as a frontier point, where γ^* is the optimal value to (11.19).

[Proof]: The proof is similar to that of Theorem 11.13 by noting that $\sum_{j \neq o} \lambda_j = 1$. ■

If $\mathbf{O} = \emptyset$, then (11.19) only considers absolute changes in inputs. If $\mathbf{I} = \emptyset$, then (10) only considers absolute changes in output. For different choices of subsets \mathbf{I} and \mathbf{O} , we can determine the sensitivity of DMU_o to the absolute changes of different sets of inputs or (and) outputs when DMU_o 's efficiency is deteriorating and DMU_j 's ($j \neq o$) efficiencies are improving.

We may change the objective function of (11.19) to “minimize $\sum_{i \in \mathbf{I}} r_i^- + \sum_{r \in \mathbf{O}} \gamma_r^+$ ” and obtain the following super-efficiency DEA model

$$\begin{aligned} \min \quad & \sum_{i \in \mathbf{I}} \gamma_i^- + \sum_{r \in \mathbf{O}} \gamma_r^+ \\ \text{subject to} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} + \gamma_i^- & i \in \mathbf{I} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} & i \notin \mathbf{I} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} - \gamma_r^+ & r \in \mathbf{O} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} & r \notin \mathbf{O} \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1 \\ & \gamma_i^-, \gamma_r^+, \lambda_j (j \neq o) \geq 0 \end{aligned} \tag{11.20}$$

We then obtain a generalized model under L_1 norm. The results in Charnes, Haag, Jaska and Semple (1992) are generalized to the situation of data changes in all DMUs by the following Theorem.

Theorem 11.16 Suppose DMU_o is a frontier point. If $0 \leq \alpha_i + \tilde{\alpha}_i \leq \gamma_i^{*+}$ ($i \in \mathbf{I}$), $0 \leq \beta_r + \tilde{\beta}_r \leq \gamma_r^{*+}$ ($r \in \mathbf{O}$), then DMU_o remains as a frontier point, where γ_i^{*+} ($i \in \mathbf{I}$) and γ_r^{*+} ($r \in \mathbf{O}$) are optimal values in (11.20).

[Proof]: The proof is similar to that of Theorem 11.13 and is omitted. ■

Similar to Theorem 11.13, for an extreme efficient DMU_o , if $\alpha_i + \tilde{\alpha}_i > \gamma_i^{*+}$ and $\beta_r + \tilde{\beta}_r > \gamma_r^{*+}$ then DMU_o will not remain extreme efficient.

11.4.3 Spreadsheet Models for Sensitivity Analysis

The current chapter presents a new approach for the sensitivity analysis of DEA models by using various super-efficiency DEA models. The sensitivity analysis approach simultaneously considers the data perturbations in all DMUs, namely, the change of the test DMU and the changes of the remaining DMUs. The data perturbations in the test DMU and the remaining DMUs can be different when all remaining DMUs work at improving their efficiencies against the deteriorating of the efficiency of the test efficient DMU. It is obvious that larger (smaller) optimal values to the input-oriented (output-oriented) super-efficiency DEA models presented in the current study correspond to greater stability of the test DMU in preserving efficiency when the inputs and outputs of all DMUs are changed simultaneously and unequally.

By using super-efficiency DEA models based upon the measure-specific models, the sensitivity analysis of DEA efficiency classification can be easily achieved. Since the approach uses optimal values to various super-efficiency DEA models, the results are stable and unique. By the additional constraint on $\sum_{j \neq o} \lambda_j$, the approach can easily be modified to study the sensitivity of other DEA models. Table 11.3 presents the measure-specific super-efficiency DEA models.

The stability measure is actually the optimal value to a specific measure-specific super-efficiency DEA model. Thus, the sensitivity analysis can be performed based upon the spreadsheets for related measure-specific models discussed in Chapter 3.

Figure 11.11 shows an input-oriented VRS measure-specific super-efficiency model where $\mathbf{I} = \{\text{Assets}, \text{Equity}\}$. i.e., we are interested in the sensitivity of VRS efficiency to the (proportional) data changes in Assets and Equity.

Table 11.3. Measure-specific Super-efficiency DEA Models

| Frontier Type | Input-Oriented | Output-Oriented |
|---------------|--|---|
| | $\theta_1^{o*} = \min \theta_1^o$ subject to $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq \theta_1^o x_{io} \quad i \in I$ $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i \notin I$ $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s$ $\theta_1^o, \lambda_j (j \neq o) \geq 0.$ | $\phi_o^{o*} = \max \phi_o^o$ subject to $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq \phi_o^o y_{ro} \quad r \in O$ $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin O$ $\sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m$ $\phi_o^o, \lambda_j (j \neq o) \geq 0.$ |
| CRS | | |
| VRS | | Add $\sum_{j \neq o} \lambda_j = 1$ |
| NIRS | | Add $\sum_{j \neq o} \lambda_j \leq 1$ |
| NDRS | | Add $\sum_{j \neq o} \lambda_j \geq 1$ |

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|-----------|------------|----|-----------|--------|---|-----------|------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Stability |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | | 0 | Infeasible |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | | 0 | 1.75189 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | | 0 | Infeasible |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | | 0 | Infeasible |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | | 0.71 | Infeasible |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | | 0 | Infeasible |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | | 0 | 0.70463 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 148855 | | 111052 | 2662.4 | | 0 | 0.76929 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | | 0 | Infeasible |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | | 0 | Infeasible |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | | 0.29 | 1.76516 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | | 0 | 0.48484 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | | 0 | Infeasible |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | | 0 | Infeasible |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | | 0 | 0.52449 |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Stability | | | | |
| 19 | Constraints | set | | Evaluation | | 0.524492 | | | | |
| 20 | Assets | 46618.92 | λ | 46618.922 | | | | | | |
| 21 | Equity | 9060.07 | λ | 9060.07 | | | | | | |
| 22 | Employees | 203091.7 | λ | 299300 | | | | | | |
| 23 | Revenue | 145773.2 | λ | 79609 | | | | | | |
| 24 | Profit | 955.1922 | λ | 139 | | | | | | |
| 25 | $\Sigma \lambda$ | 1 | | | | | | | | |
| 26 | λ_o | 0 | | | | | | | | |

Figure 11.11. Input Sensitivity Analysis Spreadsheet Model

In Figure 11.11, cell F19 represents θ_1^o . The formulas for this spreadsheet are

- Cell B20 =SUMPRODUCT(B2:B16,\$I\$2:\$I\$16)
- Cell B21 =SUMPRODUCT(C2:C16,\$I\$2:\$I\$16)

Cell B22 =SUMPRODUCT(D2:D16,\$I\$2:\$I\$16)

Cell B23 =SUMPRODUCT(F2:F16,\$I\$2:\$I\$16)

Cell B24 =SUMPRODUCT(G2:G16,\$I\$2:\$I\$16)

Cell B25 =SUM(I2:I16)

Cell B26 =INDEX(I2:I16,E18,1)

Cell D20 =\$F\$19*INDEX(B2:B16,E18,1)

Cell D21 =\$F\$19*INDEX(C2:C16,E18,1)

Cell D22 =INDEX(D2:D16,E18,1)

Cell D23 =INDEX(F2:F16,E18,1)

Cell D24 =INDEX(G2:G16,E18,1)

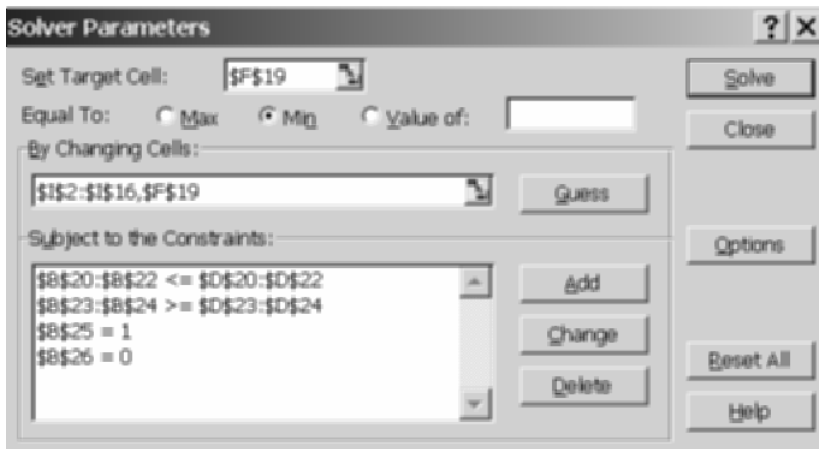


Figure 11.12. Solver Parameters for Input Sensitivity Analysis

Figure 11.12 shows the Solver parameters for the spreadsheet shown in Figure 11.11. If the optimal value in cell F19 is less than one, then this means that the associated company is VRS inefficient. The infeasibility in cells J2, J4:J7, J10:J11, and J14:J15 indicates that the corresponding companies remain VRS efficient to any simultaneous data changes in Assets and Equity across all DMUs. For DMU2 (Mitsui), we have the super-efficiency score of 1.75, indicating this DMU remains VRS efficient as long as the data variations satisfying $e \tilde{e} = 1.75$.

Next, we consider output changes. Figure 11.13 shows the spreadsheet for output-oriented VRS measure-specific super-efficiency model where $\mathbf{O} = \{\text{Revenue}\}$. In this spreadsheet, range names are used. They are, cells B2:D16 – “InputUsed”, cells F2:G16 – “OutputProduced”, cells I2:I16 – “Lambdas”, cells B20:B22 – “ReferenceSetInput”, cells B23:B24 – “ReferenceeSetOutput”, cell B25 – “SumLambdas”, cell B26 – “DMUo”,

cells D20:D22 – “DMUInput”, cells D23:D24 – “DMUOutput”, cell E18 – “DMU”, and cell F19 – “SuperEfficiency”.

| | A | B | C | D | E | F | G | H | I | J |
|----|------------------------------|-----------|---------|------------|----|-----------|--------|------|------------|-----------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | | λ | Stability |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 1 | 0.8 | 0.94407 | |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | 0.19 | 0.93499 | |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | 0 | Infeasible | |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | 0.01 | Infeasible | |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 0 | Infeasible | |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.6 | 0 | Infeasible | |
| 8 | Ford Motor | 243283 | 24547 | 346990 | | 137137 | 4139 | 0 | 1.20986 | |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | 0 | 1.46561 | |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | 0 | Infeasible | |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | 0 | Infeasible | |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | 0 | Infeasible | |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | 0 | 2.11056 | |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | 0 | Infeasible | |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | 0 | 2.16259 | |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | 0 | 2.30681 | |
| 17 | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Stability | | | | |
| 19 | Constraints | set | | Evaluation | | 2.306807 | | | | |
| 20 | Assets | 88884 | <= | 88884 | | | | | | |
| 21 | Equity | 10052.63 | <= | 17274 | | | | | | |
| 22 | Employees | 52057 | <= | 299300 | | | | | | |
| 23 | Revenue | 183642.6 | >= | 183642.58 | | | | | | |
| 24 | Profit | 139 | >= | 139 | | | | | | |
| 25 | $\sum \lambda$ | 1 | | | | | | | | |
| 26 | λ_0 | 0 | | | | | | | | |

Figure 11.13. Output Sensitivity Analysis Spreadsheet Model

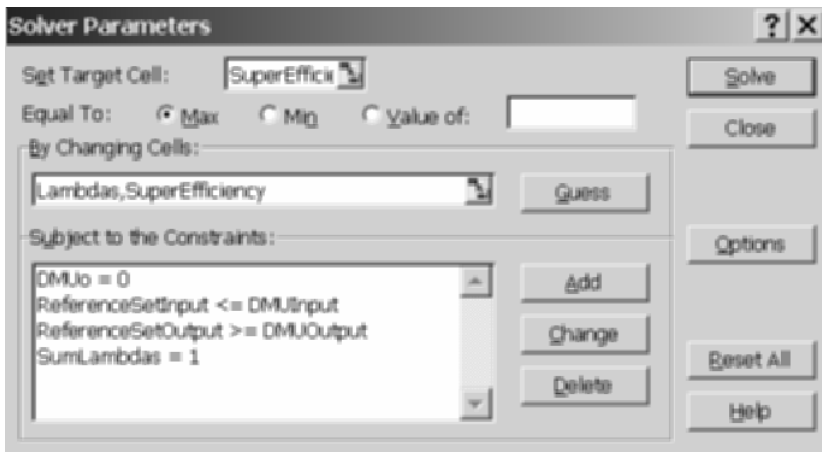


Figure 11.14. Solver Parameters for Output Sensitivity Analysis

Based upon these range names, we have the following formulas for the spreadsheet shown in Figure 11.13.

Cell B20 =SUMPRODUCT(INDEX(InputUsed,0,1),Lambdas)
 Cell B21 =SUMPRODUCT(INDEX(InputUsed,0,2),Lambdas)
 Cell B22 =SUMPRODUCT(INDEX(InputUsed,0,3),Lambdas)
 Cell B23 =SUMPRODUCT(INDEX(OutputProduced,0,1),Lambdas)
 Cell B24 =SUMPRODUCT(INDEX(OutputProduced, 0,2),Lambdas)
 Cell B25 =SUM(Lambdas)
 Cell B26 =INDEX(Lambdas,DMU,1)

Cell D20 =INDEX(InputUsed,DMU,1)
 Cell D21 =INDEX(InputUsed,DMU,2)
 Cell D22 =INDEX(InputUsed,DMU,3)
 Cell D23 =SuperEfficiency*INDEX(OutputProduced,DMU,1)
 Cell D24 =INDEX(OutputProduced,DMU,2)

Figure 11.14 shows the Solver parameters for the spreadsheet shown in Figure 11.13. The calculation is performed by the following VBA procedure that can be applied to other data sets once the proper range names are defined.

```

Sub SensitivityGeneral()
Dim NDMUs As Integer, NInputs As Integer, NOutputs As Integer
NDMUs = 15
NInputs = 3
NOutputs = 2
Dim i As Integer
For i = 1 To NDMUs
Range("DMU") = i
SolverSolve UserFinish:=True
If SolverSolve(UserFinish:=True) = 5 Then
Range("A1").Offset(i, NInputs + NOutputs + 4) = "Infeasible"
Else
Range("A1").Offset(i,NInputs+NOutputs+4)=Range("SuperEfficiency")
End If
Next i
End Sub
  
```

11.5 Solving DEA Using DEA Frontier Software

To perform the sensitivity analysis, select the “Perform Sensitivity Analysis” menu item. You will be prompted a form shown in Figure 11.15. (You will select a model from Table 11.3.) The measures that are selected

will be studied for sensitivity analysis. For example, in Figure 11.5, Assets and Employees are selected (for input-oriented CRS model). The resulting super-efficiency score measures the efficiency stability with respect to changes in both Assets and Employees.

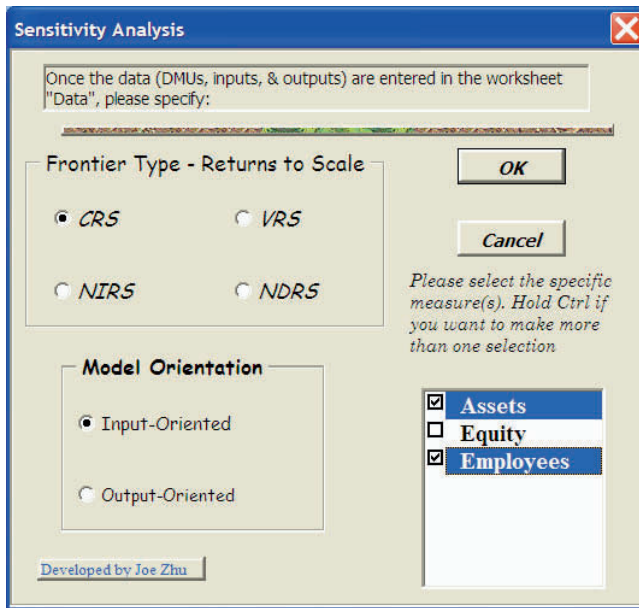


Figure 11.15. Solver Parameters for Output Sensitivity Analysis

The results are reported in the “Sensitivity Report” sheet which records the optimal values to the related measure-specific super-efficiency model.

Based upon the discussion in this chapter, we can convert these super-efficiency scores into measures for efficiency stability.

REFERENCES

1. Ahn, T.S. and L.M. Seiford (1993), Sensitivity of DEA to models and variable sets in a hypothesis test setting: the efficiency of University operations, In: Y. Ijiri, ed. *Creative and Innovative Approaches to the Science of Management*, 191-208. Quorum Books, New York.
2. Charnes, A., W.W. Cooper, A.Y. Lewin, R.C. Morey and J. Rousseau (1985), Sensitivity and stability analysis in DEA, *Annals of Operations Research*, 2, 139-156.
3. Charnes, A., W.W. Cooper, B. Golany, L.M. Seiford and J. Stutz (1985), Foundations of data envelopment analysis for Pareto-Koopman's efficient empirical production functions, *J. of Econometrics*, 30, 1-17.
4. Charnes, A. and L. Neralic (1990), Sensitivity analysis of the additive model in data envelopment analysis, *European Journal of Operational Research*, 48, 332-341.
5. Charnes, A., S. Haag, P. Jaska and J. Semple (1992), Sensitivity of efficiency classifications in the additive model of data envelopment analysis, *Int. J. Systems Sci.* 23, 789-798.
6. Charnes, A., J. Rousseau and J. Semple (1996), Sensitivity and stability of efficiency classifications in data envelopment analysis, *Journal of Productivity Analysis*, 7, 5-18.
7. Cooper, W.W., S. Li, L.M. Seiford, R.M. Thrall and J. Zhu (2001), Sensitivity and stability analysis in DEA: some recent developments, *Journal of Productivity Analysis*, 15, 217-246.
8. Neralic, L., (1994), Sensitivity analysis in data envelopment analysis: a review, in *Proceedings of the Symposium on Operations Research*, 29-42, SOR '94, V. Rupnik, and M. Bogataj (Eds.), Portoroz.
9. Seiford, L.M. and J. Zhu (1998a), Stability regions for maintaining efficiency in data envelopment analysis, *European Journal of Operational Research*, 108, No. 1, 127-139.
10. Seiford, L.M. and J. Zhu (1998b), Sensitivity analysis of DEA models for simultaneous changes in all the data, *Journal of the Operational Research Society*, 49, 1060-1071.
11. Seiford, L.M. and J. Zhu (1999), Infeasibility of super efficiency data envelopment analysis models, *INFOR*, 37, No. 2, 174-187.
12. Thompson, R.G., P.S. Dharmapala and R.M. Thrall (1994), Sensitivity analysis of efficiency measures with applications to Kansas farming and Illinois coal mining, In: A. Charnes, W.W. Cooper, A.Y. Lewin and L.M. Seiford., eds. *Data Envelopment Analysis: Theory, Methodology, and Applications*, Kluwer Academic Publishers, Boston.

13. Zhu, J. (1996), Robustness of the efficient DMUs in data envelopment analysis, *European Journal of Operational Research*, 90, No. 3, 451-460.
14. Zhu, J. (2001), Super-efficiency and DEA sensitivity analysis, *European Journal of Operational Research*, 129, No. 2, 443-455.

^a Part of the material in this section is adapted from *European Journal of Operational Research*, Vol 108, Seiford, L.M. and Zhu, J., Stability regions for Maintaining Efficiency in DEA, 127-139, 1998, with permission from Elsevier Science.

^b Part of the material in this section is adapted from *European Journal of Operational Research*, Vol 129, Zhu, J., Super-efficiency and DEA Sensitivity Analysis, 443-455, 2001, with permission from Elsevier Science.

Chapter 12

Identifying Critical Measures in DEA

12.1 Introduction

Since each DMU has its own inherent tradeoffs among the multiple measures that significantly influence the performance, it is extremely important for the management to know the critical measures. The current chapter introduces the approach of Chen and Zhu (2003) for identifying the critical measures to DMUs' performance. Note that once the DEA evaluation is done, the management needs to either (i) maintain the best practice for the efficient DMUs or (ii) achieve the best practice for the inefficient DMUs. Thus, when a set of multiple performance measures is determined, measures that are influential to maintaining and achieving the best practice should be regarded as critical to the performance of DMUs. Also, it is believed that a critical measure is signaled by whether changes in its value affect the performance, not by whether inclusion or exclusion of the measure affects the performance. Under the framework of DEA sensitivity analysis, Chen and Zhu (2003) develop an alternative approach, which is independent of identifying DEA weights or DEA multipliers, to identify such critical measures.

12.2 Performance Evaluation and DEA

Regression-based methods can be used in evaluating performance of a set of DMUs. However, they are limited to only one dependent variable. For example,

$$y = \beta_o + \sum_{i=1}^m \beta_i x_i + \varepsilon \quad (12.1)$$

where β_i are estimated coefficients which can be used to determine whether an independent variable has a positive effect on the dependent variable or makes an important contribution. i.e., by estimating the coefficients, we may identify the critical performance measures under the context of average behavior. Also, the estimated regression line can be served as the benchmark in performance evaluation.

In fact, formula (12.1) can be viewed as a performance frontier or tradeoff curve where x_i are inputs and y is the output. However, we are very likely to have multiple outputs y_r ($r = 1, \dots, s$). We may rewrite (12.1) as (Wilkens and Zhu, 2001)

$$\sum_{r=1}^s u_r y_r = \alpha + \sum_{i=1}^m v_i x_i \quad (12.2)$$

where u_r and v_i are unknown weights representing the relative importance or tradeoffs among y_r and x_i .

Suppose we can estimate u_r and v_i , then for each DMU_j , we can define

$$h_j = \frac{\alpha + \sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} \quad (12.3)$$

as a performance index, where x_{ij} , ($i = 1, 2, \dots, m$) are multiple inputs, y_{rj} , ($r = 1, 2, \dots, s$) are multiple outputs for DMU_j ($j = 1, 2, \dots, n$).

In order to estimate u_r and v_i , and further evaluate the performance of j_o th DMU, (denoted as DMU_o) by (12.2), DEA uses the following linear fractional programming problem

$$\min_{\alpha, v_i, u_r} \frac{\alpha + \sum_{i=1}^m v_i x_{io}}{\sum_{r=1}^s u_r y_{ro}} \quad (12.4)$$

subject to

$$\frac{\alpha + \sum_{i=1}^m v_i x_{ij}}{\sum_{r=1}^s u_r y_{rj}} \geq 1, j = 1, \dots, n$$

$$u_r, v_i \geq 0 \quad \forall r, i$$

where, x_{io} and y_{ro} are respectively the i th input and r th output for DMU_o under evaluation.

When $h_o^* = 1$, DMU_o is efficient or on the performance frontier. Otherwise, if $h_o^* > 1$, then DMU_o is inefficient. All the efficient DMUs constitute the performance frontier.

Note that when $h_o^* = 1$, we have

$$\sum_{r=1}^s u_r^* y_{ro} = \alpha^* + \sum_{i=1}^m v_i^* x_{io} \tag{12.5}$$

where (*) represents optimal values in model (12.4). That is, DEA estimates the “coefficients” in (12.2). It can be seen that while (12.1) estimates one set of coefficients, DEA model (12.4) estimates one set of coefficients for each DMU, resulting a piecewise linear tradeoff curve represented by several (12.5)-like equations associated with efficient DMUs. Equation (12.5) is theoretically available, but very difficult to obtain empirically.

Obviously, u_r^* and v_i^* represent the tradeoffs among various outputs and inputs. If we can obtain the exact information on u_r^* and v_i^* , the critical performance measures can be easily identified. However, the exact information on u_r^* and v_i^* cannot be obtained because of multiple optimal solutions in the multiplier models.

However, in order to solve model (4), the following transformation is used

$$t = \frac{1}{\sum_{r=1}^s u_r y_{ro}}, \quad \omega_i = t v_i, \quad \omega_o = t \alpha, \quad \mu_r = t u_r \tag{12.6}$$

Based upon (12.6), model (12.4) is solved in the following equivalent linear programming problem (VRS multiplier model, see chapter 1)

$$\begin{aligned} & \min_{\omega_o, \omega_i, \mu_r} \omega_o + \sum_{i=1}^m \omega_i x_{io} \\ & \text{subject to} \\ & \sum_{r=1}^s \mu_r y_{rj} - \sum_{i=1}^m \omega_i x_{ij} - \omega_o \leq 0 \quad \forall j \tag{12.7} \\ & \sum_{r=1}^s \mu_r y_{ro} = 1 \\ & \mu_r, \omega_i \geq 0 \quad \forall r, i \end{aligned}$$

or the dual to model (12.7) (VRS envelopment model, see chapter 1)

$$\begin{aligned}
 \varphi_o^* &= \max \varphi_o \\
 &\text{subject to} \\
 \sum_{j=1}^n \lambda_j x_{ij} &\leq x_{io} && i = 1, 2, \dots, m; \\
 \sum_{j=1}^n \lambda_j y_{rj} &\geq \varphi_o y_{ro} && r = 1, 2, \dots, s; \\
 \sum_{j=1}^n \lambda_j &= 1 \\
 \lambda_j &\geq 0 && j = 1, \dots, n.
 \end{aligned}
 \tag{12.8}$$

Based upon (12.6), we have $\frac{\omega_i}{\omega_k} = \frac{v_i}{v_k}$ and $\frac{\mu_r}{\mu_d} = \frac{u_r}{u_d}$. Thus, μ_r^* and ω_i^* are not the exact weights representing the tradeoffs in model (4). In addition, for efficient DMUs, model (12.7) often yields multiple optimal solutions on multipliers μ_r and ω_i . Also, $\sum_{r=1}^s \mu_r^* y_{rj} - \sum_{i=1}^m \omega_i^* x_{ij} - \omega_o^* = 0$ may only represent supporting hyperplanes rather than the performance frontier in empirical studies. This further leads to an incomplete tradeoff information. Because of possible multiple optimal solutions in (12.7) and the transformation in (12.6), it is very difficult to back out the tradeoffs represented by u_r^* and v_i^* in model (12.4), i.e., the performance frontier expressed by (12.5) is very difficult to obtain in empirical applications. Chen and Zhu (2003) therefore develop an alternative approach to identifying the critical measures.

Suppose that we obtain the performance frontier. In this case, for example $v_k^* > v_i^*$ indicates that the k th input measure is more influential in order for DMU_o to achieve the best-practice. i.e., the k th input is more important to DMU_o 's performance which is characterized by the efficiency score (h_o^*). Note also that the DEA model (12.4) always tries to assign larger v_i and u_r to smaller x_{io} and larger y_{ro} respectively in order to achieve the optimality. This indicates that when a set of multiple performance measures (inputs and outputs) is determined, the relative importance or tradeoffs is determined by the magnitudes of the inputs and outputs.

It can be seen from model (12.4) that for a specific DMU under evaluation, when a specific input increases, the associated input weight will not increase and when a specific output decreases, the associated output weight will not increase. Consider the frontier represented by ABC in Figure 1 with two inputs and a single output. In Figure 12.1, $v_1 > v_2$ remains true for facet AB if DMU A's x_2 (uncritical one) changes its value, and $v_2 > v_1$

remains true for facet BC if DMU C's x_1 (uncritical one) changes its value. Meanwhile, DMUs A and C remain efficient when the uncritical inputs changes their value, respectively¹. However, if we increase the x_1 of DMU A or x_2 of DMU C to a certain level, DMU A or DMU C becomes inefficient.

The example in Figure 12.2 indicates that (a) for efficient DMUs, the performance is determined and characterized by the best-practice status, and (b) for inefficient DMUs, the performance is determined and characterized by the distance to the frontier. Thus, a measure that is critical to the performance should be characterized by whether the measure is critical to (i) maintaining the best-practice for efficient DMUs and (ii) achieving the best-practice for inefficient DMUs.

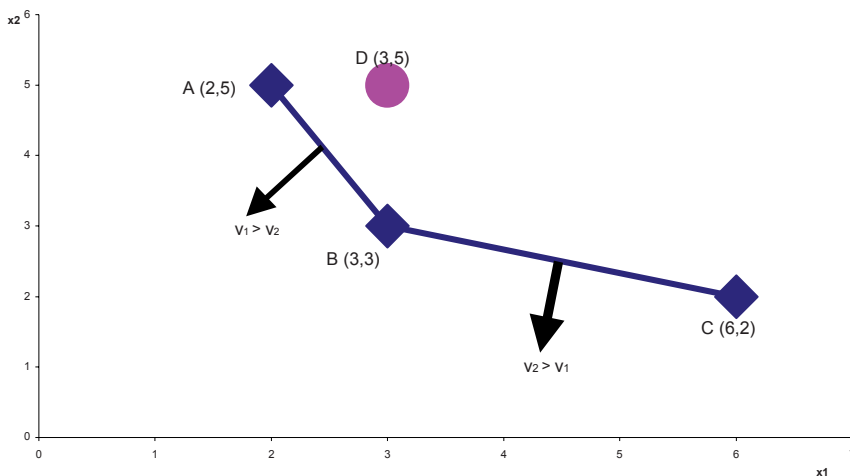


Figure 12.1. Critical Measures and Tradeoffs

The example in Figure 12.1 indicates that (a) for efficient DMUs, the performance is determined and characterized by the best-practice status, and (b) for inefficient DMUs, the performance is determined and characterized by the distance to the frontier. Thus, a measure that is critical to the performance should be characterized by whether the measure is critical to (i) maintaining the best-practice for efficient DMUs and (ii) achieving the best-practice for inefficient DMUs.

¹ Note that for example, if the second input of DMU A decreases its current level to 3, the level used by DMU B, then we no longer have the efficient facet AB. Since DMU B becomes inefficient.

Because a set of multiple performance measures is given prior to the evaluation, a critical measure is signaled by whether changes in its value affect the performance, not by whether inclusion or exclusion of the measure affects the performance.

Definition 12.1 When a set of multiple performance measures is given, a specific measure is said to be critical if changes in its value may alter the efficiency status of a specific DMU.

For efficient DMUs, the performance is determined and characterized by the best practice status. For inefficient DMUs, the performance is determined and characterized by the distance to the frontier. Thus, a measure that is critical to the performance should be characterized by whether the measure is critical to (i) maintaining the best practice for efficient DMUs and (ii) achieving the best practice for inefficient DMUs.

12.3 Identifying Critical Output Measures

Consider the following super-efficiency model where the d th output is given the pre-emptive priority to change

$$\begin{aligned}
 & \max \sigma_d \\
 & \text{subject to} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{dj} \geq \sigma_d y_{do} \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \geq y_{ro} \quad r \neq d \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1
 \end{aligned} \tag{12.6}$$

Four possible cases are associated with (12.6): (i) $\sigma_d^* > 1$, (ii) $\sigma_d^* = 1$, (iii) $\sigma_d^* < 1$ and (iv) model (12.6) is infeasible. When $\sigma_d^* > 1$, DMU_o has inefficiency in its d th output, since potential output increase can be achieved by DMU_o . Cases (ii), (iii) and (iv) indicate that no inefficiency exists in d th output.

Now, we consider the efficient DMUs and assume that DMU_o is efficient. Based upon model (12.6) the set of s outputs can be grouped into two subsets: set $O = \{d: \sigma_d^* \leq 1\}$ and set $\bar{O} = \{d: \text{model (12.6) is infeasible for } d\text{th output}\}$.

We have when model (12.6) is infeasible, the magnitude of the d th output across all DMUs has nothing to do with the efficiency status of DMU_o .

This indicates that the outputs in set \bar{O} are not critical to the efficiency status of DMU_o , since changes in the outputs in set \bar{O} do not change the efficiency classification of DMU_o . The efficiency classification of DMU_o is stable to any changes in the d th output across all DMUs when d belongs to set \bar{O} .

However, decreases in outputs in set O to certain magnitudes result in a change of efficiency status (performance) of DMU_o . For example, when the d th output of DMU_o is decreased from the current level y_{do} to a level which is less than $\sigma_d^* y_{do}$ ($\sigma_d^* < 1$), then DMU_o becomes inefficient. This in turn indicates that the outputs in set O are critical to the performance of DMU_o .

Now, let $P_{d^*} = \max\{\sigma_d^*\}$ for the outputs in set O . From the above discussion, we conclude that the d^* th output is the most critical output measure to the efficiency of DMU_o . Because, DMU_o 's efficiency status is most sensitive to changes in the d^* th output.

Next, we consider inefficient DMUs and assume that DMU_o is inefficient. For inefficient DMUs, the issue is how to improve the inefficiency to achieve the best practice. Since the focus here is how each individual output measure contributes to the performance of DMU_o , we solve model (12.6) for each d and obtain $\sigma_d^* > 1$ ($d = 1, \dots, d$), where σ_d^* measures how far DMU_o is from the frontier in terms of d th output.

As a matter of fact, model (12.6) provides an alternative way to characterize the inefficiency of DMU_o . Each σ_d^* indicates possible inefficiency existing in each associated output when other outputs and inputs are fixed at their current levels. We then can rank the inefficiency by each optimal σ_d^* . Let $G_{d^*} = \min\{\sigma_d^*\}$. That is, the d^* th output indicates the least inefficiency. If the DMU_o is to improve its performance through single output improvement, the d^* th output will yield the most effective way. Because G_{d^*} represents the shortest path onto the best practice frontier when each output is given the pre-emptive priority to improve. We therefore define that the d^* th output is the most critical output to reach the performance frontier and to DMU_o 's performance.

In summary, the critical output is identified as the output associated with $\max\{\sigma_d^*\}$ for efficient DMUs and $\min\{\sigma_d^*\}$ for inefficient DMUs.

12.4 Identifying Critical Input Measures

Consider the following super-efficiency model when the k th input measure is of interest.

min τ_k

subject to

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{kj} &\leq \tau_k x_{ko} \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} &\leq x_{io} \quad i \neq k \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} &\geq y_{ro} \quad r = 1, \dots, s \\ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j &= 1 \end{aligned} \quad (12.7)$$

Based upon model (12.7), we have (i) $\tau_k^* < 1$, (ii) $\tau_k^* = 1$, (iii) $\tau_k^* > 1$, and (iv) (12.7) is infeasible. Case (i) indicates that inefficiency exists in DMU_o 's k th input, since DMU_o needs to decrease its k th input to $\tau_k^* x_{ko}$ in order to reach the performance frontier. Cases (ii), (iii) and (iv) indicate that no inefficiency exists in DMU_o 's k th input.

Now, suppose DMU_o is efficient. Based upon model (12.7), the set of m inputs can be grouped into two subsets: set $I = \{k: \tau_k^* \geq 1\}$ and set $\bar{I} = \{k: \text{model (12.7) is infeasible for } k\text{th input}\}$.

We have when model (12.7) is infeasible, the magnitude of the k th input across all DMUs has nothing to do with the efficiency status of DMU_o .

This indicates that the inputs in set \bar{I} are not critical to the efficiency status of DMU_o , since changes in the inputs in set \bar{I} do not change the efficiency classification of DMU_o . Let $T_{k^*} = \min \{\tau_k^*\}$ for inputs in set I . We conclude that the k^* th input is the most critical input measure to the efficiency of DMU_o . Because, DMU_o 's efficiency status is most sensitive to changes in the k^* th input.

Next, suppose DMU_o is inefficient. We solve model (12.7) for each k and obtain $\tau_k^* < 1$ ($k = 1, \dots, m$), where τ_k^* measures how far DMU_o is from the frontier in terms of k th input. Each τ_k^* indicates possible inefficiency existing in each associated input when other inputs and outputs are fixed at their current levels. We then can rank the inefficiency by each optimal τ_k^* . Let $H_{k^*} = \max \{\tau_k^*\}$. Similar to the discussion on identifying the critical output measure, we say that the k^* th input is the most critical input to reach the performance frontier and to DMU_o 's performance, since the k^* th input indicates the least inefficiency.

In summary, the critical input is identified as the input associated with $\min \{\tau_k^*\}$ for efficient DMUs and $\max \{\tau_k^*\}$ for inefficient DMUs.

12.5 Numerical Example and Extension

To further illustrate the rationale of the approach, consider again the four DMUs shown in Figure 12.1. Table 12.1 reports the optimal value to model (12.7). It can be seen that for DMU D, the first input is the critical measure since DMU D’s efficiency can be easily improved if the first input is given the pre-emptive priority to change. For DMU A, the infeasibility associated with the second input indicates that the first input is the critical measure. Note that the efficient facet AB shows that the first input is more important than the second one, since $v_1 > v_2$. Our approach also indicates that the second input is the critical measure to DMU C’s performance. This finding is confirmed by the fact that $v_2 > v_1$ in BC. As for DMU B, since it is located at the intersect of AB and BC, it is very difficult to determine which input is the critical factor by looking at the coefficients of efficient facets. Our approach indicates that the second input is the critical one for DMU B, since $\tau_2^* < \tau_1^*$ ($17/12 < 14/9$).

Table 12.1. Critical Measures for the Numerical Example

| DMU | τ_1^* | τ_2^* |
|-----|------------|------------|
| A | 3/2 | infeasible |
| B | 14/9 | 17/12 |
| C | infeasible | 2 |
| D | 2/3 | 3/5 |

The above discussion assumes that DMUs are able to adjust each input and each output while other inputs and outputs are fixed. Situations when some measures are strongly related with each other may occur. In that case, a set of inputs or outputs has to be adjusted simultaneously and we need to consider the measures in groups. We use the following models.

$$\begin{aligned}
 & \min T_M \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq T_M x_{io} \quad i \in M \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} \leq x_{io} \quad i \notin M \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} \leq y_{ro} \quad r = 1, \dots, s \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1
 \end{aligned} \tag{12.8}$$

and

$$\begin{aligned}
 & \max \Omega_Q \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j y_{rj} \geq \Omega_Q y_{ro} \quad r \in Q \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r \notin Q \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1
 \end{aligned} \tag{12.9}$$

where inputs represented by set M and outputs represented by set Q are of interest.

Similar to the previous discussions, when DMU_o is inefficient, we use $\max\{T_M^*\}$ and $\min\{\Omega_Q^*\}$ to identify the most critical input and output measures, respectively. When DMU_o is efficient, infeasibility associated with (12.8) and (12.9) indicates the non-critical inputs and outputs.

The above discussion is based upon the assumption that the DEA frontier exhibits VRS. The development can be applied to other DEA models with non-VRS frontiers discussed in Chapter 11.

12.6 Application to Fortune E-Companies

To capture the Internet's effect on the economy, at the end of year 1999, Fortune magazine launched the Fortune e-50 index which consists of 50 corporations who integrate the Internet, computers and enterprise softwares to do the business. As stated in the 1999 December Fortune issue, each of

the e-50 is or has the potential to be a major player in the Internet economy. The list of e-corporation is decided by that a company must have been public for at least six months and must have a market capital value that exceeds \$100 million. Table 12.2 provides the list of the e-50.

Table 12.2. Fortune's e-corporations

| DMU No. | Name | Revenue \$ millions | Profits \$ millions | Employees | Market Capital \$ millions | Year Founded |
|---|-----------------------------|------------------------|------------------------|-----------|-------------------------------|-----------------|
| E-COMPANIES | | | | | | |
| 1 | America Online | 4777 | 762 | 12100 | 164308 | 1985 |
| 2 | Charles Schwab | 4113 | 498 | 13300 | 34194 | 1986 |
| 3 | Amazon.com | 1015 | -291 | 2100 | 21202 | 1994 |
| 4 | E*Trade Group | 621 | -54 | 1735 | 8341 | 1982 |
| 5 | Knight/Trimark Group | 618 | 119 | 446 | 4389 | 1995 |
| 6 | Yahoo | 341 | 22 | 803 | 47946 | 1995 |
| 7 | Ameritrade Holding | 301 | 12 | 985 | 3740 | 1992 |
| 8 | EarthLink Network | 254 | -88 | 1343 | 1409 | 1994 |
| 9 | Priceline.com | 189 | -125 | 194 | 7963 | 1998 |
| 10 | CMGI | 176 | 476 | 1024 | 12567 | 1986 |
| 11 | Lycos | 136 | -52 | 456 | 5687 | 1995 |
| 12 | Excite@Home | 129 | -324 | 570 | 14647 | 1995 |
| 13 | eBay | 125 | 7 | 138 | 17106 | 1995 |
| 14 | DoubleClick | 103 | -22 | 482 | 5947 | 1996 |
| 15 | RealNetworks | 89 | -4 | 434 | 9148 | 1994 |
| 16 | CNet | 79 | 40 | 491 | 3481 | 1995 |
| 17 | Healtheon | 68 | -68 | 648 | 2347 | 1995 |
| 18 | eToys | 38 | -47 | 306 | 6276 | 1996 |
| 19 | VerticalNet | 8 | -21 | 220 | 2515 | 1995 |
| NET SOFTWARE AND SERVICE COMPANIES | | | | | | |
| 20 | Microsoft | 19747 | 7785 | 31396 | 471573 | 1975 |
| 21 | Oracle | 9063 | 1332 | 44000 | 85776 | 1977 |
| 22 | Intuit | 848 | 377 | 3675 | 5942 | 1983 |
| 23 | Network Associates | 785 | -127 | 2700 | 2871 | 1992 |
| 24 | Cambridge Tech. Partners | 628 | 35 | 4444 | 726 | 1991 |
| 25 | TMP Worldwide | 585 | 10 | 5200 | 2976 | 1967 |
| 26 | Ariba | 45.4 | * | * | * | 1996 |
| 27 | Citrix Systems | 323 | 93 | 620 | 7169 | 1989 |
| 28 | Macromedia | 167 | 24 | 553 | 2690 | 1992 |
| 29 | Network Solutions | 142 | 17 | 385 | 4801 | 1979 |
| 30 | Concentric Network | 110 | -82 | 508 | 1054 | 1991 |
| 31 | Exodus Communications | 108 | -82 | 472 | 7080 | 1992 |
| 32 | BroadVision | 71 | 10 | 271 | 6777 | 1993 |
| 33 | Inktomi | 71 | -24 | 185 | 5709 | 1996 |
| 34 | Security First Technologies | 44 | -19 | 312 | 1345 | 1995 |
| 35 | Razorfish | 36 | 2 | 414 | 1896 | 1995 |

Table 12.2 Fortune's e-corporations (continued)

| NET HARDWARE COMPANIES | | | | | | |
|-----------------------------|----------------------|-------|------|--------|--------|------|
| 36 | IBM | 87448 | 7701 | 291067 | 167567 | 1911 |
| 37 | Lucent Technologies | 38303 | 4766 | 153000 | 211415 | 1995 |
| 38 | Intel | 28194 | 7371 | 64500 | 285803 | 1968 |
| 39 | Dell Computer | 21670 | 1750 | 24400 | 110530 | 1984 |
| 40 | Cisco Systems | 12154 | 2096 | 21000 | 237215 | 1984 |
| 41 | Sun Microsystems | 11726 | 1031 | 29700 | 85861 | 1982 |
| 42 | EMC | 4459 | 967 | 9700 | 75371 | 1979 |
| 43 | Qualcomm | 3937 | 201 | 11600 | 43919 | 1981 |
| 44 | Network Appliance | 335 | 42 | 816 | 6327 | 1992 |
| 45 | Broadcom | 335 | 40 | 436 | 15994 | 1991 |
| 46 | Juniper Networks | 31 | -30 | 190 | 14455 | 1992 |
| NET COMMUNICATION COMPANIES | | | | | | |
| 47 | AT&T | 56968 | 6037 | 107800 | 154791 | 1875 |
| 48 | MCI WorldCom | 30720 | -883 | 77000 | 162492 | 1983 |
| 49 | Qwest Communications | 3424 | -5 | 8700 | 27404 | 1997 |
| 50 | Global Crossing | 691 | 79 | 10000 | 26109 | 1997 |

Market capital, profit, revenue and number of employees are provided by the Fortune as the four standard measures to fully characterize the performance of the e-50 corporations. We therefore use them as a set of multiple performance measures. The data on profit, employee and market capital are not available for Ariba (DMU26), and therefore Ariba is excluded from the following analysis.

Because we are interested in the contribution of revenue, profit and employee to the market value, we select the market capital as the DEA output and the other measures as the DEA inputs. Output-oriented DEA model is used, because higher market values are desirable given the current levels of revenue, profit and the number of employees.

The third column of Table 12.3 reports the optimal value to the output-oriented VRS envelopment model. Ten e-corporations are on the performance frontier.

Next, we apply the newly developed method to identify the critical input measures to the market capital under the context of best-practice. Columns 3, 4 and 5 of Table 12.4 report the results from model (12.7).

We use the DEAFrontier software to do the calculation. Once the data are entered into the "data" sheet, we select "Perform Sensitivity Analysis" and then select the input as shown in Figure 12.2. The results are reported in the "Sensitivity Report" sheet. We can select one input at a time.

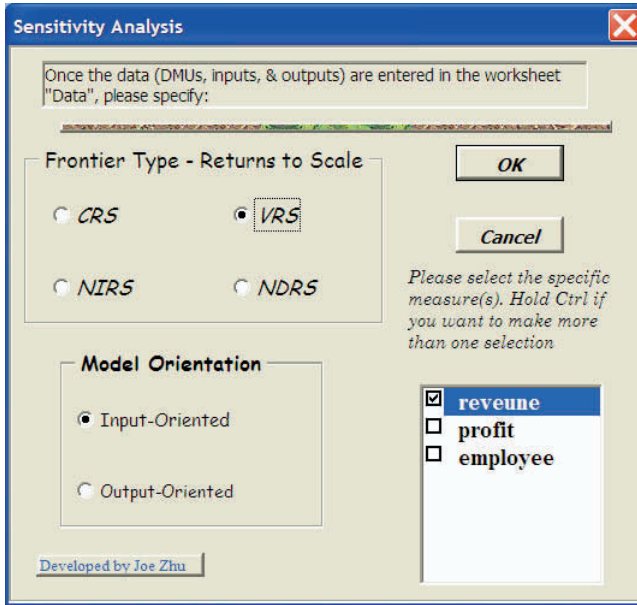


Figure 12.2. Critical Measures and Tradeoffs

For example, consider MCI WorldCom (DMU48), model (12.7) is infeasible when revenue and employee are under consideration (selected) respectively and model (12.7) yields the optimal value of 96.98 when profit is under consideration. This indicates that once the three input measures are determined, the magnitudes of revenue and employee do not affect the efficiency status of MCI WorldCom. However, the value of profit affects MCI WorldCom’s efficiency status given the current levels of market value, revenue and employee. Thus, profit is the critical factor to MCI WorldCom’s performance.

Consider Charles Schwab (DMU2) which is an inefficient unit. The optimal values to model (12.7) indicate that the profit measure is the critical one for Charles Schwab to achieve the performance frontier.

The sixth column of Table 4 reports the critical measure identified on the basis of model (12.7). However, for efficient DMUs, it is likely that model (12.7) is infeasible for each input measure. Samples can be found in America Online (DMU1), Yahoo (DMU6) and Microsoft (DMU20). This may imply that some measures must be considered in groups. We therefore employ model (12.8) for all possible combinations of the three input measures. The last column of Table 4 reports the results based upon model (12.8). Note that model (12.8) is not applied to the inefficient DMUs.

Table 12.3. Performance evaluation of Fortune's e-corporations

| DMU No. | Name | VRS |
|---------|-----------------------------|----------|
| 1 | America Online | 1.00000 |
| 2 | Charles Schwab | 3.83409 |
| 3 | Amazon.com | 1.05723 |
| 4 | E*Trade Group | 5.31514 |
| 5 | Knight/Trimark Group | 7.15192 |
| 6 | Yahoo | 1.00000 |
| 7 | Ameritrade Holding | 11.63487 |
| 8 | EarthLink Network | 25.09020 |
| 9 | Priceline.com | 1.00000 |
| 10 | CMGI | 2.39677 |
| 11 | Lycos | 4.30319 |
| 12 | Excite@Home | 1.00000 |
| 13 | eBay | 1.00000 |
| 14 | DoubleClick | 3.71566 |
| 15 | RealNetworks | 2.26509 |
| 16 | CNet | 5.64226 |
| 17 | Healtheon | 7.23136 |
| 18 | eToys | 2.32675 |
| 19 | VerticalNet | 1.00000 |
| 20 | Microsoft | 1.00000 |
| 21 | Oracle | 2.31196 |
| 22 | Intuit | 10.30718 |
| 23 | Network Associates | 13.81890 |
| 24 | Cambridge Tech. Partners | 72.12135 |
| 25 | TMP Worldwide | 16.68021 |
| 27 | Citrix Systems | 5.50414 |
| 28 | Macromedia | 10.83562 |
| 29 | Network Solutions | 5.34769 |
| 30 | Concentric Network | 19.73886 |
| 31 | Exodus Communications | 2.90959 |
| 32 | BroadVision | 2.74452 |
| 33 | Inktomi | 2.77061 |
| 34 | Security First Technologies | 11.79142 |
| 35 | Razorfish | 7.90885 |
| 36 | IBM | 2.79636 |
| 37 | Lucent Technologies | 1.72137 |
| 38 | Intel | 1.59834 |
| 39 | Dell Computer | 2.03503 |
| 40 | Cisco Systems | 1.00000 |
| 41 | Sun Microsystems | 2.24478 |
| 42 | EMC | 1.94255 |
| 43 | Qualcomm | 2.22318 |
| 44 | Network Appliance | 1.93360 |
| 45 | Broadcom | 7.47555 |
| 46 | Juniper Networks | 1.00000 |
| 47 | AT&T | 2.64384 |
| 48 | MCI WorldCom | 1.00000 |
| 49 | Qwest Communications | 2.61124 |
| 50 | Global Crossing | 2.18328 |

Table 12.4. Critical measures for Fortune’s e-corporations

| DMU No. | Name | revenue | profit | employee | Critical measures | |
|---------|----------------------|------------|------------|------------|-------------------|---------------------------|
| | | | | | (12.7) | (12.8) |
| 1 | America Online | infeasible | infeasible | infeasible | | {profit, revenue} |
| 2 | Charles Schwab | 0.0520 | 0.3619 | 0.0381 | {profit} | |
| 3 | Amazon.com | 0.5563 | 0.9817 | 0.7884 | {profit} | |
| 4 | E*Trade Group | 0.0463 | 0.6708 | 0.0945 | {profit} | |
| 5 | Knight/Trimark | 0.0188 | 0.6297 | 0.3094 | {profit} | |
| 6 | Yahoo | infeasible | infeasible | infeasible | | {profit,revenue,employee} |
| 7 | Ameritrade Holding | 0.0344 | 0.6283 | 0.1401 | {profit} | |
| 8 | EarthLink Network | 0.1368 | 0.7066 | 0.1328 | {profit} | |
| 9 | Priceline.com | infeasible | 1.1200 | 1.5994 | {profit} | {profit, revenue} |
| 10 | CMGI | 0.1555 | 0.4180 | 0.1348 | {profit} | |
| 11 | Lycos | 0.1736 | 0.7581 | 0.3700 | {profit} | |
| 12 | Excite@Home | infeasible | 1.4459 | infeasible | {profit} | {profit, revenue} |
| 13 | eBay | infeasible | infeasible | 1.7284 | {employee} | infeasible |
| 14 | DoubleClick | 0.1419 | 0.7375 | 0.3361 | {profit} | |
| 15 | RealNetworks | 0.2335 | 0.7661 | 0.3639 | {profit} | |
| 16 | CNet | 0.1248 | 0.7460 | 0.3329 | {profit} | |
| 17 | Healthcon | 0.3937 | 0.8759 | 0.3337 | {profit} | |
| 18 | eToys | 0.6037 | 0.9469 | 0.6886 | {profit} | |
| 19 | VerticalNet | 3.8750 | infeasible | infeasible | {revenue} | {profit, revenue} |
| 20 | Microsoft | infeasible | infeasible | infeasible | | {profit,revenue,employee} |
| 21 | Oracle | 0.1968 | 0.3002 | 0.0803 | {profit} | |
| 22 | Intuit | 0.0172 | 0.4408 | 0.0376 | {profit} | |
| 23 | Network Associates | 0.0641 | 0.7296 | 0.0733 | {profit} | |
| 24 | Cambridge Tech. | 0.0127 | 0.6063 | 0.0311 | {profit} | |
| 25 | TMP Worldwide | 0.0152 | 0.6238 | 0.0265 | {profit} | |
| 27 | Citrix Systems | 0.0525 | 0.5797 | 0.2226 | {profit} | |
| 28 | Macromedia | 0.0499 | 0.6331 | 0.2496 | {profit} | |
| 29 | Network Solutions | 0.0874 | 0.7455 | 0.3584 | {profit} | |
| 30 | Concentric Network | 0.2942 | 0.7667 | 0.3922 | {profit} | |
| 31 | Exodus Comm. | 0.3326 | 0.7938 | 0.4261 | {profit} | |
| 32 | BroadVision | 0.2283 | 0.8749 | 0.6195 | {profit} | |
| 33 | Inktomi | 0.5639 | 0.9775 | 0.9381 | {profit} | |
| 34 | Security First Tech. | 0.1818 | 0.9023 | 0.5859 | {profit} | |
| 35 | Razorfish | 0.2222 | 0.8968 | 0.4523 | {profit} | |
| 36 | IBM | 0.0564 | 0.0185 | 0.0324 | {revenue} | |
| 37 | Lucent Technologies | 0.1846 | 0.2452 | 0.0824 | {profit} | |
| 38 | Intel | 0.3794 | 0.4202 | 0.2788 | {profit} | |
| 39 | Dell Computer | 0.1258 | 0.3242 | 0.2181 | {profit} | |
| 40 | Cisco Systems | infeasible | 1.0754 | infeasible | {profit} | {profit, revenue} |
| 41 | Sun Microsystems | 0.1524 | 0.2609 | 0.1192 | {profit} | |
| 42 | EMC | 0.3110 | 0.4847 | 0.2870 | {profit} | |
| 43 | Qualcomm | 0.0772 | 0.5508 | 0.0617 | {profit} | |
| 44 | Network Appliance | 0.1351 | 0.7314 | 0.3165 | {profit} | |
| 45 | Broadcom | 0.0458 | 0.6096 | 0.1691 | {profit} | |
| 46 | Juniper Networks | 3.5327 | infeasible | infeasible | {revenue} | {profit, revenue} |
| 47 | AT&T | 0.0775 | 0.0025 | 0.0790 | {employee} | |
| 48 | MCI WorldCom | infeasible | 96.9787 | infeasible | {profit} | {profit, revenue} |
| 49 | Qwest Comm. | 0.0441 | 0.5771 | 0.0443 | {profit} | |
| 50 | Global Crossing | 0.2010 | 0.6601 | 0.0332 | {revenue} | |

For Yahoo and Microsoft, model (12.8) is feasible (has optimal solutions) when only all three inputs are in set M. For America Online, model (12.7) is feasible (has optimal solutions) when profit and revenue are in set M.

Model (12.8) is also applied to the remaining 7 efficient e-corporations, namely, Excite@Home (DMU12), Vertical Net (DMU19), Cisco System (DMU40), Juniper Networks (DMU46) and MCI WorldCom (DMU48). Model (12.7) is feasible when profit and revenue are in set M.

Except for America Online, Yahoo, eBay, Vertical Net, Microsoft, IBM, Juniper Networks, AT&T and Global Crossing, all the e-corporations indicate profit as their critical measure. This confirms that for the majority of the e-corporations that are rely on the Internet for business, revenue does not necessarily mean profit. In fact, about 40% of the e-corporations had negative profit in year 1999. (The negative values are treated by the translation invariance property in DEA. See Chapter 5's Appendix.)

A closer look at Table 4 indicates that America Online, Yahoo and Microsoft have distinguished themselves from the e-corporations, because the results from model (12.8) imply that their high revenue means profit. Note that among the inefficient units, employee is identified as the critical measure for eBay and AT&T, and revenue is identified as the critical measure for IBM.

The e-corporations actually represent the 21st century new economy where the electronic and information technologies are heavily used. To further illustrate the approach, we next apply models (12.7) and (12.8) to the Fortune 1000 companies in 1995 who represent old economy where the companies design, build and deliver physical, molecular-based products to customer. The purpose is to see whether the new economy e-corporations behave differently compared to the old economy companies in terms of the critical measures.

Since the e-corporations belong to computer and telecommunication industries, we exclude all those Fortune's 1000 companies who are in the computer and telecommunication industries from the analysis. We also exclude those Fortune 1000 companies who do not have complete data on the four performance measures. As a result, we have 51 industries with 760 companies which are different from the e-corporations (see the first column in Table 12.5).

Table 5 summarizes the results from the new approach. The second column reports the number of companies in each industry. The third, fourth and fifth columns report how many companies indicate revenue, profit and employee as their critical measures respectively. For example, the second row in Table 5 indicates that (i) there are 4 companies in the advertising and marketing industry, and (ii) revenue is identified as the critical measure for all companies. In the motor vehicle industry, only two companies (General

Motor and Ford) (9.52%; two out of 21) indicate that profit is the critical measure while other 19 companies indicate that revenue is the critical measure.

Our approach indicates that revenue is the critical factor to 95% of the 760 companies in the Fortune's top 1000 list. In fact, these "old-economy" companies sever relatively mature market or command a lead in markets where they compete. Our finding is consistent with the belief that revenue means a stable proportion of the profit for the old economy companies. Also, our approach does indicate that the e-corporations and the Fortune's 1000 companies behave differently.

Table 12.5. Critical measures for Fortune's 1000 companies

| Industry | Companies | Revenue | Profit | Employee |
|------------------------------------|-----------|---------|--------|----------|
| Advertising, marketing | 4 | 100% | 0% | 0% |
| Aerospace | 11 | 90.91% | 9.09% | 0% |
| Airlines | 9 | 100% | 0% | 0% |
| Apparel | 5 | 100% | 0% | 0% |
| Beverages | 7 | 100% | 0% | 0% |
| Brokerage | 7 | 100% | 0% | 0% |
| Building materials, glass | 4 | 100% | 0% | 0% |
| Chemicals | 39 | 97.44% | 2.56% | 0% |
| Commercial banks | 55 | 98.18% | 1.82% | 0% |
| Diversified financials | 14 | 92.86% | 7.14% | 0% |
| Electric and gas utilities | 73 | 98.63% | 0% | 1.37% |
| Electronics, electrical equipment | 41 | 95.12% | 4.88% | 0% |
| Engineering, construction | 11 | 90.91% | 0% | 9.09% |
| Entertainment | 3 | 33.33% | 33.33% | 33.33% |
| Food | 27 | 92.59% | 0% | 7.41% |
| Food and drug stores | 20 | 100% | 0% | 0% |
| Food services | 5 | 80.00% | 20.00% | 0% |
| Forest and paper products | 30 | 100% | 0% | 0% |
| Furniture | 5 | 100% | 0% | 0% |
| General merchandisers | 16 | 87.50% | 12.50% | 0% |
| Health care | 18 | 100% | 0% | 0% |
| Hotels, casinos, resorts | 7 | 100% | 0% | 0% |
| Industrial and farm equipment | 27 | 100% | 0% | 0% |
| Insurance: life & health | 19 | 94.74% | 5.26% | 0% |
| Insurance: prop. & casualty | 24 | 87.50% | 12.50% | 0% |
| Mail, package and freight delivery | 3 | 100% | 0% | 0% |
| Marine services | 2 | 100% | 0% | 0% |
| Metal products | 11 | 100% | 0% | 0% |
| Metals | 21 | 100% | 0% | 0% |
| Mining, crude-oil production | 7 | 100% | 0% | 0% |
| Motor vehicles and parts | 21 | 90.48% | 9.52% | 0% |

Table 12.5 Critical measures for Fortune's 1000 companies (continued)

| Industry | Companies | Revenue | Profit | Employee |
|------------------------------------|------------|---------------|--------------|--------------|
| Petroleum refining | 18 | 50.00% | 33.33% | 16.67% |
| Pharmaceuticals | 14 | 85.71% | 14.29% | 0% |
| Pipelines | 10 | 80.00% | 0% | 20.00% |
| Publishing, printing | 17 | 100% | 0% | 0% |
| Railroads | 5 | 100% | 0% | 0% |
| Rubber and plastic products | 8 | 100% | 0% | 0% |
| Savings institutions | 8 | 100% | 0% | 0% |
| Scientific, photo., control equip. | 18 | 94.44% | 5.56% | 0% |
| Soaps, cosmetics | 8 | 87.50% | 12.50% | 0% |
| Specialist retailers | 30 | 100% | 0% | 0% |
| Temporary help | 5 | 100% | 0% | 0% |
| Textiles | 6 | 100% | 0% | 0% |
| Tobacco | 4 | 75.00% | 25.00% | 0% |
| Toys, sporting goods | 3 | 100% | 0% | 0% |
| Transportation equipment | 5 | 100% | 0% | 0% |
| Truck leasing | 2 | 100% | 0% | 0% |
| Trucking | 3 | 100% | 0% | 0% |
| Waste management | 3 | 100% | 0% | 0% |
| Wholesalers | 40 | 90.00% | 0% | 10.00% |
| Miscellaneous | 7 | 100% | 0% | 0% |
| Total | 760 | 94.61% | 3.55% | 1.84% |

REFERENCES

1. Chen, Y. and J. Zhu (2003), DEA models for identifying critical performance measures, *Annals of Operations Research*, Vol.124, No. 1-4, 225-244.
2. Wilkens, K. and J. Zhu (2001), Portfolio evaluation and benchmark selection: a mathematical programming approach, *Journal of Alternative Investments*, 4, No. 1, 9-19.

Chapter 13

Returns-to-Scale

13.1 Introduction

As demonstrated in Figure 1.3, the VRS envelopment model identifies the VRS frontier with DMUs exhibiting IRS (increasing returns to scale), CRS (constant returns to scale), and DRS (decreasing returns to scale). In fact, the economic concept of RTS (returns to scale) has been widely studied within the framework of DEA. RTS have typically been defined only for single output situations. DEA generalizes the notion of RTS to the multiple-output case. This, in turn, further extended the applicability of DEA.

Seiford and Zhu (1999a) demonstrate that there are at least three equivalent *basic* methods of testing a DMU's RTS nature which have appeared in the DEA literature. Based upon the VRS multiplier models, the sign of the optimal free variable (μ^* or ν^*) indicates the RTS (Banker, Charnes and Cooper, 1984). Based upon the CRS envelopment models, the magnitude of optimal $\sum_j \lambda_j^*$ indicates the RTS (Banker, 1984). These two methods may fail when DEA models have alternate optimal solutions. The third method is based upon the scale efficiency index (Färe, Grosskopf and Lovell, 1994). The scale efficiency index method does not require information on μ^* or ν^* or $\sum_j \lambda_j^*$, and is robust even when there exist multiple optima. However, the scale efficiency index method requires the calculation of three DEA models.

Seiford and Zhu (1999b) and Seiford and Zhu (2005) study the sensitivity of RTS classification. Seiford and Zhu (1999c) provide a use of RTS sensitivity analysis in improving performance of a two-stage process.

13.2 RTS Regions

It is meaningful to discuss RTS for DMUs located on the VRS frontier. We discuss the RTS for non-frontier DMUs by their VRS efficient targets as indicated in Table 1.1. Because a VRS envelopment model can be either input-oriented or output-oriented, we may obtain different efficient targets and RTS classifications for a specific non-frontier DMU.

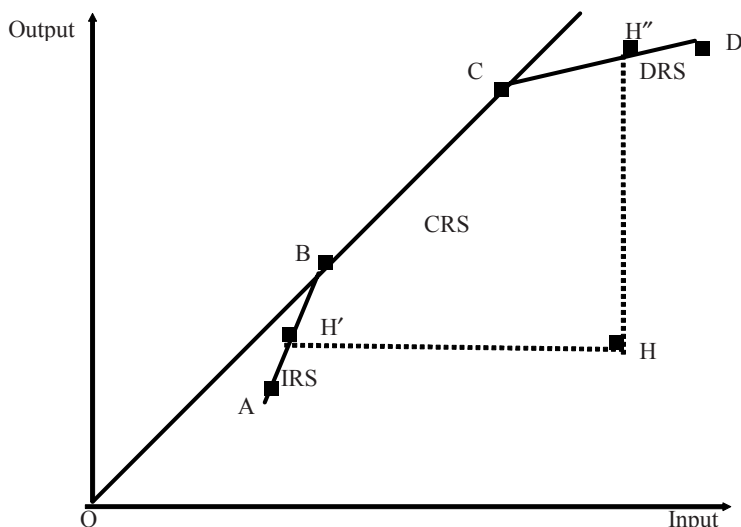


Figure 13.1. RTS and VRS Efficient Target

Suppose we have five DMUs, A, B, C, D, and H as shown in Figure 13.1. Ray OBC is the CRS frontier. AB, BC and CD constitute the VRS frontier, and exhibit IRS, CRS and DRS, respectively. B and C exhibit CRS. On the line segment AB, IRS prevail to the left of B. On the line segment CD, DRS prevail to the right of C.

Consider non-frontier DMU H. If the input-oriented VRS envelopment model is used, then H' is the efficient target, and the RTS classification for H is IRS. If the output-oriented VRS envelopment model is used, then H'' is the efficient target, and the RTS classification for H is DRS.

However some IRS, CRS and DRS regions are uniquely determined no matter which VRS model is employed. They are region 'I' – IRS, region 'II' – CRS, and region 'III' – DRS. In fact, we have six RTS regions as shown in

Figure 13.2. Two RTS classifications will be assigned into the remaining regions IV, V and VI. Region ‘IV’ is of IRS (input-oriented) and of CRS (output-oriented). Region ‘V’ is of CRS (input-oriented) and of DRS (output-oriented). Region ‘VI’ is of IRS (input-oriented) and of DRS (output-oriented).

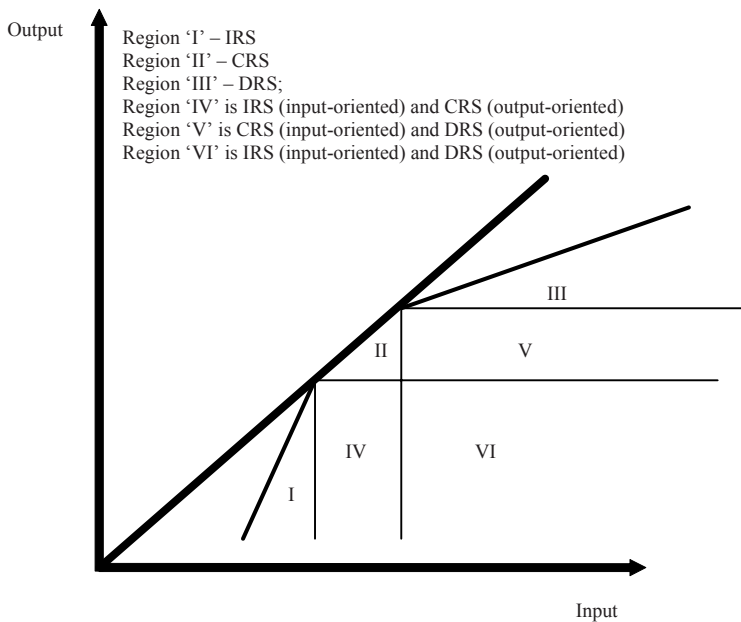


Figure 13.2. RTS Region

The RTS regions can provide a DMU classification. See also Gregoriou and Zhu (2005).

13.3 RTS Estimation

3.3.1 VRS and CRS RTS Methods

Let μ^* represent the optimal value of μ in the input-oriented VRS multiplier model, and ν^* the optimal value of ν in the output-oriented VRS multiplier model, then we have the VRS RTS method.

Theorem 13.1

- (i) If $\mu^* = 0$ (or $\nu^* = 0$) in *any* alternate optima, then CRS prevail on DMU_o .
- (ii) If $\mu^* > 0$ (or $\nu^* < 0$) in *all* alternate optima, then IRS prevail on DMU_o .
- (iii) If $\mu^* < 0$ (or $\nu^* > 0$) in *all* alternate optima, then DRS prevail on DMU_o .

Note that the VRS frontier can be expressed as $\sum_{r=1}^s \mu_r y_{rj} = \sum_{i=1}^m \nu_i x_{ij} - \mu$ (or $\sum_{r=1}^s \mu_r y_{rj} = \sum_{i=1}^m \nu_i x_{ij} + \nu$). Thus, geometrically, in the case of single output, $-\mu^*$ (or ν^*) represents the y-intercept on the output axis. Consider Figure 13.1. The intercept is positive for line segment CD so $\mu^* < 0$ (or $\nu^* > 0$) and RTS is decreasing for any DMU on CD (excluding C), whereas the intercept is negative for line segment AB so $\mu^* > 0$ (or $\nu^* < 0$) and RTS is increasing for any DMU on AB (excluding B). The intercept for line OBC is zero so $\mu^* = 0$ (or $\nu^* = 0$) and RTS is constant. However, in computation, we may not obtain the unique optimal solution (the frontier), and we may obtain supporting hyperplanes at VRS frontier DMUs. Consequently, we have to check all optimal solutions as indicated in Theorem 13.1.

Table 13.1 presents five VRS frontier DMUs with two inputs and one output. The last column indicates the RTS classification.

Table 13.1. DMUs for RTS Estimation

| DMU | input 1 (x1) | input 2 (x2) | output (y) | RTS |
|-----|--------------|--------------|------------|-----|
| 1 | 2 | 5 | 2 | CRS |
| 2 | 2 | 2 | 1 | CRS |
| 3 | 4 | 1 | 1 | CRS |
| 4 | 2 | 1 | 1/2 | IRS |
| 5 | 6 | 5 | 5/2 | DRS |

Table 13.2. Optimal Values for RTS Estimation

| DMU | $\mu^* \in [\mu^-, \mu^+]$ | λ_j^* |
|-----|----------------------------|--|
| 1 | [-7, 1] | $\lambda_1^* = 1; \sum_{j=1}^6 \lambda_j^* = 1$ |
| 2 | [0, 1] | solution 1: $\lambda_2^* = 1; \sum_{j=1}^6 \lambda_j^* = 1$ solution 2: $\lambda_1^* = 1/3, \lambda_3^* = 1/3; \sum_{j=1}^6 \lambda_j^* = 2/3$ |
| 3 | [-5/3, 1] | $\lambda_3^* = 1; \sum_{j=1}^6 \lambda_j^* = 1$ |
| 4 | [1/2, 1] | $0 \leq \lambda_1^* \leq 1/12, \lambda_2^* = 1/4 - 3\lambda_1^*, \lambda_3^* = 1/4 + \lambda_1^*$ $5/12 \leq \sum_{j=1}^6 \lambda_j^* \leq 1/2$ |
| 5 | $(-\infty, -3/37]$ | $\lambda_1^* = 35/48 - \lambda_2^*/3, 0 \leq \lambda_2^* \leq 35/16, \lambda_3^* = 25/24 - \lambda_2^*/3$ $85/48 \leq \sum_{j=1}^6 \lambda_j^* \leq 15/6$ |

The second column of Table 13.2 reports the optimal μ^* . μ^* can take all the optimal values in the interval $[\mu^-, \mu^+]$. $\mu^* = 0$ is found in DMUs 1, 2,

and 3, therefore the three DMUs exhibit CRS. All μ^* are positive and negative in DMU5 and DMU6, respectively, therefore IRS and DRS prevail on DMU5 and DMU6, respectively.

The above RTS method uses the VRS multiplier models. In fact, we can use CRS envelopment models to estimate the RTS classification (Zhu, 2000a). Let λ_j^* be the optimal values in CRS envelopment models. We have

Theorem 13.2

- (i) If $\sum_j^n \lambda_j^* = 1$ in *any* alternate optima, then CRS prevail on DMU_o .
- (ii) If $\sum_j^n \lambda_j^* < 1$ for *all* alternate optima, then IRS prevail on DMU_o .
- (iii) If $\sum_j^n \lambda_j^* > 1$ for *all* alternate optima, then DRS prevail on DMU_o .

From Table 13.2, we see that DMU2 has alternate optimal λ_j^* . Nevertheless, there exists an optimal solution such that $\sum_j^n \lambda_j^* = 1$ indicating CRS. DMU4 exhibits IRS because $\sum_j^n \lambda_j^* < 1$ in all optima, and DMU5 exhibits DRS because $\sum_j^n \lambda_j^* > 1$ in all optima.

3.3.2 Improved RTS Method

In real world applications, the examination of alternative optima is a laborious task, and one may attempt to use a single set of resulting optimal solutions in the application of the RTS methods. However, this may yield erroneous results. For instance, if we obtain $\lambda_1^* = \lambda_3^* = 1/3$, or $\mu^* = 1$ for DMU2, then DMU2 may erroneously be classified as having IRS because $\sum \lambda_j^* < 1$ or $\mu^* > 0$ in one particular alternate solution.

A number of methods have been developed to deal with multiple optimal solutions in the VRS multiplier models and the CRS envelopment models. Seiford and Zhu (1999a) show the following results with respect to the relationship amongst envelopment and multiplier models, respectively.

Theorem 13.3

- (i) The CRS efficiency score is equal to the VRS efficiency score *if and only if* there exists an optimal solution such that $\sum_j^n \lambda_j^* = 1$. If The CRS efficiency score is not equal to the VRS efficiency score, then
- (ii) The VRS efficiency score is greater than the NIRS efficiency score *if and only if* $\sum_j^n \lambda_j^* < 1$ in all optimal solutions of the CRS envelopment model.
- (iii) The VRS efficiency score is equal to the NIRS efficiency score *if and only if* $\sum_j^n \lambda_j^* > 1$ in all optimal solutions of the CRS envelopment model.

Theorem 13.4

- (i) The CRS efficiency score is equal to the VRS efficiency score *if and only if* there exists an optimal solution $\mu^* = 0$ (or $\nu^* = 0$). If The CRS efficiency score is not equal to the VRS efficiency score, then
- (ii) The VRS efficiency score is greater than the NIRS efficiency score *if and only if* $\mu^* > 0$ (or $\nu^* < 0$) in all optimal solutions.
- (iii) The VRS efficiency score is equal to the NIRS efficiency score *if and only if* $\mu^* < 0$ (or $\nu^* > 0$) in all optimal solutions.

Based upon Theorems 13.3 and 13.4, we have

Theorem 13.5

- (i) If DMU_o exhibits IRS, then $\sum_j \lambda_j^* < 1$ for *all* alternate optima.
- (ii) If DMU_o exhibits DRS, then $\sum_j \lambda_j^* > 1$ for *all* alternate optima.

The significance of Theorem 13.5 lies in the fact that the possible alternate optimal λ_j^* obtained from the CRS envelopment models only affect the estimation of RTS for those DMUs that truly exhibit CRS, and have nothing to do with the RTS estimation on those DMUs that truly exhibit IRS or DRS. That is, if a DMU exhibits IRS (or DRS), then $\sum_j \lambda_j^*$ must be less (or greater) than one, no matter whether there exist alternate optima of λ_j .

Further, we can have a very simple approach to eliminate the need for examining all alternate optima.

Theorem 13.6

- (i) The CRS efficiency score is equal to the VRS efficiency score *if and only if* CRS prevail on DMU_o . Otherwise,
- (ii) $\sum_j \lambda_j^* < 1$ *if and only if* IRS prevail on DMU_o .
- (iii) $\sum_j \lambda_j^* > 1$ *if and only if* DRS prevail on DMU_o .

Thus, in empirical applications, we can explore RTS in two steps. First, select all the DMUs that have the same CRS and VRS efficiency scores regardless of the value of $\sum_j \lambda_j^*$. These DMUs are in the CRS region. Next, use the value of $\sum_j \lambda_j^*$ (in any CRS envelopment model outcome) to determine the RTS for the remaining DMUs. We observe that in this process we can safely ignore possible multiple optimal solutions of λ_j .

Similarly, based upon VRS multiplier models, we have

Theorem 13.7

- (i) The CRS efficiency score is equal to the VRS efficiency score *if and only if* CRS prevail on DMU_o . Otherwise,
- (ii) $\mu^* > 0$ (or $\nu^* < 0$) *if and only if* IRS prevail on DMU_o .

(iii) $\mu^* < 0$ (or $\nu^* > 0$) if and only if DRS prevail on DMU_o .

3.3.3 Spreadsheets for RTS Estimation

We here develop spreadsheet models for RTS estimation based upon Theorem 13.6. The RTS spreadsheet model uses VRS and CRS envelopment spreadsheets. Figure 13.3 shows a spreadsheet for the input-oriented CRS envelopment model where CRS efficiency scores and the optimal $\sum_j \lambda_j^*$ are recorded in columns J and K, respectively. The button “Input-oriented CRS (RTS)” is linked to a VBA procedure “RTS”.

```

Sub RTS()
    Dim i As Integer
    For i = 1 To 15
        'set the value of cell E18 equal to i (1, 2,..., 15)
        Range("E18") = i
        'Run the Solver model. The UserFinish is set to True so that
        'the Solver Results dialog box will not be shown
        SolverSolve UserFinish:=True
        'Place the efficiency into column J
        Range("J" & i + 1) = Range("F19")
        'Place the sum of lambdas into column K
        Range("K" & i + 1) = Range("B25")
    Next i
End Sub

```

| | A | B | C | D | E | F | G | H | I | J | K |
|----|------------------------------|-----------|---------|------------|----|------------|--------|-----------|---|----------------|-----------------|
| 1 | Company | Assets | Equity | Employees | | Revenue | Profit | λ | | CRS Efficiency | $\Sigma\lambda$ |
| 2 | Mitsubishi | 91920.6 | 10950 | 36000 | | 184365.2 | 346.2 | 0 | | 0.662831738 | 1.101942 |
| 3 | Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | 0 | 1 | | 1 |
| 4 | Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | 0 | 1 | | 1 |
| 5 | General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | 0 | 1 | | 1 |
| 6 | Sumitomo | 50268.9 | 6681 | 6193 | | 167530.7 | 210.5 | 0.47 | 1 | | 1 |
| 7 | Marubeni | 71439.3 | 5239.1 | 6702 | | 161057.4 | 156.8 | 0 | | 0.971966637 | 0.956252 |
| 8 | Ford Motor | 243263 | 24547 | 346990 | | 137137 | 4139 | 0 | | 0.737166307 | 0.99751 |
| 9 | Toyota Motor | 106004.2 | 49691.6 | 148855 | | 111052 | 2662.4 | 0 | | 0.524557613 | 0.819264 |
| 10 | Exxon | 91296 | 40436 | 82000 | | 110009 | 6470 | 0 | 1 | | 1 |
| 11 | Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | 0 | | 0.841423731 | 1.067172 |
| 12 | Wal-Mart | 37871 | 14762 | 675000 | | 93627 | 2740 | 0.01 | 1 | | 1 |
| 13 | Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | 0 | | 0.386057261 | 0.62692 |
| 14 | Nippon Life Insurance | 364762.5 | 2241.9 | 89690 | | 83206.7 | 2426.6 | 0 | 1 | | 1 |
| 15 | Nippon Telegraph & Telephone | 127077.3 | 42240.1 | 231400 | | 81937.2 | 2209.1 | 0 | | 0.348577853 | 0.619252 |
| 16 | AT&T | 88884 | 17274 | 299300 | | 79609 | 139 | 0 | | 0.270381772 | 0.481746 |
| 17 | | | | | | | | | | | |
| 18 | | Reference | | DMU under | 15 | Efficiency | | | | | |
| 19 | Constraints | set | | Evaluation | | 0.270382 | | | | | |
| 20 | Assets | 24032.61 | \leq | 24032.613 | | | | | | | |
| 21 | Equity | 3336.645 | \leq | 4670.5747 | | | | | | | |
| 22 | Employees | 12922.98 | \leq | 80925.264 | | | | | | | |
| 23 | Revenue | 79609 | $>$ | 79609 | | | | | | | |
| 24 | Profit | 139 | $>$ | 139 | | | | | | | |
| 25 | $\Sigma\lambda$ | 0.481746 | | | | | | | | | |

Figure 13.3. Input-oriented RTS Classification Spreadsheet Model

In order to obtain the RTS classification, we need also to calculate the input-oriented VRS envelopment model. This can be achieved by using the spreadsheet model shown in Figure 1.8 (Chapter 1). We then copy the VRS efficiency scores into column L, as shown in Figure 13.4. Cells M2:M16 contain formulas based upon Theorem 13.6. The formula for cell M2 which is copied into cells M3:M16 is

$$=IF(J2=L2,"CRS",IF(AND(J2<>L2,K2<1),"IRS",IF(AND(J2<>L2,K2>1),"DRS")))$$

To obtain the output-oriented RTS classification, we use the spreadsheet for output-oriented CRS envelopment model. Figure 13.5 shows the spreadsheet, and Figure 13.6 shows the Solver parameters. Note that range names are used in the spreadsheet shown in Figure 13.5 as in the spreadsheet for output-oriented VRS envelopment model shown in Figure 1.27. For example, cell E18 is named as “DMU”, cell F19 is named as “Efficiency”, and cell B25 is named as “SumLambda”. The button “Output-oriented CRS” is linked to a VBA procedure “GeneralRTS” which automates the calculation, and records the efficiency score and $\sum_j \lambda_j^*$ into columns J and K, respectively.

```
Sub GeneralRTS()
Dim NDMUs As Integer, NInputs As Integer, NOutputs As Integer
```

```

NDMUs = 15
NInputs = 3
NOutputs = 2
Dim i As Integer
For i = 1 To NDMUs
Range("DMU") = i
SolverSolve UserFinish:=True
Range("A1").Offset(i,NInputs+NOutputs+4) = Range("Efficiency")
Range("A1").Offset(i, NInputs+NOutputs+5) = Range("SumLambda")
Next
End Sub
    
```

| M2 | J | K | L | M | N | O | P |
|----|----------------|----------|----------------|-----|------------------------------|---|---|
| 1 | CRS Efficiency | Σλ | VRS Efficiency | RTS | Company | | |
| 2 | 0.662831738 | 1.101942 | 1 | DRS | Mitsubishi | | |
| 3 | 1 | 1 | 1 | CRS | Mitsui | | |
| 4 | 1 | 1 | 1 | CRS | Itochu | | |
| 5 | 1 | 1 | 1 | CRS | General Motors | | |
| 6 | 1 | 1 | 1 | CRS | Sumitomo | | |
| 7 | 0.971986637 | 0.956252 | 1 | IRS | Marubeni | | |
| 8 | 0.737166307 | 0.99751 | 0.737555958 | IRS | Ford Motor | | |
| 9 | 0.524557613 | 0.819264 | 0.603245345 | IRS | Toyota Motor | | |
| 10 | 1 | 1 | 1 | CRS | Exxon | | |
| 11 | 0.841423731 | 1.067172 | 1 | DRS | Royal Dutch/Shell Group | | |
| 12 | 1 | 1 | 1 | CRS | Wal-Mart | | |
| 13 | 0.386057261 | 0.62692 | 0.557595838 | IRS | Hitachi | | |
| 14 | 1 | 1 | 1 | CRS | Nippon Life Insurance | | |
| 15 | 0.348577853 | 0.619252 | 0.470610997 | IRS | Nippon Telegraph & Telephone | | |
| 16 | 0.270381772 | 0.481746 | 0.533543522 | IRS | AT&T | | |

Figure 13.4. Input-oriented RTS Classification

Note that we can assign “RTS” to the button “Output-oriented CRS (RTS)”. In fact, when the range names are used, Range(“DMU”), Range(“Efficiency”), and Range(“SumLambda”) are equivalent to Range(“E18”), Range(“F19”), and Range(“B25”), respectively. The procedure “GeneralRTS” can be applied to other data sets with the range names.

With the output-oriented VRS efficiency scores and Theorem 13.6, we can obtain the output-oriented RTS classification shown in Figure 13.7.

Based upon Figures 13.4 and 13.7, we obtain the RTS regions (see column O in Figure 13.7).

| A | B | C | D | E | F | G | H | I | J | K |
|---------------------------------|---------------|---------|----------------------|--|------------|----------|-----------|--|-----------------|---|
| 1 Company | Assets | Equity | Employees | | Revenue | Profit | λ | CRS Efficiency | $\Sigma\lambda$ | |
| 2 Mitsubishi | 91920.6 | 10950 | 38000 | | 184365.2 | 1 | 0 | 58488945 | 1.74138 | |
| 3 Mitsui | 68770.9 | 5553.9 | 80000 | | 181518.7 | 314.8 | 0 | 1 | 1 | |
| 4 Itochu | 65708.9 | 4271.1 | 7182 | | 169164.6 | 121.2 | 0 | 1 | 1 | |
| 5 General Motors | 217123.4 | 23345.5 | 709000 | | 168828.6 | 6880.7 | 0 | 1 | 1 | |
| 6 Sumitomo | 50288.9 | 8681 | 6193 | | 167530.7 | 210.5 | 1.73 | 1 | 1 | |
| 7 Marubeni | 71439.3 | 5239.1 | 8702 | | 161057.4 | 156.6 | 0 | 1.028841898 | 0.983832 | |
| 8 Ford Motor | 243283 | 24547 | 348990 | | 137137 | 4139 | 0 | 1.356545993 | 1.353168 | |
| 9 Toyota Motor | 106004.2 | 49691.6 | 146855 | | 111052 | 2662.4 | 0 | 1.906368292 | 1.561819 | |
| 10 Exxon | 91296 | 40436 | 82000 | | 110009 | 8470 | 0 | 1 | 1 | |
| 11 Royal Dutch/Shell Group | 118011.6 | 58986.4 | 104000 | | 109833.7 | 6904.6 | 0 | 1.188461846 | 1.268293 | |
| 12 Wal-Mart | 37871 | 14762 | 875000 | | 93627 | 2740 | 0.05 | 1 | 1 | |
| 13 Hitachi | 91620.9 | 29907.2 | 331852 | | 84167.1 | 1468.8 | 0 | 2.590289318 | 1.623905 | |
| 14 Nippon Life Insurance | 364762.5 | 224 | | Reserved to indicate the DMU under evaluation. | 63206.7 | 2426.0 | 0 | 1 | 1 | |
| 15 Nippon Telegraph & Telephone | 127077.3 | 4224 | | | 81937.2 | 2209.1 | 0 | 2.888799584 | 1.77851 | |
| 16 AT&T | 88884 | 172 | | | 79609 | 139 | 0 | 3.698474165 | 1.781727 | |
| 17 | | | | | | | | | | |
| 18 | Reference set | | DMU under Evaluation | 15 | Efficiency | 3.698474 | | Efficiency: # A changing cell; Target cell in Solver | | |
| 19 Constraints | | | | | | | | | | |
| 20 Assets | 88884 | <= | 88884 | | | | | | | |
| 21 Equity | 12347.89 | <= | 17274 | | | | | | | |
| 22 Employees | 47795.32 | <= | 299300 | | | | | | | |
| 23 Revenue | 294431.8 | >= | 294431.83 | | | | | | | |
| 24 Profit | 518.9873 | >= | 514.08791 | | | | | | | |
| 25 $\Sigma\lambda$ | 1.781727 | | | | | | | | | |

Figure 13.5. Output-oriented RTS Classification Spreadsheet Model

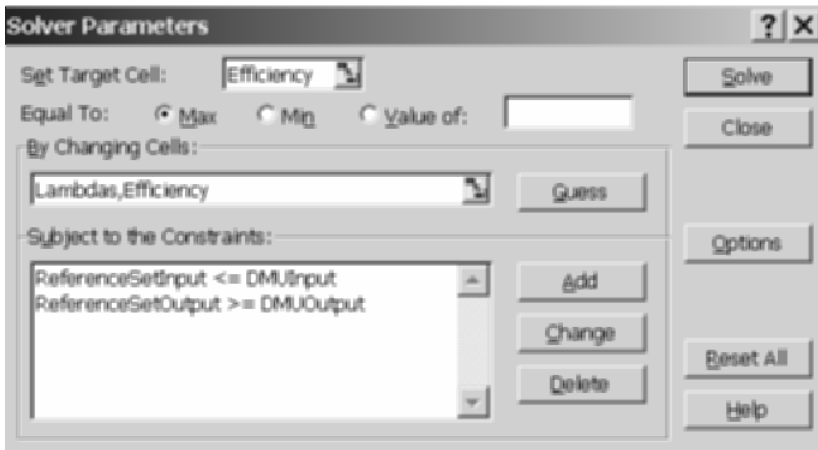


Figure 13.6. Solver Parameters for Output-oriented CRS Envelopment Model

| | M2 | | =IF(J2=L2,"CRS",IF(AND(J2<>L2,K2<1),"IRS",IF(AND(J2<>L2,K2>1),"DRS"))) | | | | |
|----|----------------|------------------|--|-----|------------------------------|------------|---|
| | J | K | L | M | N | O | P |
| 1 | CRS Efficiency | $\Sigma \lambda$ | VRS Efficiency | RTS | Company | RTS Region | |
| 2 | 1.58488945 | 1.74138 | 1 | DRS | Mitsubishi | Region III | |
| 3 | 1 | 1 | 1 | CRS | Mitsui | Region II | |
| 4 | 1 | 1 | 1 | CRS | Itochu | Region II | |
| 5 | 1 | 1 | 1 | CRS | General Motors | Region II | |
| 6 | 1 | 1 | 1 | CRS | Sumitomo | Region III | |
| 7 | 1.028841898 | 0.983832 | 1 | IRS | Marubeni | Region I | |
| 8 | 1.356545993 | 1.353168 | 1.158414974 | DRS | Ford Motor | Region VI | |
| 9 | 1.906368292 | 1.581819 | 1.371588284 | DRS | Toyota Motor | Region VI | |
| 10 | 1 | 1 | 1 | CRS | Exxon | Region II | |
| 11 | 1.188461846 | 1.268293 | 1 | DRS | Royal Dutch/Shell Group | Region III | |
| 12 | 1 | 1 | 1 | CRS | Wal-Mart | Region II | |
| 13 | 2.590289318 | 1.623905 | 1.898938938 | DRS | Hitachi | Region VI | |
| 14 | 1 | 1 | 1 | CRS | Nippon Life Insurance | Region II | |
| 15 | 2.868799584 | 1.77651 | 1.892916538 | DRS | Nippon Telegraph & Telephone | Region VI | |
| 16 | 3.698474165 | 1.781727 | 2.311193684 | DRS | AT&T | Region VI | |

Figure 13.7. Output-oriented RTS Classification

13.4 Scale Efficient Targets

By using the most productive scale size (MPSS) concept (Banker, 1984), we can develop linear programming problems to set unique scale efficient target. Consider the following linear program when the input-oriented CRS envelopment model is solved (Zhu, 2000b).

$$\begin{aligned}
 & \min \sum_{j=1}^n \lambda_j \\
 & \text{subject to} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta^* x_{io} \quad i = 1, 2, \dots, m; \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\
 & \lambda_j \geq 0. \quad j = 1, 2, \dots, n.
 \end{aligned}
 \tag{13.1}$$

where θ^* is the input-oriented CRS efficiency score.

Based upon the optimal values from (13.1) (i.e., $\sum \lambda_j^*$), the MPSS concept yields the following scale-efficient target for DMU_o corresponding to the largest MPSS

$$\text{MPSS}_{\max} : \begin{cases} \tilde{x}_{io} = \theta^* x_{io} / \sum \lambda_j^* \\ \tilde{y}_{ro} = y_{ro} / \sum \lambda_j^* \end{cases}
 \tag{13.2}$$

where (\sim) represents the target value.

If we change the objective of (13.1) to maximization,

$$\begin{aligned}
& \max \sum_{j=1}^n \hat{\lambda}_j \\
& \text{subject to} \\
& \sum_{j=1}^n \hat{\lambda}_j x_{ij} \leq \theta^* x_{io} \quad i = 1, 2, \dots, m; \\
& \sum_{j=1}^n \hat{\lambda}_j y_{rj} \geq y_{ro} \quad r = 1, 2, \dots, s; \\
& \hat{\lambda}_j \geq 0. \quad j = 1, 2, \dots, n.
\end{aligned} \tag{13.3}$$

then we have the scale efficient target corresponding to the smallest MPSS.

$$\text{MPSS}_{\min} : \begin{cases} \tilde{x}_{io} = \theta^* x_{io} / \sum \hat{\lambda}_j^* \\ \tilde{y}_{ro} = y_{ro} / \sum \hat{\lambda}_j^* \end{cases} \tag{13.4}$$

Note that models (13.1) and (13.3) are based upon the input-oriented CRS envelopment model. However, by using the relationship between the input-oriented and output-oriented CRS envelopment models (see Lemma 13.2), it is trivial to show that MPSS_{\max} (MPSS_{\min}) remains the same under both orientations. Consequently, MPSS_{\max} and MPSS_{\min} are uniquely determined by θ^* and $\sum \hat{\lambda}_j^*$ ($\sum \hat{\lambda}_j^*$).

We can select the largest or the smallest MPSS target for a particular DMU under consideration based upon the RTS preference over performance improvement. For example, one may select the smallest MPSS for an IRS DMU and the largest MPSS for a DRS DMU. Further, if the CRS envelopment models yield the unique optimal solutions, then the MPSS_{\max} and MPSS_{\min} are the same.

The spreadsheet model for calculating the scale efficient target involves (i) calculating CRS envelopment model, and (ii) calculating model (13.1). We demonstrate (ii) using the input-oriented CRS envelopment model shown in Figure 13.3.

In Figure 13.8, the target cell is B25, and contains the formula “=SUM(I2:I16)”, representing the $\sum \hat{\lambda}_j^*$. Cell F19 is no longer a changing cell, and contains the formula “=INDEX(J2:J16,E18,1)”. This formula returns the CRS efficiency score of a DMU under evaluation from column J.

The changing cells are I2:I16. The constraints in the Solver parameters for the input-oriented CRS envelopment model shown in Figure 1.24 remain the same. Figure 13.8 also shows the Solver parameters for calculating the model (13.1). Select “Max” if model (13.3) is used.

To automate the computation, we remove the statement $\text{Range}(\text{“J”} \& i+1)=\text{Range}(\text{“F19”})$ from the procedure “RTS”, and name the new procedure “MPSS”.

```

Sub MPSS()
  Dim i As Integer
  For i = 1 To 15
  'set the value of cell E18 equal to i (1, 2,..., 15)
    Range("E18") = i
  'Run the Solver model. The UserFinish is set to True so that
  'the Solver Results dialog box will not be shown
    SolverSolve UserFinish:=True
  'Place the sum of lambdas into column K
    Range("K" & i + 1) = Range("B25")
  Next i
End Sub
    
```

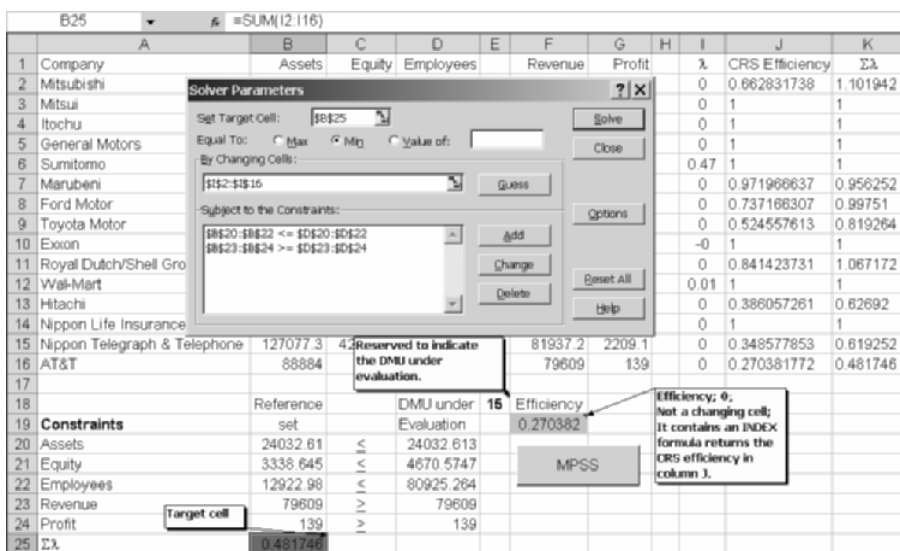


Figure 13.8. Largest MPSS Spreadsheet Model

It can be seen that the maximum $\sum \lambda_j^*$ is the same as that obtained from the input-oriented CRS envelopment model shown in Figure 13.3. This is due to the fact that we have unique optimal solutions on λ_j^* . As a result, minimum $\sum \hat{\lambda}_j^* = \text{maximum } \sum \lambda_j^*$. We can apply (13.2) or (13.4) to obtain the scale efficient targets for the 15 DMUs.

13.5 Solving DEA Using DEA Frontier Software

RTS Estimation can be found at the Returns-to-Scale menu item, as shown in Figure 13.9.

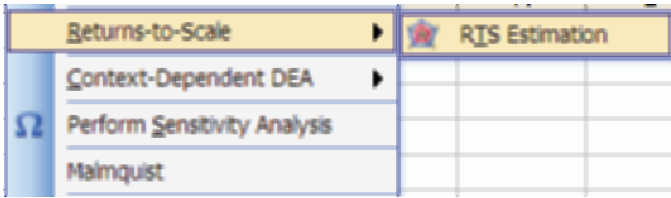


Figure 13.9. Returns-to-Scale Menu

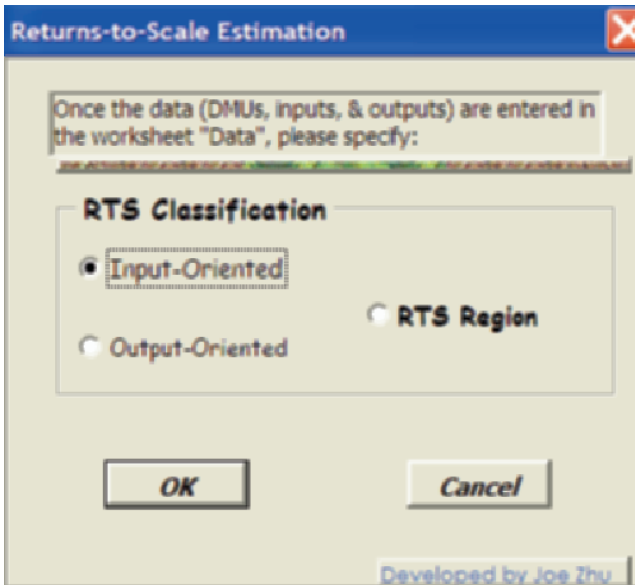


Figure 13.10. RTS Estimation

The RTS Estimation menu will provide (i) the RTS classifications, and (ii) RTS regions as shown in Figure 13.2. (see Figure 13.10).

If RTS Region is selected, the software will run both the input-oriented and output-oriented envelopment models. The results are reported in the “RTS Region” sheet.

If Input-Oriented is selected, then the software will generate the RTS classification based upon the input-oriented envelopment models and report the results in the sheet “RTS Report”. If Output-Oriented is selected, then the software will generate the RTS classification based upon the output-oriented envelopment models.

REFERENCES

1. Banker, R.D. (1984), Estimating most productive scale size using data envelopment analysis, *European Journal of Operational Research*, 17, 35-44.
2. Banker, R.D., A. Charnes and W.W. Cooper (1984), Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, 30, 1078-1092.
3. Färe, R., S. Grosskopf and C.A.K. Lovell (1994), *Production Frontiers*. Cambridge University Press.
4. Gregoriou, G.N. and J. Zhu (2005), *Evaluating Hedge Fund and CTA Performance: Data Envelopment Analysis Approach*, John Wiley & Sons, New York.
5. Seiford, L.M. and J. Zhu (1999a), An investigation of returns to scale under data envelopment analysis, *OMEGA*, 27, No. 1, 1-11.
6. Seiford, L.M. and J. Zhu (1999b), Sensitivity and stability of the classification of returns to scale in data envelopment analysis, *Journal of Productivity Analysis*, 12, No. 1, 55-75.
7. Seiford, L.M. and J. Zhu (1999c), Profitability and marketability of the top 55 US commercial banks, *Management Science*, 45, No. 9, 1270-1288.
8. Seiford, L.M. and J. Zhu (2005), Notes on sensitivity and stability of the classifications of returns to scale in data envelopment analysis: A comment, *Journal of Productivity Analysis*, Vol. 23, No. 3, 315-316.
9. Zhu, J. (2000a), Further discussion on DEA and linear production function, *European Journal of Operational Research*, 127, No. 3, 611-618.
10. Zhu, J. (2000b), Setting scale efficient targets in DEA via returns to scale estimation method, *Journal of Operational Research Society*, 51, No. 3, 376-378.

Chapter 14

DEA Models for Two-Stage Processes

14.1 Introduction

While the definition of a DMU is generic and DMUs can be in various forms such as hospitals, products, universities, cities, courts, business firms, and others, DMUs can have a two-stage structure in many cases. For example, banks use labor and assets to generate deposits which are in turn used to generate loan incomes. Seiford and Zhu (1999) use a two-stage process to measure the profitability and marketability of US commercial banks. In their study, profitability is measured using labor and assets as inputs, and the outputs are profits and revenue. In the second stage for marketability, the profits and revenue are then used as inputs, while market value, returns and earnings per share are used as outputs. Chilingerian and Sherman (2004) describe another two-stage process in measuring physician care. Their first stage is a manager-controlled process with inputs including registered nurses, medical supplies, and capital and fixed costs. These inputs generate the outputs or intermediate measures (inputs to the second stage), including patient days, quality of treatment, drug dispensed, among others. The outputs of the second (physician controlled) stage include research grants, quality of patients, and quantity of individuals trained, by specialty.

In these settings, a DMU represents a two-stage process and intermediate measures exist in-between the two stages. The first stage uses inputs to generate outputs which become the inputs to the second stage. The first stage outputs are therefore called intermediate measures. The second stage then uses these intermediate measures to produce outputs. A key feature here is that the first stage's outputs are the only inputs to the second stage. i.e., in addition to the intermediate measures, the first stage does not have its own outputs and the second stage does not have its own inputs.

These two-stage processes are different from the supply chains discussed in chapter 8 where the second stage also has its own independent inputs. An usual attempt to deal with such two-stage processes is to apply the standard DEA model to each stage (see, e.g., Seiford and Zhu (1999)). However, as noted in chapter 8 and Chen and Zhu (2004), such an approach may conclude that two inefficient stages lead to an overall efficient DMU with the inputs of the first stage and outputs of the second stage. Consequently, improvement to the DEA frontier can be distorted. i.e., the performance improvement of one stage affects the efficiency status of the other, because of the presence of intermediate measures.

Based upon the variable returns to scale (VRS) envelopment model, Chen and Zhu (2004) develop a linear DEA type model where each stage's efficiency is defined on its own production possibility set. The two production possibility sets are linked with the intermediate measures which are set as decision variables for each DMU under evaluation. Chen and Zhu's (2004) model guarantees an overall efficient two-stage process when each stage is efficient. For inefficient DMUs, Chen and Zhu's (2004) model provides a DEA projection with a set of optimal intermediate measures.

Kao and Hwang (2008), on the other hand, modify the standard DEA model by taking into account the series relationship of the two stages within the whole process. Under their framework, the efficiency of the whole process can be decomposed into the product of the efficiencies of the two sub-processes. Yet, their approach cannot be directly applied to the VRS assumption.

Chen, Liang and Zhu (2008) develop an equivalence between Chen and Zhu (2004) and Kao and Hwang (2008) under the condition of constant returns to scale (CRS).

The current chapter presents the models of Chen and Zhu (2008) and Kao and Hwang (2008) and their relations.

14.2 VRS Two-Stage Model

Chen and Zhu (2004) consider the indirect impact of information technology (IT) on firm performance where IT directly impacts certain intermediate measures which in turn are transformed to realize firm performance. Figure 14.1 describes the indirect impact of IT on firm performance where the first stage uses inputs x_i ($i=1, \dots, m$) to produce outputs z_d ($d=1, \dots, D$), and then these z_d are used as inputs in the second stage to produce outputs y_r ($r=1, \dots, s$). It can be seen that z_d (intermediate measures) are outputs in stage 1 and inputs in stage 2. The first stage is viewed as an IT-related value-added activity where deposit is

generated and then used as the input to the second stage where revenue is generated.

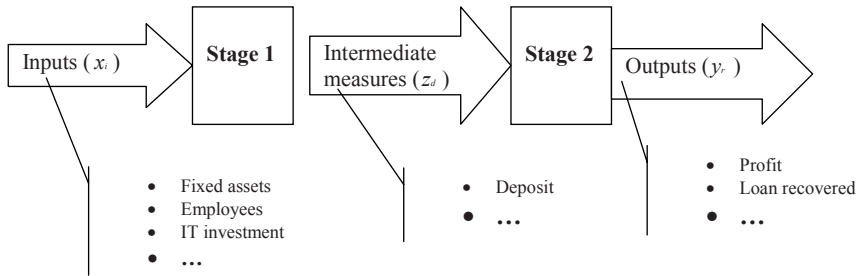


Figure 14.1. IT Impact on Firm Performance

Based upon the VRS envelopment model, Chen and Zhu (2004) develop the following model

$$\begin{aligned}
 & \min_{\alpha, \beta, \lambda_j, \mu_j, \tilde{z}} w_1 \alpha - w_2 \beta \\
 & \text{subject to} \\
 & \text{(stage 1)} \\
 & \sum_{j=1}^n \lambda_j x_{ij} \leq \alpha x_{ij_0} \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j z_{dj} \geq \tilde{z}_{dj_0} \quad d = 1, \dots, D \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & \alpha \leq 1 \\
 & \text{(stage 2)} \\
 & \sum_{j=1}^n \mu_j z_{dj} \leq \tilde{z}_{dj_0} \quad d = 1, \dots, D \\
 & \sum_{j=1}^n \mu_j y_{rj} \geq \beta y_{rj_0} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \mu_j = 1 \\
 & \mu_j \geq 0, \quad j = 1, \dots, n \\
 & \beta \geq 1
 \end{aligned} \tag{14.1}$$

where w_1 and w_2 are user-specified weights reflecting the preference over the two stages' performance, and symbol “ \sim ” represents unknown decision variables.

The rationale of model (14.1) is as follows: (i) when we evaluate the impact of IT investment on the intermediate measures, we want to minimize the input usage given the intermediate measures. For example, given the deposits generated, our objective is to examine whether a bank can reduce its input consumption (including IT investment) compared to the best practice, and (ii) when we evaluate the firm performance as a result of the intermediate measures, we want to maximize the performance given the intermediate measures. For example, given the deposits it generated, our objective is to examine whether a bank can increase its profit. Model (14.1) characterizes the indirect impact of IT on firm performance in a single linear programming problem.

If $\alpha^* = \beta^* = 1$, the two-stage achieves efficient performance when the two-stage process is viewed as a whole.

If $\alpha^* = 1$ and $\beta^* > 1$ (or $\alpha^* < 1$ and $\beta^* = 1$), then model (14.1) indicates that one of the stages can achieve 100% efficiency given a set of optimal intermediate measures.

A DMU must be a frontier point in both stages with respect to $\alpha^* x_{ij_o}$ ($i = 1, \dots, m$), $\tilde{z}_{dj_o}^*$ ($d = 1, \dots, D$), and $\beta^* y_{rj_o}$ ($r = 1, \dots, s$), where (*) represents optimal value in model (14.1).

In model (14.1), the intermediate measures for a specific DMU_o under evaluation are set as unknown decision variables, \tilde{z}_{dj_o} . As a result, additional constraints can be imposed on the intermediate measures. This can further help in correctly characterizing the indirect impact of IT on firm performance.

To illustrate model (14.1), Figure 14.2 shows the spreadsheet model of (14.1) with the data in Table 8.1 (chapter 8). Since the intermediate measures are set as decision variables, cell E6 is reserved to represent the Revenue variable. Cell D7 indicates the DMU under evaluation. Cells E8 and F8 represent α and β , respectively. Cell G8 is the objective function and contains the formula “=E8-F8”.

The changing cells are cells H2:H4, cells I2:I4, cells E8:F8, and cell E6. The formulas for cells B9:B15 are

Cell B9 =SUMPRODUCT(B2:B4,H2:H4)
 Cell B10 =SUMPRODUCT(C2:C4,H2:H4)
 Cell B11 =SUMPRODUCT(E2:E4,H2:H4)
 Cell B12 =SUMPRODUCT(E2:E4,I2:I4)
 Cell B13 =SUMPRODUCT(G2:G4,I2:I4)

Cell B14 =SUM(H2:H4)

Cell B15 =SUM(I2:I4)

| | A | B | C | D | E | F | G | H | I | J | K | L |
|----|------------------|----------------|--------|---------|----------|---------|----------|-----------|-------|----------|---------|-----------------|
| 1 | Supply Chain | Shipping costs | Labor | | Revenue | | Profit | λ | μ | α | β | Optimal Revenue |
| 2 | A | 7 | 9 | | 4 | | 16 | 0.14925 | 0 | 1 | 1.4375 | 4 |
| 3 | B | 9 | 4 | | 6 | | 14 | 0.85075 | 0 | 1 | 1.6429 | 6 |
| 4 | C | 11 | 6 | | 3 | | 23 | 0 | 1 | 0.79104 | 1 | 3 |
| 5 | | | | | | | | | | | | |
| 6 | | | | | 3 | | | | | | | |
| 7 | | | | | α | β | | | | | | |
| 8 | Constraints | | | | 0.79104 | 1 | 80.20898 | | | | | |
| 9 | Shipping costs | 8.701492537 | \leq | 8.70149 | | | | | | | | |
| 10 | Labor | 4.746268657 | \leq | 4.74627 | | | | | | | | |
| 11 | Revenue | 5.701492537 | \geq | 3 | | | | | | | | |
| 12 | Revenue | 3 | \leq | 3 | | | | | | | | |
| 13 | Profit | 23 | \geq | 23 | | | | | | | | |
| 14 | $\Sigma \lambda$ | 1 | $=$ | 1 | | | | | | | | |
| 15 | $\Sigma \mu$ | 1 | $=$ | 1 | | | | | | | | |

Figure 14.2. Spreadsheet Model for Model (14.1)

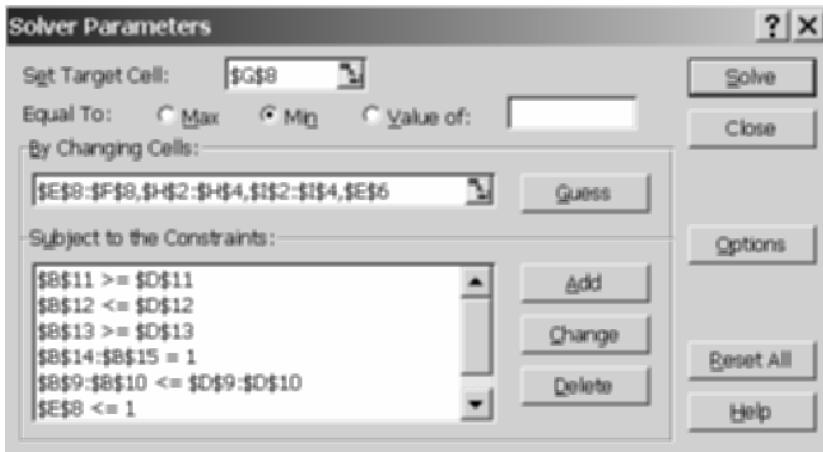


Figure 14.3. Solver Parameters for Model (14.1)

The formulas for cells D9:D13 are

Cell D9 =E8*INDEX(B2:B4,D7,1)

Cell D10 =E8*INDEX(C2:C4,D7,1)

Cell D11 =E6

Cell D12 =E6

Cell D13 =F8*INDEX(G2:G4,D7,1)

Figure 14.3 shows the Solver parameters for the spreadsheet model shown in Figure 14.2. We have “\$E\$8 <=1” and “\$F\$8 >=1” in the Constraints, representing $\alpha \leq 1$ and $\beta \geq 1$, respectively.

In Figure 14.2, cells J2:K4 record the efficiency scores. Cells L2:L4 report the optimal values on Revenue. Since optimal Revenue values are equal to the original values, α^* and β^* must be equal to the θ^* and ϕ^* of the VRS envelopment models, respectively. In this case, multiple optimal solutions on Revenue exist. (Click the “Run” button several time, you may get a set of different optimal values on Revenue.)

We next apply model (14.1) to 15 DMUs in a data set used by Wang, Gopal and Zionts (1997) which consists observations on 22 firms in the banking industry in the years 1987 – 1989¹. The data can be found in the file “two stage spreadsheet.xls”. The inputs, intermediate measure and outputs are given in the Figure 14.1.

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|----|------------------|--------------|-----------|----------------|----|----------|---------|---|--------|----------------|-----------|---------|----------|---------|
| 1 | DMU | Fixed assets | IT budget | # of employees | | Deposits | | | Profit | load recovered | λ | μ | α | β |
| 2 | 1 | 0.713 | 0.15 | 13.3 | | 14.478 | | | 0.232 | 0.986 | 0.2322 | 0 | 1 | 1 |
| 3 | 2 | 1.071 | 0.17 | 16.9 | | 19.502 | | | 0.34 | 0.986 | 0 | 0 | 0.94516 | 1 |
| 4 | 3 | 1.224 | 0.235 | 24 | | 20.952 | | | 0.363 | 0.986 | 0 | 0.71632 | 0.71639 | 1 |
| 5 | 4 | 0.963 | 0.211 | 15.6 | | 13.902 | | | 0.211 | 0.982 | 0 | 0 | 1 | 1.00388 |
| 13 | 12 | 0.874 | 0.05 | 13.1 | | 14.342 | | | 0.177 | 0.985 | 0 | 0 | 1 | 1.00097 |
| 14 | 13 | 6.918 | 0.37 | 12.5 | | 32.491 | | | 0.648 | 0.945 | 0.3898 | 0.03809 | 1 | 1 |
| 15 | 14 | 4.432 | 0.44 | 41.9 | | 47.653 | | | 0.639 | 0.979 | 0 | 0 | 0.60513 | 1 |
| 16 | 15 | 4.504 | 0.431 | 41.1 | | 52.63 | | | 0.741 | 0.981 | 0 | 0 | 0.71259 | 1 |
| 17 | | | | | | | | | | | | | | |
| 18 | | | | | | 46.7158 | | | | | | | | |
| 19 | | | | | 15 | α | β | | | | | | | |
| 20 | Constraints | | | | | 0.71259 | 1 | | 0.2874 | | | | | |
| 21 | Fixed assets | 3.2095005 | ≤ | 3.2095005 | | | | | | | | | | |
| 22 | IT budget | 0.20174686 | ≤ | 0.3071258 | | | | | | | | | | |
| 23 | # of employees | 29.2874047 | ≤ | 29.287405 | | | | | | | | | | |
| 24 | | 0 | | | | | IT | | | | | | | |
| 25 | Deposits | 46.7158495 | ≥ | 46.71585 | | | | | | | | | | |
| 26 | Deposits | 46.7158495 | ≤ | 46.71585 | | | | | | | | | | |
| 27 | | | | | | | | | | | | | | |
| 28 | Profit | 0.741 | ≥ | 0.741 | | | | | | | | | | |
| 29 | load recovered | 0.981 | ≥ | 0.981 | | | | | | | | | | |
| 30 | $\Sigma \lambda$ | 1 | = | 1 | | | | | | | | | | |
| 31 | $\Sigma \mu$ | 1 | = | 1 | | | | | | | | | | |

Figure 14.4. IT Spreadsheet Model

Figure 14.4 shows the spreadsheet. Range names are used in the spreadsheet shown in Figure 14.4. Cells B2:D16 are named as “Stage1Input”. Cells F2:F16 are named as “Intermediate”. Cells I2:J16 are named as “Stage2Output”. These cells represent the performance measures for the 15 banks.

Cells K2:K16 are named as “Lambdas” and cells L2:L16 are named as “Mus”. These cells are changing cells in the Solver parameters. Other changing cells include cell F19 – “Deposits”, representing the decision

¹ There are 36 observations in Wang, Gopal and Zionts (1997). The data on IT budgets are obtained from the annual survey by Computer World on top 100 effective users of information systems. The data on the percentage of loans recovered and the dollar value of deposits are generated from Standard and Poor’s Industry Surveys. The remaining data are obtained from the Compustat database.

variables for the intermediate measures, cell F20 – “Efficiency1”, and cell G20 – “Efficiency2”, representing α and β in model (14.1).

Cell I20 is the objective function of model (14.1). It contains the formula “=Efficiency1-Efficiency2”, i.e., “=F20-G20”. Cell I20 is named as “Efficiency”.

Based upon these range names and the related cells, we have the formulas for the constraints

Cell B21 =SUMPRODUCT(Lambdas,INDEX(Stage1Inputs,0,1))

Cell B22 =SUMPRODUCT(Lambdas,INDEX(Stage1Inputs,0,2))

Cell B23 =SUMPRODUCT(Lambdas,INDEX(Stage1Inputs,0,3))

Cell B25 =SUMPRODUCT(Lambdas,INDEX(Intermediate,0,1))

Cell B26 =SUMPRODUCT(Mus,INDEX(Intermediate,0,1))

Cell B28 =SUMPRODUCT(Mus,INDEX(Stage2Outputs,0,1))

Cell B29 =SUMPRODUCT(Mus,INDEX(Stage2Outputs,0,2))

Cell B30 =SUM(Lambdas)

Cell B31 =SUM(Mus)

Cell D21 =Efficiency1*INDEX(Stage1Inputs,E19,1)

Cell D22 =Efficiency1*INDEX(Stage1Inputs,E19,2)

Cell D23 =Efficiency1*INDEX(Stage1Inputs,E19,3)

Cell D25 =Deposits

Cell D26 =Deposits

Cell D28 =Efficiency2*INDEX(Stage2Outputs,E19,1)

Cell D29 =Efficiency2*INDEX(Stage2Outputs,E19,2)

We then apply the following range names to the constraints

Cells B21:B23 – ReferenceSetInput

Cells B25 – ReferenceSetInter1

Cells B26 – ReferenceSetInter2

Cells B28:B29 – ReferenceSetOutput

Cell B30 – SumLambda

Cell B31 – SumMu

Cells D21:D23 – DMUInput

Cells D25 – DMUInter1

Cells B26 – DMUInter2

Cells D28:D29 – DMUOutput

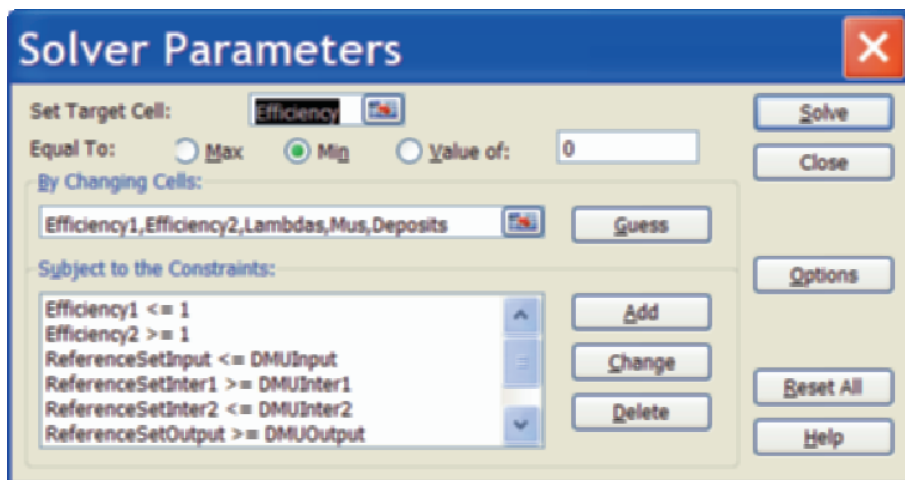


Figure 14.5. Solver Parameters for IT Spreadsheet

Figure 15.5 shows the Solver Parameters. (“SumLambda=1” and “SumMu=1” are not shown in Figure 8.5.) The calculation is performed by the following VBA procedure

```

Sub IT()
Dim i As Integer
For i = 1 To 15
Range("E19") = i
SolverSolve UserFinish:=True
'Place the efficiency into column J and column K
Range("M" & i + 1) = Range("Efficiency1")
Range("N" & i + 1) = Range("Efficiency2")
Next i
End Sub

```

14.3 CRS Two-Stage Model

Based upon the CRS DEA model, the efficiency scores of the two-stage process and the two individual stages can be expressed as

$$\theta_j = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}, \quad \theta_j^1 = \frac{\sum_{d=1}^D w_d z_{dj}}{\sum_{i=1}^m v_i x_{ij}} \quad \text{and} \quad \theta_j^2 = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{d=1}^D \tilde{w}_d z_{dj}} \quad (14.2)$$

where v_i, w_d, \tilde{w}_d , and u_r are unknown non-negative weights. Note that w_d can be equal to \tilde{w}_d .

Note that the intermediate measures of z_{dj} do not appear in θ_j . Kao and Hwang (2008) assume that $w_d = \tilde{w}_d$. As a result, for a specific DMU_{j_0} ,

$$\theta_{j_0}^1 \bullet \theta_{j_0}^2 \text{ becomes } \frac{\sum_{r=1}^s u_r y_{rj_0}}{\sum_{i=1}^m v_i x_{ij_0}} \text{ which is the overall efficiency defined in the}$$

Kao and Hwang (2008) model:

$$\begin{aligned} \text{Max } \theta_o^1 \bullet \theta_o^2 &= \frac{\sum_{r=1}^s u_r y_{rj_0}}{\sum_{i=1}^m v_i x_{ij_0}} \\ \text{s.t. } \theta_j^1 &\leq 1 \text{ and } \theta_j^2 \leq 1 \text{ for all } j & (14.3) \\ w_d &= \tilde{w}_d \text{ for all } d \end{aligned}$$

Model (14.3) indicates an efficiency decomposition. That is, efficiency of the whole process can be decomposed into the product of the efficiencies of the two sub-processes

Note that model (14.3) is an input-oriented model. The equivalent output-oriented model can be expressed as

$$\begin{aligned} \text{Min } &\frac{\sum_{i=1}^m v_i x_{ij_0}}{\sum_{r=1}^s u_r y_{rj_0}} \\ \text{s.t. } \theta_j^1 &\leq 1 \text{ and } \theta_j^2 \leq 1 \text{ for all } j & (14.4) \\ w_d &= \tilde{w}_d \text{ for all } d \end{aligned}$$

Model (14.4) is equivalent to the following linear program

$$\begin{aligned} \text{Min } &\sum_{i=1}^m v_i x_{ij_0} \\ \text{s.t. } &\sum_{r=1}^s u_r y_{rj} - \sum_{d=1}^D w_d z_{dj} \leq 0 \quad j = 1, 2, \dots, n \\ &\sum_{d=1}^D w_d z_{dj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad j = 1, 2, \dots, n & (14.5) \end{aligned}$$

$$\sum_{r=1}^s u_r y_{rj_0} = 1$$

$$w_d, d = 1, 2, \dots, D; v_i, i = 1, 2, \dots, m; u_r, r = 1, 2, \dots, s \geq 0$$

The above model is an output-oriented version of Kao and Hwang’s (2008) model.

We next present the Chen and Zhu (2004) under the CRS assumption by removing the convexity constraints of $\sum \lambda_j = \sum \mu_j = 1$. We obtain

$$\min_{\alpha, \beta, \lambda_j, \mu_j, \tilde{z}} \alpha - \beta$$

subject to

(stage 1)

$$\sum_{j=1}^n \lambda_j x_{ij} \leq \alpha x_{ij_0} \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \lambda_j z_{dj} \geq \tilde{z}_{dj_0} \quad d = 1, \dots, D$$

$$\lambda_j \geq 0, \quad j = 1, \dots, n$$

$$\alpha \leq 1$$

(stage 2)

$$\sum_{j=1}^n \mu_j z_{dj} \leq \tilde{z}_{dj_0} \quad d = 1, \dots, D$$

$$\sum_{j=1}^n \mu_j y_{rj} \geq \beta y_{rj_0} \quad r = 1, \dots, s$$

$$\mu_j \geq 0, \quad j = 1, \dots, n$$

$$\beta \geq 1$$

(14.6)

Chen, Liang and Zhu (2008) show (i) that model (14.6) is equivalent to the Kao and Hwang (2008) model (14.5), and (ii) at optimality $\alpha^* = 1$ and β^* is the optimal value to model (14.5), representing the overall efficiency.

To establish the equivalence between models (14.5) and (14.6), Chen, Liang and Zhu (2008) first consider the following linear program

$$\begin{aligned}
& \min_{\alpha, \beta, \lambda_j, \mu_j} \alpha - \beta \\
& \text{subject to} \\
& \sum_{j=1}^n \lambda_j x_{ij} \leq \alpha x_{ij_0} \quad i = 1, \dots, m \\
& \sum_{j=1}^n \mu_j y_{rj} \geq \beta y_{rj_0} \quad r = 1, 2, \dots, s \\
& \sum_{j=1}^n (\lambda_j - \mu_j) z_{dj} \geq 0 \quad d = 1, 2, \dots, D \\
& \lambda_j, \mu_j \geq 0, j = 1, 2, \dots, n \\
& \alpha \leq 1, \beta \geq 1
\end{aligned} \tag{14.7}$$

Chen, Liang and Zhu (2008) then show that model (14.7)'s optimal solutions are optimal in model (5).

Let $\lambda'_j = \frac{\lambda_j}{\alpha}$, $\mu'_j = \frac{\mu_j}{\alpha}$, and $\sigma = \beta / \alpha$, then model (14.7) becomes

$$\begin{aligned}
& \text{Max } \alpha(\sigma - 1) \\
& \text{s.t. } \sum \lambda'_j x_{ij} \leq x_{ij_0} \\
& \sum \mu'_j y_{rj} \geq \sigma y_{rj_0} \\
& \sum (\lambda'_j - \mu'_j) z_{dj} \geq 0 \\
& \lambda'_j, \mu'_j \geq 0, \alpha \leq 1, \alpha\sigma \geq 1
\end{aligned} \tag{14.8}$$

Next, consider the following model

$$\begin{aligned}
 & \text{Max } \alpha(\sigma - 1) \\
 & \text{s.t. } \sum \lambda'_j x_{ij} \leq x_{ij_0} \\
 & \sum \mu'_j y_{rj} \geq \sigma y_{rj_0} \\
 & \sum (\lambda'_j - \mu'_j) z_{dj} \geq 0 \\
 & \lambda'_j, \mu'_j \geq 0, \alpha \leq 1, \sigma \geq 1
 \end{aligned} \tag{14.9}$$

The only difference between models (14.8) and (14.9) is that model (14.9) sets $\alpha = 1$ in the constraint of $\alpha\sigma \geq 1$ in model (14.8). We have Model (14.9) optimal solutions are optimal in model (14.8) (see Chen, Liang and Zhu (2008).)

Note that $\alpha \leq 1$ does not appear in other constraints of model (14.9). Therefore, at optimality, $\alpha^* = 1$ and model (14.9) is equivalent to the following linear program

$$\begin{aligned}
 & \text{Max } \sigma \\
 & \text{s.t. } \sum \lambda'_j x_{ij} \leq x_{ij_0} \quad i = 1, 2, \dots, m \\
 & \sum \mu'_j y_{rj} \geq \sigma y_{rj_0} \quad r = 1, 2, \dots, m \\
 & \sum (\lambda'_j - \mu'_j) z_{dj} \geq 0 \quad d = 1, 2, \dots, D \\
 & \lambda'_j, \mu'_j \geq 0, \sigma \geq 1
 \end{aligned} \tag{14.10}$$

Model (14.10) is actually the dual to the model (14.5). Therefore, we have that at optimality $\alpha^* = 1$ and $\beta^* = \sigma^*$, where α^* and β^* are optimal values of α and β in model (14.6) and σ^* is the optimal value of σ in model (14.10).

This further indicates that the optimal α^* and β^* in model (14.6) do not represent the efficiency scores of individual stages under the CRS condition. In fact, α^* is always equal to unity and β^* represents the overall efficiency of the two-stage process. i.e., model (14.5) can be used to measure the overall efficiency of the two-stage process under the CRS condition.

We finally note that under the VRS condition $\theta_{j_0}^1 \bullet \theta_{j_0}^2$ no longer equals

to $\frac{\sum_{r=1}^s u_r y_{rj_0}}{\sum_{i=1}^m v_i x_{ij_0}}$, because of the free variable in the related DEA model. As a

result, the VRS version of Kao and Hwang’s (2008) model cannot be modeled as in model (14.3). The proven equivalence between the two approaches sheds lights on possible ways to developing the VRS version of Kao and Hwang’s (2008) model.

14.4 Solving DEA Using DEA Frontier Software

Since intermediate measures are present, the DMUs in the data sheet are set in a format shown in Figure 14.6. The inputs are entered first and followed by a blank column, and then the intermediate measures are entered followed by a blank column and the outputs.

Select the “Two Stage (VRS)” menu item to calculate the model (14.1). The results are reported in the “Efficiency1” (for stage 1), “Efficiency2” (for stage 2) and “Intermediate” (for optimal intermediate measures) sheets.

| | A | B | C | D | E | F | G | H | I |
|----|-----|------------|-----------|----------------|---|----------|---|--------|----------------|
| 1 | DMU | Fixed asse | IT budget | # of employees | | Deposits | | Profit | load recovered |
| 2 | 1 | 0.713 | 0.15 | 13.3 | | 14.478 | | 0.232 | 0.986 |
| 3 | 2 | 1.071 | 0.17 | 16.9 | | 19.502 | | 0.34 | 0.986 |
| 4 | 3 | 1.224 | 0.235 | 24 | | 20.952 | | 0.363 | 0.986 |
| 5 | 4 | 0.363 | 0.211 | 15.6 | | 13.902 | | 0.211 | 0.982 |
| 6 | 5 | 0.409 | 0.133 | 18.485 | | 15.206 | | 0.237 | 0.984 |
| 7 | 6 | 5.846 | 0.497 | 56.42 | | 81.186 | | 1.103 | 0.955 |
| 8 | 7 | 0.918 | 0.06 | 56.42 | | 81.186 | | 1.103 | 0.986 |
| 9 | 8 | 1.235 | 0.071 | 12 | | 11.441 | | 0.199 | 0.985 |
| 10 | 9 | 18.12 | 1.5 | 89.51 | | 124.072 | | 1.858 | 0.972 |
| 11 | 10 | 1.821 | 0.12 | 19.8 | | 17.425 | | 0.274 | 0.983 |

Figure 14.6. Data Sheet Format For Two-Stage

REFERENCES

1. Chen, Y., L. Liang and J. Zhu (2008), Equivalence in two-stage DEA approaches, *European Journal of Operational Research*, (in press)
2. Chen, Y. and J. Zhu, (2004), Measuring information technology's indirect impact on firm performance, *Information Technology & Management Journal*, Vol. 5, Issue 1-2, 9-22.
3. Chilingirian, J.A. and H.D. Sherman (2004), Health care applications: from hospitals to physicians, from productive efficiency to quality frontiers, in *Handbook on Data Envelopment Analysis*, Chapter 17, eds W.W. Cooper, L.M. Seiford and J. Zhu, Kluwer Academic Publishers, Boston.
4. Kao, C. and S-N. Hwang (2008), Efficiency decomposition in two-stage data envelopment analysis: an application to non-life insurance companies in Taiwan, *European Journal of Operational Research*. Vol. 185, No. 1, 418-429.
5. Seiford, L.M. and J. Zhu (1999), Profitability and marketability of the top 55 US commercial banks, *Management Science*, 45, No. 9, 1270-1288.

Author Index

A

Ahn, 227, 266
Ali, 52, 61, 98, 105, 107
Andersen, 205, 225
Anderson, 4, 41

B

Banker, 4, 5, 41, 63, 76, 289, 299, 303
Barr, 4, 41
Belton, 86, 95
Billington, 163, 186
Brockett, 195, 196, 197, 203
Byrnes, 190, 203

C

Camp, 2, 41, 131, 160
Charnes, 4, 5, 41, 44, 51, 61, 85, 86, 95, 182, 205, 212, 225, 227, 228, 245, 257, 258, 259, 260, 266, 289, 303

Chen, 86, 95, 161, 186, 208, 209, 269, 272, 287, 306, 307, 314, 315, 316, 318
Cheung, 164, 186
Chilingerian, 4, 41, 305, 318
Cook, 4, 5, 41, 42, 132, 143, 160, 174, 186
Cooper, 4, 5, 41, 42, 51, 61, 85, 86, 91, 95, 182, 187, 195, 196, 197, 203, 212, 225, 227, 228, 266, 289, 303, 318

D

Desai, 4, 42, 100, 107, 153, 154, 160
Dharmapala, 228, 266
Doyle, 86, 95

F

Färe, 4, 42, 77, 95, 161, 186, 187, 188, 190, 192, 195, 197, 203, 289, 303

G

Golany, 51, 61, 85, 95, 161, 186,
228, 266
Green, 86, 95
Griffin, 165
Grosskopf, 4, 42, 161, 186, 187,
188, 190, 192, 195, 197, 203,
289, 303

H

Haag, 205, 225, 227, 228, 245,
259, 260, 266
Hausman, 164

J

Jaska, 205, 225, 227, 228, 245,
259, 260, 266
Johnson, 4, 42
Joro, 86, 95

K

Kazakov, 4, 41
Kemerer, 4, 41
Kleinsorge, 4, 42
Korhonen, 86, 95

L

Lee, 61, 163, 186
Lerme, 52, 61
Lewin, 4, 41, 227, 266
Li, 174, 182, 186, 228, 266
Logan, 4, 42
Lovell, 4, 42, 77, 95, 98, 107,
188, 192, 195, 197, 203, 289,
303

M

Malmquist, 37
Morey, 63, 76, 227, 266
Morrison, 153

N

Neralic, 227, 229, 266

P

Parlar, 165
Pastor, 98, 107
Petersen, 205, 225

R

Ray, 4, 42, 195, 203, 290
Rhodes, 4, 41, 86, 95
Roll, 4, 41
Rousseau, 227, 228, 257, 258,
266

S

Schary, 4, 42
Seiford, 4, 5, 41, 42, 1, 51, 52, 61,
82, 85, 90, 91, 95, 97, 98, 105,
107, 110, 129, 132, 135, 143,
160, 161, 163, 186, 195, 203,
205, 213, 225, 227, 228, 229,
233, 252, 253, 266, 267, 289,
293, 303, 305, 306, 318
Semple, 205, 225, 227, 228, 245,
257, 258, 259, 260, 266
Sharp, 4, 41
Sherman, 2, 4, 41, 42, 208, 305,
318
Shin, 195, 196, 197, 203
Siems, 4, 41

Simonson, 109, 129
Stewart, 86
Storbeck, 4, 42
Stutz, 51, 61, 85, 95, 228, 266

T

Tanner, 4, 42
Thomas, 165
Thompson, 44, 61, 187, 203, 228, 266
Thrall, 1, 61, 187, 203, 205, 211, 212, 213, 225, 228, 266
Tone, 5, 42, 91, 95
Tversky, 109, 129

V

Vickers, 86, 95

W

Wallenius, 86, 95

Wang, 195, 196, 197, 203, 310
Weber, 100, 107, 153, 154, 160
Weng, 165
Wilkens, 105, 107, 270, 287
Wilson, 205, 225
Wise, 153

Z

Zhu, 4, 5, 41, 42, 44, 52, 61, 66, 68, 75, 76, 79, 80, 82, 90, 95, 97, 98, 105, 107, 110, 129, 132, 135, 143, 154, 160, 161, 163, 174, 186, 195, 203, 205, 208, 212, 213, 225, 227, 228, 229, 230, 232, 233, 234, 240, 246, 252, 253, 266, 267, 269, 270, 272, 287, 289, 291, 293, 299, 303, 305, 306, 307, 314, 315, 316, 318

Topic Index

A

a priori, 2, 4, 52, 63
analytic hierarchy process (AHP),
44, 80, 131
assume linear model, 19
assume non-negative, 19, 47
assuming linear model, 19
attractiveness, 109, 110, 115, 117,
118, 119, 123, 124, 125, 126,
128, 129
automatic scaling, 19

B

bank, 2, 308
benchmark, 2, 3, 6, 13, 24, 37, 40,
72, 73, 74, 106, 107, 131, 132,
133, 134, 135, 136, 137, 138,
139, 140, 141, 142, 143, 144,
145, 146, 148, 149, 150, 151,
152, 153, 155, 156, 157, 158,
159, 174, 270, 287
fixed, 144, 147, 148, 149, 150, 151,
153, 158

ideal, 149, 150, 153, 155, 156, 157,
158, 159
share, 72, 73, 74
variable, 132, 133, 134, 137, 138, 140,
141, 142, 143, 158
benchmarking, 1, 2, 5, 41, 73,
131, 134, 138, 139, 140, 141,
142, 143, 146, 148, 154, 158,
160
best practice, 2, 3, 66, 68, 72, 97,
109, 110, 111, 117, 120, 121,
132, 138, 143, 162, 164, 169,
173, 174, 269, 272, 273, 274,
275, 280, 308
business operation, 1, 2, 3
business-to-business (B2B), 153
buyer, 1, 107, 153, 154, 160, 174,
175, 177, 181, 182, 184, 186

C

changing cell, 18
competition, 1
congestion, 187, 188, 190, 191,
192, 193, 194, 195, 196, 197,
199, 200, 201, 202, 203, 213

slack, 196, 199, 200, 201, 202, 213
 continuous improvement, 1
 coordination, 164, 165
 critical measure, 269, 272, 274,
 277, 281, 284

D

data envelopment analysis (DEA),
 4, 41, 42, 95, 109, 266, 303,
 318
 add-in, 146
 context-dependent, 115, 119, 123, 125,
 126, 127, 128, 138
 DEA/PS model, 79, 80, 81, 82, 83, 84,
 86, 89, 91
 Excel Solver, 146, 152
 non-radial, 77, 78, 79, 88, 89
 projection, 13, 121, 306
 stratification, 111, 115, 127
 decision making unit (DMU), 3,
 41, 95
 decision variable, 14, 18, 45, 170,
 306, 308, 311
 desirable, 29, 88, 97, 99, 100,
 110, 121, 280

E

earnings per share (EPS), 66, 69,
 71, 73, 74, 305
 efficiency
 cost, 37, 81, 82, 91, 93
 industry, 70, 72
 minimum, 151, 152, 153, 157, 159
 profit, 38, 91, 93, 94
 radial, 77
 ratio, 146, 147
 relative, 4, 41
 revenue, 38, 79, 82, 83, 91, 93
 scale, 289

super, 109, 135, 205, 206, 208, 209,
 210, 211, 212, 213, 214, 215, 216,
 217, 218, 219, 220, 221, 222, 223,
 227, 228, 229, 234, 246, 247, 248,
 252, 253, 259, 260, 262, 265, 274,
 275

efficient

extreme, 194, 195, 205, 212, 213, 214,
 218, 222, 223, 228, 229, 232, 233,
 234, 235, 251, 252, 253, 254, 255,
 256, 258, 260
 frontier, 2, 3, 4, 5, 8, 10, 11, 29, 79,
 88, 127, 128, 167, 195, 213
 target, 8, 13, 40, 63, 79, 100, 104, 121,
 124, 155, 290, 299, 300, 301, 303
 local, 121, 125
 target
 global, 121
 target
 global, 121
 target
 global, 121
 target
 global, 125
 target
 global, 125
 weakly, 8, 10, 167, 193, 194, 208, 213,
 229, 256

envelopment model

CRS, 11, 28, 29, 34, 79, 86, 88, 111,
 123, 132, 138, 154, 166, 195, 216,
 244, 246, 289, 293, 294, 295, 296,
 299, 300, 301
 NDRS, 218
 NIRS, 222
 VRS, 8, 9, 14, 18, 27, 29, 66, 97, 98,
 99, 101, 102, 155, 162, 187, 191,
 195, 196, 197, 209, 214, 218, 219,
 222, 271, 280, 289, 290, 296, 307,
 310

EOQ, 164

equity, 14, 64, 66, 69, 70, 73, 74,
260, 262

F

flexible manufacturing system
(FMS), 112, 114
format, 38, 127, 158, 317
Fortune, 14, 66, 68, 73, 75, 76,
154, 278, 280, 284

G

gap analysis, 2, 131

I

infeasible (infeasibility), 44, 59,
126, 135, 136, 137, 143, 144,
158, 205, 210, 211, 212, 213,
214, 215, 216, 217, 218, 219,
221, 222, 223, 228, 242, 244,
245, 246, 249, 251, 252, 262,
274, 275, 276, 277, 278, 281,
283
information technology (IT), 306,
307, 308, 310, 312
 impact, 306, 308
intermediate measure, 163, 165,
167, 182, 305, 306, 308, 311,
313, 317
invalid data, 39
invariance
 classification, 97, 98
 solution, 98

J

just-in-time, 154

L

linear programming, 9, 19, 1, 82,
86, 87, 88, 115, 116, 119, 120,
124, 149, 183, 227, 230, 231,
234, 258, 271, 299, 308

M

manufacturer, 161, 162, 163, 164,
165, 166, 168, 169, 170, 173,
174, 182
market value, 66, 280, 281, 305
marketability, 66, 67, 75, 186,
303, 305, 318
measure-specific model, 63, 64,
68, 75, 229, 236, 248, 260
most productive scale size
(MPSS), 299, 303
 largest, 299, 300
 smallest, 300
multiple criteria decision making
(MCDM), 85, 95
multiple objective linear
 programming (MOLP), 85, 86,
88, 89
multiplier model, 1, 44, 48, 57,
143, 271, 289, 293, 294
multiplier model
 CRS spreadsheet, 45
multiplier model
 VRS spreadsheet, 48

N

normalization, 1, 81

O

objective function, 8, 14, 18, 23,
79, 147, 150, 170, 239, 250,
255, 259, 308, 311

P

performance

evaluation, 1, 2, 3, 5, 131, 163, 164,
186, 270, 282

financial, 2, 66, 74, 75

piecewise linear, 3, 4, 6, 271

preference structure, 61, 63, 79,
91, 95

price

input, 81, 91, 92

output, 82, 91, 92

priority, 63, 79, 177, 274, 275,
277

process, 1, 2, 3, 8, 9, 21, 29, 44,
66, 97, 99, 112, 131, 161, 174,
289, 294, 305, 306, 308, 312,
313, 316

productivity, 1, 4, 42, 52, 61, 95,
146

profitability, 66, 67, 75, 305

progress, 110, 115, 119, 120, 121,
122, 123, 124, 125, 128, 129

R

range name, 29, 30, 46, 56, 262,
263, 264, 296, 297, 311

reference set, 13, 14, 15, 19, 30,
110, 118, 193, 205, 206, 207,
208, 209, 213, 230

returns to scale, 8, 163, 186, 289,
301, 302, 303, 306

classification, 213, 289, 290, 291, 292,
293, 296, 297, 302

constant (CRS), 8

decreasing (DRS), 8, 12

estimation, 294, 295

increasing (IRS), 8, 12

region, 290, 291, 297, 302

variable (VRS), 8, 306

VRS method, 291

run Macro, 23, 25, 32, 47

S

seller, 1, 153, 154, 160, 174, 175,
177, 179, 181, 182, 184, 186

sensitivity analysis, 163, 205,
223, 227, 228, 229, 233, 235,
242, 244, 246, 251, 258, 260,
264, 267, 269, 289

slack, 7, 8, 9, 10, 25, 26, 27, 40,
44, 51, 53, 54, 56, 59, 65, 78,
79, 90, 193, 194, 195, 198, 200,
201, 213, 225, 229, 245, 248,
250, 251, 258

DEA, 8, 40, 193, 194, 195, 197, 198,
199, 200, 213

slack-based model

CRS, 52, 53

DEA, 56, 90

VRS, 56

solver

add-in, 16, 21, 22

invoke, 16

parameter, 17, 20

premium, 46, 47

reference, 22

results, 20, 21, 23, 25, 295, 301

solver

Excel, 14

solver

option, 19

solver

option, 19

solver

Excel, 35

solver
 Excel, 35

solver
 Excel, 36

solver
 Excel, 36

solver
 Excel, 36

stability region, 205, 228, 229,
 231, 233, 234, 235, 242, 244

supplier, 161, 162, 163, 164, 166,
 169, 174, 177, 186

supply chain, 1, 2, 4, 5, 161, 162,
 163, 164, 165, 166, 167, 168,
 169, 173, 174, 175, 177, 180,
 181, 182, 184, 185, 186, 306

T

total return to investors (TRI), 66,
 69, 71, 73, 74

tradeoff, 2, 4, 129, 147, 148, 150,
 151, 152, 269, 270, 271, 272

transformations, 1, 66, 97, 98, 99,
 182, 183, 271, 272

U

undesirable, 97, 99, 100, 104, 107

US commercial bank, 186, 303,
 305, 318

V

vector, 86, 87, 88, 99, 101, 135,
 175, 214

visual basic for applications, 16

W

weak disposability, 188, 201

LICENSING AND WARRANTY AGREEMENT

READ THIS: Do not install or use the CD-ROM until you have read and agreed to this agreement. By opening the accompanying software packet, you acknowledge that you have read and accepted the following terms and conditions. If you do not agree and do not want to be bounded by such terms and conditions, do not install or use the software “DEAFrontier”.

License: The copyright to the software in the CD-ROM (the “DEAFrontier”) is owned by the author. The “DEAFontier” is protected by the United States copyright laws and international treaty provisions. No part of the DEA Excel Solver may be reproduced, stored in a retrieval systems, distributed (including but not limited to over the Internet), modified, decompiled, reverse engineered, reconfigured, transmitted, or transcribed, in any form or by any means without the permission of the author. The DEAFontier may not, under any circumstances, be reproduced for sale. This license allows you to use the DEAFontier for educational and research purposes only, not for commercial purposes, including consulting. You may only (i) make one copy of the DEAFontier for backup or archival purposes, or (ii) transfer the DEAFontier to a single hard disk, provided that you keep the original for backup or archival purposes. You may not (i) rent or lease the DEAFontier, (ii) copy or reproduce the DEAFontier through a LAN or other network system or through any computer subscriber system, or (iii) modify, adapt, or create derivative works based upon the DEAFontier. You may be held legally responsible for any copying or copyright infringement which is caused by your failure to abide by the above terms and conditions.

Limited Warranty: The warranty for the media on which the DEA Excel Solver is provided is for sixty (60) days from the original purchase and valid only if the packaging for the DEAFontier was purchased unopened. If the Publisher receives notification within the warranty period of defects in materials or workmanship, the Publisher will replace the defective software media. The Publisher and the author provide no other warranties, expressed or implied, including without limitation implied warranties of merchantability and fitness for a particular purpose, and shall not be liable for any damages, including direct, special, indirect, incidental, consequential, or otherwise. The Publisher does not warrant that the functions contained in the DEAFontier will meet your requirements or that the operation of the DEAFontier will be error-free.

ABOUT THE CD-ROM

This CD contains the Excel files and a student version of DEAFrontier software that can solve up to 100 DMUs.

DEAFontier Installation Instructions

The CD-ROM contains the DEAFontier* which is a DEA Add-In for Microsoft® Excel. There are two versions of this software: one for Excel 97-2003 and the other for Excel 2007. You may follow the following steps to locate the specific version:

Step 1. Insert the CD-ROM into your computer's CD-ROM drive.

Step 2. Launch Windows Explore.

Step 3. Click Browse to browse the CD and find the file "*DEAFontier97.xla*" for Excel 97-2003 and "*DEAFontier2007.xlam*" for Excel 2007. Copy this file to your hard drive

For more information about using the DEAFrontier software, please see Chapter 1.

* May not work on a Macintosh®.

Early Titles in the

**INTERNATIONAL SERIES IN
OPERATIONS RESEARCH & MANAGEMENT SCIENCE**

Frederick S. Hillier, Series Editor, Stanford University

- Saigal/ *A MODERN APPROACH TO LINEAR PROGRAMMING*
Nagurney/ *PROJECTED DYNAMICAL SYSTEMS & VARIATIONAL INEQUALITIES WITH APPLICATIONS*
Padberg & Rijal/ *LOCATION, SCHEDULING, DESIGN AND INTEGER PROGRAMMING*
Vanderbei/ *LINEAR PROGRAMMING*
Jaiswal/ *MILITARY OPERATIONS RESEARCH*
Gal & Greenberg/ *ADVANCES IN SENSITIVITY ANALYSIS & PARAMETRIC PROGRAMMING*
Prabhu/ *FOUNDATIONS OF QUEUEING THEORY*
Fang, Rajasekera & Tsao/ *ENTROPY OPTIMIZATION & MATHEMATICAL PROGRAMMING*
Yu/ *OR IN THE AIRLINE INDUSTRY*
Ho & Tang/ *PRODUCT VARIETY MANAGEMENT*
El-Taha & Stidham/ *SAMPLE-PATH ANALYSIS OF QUEUEING SYSTEMS*
Miettinen/ *NONLINEAR MULTIOBJECTIVE OPTIMIZATION*
Chao & Huntington/ *DESIGNING COMPETITIVE ELECTRICITY MARKETS*
Weglarz/ *PROJECT SCHEDULING: RECENT TRENDS & RESULTS*
Sahin & Polatoglu/ *QUALITY, WARRANTY AND PREVENTIVE MAINTENANCE*
Tavares/ *ADVANCES MODELS FOR PROJECT MANAGEMENT*
Tayur, Ganeshan & Magazine/ *QUANTITATIVE MODELS FOR SUPPLY CHAIN MANAGEMENT*
Weyant, J./ *ENERGY AND ENVIRONMENTAL POLICY MODELING*
Shanthikumar, J.G. & Sumita, U./ *APPLIED PROBABILITY AND STOCHASTIC PROCESSES*
Liu, B. & Esogbue, A.O./ *DECISION CRITERIA AND OPTIMAL INVENTORY PROCESSES*
Gal, T., Stewart, T.J., Hanne, T. / *MULTICRITERIA DECISION MAKING: Advances in MCDM Models, Algorithms, Theory, and Applications*
Fox, B.L. / *STRATEGIES FOR QUASI-MONTE CARLO*
Hall, R.W. / *HANDBOOK OF TRANSPORTATION SCIENCE*
Grassman, W.K. / *COMPUTATIONAL PROBABILITY*
Pomeroy, J.-C. & Barba-Romero, S. / *MULTICRITERION DECISION IN MANAGEMENT*
Axsäter, S. / *INVENTORY CONTROL*
Wolkowicz, H., Saigal, R., & Vandenberghe, L. / *HANDBOOK OF SEMI-DEFINITE PROGRAMMING: Theory, Algorithms, and Applications*
Hobbs, B.F. & Meier, P. / *ENERGY DECISIONS AND THE ENVIRONMENT: A Guide to the Use of Multicriteria Methods*
Dar-El, E. / *HUMAN LEARNING: From Learning Curves to Learning Organizations*
Armstrong, J.S. / *PRINCIPLES OF FORECASTING: A Handbook for Researchers and Practitioners*
Balsamo, S., Personé, V., & Onvural, R./ *ANALYSIS OF QUEUEING NETWORKS WITH BLOCKING*
Bouyssou, D. et al. / *EVALUATION AND DECISION MODELS: A Critical Perspective*
Hanne, T. / *INTELLIGENT STRATEGIES FOR META MULTIPLE CRITERIA DECISION MAKING*
Saaty, T. & Vargas, L. / *MODELS, METHODS, CONCEPTS AND APPLICATIONS OF THE ANALYTIC HIERARCHY PROCESS*
Chatterjee, K. & Samuelson, W. / *GAME THEORY AND BUSINESS APPLICATIONS*
Hobbs, B. et al. / *THE NEXT GENERATION OF ELECTRIC POWER UNIT COMMITMENT MODELS*
Vanderbei, R.J. / *LINEAR PROGRAMMING: Foundations and Extensions, 2nd Ed.*
Kimms, A. / *MATHEMATICAL PROGRAMMING AND FINANCIAL OBJECTIVES FOR SCHEDULING PROJECTS*
Baptiste, P., Le Pape, C. & Nuijten, W. / *CONSTRAINT-BASED SCHEDULING*
Feinberg, E. & Shwartz, A. / *HANDBOOK OF MARKOV DECISION PROCESSES: Methods and Applications*
Ramík, J. & Vlach, M. / *GENERALIZED CONCAVITY IN FUZZY OPTIMIZATION AND DECISION ANALYSIS*
Song, J. & Yao, D. / *SUPPLY CHAIN STRUCTURES: Coordination, Information and Optimization*
Kozan, E. & Ohuchi, A. / *OPERATIONS RESEARCH/ MANAGEMENT SCIENCE AT WORK*

**Early Titles in the
INTERNATIONAL SERIES IN
OPERATIONS RESEARCH & MANAGEMENT SCIENCE**

(Continued)

- Bouyssou et al. / *AIDING DECISIONS WITH MULTIPLE CRITERIA: Essays in Honor of Bernard Roy*
- Cox, Louis Anthony, Jr. / *RISK ANALYSIS: Foundations, Models and Methods*
- Dror, M., L'Ecuyer, P. & Szidarovszky, F. / *MODELING UNCERTAINTY: An Examination of Stochastic Theory, Methods, and Applications*
- Dokuchaev, N. / *DYNAMIC PORTFOLIO STRATEGIES: Quantitative Methods and Empirical Rules for Incomplete Information*
- Sarker, R., Mohammadian, M. & Yao, X. / *EVOLUTIONARY OPTIMIZATION*
- Demeulemeester, R. & Herroelen, W. / *PROJECT SCHEDULING: A Research Handbook*
- Gazis, D.C. / *TRAFFIC THEORY*
- Zhu/ *QUANTITATIVE MODELS FOR PERFORMANCE EVALUATION AND BENCHMARKING*
- Ehr Gott & Gandibleux/ *MULTIPLE CRITERIA OPTIMIZATION: State of the Art Annotated Bibliographical Surveys*
- Bienstock/ *Potential Function Methods for Approx. Solving Linear Programming Problems*
- Matsatsinis & Siskos/ *INTELLIGENT SUPPORT SYSTEMS FOR MARKETING DECISIONS*
- Alpern & Gal/ *THE THEORY OF SEARCH GAMES AND RENDEZVOUS*
- Hall/ *HANDBOOK OF TRANSPORTATION SCIENCE - 2nd Ed.*
- Glover & Kochenberger/ *HANDBOOK OF METAHEURISTICS*
- Graves & Ringuest/ *MODELS AND METHODS FOR PROJECT SELECTION: Concepts from Management Science, Finance and Information Technology*
- Hassin & Haviv/ *TO QUEUE OR NOT TO QUEUE: Equilibrium Behavior in Queuing Systems*
- Gershwin et al/ *ANALYSIS & MODELING OF MANUFACTURING SYSTEMS*
- Maros/ *COMPUTATIONAL TECHNIQUES OF THE SIMPLEX METHOD*
- Harrison, Lee & Neale/ *THE PRACTICE OF SUPPLY CHAIN MANAGEMENT: Where Theory and Application Converge*
- Shanthikumar, Yao & Zijm/ *STOCHASTIC MODELING AND OPTIMIZATION OF MANUFACTURING SYSTEMS AND SUPPLY CHAINS*
- Nabrzyski, Schopf & Weglarz/ *GRID RESOURCE MANAGEMENT: State of the Art and Future Trends*
- Thissen & Herder/ *CRITICAL INFRASTRUCTURES: State of the Art in Research and Application*
- Carlsson, Fedrizzi, & Fullér/ *FUZZY LOGIC IN MANAGEMENT*
- Soyer, Mazzuchi & Singpurwalla/ *MATHEMATICAL RELIABILITY: An Expository Perspective*
- Chakravarty & Eliashberg/ *MANAGING BUSINESS INTERFACES: Marketing, Engineering, and Manufacturing Perspectives*
- Talluri & van Ryzin/ *THE THEORY AND PRACTICE OF REVENUE MANAGEMENT*
- Kavadias & Loch/ *PROJECT SELECTION UNDER UNCERTAINTY: Dynamically Allocating Resources to Maximize Value*
- Brandeau, Sainfort & Pierskalla/ *OPERATIONS RESEARCH AND HEALTH CARE: A Handbook of Methods and Applications*
- Cooper, Seiford & Zhu/ *HANDBOOK OF DATA ENVELOPMENT ANALYSIS: Models and Methods*
- Luenberger/ *LINEAR AND NONLINEAR PROGRAMMING, 2nd Ed.*
- Sherbrooke/ *OPTIMAL INVENTORY MODELING OF SYSTEMS: Multi-Echelon Techniques, Second Edition*
- Chu, Leung, Hui & Cheung/ *4th PARTY CYBER LOGISTICS FOR AIR CARGO*
- Simchi-Levi, Wu & Shen/ *HANDBOOK OF QUANTITATIVE SUPPLY CHAIN ANALYSIS: Modeling in the E-Business Era*
- Gass & Assad/ *AN ANNOTATED TIMELINE OF OPERATIONS RESEARCH: An Informal History*
- Greenberg/ *TUTORIALS ON EMERGING METHODOLOGIES AND APPLICATIONS IN OPERATIONS RESEARCH*

Early Titles in the
**INTERNATIONAL SERIES IN
OPERATIONS RESEARCH & MANAGEMENT SCIENCE**
(Continued)

- Weber/ *UNCERTAINTY IN THE ELECTRIC POWER INDUSTRY: Methods and Models for Decision Support*
- Figueira, Greco & Ehrgott/ *MULTIPLE CRITERIA DECISION ANALYSIS: State of the Art Surveys*
- Reveliotis/ *REAL-TIME MANAGEMENT OF RESOURCE ALLOCATIONS SYSTEMS: A Discrete Event Systems Approach*
- Kall & Mayer/ *STOCHASTIC LINEAR PROGRAMMING: Models, Theory, and Computation*
- Sethi, Yan & Zhang/ *INVENTORY AND SUPPLY CHAIN MANAGEMENT WITH FORECAST UPDATES*
- Cox/ *QUANTITATIVE HEALTH RISK ANALYSIS METHODS: Modeling the Human Health Impacts of Antibiotics Used in Food Animals*
- Ching & Ng/ *MARKOV CHAINS: Models, Algorithms and Applications*
- Li & Sun/ *NONLINEAR INTEGER PROGRAMMING*
- Kaliszewski/ *SOFT COMPUTING FOR COMPLEX MULTIPLE CRITERIA DECISION MAKING*
- Bouyssou et al/ *EVALUATION AND DECISION MODELS WITH MULTIPLE CRITERIA: Stepping stones for the analyst*
- Blecker & Friedrich/ *MASS CUSTOMIZATION: Challenges and Solutions*
- Appa, Pitsoulis & Williams/ *HANDBOOK ON MODELLING FOR DISCRETE OPTIMIZATION*
- Herrmann/ *HANDBOOK OF PRODUCTION SCHEDULING*

** A list of the more recent publications in the series is at the front of the book **

© 2009 Springer Science+Business Media, LLC

This electronic component package is protected by federal copyright law and international treaty. If you wish to return this book and the electronic component package to Springer Science+Business Media, LLC, do not open the disc envelope or remove it from the book. Springer Science+Business Media, LLC, will not accept any returns if the package has been opened and/or separated from the book. The copyright holder retains title to and ownership of the package. U.S. copyright law prohibits you from making any copy of the entire electronic component package for any reason without the written permission of Springer Science+Business Media, LLC, except that you may download and copy the files from the electronic component package for your own research, teaching, and personal communications use. Commercial use without the written consent of Springer Science+Business Media, LLC, is strictly prohibited. Springer Science+Business Media, LLC, or its designee has the right to audit your computer and electronic components usage to determine whether any unauthorized copies of this package have been made.

Springer Science+Business Media, LLC, or the author(s) makes no warranty or representation, either express or implied, with respect to this electronic component package or book, including their quality, merchantability, or fitness for a particular purpose. In no event will Springer Science+Business Media, LLC, or the author(s) be liable for direct, indirect, special, incidental, or consequential damages arising out of the use or inability to use the electronic component package or book, even if Springer Science+Business Media, LLC, or the author(s) has been advised of the possibility of such damages.