Chapter 6
The Measurement of Multidimensional Poverty

6.1 Introduction

In Chap. 2, we have presented a detailed and analytical discussion on the measurement of poverty using income as the only attribute of well-being. But as we have argued in Chap. 5, income is simply one of the many dimensions of well-being Therefore, poverty being a manifestation of insufficient well-being, should as well be regarded as a multidimensional phenomenon. In fact, there are many reasons for viewing poverty from a multidimensional perspective. The basic-needs approach regards poverty as lack of basic needs, and hence poverty is intrinsically multidimensional from this perspective. The importance of low income as a determinant of undernutrition is a debatable issue. (*See* Behrman and Deolikar, 1988; Dasgupta, 1993; Lipton and Ravallion, 1995; Ravallion, 1990, 1992.)

In the capability-functioning approach, poverty is regarded as a problem of capability failure. As Sen (1999) argued, capability failure captures the notion of poverty that people experience in day-to-day living condition. This approach constitutes a very sensible way of conceptualizing poverty since capability failure is generated from inability of possession of a wide range of characteristics related to the living standard rather than simply from the lowness of income. (*See also* Lewis and Ulph, 1988; Sen, 1985a, 1992; Townsend, 1979.)

An alternative way of looking at multidimensional poverty is in terms of social exclusion, which refers to exclusion of individuals from standard way of living and basic social activities (Townsend, 1979). A frequently used definition of social exclusion is ”the process through which individuals or groups are wholly or partially excluded from full participation in society in which they live” (European Foundation, 1995, p. 4). According to Atkinson (1998), it is a relative concept, in order to say whether a person is socially excluded or not, it is necessary to look at the positions of the others in the society as well. It is a dynamic process in which exclusion of individuals from full participation can be taken as the end product. Since social exclusion refers to exclusion of individuals from economic and social activities, it is a multidimensional phenomenon. We may, therefore, say that it incorporates the process aspect of capability failure (Sen, 2002). As Sen (2000) argued, if capability

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poverty is a consequence of lack of freedom, social exclusion curtails the freedom additionally. Thus, there is a close relationship between the two notions of poverty.

The human poverty index suggested by the UNDP (1997) can be regarded as a multidimensional index of poverty in the capability-failure framework. It is a summary indicator of the country level deprivations in the living standard of a population in the three basic dimensions of life, namely, decent living standard, life expectancy at birth, and educational attainment rate. Since an index of this type aggregates failures at the national level, it does not take into account the individual failures.

In this chapter, we assume that each person possesses a vector of attributes of well-being and a direct way of identification of the poor checks whether he has "minimally acceptable levels" (Sen, 1992 p. 139) of different attributes. These minimally acceptable quantities of the attributes represent their threshold levels that are necessary for maintaining a subsistence living standard. Indices of multidimensional poverty that are based on individual failures or shortfalls of attribute quantities from respective thresholds have been suggested, among others, by Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003), Tsui (2002), Alkire and Foster (2007), and Lugo and Maasoumi (2008a, b). Bourguignon and Chakravarty (2008) also investigated the issue whether one distribution of multidimensional attributes exhibits less poverty than another for all multidimensional poverty indices satisfying certain postulates (*see also* Duclos et al., 2006a, b).

In different sections of this chapter, we discuss a set of desirable axioms for multidimensional poverty indices, analyze their implications, and examine alternative multidimensional poverty indices and the poverty dominance criteria.

6.2 Postulates for an Index of Multidimensional Poverty

We follow the notation adopted in Chap.5. The number of attributes of well-being is d and the number of persons in the society is n. Each attribute is assumed to be measurable on a ratio scale. Thus, we rule out the possibility of including a variable of the type that says whether a person is ill or not (*see* Sect. 5.2). The matrix $X = (x_{ij})_{n \times d}$ is a typical distribution matrix whose (i, j)th entry x_{ij} shows the quantity of attribute j possessed by person i, $1 \leq i \leq n, 1 \leq j \leq d$. We assume that $X \in M \in \{M_1', M_2, M_3\}$ is arbitrary, where M_1' is the set of all $n \times d$ matrices with nonnegative entries and, M_2 andM_3 are the same as in Chap. 5 (*see* Sect. 5.2).

In the present multivariate setup, a poverty threshold or cutoff is defined for each attribute. These cutoffs represent the minimal quantities of the d attributes necessary for maintaining a subsistence standard of living. Let $\underline{z} = (z_1, z_2, \ldots, z_d) \in Z$ be the vector of poverty thresholds, where Z is a nonempty subset of Γ_+^d, the strictly positive part of the d dimensional Euclidean space. The censored distribution matrix associated with X is denoted by X^*, where the (i, j)th entry x_{ij}^* of X^* is defined as $x_{ij}^* = \min\{x_{ij}, z_j\}$.

In this framework, person i is regarded as poor or deprived with respect to attribute j if $x_{ij} < z_j$. Otherwise, he is called nonpoor in attribute j. Thus, we are using the weak definition to identify a poor person in a dimension. The deprivation score of a person is the total number of dimensions in which he is poor. If a person is poor in a dimension then we say that it is a meager dimension for him. Person i is called rich if $x_{ij} \geq z_j$ holds for all $1 \leq j \leq d$. Each individual is regarded as either poor or nonpoor in a dimension. But there can be a wide range of cutoffs for the attributes that coexist in a reasonable harmony (*see* Thorbecke, 2006). The possibility that the relevant information is missing may lead to an ambiguity in the concept of poverty. This may as well arise from insufficiency of information on consumption quantities of the attributes. With a view to tackling problems of this type in which indefiniteness arises from ambiguity, the fuzzy set approach appears to be quite justifiable.[1]

In this chapter, we assume that there is complete information on quantities of the attributes and thresholds. Let $\mathrm{SP}_j(X)$ (or SP_j) be the set of persons who are poor with respect to attribute j in any given $X \in M$. Bourguignon and Chakravarty (2003) argued that a very simple way of counting the number of poor in the multiattribute structure is to define the poverty indicator variable $\mathrm{ID}(x_{i.}, \underline{z})$ which takes on the value one if there is at least one dimension j in which person i is poor, where the row vector $x_{i.}$, the ith row of X, shows the quantities of d attributes possessed by person i. Otherwise, it takes on the value zero. Formally,

$$\mathrm{ID}(x_{i.}, \underline{z}) = \begin{cases} 1 & \text{if } \exists\, j \in \{1, 2, ..., d\} : x_{ij} < z_j \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Then the total number of poor in the multidimensional framework is given by $n_p(X) = \sum_{i=1}^{n} \mathrm{ID}(x_{i.}, \underline{z})$. Hence, the multidimensional headcount ratio is given by n_p/n. This method of identifying the set of multidimensional poor persons is referred to as the union method of identification. An alternative identification approach is the intersection method which says that a person is poor if he is poor in all dimensions and this leads us to identify the number of poor as the total number of persons who are poor in all dimensions. But if a person is poor in one dimension and nonpoor in another, then trade-off between these dimensions may not be possible, which in turn rules out the possibility that he becomes nonpoor in both the dimensions. An example is an old beggar who cannot trade-off his high age for low income to become rich in both income and life expectancy. Such a person cannot be regarded as rich simply because of his high longevity. Therefore, this definition does not appear to be appropriate. Alkire and Foster (2007) defined person i as multidimensionally poor if $x_{ij} < z_j$ holds for $\bar{l}$ many values of j, where $\bar{l}$ is some integer between 1 and d. Clearly, this intermediate identification method coincides with the union or the intersection method as $\bar{l} = 1$ or d (*see also* Gordon et al., 2003; Mack and Lindsay, 1985).

A multidimensional poverty index P is a nonconstant real-valued function defined on the Cartesian product $M \times Z$. For any $X \in M$ and $\underline{z} \in Z$, $P(X, \underline{z})$ determines the intensity of poverty associated with the attribute matrix X and the threshold vector $\underline{z}$.

[1] See Cerioli and Zani (1990), Cheli and Lemmi (1995), Chiappero Martinetti (1996, 2006), Balestrino (1998), Qizilbash (2003, 2006), Deutsch and Silber (2005), Betti et al. (2006, 2008) and Chakravarty (2006).

Most of the postulates we consider below for an arbitrary P are generalizations of income-based poverty axioms. They are stated in terms of an arbitrary population size n. In presenting these axioms, we follow Chakravarty et al. (1998), Bourguignon and Chakravarty (1999, 2003, 2008), Tsui (2002), and Chakravarty and Silber (2008).

Focus Axiom: For any $(X,\underline{z}) \in M \times Z$ and for any person i and attribute j such that $x_{ij} \geq z_j$, an increase in x_{ij}, such that all other attribute quantities in X remain unchanged, does not change the extent of poverty $P(X,\underline{z})$.

Normalization Axiom: For any $(X,\underline{z}) \in M \times Z$ if $x_{ij} \geq z_j$ for all i and j, then $P(X,\underline{z}) = 0$.

Monotonicity Axiom: For any $(X,\underline{z}) \in M \times Z$, any person i and attribute j such that $x_{ij} < z_j$, an increase in x_{ij}, given that other attribute levels in X remain unaltered, decreases the poverty value $P(X,\underline{z})$.

Population Replication Invariance Axiom: For any $(X,\underline{z}) \in M \times Z$, $P(X,\underline{z}) = P(X^{(l)},\underline{z})$, where $X^{(l)}$ is the l-fold replication of X, that is, $X^{(l)} = (X^1, X^2, \ldots X^l)$ with each $X^i = X$, and $l \geq 2$ is arbitrary.

Symmetry Axiom: For any $(X,\underline{z}) \in M \times Z$, $P(X,\underline{z}) = P(\Pi X,\underline{z})$, where Π is any $n \times n$ permutation matrix.

Continuity Axiom: $P(X,\underline{z})$ is continuous in $(X,\underline{z})$.

Subgroup Decomposability Axiom: Let $X^1, X^2, \ldots, X^J$ are J distribution matrices of d attributes over population sizes $n_1, n_2, \ldots, n_J$ such that $\sum_{i=1}^{J} n_i = n$. Then for $\underline{z} \in Z, P(X,\underline{z}) = \sum_{i=1}^{J} \frac{n_i}{n} P(X^i,\underline{z})$, where $X = (X^1, \ldots, X^J) \in M$.

Transfer Axiom: For any $\underline{z} \in Z$, $X, Y \in M$ if X is obtained from Y by the Uniform Majorization Principle or the Uniform Pigou-Dalton Transfers Principle, where the transfers are among the poor, then $P(X,\underline{z}) \leq P(Y,\underline{z})$.

Increasing Threshold Levels Axiom: For any $X \in M$, $P(X,\underline{z})$ is increasing in z_j for all j.

Nonpoverty Growth Axiom: For any $(Y,\underline{z}) \in M \times Z$, if X is obtained from Y by adding a rich person to the society, then $P(X,\underline{z}) \leq P(Y,\underline{z})$.

Scale Invariance Axiom: For all $(X^1,\underline{z}^1) \in M \times Z$, $P(X^1,\underline{z}^1) = P(X^2,\underline{z}^2)$, where $X^2 = X^1\Omega$, $\underline{z}^2 = \underline{z}^1\Omega$, and $\Omega = \text{diag}(\omega_1, \omega_2, \ldots, \omega_d)$, $\omega_i > 0$ for all i.

The Normalization, Population Replication Invariance, Continuity, Subgroup Decomposability, Increasing Threshold Levels, Nonpoverty Growth, and Scale Invariance Axioms are multidimensional versions of the corresponding income-based poverty axioms. The Monotonicity Axiom says that poverty decreases if the condition of a person who is poor in a dimension improves. It parallels the Strong Monotonicity Axiom discussed in Chap. 2 and implies the *Dimensional Monotonicity Axiom* of Alkire and Foster (2007) which demands that poverty should fall if improvement makes the person rich in the attribute. The Transfer Axiom is the poverty counterpart to the majorization criteria of multidimensional inequality indices. According to the Focus Axiom, if a person is nonpoor with respect to an attribute, then improving his position in the attribute does not change the level of poverty, even if he/she is poor in the other attributes. That is, poverty is independent of quantities of attributes that are above thresholds. If one views poverty in terms of shortfalls of attribute quantities from thresholds then this axiom is quite sensible. It rules out trade-off between two attributes of a person who is in poverty with respect to one but not in poverty with respect to the other. For instance, if education and a composite good are two attributes, more education above the threshold cannot be traded off to compensate the lack of composite good whose quantity is below its threshold. Equivalently, we say that above the threshold level of an attribute, the isopoverty contour of an individual becomes parallel to the axis that represents the quantities of the attribute. This, however, does not exclude the possibility of a trade-off between the attributes if a person is poor with respect to both of them. We can also consider a weak version of the axiom which says that the poverty index is independent of attribute quantities of rich persons only. Clearly, in this case the trade-off of the type we have discussed above is permissible because we do not assume that the poverty index is independent of the quantities of attributes in which a person is nonpoor. But although trade-off is allowed, poverty is never eliminated. This means that there is a positive lower bound of the poverty index. Consequently, the isopoverty contour becomes a line asymptotically.

The next axiom, suggested by Bourguignon and Chakravarty (2003), is concerned with poverty change under a correlation increasing switch (*see* Chap. 5). If the two attributes involved in the correlation increasing switch are close to each other, that is, they are substitutes, then one can compensate the smallness of the other in the definition of individual poverty. Then increasing correlation between the attributes will not reduce poverty. The reason behind this is that because of closeness between the attributes, the switch can be regarded as a regressive rearrangement in the sense that the richer poor is becoming even better-off after the switch. This in turn makes the poorer poor worse-off. Assuming that the poverty index is subgroup decomposable, the following Bourguignon-Chakravarty axiom can be stated:

Nondecreasing Poverty Under Correlation Increasing Switch Axiom: For any $(X,\underline{z}) \in M \times Z$, if $Y \in M$ is obtained from X by a correlation increasing switch between two poor persons who are poor in the two concerned attributes, then $P(X,\underline{z}) \leq P(Y,\underline{z})$ if the two attributes are substitutes.

The corresponding property, when the attributes are complements, demands that poverty should not increase under such a switch. Note that for these two properties to be well defined, the two persons should be poor in both the attributes involved in the switch. If the poverty index is insensitive to a correlation increasing switch, then the underlying attributes are independent.

6.3 Indices of Multidimensional Poverty

The objective of this section is to discuss some important indicators for multidimensional poverty and analyze their properties. We begin with the observation that by repeated application of the Subgroup Decomposability Axiom, the poverty index can be written as

$$P(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n} \zeta(x_{i.},\underline{z}), \tag{6.2}$$

where $\zeta(x_{i.},\underline{z}) = P(x_{i.},\underline{z})$ is the individual multidimensional poverty function (*see* 2.2). Thus, the symmetric and population replication invariant index P in (6.2) is simply the average of individual poverty levels.

While the Subgroup Decomposability Axiom deals with nonoverlapping subgroups of the population, we can have an analogous postulate for attributes, which we refer to as Factor Decomposability Axiom. According to the Factor Decomposability Axiom, overall poverty is a weighted average of poverty levels for individual attributes. Formally,

Factor Decomposability Axiom: For any $(X,\underline{z}) \in M \times Z$,

$$P(X,\underline{z}) = \sum_{j=1}^{d} \hat{b}_j P(x_{.j}, z_j), \tag{6.3}$$

where the nonnegative sequence $\{\hat{b}_j\}$ satisfies the restriction that $\sum_{j=1}^{d} \hat{b}_j = 1$ and $x_{.j}$ is the jth column of the distribution matrix X. That is, $x_{.j}$ gives the distribution of attribute j among n persons. The weight $\hat{b}_j \geq 0$ assigned to attribute j reflects the importance of this attribute in the aggregation defined in (6.3). It may also be interpreted as the priority that the government assigns for removing poverty from the jth dimension of well-being. The contribution of dimension j to overall poverty is given by the amount $\hat{b}_j P(x_{.j}, z_j)$. Complete elimination of poverty from dimension j will reduce total poverty exactly by this quantity. Thus, the percentage contribution of dimension j to overall poverty becomes $100(\hat{b}_j P((x_{.j}, z_j))/P(X,z)$ (*see* Alkire and Foster, 2007; Chakravarty and Silber, 2008; Chakravarty et al., 1998).

If the two decomposition postulates are employed simultaneously, we can calculate each subgroup's contribution for each dimension. To see this, suppose that there are only two subgroups with population sizes n_1 and n_2, and the corresponding components of the distribution matrix X are X^1 and X^2 so that $X = (X^1, X^2)$. Then by the Subgroup Decomposability Axiom $P(X,\underline{z}) = (n_1/n)P(X^1,\underline{z}) + (n_1/n)(X^2,\underline{z})$, which in view of (6.3), for $d = 2$, becomes

$$P(X,\underline{z}) = \frac{n_1}{n}[\hat{b}_1 P(x^1_{.1}, z_1) + \hat{b}_2 P(x^1_{.2}, z_2)] + \frac{n_2}{n}[\hat{b}_1 P(x^2_{.1}, z_1) + \hat{b}_2 P(x^2_{.2}, z_2)], \tag{6.4}$$

where $x^i_{.j}$ is the jth column of the matrix X^i and $P(x^i_{.j}, z_j)$ is the poverty level in subgroup i for dimension $j, i, j = 1,2$. By looking at the individual components of the micro-breakdown of poverty, as shown in (6.4), we can identify simultaneously the population subgroup(s) as well as dimensions(s) for which poverty levels are very high. For instance, suppose we first note that between the two subgroups, the poverty level for subgroup 1 is higher. Next, it is observed that this subgroup's poverty for dimension 2 is more, that is, $\hat{b}_2 P(x^1_{.2}, z_2) > \hat{b}_1 \mathrm{P}(x^1_{.1}, z_1)$. Therefore, the subgroup-attribute combination (1,2) of the population should get maximum attention from antipoverty perspective. This type of two-way splitting of poverty becomes especially helpful when the limited resources of the society may not be sufficient for removal of poverty from one entire subgroup or for one dimension of the entire population (*see* Chakravarty and Silber, 2008; Chakravarty et al., 1998).

The general form of the poverty index fulfilling the two decomposability postulates is given by

$$P(X,\underline{z}) = \frac{1}{n}\sum_{j=1}^{d}\hat{b}_j \sum_{i\in SP_j}\zeta(x_{ij},z_j). \tag{6.5}$$

Under the Scale Invariance, Focus, Normalization, Monotonicity, and Transfer Axioms, we can rewrite $\zeta(x_{ij},z_j)$ as $h(x_{ij}/z_j)$, where $h : R^1_+ \to R^1$ is continuous, decreasing, convex, and $h(x_{ij}/z_j) = 0$ for all $x_{ij} \geq z_j$ (*see* 2.19). In view of the assumption that $h(x_{ij}/z_j) = 0$ for all $x_{ij} \geq z_j$, we can restrict attention on the censored matrix X^*. By assumptions on $h, P(X,\underline{z})$ in (6.5) is increasing in threshold limits and satisfies the Nonpoverty Growth Axiom. However, the entire family of indices given by (6.5) is insensitive to a correlation increasing switch.

To illustrate the formula (6.5), let $h(x^*_{ij}/z_j) = -\log(x^*_{ij}/z_j)$, where $x^*_{ij} > 0$. Then the resulting index becomes the multidimensional Watts index of poverty

$$P_{\mathrm{WM}}(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\hat{b}_j \log\left(\frac{z_j}{x^*_{ij}}\right), \tag{6.6}$$

where $X \in M_3$. Tsui (2002) and Chakravarty and Silber (2008) characterized a more general form of P_{WM} which requires that $\hat{b}_j \geq 0$, with $>$ for some j, without the restriction $\sum\limits_{j=1}^{d}\hat{b}_j = 1$. It is a normalized version of the Lugo and Maasoumi (2008a) first class of IT poverty indices based on the "aggregate poverty line approach." Chakravarty et al. (2008) employed this index to investigate different causal factors of poverty. The transfer sensitivity property of P_{WM} is similar to its single-dimensional sister.

Next, suppose that $h(x^*_{ij}/z_j) = (1 - x^*_{ij}/z_j)^{\alpha_j}$, where $\alpha_j \geq 1$ is a parameter. Then the resulting index is a multidimensional generalization of the Foster et al. index (Foster et al., 1984):

$$P_{\mathrm{FGTM}}(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\hat{b}_j\left(1 - \frac{x^*_{ij}}{z_j}\right)^{\alpha_j}. \tag{6.7}$$

If $\alpha_j = 1$ for all j, then P_{FGTM} becomes a weighted average of the product of $\mathrm{PD}_j = q_j/n$, the proportion of population in poverty in dimension j, and the average of the relative gaps $\mathrm{RG}_j = \sum_{i\in SP_j}(1 - x^*_{ij})/q_j z_j$, across all dimensions. If $\alpha_j = 2$ for all j, then the formula can be written as

$$P_{\mathrm{FGTM}}(X,\underline{z}) = \frac{1}{n}\sum_{j=1}^{d}\hat{b}_j \mathrm{PD}_j(\mathrm{RG}^2_j + (1 - \mathrm{RG}^2_j)(I^j_{\mathrm{CV}})^2), \tag{6.8}$$

where I^j_{CV} is the coefficient of variation of the distribution of attribute j among the associated deprived persons. Given other things, an increase in I^j_{CV}, say through a rank-preserving regressive transfer between two persons for whom dimension j is meager, increases the poverty index. Thus, the decomposition in (6.8) shows that the poverty index is increasingly related to the dimension-wise inequality levels of the poor.

Finally, for the specification $1-(x^*_{ij}/z_j)^{e_j}$, where $0 < e_j \leq 1$, the associated poverty index turns out to be

$$P_{\mathrm{CM}}(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{d}\hat{b}_j\left(1-\left(\frac{x^*_{ij}}{z_j}\right)^{e_j}\right). \tag{6.9}$$

This form of the multidimensional Chakravarty index was considered by Chakravarty et al. (1998). Given other things, the index is increasing in e_j for all j. For $e_j = 1$, it coincides with the particular case of P_{FGTM} when $\alpha_j = 1, 1 \leq j \leq d$. On the other hand, as $e_j \to 0$ for all j, P_{CM} approaches its lower bound, zero. As the value of e_j decreases over the interval $(0,1]$, P_{CM} shows greater sensitivity to transfers at lower down the scale of the distribution of attribute j.

We derive formula (6.5) from (6.2) assuming that $\zeta(x_{i.},\underline{z})$ satisfies an additivity condition across dimensions. A more general representation of subgroup decomposable indices can be made by defining transformations (not necessarily additive) of dimension-wise poverty gaps of the individuals in different subgroups. The Bourguignon-Chakravarty general form of the multidimensional poverty index is based on this type of aggregation:

$$P_{\mathrm{BC}}(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\bar{h}\left(\left(\sum_{j=1}^{d}\left(\bar{b}_j\left(1-\frac{x^*_{ij}}{z_j}\right)^{\bar{\eta}}\right)\right)^{1/\bar{\eta}}\right), \tag{6.10}$$

where $\bar{h}$ is increasing, convex, and $\bar{h}(0) = 0$, $\bar{b}_j$ is the positive weight assigned to poverty gaps in dimension j and $\bar{\eta} > 1$ is a parameter that enables us to parameterize the elasticity of substitution between relative shortfalls in different dimensions. If $\bar{h}$ is the identity function then for $\bar{\eta} = 1$, at the first stage P_{BC} adds the dimension-wise relative gaps $(1-x^*_{ij}/z_j)$ weighted by the sequence $\{\bar{b}_j\}$ and then these weighted gaps are averaged across individuals. In this case, we have straight-line individual isopoverty contours and the relative gaps are perfectly substitutable.

If $\bar{\eta} \to \infty$, then the corresponding limiting form of P_{BC} is given by

$$P_{\mathrm{BC}}(X,\underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\bar{h}\left(\max_j\left\{1-\frac{x^*_{ij}}{z_j}\right\}\right). \tag{6.11}$$

Since the two-dimensional individual isopoverty curves associated with the functional form (6.11) are of rectangular shape, there is no scope for substitutability between the two relative shortfalls in this case. The informational requirement of this index is minimal, we only need information on the relative shortfalls $(1-x^*_{ij}/z_j)$ and a functional form for $\bar{h}$ to perform the aggregation. This index is nonincreasing under a correlation increasing switch. [*See* Bourguignon and Chakravarty (1999), for further discussion.]

An alternative of interest arises from the Foster et al. (1984) type specification $\bar{h}(v) = v^{\alpha}$, where $\alpha > 0$. The corresponding member of the family P_{BC} in (6.10) is given by

$$P_{\mathrm{BC}}(X, \underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=1}^{d}\bar{b}_j\left(1 - \frac{x_{ij}^*}{z_j}\right)^{\bar{\eta}}\right]^{\alpha/\bar{\eta}}. \qquad (6.12)$$

The stages of aggregation employed in (6.12) are as follows. We first aggregate the transformed relative poverty shortfalls $(1 - (x_{ij}^*/z_j)^{\bar{\eta}})$ of each person across dimensions into an aggregate relative shortfall using the coefficients $\bar{b}_j$. At the second stage, we take the average of such shortfalls, raised to the power α, over the whole population, to define multidimensional poverty. The index in (6.12) is the symmetric mean of power α of aggregated transformed relative poverty shortfalls of individuals in different dimensions. Therefore, it may be regarded as an alternative multidimensional generalization of the Foster et al. (1984) index. As the value of α increases, it becomes more sensitive toward extreme poverty. It is nondecreasing or nonincreasing under a correlation increasing switch depending on whether α is greater or less than $\bar{\eta}$.

Tsui (2002) characterized a family of multidimensional poverty indices using the multidimensional version of the Subgroup Consistency Axiom. This family turns out to be a generalization of the Chakravarty (1983c) index. The functional form of the Tsui family of indices is given by

$$P_{\mathrm{TCM}}(X, \underline{z}) = \frac{1}{n}\sum_{i=1}^{n}\left[\prod_{j=1}^{d}\left(\frac{z_j}{x_{ij}^*}\right)^{\bar{e}_j} - 1\right], \qquad (6.13)$$

where $X \in M_3$ and the nonnegative parameters $\bar{e}_j$'s have to be chosen such that different postulates are satisfied. For instance, if $d = 2$, the restrictions $\bar{e}_1(\bar{e}_1 + 1) \geq 0$ and $\bar{e}_1\bar{e}_2(\bar{e}_1\bar{e}_2 + 1) \geq 0$ are necessary for fulfillment of the Transfers Axiom. These two conditions are guaranteed by nonnegativity of $\bar{e}_1$ and $\bar{e}_2$. Nonnegativity of $\bar{e}_1\bar{e}_2$ ensures that P_{TCM} is nondecreasing under a correlation increasing switch.

Lugo and Maasoumi (2008b) employed an information theory-based approach to the design of multidimensional poverty indices. Their index is subgroup decomposable and the individual poverty function relies on the same aggregation rule, as employed in Maasoumi (1986), for aggregating the attributes of a person (*see* Sect. 5.4). Then a Foster et al. (1984) type transformation is used to aggregate the individual indices into an overall index. More precisely, the Lugo-Maasoumi index of poverty is given by

$$P_{\mathrm{LMM}}(X, \underline{z}) = \begin{cases} \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\left(\sum_{j=1}^{d} a_j''(x_{ij}^*)^{-\hat{\delta}}\right)^{-1/\hat{\delta}}}{\left(\sum_{j=1}^{d} a_j''(z_j)^{-\hat{\delta}}\right)^{-1/\hat{\delta}}}\right)^{\alpha}, & \hat{\delta} \neq 0, \\ \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\prod_{j=1}^{d}(x_{ij}^*)^{a_j''}}{\prod_{j=1}^{d}(z_j)^{a_j''}}\right)^{\alpha}, & \hat{\delta} = 0, \end{cases} \qquad (6.14)$$

where $X \in M_3$, a''_j and $\hat{\delta}$ are the same as in σ_i used in (5.14) and $\alpha > 0$. Note that in this case, the poverty thresholds are also aggregated using the same transformation. By construction, this index is independent of the attribute quantities that are above the corresponding thresholds and hence it is focused. Lugo and Maasoumi (2008b) also considered a variant of the index that meets the weak version of the Focus Axiom.

Alkire and Foster (2007) suggested an index that relies on the intermediate identification method. For any distribution matrix X, they defined the deprivation function of person i in dimension j as $d_{ij}^{\alpha} = (1 - x_{ij}/z_j)^{\alpha}$ if $x_{ij} < z_j$ while $d_{ij}^{\alpha} = 0$ if $x_{ij} \geq z_j$ and $\alpha > 0$. This function is then used to identify the poor persons in their framework as follows: define $d_{ij}^{\alpha}(\bar{l}) = d_{ij}^{\alpha}$ if the deprivation score of person i is at least $\bar{l}$, while $d_{ij}^{\alpha}(\bar{l}) = 0$ if the deprivation score is less than $\bar{l}$. That is, we consider the transformed relative shortfalls $(1 - x_{ij}/z_j)^{\alpha}$ of persons i in different dimensions and check if he has positive shortfalls in at least $\bar{l}$ dimensions, in which case he is treated as multidimensionally poor. Equivalently, we say that person i is deprived in the Alkire-Foster sense if his deprivation score is at least $\bar{l}$. The Alkire-Foster multidimensional poverty index is then defined as

$$P_{\mathrm{AFM}}(X, \underline{z}) = \frac{1}{nd} \sum_{i=1}^{n} \sum_{j=1}^{d} d_{ij}^{\alpha}(\bar{l}). \tag{6.15}$$

P_{AFM} is the sum of α powers of the relative poverty gaps of the poor divided by the maximum possible value that the sum can take. Note that in (6.7), if we assume that $\hat{b}_j = 1/d$, $\alpha_j = \alpha$ for all $1 \leq j \leq d$ and adopt the intermediate notion of identification, then it coincides with P_{AFM}.

As we have mentioned in Chap. 5, some of the dimensions of well-being may be of ordinal type. Therefore, each variable representing a dimension can be transformed using an increasing function which need not be identical across dimensions. Let $\mathrm{TR}_j : R_+^1 \to R_+^1$ be an arbitrary increasing function. Thus, for each j, x_{ij} gets transformed into $\mathrm{TR}_j(x_{ij})$. Likewise, for each j, z_j becomes $\mathrm{TR}_j(z_j)$. Now, measurability information invariance requires that the poverty level based on x_{ij} and z_j values should be same as that calculated using $\mathrm{TR}_j(x_{ij})$ and $\mathrm{TR}_j(z_j)$ values, where $1 \leq i \leq n$ and $1 \leq j \leq d$. Clearly, the indices based on shortfalls of the type $(1 - x_{ij}/z_j)$ may not fulfill the required information invariance assumption. The reason is that $(1 - \mathrm{TR}_j(x_{ij})/\mathrm{TR}_j(z_j))$ may not be equal to $(1 - x_{ij}/z_j)$, $1 \leq i \leq n$, and $1 \leq j \leq d$. However, the headcount ratio remains unaltered under this type of transformations. Thus, if some of the dimensions are ordinally measurable and the remaining dimensions are measurable on ratio scales, then the headcount ratio is a suitable index of poverty. Another index that survives this requirement is the Alkire and Foster (2007) dimension adjusted headcount ratio. It is given by the total number of deprivation scores of the poor in the Alkire-Foster sense divided by nd, the maximum deprivation score that could be experienced by all people. This index is obtained as the limiting case of P_{AFM} as $\alpha \to 0$. It satisfies the Dimensional Monotonicity Axiom, that is, a reduction in the deprivation score of a person decreases the index. However, the headcount ratio does not fulfill this axiom.

6.4 Multidimensional Poverty Orderings

In this section, we are concerned with the ranking of distribution matrices by a given set of poverty indices assuming that the threshold limits are the same. For the sake of simplicity, we assume that there are only two attributes of well-being. That is, our objective is to deal with two-dimensional poverty-measure ordering.

In order to simplify the exposition, a continuous representation of the bivariate distribution is considered. The cumulative distribution function $G(x_1,x_2)$ is defined on the range $[0,\hat{v}_1]\times[0,\hat{v}_2]$. (Since the formulation involves a continuum of population, the suffix i in $x_{i.}$ is dropped.) The marginal distribution function for attribute i is denoted by G_i, $i=1,2$. Our objective now is to compare two distributions represented by the distribution functions G and G^*. The difference $G(x_1,x_2)-G^*(x_1,x_2)$ will be denoted by $\Delta G(x_1,x_2)$. Assuming that the poverty index is subgroup decomposable, we can write it as

$$P(G,\underline{z}) = \int_0^{\hat{v}_1}\int_0^{\hat{v}_2} \zeta(x_1,x_2,z_1,z_2)\mathrm{d}G. \tag{6.16}$$

If ζ is twice differentiable then positivity of ζ_{12}, the cross partial derivative of ζ with respect to attribute quantities, means that the two attributes are substitutes. If ζ_{12} is negative, then the attributes are complements. The intermediate situation $\zeta_{12}=0$ means that they are independent. We write $\Delta\mathrm{PG}(G,G^*,\underline{z})$ for the poverty difference $P(G,\underline{z})-P(G^*,\underline{z})$.

The following theorem of Bourguignon and Chakravarty (2008) can now be stated:

Theorem 6.1. *Let G and G^* be two bivariate distribution functions on the same range $[0,\hat{v}_1]\times[0,\hat{v}_2]$.*

Assume that the poverty index is twice differentiable. Then the following conditions are equivalent:

(i) $\Delta P(G,G^*,\underline{z}) \leq 0$ *for all poverty indices that satisfy the Focus, Symmetry, Population Replication Invariance, Subgroup Decomposability, Monotonicity, and Nondecreasingness of Poverty under Correlation Increasing Switch Axioms.*

(ii) $\Delta G_i(x_i) \leq 0$ *for all $x_i < z_i$ for $i=1,2$, and* $\Delta G(x_1,x_2) \leq 0$ *for all $x_1 < z_1$ and $x_2 < z_2$.*

Theorem 6.1 demands that poverty dominance under properties stated in condition (i) requires the headcount ratio in each dimension not to be higher for all threshold limits below the thresholds z_i and the headcount ratio in the two-dimensional space, defined by any combination of poverty lines below the threshold values (z_1,z_2), not to be higher. That is, weak single dimensional dominance in each dimension and weak two dimensional dominance on the set of poor persons are required simultaneously. This two-dimensional dominance simply means that the headcount ratio should not be higher in the intersection of the two sets in which the individuals are poor attribute-wise. Note that this situation arises when the two attributes are substitutes.

If in condition (i) of Theorem 6.1 we replace nondecreasingness of poverty under correlation increasing switch by its nonincreasingness counterpart and retain all other assumptions, then the corresponding equivalent condition becomes $\Delta G(x_1) + \Delta G_2(x_2) - \Delta G(x_1, x_2) \leq 0$, for all $x_1 < z_1$ and/or $x_2 < z_2$. When evaluated at $x_1 = 0$ and $x_2 = 0$, this condition implies weak single-dimensional headcount ratio dominances. Dominance in the two-dimensional space thus requires weak single-dimensional dominances. The additional condition that the headcount ratio should not be higher in the union of the two sets in which people are poor dimension-wise has to be fulfilled. In this case, the two attributes are complements.

If the two attributes are neither substitutes nor complements, then instead of nondecreasingness of poverty under a correlation increasing switch, we assume in condition (i) of Theorem 6.1 that poverty does not change with respect to such a switch and maintain other assumptions. The equivalent dominance condition becomes $\Delta G_i(x_i) \leq 0$ for all $x_i < z_i$ for $i = 1, 2$. This means that the individual poverty function is additive across components. In this case, we simply have weak single-dimensional headcount ratio dominances. Equivalently weak first-order stochastic dominance for each marginal distribution is required. The reason behind this is that because of independence between the attributes we simply need to check attribute-wise dominance. Since in the case of independence, the dominance condition reduces to the single-dimensional ordering, the attribute-wise second-order stochastic dominance can also be employed. Duclos et al. (2006a) considered bivariate poverty orderings using an alternative set of assumptions. Their framework treats the attributes only as substitutes. There is a major difference between the Bourguignon-Chakravarty and the Duclos et al. frameworks because the latter assumes the dependence of the threshold limit of one dimension on that of the other and vice versa.