# Chapter 3
# Measuring Income Deprivation

## 3.1 Introduction

A person's feeling of deprivation with respect to an attribute of well-being arises from the comparison of his situation in the society with those of the persons that are better-off in the attribute. Evidently, high deprivation may generate tensions in the society which ultimately may lead to conflicts. A natural objective of the society should, therefore, be to make deprivation as low as possible. In this chapter, for simplicity, we will study only income deprivation.

The concept of deprivation was introduced into the income distribution literature by Sen (1973, 1976a). According to Sen (1973), in any pairwise comparison, the person with lower income may have a feeling of depression on finding that his income is lower. Assuming that the extent of depression suffered by an individual is proportional to the difference between the two incomes concerned, the average of all such depressions in all pairwise comparisons becomes the Gini index. A more formal treatment of this result was provided by Hey and Lambert (1980). Kakwani (1980a) interpreted the coefficient of variation from a similar perspective under the assumption that an individual's extent of depression is proportional to the square of the income difference. Tsui and Wang (2000) characterized a transformation of the Donaldson and Weymark (1980, 1983) S-Gini indices as a deprivation index using the concept of "net marginal deprivation." Net marginal deprivation demands that a rank-preserving increase in a person's income will generate two effects: (1) the feeling of deprivation among those poorer than him will increase and (2) his deprivation with respect to those richer than him will decrease. This approach bears some similarity with the Berrebi and Silber (1981) formulation.

A person in subgroup $i$ of persons with $i$ lowest incomes in the society may regard the subgroup highest income as his source of envy and the sum of gaps between the subgroup highest income and all lower incomes can be taken as an aggregate depression index of the subgroup. Aggregation of depressions across subgroups generates the absolute Bonferroni inequality index as the summary index of depression for the population as a whole (Chakravarty, 2007).

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Sen (1976a) argued that for any person, an increasing function of the number or share of the persons who have higher incomes can be taken as the level of deprivation. Alternatively, one might use the individual's income shortfall from a reference income level as an indicator of his deprivation. Yitzhaki (1979) considered the former notion and showed that one plausible index of average deprivation in a society is the absolute Gini index (*see also* Hey and Lambert, 1980). In either case, the position of the individual on income hierarchy plays an important role in the determination of his deprivation. Runciman (1966) discussed these two notions of deprivation earlier in a more general context (*see also* Weiss and Fershatman, 1998). In this general framework, an individual's assessment of a social state depends on the positions of those who are more favorably treated than him.

Bossert and D'Ambrosio (2007) considered time as a dimension in the determination of individual deprivation. In their framework, individual deprivation depends on two components, the average income shortfall of a person from all persons who are richer than him in the current period and the number of persons who were not richer than him in the previous period but are now better-off than him. Thus, this approach incorporates the idea that a person feels deprived not only because he is poor now but also because he was not poorer in the earlier period. They also developed axiomatic characterizations of deprivation indices that capture these ideas.

Chakravarty et al. (1995), Chakravarty (1997b, 2008b), and Chakravarty and Mukherjee (1999) looked at alternative implications of deprivation dominance induced by Kakwani's (1984a) relative deprivation curve (RDC), which is obtained by plotting the normalized cumulative sum of income shortfalls of different individuals from richer individuals against the corresponding cumulative population proportions. Chakravarty (1997b, 2008b) also studied satisfaction dominance in details, where the notion "satisfaction" may be regarded as the dual of the notion of deprivation. These issues have been examined further, among others, by Zoli (2000), Chakravarty and Moyes (2003), Chateauneuf and Moyes (2004, 2006), Moyes (2007), and Zheng (2007b).

Marshall et al. (1967) and Marshall and Olkin (1979) developed conditions on pairwise absolute and relative (ratio) income differences that are sufficient for Lorenz dominance. Preston (1990) provided some characterizations of these conditions along with an empirical illustration. The absolute difference and ratio criteria are, in fact, special cases of Zheng's (2007b) general utility gap dominance. He investigated a weak dominance concept which imposes conditions only on the gap between each person's utility and some reference utility.

According to Temkin (1986, 1993), a person has a complaint if he has lower income than others and inequality can be viewed in terms of such complaints. The greater is the difference between the income of a person and income of those richer than him, the greater will be his complaint. Similarly, the higher is the number of persons richer than him, the higher is his complaint. Social inequality then aggregates the complaints of different individuals concerning the income gaps and the numbers of persons. More precisely, inequality is defined as an increasing function of the total numbers and sizes of complaints of different individuals in the society. An important case here is that the highest income of the society is the reference point for all and everybody except the richest has a legitimate complaint. Cowell and Ebert (2004) used this structure to derive a complaint-based dominance criterion and a new class of inequality indices (*see also* Cowell, 2008, and Cowell and Ebert, 2008). Some implications of the complaint dominance relation have also been examined.

This chapter provides a comprehensive and analytical treatment of alternative notions of deprivation. Particularly, we examine the alternative notions of redistributive principles that take us from a more deprived distribution to a less deprived one under general assumptions about the mean income and the population size.

## 3.2 Deprivation and Satisfaction

For a population of size $n > 2$, a typical income distribution is given by $x = (x_1, \ldots, x_n)$, where $x_i > 0$ is the income of person $i$. Assuming that all income distributions are illfare-ranked, the set of income distributions in this $n$-person economy is $D^n_+$ and the set of all possible income distributions is $D_+ = \underset{n \in N}{\cup} D^n_+$, where $N$ is the set of natural numbers.

Let us now combine the two notions of deprivation explored in the introduction to arrive at a single indicator. Essential to the construction of this indicator is the existence of higher incomes than the income of the person under consideration and they constitute a source of frustration for the person. Given that $x \in D^n_+$ is illfare-ranked, according to the first notion, a measure of deprivation felt by person $i$ is $(n-i)/n$. An alternative measure of deprivation for person $i$ can be $(\lambda_{n-i}(x) - x_i)/\lambda(x)$, where $\lambda_{n-i}(x)$ is the mean income of the $(n-i)$ persons richer than $i$ in the distribution $x$. We can arrive at a combined indicator from these two measures by a multiplicative aggregation. The resulting indicator then becomes

$$\mathrm{RD}_i(x) = \left(\frac{n-i}{n}\right)\left(\frac{\lambda_{n-i}(x) - x_i}{\lambda(x)}\right) = \left(\frac{n-i}{n}\right)\sum_{j=i+1}^{n}\frac{(x_j - x_i)}{\lambda(x)(n-i)} = \sum_{j=i+1}^{n}\frac{(x_j - x_i)}{n\lambda(x)}. \tag{3.1}$$

This is the Kakwani (1984a) measure of deprivation of person $i$. It determines the sum of income share shortfalls of person $i$ from all persons who are not poorer than him.

Note that $\mathrm{RD}_i$ is homogeneous of degree zero in incomes, that is, it is a relative indicator of individual deprivation. Alternatively, we may assume that the individual deprivation indicator is an absolute measure. Multiplying $\mathrm{RD}_i$ by the mean we arrive at the following simple specification, which looks at deprivation in terms of absolute income differentials:

$$\mathrm{AD}_i(x) = \sum_{j=i+1}^{n}\frac{(x_j - x_i)}{n}. \tag{3.2}$$

This absolute counterpart to $\mathrm{RD}_i$ is the Yitzhaki measure of deprivation of person $i$ (Yitzhaki, 1979). It indicates the total income shortfall of person $i$ from all those who are not worse-off, as a fraction of the population size $n$.

The following are some of the properties of the functions $RD_i$ and $AD_i$ (*see* Chakravarty, 1997b, 2008b; Ebert and Moyes, 2000).

1. They are continuous, symmetric, population replication invariant, and nonnegative, where the lower bound zero is achieved whenever there is no feeling of deprivation.
2. When deprivation is measured by these two indicators, the richest individual with income $x_n$ does not feel deprived at all.
3. They are decreasing under a rank-preserving increase in $x_i$.
4. An increase in any income higher than $x_i$ that does not change the income ranks increases them.
5. An increase in any income lower than $x_i$, keeping the income hierarchy positions unaltered, does not change $AD_i$ but decreases $RD_i$.
6. They decrease under a rank-preserving income transfer of income from a person with income higher than $x_i$ to someone with income lower than $x_i$.
7. They remain unaltered if a rank-preserving income transfer takes place among persons richer/poorer than person $i$.

Note that we can rewrite $RD_i(x)$ in (3.1) as

$$RD_i(x) = \frac{n\lambda(x) - \left(\sum_{j=1}^{i} x_j + (n-i)x_i\right)}{n\lambda(x)} = 1 - LC\left(x, \frac{i}{n}\right) - \frac{(n-i)x_i}{n\lambda(x)}, \quad (3.3)$$

where $LC(x,(i/n))$ is the ordinate of the Lorenz curve of $x$ at the cumulative population proportion $i/n$. We define the complement

$$RS_i(x) = \frac{\sum_{j=1}^{i} x_j + (n-i)x_i}{n\lambda(x)} \quad (3.4)$$

of $RD_i(x)$ in (3.3) from unity as the relative satisfaction function of person $i$. The function $RS_i$ can be interpreted as follows. Person $i$ does not have any feeling of frustration if he compares his income $x_i$ with the lower incomes $x_1,\ldots,x_{i-1}$. This justifies the inclusion of first term $\sum_{j=1}^{i} x_j$, which depends on $x_1,\ldots,x_{i-1},x_i$, in the numerator of the right-hand side of (3.4). Next, we can eliminate person $i$'s frustration about the higher incomes $x_{i+1},\ldots,x_n$ by replacing each of them by $x_i$. This then generates the distribution $(x_1,x_2,\ldots,x_i,x_i,\ldots,x_i)$ censored at $x_i$. In the censored income distribution $(x_1,x_2,\ldots,x_i,x_i,\ldots,x_i)$ corresponding to $(x_1,x_2,\ldots,x_{i-1},x_i,\ldots,\ldots,x_n)$, person $i$ does not feel frustrated because of absence of incomes that are higher than $x_i$. Given the position of an individual in the income distribution ladder, he can be regarded as being either satisfied or frustrated. Since in the censored distribution in addition to person $i$ there are $(n-i)$ persons with income $x_i$ and they are all treated in a symmetric manner, we simply add $(n-i)x_i$ to $\sum_{j=1}^{i} x_j$ to arrive at the numerator of $RS_i$. Thus, the definition of $RS_i$ relies on the assumption that an individual derives satisfaction from the observation that nobody in the society is richer than him and there are people who are as well-off as he is. By multiplying $RS_i$

with the mean income, we get the generalized satisfaction function $\mathrm{GS}_i$. That is, $\mathrm{GS}_i(x) = \lambda(x)\mathrm{RS}_i(x) = 1/n(\sum_{j=1}^{i} x_j + (n-i)x_i) = \mathrm{GL}(x,(i/n)) + [((n-i)x_i)/n]$, where $\mathrm{GL}(x, i/n)$ is the ordinate of the generalized Lorenz curve of $x$ at $i/n$. $\mathrm{RS}_i$ and $\mathrm{GS}_i$ defined this way may be regarded as indicators of individual well-being. Note that $\mathrm{GS}_i$ is continuous, increasing in $x_i$ (assuming that income ranks are unaltered), linear homogeneous, unit translatable, and population replication invariant. For any $x \in D^n$, $x_1 = \mathrm{GS}_1(x) \leq \mathrm{GS}_2(x) \leq \ldots.. \leq \mathrm{GS}_n(x) = \lambda(x)$. If incomes are equally distributed, then $\mathrm{RS}_i$ and $\mathrm{GS}_i$ become respectively one and the common income itself. [Further discussion along this line can be found in Yitzhaki (1979), Hey and Lambert (1980), Stark and Yitzhaki (1988), Chakravarty (1997b, 2008b), and Chakravarty and Mukherjee (1999).]

For any income distribution $x$, $\mathrm{RD}_i(x)$ is, in fact, the ordinate $\mathrm{RD}(x, i/n)$ of the RDC corresponding to the cumulative population proportion $i/n$ (*see* Kakwani, 1984a). The RDC of $x$, $\mathrm{RD}(x,t)$, where $t \in [0,1]$, is completed by assuming $\mathrm{RD}(x,0) = 1$ and by defining

$$\mathrm{RD}\left(x, \frac{i+\tau}{n}\right) = (1-\tau)\mathrm{RD}\left(x, \frac{i}{n}\right) + \tau\mathrm{RD}\left(x, \frac{i+1}{n}\right), \tag{3.5}$$

for all $0 \leq \tau \leq 1$ and $1 \leq i \leq (n-1)$. Clearly, the RDC is downward sloping, which means that for any two persons, the richer person has a lower level of deprivation than the poorer person. If all the incomes are equal, then there is no feeling of deprivation by any person ($\mathrm{RD}(x,t) = 0$ for all $t$). In this case, the curve coincides with the horizontal axis. In contrast, maximum deprivation arises if the entire income is monopolized by the richest person and the curve coincides with the line BC shown in the Fig. 3.1. Equation (3.3) shows how we can generate the RDC from the Lorenz curve.

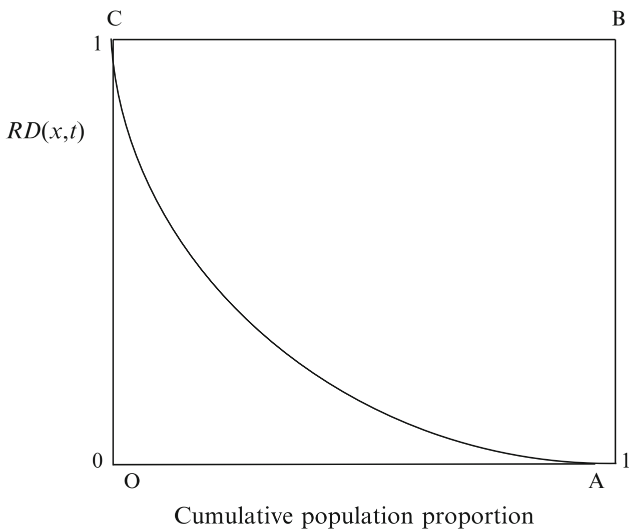


**Fig. 3.1** Relative deprivation curve

The absolute deprivation curve (ADC) of $x$, $\mathrm{AD}(x,t)$, where $t \in [0,1]$, is obtained by multiplying the RDC of $x$ by the mean. Formally, $\mathrm{AD}(x,t) = \lambda(x)\mathrm{RD}(x,t)$, where $t \in [0,1]$. We now define the absolute deprivation dominance (relative deprivation dominance) rule using the ADC (RDC) as follows. Given $x, y \in D_+^n$, we say that $y$ absolute deprivation dominates (relative deprivation dominates) $x$, what we write $y \geq_{\mathrm{AD}} x(y \geq_{\mathrm{RD}} x)$, if we have $\mathrm{AD}(y,t) \geq \mathrm{AD}(x,t)(\mathrm{RD}(y,t) \geq \mathrm{RD}(x,t))$ for all $t \in [0,1]$, with $>$ for some $t$.

We can use $\mathrm{RS}_i(x)$ values to define the relative satisfaction curve (RSC), $\mathrm{RS}(x,t)$ associated with $x$, where $t \in [0,1]$. More precisely, assuming that the ordinate of the curve at the cumulative population proportion $i/n$ is given by $\mathrm{RS}_i(x)$, it is drawn under the assumption that $\mathrm{RS}(x,0) = 0$ and by defining

$$\mathrm{RS}\left(x, \frac{i+\tau}{n}\right) = (1-\tau)\mathrm{RS}\left(x, \frac{i}{n}\right) + \tau\mathrm{RS}\left(x, \frac{i+1}{n}\right), \tag{3.6}$$

for all $0 \leq \tau \leq 1$ and $1 \leq i \leq (n-1)$. This curve is upward sloping. The generalized satisfaction curve (GSC) of the distribution $x$, $\mathrm{GS}(x,t)$ is produced by scaling up its RSC by the mean. That is, $\mathrm{GS}(x,t) = \lambda(x)\mathrm{RS}(x,t)$, $0 \leq t \leq 1$. It should now be clear the RSC (GSC) of a distribution can be generated by taking complement of the RDC (ADC) from unity (the mean). Given the relationship of $\mathrm{GS}_i$ with $\mathrm{GL}(x, i/n)$, we can say that the generalized Lorenz curve of a distribution never lies above its positively sloped GSC. Like the generalized Lorenz curve, the satisfaction curves, which show the levels of satisfactions enjoyed by different fractions of the population, may be interpreted as measures of social welfare. Thus, while deprivation has a negative impact on individual well-being, satisfaction makes a positive contribution to it (Fig. 3.2).

We can define the generalized and relative satisfaction dominance relations $\geq_{\mathrm{GS}}$ and $\geq_{\mathrm{RS}}$ using the GSC and the RSC curves, respectively, in the same way we employed the ADC and the RDC curves to define $\geq_{\mathrm{AD}}$ and $\geq_{\mathrm{RD}}$, respectively.

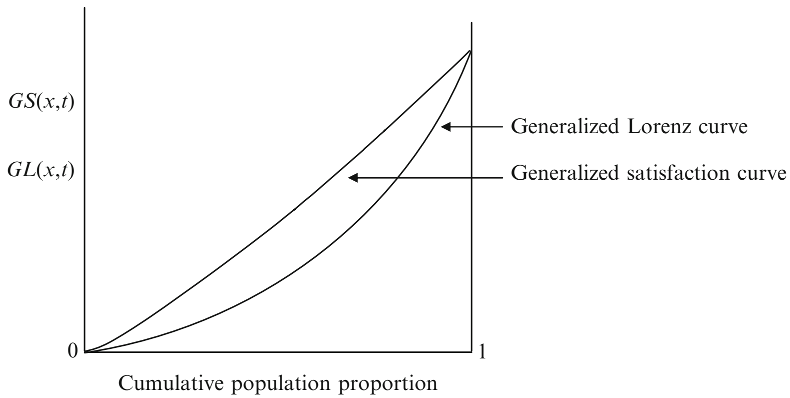


**Fig. 3.2** Generalized satisfaction curve and generalized Lorenz curve

The following two theorems, which were established in Hey and Lambert (1980), Chakravarty et al. (1995), Chakravarty (1997b, 2008b), and Chateauneuf and Moyes (2004, 2006), show some implications of the relations $\geq_{\mathrm{AD}}$ and $\geq_{\mathrm{GS}}$.

**Theorem 3.1.** *Let $x, y \in D_+^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then $y \geq_{\mathrm{RD}} x$ implies that $x$ is Lorenz superior to $y$ (that is, $x \geq_{\mathrm{LC}} y$). But the converse is not true.*

*Proof.* $y \geq_{\mathrm{RD}} x$ along with $\lambda(x) = \lambda(y)$, in view of (3.3), implies that

$$\sum_{j=1}^{i} x_j + (n-i)x_i \geq \sum_{j=1}^{i} y_j + (n-i)x_i \tag{3.7}$$

for all $1 \leq i \leq n$, with $>$ for some $i < n$. For $i = 1$, the above inequality becomes $nx_1 \geq ny_1$ which gives $x_1 \geq y_1$. Suppose that the result is true for $i = l$, that is, $\sum_{j=1}^{l} x_j \geq \sum_{j=1}^{l} y_j$ . We will show that it is true for $i = l+1$ also. Now, for $i = l+1$, inequality (3.7) becomes $\sum_{j=1}^{l+1} x_j + (n-l-1)x_{l+1} \geq \sum_{j=1}^{l+1} y_j + (n-l-1)y_{l+1}$. Adding $(n-l-1)\sum_{j=1}^{l} x_j((n-l-1)\sum_{j=1}^{l} y_j)$ to the left- (right-) hand side of this inequality, we get $(n-l)(\sum_{j=1}^{l} x_j + x_{l+1}) \geq (n-l)(\sum_{j=1}^{l} y_j + y_{l+1})$, from which it follows that $\sum_{j=1}^{l+1} x_j \geq \sum_{j=1}^{l+1} y_j$. This shows that the result is true for $i = l+1$ also. Hence, by the method of mathematical induction, the inequality $\sum_{j=1}^{i} x_j \geq \sum_{j=1}^{i} y_j$ holds for all $1 \leq i \leq n$. Given that there is strict inequality in $\geq_{\mathrm{RD}}$ for some $i < n$, there will be similar strict inequality in $\geq_{\mathrm{LC}}$ as well. For instance, if the inequality in (3.7) is strict for $i = l+1$, then the corresponding inequality in $\geq_{\mathrm{LC}}$ will be strict, that is, we will have $\sum_{j=1}^{l+1} x_j > \sum_{j=1}^{l+1} y_j$. Hence we have $x \geq_{\mathrm{LC}} y$.

To demonstrate that the reverse implication does not follow, consider the distribution $y = (5, 10, 15, 20)$. Then $x = (5, 11, 14, 20)$ is derived from $y$ by transferring one unit of income from the person with income 15 to the one with income 10. By the Hardy et al. (1934) theorem, this transfer ensures that $x \geq_{\mathrm{LC}} y$ holds, but $y \geq_{\mathrm{RD}} x$ does not hold. This completes the proof of the theorem. □

To understand why $y \geq_{\mathrm{RD}} x$ does not hold in the example taken above, note that while the $\mathrm{RD}_i$ measure for the recipient decreases, that of the donor increases, making the net effect ambiguous. It is evident that in view of the equality of the means, in Theorem 3.1, we can replace $y \geq_{\mathrm{RD}} x$ by $y \geq_{\mathrm{AD}} x$ or by $x \geq_{\mathrm{RS}} y$.

**Theorem 3.2.** *Let $x, y \in D_+^n$ be arbitrary. Then $x \geq_{\mathrm{GS}} y$ implies that $x$ is generalized Lorenz superior to $y$ (i.e., $x \geq_{\mathrm{GL}} y$). But the converse is not true.*

*Proof.* In this case, we compare $\mathrm{GS}_i(x) = 1/n(\sum_{j=1}^{i} x_j + (n-i)x_i)$ with the corresponding expression for $\mathrm{GS}_i(y)$ for all $1 \leq i \leq n$. Since the structure of the proof of the part that $x \geq_{\mathrm{GS}} y$ implies $x \geq_{\mathrm{GL}} y$ is similar to the demonstration of the claim that $y \geq_{\mathrm{RD}} x$ implies $x \geq_{\mathrm{LC}} y$, we are omitting the proof. To see that the converse is not true, consider the distributions $y' = (2, 3, 6)$ and $\bar{y} = (1, 4, 5)$. Then we have $y' \geq_{\mathrm{GL}} \bar{y}$ but not $y' \geq_{\mathrm{GS}} \bar{y}$. □

To understand the reason for not having $y' \geq_{GS} \bar{y}$ in the proof of Theorem 3.2, note that by increasingness of any increasing, strictly S-concave social welfare function $W$, we get $W(\hat{y}) > W(\bar{y})$, where $\hat{y} = (1,4,6)$. Now, we get $y'$ from $\hat{y}$ by transferring one unit of income from the second richest person to the poorest person. Hence by strict S-concavity of $W$, $W(y') > W(\hat{y})$, from which it follows that $W(y') > W(\bar{y})$. Thus, by the Shorrocks (1983a) theorem, we have $y' \geq_{GL} \bar{y}$. But while the increase in the richest person's income from 5 to 6 increases his satisfaction, the progressive transfer reduces the satisfaction of the donor and increases that of the recipient, generating an intersection between the GSCs of $y'$ and $\bar{y}$.

Given equivalence of the generalized Lorenz relation with second-order stochastic dominance, it follows from Theorem 3.2 that the generalized satisfaction dominance is a sufficient condition for second-order stochastic dominance as well.

In view of Theorem 3.1, it is clear that we need redistributive principles other than the Pigou-Dalton condition that will be consistent with the dominance principles introduced in this chapter. As a first step, following Chateauneuf and Moyes (2006) and Moyes (2007), we say that for $x, y \in D^n_+$, where $\lambda(x) = \lambda(y)$, $x$ is obtained from $y$ by a $T_2$-transformation if there exist $\hat{\sigma}, \hat{\rho} > 0$ and two individuals $j, l (1 \leq j < l \leq n)$ such that:

$$\begin{aligned} &x_i = y_i \quad \text{for all} \quad i \in \{1,2,\ldots,j-1\} \cup \{j+1,\ldots,l-1\}; \\ &x_j = y_j + \hat{\sigma}; \\ &x_i = y_i - \hat{\rho} \quad \text{for all} \quad i \in \{l,\ldots,n\} \hat{\sigma} = (n-l+1)\hat{\rho}. \end{aligned} \tag{3.8}$$

The essential idea underlying a Chateauneuf-Moyes transformation of type $T_2$ is that if some amount of income is taken from an individual $l$, then the same amount of income should be taken from all the persons who are richer than $l$. The entire rank-preserving transfer is received by person $j$, who is poorer than $l$. However, individuals in the set $\{1,2,..j-1\}$ who are poorer than individual $j$ do not benefit from the redistribution.

We can look at the transformation $T_2$ from a more general perspective. Let us rewrite $x$ as $y+b$, where $b_i = 0$ for all $i \in \{1,2,\ldots,j-1\} \cup \{j+1,\ldots,l-1\}$, $b_j = \hat{\sigma}$, and $b_i = -\hat{\rho}$ for all $i \in \{l,\ldots,n\}$. The condition $\hat{\sigma} = (n-l+1)\hat{\rho}$ shows that $\sum_{i=1}^n b_i = 0$. Further, $b_i \geq \sum_{j=i+1}^n b_j/(n-i)$ with $>$ for at least one $i < n$. We may verify this claim using the example $x = (10,24,30,38,48)$ and $y = (10,20,30,40,50)$. That is, in going from $y$ to $x$, if person $i$ has to forgo some amount of money ($b_i < 0$), then this amount should be less than the average net giving up (total giving ups in excess of receiving) of all who are richer than him. Likewise, if the redistribution enables him to get some amount of money ($b_i > 0$), then his receipt should be greater than the average net receipt (total receipt in excess of giving up) of all who are richer than him. One possible way in which such a situation can arise is that a progressive transfer is shared by the recipients, starting from the poorest, in decreasing order of income without destroying incentive preservation. Incentive preservation of a scheme requires that it does not alter rank orders of the individuals. This scheme has a lexicographic flavor in the sense that a person cannot receive his share of the donation unless all persons poorer than him have received their shares. Since the general scheme is a fair way of redistribution, we can refer to it as a "fair redistributive program." We may relate this condition with a balanced fiscal program $(y,x)$ which is minimally progressive and incentive preserving, where $y$ is the pretax income distribution and $x$ is the after tax distribution. Balancedness of the program means that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, that is, $\sum_{i=1}^{n} b_i = 0$. Since $x$ and $y$ are nondecreasingly ordered, the fiscal program is incentive preserving. Minimal progressivity requires that if $y_i \geq y_j$ then $b_i \geq b_j$. Incentive preservation and minimal progressivity of a tax function are necessary and sufficient for the after tax distribution to be more equally distributed than the pretax distribution by the absolute Lorenz criterion (Moyes, 1988, 1994). Note that fairness does not need $b_1 \geq b_2 \geq \ldots \geq b_n$. Hence, fairness is weaker than minimal progressivity (*see* Chakravarty, 1997b, 2008b; Chakravarty et al., 1995; Moyes, 2007; Zheng, 2007b).

One can see that if we have $y \geq_{\mathrm{RD}} x$ (or $x \geq_{\mathrm{RS}} y$) under the equality of the means, then we can arrive at $x$ from $y$ by a fair redistribution. Conversely, we can start with fairness, that is, $x_i - y_i = b_i \geq \sum_{j=i+1}^{n} b_j/(n-i) = \sum_{j=i+1}^{n} (x_j - y_j)/(n-i)$ with $>$ for some $i < n$. Then we can verify easily that $y \geq_{\mathrm{RD}} x$ holds. The following theorem can now be stated (*see* Chakravarty, 1997b, 2008b).

**Theorem 3.3.** *Let $x, y \in D_{+}^{n}$, where $\lambda(x) = \lambda(y)$, are arbitrary. Then the following conditions are equivalent:*

*(i) $y \geq_{\mathrm{RD}} x$ (or $x \geq_{\mathrm{RS}} y$).*
*(ii) $x$ can be obtained from $y$ by a fair redistributive program.*

Essentially Theorem 3.3 says that $x$ has less deprivation than $y$ if and only if the former is obtainable from the latter through a fair redistribution of incomes. Given that the means are the same, we can replace $\geq_{\mathrm{RD}}$ by $\geq_{\mathrm{AD}}$ and $\geq_{\mathrm{RS}}$ by $\geq_{\mathrm{GS}}$ in the theorem. We can also say that if condition (i) in the theorem is satisfied, then $x$ is regarded as less deprived than $y$ by all symmetric deprivation indices whose values reduce under a $T_2$- transformation/fair redistribution. More precisely, dominance of relative satisfaction of one distribution over that of another distribution is sufficient to guarantee that they can be ranked unambiguously by deprivation indices of the specified type. Furthermore, the converse is also true. If we assume that the means are unequal and population sizes are also not the same, then in addition to population replication invariance and these postulates, we need scale or translation invariance of the indices according as we use $\geq_{\mathrm{RD}}$ or $\geq_{\mathrm{AD}}$. We can develop similar results for relative satisfaction indices using $\geq_{\mathrm{RS}}$.

In addition to the Gini index, the area under the RDC, the following is an example of a deprivation index which corresponds to the relation $\geq_{\mathrm{RD}}$:

$$C_{\bar{\theta}}(x) = \begin{cases} 1 - \dfrac{1}{\lambda(x)} \left( \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \left( \sum_{j=1}^{i} \dfrac{x_j}{n} + \dfrac{n-i}{n} x_i \right)^{\bar{\theta}} \right)^{1/\bar{\theta}}, & \bar{\theta} \leq 1, \bar{\theta} \neq 0, \\ 1 - \dfrac{1}{\lambda(x)} \displaystyle\prod_{i=1}^{n} \left( \sum_{j=1}^{i} \left( \dfrac{x_j}{n} + \dfrac{n-i}{n} x_i \right) \right)^{1/n}, & \bar{\theta} = 1. \end{cases} \tag{3.9}$$

This index is the shortfall of the ratio between the symmetric mean of order $\bar{\theta}$ of the individual satisfactions and the mean from unity. Since none of the individual satisfaction levels exceeds the mean, the index is bounded between zero and one, where the lower bound is achieved for a perfectly egalitarian distribution. Evidently $(1-C_{\bar{\theta}})$ can be regarded as a relative satisfaction index. A decrease in the value of $\bar{\theta}$ makes $C_{\bar{\theta}}(1-C_{\bar{\theta}})$ more sensitive to the deprivation (satisfaction) of the poorer persons. Likewise, one minus the Gini index, the area under the RSC, can also be used as an index of relative satisfaction. An example of an absolute deprivation index is the absolute Gini index, the area under the ADC.

In Theorem 3.3, the RSC makes distributional judgments independently of the size of the distributions, that is, over distributions with a fixed total. Thus, efficiency considerations are absent in RSC comparison. In most circumstances of distributional comparisons, total income is likely to vary. This is likely to be true for intertemporal and intercountry comparisons. For ordering of income distributions with differing totals, we use the GSC.

Note that the area under the GSC is the (abbreviated) Gini welfare function. This is consistent with our observation that $\mathrm{GS}_i$ values may be used as indicators of individual well-being. Therefore, it should be clear that the GSC should be helpful in ranking income distributions in terms of welfare. The following theorem may be regarded as a step toward this direction (Chakravarty, 1997b).

**Theorem 3.4.** *Let $x,y \in D_+^n$ be arbitrary. Then the following conditions are equivalent.*

*(i) x is weakly generalized satisfaction dominant over y, that is,* $\mathrm{GS}(x,t) \geq \mathrm{GS}(y,t)$ *for all* $0 \leq t \leq 1$.
*(ii) $W(x) \geq W(y)$ for any symmetric social welfare function $W : D_+^n \to R^1$ which is nondecreasing in individual incomes and also nondecreasing under a fair redistributive program.*

*Proof.* (i) $\Rightarrow$ (ii): Weak generalized satisfaction dominance, which we denote by $x \geq_{\mathrm{WGS}} y$, implies that $\lambda(x) \geq \lambda(y)$. Define the distribution $u \in D_+^n$ by $u_i = y_i$ and $u_n = n(\lambda(x)-\lambda(y))+y_n$. By nondecreasingness of $W$, $W(u) \geq W(y)$. Note that $\lambda(u) = \lambda(x)$ and $x \geq_{\mathrm{WGS}} u$. Given the equality $\lambda(u) = \lambda(x)$, and the fact that $\mathrm{GS}(x,t) = \lambda(x)\mathrm{RS}(x,t)$, we can say that $x$ weakly relative satisfaction dominates $u$. Hence by Theorem 3.3, $W(x) \geq W(u)$, which shows that $W(x) \geq W(y)$. Note that $W$ is symmetric since we have defined it directly on ordered distributions.

(ii) $\Rightarrow$ (i): Consider the social welfare function $W(x) = 1/n(\sum_{j=1}^{i} x_j + (n-i)x_i)$, where $1 \leq i \leq n$. This welfare function satisfies all the assumptions stipulated in condition (ii) of the theorem. Thus, $W(x) = 1/n(\sum_{j=1}^{i} x_j + (n-i)x_i) \geq W(y) = 1/n(\sum_{j=1}^{i} y_j + (n-i)y_i)$ for $1 \leq i \leq n$, which in turn implies that $x$ weakly generalized satisfaction dominates $y$. □

Theorem 3.4 indicates that an unambiguous ranking of income distributions by all nondecreasing, symmetric, and equity-oriented social welfare functions is achievable if and only if their GSCs do not intersect, where equity orientation is defined involving redistribution of income in a fair way. If we assume that the mean income is the same in the above theorem, then for weak satisfaction dominance to hold the welfare function should only be symmetric and nondecreasing under a fair transformation. This can then be regarded as the satisfaction counterpart to the Dasgupta et al. (1973) theorem, whereas with variable mean, Theorem 3.4 parallels Shorrocks' theorem (1983a) on the generalized Lorenz criterion. Note that the GSC is population replication invariant. Therefore, satisfaction ranking of distributions over differing population sizes using the real valued welfare functions (defined on $D_+$) that fulfill population replication invariance, along with the requirements specified in condition (ii) of the theorem, can be implemented by seeking GSC dominance. In addition to the Gini welfare function, the abbreviated welfare function $\lambda(1-C_\theta)$ satisfies all these postulates.

Now, if a person feels deprived when comparing himself with a better-off person, he may as well have a feeling of "contentment" when he compares his position with that of a less fortunate person. In other words, he remains contented with the existence of persons who are poorer than him in the society. This specific way of definition of contentment does not take the higher incomes into account. Formally, given the income distribution $x \in D_+^n$, following Zheng (2007b), we define the absolute contentment function of person $i$ as

$$\mathrm{AC}_i(x) = \sum_{j=1}^{i} \frac{(x_i - x_j)}{n}. \tag{3.10}$$

Although both $\mathrm{AC}_i$ and $\mathrm{GS}_i$ are increasing under rank-preserving increments in $x_i$, there are important differences between them. While the latter possesses an altruistic flavor in the sense that an order preserving increase in any income less than $x_i$ increases $\mathrm{GS}_i$, the opposite happens for $\mathrm{AC}_i$. $\mathrm{AC}_i$ is a focused index, it is based on the distribution $(x_1, x_2, \ldots, x_i)$, which is obtained by truncating $x$ form above at $x_i$. In contrast, $\mathrm{GS}_i$ is defined on the distribution in which all incomes higher than $x_i$ are censored at $x_i$. Note also that the worst-off person derives some satisfaction if he has a positive income but contentment is not a source of happiness for him even if his income is positive.

We can interpret $\mathrm{AC}_i$ from an alternative perspective. Consider the subgroup $\{1, 2, \ldots i\}$ of $i$ persons with $i$ lowest incomes in the society. Any person with income less than $x_i$ may consider the subgroup highest income as his source of envy and, therefore, $1/i \sum_{i=1}^{i} (x_i - x_j)$ may be taken to represent the average level of depression in the subgroup. Thus, $\mathrm{AC}_i$ is the product of the proportion $i/n$ of persons in the subgroup and the average depression of this proportion (*see* Chakravarty, 2007). This interpretation is quite similar to the one we have provided for the Kakwani (1984a) index.[1] If $x_i$ is taken as the poverty line for the persons in the subgroup, then $(x_i - x_j)$ is individual $j$'s poverty gap and $\sum_{j=1}^{i} (x_i - x_j)$ gives us the total amount of money necessary to put the persons in the subgroup at the poverty line itself. Then, under the strong definition of the poor, $\mathrm{AC}_i$ becomes the

[1] Chateauneuf and Moyes (2006, p. 31) used the term "measure of the absolute satisfaction felt by individual ranked $i$" for the equation in (3.10). However, we follow Zheng's (2007b) terminology "contentment."

product of two crude poverty indicators, the headcount ratio and the average poverty gap of the poor $1/\sum_{j=1}^{i}(x_i - x_j)/i$.

The society absolute contentment curve (ACC) is a plot of individual contentment functions $\mathrm{AC}_i$ 's against the cumulative population proportions $i/n$. That is, $\mathrm{ACC}(x, i/n) = \mathrm{AC}_i(x)$ and the curve is made smooth throughout assuming that $\mathrm{ACC}(x, (i+\tau)/n) = (1-\tau)\mathrm{ACC}(x, i/n) + \tau\mathrm{ACC}(x, (i+1)/n)$, where $0 \le \tau \le 1$, $1 \le i \le (n-1)$, and $\mathrm{ACC}(x, 0) = 0$. Clearly, the ACC of a distribution has a positive slope. For $x, y \in D_+^n$, we say that $x$ is absolute contentment inferior to $y$ ($y \ge_{\mathrm{AC}} x$, for short) if $\mathrm{ACC}(x,t) \le \mathrm{ACC}(y,t)$ for all $0 \le t \le 1$ with $<$ for some $t$. That is, the relation $\ge_{\mathrm{AC}}$ stands for absolute contentment dominance. Now, $\ge_{\mathrm{AD}}$ concentrates on the distribution $(x_i, x_{i+1}, \ldots, x_n)$, which is obtained by truncating $x$ from below at $x_i$. Therefore, by definition, $\ge_{\mathrm{AC}}$ is different from $\ge_{\mathrm{AD}}$ (*see also* Chateauneuf and Moyes, 2006).

As Chateauneuf and Moyes (2006) and Zheng (2000b) noted, $\ge_{\mathrm{AC}}$ is stronger than $\ge_{\mathrm{LC}}$ (*see also* Chakravarty et al., 2003). Formally,

**Theorem 3.5.** *Let $x, y \in D_+^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then $y \ge_{\mathrm{AC}} x$ implies $x \ge_{\mathrm{LC}} y$ but the converse is not true.*

*Proof.* The $n$th inequality in $y \ge_{\mathrm{AC}} x$ can be written more explicitly as $nx_n - \sum_{j=1}^{n} x_j \le ny_n - \sum_{j=1}^{n} y_j$, which in view of the equality of the means gives $x_n \le y_n$. Therefore, we must have $\sum_{j=1}^{n-1} x_j \ge \sum_{j=1}^{n-1} y_j$. Thus, the following inequality holds for $i = 1, 2$.

$$\sum_{j=1}^{n-i+1} x_j \ge \sum_{j=1}^{n-i+1} y_j. \tag{3.11}$$

Assume that the inequality is true for $i = l$. We will show that it is true for $i = l+1$ also.

Now, by assumption

$$\sum_{j=1}^{n-l+1} x_j \ge \sum_{j=1}^{n-l+1} y_j, \tag{3.12}$$

which by the equality of the means implies

$$\sum_{j=n-l+2}^{n} x_j \le \sum_{j=n-l+2}^{n} y_j. \tag{3.13}$$

The $(n-l+1)$th inequality in $y \ge_{\mathrm{AC}} x$ gives $(n-l)x_{n-l+1} - x_{n-l} - \ldots - x_1 \le (n-l)y_{n-l+1} - y_{n-l} - \ldots - y_1$, which can be rewritten as

$$(n-l+1)x_{n-l+1} + \sum_{j=n-l+2}^{n} x_j - \sum_{j=1}^{n} x_j \le (n-l+1)y_{n-l+1} + \sum_{j=n-l+2}^{n} y_j - \sum_{j=1}^{n} y_j. \tag{3.14}$$

Given the equality of the means, inequality (3.14) implies that

$$(n-l+1)x_{n-l+1} + \sum_{j=n-l+2}^{n} x_j \le (n-l+1)y_{n-l+1} + \sum_{j=n-l+2}^{n} y_j. \quad (3.15)$$

Multiplying both sides of (3.13) by ($n$–$l$) and then adding the right- (left-) hand side of the resulting expression to the corresponding side of (3.15), we get:

$$(n-l+1)\sum_{j=n-l+1}^{n} x_j \le (n-l+1)\sum_{j=n-l+1}^{n} y_j. \quad (3.16)$$

Canceling $(n-l+1)$ from both sides of (3.16) and invoking the condition that $\lambda(x) = \lambda(y)$, we deduce that

$$\sum_{j=1}^{n-l} x_j \ge \sum_{j=1}^{n-l} y_j. \quad (3.17)$$

Hence, the inequality (3.11) is true for $i = l+1$ also. Thus, by the method of mathematical induction, (3.11) holds for $1 \le i \le n$ and a perfect equality occurs for $i = 1$ (given). The existence of $<$ for some $i$ in $\ge_{\text{AC}}$ implies that there will be similar $>$ in $\ge_{\text{LC}}$ as well. This demonstrates the claim that $x \ge_{\text{LC}} y$ holds. For the numerical income distributions considered in the proof of Theorem 3.1, we have $x \ge_{\text{LC}} y$ but not $y \ge_{\text{AC}} x$. This completes the proof of the theorem. □

Another implication of $\ge_{\text{AC}}$ is Zheng's (2007b) look-down dominance. For $x, y \in D_+^n$, we say that $y$ look-down dominates $x$, what we write $y \ge_{\text{LD}} x$, if $x_i - x_1 \le y_i - y_1$ holds for $i = 1, 2, \ldots, n$, with $<$ for some $i$. Thus, look-down dominance compares the excess of each income in a distribution over its minimum with the corresponding excess in another distribution. Evidently, in the dominated distribution, all incomes will be closer to the reference income – the minimum. For this to materialize, the minimum income should be increased. Apart from this, all other incomes can be increased or decreased such that the excesses over the minimum are lower. Formally, we have

**Theorem 3.6.** *Let $x, y \in D_+^n$ be arbitrary. Then $y \ge_{\text{AC}} x$ implies $y \ge_{\text{LD}} x$ but the converse is not true.*

*Proof.* For $i = 2$, $y \ge_{\text{AC}} x$ gives the inequality $x_2 - x_1 \le y_2 - y_1$. Thus, the result is true for $i = 1, 2$. Assume that it is true for all $i \le l$. That is, we have $x_i - x_1 \le y_i - y_1$ for all $i = 1, 2, \ldots, l$. We will show that it is true for $i = l+1$ also. Now, $(l+1)$th inequality in $y \ge_{\text{AC}} x$ implies $l x_{l+1} - x_l - \ldots x_1 \le l y_{l+1} - y_l - \ldots - y_1$. Adding the left- (right-) hand side of the latter inequality with corresponding sides of the inequalities $x_i - x_1 \le y_i - y_1$ for $i = 1, 2, \ldots, l$, it can be deduced that $x_{l+1} - x_1 \le y_{l+1} - y_1$. Hence, by the method of induction, the result is true for all $1 \le i \le n$. If for some $i$ (say, for $i = j$), strict inequality occurs in $y \ge_{\text{AC}} x$, then $x_j - x_1 < y_j - y_1$. To see that the opposite is not true, let $x = (15, 15, 35, 35, 50)$ and $y = (20, 30, 40, 50, 60)$. Then $x_i - x_1 \le y_i - y_1$ for all $i$, with three inequalities being strict. But $y \ge_{\text{AC}} x$ does not hold. □

It will now be worthwhile to identify a redistributive criterion consistent with the absolute contentment dominance principle. An attempt along this line has been made by Chateauneuf and Moyes (2006). According to these authors, for $x, y \in D^n_+$, where $\lambda(x) = \lambda(y)$, $x$ is obtained from $y$ by a $T_3$-transformation if there exist $\tilde{\sigma}, \tilde{\rho} > 0$ and two individuals $j, l (1 \leq j < l \leq n)$ such that:

$$\begin{aligned} &x_i = y_i \quad \text{for all} \quad i \in \{j+1, \ldots, l-1\} \cup \{l+1, \ldots, n\}; \\ &x_i = y_i + \tilde{\sigma} \quad \text{for all} \quad i \in \{1,, \ldots, j\}; \\ &x_l = y_l - \tilde{\rho}; \\ &j\tilde{\sigma} = \tilde{\rho}. \end{aligned} \tag{3.18}$$

A Chateauneuf-Moyes transformation of type $T_3$ demands that if a person receives some amount of income through a rank-preserving progressive transfer, then the transfer should give the same amount of income to all persons poorer than him. This is similar to the lexicographically equitable transfer defined in Chap. 1, this volume. We have stated this here for the sake of completeness and because of its alternative presentation.

Rewriting $x = y + b$, as before, it now appears that if we arrive at $x$ from $y$ by a $T_3$-transformation, then $b_i \leq \sum_{j=1}^{i-1} b_j/(i-1)$ for all $i = 2, \ldots, n$, with $<$ for some $i > 1$ (*see* Zheng, 2007b). From this, it follows that $x$ has lower contentment than $y$. The converse is true as well, that is, if we start with an inequality system of the type $b_i \leq \sum_{j=1}^{i-1} b_j/(i-1)$, then we can deduce that $y \geq_{\text{AC}} x$ holds. Hence, we can refer to $b$ as a contentment reducing transformation. The interpretation of the transformation is similar to the one provided for a fair redistributive program. This enables us to state the following:

**Theorem 3.7.** *Let $x, y \in D^n_+$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then the following conditions are equivalent:*

*(i) $y \geq_{\text{AC}} x$.*
*(ii) $x$ is obtained from $y$ by a contentment reducing transformation.*

Note that we can also have an index counterpart to Theorem 3.7, which says that the ranking of two income distributions of a given total, over a given population size, by all symmetric contentment indices that reduce under a transformation defined above is obtainable if and only if their ACCs do not intersect. An example of an index of this type can be the following:

$$C_{\hat{r}}(x) = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sum_{j=1}^{i} (x_j - x_i)^{\hat{r}} \right)^{1/\hat{r}} \tag{3.19}$$

where $\hat{r} \geq 1$ is a parameter. For $\hat{r} = 1, 2$, $C_{\hat{r}}$ corresponds respectively to the absolute Gini index and the standard deviation. An increase in the value of $\hat{r} > 2$ makes the index more sensitive to the extents of contentment of the poorer persons (Chakravarty et al., 2003).

## 3.3 Absolute and Relative Income Differentials and Deprivation

Since absolute income differentials are easy to imagine and calculate, they often constitute a natural source of envy for a person when he compares his income with higher incomes. Given $x, y \in D^n_+$, we say that $y$ dominates $x$ by absolute differentials, which we denote by $y \geq_{\text{ADI}} x$, if $x_i - y_i \geq x_{i+1} - y_{i+1}$, for all $i = 1, 2, \ldots, n-1$, with $>$ for some $i < n$. Since we can rewrite $x_i - y_i \geq x_{i+1} - y_{i+1}$ as $y_{i+1} - y_i \geq x_{i+1} - x_i$, $y \geq_{\text{ADI}} x$ simply means that differences between any two consecutive incomes are not lower in $y$ than in $x$, and will be higher in some case(s). It was first introduced by Marshall et al. (1967) and has been considered as a suitable inequality criterion by Preston (1990) and Moyes (1994, 1999). Marshall and Olkin (1979) showed that for distributions of a given total, absolute differentials dominance implies Lorenz domination. More precisely, for $x, y \in D^n_+$, where $\lambda(x) = \lambda(y)$, $y \geq_{\text{ADI}} x$ implies $x \geq_{\text{LC}} y$. This is intuitively reasonable because nondominant consecutive gaps under $x$ along with the equality of the means will ensure that $x$ has lower inequality. The numerical income distributions $x$ and $y$ taken in the proof of Theorem 3.1 show that $x \geq_{\text{LC}} y$ is true but $y \geq_{\text{ADI}} x$ is not.

If $U$ stands for the identical individual utility function, person $i$'s utility distance from person $j$ can be defined as $U(x_j) - U(x_i)$. Then we say that $y$ utility gap dominates $x$ if $U(x_i) - U(y_i) \geq U(x_{i+1}) - U(y_{i+1})$ holds for all $i = 1, 2, \ldots, n-1$, with $>$ for some $i < n$ (Zheng, 2007b). We can now imagine $\geq_{\text{ADI}}$ as utility gap dominance if $U(x_i) = x_i$. Likewise, if $U(x_i) = \log x_i$, then the utility difference inequality $U(x_i) - U(y_i) \geq U(x_{i+1}) - U(y_{i+1})$ becomes $\log(x_i/y_i) \geq \log(x_{i+1}/y_{i+1})$, which reduces to $x_i/y_i \geq x_{i+1}/y_{i+1}$. This forms the basis of Marshall and Olkin's relative or ratio differentials dominance (Marshall and Olkin, 1979). Formally, $y$ dominates $x$ by ratio differentials, which is denoted by $y \geq_{\text{RDI}} x$, if $x_i/y_i \geq x_{i+1}/y_{i+1}$ for all $i = 1, 2, \ldots, n-1$, with $>$ for some $i < n$. Moyes (1994) showed that the relations $\geq_{\text{ADI}}$ and $\geq_{\text{RDI}}$ are different.

We will now examine some implications of the relations $\geq_{\text{ADI}}$ and $\geq_{\text{RDI}}$. The following theorem shows that the former is sufficient for absolute contentment dominance (*see* Chakravarty et al., 2003; Chateauneuf and Moyes, 2006).

**Theorem 3.8.** *Let $x, y \in D^n_+$ be arbitrary. Then $y \geq_{\text{ADI}} x$ implies $y \geq_{\text{AC}} x$ but the converse is not true.*

*Proof.* From $y \geq_{\text{AC}} x$, we have

$$\sum_{j=1}^{i} (y_i - y_j) \geq \sum_{j=1}^{i} (x_i - x_j) \tag{3.20}$$

for all $i = 1, 2, \ldots, n$. Given any $i$, a sufficient condition for (3.20) to hold is that $(x_i - x_j) \leq (y_i - y_j)$ for $j = 1, 2, \ldots, i-1$. This is same as the condition that $(y_j - x_j) \leq (y_i - x_i)$. We write this more explicitly as $(y_1 - x_1) \leq (y_i - x_i)$, $(y_2 - x_2) \leq (y_i - x_i), \ldots, (y_{i-1} - x_{i-1}) \leq (y_i - x_i)$. A sufficient condition for this inequality to hold is that $(y_1 - x_1) \leq (y_2 - x_2) \leq \ldots .(y_{i-1} - x_{i-1}) \leq (y_i - x_i)$, which follows from $y \geq_{\text{ADI}} x$. Evidently, whenever there is a strict inequality for some $i$, say, for $i = l$, in $y \geq_{\text{ADI}} x$, there will be strict inequality for $i = l$ in $y \geq_{\text{AC}} x$. Falsity of the converse can be proved using the numerical example $y = (10,20,30,40)$ and $x = (15,20,25,40)$. Here we have $y \geq_{\text{AC}} x$ but not $y \geq_{\text{AD}} x$. □

For two distributions $x$ and $y$ over the population size $n$, $y \geq_{\text{AD}} x$ implies that $\sum_{j=i+1}^{n}(y_j - y_i) \geq \sum_{j=i+1}^{n}(x_j - x_i)$ for all $i = 1,2,\ldots,n$, with $>$ for some $i < n$. For any given arbitrary $i$, a sufficient condition that ensures this inequality system is $(y_{i+1} - y_i) \geq (x_{i+1} - x_i)$, $(y_{i+2} - y_i) \geq (x_{i+2} - x_i), \ldots, (y_n - y_i) \geq (x_n - x_i)$, $1 \leq i \leq n$. We rewrite this latter condition as $(y_{i+1} - x_{i+1}) \geq (y_i - x_i)$, $(y_{i+2} - x_{i+2}) \geq (y_i - x_i), \ldots, (y_n - x_n) \geq (y_i - x_i)$, $1 \leq i \leq n$. This is guaranteed if we assume that $(y_n - x_n) \geq \ldots \geq (y_{i+2} - x_{i+2}) \geq (y_{i+1} - x_{i+1}) \geq (y_i - x_i)$, $1 \leq i \leq n$. But this follows from the condition that $y \geq_{\text{ADI}} x$. Strict inequality for some $i < n$ in $y \geq_{\text{ADI}} x$ generates the corresponding condition in $y \geq_{\text{AD}} x$. Thus, $y \geq_{\text{ADI}} x$ implies $y \geq_{\text{AD}} x$. For $x = (1,3,6,6)$ and $y = (1,3,5,7)$, we have $y \geq_{\text{AD}} x$ but not $y \geq_{\text{ADI}} x$ (Moyes, 2007). These observations are summarized in the following theorem.

**Theorem 3.9.** *For arbitrary $x, y \in D_+^n$, $y \geq_{\text{ADI}} x$ implies $y \geq_{\text{AD}} x$ but the converse is not true.*

Chateauneuf and Moyes (2006) defined a $T_1$-transformation which when applied successively results in distributional improvement according to $\geq_{\text{ADI}}$. For $x, y \in D_+^n$, where $\lambda(x) = \lambda(y)$, $x$ is obtained from $y$ by a $T_1$-transformation if there exist $\sigma', \rho' > 0$ and two individuals $j, l (1 \leq j < l \leq n)$ such that:

$$\begin{aligned} &x_i = y_i \quad \text{for all} \quad i \in \{j+1,\ldots,l-1\}; \\ &x_i = y_i + \sigma' \quad \text{for all} \quad i \in \{1,,\ldots,j\}; \\ &x_i = y_i - \rho' \quad \text{for all} \quad i \in \{l,,\ldots,n\}; \\ &j\sigma' = (n-l+1)\rho'. \end{aligned} \tag{3.21}$$

A Chateauneuf-Moyes transformation of type $T_1$ says that if some amount of income is transferred progressively from a person, then the same amount of income should also be transferred from all those who are not poorer than him. Further, if the progressive transfer gives some amount of income to a person, then all those who are not richer than him are also recipients of the same amount of income. In fact, a $T_1$-transformation can be regarded as a fiscal program which is balanced, minimally progressive, and incentive preserving (*see* Moyes, 2007). The following theorem of Chateauneuf and Moyes (2006) can now be stated.

**Theorem 3.10.** *Let $x, y \in D_+^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then the following conditions are equivalent:*

*(i) $y \geq_{\text{ADI}} x$.*
*(ii) $x$ can be obtained from $y$ by a finite sequence of $T_1$-transformations.*

Theorem 3.10 establishes the connection between the absolute differentials dominance relation and the rank-preserving progressive transfers underlying a $T_1$-transformation.

People often view depression in terms of relative income differentials. Marshall et al. (1967) established that for two distributions $x$ and $y$ of a given total, $y \geq_{\mathrm{RDI}} x$ is sufficient for $x \geq_{\mathrm{LC}} y$ (see also Marshall and Olkin, 1979, p. 129). For the numerical distributions $x$ and $y$ considered in the proof of Theorem 3.1, we have $x \geq_{\mathrm{LC}} y$ and not $y \geq_{\mathrm{RDI}} x$. Hence, $x \geq_{\mathrm{LC}} y$ does not imply $y \geq_{\mathrm{RDI}} x$.

In the following theorem, we identify the relationship between the dominance based on income ratios and the relative contentment dominance relation $\geq_{\mathrm{RC}}$, which relies on the ratios $x_i/x_j$, $1 \leq j \leq i$. Formally for $x, y \in D_+^n$, $y \geq_{\mathrm{RC}} x$ means that $\sum_{j=1}^{i} (y_i - y_j)/ny_j \geq \sum_{j=1}^{i} (x_i - x_j)/nx_j$ for $1 \leq i \leq n$, with $>$ for some $i$.

**Theorem 3.11.** *For all $x, y \in D_+^n$, $y \geq_{\mathrm{RDI}} x$ implies that $y$ relative contentment dominates $x$, but the converse is not true.*

*Proof.* By $y \geq_{\mathrm{RC}} x$ we have $\sum_{j=1}^{i} (y_i - y_j)/y_j \geq \sum_{j=1}^{i} (x_i - x_j)/x_j$ for $1 \leq i \leq n$, with $>$ for some $i$. Given $i$, a sufficient condition for the above inequality to hold is that $y_i/y_j \geq x_i/x_j$ for $1 \leq j \leq i$. We rewrite this latter inequality as $y_i/x_i \geq y_j/x_j$ for $1 \leq j \leq i$. This requirement is satisfied if we assume that $y_i/x_i \geq y_{i-1}/x_{i-1} \geq \ldots. \geq y_1/x_1$, a condition implied by $y \geq_{\mathrm{RDI}} x$. Whenever there is a strict inequality in $y \geq_{\mathrm{RDI}} x$, there will be a strict inequality in $y \geq_{\mathrm{RC}} x$ also. To check that the converse is not true, consider the numerical distributions $x$ and $y$ taken in the proof of Theorem 3.8. Then we have $y \geq_{\mathrm{RC}} x$ but not $y \geq_{\mathrm{RDI}} x$. □

It may now be worthwhile to make a comparison between $\geq_{\mathrm{RC}}$ and $\geq_{\mathrm{RD}}$. Note that for the distributions $y' = (10, 20, 30, 40)$ and $y'' = (15, 20, 30, 35)$, we have both $y' \geq_{\mathrm{RC}} y''$ and $y' \geq_{\mathrm{RD}} y''$. Next, for the distributions $\bar{y} = (2, 4, 6, 8)$ and $\tilde{y} = (6, 6, 12, 16)$, $\bar{y} \geq_{\mathrm{RC}} \tilde{y}$ holds but $\bar{y} \geq_{\mathrm{RD}} \tilde{y}$ does not hold. To see the converse, consider the distribution $\hat{y} = (4, 8, 14, 14)$. One can check that $\bar{y} \geq_{\mathrm{RD}} \hat{y}$ is true but $\bar{y} \geq_{\mathrm{RC}} \hat{y}$ is not true. Finally, consider the distribution $\ddot{y} = (2, 5, 5, 8)$ and note that neither $\geq_{\mathrm{RD}}$ nor $\geq_{\mathrm{RC}}$ can rank the distributions $\ddot{y}$ and $\bar{y}$. These, observations enable us to conclude that $\geq_{\mathrm{RC}}$ and $\geq_{\mathrm{RD}}$ are different.

One can prove that the relations $\geq_{\mathrm{AC}}$ and $\geq_{\mathrm{RC}}$ are also different. To see this, note that while $\geq_{\mathrm{AC}}$ also cannot rank the distributions $\ddot{y}$ and $\bar{y}$, we have $y' \geq_{\mathrm{AC}} y''$. Next, we can verify that $\bar{y} \geq_{\mathrm{AC}} \tilde{y}$ does not hold. Finally, for the distributions $\dot{y} = (1, 1, 4, 6)$ and $\bar{y}$, we have $\bar{y} \geq_{\mathrm{AC}} \dot{y}$ but not $\bar{y} \geq_{\mathrm{RC}} \dot{y}$. These observations combined with our observations in the earlier paragraph regarding ranking of distributions by $\geq_{\mathrm{RC}}$ demonstrate that $\geq_{\mathrm{AC}}$ and $\geq_{\mathrm{RC}}$ are different (*see* Chakravarty et al., 2003).

## 3.4 Complaints and Deprivation

The central idea underlying the Temkin (1986, 1993) notion of inequality is individual complaint. Thus, like our earlier treatments in the chapter, the Temkin approach is also an individualistic approach to the assessment of income distributions. Among the various possibilities considered by Temkin (1986, 1993), the one that received principal focus is that the highest income in the society is the reference point and everybody except the richest person has a legitimate complaint. Alternatively, the average income or incomes of all better-off persons can be the reference points of different worse-off individuals (*see* Chakravarty, 1997b). By aggregating the individual complaints in an unambiguous way, we arrive at an overall inequality index. Although there appears to be similarity of this approach with the Runciman (1966) approach, there are differences as well. For instance, reference to the best-off person is one case of difference.

Cowell and Ebert (2004) considered the framework where the highest income $x_n$ is the reference point for all the persons except the richest. Then $\mathrm{SC}_i(x) = (x_n - x_i)$ is the size of complaint of person $i$. These sizes form the basis of our analysis in this section. The graph of cumulative complaints $1/n\sum_{j=0}^{i}\mathrm{SC}_j(x)$ against the corresponding cumulative population proportions $i/n$ gives us the cumulative complaint contour $\mathrm{CCC}(x,i/n)$ of the distribution $x$, where $\mathrm{SC}_0(x) = 0$ and $i = 0,1,..,n$. Segments of the curve between any two consecutive population proportions $i/n$ and $(i+1)/n$ is defined by the convex combination $\mathrm{CCC}(x,(i+\tau)/n) = (1-\tau)\mathrm{CCC}(x,i/n) + \tau\mathrm{CCC}(x,(i+1)/n)$, where $0 \le \tau \le 1$. By construction, the CCC of a distribution is upward sloping. We then say that for $x,y \in D_+^n$, $y$ complaint dominates $x$ ($y \ge_{\mathrm{CC}} x$, for brevity) if $\mathrm{CCC}(y,t) \ge \mathrm{CCC}(x,t)$ for all $0 \le t \le 1$, with $>$ for some $t$.

The following theorem of Cowell and Ebert (2004) shows the relationship between the generalized Lorenz relation $\ge_{GL}$ and the complaint dominance rule $\ge_{\mathrm{CC}}$.

**Theorem 3.12.** *Let $x,y \in D_+^n$ be arbitrary. Then $y \ge_{CC} x$ implies $(x - x_n 1^n) \ge_{GL} (y - y_n 1^n)$.*

*Proof.* From $y \ge_{\mathrm{CC}} x$, we get $\sum_{j=1}^{i}(x_n - x_j) \le \sum_{j=1}^{i}(y_n - y_j)$ for all $1 \le i \le n-1$, with $<$ for some $i$. We rewrite this inequality as $\sum_{j=1}^{i}(x_j - x_n 1^n) \ge \sum_{j=1}^{i}(y_j - y_n 1^n)$ for all $1 \le i \le n$, with $>$ for some $i$. This latter inequality gives $(x - x_n 1^n) \ge_{GL} (y - y_n 1^n)$. □

The next theorem is concerned with the relationship between $\ge_{\mathrm{AC}}$ and $\ge_{\mathrm{CC}}$.

**Theorem 3.13.** *Let $x,y \in D_+^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then $y \ge_{\mathrm{AC}} x$ implies $y \ge_{\mathrm{CC}} x$, but the converse is not true.*

*Proof.* Suppose that we have $y \ge_{CC} x$, which by Theorem 3.12 implies the condition that $(x - x_n 1^n) \ge_{GL} (y - y_n 1^n)$. We can write this latter relation explicitly as:

$$\frac{1}{n}\left(iy_n + \sum_{j=1}^{i} x_j\right) \ge \frac{1}{n}\left(ix_n + \sum_{j=1}^{i} y_j\right), \tag{3.22}$$

for all $i = 1,2,\ldots,n$ with $>$ for at least one $i$. Given the equality of the total incomes in $x$ and $y$, two sufficient conditions for (3.22) to hold are $x_n \le y_n$ and $x \ge_{\mathrm{LC}} y$. By Theorem 3.5, under the assumption of the equality of the means, $y \ge_{\mathrm{AC}} x$ implies $x \ge_{\mathrm{LC}} y$. Further, from the proof of Theorem 3.5, we know that under the given assumption $x_n \le y_n$ holds. Hence, $y \ge_{\mathrm{AC}} x$ implies $y \ge_{\mathrm{CC}} x$. Using the example that $y = (10,20,30,40)$ and $x = (10,24,26,40)$, one can check that $y \ge_{\mathrm{CC}} x$ holds but $y \ge_{\mathrm{AC}} x$ does not hold. □

Cowell and Ebert (2004) characterized weighted mean of order $\upsilon \geq 1$ of individual complaints as an index of overall complaint. Formally, the Cowell-Ebert index is given by

$$C_{\upsilon}(x) = \left[ \sum_{i=1}^{n-1} \tilde{w}_i (x_n - x_i)^{\upsilon} \right]^{1/\upsilon}, \tag{3.23}$$

where the positive weight sequence $\{\tilde{w}_i\}$, satisfying the restriction $\sum_{i=1}^{n-1} \tilde{w}_i = 1$, is nonincreasing if $\upsilon > 1$ and decreasing if $\upsilon = 1$. Members of the class $C_{\upsilon}$ decrease under a rank-preserving transfer from a person to anyone with lower income. They also demonstrated that $y \geq_{\mathrm{CC}} x$ is equivalent to the condition that $C_{\upsilon}(y) > C_{\upsilon}(x)$.

Instead of comparing the aggregated look-up complaints across distributions, we can compare them at individual levels. More precisely, following Zheng (2007b), for $x, y \in D_+^n$, we say that $y$ look-up dominates $x$, what we write $y \geq_{\mathrm{LU}} x$, if $(x_n - x_i) \leq (y_n - y_i)$ holds for all $1 \leq i \leq n$, with $<$ for some $i$. Thus, look-up dominance is an alternative dominance implication of Temkin's (1986, 1993) suggestion that the highest income is the reference point. Clearly, $\geq_{\mathrm{LU}}$ requires reduction in the highest income of the dominated distribution because all look-up differences are getting smaller in this distribution. It is easy to see that $\geq_{\mathrm{ADI}}$ implies $\geq_{\mathrm{LU}}$, which in turn implies $\geq_{\mathrm{CC}}$.

Our discussion so far has concentrated on distributions in a particular period. Let us denote the current and previous period income distributions on a set of $n$ persons by $x^1$ and $x^0$, respectively, where both $x^1, x^0 \in \Gamma_+^n$. However, they are not assumed to be illfare-ranked. Bossert and D'Ambrosio (2007) suggested the use of

$$\mathrm{BD}_i(x^0, x^1) = \frac{\bar{\alpha}^{|\hat{B}_i(x^1) - \hat{B}_i(x^0)|}}{n} \sum_{j \in \hat{B}_i(x^1)} (x_j^1 - x_i^1) \tag{3.24}$$

as an index of the extent of deprivation felt by person $i$, where $\bar{\alpha} \geq 1$ is a constant. $\hat{B}_i(x)$ is the set of persons that are better-off than $i$ in the distribution $x$, the difference $\hat{B}_i(x^1) - \hat{B}_i(x^0)$ gives the set of persons that are in $\hat{B}_i(x^1)$ but not in $\hat{B}_i(x^0)$ and for any set $S$, $|S|$ is the number of elements in $S$. If $\bar{\alpha} = 1$, the Bossert-D'Ambrosio index $\mathrm{BD}_i$ becomes the Yitzhaki (1979) index of deprivation $\mathrm{AD}_i$. Higher values of $\bar{\alpha}$ assign higher weight to the deprivation suffered from the information that there are people who were not previously richer than $i$ are now richer than him. This information takes into account the dynamic aspect of deprivation. Thus, the dynamic aspect of deprivation depends on the number of persons who were at most as well off as $i$ in the previous period but have now become more well-off than $i$. If the set of such persons is empty, then also $\mathrm{BD}_i$ coincides with $\mathrm{AD}_i$. This implies that if we regard $(x^0, x^1)$ as an incentive preserving fiscal program, then $\mathrm{BD}_i$ and $\mathrm{AD}_i$ are the same. Bossert and D'Ambrosio (2007) characterized general classes of indices that contain $\mathrm{BD}_i$ as special cases.

Our analysis in this chapter reveals that deprivation is a multifaceted phenomenon. There are many ways of incorporating components, such as envy and depression, of individual judgments into distributive justice. Furthermore, we have seen that the required alternative notions of redistributive principles are different from the one based on the Lorenz curve. In each case, our discussion makes the structure and the fundamental properties of the principle quite transparent. It should definitely be clear that the dominance relations we have investigated are incomplete – there may be situations where we have to withhold our judgment concerning superiority of one distribution over another in terms of deprivation.

We may recall here that the "deprivation profile" of Shorrocks (1998) looks at deprivation from a completely different perspective. We may also mention that Satts' (1996) study of relative deprivation in the kibbutz economy uses a completely different structure as well. It explores the equity characteristics of the ideal kibbutz economy which maintains perfect equality as the benchmark. An investigation of the issue has been made in the distributive and productive justice framework of Varian (1974).

Finally, it may be worthwhile to mention that some of the relations discussed in the chapter have also been analyzed from alternative perspectives. For instance, Jewitt (1989), Gilboa and Schmeidler (1994), and Landsberger and Meilijson (1994) used the absolute contentment dominance condition to characterize location-independent riskier prospects. Likewise, Doksum (1969) suggested the use of the absolute differential relations as a tail dominance, whereas Bickel and Lehmann (1979) used it as a dispersion dominance (*see also* Landsberger and Meilijson, 1994; Quiggin, 1993).