SPRINGER OPTIMIZATION AND ITS APPLICATIONS 17

Altannar Chinchuluun · Panos M. Pardalos
Athanasios Migdalas · Leonidas Pitsoulis
(Editors)

Pareto Optimality, Game Theory and Equilibria

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PARETO OPTIMALITY, GAME THEORY AND EQUILIBRIA

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Aims and Scope
Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics and other sciences.

The series *Optimization and Its Applications* publishes undergraduate and graduate textbooks, monographs and state-of-the-art expository works that focus on algorithms for solving optimization problems and also study applications involving such problems. Some of the topics covered include nonlinear optimization (convex and nonconvex), network flow problems, stochastic optimization, optimal control, discrete optimization, multi-objective programming, description of software packages, approximation techniques and heuristic approaches.

PARETO OPTIMALITY, GAME THEORY AND EQUILIBRIA

Edited By

ALTANNAR CHINCHULUUN
University of Florida, Gainesville, FL

PANOS M. PARDALOS
University of Florida, Gainesville, FL

ATHANASIOS MIGDALAS
Technical University of Crete, Greece

LEONIDAS PITSOULIS
Aristotle University of Thessaloniki, Greece

Springer

Altannar Chinchuluun
Department of Industrial
and Systems Engineering
University of Florida
Gainesville, FL

Panos M. Pardalos
Department of Industrial
and Systems Engineering
University of Florida
Gainesville, FL

Athanasios Migdalas
Department of Production Engineering
and Management
Technical University of Crete
Greece

Leonidas Pitsoulis
Department of Production Engineering
and Management
Aristotle University of Thessaloniki
Greece

Managing Editor:
Panos M. Pardalos
Department of Industrial
and Systems Engineering
University of Florida
Gainesville, FL 32611
pardalos@cao.ise.ufl.edu

Editor/ Combinatorial Optimization
Ding-Zhu Du
Department of Computer Science
and Engineering
University of Texas at Dallas
Richardson, TX 75083
dzdu@utdallas.edu

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Ἀεί δέ ὡς πρός εὖ βουλομένους τούς ἐναντίους ἔργῳ παρασκευαζόμεθα· καί οὐκ ἐξ ἐκείνων ὡς ἁμαρτησομένων ἔχει δεῖ τάς ἐλπίδας, ἀλλ᾽ ὡς ἡμῶν ἀσφαλῶς προνοουμένων, πολύ τε διαφέρειν οὐ δεῖ νομίζειν ἄνθρωπον ἀνθρώπου, κράτιστον δέ εἶναι ὅστις ἐν τοῖς ἀναγκαιοτάτοις παιδεύεται.

Ἀγόρευσις τοῦ Λακεδαιμονίου βασιλέως Ἀρχιδάμου
(ΘΟΥΚΙΔΙΔΟΥ: Ἱστορῖαι, Βιβλίον Α′)

We always base our preparations against an opponent on the assumption that his plans are sound; indeed, it is right to rest our hopes not on a belief in his blunders, but on the soundness of our provisions. Nor ought we to believe that there is much difference between man and man, but to think that superiority lies with him who is reared in the severest school.

Speech by the Spartan king Archidamus
according to Thoukidides' *History of the Peloponnesian War*.
Translation adapted from `www.wikipedia.org`

Preface

Humans have always been involved in situations where decisions must be made that best fit the circumstances. We read, for instance, in Homer's *Iliad*,[1] the oldest written European composition (eighth century B.C.):

> So he taunted. Deiphobus' mind was torn –
> should he pull back and call a friend to his side,
> some hardy Trojan, or take the Argive on alone?
> As he thought it out, the first way seemed the best.
> He went for Aeneas

The decision taken may or may not affect and be affected by other decision makers. The best decision may depend on one or more objectives of the decision maker. The decision may concern a static situation or a situation that evolves in time. Thus, mathematical and algorithmic tools have been developed in order to model, analyze, and resolve such decision-making processes. Mathematical programming, multiobjective optimization, optimal control theory, and static and dynamic game theory provide the language and the tools to achieve such goals. The notions of optimality, Pareto efficiency, and equilibrium are intimately related in a mathematical sense and tightly connected through the notions of Karush–Kuhn–Tucker (KKT) optimality, complementarity, variational inequalities, and fixed points. The problem underlying the search for an optimal point, an efficient point, an equilibrium, or a fixed point is essentially the same.

It is true that we can recognize in ancient texts the roots for the need of such mathematical formalism. It is hard to deny that in the words of the Lacedaemonian king Archidamus, as given by the historian Thoukidides (fifth century B.C.), we start to recognize seeds of rationality desired by game theory[2]:

[1] Translated by Robert Fagles, Penguin Classics.
[2] Translation adapted from `www.wikipedia.org`.

> We always base our preparations against an opponent on the assumption that his plans are sound; indeed, it is right to rest our hopes not on a belief in his blunders, but on the soundness of our provisions. Nor ought we to believe that there is much difference between man and man, but to think that superiority lies with him who is reared in the severest school.

Or that the following verses of the *Iliad* depict a game situation[3]:

> If you really want me to fight to the finish here,
> have all Trojans and Argives take their seats
> and pit me against Menelaus dear to Ares –
> right between the lines –
> we'll fight it out for Helen and all her wealth.
> And the one who proves the better man and wins,
> he'll take these treasures fairly, lead the woman home

However, only with the development of optimization, control, and game theory has it been possible to fully achieve the analysis of such and other far more complicated situations. The concepts of equilibrium and optimality are of immense practical importance in decision-making problems of policies and strategies, in understanding and predicting what will eventually happen in systems in different application domains, ranging from economics and engineering to military applications.

This book brings together recent developments in all these fields that support decision making as well as recent applications of these results to a wide range of modern problems. The book consists of twenty-nine chapters contributed by experts around the world who work with optimization, control, game theory, and equilibrium programming either at a theoretical level and/or at the level of using these tools in practice. Each chapter is of expository but also of scholarly nature. Each includes a state-of-the-art overview relative to its dedicated topic as well as key references in the field. The chapters can be divided into six partially overlapping groups.

The first five chapters of the book are concerned with minimax theory, fixed-points, and noncooperative game theory. The chapter by H. Tuy presents a unified framework for studying existence and stability conditions for minimax of quasiconvex-quasiconcave functions that refines several known results from game theory, optimization, and nonlinear analysis. The chapter by B. Ricceri surveys recent advances in minimax theory, including multiplicity theorems for nonlinear equations and well-posedness results for optimization problems. The chapter by J.B.G. Frenk and G. Kassay gives an overview on the theory of noncooperative games, both zero-sum and nonconstant-sum games. Based on the KKM lemma, they provide proofs of existence of saddle-point strategies in the former case as well as of Nash equilibrium strategies

[3] Translated by Robert Fagles, Penguin Classics.

in the latter case. The chapter by F. Szidarovszky gives an overview of the existence and computation of equilibrium in nonlinear n-person games. The chapter by G. Isac develops a new method for the study of existence of fixed points for nonexpansive mappings defined on unbounded sets.

Cooperative game theory is concerned with situations in which decision makers agree to cooperate in order to maximize their profits or minimize their costs. In the chapter by I. Curiel, cooperative combinatorial games are considered. Such games model situations in which the decision makers who agree to cooperate encounter a combinatorial optimization problem in order to maximize their profits or minimize their costs. Eight cooperative combinatorial games are surveyed and analyzed. The chapter by X. Deng and Q. Fang highlights the linear and integer programming approaches to cooperative combinatorial games as well as computational complexity issues. The chapter by J.M. Bilbao et al. introduces the notions of bicooperative games and bisupermodular games and describes several solution concepts for them. The chapter by Y. Marinakis et al. surveys more than thirteen cooperative combinatorial games and provides insight through numerical examples.

The next five chapters are concerned with dynamic systems, in particular with differential games and time-dependent equilibria. The chapter by A. Maugeri and C. Vitanza provides a review of the variational inequality approach to problems in a variety of fields including traffic networks, models dynamic equilibrium problems as time-dependent variational inequalities, presents existence results, and applies infinite dimensional Lagrangian duality to these inequalities. The chapter by P.M. Pardalos et al. deals with differential games of multiple agents in a hierarchical structure setting as well as in a cooperative setting. Controllability, observability, and optimality problems are studied. Maneuvers are introduced, using fiber bundles. The chapter by V. Ostapenko is devoted to developing convex analysis concepts in the context of pursuit-evasion differential games. The notion of matrix-convex sets and H-convex sets are introduced, and their properties required for the theory of differential games are studied. The chapter by A.A. Chikrii provides a general approach to solving game problems when the dynamics of the conflict-controlled process is described by a system with fractional derivatives. Solutions to such systems are derived and sufficient conditions for termination in guaranteed time are obtained. The chapter by M.-G. Cojocaru et al. establishes the equivalence between the solutions to an evolutionary variational inequality and the critical points of a projected dynamical system in infinite dimensional spaces. A convergent algorithm is derived for the solution of evolutionary variational inequalities, and it is illustrated for the case of traffic networks.

Information is crucial in the process of decision making. The next two chapters are largely concerned with the role and implications of information in audit policies and auction design. In the chapter by K. Chatterjee et al., a simple auditing model is constructed and analyzed in order to address three principle issues: the information contained in the report, the commitment to the audit policy, and the audit effort. The approach is based on the concept of perfect Bayesian equilibrium. An auction is a game with partial information where an agent's valuation of an object is hidden from other agents. The chapter by R.L. Zhan provides a thorough survey on the current auctions design literature and synthesizes the developed theories underlying traditional auctions with new elements and phenomena from emerging and rapidly growing areas, such as online auctions.

The next five chapters are concerned with multiobjective optimization, bilevel optimization, and linear complementarity problems. The chapter by G. Zhang et al. develops a fuzzy multiobjective linear bilevel model to handle hierarchical situations where uncertainty is present in the parameters of either the objective functions or the constraints of the leader and the follower and where the leader and the follower may have multiple objectives. They derive theorems characterizing the solutions and develop an approximation Kuhn–Tucker approach to solve the problem. The chapter by D.T. Luc is devoted to the theory of Pareto optimality and discusses existence, optimality in product spaces, scalarization via support functions, duality, and solution methods. Multiobjective optimization is overviewed in the chapter by M. Pappalardo. Theorems of solution existence as well as optimality conditions and solution methods are presented. In the chapter by R. Enkhbat et al., the weighted sum approach to finding Pareto optimal solutions in multiobjective optimization is studied in the context of one-parametric optimization techniques. The chapter by B. De Schutter is devoted to the linear complementarity problem and to its most general linear extension. A link is established between the extended problem and max-plus equations that allows the application of the extended model in the analysis and control of discrete-event systems such as traffic signals, manufacturing systems, railway networks, etc.

The remaining eight chapters are largely devoted to applications. Five chapters are devoted to network applications, one to military application, and two to supply chain management. The chapter by M. Florian and D. W. Hearn surveys user equilibrium formulations of static traffic assignment models based on Wardrop's first principle and presents the main solution algorithms for both deterministic and stochastic models. M. Bjørndal and K. Jørnsten demonstrate in the subsequent chapter that the famous traffic paradox, which essentially differentiates between Wardropian user equilibrium formulations and nonequilibrium formulations of congested traffic assignment models, also occurs in congested electricity networks, where flows follow Kirchhoff's juction rule and loop rule. Hence, it is demonstrated that grid investments may prove to be detrimental to social surplus. The chapter by J. Cole Smith and C. Lim explores models and algorithms applied to a class of Stackelberg (two-stage) games on networks, called network interdiction games. Two players are involved, an operator who wishes to execute some function on an existing network and an interdictor who acts first to strategically compromise certain elements of the network. Recent applications of stochastic models, valid inequalities, and bilinear programming techniques to network interdiction games are discussed and the problem is extended to a three-stage game, where the operator fortifies the network. The chapter by M. Min reviews game theoretical approaches in wireless networks, addressing mainly the issues of power control, cooperation between terminals, security, and radio channel access control. The chapter by D. Lozovanu is dedicated to time-discrete systems on networks where the dynamics of the system is controlled by several players. Nash and Pareto optimality principles are applied, existence results are derived, and dynamic programming techniques are utilized. Efficient polynomial-time algorithms are developed in order to find optimal strategies of players in dynamic games on networks. G. Isac and A. Gosselin address a military application of viability in terms of differential Lanchester type models. Set-valued analysis is utilized to introduce the notion of Lanchester type differential inclusions and to replace the Lanchester coefficients by intervals in order to overcome the difficulties associated with these coefficients and to facilitate the application of such models. In the chapter by A. Nagurney et al., static and dynamic models of global supply chains are developed as networks with three tiers of decision makers. A discrete-time algorithm is proposed that allows for the discretization of the continuous time trajectories. The proposed supernetwork framework formalizes the modeling and analysis of global supply chains. The final chapter by A. Chinchuluun et al. reviews classic game theoretical approaches to modeling and solving problems in supply chain management. Both noncooperative and cooperative single-period and multiperiod models are discussed.

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A. Chinchuluun, P.M. Pardalos
A. Migdalas, L. Pitsoulis

Contents

List of Contributors

Jesús M. Bilbao
Department of Applied
Mathematics II
University of Seville
Escuela Superior de Ingenieros
Camino de los Descubrimientos
41092 Sevilla, Spain
`mbilbao@us.es`

Mette Bjørndal
Department of Finance and
Management Science
Norwegian School of Economics and
Business Administration
Helleveien 30, N-5045 Bergen
Norway
`mette.bjorndal@nhh.no`

Kalyan Chatterjee
Department of Economics
The Pennsylvania State University
University Park, PA 16802, USA
`kchatterjee@psu.edu`

Arkadii A. Chikrii
Glushkov Cybernetics Institute NAS
Ukraine
40 Glushkov prsp., CSP 03680
Kiev 187, Ukraine
`chik@insyg.kiev.ua`

Altannar Chinchuluun
Center for Applied Optimization
Department of Industrial and
Systems Engineering
University of Florida
Gainesville, FL 32611
USA
`altannar@ufl.edu`

Monica–Gabriela Cojocaru
Department of Mathematics and
Statistics
University of Guelph
Guelph, Ontario
Canada
`mcojocar@uoguelph.ca`

Jose Cruz
Department of Operations and
Information Management
School of Business
University of Connecticut
Storrs, CT 06269-1041
USA
`jcruz@business.uconn.edu`

Imma Curiel
University of the Netherlands
Antilles,
Curaçao, Netherlands Antilles
`romali@interneeds.net`

Patrizia Daniele
Department of Mathematics and Computer Sciences
University of Catania
Viale A. Doria, 6, 95125 Catania
Italy
daniele@dmi.unict.it

Xiaotie Deng
Department of Computer Science
City University of Hong Kong
Hong Kong, P.R. China
csdeng@cityu.edu.hk

Tharam Dillon
Faculty of Information Technology
University of Technology
Sydney (UTS),
Broadway, NSW 2007, Australia
tharam@it.uts.edu.au

Rentsen Enkhbat
Department of Mathematics
School of Economics Studies
National University of Mongolia
Ulaanbaatar, Mongolia
renkhbat46@ses.edu.mn

Qizhi Fang
Department of Mathematics
Ocean University of China
Qingdao 266071, P.R. China
qfang@ouc.edu.cn

Julio R. Fernández
Department of Applied Mathematics II
University of Seville
Escuela Superior de Ingenieros
Camino de los Descubrimientos
41092 Sevilla, Spain
julio@us.es

Michael Florian
Center for Research on Transportation
University of Montreal
Montreal, Canada, H3C 3J7
mike@crt.umontreal.ca

Johannes B.G. Frenk
Econometric Institute
Erasmus University
Rotterdam, The Netherlands
frenk@few.eur.nl

Alain Gosselin
Department of Mathematics
Royal Military College of Canada
Kingston, Ontario
Canada K7K 7B4
gosselin-a@rmc.ca

Jurgen Guddat
Institute of Mathematics
Humboldt University Berlin
Berlin, Germany
guddat@mathematik.hu-berlin.de

Donald W. Hearn
Department of Industrial and Systems Engineering
University of Florida
Gainesville, FL 32611, USA
hearn@ise.ufl.edu

George Isac
Department of Mathematics
Royal Military College of Canada
STN Forces, Kingston, Ontario
Canada K7K 7B4
isac-g@rmc.ca

Nieves Jiménez
Department of Applied Mathematics II
University of Seville
Escuela Superior de Ingenieros
Camino de los Descubrimientos
41092 Sevilla, Spain
niejimjim@us.es

Kurt Jørnsten
Department of Finance and Management Science
Norwegian School of Economics and Business Administration
Helleveien 30, N-5045 Bergen
Norway
kurt.jornsten@nhh.no

Athanasia Karakitsiou
Decision Support Systems Laboratory
Department of Production Engineering and Management
Technical University of Crete
73100 Chania, Greece
atkar@verenike.ergasya.tuc.gr

Gábor Kassay
Faculty of Mathematics and Computer Science
Babeş-Bolyai University
Cluj, Romania
kassay@math.ubbcluj.ro

Churlzu Lim
Systems Engineering and Engineering Management
University of North Carolina at Charlotte
9201 University City Blvd.
Charlotte, NC 28223, USA
clim2@uncc.edu

Jorge J. López
Department of Applied Mathematics II
University of Seville
Escuela Superior de Ingenieros
Camino de los Descubrimientos
41092 Sevilla, Spain
jorlopvaz@us.es

Jie Lu
Faculty of Information Technology
University of Technology Sydney (UTS),
Broadway, NSW 2007, Australia
jielu@it.uts.edu.au

Dinh The Luc
Department of Mathematics
University of Avignon
Avignon, France
dtluc@univ.avignon.fr

Dmitrii Lozovanu
Moldova State University
Faculty of Mathematics and Informatics
Institute of Mathematics and Computer Science
Moldovan Academy of Sciences
Academy str., 5, Chisinau, MD–2028
Moldova
lozovanu@math.md

Yannis Marinakis
Decision Support Systems Laboratory
Department of Production Engineering and Management
Technical University of Crete
73100 Chania, Greece
marinakis@ergasya.tuc.gr

Antonino Maugeri
Department of Mathematics and Computer Sciences
University of Catania, Viale A. Doria, 6, 95125 Catania, Italy
maugeri@dmi.unict.it

Athanasia Mavrommati
Decision Support Systems Laboratory
Department of Production Engineering and Management
Technical University of Crete
73100 Chania, Greece
atmavr@verenike.ergasya.tuc.gr

Athanasios Migdalas
Decision Support Systems Laboratory
Department of Production Engineering and Management
Technical University of Crete, 73100 Chania, Greece
sakis@verenike.ergasya.tuc.gr

Manki Min
Department of Electrical Engineering and Computer Science
South Dakota State University
Brookings, SD, USA
manki.min@sdstate.edu

Sanford Morton
Private Consultant
Seattle, WA, USA
smorton@pobox.com

Anna Nagurney
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, MA 01003, USA
nagurney@gbfin.umass.edu

Artyom G. Nahapetyan
Center for Applied Optimization
Department of Industrial and Systems Engineering
University of Florida
Gainesville, FL 32611, USA
artyom@ufl.edu

Valentin Ostapenko
Training-Scientific Complex “Institute of Applied System Analysis”
National Technical University of Ukraine “Kyiv Polytechnical Institute”
Kiev, Ukraine
nmo@mmsa.ntu-kpi.kiev.ua

Massimo Pappalardo
Department of Applied Mathematics
University of Pisa
via Buonarroti 1/c, 56127 - Pisa
Italy
pappalardo@dma.unipi.it

Panos M. Pardalos
Center for Applied Optimization
Department of Industrial and Systems Engineering
University of Florida
Gainesville, FL 32611, USA
pardalos@ufl.edu

Biagio Ricceri
Department of Mathematics
University of Catania
Viale A. Doria 6, 95125 Catania
Italy
ricceri@dmi.unict.it

Bart De Schutter
Delft Center for Systems and Control
Delft University of Technology
Mekelweg 2, 2628CD Delft
The Netherlands
b.deschutter@dcsc.tudelft.nl

Ferenc Szidarovszky
SIE Department
The University of Arizona
Tucson, AZ, USA
szidar@sie.arizona.edu

J. Cole Smith
Department of Industrial and Systems Engineering
University of Florida
Gainesville, FL 32611, USA
cole@ise.ufl.edu

Fuminori Toyasaki
Faculty of Management
McGill University
Montreal, Quebec H3A 1G5, Canada
fuminori.toyasaki@mcgill.ca

Hoang Tuy
Institute of Mathematics
10307 Hanoi, Vietnam
htuy@math.ac.vn

Carmela Vitanza
Department of Mathematics
University of Messina
Contrada Papardo, Salita Sperone
31, 98100 Messina, Italy
vitanzac@unime.it

Vitaliy A. Yatsenko
Center for Applied Optimization
Department of Industrial and
Systems Engineering
University of Florida
Gainesville, FL 32611, USA
yatsenko@ufl.edu

Roger L. Zhan
Department of Industrial and
Systems Engineering
University of Florida
Gainesville, FL 32611, USA
zhan@ufl.edu

Guangquan Zhang
Faculty of Information Technology
University of Technology
Sydney (UTS)
Broadway, NSW 2007, Australia
zhangg@it.uts.edu.au

Part I

Game and Game Theory

Minimax: Existence and Stability

Hoang Tuy

Institute of Mathematics, Hanoi, Vietnam
htuy@math.ac.vn

Abstract A unified framework is presented for studying existence and stability conditions for minimax of quasiconvex quasiconcave functions. These theorems include as special cases refinements of several known results from game theory, optimization, and nonlinear analysis. In particular, existence conditions are developed that turn out to be sufficient also for the continuity of the saddle value and stability of the saddle point under continuous perturbation. Also, a lopsided minimax theorem is established that yields as immediate corollaries both von Neumann's classic minimax theorem and Nash's theorem on noncooperative equilibrium.

Key words: minimax theorems, quasiconvex quasiconcave functions, saddle value, existence conditions, stability conditions, lopsided minimax, cooperative equilibrium

1 Introduction

Let X, Y be two finite-dimensional Euclidean spaces. Given two closed convex sets $C \subset X,\ D \subset Y$ and a function $F(x,y) : C \times D \to \mathbb{R}$, we define

$$\gamma := \inf_{x\in C} \sup_{y\in D} F(x,y), \quad \eta := \sup_{y\in D} \inf_{x\in C} F(x,y). \tag{1}$$

We are interested in conditions under which $\gamma = \eta \in \mathbb{R}$, i.e.,

$$\inf_{x\in C} \sup_{y\in D} F(x,y) = \sup_{y\in D} \inf_{x\in C} F(x,y) \in \mathbb{R}. \tag{2}$$

If this occurs, the common value of γ and η is called a *saddle value* of the function $F(x,y)$.

Investigations on the existence of a saddle value for various classes of functions were at the beginning motivated by the theory of games. According to a classic result of von Neumann [11], later improved by Kneser [9], a saddle value exists when C, D are compact convex subsets of $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, respectively, while the function $F(x, y)$ is continuous convex in x and continuous concave in y. Subsequently, it was realized that minimax theorems also constitute a very useful tool in different areas of nonlinear analysis and optimization. This has stimulated much research activity over the years for generalizing and refining this result.

In the first period, much effort was spent on relaxing the assumptions on convexity-concavity and continuity property of $F(x, y)$ and also compactness of both C, D. The best known result in this direction was Sion's theorem [17], which only required the function $F(x, y)$ to be quasiconvex l.s.c. (lower semi-continuous) in x, quasiconcave u.s.c. (upper semi-continuous) in y, and one of the sets C, D (but not necessarily both) to be compact. The second period began with the work of Wu [25] who established the first minimax theorem in topological spaces, replacing the convexity-concavity assumption by a more general topological property. Wu's theorem required, however, rather restrictive assumptions and did not include several minimax results well-known at the time.

In 1974, by a different approach, Tuy [19, 20] (see also [15, 21, 24]) proved a topological minimax theorem in the same vein as, but much stronger than, Wu's theorem as it did contain most important results currently available in the field ([9, 12, 13]). The proof of this theorem, besides, was very simple, making use only of elementary set-theoretical arguments.

However, Tuy's theorem still required compactness of at least one of the sets C, D. This assumption turned out to be too restrictive for recent developments of mathematical programming and nonlinear analysis (see, e.g., [1, 2] also [6, 14]). To cover the cases considered in these works, weaker conditions than compactness of C or D had to be developed, and quasiconvexity-concavity of $F(x, y)$ seemed to be a convenient condition for ensuring existence of a saddle value when working in vector topological spaces. Furthermore, conceptually, all the minimax results so far available for quasiconvex quasiconcave functions look somewhat disparate, so in [22] an effort has been made to clarify the relationship between different existence conditions formulated in these theorems and on this basis strengthen and refine several known results.

Aside from existence, another important topic is stability condition for the saddle point and continuity property of the saddle value. The central result on this question, Golshtein's theorem [5] (see also [16]), though proved more than three decades ago, still remains, to our knowledge, an isolated result in this area. Although the proof of this theorem is elaborate, its assumptions are too restrictive if one only needs existence and some weak continuity of the saddle value rather than these properties for the saddle point.

The purpose of the current paper is to provide a sufficiently simple unified framework for studying existence and stability conditions for the saddle value and saddle point of quasiconvex quasiconcave functions, and to establish or to refine various strong minimax theorems known to date for this class of functions. As it turns out, most of these existence conditions are also sufficient to ensure stability, in a sense or another, of the saddle value and saddle point.

After the Introduction, in the second section we discuss fundamental minimax theorems for quasiconvex quasiconcave functions under weakest conditions. Starting from a basic lemma, established by purely set-theoretical arguments, various existence conditions are developed, mostly in more or less refined form. Section 3 deals with continuity and stability of the saddle value and saddle point under continuous perturbation. Some new results are presented that include the above-mentioned theorem of Golshtein as a corollary, while providing, as a by-product, a simple proof for this sophisticated theorem. Finally, Section 4 presents a new lopsided minimax theorem that is an extension of an ordinary minimax theorem but can also be used to derive, in a simple way, Nash theorem on cooperative equilibrium in n-person games [10].

Although for the sake of simplicity we restrict ourselves to finite dimensional spaces, the reader should be aware that many of the results to be presented can be easily extended to work in a much more general setting.

2 Existence Theorems

In this section, we discuss conditions to be imposed on the sets C, D and the function $F(x, y)$ in order to guarantee the existence of a saddle value. Note that, according to our definition of a saddle value, we require that $\gamma = \eta \in \mathbb{R}$. Because always $\gamma \geq \eta$, this excludes the cases $\gamma = -\infty$ or $\eta = +\infty$, which obviously imply that $\gamma = \eta = -\infty$ or $\gamma = \eta = +\infty$, respectively.

Following [1], we say that for a given $y \in D$ the function $x \mapsto F(x, y)$ is *l.s.c. (lower semi-continuous) in every line segment* if for every $a, b \in C$, the univariate function $\phi(\lambda) = F((1-\lambda)a + \lambda b, y)$ is l.s.c. on the segment $0 \leq \lambda \leq 1$.

The following lemma is fundamental for deriving the basic existence theorem, which includes virtually all so far known minimax theorems for quasiconvex quasiconcave functions.

Lemma 1. (Fundamental Lemma) *Assume that the function $F(x, y)$ is quasiconvex l.s.c. in x in every line segment and quasiconcave u.s.c. in y. Then for every nonempty finite set $M \subset C$ and every $\alpha < \gamma$ we have*

$$\cap_{x \in M} \{y \in D | \ F(x, y) \geq \alpha\} \neq \emptyset. \tag{3}$$

Proof. Proceeding by induction, we first prove (3) when $|M| = 2$. For every $x \in C$ let

$$D(x) := \{y \in D | \ F(x, y) \geq \alpha\}.$$

Because $\gamma > \alpha$, clearly $\sup_{y \in D} F(x, y) > \alpha \ \forall x \in C$, and it follows from the assumptions on $F(x, y)$ that every set $D(x), \ x \in C$, is nonempty and closed.

Arguing by contradiction, assume there are $a, b \in C$ such that

$$D(a) \cap D(b) = \emptyset. \tag{4}$$

Consider a point $x^\lambda = (1-\lambda)a + \lambda b$ with $0 \leq \lambda \leq 1$. If $y \in D(x^\lambda)$ then, by quasiconvexity of $F(x, y)$ in x, we have $\alpha \leq F(x^\lambda, y) \leq \max\{F(a, y), F(b, y)\}$, hence

$$D(x^\lambda) \subset D(a) \cup D(b). \tag{5}$$

Because $D(x^\lambda)$ is convex, while $D(a), D(b)$ are disjoint by (4), $D(x^\lambda)$ cannot simultaneously meet $D(a)$ and $D(b)$. Consequently, for every $\lambda \in [0, 1]$, one and only one of the following alternatives holds:

$$\text{(a) } D(x_\lambda) \subset D(a); \qquad \text{(b) } D(x_\lambda) \subset D(b).$$

Denote by $L_a(L_b$, respectively) the set of all $\lambda \in [0, 1]$ satisfying (a) (satisfying (b), respectively). Clearly $0 \in L_a, 1 \in L_b, L_a \cup L_b = [0, 1]$ and, analogously to (5):

$$D(x_\lambda) \subset D(x_{\lambda_1}) \cup D(x_{\lambda_2}) \quad \text{whenever } [\lambda_1 \leq \lambda \leq \lambda_2]. \tag{6}$$

Therefore, $\lambda \in L_a$ implies $[0, \lambda] \subset L_a$, and $\lambda \in L_b$ implies $[\lambda, 1] \subset L_b$. Let $s = \sup L_a = \inf L_b$ and assume for instance that $s \in L_a$ (the argument is similar if $s \in L_b$). We show that (4) leads to a contradiction.

We cannot have $s = 1$, for this would imply $D(b) \subset D(a)$. Therefore, $0 \leq s < 1$. Because $\alpha < \gamma \leq \sup_{y \in D} F(x_s, y)$, it follows that $F(x_s, \bar{y}) > \alpha$ for some $\bar{y} \in D$. Because $F(x_\lambda, y)$ is l.s.c. in λ, there is $\varepsilon > 0$ such that $F(x_{s+\varepsilon}, \bar{y}) > \alpha$ and so $\bar{y} \in D(x_{s+\varepsilon})$. But $\bar{y} \in D(x_s) \subset D(a)$, hence $D(x_{s+\varepsilon}) \subset D(a)$, i.e., $s + \varepsilon \in L_a$, contradicting the definition of s. Thus (4) cannot occur, and so the proposition holds when $|M| = 2$.

Assuming now that the proposition holds for $|M| = k$, let us prove it for $|M| = k + 1$. Let $M = \{x^1, \ldots, x^k, x^{k+1}\} \subset C$ and $D' = D(x^{k+1})$. From the above, for any $\alpha' \in (\alpha, \gamma)$ and any $x \in C$ we have $\{y \in D|\ F(x^{k+1}, y) \geq \alpha', F(x, y) \geq \alpha'\} \neq \emptyset$, hence $\{y \in D'|\ F(x, y) \geq \alpha'\} \neq \emptyset$, i.e.,

$$\forall x \in C \quad \exists y \in D' \qquad F(x, y) \geq \alpha',$$

which implies that $\inf_{x \in C} \sup_{y \in D'} F(x, y) \geq \alpha' > \alpha$. By the induction hypothesis, the proposition holds for k points, so by applying it, with D replaced by D', we have

$$\cap_{i=1}^k D^{'}(x^i) \neq \emptyset,$$

hence $\cap_{i=1}^{k+1} D(x^i) \neq \emptyset$. ■

Lemma 2. *If $\{E_i|\ i \in I\}$ is an arbitrary collection of closed convex sets in X whose intersection is nonempty and compact, then there is a finite set $J \subset I$ such that $\cap_{j \in J} E_j$ is nonempty and compact.*

Proof. Let K_i be the recession cone of E_i and K the recession cone of $E = \cap_{i\in I} E_i$. As is well-known, K, K_i are closed convex cones, and $K = \cap_{i\in I} K_i$. Let A, A_i be the intersection of K, K_i, resp., with the unit sphere $S = \{x \in \mathbb{R}^n |\ \|x\| = 1\}$. Clearly $A = \cap_{i\in I} A_i$ and $A_i, i \in I$, are closed subsets of the compact set S. Therefore, if E is bounded, i.e., $A = \emptyset$, there must exist a finite set $J \subset I$ such that $\cap_{j\in J} A_j = \emptyset$. Then the closed set $\cap_{j\in J} E_j$ is nonempty and bounded, hence compact. ■

An immediate consequence of the above lemmas is the following basic theorem.

Theorem 1. *Assume that the function $F(x, y)$ is quasiconvex l.s.c. in x in every line segment for fixed y, quasiconcave u.s.c. in y for fixed x, and, in addition, that*

(M) *There exist a nonempty finite set $M \subset C$ and a real number $\alpha \leq \gamma$ such that the set $D^M := \{y \in D|\ \min_{x\in M} F(x, y) \geq \alpha\}$ is nonempty and compact.*

Then

$$\max_{y\in D} \inf_{x\in C} F(x, y) = \inf_{x\in C} \sup_{y\in D} F(x, y). \tag{7}$$

Proof. Because $D^M = \cap_{k=1}^{+\infty} D_k^M$, with $D_k^M = \{y \in D|\ \min_{x\in M} F(x, y) \geq \alpha - 1/k$, and each D_k^M is a closed convex set, by Lemma 2 there exists r such that D_r^M is nonempty and compact. Therefore, by replacing α with $\alpha - 1/r$ if necessary we can assume that $\alpha < \gamma$. For a fixed natural h, take a $\gamma_h \in (\alpha, \gamma)$ and consider the sets

$$D^h(x) = \{y \in D|\ \min_{x'\in M} F(x', y) \geq \gamma_h,\ F(x, y) \geq \gamma_h\},\ x \in C.$$

These are all closed subsets of the compact set D^M, and by Lemma 1 they have the finite intersection property. Therefore, there exists $y^h \in D^M$ such that $\inf_{x\in C} F(x, y^h) \geq \gamma_h$. Noting that D^M is compact, while the function $y \mapsto \inf_{x\in C} F(x, y)$ is u.s.c., it then follows that for $\gamma_h \to \gamma$, the sequence $\{y^h\} \subset D^M$ has a cluster point $\bar{y} \in D^M$ such that $\inf_{x\in C} F(x, \bar{y}) \geq \gamma$. We cannot have $\inf_{x\in C} F(x, \bar{y}) = +\infty$ because this would imply $F(x, \bar{y}) = +\infty\ \forall x \in C$. Therefore $\gamma \leq \max_{y\in D^M} \inf_{x\in C} F(x, y) < +\infty$, and because always $\max_{y\in D} \inf_{x\in C} F(x, y) \leq \gamma = \inf_{x\in C} \sup_{y\in D} F(x, y)$, the equality (7) follows. ■

Remark 1. Condition (M) obviously holds if D is compact while $\gamma < +\infty$, because for any $a \in C$ and $M = \{a\} \subset C$, the set D^M is nonempty. This is essentially a boundedness condition for D and in fact it is present, in one form or another, in all particular minimax theorems so far known for quasiconvex quasiconcave functions.

Here are various variants of such existence conditions:

(M′) *There exists a finite set* $M \subset C$ *such that for every* $\alpha \in \mathbb{R}$*, the set* $\{y \in D|\ \min_{x\in M} F(x,y) \geq \alpha\}$ *is compact;*

($\widetilde{\mathrm{M}}$) *There exists a finite set* $M \subset C$ *such that* $\min_{x\in M} F(x,y) \to -\infty$ *as* $y \in D, \|y\| \to +\infty$;

(H) *There exist* $\alpha \in \mathbb{R}$ *such that the set* $D^\alpha := \{y \in D|\ \inf_{x\in C} F(x,y) \geq \alpha\}$ *is nonempty and compact;*

(H′) *For every* $\alpha \in \mathbb{R}$*, the set* $D^\alpha := \{y \in D|\ \inf_{x\in C} F(x,y) \geq \alpha\}$ *is compact;*

($\widetilde{\mathrm{H}}$) $\inf_{x\in C} F(x,y) \to -\infty$ as $y \in D, \|y\| \to +\infty$.

(H*) The set $D^* := \{y \in D|\ \inf_{x\in C} F(x,y) = \eta\}$ is nonempty and compact.

Theorem 2. *Any one of the above conditions implies* (M)*, while*

$$(\widetilde{\mathrm{M}}) \quad \Leftrightarrow \quad (\mathrm{M}'); \tag{8}$$

$$(\widetilde{\mathrm{H}}) \quad \Leftrightarrow \quad (\mathrm{H}') \quad \Rightarrow \quad (\mathrm{H}) \quad \Leftrightarrow \quad (\mathrm{H}^*). \tag{9}$$

Consequently, Theorem 1 still holds with condition (M) replaced by any one of the above conditions.

Proof. (M′) ⇒ (M). If (M′) holds, then, because the function $\min_{x\in M} F(x,y)$ is u.s.c. (as lower envelope of a family of u.s.c. functions in y), while for every $\alpha \in \mathbb{R}$ the set $\{y \in D|\ \min_{x\in M} F(x,y) \geq \alpha\}$ is compact, it is easily seen that this function has a maximum on D, i.e., $\max_{y\in D} \min_{x\in M} F(x,y) \in \mathbb{R}$. On the other hand, from (M′) and the fact $\inf_{x\in C} F(x,y) \leq \min_{x\in M} F(x,y)$, it follows that for any $\alpha \in \mathbb{R}$, the set $\{y \in D|\ \inf_{x\in C} F(x,y) \geq \alpha\}$ is compact. Then the u.s.c. function $\inf_{x\in C} F(x,y)$ must achieve a maximum on D, so that $\eta = \max_{y\in D} \inf_{x\in C} F(x,y)$. By taking $\tilde{y} \in D$ such that $\min_{x\in M} F(x,\tilde{y}) = \max_{y\in D} \min_{x\in M} F(x,y) \geq \eta$, we have $\tilde{y} \in D^M := \{y \in D|\ \min_{x\in M} F(x,y) \geq \eta\}$, so the set D^M, with $\eta \leq \gamma$, is nonempty and bounded, i.e., condition (M) holds.

($\widetilde{\mathrm{M}}$) ⇔ (M′). Immediate.

($\widetilde{\mathrm{M}}$) ⇒ (H). If ($\widetilde{\mathrm{M}}$) holds, then for any $\alpha \in \mathbb{R}$, the set $\{y \in D|\ \min_{x\in M} F(x,y) \geq \alpha\}$ is compact, hence its closed subset $\{y \in D|\ \inf_{x\in C} F(x,y) \geq \alpha\}$, too, is compact. On the other hand for $\alpha < \eta$, the latter set is nonempty, by the definition of η, so (H) holds.

(H) ⇒ (M). If (H) holds, i.e., for some $\alpha \in \mathbb{R}$ the set D^α is nonempty and compact, then, because $D^\alpha = \cap_{x\in C}\{y \in D|\ F(x,y) \geq \alpha\}$, by Lemma 2 there exists a nonempty finite set $M \subset C$ such that D^M is nonempty and compact; furthermore, $D^\alpha \neq \emptyset \Rightarrow \alpha \leq \eta \leq \gamma$, so (M) holds.

($\widetilde{\mathrm{H}}$) ⇔ (H′). Immediate.

(H′) $\Rightarrow$ (H). If (H′) holds, then for every $\alpha \in \mathbb{R}$, the set $D^\alpha{:=}\{y \in D|\ \inf_{x\in C} F(x,y) \geq \alpha\}$ is compact, and for $\alpha \leq \eta$ the set D^α is nonempty from the definition of η, so (H) holds.

(H) $\Leftrightarrow$ (H*). Immediate. ∎

For any given $x \in C$, we say that the function $y \mapsto F(x,y)$ is *u.s.c. in every line segment* if for any $a, b \in D$ the univariate function $\psi(\lambda) = F(x, (1-\ \lambda)a + \lambda b)$ is u.s.c. on the segment $0 \leq \lambda \leq 1$. Using the obvious relation $\inf_{x\in C} \sup_{y\in D} F(x,y) = -\sup_{x\in C} \inf_{y\in D}(-F(x,y))$, we easily deduce from Theorems 1 and 2 the following propositions:

Theorem 3. *Assume that the function $F(x,y)$ is quasiconvex l.s.c. in x for fixed y, quasiconcave u.s.c. in every line segment in y for fixed x, and, in addition:* (N) *There exist a nonempty finite set $N \subset D$ and a real number β such that the set $C^N := \{x \in C|\ \max_{y\in N} F(x,y) \leq \beta\}$ is nonempty and compact;*

Then

$$\min_{x\in C} \sup_{y\in D} F(x,y) = \sup_{y\in D} \inf_{x\in C} F(x,y). \tag{10}$$

Theorem 4. *Theorem 3 still holds with condition* (N) *replaced by any one of the conditions listed below:*

(N′) There exists a finite set $N \subset D$ such that for every $\beta \in \mathbb{R}$, the set $\{x \in C|\ \max_{y\in N}\ F(x,y) \leq \beta\}$ is compact;

($\widetilde{\text{N}}$) There exists a finite set $N \subset D$ such that $\max_{y\in D} F(x,y) \to +\infty$ *as $x \in C, \|x\| \to +\infty$;*

(K) There exist $\beta \in \mathbb{R}$ such that the set $C^\beta := \{x \in C|\ \sup_{y\in D} F(x,y) \leq \beta\}$ is nonempty and compact;

(K′) For every $\beta \in \mathbb{R}$, the set $C^\beta := \{y \in D|\ \inf_{x\in C} F(x,y) \leq \beta\}$ is compact;

($\widetilde{\text{K}}$) $\sup_{y\in D} F(x,y) \to +\infty$ *as $x \in C, \|x\| \to +\infty$.*

(K) The set $C^* := \{x \in C|\ \sup_{y\in D} F(x,y) = \eta\}$ is nonempty and compact.*

Remark 2. Most known minimax theorems for quasiconvex quasiconcave functions, including Sion's well-known result and some refined versions of it as used in nonlinear analysis (see, e.g., [1, 2]), are special cases of the above propositions.

Also note that the earliest proofs for minimax theorems used fixed point or separation arguments in one form or another. The above proof, given originally in [19, 20], was the first one using only elementary set-theoretical arguments for establishing general topological minimax theorems. The results in the mentioned papers with their proofs have been presented, partially or in full, in some books (see, e.g., [15, 24]). Nevertheless, exactly the same results were rediscovered in [4], with only a difference of notation. Also, the above simple proof was many years later rediscovered in Joo [7,8], according to Frenk and Kassay [3].

In the above propositions, $F(x,y)$ is always assumed to be l.s.c. in x, u.s.c. in y. We now prove some minimax theorems for *functions* $F(x,y)$ *l.s.c. in each variable, or u.s.c in each variable.*

Lemma 3. *Assume that the function $F(x,y)$ is quasiconvex u.s.c in x in every line segment for fixed y and quasiconcave u.s.c. in y for fixed x. If condition* (M) *in Theorem 1 is satisfied, then for any $\alpha' \in (\alpha, \gamma)$, the family of sets*

$$D(x) := \{y \in D^M |\ F(x,y) \geq \alpha'\}, \quad x \in C,$$

have the finite intersection property.

Proof. The proof is similar to that of Lemma 1, with the following change in the argument for showing that (4) cannot occur.

For every $y \in D(b)$, because $s \in L_a$, i.e., $D(x_s) \subset D(a)$, we have $y \notin D(x_s)$, hence $F(x_s,y) < \alpha'$, and by upper semi-continuity of $F(x_\lambda, y)$ in λ there exists an open interval $I_y = (s_1, s_2)$ containing s $(s_1 = s_1(y), s_2 = s_2(y))$ such that $F(x_\lambda, y) < \alpha'$ for all $\lambda \in I_y$. Then $F(x_{s_i}, y) < \alpha'$, i.e., $y \notin D(x_{s_i}),\ i = 1,2$, and using the closedness of the sets $D(x_{s_i}), i = 1,2$ we can find for each $i = 1,2$ a neighborhood $W_i(y)$ of y such that $F(x_{s_i}, z) < \alpha'\ \forall z \in W_i(y)$. Clearly $W_y = W_1(y) \cap W_2(y)$ is a neighborhood of y such that $F(x_{s_i}, z) < \alpha'$ for all $z \in W_y$, i.e., $z \notin D(x_{s_i}),\ i = 1,2$, and hence, $z \notin D(x_\lambda)$ for all $\lambda \in I_y$. Thus for every $y \in D(b)$, there exist a neighborhood W_y and an interval I_y satisfying

$$F(x_\lambda, z) < \alpha' \quad \forall \lambda \in I_y, \forall z \in W_y.$$

Because $D(b)$ is a closed subset of the compact set D^M, it is itself compact, and from the family $\{W_y, y \in D(b)\}$ one can extract a finite collection $\{W_y, y \in E\}, |E| < +\infty$, still covering $D(b)$. If $\lambda \in I := \cap_{y \in E} I_y$ and $y \in D(b)$, then $y \in W_{y'}$ for some $y' \in E$, hence $F(x_\lambda, y) < \alpha'$. Therefore, $D(x_\lambda) \subset D(a)$ for all $\lambda \in I$, i.e., $I \subset L_a$, contradicting the definition of s. ■

Theorem 5. *Assume that the function $F(x,y) : C \times D \to \mathbb{R}$ is quasiconvex u.s.c. in x in every line segment for fixed y, quasiconcave u.s.c. in y for fixed x. If condition* (M) *in Theorem 1 is satisfied, then the equality (7) holds.*

Proof. The proof is similar to that of Theorem 1, but using Lemma 3 instead of Lemma 1. Specifically, for a sequence $\gamma_k \in (\alpha, \gamma), \gamma_k \nearrow \gamma$ consider the sets

$$D^k(x) = \{y \in D |\ \min_{x' \in M} F(x', y) \geq \gamma_k,\ F(x,y) \geq \gamma_k\},\ x \in C.$$

For k fixed they are all closed subsets of the compact set D^M and by Lemma 3 they have the finite intersection property. Therefore, these sets have a nonempty intersection, i.e., there exists $y^k \in D^M$ such that $\inf_{x\in C} F(x,y^k) \geq \gamma_k$. Because D^M is compact, while the function $y \mapsto \inf_{x\in C} F(x,y)$ is u.s.c., it follows that the sequence $\{y^k\} \subset D^M$ has a cluster point $\bar{y} \in D^M$ such that $\inf_{x\in C} F(x,\bar{y}) \geq \gamma$. Consequently, $\max_{y\in D} \inf_{x\in C} F(x,y) \geq \gamma$ and because always $\max_{y\in D} \inf_{x\in C} F(x,y) \leq \gamma = \inf_{x\in C} \sup_{y\in D} F(x,y)$, the equality (7) follows. ■

Theorem 6. *Assume that the function $F(x,y) : C \times D \to \mathbb{R}$ is quasiconvex l.s.c. in x for fixed y, quasiconcave l.s.c. in y in every line segment for fixed x. If condition* (N) *in Theorem 3 is satisfied, then the equality (10) holds.*

Remark 3. Propositions analogous to Theorems 1 and 1b can also be derived from Theorems 5 and 6, for example:

If $F(x,y)$ is as in Theorem 5 but satisfies either (H) or (H*) (instead of (M)), then (10) holds.

As a special case of Theorem 6, let us mention the following result of Golshtein ([5], Theorem 2), which was established (by the way, by a rather elaborate argument) to provide a tool for the foundation of a general duality theory of convex programming:

Assume that the function $F(x,y) : C\times D \to \mathbb{R}$ is convex continuous in x for fixed y and concave in y for fixed x. If the set $C^* := \{x \in C|\ \sup_{y\in D} F(x,y) = \gamma\}$ is nonempty and compact, then the equality (10) holds.

Proof. A concave function on a convex set is always l.s.c. in every line segment, while the assumption about C^* is nothing but condition (K*) in Theorem 4, which in turn implies condition (N) in Theorem 3. ■

A point $(\bar{x},\bar{y}) \in C \times D$ is said to be a *saddle point* of $F(x,y)$ on the set $C \times D$ if it satisfies

$$F(\bar{x},y) \leq F(\bar{x},\bar{y}) \leq F(x,\bar{y}) \quad \forall x \in C,\ \forall y \in D. \tag{11}$$

As is well-known (see, e.g., [2], Proposition 1.2, Chapter VI), $F(x,y)$ possesses a saddle point on $C \times D$ if and only if

$$\min_{x\in C} \sup_{y\in D} F(x,y) = \max_{y\in D} \inf_{x\in C} F(x,y)$$

and then $(\bar{x},\bar{y})$ is a saddle point if and only if $(\bar{x},\bar{y}) \in C^* \times D^*$, where

$$C^* := \{x \in C|\ \sup_{y\in D} F(x,y) = \eta\}, \quad D^* := \{y \in D|\ \inf_{x\in C} F(x,y) = \gamma\}, \tag{12}$$

and $\gamma = \eta$ is the saddle value.

Combining Theorems 1 and 3 yields:

Theorem 7. *Let $F(x,y)$ be a function quasiconvex l.s.c in x for fixed y, quasiconcave u.s.c. in y for fixed x. Assume that*

(MN) *There exist two nonempty finite sets $M \subset C$, $N \subset D$ along with real numbers α, β such that $\alpha \leq \gamma, \eta \leq \beta$, and the following sets are nonempty and compact:*

$$C^N := \{x \in C | \max_{y \in N} F(x,y) \leq \beta\}, \quad D^M := \{y \in D | \min_{x \in M} F(x,y) \geq \alpha\}. \tag{13}$$

Then the function $F(x,y)$ possesses a saddle point on $C \times D$.

Proof. By Theorem 1, $\max_{y \in D} \inf_{x \in C} F(x,y) = \gamma$, and by Theorem 3, $\min_{x \in D} \sup_{y \in D} F(x,y) = \eta$, hence (11). ■

Corollary 1. *With $F(x,y)$ as in Theorem 7, if the sets C^* and D^* are nonempty and compact, then $F(x,y)$ has a saddle point $(\bar{x}, \bar{y})$ on $C \times D$.*

Proof. Then conditions (H*) and (K*) hold, and this implies (MN), by Theorems 2 and 4. ■

3 Stability Theorems

We now turn to conditions for the existence and continuity of the saddle value and/or the saddle point of a function depending upon a parameter.

Let C, D, X, Y be as previously specified, let Ω be a metric space and $F(u,x,y) : \Omega \times C \times D \to \mathbb{R}$, a function continuous on $\Omega \times C \times D$, quasiconvex in x for fixed (u,y) and quasiconcave in y for fixed (u,x). For every $u \in \Omega$ define

$$\gamma(u) = \inf_{x \in C} \sup_{y \in D} F(u,x,y), \quad \eta(u) = \sup_{y \in D} \inf_{x \in C} F(u,x,y). \tag{14}$$

It is convenient to begin with a simple fact that will be often needed in this section.

Lemma 4. *Let $\hat{D}$ be a compact set in Y and $g(u,y)$ be an u.s.c. function on $\Omega \times \hat{D}$ satisfying* $\max_{y \in \hat{D}} g(u^*,y) < 0$. *Then there exists an open ball U around u^* such that*

$$\max_{y \in \hat{D}} g(u,y) < 0 \quad \forall u \in U.$$

Proof. By upper semi-continuity, for fixed $y \in \hat{D}$ there exists an open ball U_y around u^* and an open ball V_y around y such that $g(u,y') < 0$ $\forall u \in U_y, \forall y' \in V_y$. Because $\hat{D}$ is compact, a finite set $J \subset \hat{D}$ exists such that $\hat{D} \subset \cup_{y \in J} V_y$. Setting $U = \cap_{y \in J} U_y$ yields an open ball U around u^* such that for every $y \in \hat{D}, u \in U$ we have $y \in V_{y'}$ for some $y' \in J$, while $u \in U_{y'}$, hence $g(u,y) < 0$. ■

The following theorems have been established in [23], under slightly weaker continuity conditions for $F(u,x,y)$. For a given $u^* \in \Omega$, we set

$$\gamma^* = \gamma(u^*), \quad \eta^* = \eta(u^*).$$

Theorem 8. *Assume condition* (M) *holds, i.e., there exist a nonempty finite set* $M \subset C$ *and a real number* $\alpha \leq \gamma^*$ *such that the set* $D^M(u^*) := \{y \in D|\ \min_{x\in M} F(u^*,x,y) \geq \alpha\}$ *is nonempty and compact. Then* $\gamma^* = \eta^*$ *and there exist a compact set* $D^0 \subset D$ *and an open ball* U *around* u^* *such that*

$$\emptyset \neq \{y \in D|\ \min_{x\in M} F(u,x,y) \geq \alpha\} \subset D^0 \quad \forall u \in U, \tag{15}$$

and the function $\eta(u) = \sup_{y\in D} \inf_{x\in C} F(u,x,y)$ *is upper semi-continuous at* u^*.

Proof. First, by Theorem 1, $\eta^* = \gamma^*$. Now, define $\psi(u,y) := \min_{x\in M} F(u,x,y)$. Clearly $D^M(u^*) = \{y \in D|\ \psi(u^*,y) \geq \alpha\} = \cap_{k=1}^{+\infty} D_k^M$ where $D_k^M(u^*) := \{y \in Y|\ \psi(u^*,y) \geq \alpha - 1/k\}$ are closed convex sets. Hence, by Lemma 2 there exists k_0 such that $D_{k_0}^M(u^*)$ is compact, and by replacing α with $\alpha' := \alpha - 1/k_0 < \alpha$ it can be assumed that $\alpha < \gamma^*$ and

$$\max_{y\in D^M(u^*)} \psi(u^*,y) > \alpha.$$

Because $\psi(u,y)$ is pointwise minimum of finitely many functions continuous in (u,y) and quasiconcave in y, it is continuous in (u,y) and quasiconcave in y. Therefore, the function $u \mapsto \max_{y\in D^M(u^*)} \psi(u,y)$ is l.s.c. and because $\max_{y\in D^M(u^*)} \psi(u^*,y) > \alpha$, there is an open ball U around u^* such that

$$\max_{y\in D^M(u^*)} \psi(u,y) > \alpha \quad \forall u \in U. \tag{16}$$

In particular, $D^M(u) := \{y \in D|\ \psi(u,y) \geq \alpha\} \neq \emptyset \quad \forall u \in U$. Let us show that the ball U can be chosen so that all sets $D^M(u),\ u \in U$, are contained in a compact set $D^0 \subset D$.

Let $\widetilde{D} = D^M(u^*)$. It suffices of course to consider the case when D is noncompact, so that $D \setminus \widetilde{D} \neq \emptyset$. For $\delta > 0$ consider the sets

$$D_\delta = \{y \in D|\ \delta \leq d(y,\widetilde{D}) \leq 2\delta\}, \quad D^0 = \{y \in D|\ d(y,\widetilde{D}) \leq 2\delta\},$$

where $d(y,\widetilde{D}) = \min_{y'\in\widetilde{D}} \|y - y'\|$ is the distance from y to the set $\widetilde{D}$.

Because $\widetilde{D}$ is nonempty by Lemma 1 and compact by assumption, D_δ and D^0 are also nonempty and compact, and we have

$$\psi(u^*,y) < \alpha \quad \forall y \in D_\delta. \tag{17}$$

But the function $\psi(u,y)$ is u.s.c. as it is pointwise minimum of a family of continuous functions, so by Lemma 4 there exists an open ball around u^* (which can be considered to be the same U) such that

$$\psi(u,y) < \alpha \quad \forall u \in U,\ \forall y \in D_\delta. \tag{18}$$

Furthermore, as $D\backslash D^0 \neq \emptyset$, we can consider the value $\psi(u,y)$ at any $u \in U$ and $y \in D \setminus D^0$. From (16), there is a point $y' \in \widetilde{D}$ such that $\psi(u,y') > \alpha$. Then the line segment $[y,y']$ joining y with y' contains at least a point $y'' \in D_\delta$. Because $\psi(u,y') > \alpha > \psi(u,y'')$ by (18), i.e., $\psi(u,y') > \psi(u,y'')$, while $y'' = \tau y + (1-\tau)y',\ 0 < \tau < 1$, it follows from the quasiconcavity of the function $y \mapsto \psi(u,y)$ that its minimum over the line segment $[y,y']$ is attained at y, i.e., $\psi(u,y) < \alpha$. Therefore,

$$\psi(u,y) < \alpha \quad \forall u \in U,\ \forall y \in D\backslash D^0, \tag{19}$$

which implies that $\{y \in D|\ \psi(u,y) \geq \alpha\} \subset D^0 \quad \forall u \in U$, proving (15). It remains to prove the upper semi-continuity of $\eta(u)$ at u^*.

By Theorem 1, $\max_{y\in D} \inf_{x\in C} F(u^*,x,y) = \eta^* \geq \alpha$, hence the maximum of $\inf_{x\in C} F(u^*,x,y)$ is achieved at a point in D^0. So $\max_{y\in D^0} \inf_{x\in C} F(u^*,x,y) = \eta^*$, and consequently, for any given $\varepsilon > 0$,

$$\max_{y\in D^0} \inf_{x\in C} F(u^*,x,y) < \eta^* + \varepsilon.$$

Because the function $(u,y) \mapsto \inf_{x\in C} F(u,x,y)$ is u.s.c. (pointwise minimum of a family of continuous functions), by Lemma 4 there exists an open ball $W \subset U$ around u^* such that

$$\max_{y\in D^0} \inf_{x\in C} F(u,x,y) < \eta^* + \varepsilon \quad \forall u \in W. \tag{20}$$

But from (19)

$$\sup_{y\in D\backslash D^0} \inf_{x\in C} F(u,x,y) \leq \sup_{y\in D\backslash D^0} \psi(u,y) \leq \alpha \quad \forall u \in W. \tag{21}$$

Hence, by noting that $\alpha \leq \gamma^* = \eta^*$,

$$\eta(u) := \sup_{y\in D} \inf_{x\in C} F(u,x,y) \leq \eta(u^*) + \varepsilon \quad \forall u \in W.$$

This means that $\eta(u)$ is u.s.c. at u^*, thereby completing the proof of the theorem. ■

Theorem 9. *Assume condition* (N) *holds, i.e., there exist a nonempty finite set $N \subset D$ and a real number $\beta \geq \eta^*$ such that the set $C^N(u^*) := \{x \in C|\ \max_{y\in N} F(u^*,x,y) \leq \beta\}$ is nonempty and compact. Then $\eta^* = \gamma^*$, and there exist a compact set $C^0 \subset C$ and an open ball U around u^* such that*

$$\emptyset \neq \{x \in C|\ \max_{y\in N} F(u,x,y) \leq \beta\} \subset C^0 \quad \forall u \in U, \tag{22}$$

and the function $\gamma(u) = \inf_{x\in C} \sup_{y\in D} F(u,x,y)$ is lower semi-continuous at u^*.

Proof. Analogous to Theorem 8. ■

As was recalled earlier, the function $F(u, x, y)$ possesses a saddle point on $C \times D$ if and only if

$$\min_{x \in C} \sup_{y \in D} F(u, x, y) = \max_{y \in D} \sup_{x \in C} F(u, x, y), \tag{23}$$

and then a point $(\bar{x}, \bar{y})$ is a saddle point if and only if $(\bar{x}, \bar{y}) \in C^*(u) \times D^*(u)$, where

$$C^*(u) := \operatorname{argmin}_{x \in C}(\sup_{y \in D} F(u, x, y)), \quad D^*(u) := \operatorname{argmax}_{y \in D}(\inf_{x \in C} F(u, x, y)).$$

Theorem 10. *Assume condition* (MN), *i.e.,*

> (MN) *There exist two nonempty finite sets $M \subset X, N \subset Y$, along with real numbers α, β such that $\alpha \leq \gamma^*, \eta^* \leq \beta$, and the sets $C^N(u^*) := \{x \in X |\ \max_{y \in N} F(u^*, x, y) \leq \beta\}$, $\quad D^M(u^*) := \{y \in\ Y |\ \min_{x \in M} F(u^*, x, y) \geq \alpha\}$ are nonempty and compact.*

Then there exists an open ball U around u^* such that for each $u \in U$, the function $(x, y) \mapsto F(u, x, y)$ possesses a saddle point on $C \times D$ with the property that the saddle value is a continuous function of u on U and the set-valued map $u \mapsto C^*(u) \times D^*(u)$ is upper semi-continuous at every $u \in U$.

Proof. As mentioned at the beginning of the proof of Theorem 8, without loss of generality we can assume that $\alpha < \gamma^*, \eta^* < \beta$. By Theorems 8 and 9, there are two compact sets $C^0 \subset C,\ D^0 \subset D$, and an open ball U around u^* such that for each $u \in U$:

$$\emptyset \neq \{x \in C |\ \max_{y \in N} F(u, x, y) \leq \beta\} \subset C^0, \tag{24}$$

$$\emptyset \neq \{y \in D |\ \min_{x \in M} F(u, x, y) \geq \alpha\} \subset D^0. \tag{25}$$

We show that U can be selected so that for each $u \in U$:

$$\min_{x \in C} \sup_{y \in D} F(u, x, y) = \eta(u). \tag{26}$$

Because $\sup_{y \in D} \inf_{x \in C} F(u^*, x, y) = \eta^* < \beta$, we have

$$\max_{y \in D^0} \inf_{x \in C} F(u^*, x, y) < \beta.$$

In view of the upper semi-continuity of the function $(u, y) \mapsto \inf_{x \in C} F(u, x, y)$, by Lemma 4 there exists an open ball around u^* (which can be considered to be the same U) such that $\max_{y \in D^0} \inf_{x \in C} F(u, x, y) < \beta \quad \forall u \in U$. Because D^0 is compact, this implies, by Theorem 1, $\inf_{x \in C} \sup_{y \in D^0} F(u, x, y) < \beta \quad \forall u \in U$, and so the set $\{x \in C |\ \sup_{y \in D} F(u, x, y) \leq \beta\}$ is nonempty. But this set is obviously contained in the set $\{x \in X |\ \max_{y \in N} F(u, x, y) \leq \beta\}$. Hence, according to (24),

$$\emptyset \neq \{x \in X | \sup_{y \in D} F(u,x,y) \leq \beta\} \subset C^0. \tag{27}$$

This means that condition (K) holds, and, consequently, by Theorem 4 we must have (26). Also, (27) implies that

$$\emptyset \neq C^*(u) = \mathrm{argmin}_{x \in C}(\sup_{y \in D} F(u,x,y)) \subset C^0.$$

Analogously, we show that the ball U can be chosen so that for each $u \in U$:

$$\max_{y \in D} \inf_{x \in C} F(u,x,y) = \gamma(u), \quad \emptyset \neq D^*(u) = \mathrm{argmax}_{y \in D}(\inf_{x \in C} F(u,x,y)) \subset D^0.$$

Hence,

$$\min_{x \in C} \sup_{y \in D} F(u,x,y) = \max_{y \in D} \inf_{x \in C} F(u,x,y) \quad \forall u \in U,$$

and so for every $u \in U$, the set $C^*(u) \times D^*(u)$ of saddle points is nonempty.

Furthermore, the saddle value $\sigma(u)$ is continuous at u^* by virtue of Theorems 8 and 9. Let us prove the upper semi-continuity of the mapping $u \mapsto C^*(u) \times D^*(u)$ at u^*. Let $(x^\nu, y^\nu) \in C^*(u^\nu) \times D^*(u^\nu)$, $x^\nu \to x^*$, $y^\nu \to y^*, u^\nu \to u^*$. Then $F(u^\nu, x^\nu, y^\nu) = \sigma(u^\nu)$, hence, by continuity, $F(u^*, x^*, y^*) = \sigma(u^*)$, i.e.,

$$\sup_{y \in D} \inf_{x \in C} F(u^*,x,y) = F(u^*,x^*,y^*) = \inf_{x \in C} \sup_{y \in D} F(u^*,x,y)$$

whence $(x^*, y^*) \in C^*(u^*) \times D^*(u^*)$. This means that the mapping $u \to C^*(u) \times D^*(u)$ is closed and hence, u.s.c. at u^*, because $C^*(u) \times D^*(u) \subset C^0 \times D^0$ with $C^0 \times D^0$ compact.

Because by Theorems 2 and 4 condition (HK) implies (MN), the state of affairs at every $u \in U$ is exactly the same as that at u^*. Therefore the saddle value is continuous, and the set-valued mapping $C^*(u) \times D^*(u)$ is upper semi-continuous, at every $u \in U$. This completes the proof of the theorem. ■

As immediate consequence of Theorem 10, we obtain the following important result of Golshtein established in [5] by a much more elaborate proof.

Corollary 2. *The conclusion of Theorem 10 remains valid if the following condition is satisfied:*

(CD) *The sets* $C^*(u^*) := \{x \in C | \ \sup_{y \in D} F(u^*, x, y) = \gamma^*\}$ *and* $D^*(u^*) := \{y \in D | \ \inf_{x \in C} F(u^*, x, y) = \eta^*\}$ *are nonempty and compact.*

Proof. Using the representations

$$C^*(u^*) = \{x \in C | \sup_{y \in D} F(u^*,x,y) \leq \eta^*\} = \cap_{y \in D}\{x \in C | \ F(u^*,x,y) \leq \eta^*\},$$
$$D^*(u^*) = \{y \in D | \inf_{x \in C} F(u^*,x,y) \geq \gamma^* = \cap_{x \in C}\{y \in D | \ F(u^*,x,y) \geq \gamma^*\},$$

one easily derives from Lemma 2 that assumption (CD) implies (MN) for some nonempty finite sets $M \subset C, N \subset D$ and real numbers $\alpha = \gamma^*, \beta = \eta^*$. ■

Corollary 3. *The conclusion of Theorem 10 remains valid if the following condition is satisfied:*

($\widetilde{\mathrm{M}}\widetilde{\mathrm{N}}$) *There exist two nonempty finite sets $M \subset C,\ N \subset D$ satisfying*

$$\max_{x\in M} F(u^*, x, y) \to -\infty \text{ as } y \in D,\ \|y\| \to +\infty.$$

$$\min_{y\in N} F(u^*, x, y) \to +\infty \text{ as } x \in C, \|x\| \to +\infty.$$

Proof. This follows from Theorems 10 and 2, 4. ■

4 Lopsided Minimax and Noncooperative Equilibrium

Consider a two-person zero-sum game $(C, D, F(x, y))$ where C, D are the strategy sets of the players, and $F(x, y)$ is the "loss" of the first player (the "gain" of the second player) when the first player chooses x and the second player chooses y. Following Theorem 7, if C, D are compact convex sets and $F(x, y)$ is a function continuous on $C \times D$, quasiconvex in x and quasiconcave in y, then the game has an equilibrium expressed by a saddle point of the function $F(x, y)$.

Suppose now that $F(x, y)$ is not quasiconvex in x while all other conditions in the just stated minimax theorem are satisfied. If for each strategy $x \in C$ the second player always responds by a strategy y maximizing $F(x, y)$, then how will things change?

An answer to this question is provided by a proposition that is a direct extension of the minimax proposition and can be termed a *lopsided minimax theorem* because of the dissymmetry between the two players. As it turns out, this extension, furthermore, includes as an immediate corollary the famous theorem of Nash on noncooperative equilibrium in n-person games.

Theorem 11. *Let C be a convex subset of $\mathbb{R}^m$, D a convex compact subset of $\mathbb{R}^n$, $F(x, y)$ a continuous function on $C \times D$, quasiconcave in y for fixed x. If $Z : D \mapsto 2^C$ is an upper semi-continuous set-valued map from D to C, such that for every $y \in D$, $Z(y)$ is a nonempty convex compact set, then*

$$\inf_{x\in C} \max_{y\in D} F(x, y) \le \max_{y\in D} \max_{x\in Z(y)} F(x, y). \tag{28}$$

Proof. We first show that the set $\bar{C} = \cup_{y\in D} Z(y)$ is compact. To this end, let $W_t,\ t \in T$, be a family of open sets covering $\bar{C}$. For each fixed $y \in D$, because $Z(y)$ is compact, there exists a finite set $I(y) \subset T$, such that $Z(y) \subset \cup_{i\in I(y)} W_i$. In view of the upper semi-continuity of the set-valued map Z, there exists an open ball $V(y)$ around y such that $Z(y') \subset \cup_{i\in I(y)} W_i\ \forall y' \in V(y)$. Then, using the compactness of D, we can find a finite set $E \subset D$ such that D is entirely covered by $\cup_{y\in E} V(y)$. Clearly the finite family $W_i,\ i \in I(y), y \in E\}$ is a covering of $\bar{C}$. Thus from any open covering of $\bar{C}$, one can extract a finite subcovering. This proves the compactness of $\bar{C}$ and hence also the compactness of its convex hull C'.

Now for every $x \in C$ define $f(x) := \{y \in D|\ F(x,y) = \max_{y' \in D} F(x,y')\}$. Clearly, for every $x \in C$, $f(x)$ is a nonempty convex compact subset of D. We contend that $x \mapsto f(x)$ is a closed set-valued map from C to D. Indeed, consider a sequence $(x^k, y^k) \in C \times D$ such that $y^k \in f(x^k), x^k \to x^0,\ y^k \to\ y^0\ (k \to +\infty)$. Because $y^k \in f(x^k)$, we have $F(x^k, y^k) \geq F(x^k, y)\ \forall y \in D$, hence, by continuity, $F(x^0, y^0) \geq F(x^0, y)\ \forall y \in D$, i.e., $y^0 \in f(x^0)$, proving the closedness of the map $x \mapsto f(x)$. In view of the compactness of D, this closed set-valued map is upper semi-continuous (see, e.g., [1], Chapter 3, Corollary 9) and hence, so is the set-valued map $\Gamma : C' \times D :\to 2^{C' \times D}$ defined by $\Gamma(x,y) = Z(y) \times f(x)$. By the Kakutani fixed point theorem, there exists a point $(x^*, y^*) \in \Gamma(x^*, y^*) = Z(y^*) \times f(x^*)$, i.e., such that $y^* \in f(x^*)$ and $x^* \in Z(y^*)$. If $\max_{y \in D} \sup_{x \in Z(y)} F(x,y) = \alpha$, i.e., $F(x,y) \leq \alpha\ \forall y \in D,\ \forall x \in Z(y)$, then, because $x^* \in Z(y^*),\ y^* \in f(x^*) \subset D$, we have $F(x^*, y^*) \leq \alpha$ and $F(x^*, y^*) = \max_{y \in D} F(x^*, y)$, hence $\inf_{x \in C} \max_{y \in D} F(x,y) \leq \alpha$. This proves (28). ■

A special case of this lopsided minimax theorem is the following minimax proposition mentioned at the beginning of this section:

Corollary 4. *(von Neumann [12]) Let $C \subset \mathbb{R}^m, D \subset \mathbb{R}^n$ be compact convex sets, and $F(x,y)$ a function continuous on $C \times D$, quasiconvex in x for fixed y and quasiconcave in y for fixed x. Then*

$$\min_{x \in C} \max_{y \in D} F(x,y) = \max_{y \in D} \min_{x \in C} F(x,y).$$

Proof. It suffices to define $Z(y) = \mathrm{argmin}_{x \in C} F(x,y)\}$ and to observe that $\max_{x \in Z(y)} F(x,y) = \min_{x \in C} F(x,y)$. Then by Theorem 11

$$\min_{x \in C} \max_{y \in D} F(x,y) \leq \max_{y \in D} \min_{x \in C} F(x,y),$$

and the reverse inequality is always true. ■

The following theorem is simply an analogue of Theorem 11.

Theorem 12. *Let C be a convex subset of $\mathbb{R}^m$, D a convex compact subset of $\mathbb{R}^n$, $F(x,y)$ a continuous function on $C \times D$, quasiconvex in y for fixed x. If $Z : D \mapsto 2^C$ is an upper semi-continuous set-valued map from D to C, such that for every $y \in D$, $Z(y)$ is a nonempty convex compact set, then*

$$\sup_{x \in C} \min_{y \in D} F(x,y) \geq \min_{y \in D} \min_{x \in Z(y)} F(x,y). \tag{29}$$

Remark 4. A weaker version of the above lopsided minimax theorem (with both C, D compact) was established many years ago as an extension of Walras' excess demand theorem in Mathematical Economics [18] and was proven to be equivalent to the Kakutani fixed point theorem. Because in this case (29) implies, for every $\alpha \in \mathbb{R}$:

$$\min_{y \in D} \inf_{x \in Z(y)} F(x,y) \geq \alpha \Rightarrow \max_{x \in C} \min_{y \in D} F(x,y) \geq \alpha,$$

the above theorem has the following transparent heuristic interpretation:

Suppose the utility function $F(x, y)$ of a company depends upon a variable $x \in C$ under its control and a variable $y \in D$ outside its control. If for every $y \in D$ there exists for the company a set $Z(y) \subset C$ every element of which guarantees a utility level no less than α, then, under suitable conditions, there exists for the company an $x^* \in C$ guaranteeing a utility level no less than α, whatever $y \in D$ may be.

No wonder that Theorem 11 (or Theorem 12), which is an extension of an ordinary minimax theorem, can also be used to directly derive Nash's noncooperative equilibrium theorem for n-person games. A link is thus established between minimax and noncooperative equilibrium concepts.

Consider an n-person game in which the strategy set C_i of the player i is a subset of a finite-dimensional Euclidean space X_i. When the player i chooses a strategy $x_i \in C_i$, the situation of the game is described by the vector $x = (x_1, \ldots, x_n) \in \prod_{i=1}^n C_i$. In that situation, the player i obtains a payoff $f_i(x)$.

Assume that each player does not know which strategy is taken by the other players. A vector $\widetilde{x} \in \prod_{i=1}^n C_i$ is then called a *Nash equilibrium* if for every $i = 1, \ldots, n$ we have

$$f_i(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n) = \max_{x_i \in C_i} f_i(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, x_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n).$$

By $x_{\bar{i}}$ denote the vector formed by the x_j with $j \neq i$:

$$x_{\bar{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

so that $x = (x_1, x_{\bar{1}}) = (x_2, x_{\bar{2}}) = \cdots = (x_n, x_{\bar{n}})$ (after rearranging the components if necessary). With this notation, the equilibrium condition can be written as

$$f_i(\widetilde{x}) = \max_{y_i \in C_i} f_i(y_i, \widetilde{x}_{\bar{i}}).$$

Theorem 13. (Nash [10]) *Assume that for each $i = 1, \ldots, n$ the set C_i is convex compact, the function f_i is continuous, and the function $y_i \mapsto f_i(y_i, x_{\bar{i}})$ is concave. Then there exists a Nash equilibrium.*

Proof. Let $C = \prod_{i=1}^n C_i$ and consider the function $F(x, y)$ defined on $C \times C$ by

$$F(x,y) = \sum_{i=1}^{n} (f_i(x) - f_i(y_i, x_{\bar{i}})).$$

The set C is convex, compact as the Cartesian product of n convex compact sets, and the function $F(x, y)$ is jointly continuous in x, y and convex in y

for fixed x. By setting $D = C$, $Z(y) := \{y\}$, the conditions of Theorem 12 are satisfied. Because $\min_{x\in Z(y)} F(x,y) = F(y,y) = 0$, it follows from (29) that $\sup_{x\in C}\min_{y\in C} F(x,y) \geq 0$, and hence, $\max_{x\in C}\min_{y\in C} F(x,y) \geq 0$, because the function $x \mapsto \min_{y\in C} F(x,y)$ is u.s.c. and the set C is compact. Consequently, there exists $\widetilde{x} \in C$ such that

$$F(\widetilde{x}, y) = \sum_{i=1}^{n}(f_i(\widetilde{x}) - f_i(y_i, \widetilde{x}_{\bar{i}})) \geq 0 \quad \forall y \in C.$$

Fixing an arbitrary i and letting $y := (y_i, \widetilde{x}_{\bar{i}})$ yields

$$f_i(\widetilde{x}) - f_i(y_i, \widetilde{x}_{\bar{i}}) + \sum_{j\neq i}(f_i(\widetilde{x}_j, \widetilde{x}_{\hat{j}}) - f_j(y_j, \widetilde{x}_{\bar{j}})) \geq 0 \quad \forall y \in C.$$

But for each $j \neq i$ we have $h_j = \widetilde{x}_j$, so $(\widetilde{x}_j, \widetilde{x}_{\bar{j}}) = (y_j, \widetilde{x}_{\bar{j}})$. Therefore, for every $i = 1, \ldots, n$:

$$f_i(\widetilde{x}) \geq f_i(y_i, \widetilde{x}_{\bar{i}}) \quad \forall y_i \in C_i,$$

which implies that $\widetilde{x}$ is a Nash equilibrium. ■

5 Conclusion

In this paper, we have developed a unified approach to existence and stability conditions for the saddle value and saddle point of a quasiconvex quasiconcave function. It turned out that, under usual assumptions, condition (M) ((N), respectively) ensures not only existence but also upper (lower, respectively) semi-continuity of the saddle value under continuous perturbation, and condition (MN) ensures both existence and upper semi-continuity of the saddle point. Also, a lopsided minimax theorem is established that yields as immediate corollaries von Neumann's minimax theorem for two-person zero-sum games as well as Nash's theorem on equilibrium in n-person games.

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Recent Advances in Minimax Theory and Applications

Biagio Ricceri

Department of Mathematics, University of Catania, Catania, Italy
ricceri@dmi.unict.it

Abstract In this chapter, we give an overview of various applications of a recent minimax theorem. Among them, there are some multiplicity theorems for nonlinear equations as well as a general well-posedness result for functionals with locally Lipschitzian derivative.

Key words: minimax theorems, multiplicity of solutions, nonlinear equations, p-Laplacian, well-posed optimization problems

1 Introduction

Let X and Y be two nonempty sets, and let $f : X \times Y \to \mathbf{R}$ be a given function.

The object of minimax theory, in its classic sense, is to find conditions on X, Y and f that are sufficient to guarantee the validity of the equality

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

In this paper, we do not intend at all to offer a comprehensive survey of the subject. For such a survey, we refer to the excellent [44].

The current paper should be rather considered, in the spirit, as a continuation of [33]. This latter was devoted to give an overview of the various applications ([25, 26, 30–32, 35]) of the following result proved in [29].

Theorem 1. *Let X be a topological space, Y a compact real interval, and $f : X \times Y \to \mathbf{R}$. Assume that, for each $\rho \in \mathbf{R}$, $x_0 \in X$, $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) \le \rho\}$$

and

$$\{y \in Y : f(x_0, y) > \rho\}$$

are connected. In addition, assume that at least one of the following three sets of conditions is satisfied:

(h_1) *$f(x,\cdot)$ is upper semicontinuous in Y for each $x \in X$, and $f(\cdot,y)$ is lower semicontinuous in X for each $y \in Y$;*
(h_2) *f is upper semicontinuous in $X \times Y$;*
(h_3) *X is compact, and f is lower semicontinuous in $X \times Y$.*

Then, one has

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

Recently, in [38], we revisited Theorem 1 extending it in the following way:

Theorem 2. *Let X be a topological space, $Y \subseteq \mathbf{R}$ an interval, and $f : X \times Y \to \mathbf{R}$ a function such that $f(x,\cdot)$ is continuous for all $x \in X$. Assume that there exist a number $\rho^* > \sup_Y \inf_X f$, a point $\hat{y} \in Y$, and two sets $D_1, D_2 \subseteq Y$, both dense in Y, such that for each $\rho \in]-\infty, \rho^*[$, the following conditions hold:*
(α) *the set $\{y \in Y : f(x,y) > \rho\}$ is an interval for all $x \in X$;*
(β) *the set $\{x \in X : f(x,y) \leq \rho\}$ is closed for all $y \in D_1$ and compact for $y = \hat{y}$, and the set $\{x \in X : f(x,y) < \rho\}$ is connected for all $y \in D_2$.*

Then, one has

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) = \inf_{x\in X} \sup_{y\in Y} f(x,y).$$

As it will be remarked later, when Y is compact, Theorem 2 holds without requiring the existence of the point $\hat{y}$ with the indicated property. Likewise, when $D_1 = Y$, it is enough to assume that $f(x,\cdot)$ is upper semicontinuous for all $x \in X$.

The aim of the current paper is just to survey some applications of Theorem 2.

Theorems 1 and 2 belong to the class of the so-called topological minimax theorems, due to the fact that the assumptions are of purely topological nature.

From a theoretical point of view, the best topological minimax theorem is, in our opinion, the following result by H. König [18]:

Theorem 3. *Let X, Y be two topological spaces, with X compact. Assume that, for each $x \in X$, the function $f(x,\cdot)$ is upper semicontinuous in Y and that, for each $y \in Y$, the function $f(\cdot,y)$ is lower semicontinuous in X. Further, assume that:*

(i_1) *for each $\rho \in \mathbf{R}$ and each nonempty finite set $H \subseteq Y$, the set*

$$\bigcap_{y\in H} \{x \in X : f(x,y) < \rho\}$$

is connected;

(i_2) *for each* $\rho \in \mathbf{R}$ *and each nonempty set* $H \subseteq X$*, the set*

$$\bigcap_{x \in H} \{y \in Y : f(x,y) > \rho\}$$

is connected.

Then, one has

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

In [44] is well described the process of successive improvements ([16, 17, 48]) of the first topological minimax theorem [50], which just culminated with Konig's result.

We want to repeat that Theorem 3 is a great theoretical result. However, if we pass to the natural question of how can assumptions (i_1) and (i_2) be satisfied, then we encounter serious difficulties. In fact, the natural general situation in which (i_1) and (i_2) are satisfied is when X and Y are convex sets in topological vector spaces and all the sets $\{x \in X : f(x,y) < \rho\}$ and $\{y \in Y : f(x,y) > \rho\}$ are convex. Out of such a setting, checking (i_1) and (i_2) becomes extremely difficult, as the intersection of two even "extremely simple" connected sets fails to be connected. In other words, we can conclude that, apart from very specific situations, Theorem 3 becomes canonically applicable when it assumes the fashion of Sion minimax theorem [46] which, in turn, improved the most classic results of the theory, due to Von Neumann [49], Ky Fan [13], and Nikaidô [27].

On the other hand, without further assumption on Y, there is no hope to achieve the optimal version of Theorem 3 coming out from removing intersection in (i_1) and (i_2) (that is, assuming that they are satisfied simply when H is a singleton). In this connection, consider the following example. Take:

$$X = \{(t,u) \in \mathbf{R}^2 : t^2 + u^2 = 1\}$$

$$Y = \{(v,z) \in \mathbf{R}^2 : v^2 + z^2 \leq 1\}$$

and, for each $(t,u) \in X, (v,z) \in Y$,

$$f(t,u,v,z) = tv + uz.$$

So, for fixed $(t,u) \in X$, we have

$$\sup_{(v,z) \in Y} f(t,u,v,z) = \sqrt{t^2 + u^2} = 1.$$

Moreover, for fixed $(v,z) \in Y$, we have

$$\inf_{(t,u) \in X} f(t,u,v,z) = -\sqrt{v^2 + z^2}.$$

Hence, it follows that

$$0 = \sup_{(v,z)\in Y} \inf_{(t,u)\in X} f(t,u,v,z) < \inf_{(t,u)\in X} \sup_{(v,z)\in Y} f(t,u,v,z) = 1.$$

Now, come back to Theorems 1 and 2. On the basis of the discussion above, they can be regarded as optimal versions of Theorem 3 when Y is a real interval. Of course, this very severe restriction on Y (that, we repeat, is necessary for being able to assume simply the connectedness of the single level sets $\{x \in X : f(x,y) < \rho\}$) prevents the use of Theorems 1 and 2 in many important instances (as the theory of duality [41], or the theory of monotone operators [45]). Nevertheless, there are likewise important cases where the second variable of the considered function f runs over an interval. In these cases, the use of Theorems 1 and 2 allows one to get results that are incomparably better than those that one could get applying Theorem 3.

In this connection, the most enlightening example is the case of an integral functional on an L^p space.

So, let $(T, \mathcal{F}, \mu)$ be a σ-finite nonatomic measure space, E a real Banach space ($E \neq \{0\}$), and p a real number greater than or equal to 1.

As usual, $L^p(T,E)$ denotes the space of all (equivalence classes of) strongly μ-measurable functions $u : T \to E$ such that $\int_T \| u(t) \|^p \, d\mu < +\infty$, equipped with the norm $\| u \|_{L^p(T,E)} = (\int_T \| u(t) \|^p \, d\mu)^{\frac{1}{p}}$.

A set $D \subseteq L^p(T,E)$ is said to be decomposable if, for every $u, v \in D$ and every $S \in \mathcal{F}$, the function $t \to \chi_S(t)u(t) + (1-\chi_S(t))v(t)$ belongs to S, where χ_S denotes the characteristic function of S.

A function $\varphi : T \times E \to \mathbf{R}$ is said to be sup-measurable if for every strongly μ-measurable function $u : T \to E$, the function $t \to \varphi(t, u(t))$ is μ-measurable.

In [42], J. Saint Raymond established the following very interesting result:

Theorem 4. *Let $\varphi : T \times E \to \mathbf{R}$ be a sup-measurable function, and let $D \subseteq L^p(T,E)$ be a decomposable set.*

Then, if we put

$$S = \{u \in D : \varphi(\cdot, u(\cdot)) \in L^1(T)\},$$

for each $\rho \in \mathbf{R}$, the set

$$\left\{ u \in S : \int_T \varphi(t, u(t)) d\mu \leq \rho \right\}$$

is arcwise connected.

Then, applying Theorem 1 via Theorem 4, we get

Theorem 5. *Let $Y \subseteq \mathbf{R}$ be a compact interval, $X \subseteq L^p(T,E)$ a decomposable set, $\varphi : T \times E \times Y \to \mathbf{R}$ a function that is sup-measurable in $T \times E$ and concave in Y. Moreover, assume that $\varphi(\cdot, u(\cdot), y) \in L^1(T)$ for all $u \in X$, $y \in Y$.*

Finally, suppose that the functional $u \to \int_T \varphi(t, u(t), y)d\mu$ *is lower semicontinuous in* X *for each* $y \in Y$*, and that the function* $y \to \int_T \varphi(t, u(t), y)d\mu$ *is upper semicontinuous in* Y *for each* $u \in X$*.*

Then, one has

$$\sup_{y \in Y} \inf_{u \in X} \int_T \varphi(t, u(t), y)d\mu = \inf_{u \in X} \sup_{y \in Y} \int_T \varphi(t, u(t), y)d\mu.$$

Note that to get Theorem 5 via Theorem 3, we would be forced, in practice, to assume two spurious assumptions: X should be convex and weakly compact, and $\varphi(t, \cdot, y)$ should be convex for each $(t, y) \in T \times Y$.

From Theorem 5, in turn, many consequences follow. Let us here recall some of them.

Theorem 6. *Let* $\varphi : T \times E \to \mathbf{R}$ *be a sup-measurable function. Assume that there exist* $\alpha \in L^1(T)$*,* $\gamma_i \in]0, 1[$ *and* $\beta_i \in L^{\frac{p}{p-\gamma_i}}(T)$ *(*$i = 1, \ldots, k$*) such that*

$$-\alpha(t) \leq \varphi(t, x) \leq \alpha(t) + \sum_{i=1}^{k} \beta_i(t) \parallel x \parallel^{\gamma_i}$$

for almost every $t \in T$ *and for every* $x \in E$*.*

Then, for every decomposable linear subspace X *of* $L^p(T, E)$ *and every closed hyperplane* V *of* X*, one has*

$$\inf_{u \in V} \int_T \varphi(t, u(t))d\mu = \inf_{u \in X} \int_T \varphi(t, u(t))d\mu.$$

Let us now observe a consequence of Theorem 6 that extends the classic fact that, for $\gamma \in]0, 1[$, the topological dual of $L^\gamma(T, E)$ reduces to zero. Precisely, we denote by $\mathcal{M}$ the set of all metrics d on $L^p(T, E)$ of the following type:

$$d(u, v) = \sum_{i=1}^{k} \int_T \beta_i(t) \parallel u(t) - v(t) \parallel^{\gamma_i} d\mu$$

where $u, v \in L^p(T, E)$, $\gamma_i \in]0, 1[$, $\beta_i \in L^{\frac{p}{p-\gamma_i}}(T)$, $\beta_i > 0$ in T $(i = 1, \ldots, k)$. Note that each $d \in \mathcal{M}$ is a metric inducing a vector topology that is weaker than the $\parallel \cdot \parallel_{L^p(T,E)}$-topology.

Theorem 7. *For every* $d \in \mathcal{M}$ *and every decomposable linear subspace* X *of* $L^p(T, E)$*, the topological dual of* (X, d) *reduces to zero.*

When we take $X = L^p(T, E)$, the conclusion of Theorem 6 can be extended to a class of functions φ with a more general growth.

Theorem 8. *Let* $\varphi : T \times E \to [0, +\infty[$ *be such that* $\varphi(\cdot, x)$ *is* μ*-measurable for each* $x \in E$ *and* $\varphi(t, \cdot)$ *is Lipschitzian with Lipschitz constant* $M(t)$ *for almost every* $t \in T$*, where* $M \in L^{\frac{p}{p-1}}(T)$*. Assume that* $\varphi(\cdot, 0) \in L^1(T)$ *and* *that there exists a sequence* $\{\lambda_n\}$ *in* $]0,+\infty[$, *with* $\lim_{n\to+\infty}\lambda_n = +\infty$, *such that, for almost every* $t \in T$ *and for every* $x \in E$, *one has*

$$\lim_{n\to+\infty} \frac{\varphi(t,\lambda_n x)}{\lambda_n} = 0.$$

Then, for every closed hyperplane V *of* $L^p(T,E)$, *one has*

$$\inf_{u\in V}\int_T \varphi(t,u(t))d\mu = \inf_{u\in X}\int_T \varphi(t,u(t))d\mu.$$

Let us recall that a multifunction $F : T \to 2^E$ is said to be measurable if, for every open set $\Omega \subseteq E$, one has $\{t \in T : F(t) \cap \Omega \neq \emptyset\} \in \mathcal{F}$. A function $u : T \to E$ is a selection of the multifunction $F : T \to 2^E$ if $u(t) \in F(t)$ for all $t \in T$. We denote by $\mathcal{S}_F$ the set of all selections of F belonging to $L^1(T,E)$. An application of Theorem 8 gives

Theorem 9. *Let* E *be separable, and let* $F : T \to 2^E$ *be a measurable multifunction, with nonempty closed values. Assume that* $\mathrm{dist}(0,F(\cdot)) \in L^1(T)$ *and that there exists a sequence* $\{\lambda_n\}$ *in* $]0,+\infty[$, *with* $\lim_{n\to+\infty}\lambda_n = +\infty$, *such that, for almost every* $t \in T$ *and for every* $x \in E$, *one has*

$$\lim_{n\to+\infty} \frac{\mathrm{dist}(\lambda_n x, F(t))}{\lambda_n} = 0.$$

Then, $\mathcal{S}_F$ *intersects each closed hyperplane of* $L^1(T,E)$.

We stress that each of the above recalled consequences of Theorem 5 is made possible just because we do not assume the convexity of $\varphi(t,\cdot,y)$.

The plan of the current paper is as follows.

In the next section, we prove Theorem 2 and derive some of its consequences among which there are Theorems 12 and 13. In Section 3, we then apply Theorem 12 to get a general multiplicity theorem for certain nonlinear equations in Hilbert spaces. Section 4 is devoted to an application of Theorem 13 to a Neumann problem for elliptic equations involving the p-Laplacian. Finally, in Section 5, using Theorem 2, we prove that the problem of minimizing locally a C^2 functional around noncritical points is well-posed.

2 Proof and Corollaries of Theorem 2

Let us start with the proof of Theorem 2.

Proof (Proof of Theorem 2). First, fix a nondecreasing sequence $\{Y_n\}$ of compact subintervals of Y, with $\hat{y} \in Y_1$, such that $\cup_{n\in\mathbf{N}} Y_n = Y$. Now, fix $n \in \mathbf{N}$. We claim that

$$\sup_{y\in Y_n}\inf_{x\in X} f(x,y) = \inf_{x\in X}\sup_{y\in Y_n} f(x,y). \tag{2.1}$$

Arguing by contradiction, suppose that

$$\sup_{y\in Y_n}\inf_{x\in X} f(x,y) < \inf_{x\in X}\sup_{y\in Y_n} f(x,y).$$

Fix ρ satisfying

$$\sup_{y\in Y_n}\inf_{x\in X} f(x,y) < \rho < \min\left\{\rho^*, \inf_{x\in X}\sup_{y\in Y_n} f(x,y)\right\}.$$

Set

$$S = \{(x,y) \in X \times Y_n : f(x,y) < \rho\}$$

as well as, for each $y \in Y_n$,

$$S^y = \{x \in X : (x,y) \in S\}.$$

Because $\sup_{Y_n} \inf_X f < \rho$, one has $S^y \neq \emptyset$ for all $y \in Y_n$. Let $Y_n = [a_n, b_n]$. Put

$$A = \left\{(x,y) \in S : y < b_n\ ,\ \sup_{s\in]y,b_n]} f(x,s) > \rho\right\}$$

and

$$B = \left\{(x,y) \in S : y > a_n\ ,\ \sup_{s\in[a_n,y[} f(x,s) > \rho\right\}.$$

Observe that A, B are nonempty. Indeed, let $x_1 \in S^{a_n}$ and $x_2 \in S^{b_n}$. Because $\rho < \inf_X \sup_{Y_n} f$, there are $t, s \in Y_n$ such that $\min\{f(x_1,t), f(x_2,s)\} > \rho$. Because $\max\{f(x_1,a_n), f(x_2,b_n)\} < \rho$, it follows that $t > a_n$ and $s < b_n$. Consequently, $(x_1,a_n) \in A$ and $(x_2,b_n) \in B$. Furthermore, observe that A, B are open in S. Let us see this for A, the other case being analogous. So, let $(x_0,y_0) \in A$. Because the function $f(x_0,\cdot)$ is lower semicontinuous, the set $\{y \in]y_0,b_n] : f(x_0,y) > \rho\}$ is nonempty and open in Y_n and hence it contains a point $y^* \in D_1$, by density. Now, by (β), the set

$$(\{x \in X : f(x,y^*) > \rho\} \times [a_n, y^*[) \cap S$$

is clearly a neighbourhood of (x_0,y_0) in S that is contained in A. We now prove that $S = A \cup B$. Indeed, let $(x,y) \in S \setminus A$. We have seen above that $S^{a_n} \times \{a_n\} \subseteq A$, and so $y > a_n$. If $y = b_n$, the fact that $(x,y) \in B$ has been likewise proved above. Suppose $y < b_n$. Thus, we have $\sup_{s\in]y,b_n]} f(x,s) \le \rho$. From this, it clearly follows that $\sup_{s\in[a_n,y[} f(x,s) > \rho$ (note that $f(x,y) < \rho$), and so $(x,y) \in B$. Furthermore, we have $A \cap B = \emptyset$. Indeed, if $(x_1,y_1) \in A \cap B$, there would be $t, s \in Y_n$, with $t < y_1 < s$, such that $\min\{f(x_1,t), f(x_1,s)\} > \rho$. By (α), the set $\{u \in Y : f(x_1,u) > \rho\}$ is an interval, and so we would have $f(x_1,y_1) > \rho$, against the fact that $(x_1,y_1) \in S$. Let $p_{\mathbf{R}}$ be the projection from $X \times \mathbf{R}$ onto $\mathbf{R}$. Now, consider the sets $p_{\mathbf{R}}(A)$ and $p_{\mathbf{R}}(B)$. Because $p_{\mathbf{R}}(S) = Y_n$, thanks to the properties of A, B seen above and to the upper semicontinuity of $f(x,\cdot)$ for all $x \in X$, they are nonempty, open in Y_n, and cover Y_n. So, by the connectedness of Y_n, we have $p_{\mathbf{R}}(A) \cap p_{\mathbf{R}}(B) \neq \emptyset$. Because D_2 is dense in Y, there exists some $y' \in D_2 \cap p_{\mathbf{R}}(A) \cap p_{\mathbf{R}}(B)$. By ($\beta$), the set $S^{y'}$ (and hence $S^{y'} \times \{y'\}$ too) is connected. But $S^{y'} \times \{y'\}$ meets both A and B, and this just contradicts its being connected. So, we have proved (2.1). Finally, let us prove the theorem. Again arguing by contradiction, suppose that

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) < \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Choose r satisfying

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) < r < \min\left\{\rho^*, \inf_{x \in X} \sup_{y \in Y} f(x,y)\right\}.$$

For each $n \in \mathbf{N}$, put

$$C_n = \left\{x \in X : \sup_{y \in Y_n} f(x,y) \leq r\right\}.$$

Note that $C_n \neq \emptyset$. Indeed, otherwise, we would have

$$r \leq \inf_{x \in X} \sup_{y \in Y_n} f(x,y) = \sup_{y \in Y_n} \inf_{x \in X} f(x,y) \leq \sup_{y \in Y} \inf_{x \in X} f(x,y).$$

Furthermore, for each $x \in X$, we have

$$\sup_{y \in Y_n} f(x,y) = \sup_{y \in D_1 \cap Y_n} f(x,y)$$

as $f(x, \cdot)$ is lower semicontinuous and D_1 is dense in Y. So, we have

$$C_n = \bigcap_{y \in D_1 \cap Y_n} \{x \in X : f(x,y) \leq r\}.$$

Consequently, $\{C_n\}$ is a nonincreasing sequence of nonempty closed subsets of the compact set $\{x \in X : f(x, \hat{y}) \leq \rho^*\}$. Therefore, one has $\cap_{n \in \mathbf{N}} C_n \neq \emptyset$. Let $x^* \in \cap_{n \in \mathbf{N}} C_n$. Then, one has

$$\sup_{y \in I} f(x^*, y) = \sup_{n \in \mathbf{N}} \sup_{y \in Y_n} f(x^*, y) \leq r$$

and so

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq r,$$

a contradiction. The proof is complete. ■

Remark 1. It is clear from the proof that when Y is compact, Theorem 2 holds without requiring the existence of the point $\hat{y}$ with the indicated property. Likewise, when $D_1 = Y$, it is enough to assume that $f(x, \cdot)$ is upper semicontinuous for all $x \in X$.

It is important to note the next result, which is a consequence of Theorem 2.

If (X, τ) is a topological space, for any $f : X \to \mathbf{R}$, we denote by τ_f the smallest topology on X that contains both τ and the family of sets $\{f^{-1}(]-\infty, r[)\}_{r\in\mathbf{R}}$.

Theorem 10. *Let (X, τ) be a Hausdorff topological space, $Y \subseteq \mathbf{R}$ an interval, and $f : X \times Y \to \mathbf{R}$ a function such that $f(x, \cdot)$ is continuous for all $x \in X$. Assume that there exist a number $\rho^* > \sup_Y \inf_X f$ and a set $D \subseteq Y$, dense in Y, such that, for each $\rho \in]-\infty, \rho^*[$ and each $y \in D$, the following conditions hold:*

(i) the set $\{s \in Y : f(x, s) > \rho\}$ is an interval for all $x \in X$;
(ii) the set $\{x \in X : f(x, y) \leq \rho\}$ is compact and sequentially compact;
(iii) there exist a function $\Phi_y : X \to \mathbf{R}$, bounded below on the set $\{x \in X : f(x, y) \leq \rho^*\}$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the function $f(\cdot, y)+\lambda\Phi_y(\cdot)$ is sequentially lower semicontinuous, and, for each $n \in \mathbf{N}$, the function $f(\cdot, y) + \mu_n\Phi_y(\cdot)$ has at most one $\tau_{f(\cdot,y)}$-local minimum lying in the set $\{x \in X : f(x, y) < \rho^*\}$.

Then, one has

$$\sup_{y\in Y}\inf_{x\in X} f(x, y) = \inf_{x\in X}\sup_{y\in Y} f(x, y).$$

Before deriving Theorem 10 from Theorem 2, we establish the following result [36]:

Theorem 11. *Let (X, τ) be a Hausdorff topological space, and $\Phi, f : X \to \mathbf{R}$ two functions. Assume that there is $\rho > \inf_X f$ such that the set $\overline{f^{-1}(]-\infty, \rho[)}$ is compact and sequentially compact and has at least k connected components. Moreover, suppose that the function Φ is bounded below in $\overline{f^{-1}(]-\infty, \rho[)}$ and that the function $f + \lambda\Phi$ is sequentially lower semicontinuous for each $\lambda > 0$ small enough.*

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the function $f+\lambda\Phi$ has at least k τ_f-local minima lying in $f^{-1}(]-\infty, \rho[)$.

Proof. Denote by $\mathcal{C}$ the family of all connected components of $\overline{f^{-1}(]-\infty, \rho[)}$. Note that these sets are closed in X because they are closed in $\overline{f^{-1}(]-\infty, \rho[)}$, which is, in turn, closed in X. We now observe that there are k pairwise disjoint closed nonempty sets $C_1, \dots, C_k$ such that

$$\overline{f^{-1}(]-\infty, \rho[)} = \bigcup_{i=1}^{k} C_i.$$

We distinguish two cases. First, assume that $\mathcal{C}$ is finite. Let h be its cardinality. Let $B_1, \dots, B_h$ be the members of $\mathcal{C}$. Then, if we choose $C_i = B_i$ for $i = 1, \dots, k-1$ and $C_k = \cup_{i=k}^{h} B_i$, we are clearly done. Now, assume that $\mathcal{C}$ is infinite. In this case, we prove our claim by induction. The claim is true, of course, if $k = 1$. Assume that it is true if $k = p$. So, we are assuming that there are p pairwise disjoint closed nonempty sets $D_1, \dots, D_p$, such that

$$\overline{f^{-1}(]-\infty,\rho[)} = \bigcup_{i=1}^{p} D_i.$$

Notice that at least one of the sets D_i must be disconnected, as, otherwise, we would have $\{D_1,\ldots,D_p\} = \mathcal{C}$, contrary to the assumption that $\mathcal{C}$ is infinite. Then, if D_{i^*} is disconnected, there are two disjoint closed nonempty sets E_1, E_2 such that $D_{i^*} = E_1 \cup E_2$. So, $D_1,\ldots,D_{i^*-1},D_{i^*+1},\ldots,D_p,E_1,E_2$ are $p+1$ pairwise disjoint closed nonempty sets whose union is $\overline{f^{-1}(]-\infty,\rho[)}$. So, our claim is true for $k = p+1$, and hence, by induction, for any k.

Now, fix i $(1 \leq i \leq k)$. By compactness and Hausdorffness, it is clear that there exists an open set $A_i \subset X$ such that $C_i \subset A_i$ and $A_i \cap \cup_{j=1,j\neq i}^{k} C_j = \emptyset$. Furthermore, it is easily seen that, if we put

$$G_i = \{x \in A_i : f(x) < \rho\},$$

we have

$$\overline{G_i} = C_i.$$

Taken into account that, by assumption, $\inf_{C_i} \Phi$ is finite, put

$$\mu_i = \inf_{x \in G_i} \frac{\Phi(x) - \inf_{C_i} \Phi}{\rho - f(x)}.$$

Let $\lambda' > 0$ be such that $f + \lambda\Phi$ is sequentially lower semicontinuous for each $\lambda \in]0,\lambda']$. Fix $\mu > \max\{\mu_i, \frac{1}{\lambda'}\}$. Then, there exists $y \in G_i$ such that

$$\mu\rho > \mu f(y) + \Phi(y) - \inf_{C_i} \Phi.$$

Moreover, because C_i is sequentially compact, there exists $x_i^* \in C_i$ such that

$$\Phi(x_i^*) + \mu f(x_i^*) \leq \Phi(x) + \mu f(x)$$

for all $x \in C_i$. We claim that $x_i^* \in G_i$. Arguing by contradiction, assume that $f(x_i^*) \geq \rho$. We then have

$$\Phi(x_i^*) + \mu f(x_i^*) \geq \Phi(x_i^*) + \mu\rho > \Phi(x_i^*) + \Phi(y) + \mu f(y) - \inf_{C_i} \Phi \geq \Phi(y) + \mu f(y)$$

which is absurd. Now, let i vary. Put $\mu^* = \max\{\mu_1,\ldots,\mu_k,\frac{1}{\lambda'}\}$. Clearly, each set G_i is τ_f-open, and hence each x_i^* is a τ_f-local minimum of $\Phi + \mu f$ for all $\mu > \mu^*$. Consequently, the points $x_1^*,\ldots,x_k^*$ satisfy the conclusion, taking $\lambda^* = \frac{1}{\mu^*}$, and the proof is complete. ■

Proof (Proof of Theorem 10). We have only to check that f satisfies the hypotheses of Theorem 2. So, let $y \in D$, and $r < \sigma < \rho^*$. By (ii), it clearly follows that the set $\{x \in X : f(x,y) \leq \sigma\}$ is closed (because X is Hausdorff) and that the set $\overline{\{x \in X : f(x,y) < \sigma\}}$ is compact and sequentially compact.

From (iii), it follows that the functions $f(\cdot, y)$, Φ_y do not satisfy the conclusion of Theorem 11 with $k = 2$, and so, because function Φ_y is bounded below in $\overline{\{x \in X : f(x,y) < \sigma\}}$ and the function $f(\cdot, y) + \lambda\Phi_y(\cdot)$ is sequentially lower semicontinuous for each $\lambda > 0$ small enough, it necessarily follows that the set $\overline{\{x \in X : f(x,y) < \sigma\}}$ is connected. Now, observe that, because $\overline{\{x \in X : f(x,y) < \sigma\}} \subseteq \{x \in X : f(x,y) \leq \sigma\}$, one has

$$\{x \in X : f(x,y) \leq r\} = \bigcap_{r<\sigma<\rho^*} \overline{\{x \in X : f(x,y) < \sigma\}}.$$

Therefore, the closed set $\{x \in X : f(x,y) \leq r\}$, as the intersection of a non-increasing sequence of compact and connected sets, is connected too. Finally, let $\rho \in]-\infty, \rho^*[$. Because

$$\{x \in X : f(x,y) < \rho\} = \bigcup_{r<\rho} \{x \in X : f(x,y) \leq r\},$$

it follows that the set $\{x \in X : f(x,y) < \rho\}$ is connected. So, all the assumptions of Theorem 2 are satisfied, and the conclusion follows. ■

Remark 2. We do not know whether, in Theorem 10, condition (iii) can be improved replacing $\tau_{f(\cdot,y)}$ with τ. However, this is the case when we are allowed to take $\Phi_y = 0$. To see this, we first establish the following

Proposition 1. *Let X be a Hausdorff topological space and $f : X \to \mathbf{R}$ a function. Assume that, for some $r > \inf_X f$, f has at most one local minimum lying in $f^{-1}(]-\infty, r])$ and that $f^{-1}(]-\infty, \rho])$ is compact for all $\rho \in]-\infty, r]$. Then, the set $f^{-1}(]-\infty, r])$ is connected.*

Proof. Assume that the set $f^{-1}(]-\infty, r])$ is disconnected. Then, because it is closed, there would be two nonempty, closed and disjoint sets A, B such that

$$f^{-1}(]-\infty, r]) = A \cup B.$$

Because the restriction of f to $f^{-1}(]-\infty, r])$ is lower semicontinuous and A, B are compact, there are $x_1 \in A$ and $x_2 \in B$ such that $f(x_1) = \inf_{x\in A} f$ and $f(x_2) = \inf_{x\in B} f$. Now, choose two open and disjoint sets $\Omega_1, \Omega_2 \in X$ such that $A \subseteq \Omega_1$ and $B \subseteq \Omega_2$. It is readily seen that $f(x_1) \leq f(x)$ for all $x \in \Omega_1$ and that $f(x_2) \leq f(x)$ for all $x \in \Omega_2$. Therefore, x_1 and x_2 would be two distinct local minima of f lying in $f^{-1}(]-\infty, r])$, against one of the hypotheses. ■

Theorem 12. *Let X be a Hausdorff topological space, $Y \subseteq \mathbf{R}$ an interval, and $f : X \times Y \to \mathbf{R}$ a function such that $f(x, \cdot)$ is continuous for all $x \in X$. Assume that there exist a number $\rho^* > \sup_Y \inf_X f$ and a set $D \subseteq Y$, dense in Y, such that, for each $\rho \in]-\infty, \rho^*[$ and each $y \in D$, the following conditions hold:*

(i') *the set* $\{s \in I : f(x,s) > \rho\}$ *is an interval for all* $x \in X$*;*
(ii') *the set* $\{x \in X : f(x,y) \leq \rho\}$ *is compact;*
(iii') *the function* $f(\cdot,y)$ *has at most one local minimum lying in the set* $\{x \in X : f(x,y) < \rho^*\}$.

Then, one has

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Proof. The proof is exactly the same as that of Theorem 10, with the only change of using Proposition 1 instead of Theorem 11. ■

The following consequence of Theorem 10 will be applied later to nonlinear differential equations:

Theorem 13. *Let* (X,τ) *be a Hausdorff topological space,* $Y \subseteq \mathbf{R}$ *an interval, and* $f : X \times Y \to \mathbf{R}$ *a function such that* $f(x,\cdot)$ *is continuous for all* $x \in X$ *and*

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) < \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Assume that there exist a number $\rho^* > \sup_Y \inf_X f$ *and an open set* $D \subseteq Y$*, dense in* Y*, such that, for each* $\rho \in]-\infty, \rho^*[$*, the set* $\{y \in Y : f(x,y) > \rho\}$ *is an interval for all* $x \in X$*, and the set* $\{x \in X : f(x,y) \leq \rho\}$ *is compact and sequentially compact for all* $y \in D$.

Then, there exist a nonempty open set $A \subset Y$ *such that, for every* $y \in A$ *and for every function* $\Phi : X \to \mathbf{R}$*, bounded below on the set* $\{x \in X : f(x,y) \leq \rho^*\}$ *and such that, for each* $\lambda > 0$ *small enough, the function* $f(\cdot,y) + \lambda\Phi(\cdot)$ *is sequentially lower semicontinuous, there exists* $\delta > 0$ *such that, for each* $\mu \in]0,\delta]$*, the function* $f(\cdot,y) + \mu\Phi(\cdot)$ *has at least two* $\tau_{f(\cdot,y)}$*-local minima lying in the set* $\{x \in X : f(x,y) < \rho^*\}$.

Proof. Denote by D' the set of all $y \in Y$ such that there exist a function $\Phi_y : X \to \mathbf{R}$, bounded below on the set $\{x \in X : f(x,y) \leq \rho^*\}$, and a sequence $\{\mu_n\}$ in $\mathbf{R}^+$ converging to 0 such that, for each $\lambda > 0$ small enough, the function $f(\cdot,y) + \lambda\Phi_y(\cdot)$ is sequentially lower semicontinuous, and, for each $n \in \mathbf{N}$, the function $f(\cdot,y) + \mu_n\Phi_y(\cdot)$ has at most one $\tau_{f(\cdot,y)}$-local minimum lying in the set $\{x \in X : f(x,y) < \rho^*\}$. By Theorem 10, the set $D \cap D'$ is not dense in Y. Consequently, because D is open and dense in Y, the set D' is not dense in Y, and so the set $A = \mathrm{int}(Y \setminus D')$ satisfies the conclusion. ■

Analogously, from Theorem 12 we get

Theorem 14. *Let* X *be a Hausdorff topological space,* $Y \subseteq \mathbf{R}$ *an interval, and* $f : X \times Y \to \mathbf{R}$ *a function such that* $f(x,\cdot)$ *is continuous for all* $x \in X$ *and*

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) < \inf_{x \in X} \sup_{y \in Y} f(x,y).$$

Assume that there exist a number $\rho^* > \sup_Y \inf_X f$ and an open set $D \subseteq Y$, dense in Y, such that, for each $\rho \in]-\infty, \rho^*[$, the set $\{y \in Y : f(x,y) > \rho\}$ is an interval for all $x \in X$, and the set $\{x \in X : f(x,y) \leq \rho\}$ is compact for all $y \in D$.

Then, there exist a nonempty open set $A \subset Y$ such that, for every $y \in A$, the function $f(\cdot, y)$ has at least two local minima lying in the set $\{x \in X : f(x,y) < \rho^*\}$.

Before ending this section, let us recall the important defintion of a saddle-point.

A saddle-point of $f : X \times Y \to \mathbf{R}$ is any $(x^*, y^*) \in X \times Y$ such that

$$f(x^*, y^*) = \inf_{x \in X} f(x, y^*) = \sup_{y \in Y} f(x^*, y).$$

The characterization of saddle-points is as follows:

Proposition 2. *(x^*, y^*) is a saddle-point of f if and only if the following three conditions hold:*

$$\sup_Y \inf_X f = \inf_X \sup_Y f,$$

$$\inf_{x \in X} f(x, y^*) = \sup_Y \inf_X f,$$

$$\sup_{y \in Y} f(x^*, y) = \inf_X \sup_Y f.$$

It is worth noticing that the mere validity of the condition

$$f(x^*, y^*) = \sup_Y \inf_X f = \inf_X \sup_Y f$$

is not enough to ensure that (x^*, y^*) is a saddle-point of f. For instance, take $X =]0,1]$, $Y = [0,1]$, and $f(x,y) = xy$. In this case, for each $x \in X$, we have

$$f(x, 0) = \sup_Y \inf_X f = \inf_X \sup_Y f = 0$$

but, clearly, f has no saddle-point, as the function $\sup_{y \in Y} f(\cdot, y)$ does not attain its infimum in X.

3 A General Multiplicity Theorem for Certain Nonlinear Equations in Hilbert Spaces

In the current section, we apply Theorem 12 to get the following result [37]:

Theorem 15. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \leq 0. \tag{3.1}$$

Then, for each $r \in]\inf_X J, \sup_X J[$ *and each* $x_0 \in J^{-1}(]-\infty, r[)$*, at least one of the following assertions holds:*

(a) *There exists* $\lambda > 0$ *such that the equation*

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

(b) *There exists a unique* $y \in J^{-1}([r, +\infty[)$ *such that*

$$\|x_0 - y\| = \mathrm{dist}(x_0, J^{-1}([r, +\infty[)) = \mathrm{dist}(x_0, J^{-1}(r)).$$

Among the most significant consequences of Theorem 15, there is the general multiplicity theorem announced in the title of the section. It reads as follows:

Theorem 16. *Let* X *be a real Hilbert space and let* $J : X \to \mathbf{R}$ *be a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\limsup_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} \leq 0.$$

Then, for each $r \in]\inf_X J, \sup_X J[$ *for which the set* $J^{-1}([r, +\infty[)$ *is not convex and for each convex set* $S \subseteq X$ *dense in* X*, there exist* $x_0 \in S \cap J^{-1}(]-\infty, r[)$ *and* $\lambda > 0$ *such that the equation*

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

To derive Theorem 16 from Theorem 15, we use a very recent result by I. G. Tsar'kov [47]. We state it below in a form that is enough for our purposes.

Theorem 17. *Let* X *be a real Hilbert space and* $C \subset X$ *a sequentially weakly closed and nonconvex set.*

Then, for each convex set $S \subseteq$ *dense in* X*, there exists* $x_0 \in S \setminus C$ *such that the set* $\{y \in C : \|x_0 - y\| = \mathrm{dist}(x_0, C)\}$ *has at least two points.*

In practice, when $\dim(X) = \infty$, Theorem 17 is a more precise version of the celebrated, classic result of Efimov and Stechkin on Chebyshev sets [12] (see also [52] for a proof based on convex analysis methods).

Now, the way of drawing Theorem 16 from Theorem 15 is transparent. Let us formalize it.

Proof (Proof of Theorem 16). Let $r \in]\inf_X J, \sup_X J[$ be such that the set $J^{-1}([r, +\infty[)$ is not convex and let $S \subseteq X$ be a convex set dense in X. Because J' is compact, the functional J turns out to be sequentially weakly continuous [54], Corollary 41.9). Hence, the set $J^{-1}([r, +\infty[)$ is sequentially weakly closed (possibly not weakly closed). Consequently, by Theorem 17, there exists $x_0 \in S \cap J^{-1}(]-\infty, r[)$ such that (b) of Theorem 15 does not hold. Hence, (a) of the same theorem holds, which is the conclusion. ∎

We are going to prove Theorem 15. We first recall that a Gâteaux differentiable functional J on a real Banach space X is said to satisfy the Palais–Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n\in\mathbf{N}} |J(x_n)| < +\infty$ and $\lim_{n\to+\infty} \|J'(x_n)\|_{X^*} = 0$ admits a strongly converging subsequence.

We also recall the following three critical points theorem [28]:

Theorem 18. *Let X be a real Banach space and let $J : X \to \mathbf{R}$ be a continuously Gâteaux differentiable functional satisfying the Palais–Smale condition and having at least two local minima.*

Then J has at least three critical points.

We now are in a position to prove Theorem 15.

Proof (Proof of Theorem 15). Let $r \in]\inf_X J, \sup_X J[$ and $x_0 \in J^{-1}(]-\infty, r[)$. Assume that assertion (a) does not hold. So, let us suppose that, for each $\lambda > 0$, the equation

$$x = \lambda J'(x) + x_0 \qquad (E_\lambda)$$

has at most two solutions. Now, define the function $f : X \times [0, +\infty[\to \mathbf{R}$ by setting

$$f(x, \lambda) = \frac{1}{2}\|x - x_0\|^2 + \lambda(r - J(x))$$

for all $(x, \lambda) \in X \times [0, +\infty[$. Let us check that f satisfies the hypotheses of Theorem 12, the space X being endowed with the weak topology. It is clear that (i') is satisfied. So, fix $\lambda \in [0, +\infty[$. As we have already observed, the functional J is sequentially weakly continuous. Hence, the functional $f(\cdot, \lambda)$ is sequentially weakly lower semicontinuous. Fix $\epsilon > 0$ so that $\frac{1}{2} - \epsilon\lambda > 0$. By (3.1), there is $\delta > 0$ such that

$$\sup_{\|x\|>\delta} \frac{J(x)}{\|x\|^2} < \epsilon.$$

Thus, we have

$$f(x, \lambda) > \left(\frac{1}{2} - \epsilon\lambda\right)\|x\|^2 - \|x_0\|\|x\| + \frac{1}{2}\|x_0\|^2 + \lambda r$$

for all $x \in X$, with $\|x\| > \delta$. Hence, we get

$$\lim_{\|x\|\to+\infty} f(x, \lambda) = +\infty.$$

From this, by the reflexivity of X, by the Eberlein–Smulyan theorem and by a classic result ([54], Example 38.25) we infer that $f(\cdot, \lambda)$ has weakly compact sublevels, has a global minimum, and satisfies the Palais–Smale condition. On the other hand, the critical points of $f(\cdot, \lambda)$ are exactly the solutions of (E_λ). Hence, by assumption, $f(\cdot, \lambda)$ has at most two critical points. Then, thanks to Theorem 18, $f(\cdot, \lambda)$ has exactly one global minimum and no other local minimum in the strong topology, and so, *a fortiori*, in the weak topology. Hence, also conditions (ii') and (iii') are satisfied. Therefore, Theorem 12 ensures that

$$\sup_{\lambda \geq 0} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda). \tag{3.2}$$

Clearly, one has

$$\inf_{x \in X} \sup_{\lambda \geq 0} f(x, \lambda) = \frac{1}{2} \inf_{x \in J^{-1}([r,+\infty[)} \|x - x_0\|^2. \tag{3.3}$$

Furthermore, observe that, because $J^{-1}([r, +\infty[)$ is sequentially weakly closed, there exists $y \in J^{-1}([r, +\infty[)$ such that

$$\|x_0 - y\| = \mathrm{dist}(x_0, J^{-1}([r, +\infty[)).$$

We claim that $y \in J^{-1}(r)$. Indeed, if $J(y) > r$, because J is continuous and $J(x_0) < r$, there would exist a point z in the line segment joining x_0 and y such that $J(z) = r$. So, we would have $\|x_0 - z\| <\mathrm{dist}(x_0, J^{-1}([r, +\infty[))$, an absurdity. In particular, this implies that

$$\mathrm{dist}(x_0, J^{-1}([r, +\infty[)) = \mathrm{dist}(x_0, J^{-1}(r)).$$

Now, observe that the function $\inf_{x\in X} f(x, \cdot)$ is upper semicontinuous in $[0, +\infty[$ and that $\lim_{\lambda\to+\infty} \inf_{x\in X} f(x, \lambda) = -\infty$, as $r < \sup_X J$. Hence, there is $\lambda^* \geq 0$ such that

$$\inf_{x \in X} f(x, \lambda^*) = \sup_{\lambda \geq 0} \inf_{x \in X} f(x, \lambda).$$

So, from (3.2) and (3.3), we get

$$\inf_{x \in X} \left(\frac{1}{2} \|x - x_0\|^2 - \lambda^* J(x) \right) = \inf_{x \in J^{-1}(r)} \left(\frac{1}{2} \|x - x_0\|^2 - \lambda^* J(x) \right).$$

From this, we infer that $\lambda^* > 0$, as $J(x_0) < r$, and that each global minimum of the restriction of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda^* J(x)$ to $J^{-1}(r)$ is a global minimum of the same functional on X. But, as we have seen above, for each $\lambda > 0$, the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda J(x)$ has exactly one global minimum in X. On the other hand, a point $y \in J^{-1}(r)$ is a global minimum for the restriction of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \lambda^* J(x)$ to $J^{-1}(r)$ if and only if $\|y - x_0\| =\mathrm{dist}(x_0, J^{-1}(r))$, and so (b) follows. ■

Remark 3. The conclusion of Theorem 15 can be false when (3.1) is not satisfied. To see this, take, for instance, $X = \mathbf{R}$, $J(x) = x^3 - x$, $r = 0$ and $x_0 = \frac{1}{2}$. We also believe that some more sophisticated example should show that the assumption about the compactness of J' cannot be omitted.

Remark 4. In [14], Theorem 15 has been extended to a broader class of Banach spaces, and [15] is devoted to a nonsmooth version of it.

Further, observe that, applying Theorem 15 to both J and $-J$, we get the following result

Theorem 19. *Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} = 0.$$

Then, for each $r \in]\inf_X J, \sup_X J[$ and each $x_0 \in X \setminus J^{-1}(r)$, at least one of the following assertions holds:

(*i*) *There exists $\lambda \in \mathbf{R}$ such that the equation*

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

(*ii*) *There exists a unique $y \in J^{-1}(r)$ such that*

$$\|x_0 - y\| = \operatorname{dist}(x_0, J^{-1}(r)).$$

Reasoning as in the proof of Theorem 16, we obtain the following consequence of Theorem 19:

Theorem 20. *Let X be a real Hilbert space and $J : X \to \mathbf{R}$ a continuously Gâteaux differentiable, nonconstant functional, with compact derivative, such that*

$$\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|^2} = 0.$$

Then, for each $r \in]\inf_X J, \sup_X J[$ for which the set $J^{-1}(r)$ is not convex and for each convex set $S \subseteq X$ dense in X, there exist $x_0 \in S \setminus J^{-1}(r)$ and $\lambda \in \mathbf{R}$ such that the equation

$$x = \lambda J'(x) + x_0$$

has at least three solutions.

We conclude this section presenting an application of Theorem 16 to a two-point boundary value problem.

Theorem 21. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous nonconstant and nondecreasing function satisfying*

$$\lim_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2} = 0. \tag{3.4}$$

Then, for each $r \in]\inf_{\xi\in\mathbf{R}} \int_0^\xi f(t)dt, \sup_{\xi\in\mathbf{R}} \int_0^\xi f(t)dt[$ and for each convex set $S \subseteq C_0^\infty(]0,1[)$ dense in $W_0^{1,2}(]0,1[)$, there exist $w \in S \cap J_f^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the problem

$$\begin{cases} -u'' = \lambda f(u) - w''(t) \text{ in } [0,1], \\ u(0) = u(1) = 0 \end{cases}$$

has at least three (classic) solutions.

Proof. Consider the Sobolev space $H_0^1(0,1)$ endowed with the inner product

$$\langle u, v\rangle = \int_0^1 u'(t)v'(t)dt.$$

Define the functional $J_f : H_0^1(0,1) \to \mathbf{R}$ putting

$$J_f(u) = \int_0^1 F(u(t))dt$$

for all $u \in H_0^1(0,1)$, where

$$F(\xi) = \int_0^\xi f(s)ds.$$

The functional J_f is continuously Gâteaux differentiable on $H_0^1(0,1)$ with compact derivative, and one has

$$\langle J_f'(u), v\rangle = \int_0^1 f(u(t))v(t)dt$$

for all $u, v \in H_0^1(0,1)$. Further, from (3.4), it readily follows that

$$\lim_{\|u\|\to+\infty} \frac{J_f(u)}{\|u\|^2} = 0.$$

So, J_f satisfies the assumption of Theorem 16. Fix $r \in]\inf_{\mathbf{R}} F, \sup_{\mathbf{R}} F[$ (note that r is in the interior of the range of J_f (see the argument below)). Now, let us show that the set $J_f^{-1}(r)$ is not convex. First, we note that, for each $a \in \mathbf{R}$, there exists $u \in H_0^1(0,1)$ such that $u\left(\frac{1}{2}\right) = a$ and $\int_0^1 F(u(t))dt = r$. Indeed, set

$$A = \left\{u \in H_0^1(0,1) : u\left(\frac{1}{2}\right) = a\right\}.$$

Fix r_1, r_2 satisfying $\inf_{\mathbf{R}} F < r_1 < r < r_2 < \sup_{\mathbf{R}} F$, and pick ξ_1, ξ_2 so that $F(\xi_1) = r_1$, $F(\xi_2) = r_2$. Next, choose $\epsilon > 0$ such that

$$r_1(1-4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r < r_2(1-4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F|,$$

where $\rho = \max\{|\xi_1|, |\xi_2|, |a|\}$. Finally, fix two functions $u_1, u_2 \in A$ so that

$$\max\left\{\sup_{[0,1]} |u_1|, \sup_{[0,1]} |u_2|\right\} \le \rho,\ u_1(t) = \xi_1,\ u_2(t) = \xi_2$$

for all $t \in [\epsilon, \frac{1}{2} - \epsilon] \cup [\frac{1}{2} + \epsilon, 1 - \epsilon]$. Then, we have

$$\int_0^1 F(u_1(t))dt \le r_1(1-4\epsilon) + 4\epsilon \sup_{[-\rho,\rho]} |F| < r$$

as well as

$$\int_0^1 F(u_2(t))dt \geq r_2(1-4\epsilon) - 4\epsilon \sup_{[-\rho,\rho]} |F| > r.$$

Because A is connected (being convex) and the functional $u \to \int_0^1 F(u(t))dt$ is continuous, there is $u \in A$ such that $\int_0^1 F(u(t))dt = r$, as claimed. Now, because f is not constant, we can fix $a, b \in \mathbf{R}$ so that $f(a) \neq f(b)$. According to the previous claim, there are $u, v \in H_0^1(0,1)$ such that $u\left(\frac{1}{2}\right) = a$, $v\left(\frac{1}{2}\right) = b$ and $\int_0^1 F(u(t))dt = \int_0^1 F(v(t))dt = r$. Finally, we claim that, for some $\mu \in]0,1[$, we have

$$\int_0^1 F(u(t) + \mu(v(t) - u(t)))dt \neq r.$$

Arguing by contradiction, assume the contrary. Hence, the derivative of the function $\mu \to \int_0^1 F(u(t) + \mu(v(t) - u(t)))dt$ is zero in $[0,1]$. That is,

$$\int_0^1 f(u(t) + \mu(v(t) - u(t)))(v(t) - u(t))dt = 0$$

for all $\mu \in [0,1]$. From this, it clearly follows that

$$\int_0^1 (f(v(t)) - f(u(t)))(v(t) - u(t))dt = 0.$$

Then, because f is nondecreasing, we infer that

$$(f(v(t)) - f(u(t)))(v(t) - u(t)) = 0$$

for all $t \in [0,1]$. So, because $u\left(\frac{1}{2}\right) \neq v\left(\frac{1}{2}\right)$, we get

$$f\left(u\left(\frac{1}{2}\right)\right) = f\left(v\left(\frac{1}{2}\right)\right),$$

a contradiction. Now, observe that J_f is convex, as f is nondecreasing. Consequently, $J^{-1}(]-\infty, r])$ is convex. Then, because $J^{-1}(r)$ is not convex, $J^{-1}([r, +\infty[)$ is not convex, too. Now, let $S \subseteq C_0^\infty(]0,1[)$ be a convex set dense in $H_0^1(0,1)$. Theorem 16 ensures the existence of $w \in S \cap J_f^{-1}(]-\infty, r[)$ and $\lambda > 0$ such that the equation

$$v = \lambda J_f'(v) + w$$

has at least three solutions in $H_0^1(0,1)$. Note that v is one of them if and only if

$$\begin{aligned}\int_0^1 v'(t)\omega'(t)dt &= \lambda \int_0^1 f(v(t))\omega(t)dt + \int_0^1 w'(t)\omega'(t)dt \\ &= \int_0^1 (\lambda f(v(t)) - w''(t))\omega(t)dt\end{aligned}$$

for all $\omega \in H_0^1(0,1)$. This clearly implies that $v \in C^2([0,1])$, with

$$-v''(t) = \lambda f(v(t)) - w''(t)$$

for all $t \in [0,1]$. Consequently, the function v is a classic solution of the problem

$$\begin{cases} -u'' = \lambda f(u) - w''(t) \ in \ [0,1], \\ u(0) = u(1) = 0. \end{cases}$$

So, this problem has at least three solutions, and the proof is complete. ■

Another application of Theorem 16 can be found in [21].

4 An Application of Theorem 13

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set, with boundary of class C^1, and let p be a real number greater than n. Consider the Sobolev space $W^{1,p}(\Omega)$ with the usual norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{\frac{1}{p}}.$$

Because $p > n$, $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. So, we have

$$c := \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}} < +\infty.$$

Let $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function such that $\sup_{|\xi|\le s} |\varphi(\cdot, \xi)| \in L^1(\Omega)$ for all $s > 0$.

For each $u \in W^{1,p}(\Omega)$, put

$$J_\varphi(u) = \int_\Omega \left(\int_0^{u(x)} \varphi(x,t) dt \right) dx.$$

The functional J_φ is (well-defined and) continuously Gâteaux differentiable on $W^{1,p}(\Omega)$, with compact derivative (so, J_φ is sequentially weakly continuous), and one has

$$J'_\varphi(u)(v) = \int_\Omega \varphi(x, u(x)) v(x) dx$$

for all $u, v \in W^{1,p}(\Omega)$.

Consider now the following Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \varphi(x,u) \text{ in } \Omega \\ \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

where ν is the outer unit normal to $\partial\Omega$. Let us recall that a weak solution of the problem is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Omega |u(x)|^{p-2}u(x)v(x)dx - \int_\Omega \varphi(x,u(x))v(x)dx = 0$$

for all $v \in W^{1,p}(\Omega)$.

Hence, the weak solutions of the problem are precisely the critical points in $W^{1,p}(\Omega)$ of the functional $u \to \frac{1}{p}\|u\|^p - J_\varphi(u)$.

The current section is devoted to get a multiplcity theorem for the above problem as an application of Theorem 13.

The result is as follows:

Theorem 22. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function for which there are $r > 0$ and $\xi_1 \in \mathbf{R}$, with $\mathrm{meas}(\Omega)|\xi_1|^p > pr$, such that*

$$\sup_{|\xi|\leq c(pr)^{\frac{1}{p}}} \int_0^\xi f(t)dt < \frac{pr}{\mathrm{meas}(\Omega)|\xi_1|^p}\int_0^{\xi_1} f(t)dt. \tag{4.1}$$

Assume also that

$$\limsup_{\|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{|\xi|^p} \leq 0. \tag{4.2}$$

Then, there exist $\rho > 0$ and a nonempty open set $B \subset]0,+\infty[$ with the following property: for each $\lambda \in B$ and for each Carathéodory function $g : \Omega \times \mathbf{R} \to \mathbf{R}$, with $\sup_{|\xi|\leq s} |g(\cdot,\xi)| \in L^1(\Omega)$ for all $s > 0$, there exists $\delta > 0$ such that, for every $\mu \in [0,\delta]$, the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \lambda f(u) + \mu g(x,u) \ \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \ \text{on } \partial\Omega \end{cases}$$

has at least two weak solutions whose norms in $W^{1,p}(\Omega)$ are less than ρ.

In the proof of Theorem 22, we will also use the following

Proposition 3. *Let X be a nonempty set and Ψ, J two real functions on X. Assume that there are $r > 0$, $x_0, x_1 \in X$ such that*

$$\Psi(x_0) = J(x_0) = 0,$$

$$\Psi(x_1) > r,$$

$$\sup_{x\in\Psi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Psi(x_1)}.$$

Then, for each σ satisfying

$$\sup_{x \in \Psi^{-1}(]-\infty, r])} J(x) < \sigma < r \frac{J(x_1)}{\Psi(x_1)} \tag{4.3}$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Psi(x) + \lambda(\sigma - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Psi(x) + \lambda(\sigma - J(x))).$$

Proof. First of all, observe that

$$\inf_{x \in X} \sup_{\lambda \geq 0} (\Psi(x) + \lambda(\sigma - J(x))) = \inf_{x \in J^{-1}([\sigma, +\infty[)} \Psi(x).$$

Next, note that, by (4.3), one has

$$r \leq \inf_{x \in J^{-1}([\sigma, +\infty[)} \Psi(x).$$

Moreover, because $\Psi(x_1) > r$, from (4.3), we infer that $J(x_1) > \sigma$. This implies that the function $\lambda \to \inf_{x \in X}(\Psi(x) + \lambda(\sigma - J(x))$ tends to $-\infty$ as $\lambda \to +\infty$. But, this function is upper semicontinuous in $[0, +\infty[$, and hence it attains its supremum at a point $\overline{\lambda}$. We now distinguish two cases. If $0 \leq \overline{\lambda} < \frac{r}{\sigma}$ (note that $\sigma > 0$ because $\Psi(x_0) = J(x_0) = 0$), then $\Psi(x_0) + \overline{\lambda}(\rho - J(x_0)) = \overline{\lambda}\sigma < r$. If $\frac{r}{\sigma} \leq \overline{\lambda}$, then, because (by (4.3) again) $\frac{r - \Psi(x_1)}{\sigma - J(x_1)} < \frac{r}{\sigma}$, we have $\Psi(x_1) + \overline{\lambda}(\sigma - J(x_1)) < r$, and the proof is complete. ■

Remark 5. Let X be a nonempty set and Ψ, J two real functions on X having a common zero. Consider the function $\eta :]0, +\infty[\to [0, +\infty]$ defined by putting

$$\eta(t) = \frac{\sup_{x \in \Psi^{-1}(]-\infty, t])} J(x)}{t}$$

for all $t > 0$. Then, it is easy to check that the following conditions are equivalent:

(i) The function η is not nonincreasing.
(ii) There exist $r > 0$ and $x_1 \in X$, with $\Psi(x_1) > r$, such that

$$\sup_{x \in \Psi^{-1}(]-\infty, r])} J(x) < r \frac{J(x_1)}{\Psi(x_1)}.$$

Proof (Proof of Theorem 22). For each $u \in W^{1,p}(\Omega)$, put

$$\Psi(u) = \frac{1}{p} \|u\|^p.$$

Note that if $\Psi(u) \leq r$, then $\sup_\Omega |u| \leq c(pr)^{\frac{1}{p}}$, and so, by (4.1), if u_1 denotes the constant function in Ω taking the value ξ_1, we have

$$\sup_{u\in\Psi^{-1}(]-\infty,r])} J_f(u) \le \mathrm{meas}(\Omega) \sup_{|\xi|\le c(pr)^{\frac{1}{p}}} \int_0^{\xi} f(t)dt$$
$$< \frac{pr}{|\xi_1|^p}\int_0^{\xi_1} f(t)dt = r\frac{J_f(u_1)}{\Psi(u_1)}.$$

Hence, by Proposition 3, for a suitable constant σ, we have

$$\sup_{\lambda\ge 0}\inf_{u\in W^{1,p}(\Omega)}(\Psi(u)+\lambda(\sigma-J_f(u))) < \inf_{u\in W^{1,p}(\Omega)}\sup_{\lambda\ge 0}(\Psi(u)+\lambda(\sigma-J_f(u))).$$

Observe that, by (4.2), we have for each $\lambda \ge 0$

$$\lim_{\|u\|\to+\infty}(\Psi(u)-\lambda J_f(u)) = +\infty$$

and so the functional $\Psi - \lambda J_f$ has weakly compact sublevels. Hence, the functional $(u,\lambda)\to\Psi(u)+\lambda(\sigma-J_f(u))$ $((u,\lambda)\in W^{1,p}(\Omega)\times[0,+\infty[)$ satisfies all the assumptions of Theorem 13, the space $W^{1,p}(\Omega)$ being endowed with the weak topology. Fix $s^* > \sup_{\lambda\ge 0}\inf_{u\in W^{1,p}(\Omega)}(\Psi(u)+\lambda(\sigma-J_f(u)))$. Let $A \subset]0,+\infty[$ be a nonempty open set with the property declared in Theorem 13. Let $g : \Omega\times\mathbf{R}\to\mathbf{R}$ be a Carathéodory function with $\sup_{|\xi|\le s}|g(\cdot,\xi)| \in L^1(\Omega)$ for all $s > 0$. Fix $a, b \in A$, with $a < b$. Then, for every $\lambda\in[a,b]$, there exists $\delta > 0$ such that, for each $\mu\in[0,\delta]$, the functional $\Psi+\lambda(\sigma-J_f)-\mu J_g$ has at least two local minima lying in the set $\{u\in W^{1,p}(\Omega) : \Psi(u)+\lambda(\sigma-J_f(u)) < s^*\}$. Such local minima are critical points of the functional and so weak solutions of problem $(P_{\lambda,\mu})$. Finally, observe that

$$\bigcup_{\lambda\in[a,b]}\{u\in W^{1,p}(\Omega) : \Psi(u)+\lambda(\sigma-J_f(u)) < s^*\}$$
$$\subseteq \{u\in W^{1,p}(\Omega) : \Psi(u)+a(\sigma-J_f(u)) < s^*\}$$
$$\cup\{u\in W^{1,p}(\Omega) : \Psi(u)+b(\sigma-J_f(u)) < s^*\}.$$

But the set on the right-hand side is bounded, and hence we can choose as ρ the radius of a ball, centered at 0, containing this latter. The proof is complete. ■

Remark 6. Other recent applications of Theorem 13 can be found in [6, 7], and [8]. We also recall that the minimax result in [34] (which is very close to Theorem 14) was the starting point for a long series of applications to nonlinear differential equations (see, for instance, [1–5, 9, 10, 19, 20, 22–24, 43, 51]).

5 An Application of Theorem 2 to Locally Minimizing Functionals with Locally Lipschitzian Derivative

In the sequel, $(X, \langle\cdot,\cdot\rangle)$ is a real Hilbert space. For each $x \in X$, $r > 0$, we set

$$B(x,r) = \{y\in X : \|y-x\| \le r\}$$

and

$$S(x,r) = \{y \in X : \|y - x\| = r\}.$$

Given a functional $J : X \to \mathbf{R}$ and a set $C \subseteq X$, we say that the problem of minimizing J over C is well-posed if the following two conditions hold:

- the restriction of J to C has a unique global minimum, say $\hat{x}$;
- for every sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} J(x_n) = J(\hat{x})$, one has $\lim_{n\to\infty} \|x_n - \hat{x}\| = 0$.

The aim of this section is to prove, making use of Theorem 2, the following general result:

Theorem 23. *Let $J : X \to \mathbf{R}$ be a C^1 functional with locally Lipschitzian derivative.*

Then, for each $x_0 \in X$ with $J'(x_0) \neq 0$, there exists $\delta > 0$ such that, for every $r \in]0, \delta[$, one has

$$\inf_{B(x_0,r)} J = \inf_{S(x_0,r)} J$$

and the problems of minimizing J over $S(x_0, r)$ and over $B(x_0, r)$ are well-posed.

Proof. Fix $x_0 \in X$ with $J'(x_0) \neq 0$. Fix also $\rho > 0$ so that

$$J'(x) \neq 0$$

for all $x \in B(x_0, \rho)$ and

$$L := \sup_{x,y\in B(x_0,\rho),x\neq y} \frac{\|J'(x) - J'(y)\|}{\|x - y\|} < +\infty.$$

For each $\lambda > 0$, $x \in X$, set

$$I_\lambda(x) = \frac{\lambda}{2}\|x - x_0\|^2 + J(x).$$

Let $\lambda \geq L$. For each $x, y \in B(x_0, \rho)$, we have

$$\langle I'_\lambda(x) - I'_\lambda(y), x - y\rangle = \langle \lambda(x - x_0) + J'(x) - \lambda(y - x_0) - J'(y), x - y\rangle$$

$$\geq \lambda\|x - y\|^2 - \|J'(x) - J'(y)\|\|x - y\| \geq (\lambda - L)\|x - y\|^2. \tag{5.1}$$

From (5.1), via a classic result ([53], Proposition 25.10), we then get that the functional I_λ is striclty convex (resp. convex) in $B(x_0, \rho)$ if $\lambda > L$ (resp. $\lambda = L$). Denote by Γ the set of all global minima of the restriction of I_L to $B(x_0, \rho)$ and set

$$\delta = \inf_{x\in\Gamma} \|x - x_0\|.$$

Observe that $\delta > 0$. Indeed, if $\delta = 0$, then x_0 would be a local minimum in X for I_L, and so

$$0 = I'_L(x_0) = J'(x_0)$$

against an assumption. Now, fix $r \in]0, \delta[$ and consider the the function $\Phi : B(x_0, \rho) \times [L, +\infty[\to \mathbf{R}$ defined by

$$\Phi(x, \lambda) = I_\lambda(x) - \frac{\lambda r^2}{2}$$

for all $(x, \lambda) \in B(x_0, \rho) \times [L, +\infty[$. As we have seen above, $\Phi(\cdot, \lambda)$ is continuous and convex in $B(x_0, \rho)$ for all $\lambda \geq L$, and $\Phi(x, \cdot)$ is continuous and concave for all $x \in B(x_0, \rho)$, with $\lim_{\lambda \to +\infty} \Phi(x_0, \lambda) = -\infty$. So, applying jointly Theorem 2 and Proposition 2 to Φ, we get the existence of $(\hat{x}, \hat{\lambda}) \in B(x_0, \rho) \times [L, +\infty[$ such that

$$J(\hat{x}) + \frac{\hat{\lambda}}{2}(\|\hat{x} - x_0\|^2 - r^2) = \inf_{x \in B(x_0,\rho)} \left(J(x) + \frac{\hat{\lambda}}{2}(\|x - x_0\|^2 - r^2) \right)$$

$$= J(\hat{x}) + \sup_{\lambda \geq L} \frac{\lambda}{2}(\|\hat{x} - x_0\|^2 - r^2).$$

Of course, we have $\|\hat{x} - x_0\| \leq r$, because the sup is finite. But, if it were $\|\hat{x} - x_0\| < r$, we would have $\hat{\lambda} = L$. This, in turn, would imply that $\hat{x} \in S$, against the fact that $r < \delta$. Hence, we have $\|\hat{x} - x_0\| = r$. Consequently

$$J(\hat{x}) + \frac{\hat{\lambda} r^2}{2} = \inf_{x \in B(x_0,\rho)} \left(J(x) + \frac{\hat{\lambda}}{2}\|x - x_0\|^2 \right).$$

From this, we infer that $\hat{\lambda} > L$ (because $r < \delta$), that $\hat{x}$ is a global minimum of $J_{|S(x_0,r)}$, and that each global minimum of $J_{|S(x_0,r)}$ is a global minimum of $I_{\hat{\lambda}|B(x_0,\rho)}$. Because $\hat{\lambda} > L$, this latter functional is strictly convex, and so $\hat{x}$ is its unique global minimum in $B(x_0, \rho)$ toward which every minimizing sequence weakly converges ([11], p. 3). In particular, note that if $\{y_n\}$ is a sequence in $B(x_0, \rho)$ such that $\lim_{n\to\infty} J(y_n) = J(\hat{x})$ and $\lim_{n\to\infty} \|y_n - x_0\| = r$, then

$$\lim_{n\to\infty} \left(J(y_n) + \frac{\hat{\lambda}}{2}\|y_n - x_0\|^2 \right) = \inf_{x \in B(x_0,\rho)} \left(J(x) + \frac{\hat{\lambda}}{2}\|x - x_0\|^2 \right),$$

and so $\{y_n\}$ converges weakly to $\hat{x}$. Because $\lim_{n\to\infty} \|y_n - x_0\| = \|\hat{x} - x_0\|$ and X is a Hilbert space, it follows that $\lim_{n\to\infty} \|y_n - \hat{x}\| = 0$. This shows that, for each $r \in]0, \delta[$, the problem of minimizing J over $S(x_0, r)$ is well-posed.

Fix again $r \in]0, \delta[$. Now, let us show that $\inf_{B(x_0,r)} J = \inf_{S(x_0,r)} J$. To this end, for each $t \in [0, r]$, put

$$\varphi(t) = \inf_{S(x_0,t)} J$$

and denote by x_t the unique global minimum of $J_{|S(x_0,t)}$. Clearly, we have

$$\inf_{B(x_0,r)} J = \inf_{[0,r]} \varphi.$$

Note also that, by the mean value theorem, J is Lipschitzian in $B(x_0, \rho)$, with Lipschitz constant $L_1 := \|J'(x_0)\| + L\rho$. Fix $t, s \in [0, r]$. We have

$$\varphi(s) - \varphi(t) \leq J\left(x_0 + \frac{s}{t}(x_t - x_0)\right) - J(x_t) \leq L_1|t - s|$$

as well as

$$\varphi(t) - \varphi(s) \leq J\left(x_0 + \frac{t}{s}(x_s - x_0)\right) - J(x_s) \leq L_1|t - s|.$$

Thus, φ is Lipschitzian and so it attains its infimum in $[0, r]$ at a point $\hat{t}$. In other words, we have

$$\inf_{B(x_0,r)} J = J(x_{\hat{t}}).$$

Recalling that $J'(x) \neq 0$ for all $x \in B(x_0, r)$, we then infer that $\hat{t} = r$. So, x_r is also the unique global minimum of $J_{|B(x_0,r)}$. Finally, let $\{y_n\}$ be a sequence in $B(x_0, r)$ such that $\lim_{n\to\infty} J(y_n) = J(x_r)$. By a remark above, to get that $\lim_{n\to\infty} \|y_n - x_r\| = 0$, we have to show that $\lim_{n\to\infty} \|y_n - x_0\| = r$. Argue by contradiction. If it was

$$\liminf_{n\to\infty} \|y_n - x_0\| < r,$$

then, for some $\gamma \in]0, r[$, we would have $\|y_n - x_0\| < \gamma$ for infinitely many n, and so

$$\inf_{B(x_0,r)} J = \inf_{B(x_0,\gamma)} J = J(x_\gamma)$$

against the fact that $J'(x_\gamma) \neq 0$. Thus, also the problem of minimizing J over $B(x_0, r)$ is well-posed, and the proof is complete. ■

Observe that, in Theorem 23, the condition $J'(x_0) \neq 0$ is essential. In fact, consider the case where J is even (and so $J'(0) = 0$ because J' is odd). Then, for any $r > 0$, $J_{|S(0,r)}$ has either none or at least two global minima.

Also, the local Lipschitzianity of J' is essential. In fact, if J' is not locally Lipschitzian at x_0 (and $J'(x_0) \neq 0$ as well), it may occur either that $J_{|S(x_0,r)}$ has at least two global minima for each $r > 0$ or that $J_{|S(x_0,r)}$ has no global minima for each $r > 0$. In this connection, consider the two following examples.

Example 1. Take $X = \mathbf{R}^2$ and

$$J(x, y) = x - |y|^q$$

where $1 < q < 2$. Note that $J \in C^1(\mathbf{R}^2)$ and $\nabla J(0) \neq 0$. Let $r > 0$. Because

$$\lim_{n\to\infty} n^{q-1}(n - \sqrt{n^2-1}) = 0,$$

for $n \in \mathbf{N}$ large enough, we have

$$J\left(-\sqrt{r^2 - \frac{r^2}{n^2}}, \frac{r}{n}\right) = -\sqrt{r^2 - \frac{r^2}{n^2}} - \left(\frac{r}{n}\right)^q < -r = J(-r, 0).$$

Now, observe that $J_{|S(0,r)}$ attains its infimum at some point (x_0, y_0) with $x_0 \leq\ 0$. The above inequality shows that $x_0 > -r$ (and so $y_0 \neq 0$). Consequently, also $(x_0, -y_0)$ is a global minimum of $J_{|S(0,r)}$.

Example 2. Take $X = l^2$ and

$$J(x) = x_1 - \left(\sum_{n=2}^{\infty} a_n^2 x_n^2\right)^p$$

where $\frac{1}{2} < p < 1$ and $\{a_n\}$ is a strictly increasing sequence of positive numbers converging to 1. Note that $J \in C^1(l^2)$ and $J'(0) \neq 0$. Fix $r > 0$. Let $\{e_n\}$ be the canonical basis of l^2. Moreover, set

$$I = \{x \in l^2 : x_1 = 0\}$$

and let $A : l^2 \to l^2$ the operator defined by

$$A(x) = \{a_n x_n\}$$

for all $x \in l^2$. Note that $\|A(e_n)\| = a_n$ and so $\sup_{n\in\mathbf{N}} \|A(e_n)\| = 1$. Note also that $\|A(y)\| < 1$ for all $y \in I \cap S(0,1)$. Further, it is easy to see that

$$S(0,r) = \{-r\sqrt{1-\lambda^2}e_1 + \lambda r y : \lambda \in [0,1], y \in I \cap S(0,1)\}.$$

Consequently, we have

$$\inf_{S(0,r)} J = \inf_{y\in I\cap S(0,1)} \inf_{\lambda\in[0,1]} -r\left(\sqrt{1-\lambda^2} + r^{2p-1}\|A(y)\|^{2p}\lambda^{2p}\right).$$

Now, let $\eta : [0, +\infty[\to \mathbf{R}$ be the continuous function defined by

$$\eta(t) = \sup_{\lambda\in[0,1]} (\sqrt{1-\lambda^2} + t\lambda^{2p})$$

for all $t \geq 0$. Because $p < 1$, one readily sees that η is strictly increasing. Hence, we have

$$\begin{aligned}\inf_{S(0,r)} J &= -r \sup_{y\in I\cap S(0,1)} \sup_{\lambda\in[0,1]} \left(\sqrt{1-\lambda^2} + r^{2p-1}\|A(y)\|^{2p}\lambda^{2p}\right)\\ &= -r \sup_{y\in I\cap S(0,1)} \eta(r^{2p-1}\|A(y)\|^{2p}) = -r\eta(r^{2p-1}).\end{aligned}$$

But, for every $\lambda \in [0,1]$ and $y \in I \cap S(0,1)$, we have

$$J(-r\sqrt{1-\lambda^2}e_1 + \lambda r y) \geq -r\eta(r^{2p-1}\|A(y)\|^{2p}) > -r\eta(r^{2p-1})$$

and hence $J_{|S(0,r)}$ has no global minima.

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On Noncooperative Games, Minimax Theorems, and Equilibrium Problems

Johannes B.G. Frenk[1] and Gábor Kassay[2]

[1] Econometric Institute, Erasmus University, Rotterdam, The Netherlands
`frenk@few.eur.nl`

[2] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj, Romania
`kassay@math.ubbcluj.ro`*

Abstract In this chapter, we give an overview on the theory of noncooperative games. In the first part, we consider in detail zero-sum (constant-sum) games with two players having arbitrary strategy sets, under which necessary and sufficient conditions on the payoff function and the different (extended strategy) sets an equilibrium (saddle-point) strategy (for both players) exists. The existence of such an equilibrium strategy is equivalent to whether a so-called minimax theorem for the payoff function holds. The proof of such a result uses either the separation result for disjoint convex sets in finite dimensional linear spaces or the strong duality theorem for linear programming in combination with some elementary properties of compact sets. Both proof techniques are given together with a discussion of the most well-known minimax theorems that appeared in the literature. Also for the most famous minimax result given by Sion, we separately show an elementary proof avoiding the KKM lemma and using only the definition of connectedness. In the final part, we also consider n-person nonzero-sum noncooperative games. It is shown by a simple application of the same KKM lemma that a Nash equilibrium strategy (a generalization of a (saddle-point) equilibrium strategy for two players) exists under certain conditions on the payoff functions and the strategy sets. The main goal of this chapter is to discuss in detail and full generality the most elementary mathematical techniques for proving the existence of equilibrium points in noncooperative games (with an emphasis on two players).

Key words: noncooperative game theory, extended strategy set, saddle-point equilibrium, minimax theorems, Nash equilibrium, linear programming duality, separation result for disjoint convex sets

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1 Introduction to Noncooperative Game Theory

To introduce a static two-player zero-sum (noncooperative) game (for more details and examples see [4, 18, 25, 40] or [2]) and its relation to a minimax theorem, we consider two players called 1 and 2 and assume that the set of pure strategies (also called actions) of player 1 is given by some nonempty set A, and the set of pure strategies of player 2 is given by a nonempty set B. Without loss of generality, we may assume that the sets A and B are topological spaces with Borel σ-algebras $\mathcal{B}(A)$, respectively $\mathcal{B}(B)$. By definition, a Borel σ-algebra is the smallest σ-algebra generated by the open sets ([33]). If player 1 chooses the pure strategy $a \in A$ and player 2 chooses the pure strategy $b \in B$, then player 2 has to pay player 1 an amount $f(a,b)$ with $f : A \times B \to R$ a given function. This function is called the payoff function of player 1. Because the gain of player 1 is the loss of player 2 (this is a so-called zero-sum game), the payoff function of player 2 is $-f$. Clearly player 1 likes to gain as much profit as possible. However, at the moment he does not know how to achieve this and so he first decides to compute a lower bound on his profit. To compute this lower bound, player 1 argues as follows: if he decides to choose action $a \in A$, then it follows that his profit is at least $\inf_{b \in B} f(a,b)$, irrespective of the action of player 2. Therefore, a lower bound on the profit for player 1 is given by

$$r_* := \sup_{a \in A} \inf_{b \in B} f(a,b). \tag{1}$$

Similarly, player 2 likes to minimize his losses but as he does not know how to achieve this, he also decides to compute first an upper bound on his losses. To do so, player 2 argues as follows. If he decides to choose action $b \in B$, it follows that he loses at most $\sup_{a \in A} f(a,b)$, and this is independent of the action of player 1. Therefore, an upper bound on his losses is given by

$$r^* := \inf_{b \in B} \sup_{a \in A} f(a,b). \tag{2}$$

Because the profit of player 1 is at least r_* and the loss of player 2 is at most r^* and the losses of player 2 are the profits of player 1, it follows directly that $r_* \leq r^*$. In general $r_* < r^*$, but under some properties on the pure strategy sets and payoff function, one can show that $r_* = r^*$. If this equality holds and in relations (1) and (2) the suprema and infima are attained, an optimal strategy for both players is obvious. By the interpretation of r_* for player 1 and the interpretation of r^* for player 2 and $r^* = r_* := v$, both players will choose an action that achieves the value v and so player 1 will choose that action $a_0 \in A$ satisfying

$$\inf_{b \in B} f(a_0, b) = \max_{a \in A} \inf_{b \in B} f(a,b).$$

Moreover, player 2 will choose that strategy $b_0 \in B$ satisfying

$$\sup_{a \in A} f(a, b_0) = \min_{b \in B} \sup_{a \in A} f(a,b).$$

In case only $r_* = r^*$ or equivalently

$$\inf_{b\in B} \sup_{a\in A} f(a,b) = \sup_{a\in A} \inf_{b\in B} f(a,b) \tag{3}$$

both players can approximate their optimal pure strategies by so-called ϵ-optimal pure strategies. A pure strategy $a_0 \in A$ for player 1 is called an ϵ-optimal pure strategy if

$$\inf_{b\in B} f(a_0,b) \geq v - \epsilon.$$

A similar definition applies to an ϵ-optimal pure strategy for player 2. By these observations, it is now important to know for which payoff functions and pure strategy sets the so-called minimax result $r_* = r^*$ holds and under which conditions the supremum in relation (1) and the infimum in relation (2) are attained. Before discussing this, we give an example for which the equality $r^* = r_*$ does not hold.

Example 1. Consider the continuous payoff function $f : [0,1] \times [0,1] \to [0,\infty)$ given by

$$f(a,b) = (a-b)^2.$$

For this function, it holds for every $0 \leq a \leq 1$ that $\inf_{b\in[0,1]}(a-b)^2 = 0$ and so

$$r_* := \sup_{0\leq a\leq 1} \inf_{0\leq b\leq 1} (a-b)^2 = 0.$$

Moreover, it follows that

$$\sup_{0\leq a\leq 1}(a-b)^2 = (1-b)^2$$

for every $0 \leq b < \frac{1}{2}$ and

$$\sup_{0\leq a\leq 1}(a-b)^2 = b^2$$

for every $\frac{1}{2} \leq b \leq 1$. This shows

$$r^* := \inf_{0\leq b\leq 1} \sup_{0\leq a\leq 1}(a-b)^2 = 4^{-1}$$

and so r_* does not equal r^*.

The above example shows a particular case for which it is not clear how the players should select their strategies. A possible solution to this problem is to extend the set of pure strategies to the larger set of so-called mixed strategies. Recall in the next definition that a Borel finite measure on a topological space D is a finite measure defined on the Borel σ-algebra $\mathcal{B}(D)$ of D (for more details on Borel measures see [7, 8, 34]). Moreover, we also need in this definition the unit simplex $\Delta_k \subseteq R^k$ given by

$$\Delta_k := \{\alpha^\top = (\alpha_1, ..., \alpha_k) \in R^k : \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0, 1 \leq i \leq k\}. \tag{4}$$

Definition 1. *Let D be a nonempty topological space and $\mathcal{B}(D)$ its Borel σ-algebra. A Borel finite measure $\epsilon_d : \mathcal{B}(D) \to [0, \infty)$ is called a one-point Borel probability measure concentrated on the set $\{d\}$ if $\epsilon_d(D_0) = 1$ for $D_0 \in \mathcal{B}(D)$ containing d and $\epsilon_d(D_0) = 0$ otherwise. A Borel finite measure $\nu : \mathcal{B}(D) \to [0, 1]$ is called a Borel probability measure with finite support if there exists some finite set $\{d_1, \ldots, d_k) \subseteq D$ and some vector $s(\nu)^\top \in \Delta_k$ with $s_i(\nu) > 0, 1 \le i \le k$ such that*

$$\nu = \sum\nolimits_{i=1}^{k} s_i(\nu)\epsilon_{d_i}.$$

If we denote by $\mathcal{P}_F(D)$ the set of all Borel probability measures on D with a finite support, then within game theory, any element ν belonging to $\mathcal{P}_F(D)$ is called a mixed strategy and it has the following interpretation. If a player with pure strategy set D selects the mixed strategy

$$\nu = \sum\nolimits_{i=1}^{k} s_i(\nu)\epsilon_{d_i},$$

then with probability $s_i(\nu), 1 \le i \le k$ this player will use the pure strategy $d_i \in D$. By this interpretation, it is clear that the set D of pure strategies can be identified within the set of mixed strategies by the one-point Borel probability measures $\{\epsilon_d : d \in D\}$. We now assume that player 1, respectively player 2 are using their sets of mixed strategies. This means that the payoff function f should be extended to a function $f_e : \mathcal{P}_F(A) \times \mathcal{P}_F(B) \to R$. This extension is defined by

$$f_e(\lambda, \mu) := \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{m} s_i(\lambda)s_j(\mu)f(a_i, b_j) \tag{5}$$

with $\lambda = \sum_{i=1}^{k} s_i(\lambda)\epsilon_{a_i} \in \mathcal{P}_F(A)$ and $\mu = \sum_{j=1}^{m} s_j(\mu)\epsilon_{b_j} \in \mathcal{P}_F(B)$, and it represents the expected profit for player 1 or expected loss of player 2 if player 1 selects the mixed strategy $\lambda \in \mathcal{P}_F(A)$ and player 2 selects the mixed strategy $\mu \in \mathcal{P}_F(B)$. Under some topological/algebraic conditions on the function f and the sets A and B of pure strategies, it can be shown that the game represented by f_e and the mixed strategy sets $\mathcal{P}_F(A)$ and $\mathcal{P}_F(B)$ has a solution. This means that we need to investigate under which necessary and sufficient conditions the following minimax result holds:

$$\inf\nolimits_{\mu \in \mathcal{P}_F(B)} \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \mu) = \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} \inf\nolimits_{\mu \in \mathcal{P}_F(B)} f_e(\lambda, \mu). \tag{6}$$

In case player 2 is only allowed to use his pure strategy set B, we will also investigate under which necessary and sufficient conditions the game represented by f_e and the sets B and $\mathcal{P}_F(A)$ has a solution. Hence for this case, we need to check under which conditions the minimax result

$$\inf\nolimits_{b \in B} \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \epsilon_b) = \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} \inf\nolimits_{b \in B} f_e(\lambda, \epsilon_b) \tag{7}$$

holds. Finally, if player 1 and player 2 are only allowed to use their pure strategy sets, we again pose the same question and investigate under which necessary and sufficient conditions the game represented by f and the sets A and B has a solution or equivalently under which condition the classic minimax result

$$\inf_{b \in B} \sup_{a \in A} f(a,b) = \sup_{a \in A} \inf_{b \in B} f(a,b). \tag{8}$$

holds. A slight extension of a two-player zero-sum game is given by a so-called two-player constant-sum game. In this case, each player has a payoff function $f_i, i = 1,2$ and for these payoff functions there exists some $c \in R$ such that

$$f_1(a,b) + f_2(a,b) = c$$

for every $a \in A$ and $b \in B$. As in a zero-sum game, the gain for player 1, respectively player 2 is given by $f_1(a,b)$, respectively $f_2(a,b)$ when both players select independently the strategies a, respectively b. Introducing for this game the payoff functions $\widetilde{f}_i, i = 1,2$ given by

$$\widetilde{f}_i = f_i - c_i$$

with $c_1 + c_2 = c$, it is easy to see that the analysis of the original constant-sum game reduces to the analysis of a zero-sum game with payoff function $\widetilde{f}_1$ for player 1. The above two-player zero-sum (constant-sum) noncooperative game can also be extended to a nonconstant-sum noncooperative game involving $n \geq 2$ players. In this model we have n players, $n \geq 2$ and player i, $1 \leq i \leq n$ has a pure strategy set X_i and a payoff function $f_i : X \to R$ with $X = \Pi_{i=1}^n X_i$ (for a detailed definition of these games, the reader is referred to [4, 40] or [38]). Embedding the two-player zero-sum game into this more general framework, we observe that in this case player 1 has payoff function $f_1 = f$, and player 2 has payoff function $f_2 = -f$. For the nonconstant-sum case and $n \geq 2$, we use the notation X_i to distinguish between the two different models, and as before the pure strategy sets $X_i, i = 1, \ldots, n$ are topological spaces. For the more general n-player nonzero-sum noncooperative games, the concept of a minimax or saddle-point approach used within a two-player zero-sum game is generalized and replaced by a so-called Nash equilibrium point ([28, 29]). In Section 6, these more general games will be explained in detail. To analyse the minimax relations given in (6) up to (8) for a two-player zero-sum (noncooperative) game, we start in Section 2 with a discussion of Wald's minimax theorem. This theorem plays a key role in deriving necessary and sufficient conditions and will be proved using two different methods. The first proof uses the separation result of disjoint convex sets in convex analysis, whereas the second one uses strong linear programming duality and some elementary properties of compact sets. In Section 3, these conditions together with an overview of important sufficient conditions that have appeared in the literature are discussed. Also, we show that the sufficient conditions discussed in the literature can be easily verified using our necessary and sufficient conditions. In Section 4, we then give the relations between the different minimax theorems, and in Section 5 we consider the famous minimax result of the form (8) derived by Sion ([37]). Unfortunately, it remains an open question whether this minimax result can be derived directly from our necessary and sufficient conditions discussed in Section 3. Although it is not well-known, a primitive version of Sion's minimax theorem already appeared in the classic paper by von Neumann ([21, 30]). The proof of Sion's theorem given here is completely elementary and uses a proof technique originated by Joó ([11,19]), which differs from the original proof using the so-called KKM (Knaster–Kuratowski–Mazurkiewicz) lemma. Observe the KKM lemma is equivalent to the Brouwer fixed point theorem ([43]) and is discussed in Section 6. Also in Section 6, we introduce the extension of a two-player zero-sum game to a n-player nonzero-sum (noncooperative) game and introduce the concept of a Nash equilibrium point. Moreover, we prove that under certain conditions, a n-player nonzero-sum (noncooperative) game indeed has a Nash equilibrium point using a simple proof that applies the aforementioned KKM lemma. Unfortunately, it remains an open question whether it is possible to prove the existence of a Nash equilibrium point by the elementary techniques used for the two-player zero-sum model.

2 On Wald's Minimax Theorem

We assume in this section that the reader is familiar with the basic notions in set theory, analysis, and some elementary function theory (for more details see [35]). Besides this basic knowledge, this section will be self-contained. To show for the different minimax results listed in relation (6) up to (8) necessary and sufficient conditions on the payoff function f and the sets A and B, we first need to discuss in detail Wald's minimax theorem, and this will be the topic of this section. The derivation of the necessary and sufficient conditions will be postponed until Section 3. For readers familiar with convex analysis, a proof of Wald's minimax theorem will be given using the (finite dimensional) separating hyperplane result, whereas for readers more familiar with linear programming, we will show Wald's minimax result using the strong duality theorem of linear programming and some elementary properties of compact sets. We first start with a proof using tools from convex analysis. To do so, we first need to recall some well-known definitions and introduce the proper notation.

Definition 2. *A subset C of a linear space is called convex if for every $0 < \beta < 1$ and $x, y \in C$, it follows that $\beta x + (1-\beta)y$ belongs to C.*

In set notation, this means that $\beta C + (1-\beta)C \subseteq C$ for every $0 < \beta < 1$.

Definition 3. *A real-valued function $k : C \to R$ is called convex on the (convex) subset C if*

$$k(\beta x + (1-\beta)y) \le \beta k(x) + (1-\beta)k(y)$$

for every $0 < \beta < 1$ and $x, y \in C$, and it is called concave on C if $-k$ is convex. The function $k : C \to R$ is called affine on C if it is both convex and concave on C.

Introducing the set

$$R^n_- := \{x = (x_1, \ldots, x_n) \in R^n : x_i \le 0, 1 \le i \le n\}$$

and

$$x^\top y = \sum\nolimits_{i=1}^n x_i y_i$$

the inner product of the vectors $x^\top = (x_1, \ldots, x_n) \in R^n$ and $y^\top = (y_1, \ldots, y_n) \in R^n$ (by $x^\top$ we denote the transpose of the column vector x), the most elementary minimax result is given by the following.

Theorem 1. *If $C \subseteq R^n$ is a convex set, then it follows that*

$$\inf_{x\in C} \max_{\alpha\in\Delta_n} \alpha^\top x = \max_{\alpha\in\Delta_n} \inf_{x\in C} \alpha^\top x.$$

Proof. It is obvious that

$$\inf_{x\in C} \max_{\alpha\in\Delta_n} \alpha^\top x \ge \max_{\alpha\in\Delta_n} \inf_{x\in C} \alpha^\top x. \tag{9}$$

To show that we actually have an equality in relation (9), we assume by contradiction that

$$\inf_{x\in C} \max_{\alpha\in\Delta_n} \alpha^\top x > \max_{\alpha\in\Delta_n} \inf_{x\in C} \alpha^\top x := \gamma. \tag{10}$$

By relation (10), there exists some β satisfying

$$\inf_{x\in C} \max_{\alpha\in\Delta_n} \alpha^\top x > \beta > \gamma. \tag{11}$$

Introduce now the mapping $H : C \to R^n$ given by

$$H(x) := x - \beta e$$

with $e^\top = (1, 1, \ldots, 1) \in R^n$. If the set $H(C) \cap R^n_-$ is nonempty, there exists some $x_0 \in C$ satisfying $x_0 - \beta e \le 0$. This implies $\max_{\alpha\in\Delta_n} \alpha^\top x_0 \le \beta$, and we obtain a contradiction with relation (11). Hence the set $H(C) \cap R^n_-$ is empty, and by the separation result for disjoint convex sets ([32]) one can find some $\alpha_0 \in \Delta_n$ satisfying $\inf_{x\in C} \alpha_0^\top x \ge \beta$. This implies by the definition of γ that $\gamma \ge \inf_{x\in C} \alpha_0^\top x \ge \beta$ contradicting relation (11), and the desired result is proved. ∎

Let us introduce the following notation. The set $\mathcal{F}(A_0)$ represents the set of all finite subsets of the set $A_0 \subseteq A$, and for every $I \in \mathcal{F}(A_0)$ the set $\mathcal{P}(I)$ denotes the set of all Borel probability measures concentrated on I. This means for $I = \{a_1, \ldots, a_{|I|}\} \subseteq A$ and $|I| < \infty$ denoting the cardinality of the set I that λ belongs to $\mathcal{P}(I)$ if and only if

$$\lambda = \sum\nolimits_{i=1}^{|I|} s_i(\lambda)\epsilon_{a_i} \tag{12}$$

for some $s(\lambda)^\top \in \Delta_{|I|}$. By relation (12), it is clear that the set $\mathcal{P}(I)$ is convex and in particular

$$\mathcal{P}(I) = co(\{\epsilon_a\}_{a\in I}) \tag{13}$$

with $co(C)$ denoting the convex hull of a set C. Remember $co(C)$ represents the set of all finite convex combinations of elements of the set C ([32]). By the definition of $\mathcal{P}_F(A_0)$ with $A_0 \subseteq A$, we also obtain that

$$\mathcal{P}_F(A_0) = co(\{\epsilon_a\}_{a\in A_0}) = \cup_{I\in\mathcal{F}(A_0)}\mathcal{P}(I) \tag{14}$$

and this set is also convex. In the next theorem, we will prove Wald's minimax result. This result was proved in 1945 ([41]) using a more complicated approach.

Theorem 2. *For any payoff function $f : A\times B \to R$ and every set I belonging to $\mathcal{F}(A)$,*

$$\inf\nolimits_{\mu\in\mathcal{P}_F(B)} \max\nolimits_{\lambda\in\mathcal{P}(I)} f_e(\lambda,\mu) = \max\nolimits_{\lambda\in\mathcal{P}(I)} \inf\nolimits_{\mu\in\mathcal{P}_F(B)} f_e(\lambda,\mu).$$

Proof. Let I belong to $\mathcal{F}(A)$ and introduce the mapping $L : \mathcal{P}_F(B) \to R^{|I|}$ given by

$$L(\mu) := (f_e(\epsilon_a,\mu))_{a\in I}.$$

By the definition of the mapping L and the function f_e, we obtain

$$\inf\nolimits_{\mu\in\mathcal{P}_F(B)} \max\nolimits_{\lambda\in\mathcal{P}(I)} f_e(\lambda,\mu) = \inf\nolimits_{x\in L(\mathcal{P}_F(B))} \max\nolimits_{s(\lambda)\in\Delta_{|I|}} s(\lambda)^\top x \tag{15}$$

and

$$\max\nolimits_{\lambda\in\mathcal{P}(I)} \inf\nolimits_{\mu\in\mathcal{P}_F(B)} f_e(\lambda,\mu) = \max\nolimits_{s(\lambda)\in\Delta_{|I|}} \inf\nolimits_{x\in L(\mathcal{P}_F(B))} s(\lambda)^\top x. \tag{16}$$

Also by relation (5), it follows for every $a \in I$ that the function

$$\mu \longmapsto f_e(\epsilon_a,\mu)$$

is affine on $\mathcal{P}_F(B)$. This shows by the convexity of the set $\mathcal{P}_F(B)$ that the range $L(\mathcal{P}_F(B)) \subseteq R^{|I|}$ is a convex set. Applying now Theorem 1, we obtain

$$\inf\nolimits_{x\in L(\mathcal{P}_F(B))} \max\nolimits_{s(\lambda)\in\Delta_{|I|}} s(\lambda)^\top x = \max\nolimits_{s(\lambda)\in\Delta_{|I|}} \inf\nolimits_{x\in L(\mathcal{P}_F(B))} s(\lambda)^\top x,$$

and by relations (15) and (16), the desired result follows. ■

A symmetrical version of Wald's minimax theorem needed in the proof of Lemma 4 is given by

$$\sup_{\lambda \in \mathcal{P}_F(A)} \min_{\mu \in \mathcal{P}(J)} f_e(\lambda, \mu) = \min_{\mu \in \mathcal{P}(J)} \sup_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \mu) \qquad (17)$$

for any J belonging to $\mathcal{F}(B)$. This can be easily derived from Theorem 2 (replace $f_e(\lambda, \mu)$ by $-f_e(\lambda, \mu)$ and reverse the sets A and B!). Using the next lemma, it is also possible to give different equivalent representations of Wald's minimax theorem.

Lemma 1. *Let $f : A \times B \to R$ be a given payoff function. For any $\mu \in \mathcal{P}_F(B)$ and $A_0 \subseteq A$*

$$\sup_{\lambda \in \mathcal{P}_F(A_0)} f_e(\lambda, \mu) = \sup_{a \in A_0} f_e(\epsilon_a, \mu),$$

and for any $B_0 \subseteq B$ and $\lambda \in \mathcal{P}_F(A)$

$$\inf_{\mu \in \mathcal{P}_F(B_0)} f_e(\lambda, \mu) = \inf_{b \in B_0} f_e(\lambda, \epsilon_b).$$

Proof. We only give a proof of the first equality because the second one can be verified in a similar way. Because the set $A_0 \subseteq A$ can be identified with the set of one-point Borel probability measures $\epsilon_a, a \in A_0$, it is obvious for every μ belonging to $\mathcal{P}_F(B)$ that

$$\sup_{\lambda \in \mathcal{P}_F(A_0)} f_e(\lambda, \mu) \geq \sup_{a \in A_0} f_e(\epsilon_a, \mu).$$

Consider now an arbitrary λ belonging to $\mathcal{P}_F(A_0)$. By definition there exists a finite set $\{a_1, \ldots, a_k\} \subseteq A_0$ and $s(\lambda)^\top \in \Delta_k$ such that $\lambda = \sum_{i=1}^k s_i(\lambda)\epsilon_{a_i}$, and hence we obtain

$$f_e(\lambda, \mu) = \sum_{i=1}^k s_i(\lambda) f_e(\epsilon_{a_i}, \mu) \leq \sup_{a \in A_0} f_e(\epsilon_a, \mu).$$

Because λ belonging to $\mathcal{P}_F(A_0)$ is arbitrary, this implies

$$\sup_{\lambda \in \mathcal{P}_F(A_0)} f_e(\lambda, \mu) \leq \sup_{a \in A_0} f_e(\epsilon_a, \mu)$$

and the desired result is verified. ∎

By Lemma 1, it follows with A_0 replaced by $I \in \mathcal{F}(A)$ and $\mathcal{P}_F(A_0)$ by $\mathcal{P}(I)$ that

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{\lambda \in \mathcal{P}(I)} f_e(\lambda, \mu) = \inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) \qquad (18)$$

By a similar argument, we obtain

$$\max_{\lambda \in \mathcal{P}(I)} \inf_{\mu \in \mathcal{P}_F(B)} f_e(\lambda, \mu) = \max_{\lambda \in \mathcal{P}(I)} \inf_{b \in B} f_e(\lambda, \epsilon_b), \qquad (19)$$

and combining relations (18), (19) and Theorem 2, one can give different equivalent representations of Wald's minimax theorem. For its proof using the strong duality theorem of linear programming, we need some elementary properties of closed and compact sets.

Definition 4. *A topological space X is called compact if every collection of open subsets of X that covers X contains a finite subcollection covering X.*

It is well-known that $X \subseteq R^n$ is compact if and only if it is bounded and closed. ([35]). Moreover, an easy consequence of the above definition is the so-called finite intersection property of compact sets given by the following ([33]): any collection of closed subsets of a compact topological space X, for which any finite subcollection has a nonempty intersection, must have a nonempty intersection.

Definition 5. *A function $k : X \to R$ with X a topological space is called lower semicontinuous if all its lower level sets $\{x \in X : k(x) \leq r\}, r \in R$ are closed subsets of X. It is called upper semicontinuous if all its upper level sets $\{x \in X : k(x) \geq r\}, r \in R$ are closed subsets of X, and it is called continuous if it is both upper and lower semicontinuous.*

One can now show the following so-called Weierstrass–Lebesgue lemma ([33]). For completeness, a proof is listed.

Lemma 2. *If the function $k : X \to (-\infty, \infty]$ is lower semicontinuous and X is a compact topological space, then the function k is bounded from below and attains its minimum on X.*

Proof. Because k is a lower semicontinuous function with values $> -\infty$, it follows that the decreasing sequence $O_n := \{x \in X : k(x) > n\}, n \in Z$ of open sets covers X. This implies by the compactness of X that there exist a finite subcover and as $O_{n+1} \subseteq O_n$, one can find some $m \in Z$ satisfying $X \subseteq O_m$ and so the function k is bounded from below. To show that the function k attains its minimum, introduce $\beta := \inf_{x \in X} k(x)$. If $\beta = \infty$ we are done. Hence we assume that $\beta < \infty$, and by the first part β is finite. Consider now the collection of nonempty closed sets $F_n = \{x \in X : k(x) \leq \beta + n^{-1}\}, n \in N$. Because $F_{n+1} \subseteq F_n$, it follows that by the definition of β, $\cap_{n \in I} F_n$ is nonempty for every finite subset I of N. Hence by the finite intersection property of compact sets, we obtain that the intersection $\cap_{n \in N} F_n$ is nonempty, and this shows that k attains its minimum on X. ■

A symmetrical version of the above result is given by the following. If the function $k : X \to [-\infty, \infty)$ is upper semicontinuous and X is a compact topological space, then the function k is bounded from above and attains its maximum on X. As shown by the next observation, the above result is useful in determining whether an optimal pure strategy for player 2 exists if the minimax relations (7) or (8) hold. Because for any payoff function $f : A \times B \to R$ it follows for r finite that

$$\{b \in B : \sup_{a \in A} f(a, b) \leq r\} = \cap_{a \in A} \{b \in B : f(a, b) \leq r\}, \tag{20}$$

we obtain immediately for $b \longmapsto f(a, b), a \in A$ lower semicontinuous that the function $b \longmapsto \sup_{a \in A} f(a, b)$ is also lower semicontinuous. This implies by

Lemma 2 for B a compact topological space and using Lemma 1 that there exists some $b_0 \in B$ satisfying

$$\sup_{a\in A} f(a, b_0) = \inf_{b\in B} \sup_{a\in A} f(a, b) = \inf_{b\in B} \sup_{\lambda\in\mathcal{P}_F(A)} f_e(\lambda, \epsilon_b).$$

By a symmetry argument, a similar observation holds for player 1 if A is a compact topological space and $a \longmapsto f(a, b)$ is upper semicontinuous for every $b \in B$.

Definition 6. *A function $k : X \to R$ with X a topological space is called inf-compact if all its lower level sets $\{x \in X : k(x) \le r\}, r \in R$ are compact. It is called sup-compact if all its upper level sets $\{x \in X : k(x) \ge r\}, r \in R$ are compact.*

If B is a Hausdorff space, it is shown in Chapter 9 of [33] that a compact subset of B is closed. This proves for B Hausdorff that every inf-compact (sup-compact) function is actually lower semicontinuous (upper semicontinuous). Using now Lemma 2 and Definition 6, one can prove the following important result.

Lemma 3. *If the pure strategy set B is a topological space and there exist some $I_0 \in \mathcal{F}(A)$ such that the function $b \longmapsto \max_{a\in I_0} f(a, b)$ is inf-compact and $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, then*

$$\sup_{I\in\mathcal{F}(A)} \inf_{b\in B} \sup_{a\in I} f(a, b) = \inf_{b\in B} \sup_{a\in A} f(a, b).$$

Moreover, the inf in the last expression is attained, and so, it can be replaced by min.

Proof. Introducing $\beta := \sup_{I\in\mathcal{F}(A)} \inf_{b\in B} \sup_{a\in I} f(a, b)$, we first verify that

$$\inf_{b\in B} \sup_{a\in A} f(a, b) \le \beta + \epsilon$$

for every $\epsilon > 0$. Consider for $\epsilon > 0$ the nonempty set

$$F_\alpha(\epsilon) := \{b \in B : \max_{a\in I_0\cup\{\alpha\}} f(a, b) \le \beta + \epsilon\}, \alpha \in A\backslash I_0.$$

Because the function $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, it follows by relation (20) that the nonempty set $F_\alpha(\epsilon)$ is closed for every $\alpha \in A\backslash I_0$. Moreover, for every finite set $I \subseteq A\backslash I_0$, we obtain by the definition of β that $\cap_{\alpha\in I} F_\alpha(\epsilon)$ is nonempty and

$$F_\alpha(\epsilon) \subseteq \{b \in B : \max_{a\in I_0} f(a, b) \le \beta + \epsilon\} \tag{21}$$

for any $\alpha \in A\backslash I_0$. By assumption, the last set in relation (21) is compact, and we have shown that the collection $F_\alpha(\epsilon), \alpha \in A\backslash I_0$ of closed sets satisfies the finite intersection property. This shows that $\cap_{\alpha\in A\backslash I_0} F_\alpha(\epsilon)$ is nonempty, and because

$$\cap_{\alpha \in A \setminus I_0} F_\alpha(\epsilon) = \{b \in B : \sup_{a \in A} f(a,b) \leq \beta + \epsilon\}, \tag{22}$$

we obtain

$$\inf_{b \in B} \sup_{a \in A} f(a,b) \leq \beta + \epsilon.$$

Because $\epsilon > 0$ is arbitrary, this implies $\inf_{b \in B} \sup_{a \in A} f(a,b) = \beta$, and to show that the infimum is actually attained, we observe the following. Because by relation (22) we obtain for every $\epsilon > 0$ that

$$G(\epsilon) := \cap_{\alpha \in A \setminus I_0} F_\alpha(\epsilon)$$

is a closed nonempty set of the compact set $\{b \in B : \max_{a \in I_0} f(a,b) \leq \beta + \epsilon\}$, the finite intersection property also holds for the decreasing collection $G(\epsilon), \epsilon > 0$. This shows that

$$\inf_{b \in B} \sup_{a \in A} f(a,b) = \min_{b \in B} \sup_{a \in A} f(a,b)$$

and so the infimum can be replaced by min. ■

An important special case of Lemma 3 is given by B a compact topological space, and $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$. Because every closed subset of a compact set is compact (see Chapter 9 of [33]), it is obvious that the conditions of Lemma 3 are satisfied. A symmetrical version of Lemma 3 needed in the next proof of Wald's minimax theorem is given by

$$\inf_{J \in \mathcal{F}(B)} \sup_{a \in A} \min_{b \in J} f(a,b) = \max_{a \in A} \inf_{b \in B} f(a,b), \tag{23}$$

and this holds if the function $a \longmapsto f(a,b)$ is upper semicontinuous for every $b \in B$ and there exist some $J_0 \in \mathcal{F}(B)$ such that the function $a \longmapsto \min_{b \in J_0} f(a,b)$ is sup-compact. A sufficient condition for this is given by A a compact topological space, and $a \longmapsto f(a,b)$ is upper semicontinuous for every $b \in B$. We are now able to give a proof of Wald's minimax result using the strong duality theorem for linear programming and relation (23).

Proof. (Alternative proof of Wald's minimax theorem)

By relation (14) with A_0 replaced by B, it follows for I belonging to $\mathcal{F}(A)$ that

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \inf_{J \in \mathcal{F}(B)} \min_{\mu \in \mathcal{P}(J)} \max_{a \in I} f_e(\epsilon_a, \mu). \tag{24}$$

For every $J \in \mathcal{F}(B)$, the optimization problem

$$\min_{\mu \in \mathcal{P}(J)} \max_{a \in I} f_e(\epsilon_a, \mu) = \min\{z : z \geq f_e(\epsilon_a, \mu), a \in I, \mu \in \mathcal{P}(J)\}$$

is a linear programming problem with a finite optimal solution. Hence by the strong duality theorem for linear programming ([6]), we obtain the minimax result given by

$$\min_{\mu \in \mathcal{P}(J)} \max_{a \in I} f_e(\epsilon_a, \mu) = \max_{\lambda \in \mathcal{P}(I)} \min_{b \in J} f_e(\lambda, \epsilon_b). \tag{25}$$

Applying now relations (24) and (25) yields

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \inf_{J \in \mathcal{F}(B)} \max_{\lambda \in \mathcal{P}(I)} \min_{b \in J} f_e(\lambda, \epsilon_b). \tag{26}$$

Moreover, because the set I is finite and hence $\Delta_{|I|} \subseteq R^{|I|}$ being closed and bounded and hence compact (in the Euclidean topology) and $\lambda \longmapsto f_e(\lambda, \epsilon_b)$ is continuous on $\mathcal{P}(I)$ for every $b \in B$, we may use relation (23) with the set A replaced by $\mathcal{P}(I)$ and the function $f(a, b)$ by $f_e(\lambda, \epsilon_b)$. This shows

$$\inf_{J \in \mathcal{F}(B)} \max_{\lambda \in \mathcal{P}(I)} \min_{b \in J} f_e(\lambda, \epsilon_b) = \max_{\lambda \in \mathcal{P}(I)} \inf_{b \in B} f_e(\lambda, \epsilon_b)$$

and so we obtain by relation (26) that

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \max_{\lambda \in \mathcal{P}(I)} \inf_{b \in B} f_e(\lambda, \epsilon_b).$$

Finally by Lemma 1 (replace B_0 by B), Wald's minimax result is verified. ∎

Actually the minimax result

$$\min_{\mu \in \mathcal{P}(J)} \max_{a \in I} f_e(\epsilon_a, \mu) = \max_{\lambda \in \mathcal{P}(I)} \min_{b \in J} f_e(\lambda, \epsilon_b) \tag{27}$$

was first proved by von Neumann in 1928 ([30]). In fact in this paper, a more general minimax result for a continuous payoff function defined on the Cartesian product of compact simplices that is quasiconvex in B and quasiconcave in A was shown. This result seems to have been forgotten in the literature (the special case in relation (27) was published in [31]) and was later independently generalized by Sion ([37]) in 1958. A useful consequence of Lemma 3 and Wald's minimax result is given by Kneser's minimax result ([22]).

Lemma 4. *If the set A is a compact convex subset of a linear topological space, B is a convex subset of a linear space, the payoff function $f : A \times B \to R$ is affine in both variables, and $a \longmapsto f(a, b)$ is upper semicontinuous for every $b \in B$, then*

$$\sup_{a \in A} \inf_{b \in B} f(a, b) = \inf_{b \in B} \sup_{a \in A} f(a, b)$$

and in both expressions the sup can be replaced by max.

Proof. Because A is a compact convex topological space and the function $a \longmapsto f(a, b)$ is upper semicontinuous for every $b \in B$, we obtain by relation (23) that

$$\max_{a \in A} \inf_{b \in B} f(a, b) = \inf_{J \in \mathcal{F}(B)} \max_{a \in A} \min_{b \in J} f(a, b). \tag{28}$$

Considering now any λ belonging to $\mathcal{P}_F(A)$ and $b \in B$, it follows that there exists some finite set $\{a_1, ..., a_k\} \subseteq A$ and $s(\lambda)^\top \in \Delta_k$ such that

$$\lambda = \sum_{i=1}^{k} s_i(\lambda) \epsilon_{a_i}.$$

This implies, using $a \longmapsto f(a,b)$ is affine for every $b \in B$ and A is a convex set, that

$$\max_{a\in A}\min_{b\in J} f(a,b) \geq \min_{b\in J} f(\sum_{i=1}^{k} s_i(\lambda)a_i, b) = \min_{b\in J} f_e(\lambda, \epsilon_b). \tag{29}$$

Because $\lambda \in \mathcal{P}_F(A)$ is arbitrary, relation (29) yields

$$\max_{a\in A}\min_{b\in J} f(a,b) \geq \sup_{\lambda\in\mathcal{P}_F(A)}\min_{b\in J} f_e(\lambda, \epsilon_b)$$

and by Lemma 1 and relation (14) with A_0 replaced by A, this implies

$$\max_{a\in A}\min_{b\in J} f(a,b) = \sup_{\lambda\in\mathcal{P}_F(A)}\min_{\mu\in\mathcal{P}(J)} f_e(\lambda, \mu). \tag{30}$$

Applying now the symmetrical version of Wald's minimax theorem listed in relation (17) to the last part of relation (30) yields

$$\max_{a\in A}\min_{b\in J} f(a,b) = \min_{\mu\in\mathcal{P}(J)}\sup_{\lambda\in\mathcal{P}_F(A)} f_e(\lambda, \mu). \tag{31}$$

Hence by relations (28), (31), (14) and Lemma 1, we obtain

$$\max_{a\in A}\inf_{b\in B} f(a,b) = \inf_{\mu\in\mathcal{P}_F(B)}\sup_{a\in A} f_e(\epsilon_a, \mu). \tag{32}$$

Because the function $b \longmapsto f(a,b)$ is affine for every $a \in A$ and the set B is convex, we obtain as in the first part of this proof that

$$\inf_{\mu\in\mathcal{P}_F(B)}\sup_{a\in A} f_e(\epsilon_a, \mu) = \inf_{b\in B}\sup_{a\in A} f(a,b) = \inf_{b\in B}\max_{a\in A} f(a,b)$$

and in combination with relation (32), the desired result follows. ∎

Actually one can show that the minimax results of Wald, von Neumann, and Kneser can be easily derived from each other. For more equivalent minimax results, the reader is referred to ([15]). An easy consequence of Wald's minimax theorem useful in Section 2 is given by the following.

Theorem 3. *For any payoff function $f : A \times B \to R$*

$$\sup_{I\in\mathcal{F}(A)}\inf_{\mu\in\mathcal{P}_F(B)}\max_{a\in I} f_e(\epsilon_a, \mu) = \sup_{\lambda\in\mathcal{P}_F(A)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda, \mu).$$

Proof. By Lemma 1 and Wald's minimax theorem, we obtain for every I belonging to $\mathcal{F}(A)$ that

$$\inf_{\mu\in\mathcal{P}_F(B)}\max_{a\in I} f_e(\epsilon_a, \mu) = \max_{\lambda\in\mathcal{P}(I)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda, \mu). \tag{33}$$

Because by relation (14)

$$\sup_{I\in\mathcal{F}(A)}\max_{\lambda\in\mathcal{P}(I)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda, \mu) = \sup_{\lambda\in\mathcal{P}_F(A)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda, \mu)$$

the desired result follows using relation (33). ∎

3 On Necessary and Sufficient Conditions for Minimax Theorems

In this section, we will derive necessary and sufficient conditions for the different minimax equalities listed in relations (6) up to (8) by means of the extension of Wald's minimax result listed in Theorem 3. Observe these minimax results are equivalent to the existence of "optimal" strategies for two-player zero-sum noncooperative games under different conditions on the use of the strategy sets of the two players. To derive these conditions for relation (6), we introduce the following class of functions.

Definition 7. *The payoff function $f : A \times B \to R$ belongs to the set $\mathcal{U}_0$ if*

$$\sup\nolimits_{I \in \mathcal{F}(A)} \inf\nolimits_{\mu \in \mathcal{P}_F(B)} \max\nolimits_{a \in I} f_e(\epsilon_a, \mu) = \inf\nolimits_{\mu \in \mathcal{P}_F(B)} \sup\nolimits_{a \in A} f_e(\epsilon_a, \mu).$$

A game theoretic interpretation of a payoff function f belonging to the set $\mathcal{U}_0$ is given by the observation that for player 2 using the mixed strategy set $\mathcal{P}_F(B)$ and the minimax approach, it does not make any difference whether his opponent given by player 1 selects a pure strategy from the set A or first considers all finite subsets of A and then selects from one of these finite subsets his pure strategy. However, it might be possible that the value for player 2 cannot be achieved if he uses the set $\mathcal{P}_F(B)$ of mixed strategies.

Theorem 4. *The minimax result in relation (6), given by*

$$\inf\nolimits_{\mu \in \mathcal{P}_F(B)} \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \mu) = \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} \inf\nolimits_{\mu \in \mathcal{P}_F(B)} f_e(\lambda, \mu),$$

holds if and only if the function f belongs to the set $\mathcal{U}_0$.

Proof. By Theorem 3 and the definition of $\mathcal{U}_0$, the result follows immediately. ∎

The importance of the above theorem is that the minimax equality in relation (6) is replaced by an easier condition. Notice that $\mathcal{U}_0$ is automatically satisfied if A is a finite set. In this way, Wald's minimax theorem is a direct consequence of Theorem 4. Moreover, we will show at the end of this section that a minimax result derived by Ville ([39]) is an easy consequence of Theorem 4. We do this by showing that the conditions imposed on the payoff function f and the pure strategy sets A and B imply that the function f should belong to the set $\mathcal{U}_0$. Actually by a symmetric argument (replace f by $-f$ and reverse the sets A and B!), one can also introduce the following class of functions.

Definition 8. *The payoff function $f : A \times B \to R$ belongs to the set $\mathcal{V}_0$ if*

$$\inf\nolimits_{J \in \mathcal{F}(B)} \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} \min\nolimits_{b \in J} f_e(\lambda, \epsilon_b) = \sup\nolimits_{\lambda \in \mathcal{P}_F(A)} \inf\nolimits_{b \in B} f_e(\lambda, \epsilon_b).$$

Using the same symmetry argument, the next corollary is an easy consequence of Theorem 4.

Corollary 1. *The minimax result in relation (6), given by*

$$\inf_{\mu\in\mathcal{P}_F(B)}\sup_{\lambda\in\mathcal{P}_F(A)} f_e(\lambda,\mu)=\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda,\mu),$$

holds if and only if the function f belongs to $\mathcal{V}_0$.

To derive a necessary and sufficient condition for the minimax equality in relation (7), we introduce the following class of functions.

Definition 9. *The function $f : A\times B\to R$ belongs to the set $\mathcal{U}_1$ if*

$$\sup_{I\in\mathcal{F}(A)}\inf_{\mu\in\mathcal{P}_F(B)}\max_{a\in I} f_e(\epsilon_a,\mu)=\inf_{b\in B}\sup_{a\in A} f(a,b).$$

A game theoretic interpretation of the payoff function f belonging to the set $\mathcal{U}_1$ is given by the observation that for player 2 using the mixed strategy set $\mathcal{P}_F(B)$ and the minimax approach, it does not make any difference whether his opponent given by player 1 selects a pure strategy from the set A or first considers all finite subsets of A and then selects from one of these finite subsets his pure strategy. Moreover, the payoff function for player 2 is such that his mixed strategy set is always dominated by his pure strategy set. A sufficient condition for the listed minimax result was discussed in [20].

Theorem 5. *The minimax result in relation (7), given by*

$$\inf_{b\in B}\sup_{\lambda\in\mathcal{P}_F(A)} f_e(\lambda,\epsilon_b)=\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b),$$

holds if and only if the function f belongs to $\mathcal{U}_1$.

Proof. By Lemma 1, the minimax result listed in relation (7) is the same as

$$\inf_{b\in B}\sup_{a\in A} f(a,b)=\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{\mu\in\mathcal{P}_F(B)} f_e(\lambda,\mu).$$

Hence by Theorem 3 and the definition of $\mathcal{U}_1$, the desired result follows. ■

Finally we derive a necessary and sufficient condition for the minimax equality listed in relation (8) involving the pure strategy sets A and B.

Definition 10. *The function $f : A\times B\to R$ belongs to the set $\mathcal{U}_2$ if*

$$\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b)=\sup_{a\in A}\inf_{b\in B} f(a,b).$$

A game theoretic interpretation of the payoff function f belonging to the set $\mathcal{U}_2$ is given by the observation that for player 1 using the mixed strategy set $\mathcal{P}_F(A)$ and the minimax approach, his mixed strategy set is always dominated by his pure strategy set. This means that player 1 can restrict himself to the set of pure strategies instead of using the set of mixed strategies. One can now show the most well-known minimax result.

Theorem 6. *The minimax result in relation (8), given by*

$$\inf_{b\in B}\sup_{a\in A} f(a,b) = \sup_{a\in A}\inf_{b\in B} f(a,b),$$

holds if and only if the function f belongs to the set $\mathcal{U}_1 \cap \mathcal{U}_2$.

Proof. If the function f belongs to the set $\mathcal{U}_1 \cap \mathcal{U}_2$, then by Lemma 1 (replace A_0 by A) and Theorem 5 we obtain

$$\inf_{b\in B}\sup_{a\in A} f(a,b) = \sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b).$$

By the definition of the set $\mathcal{U}_2$, this implies that relation (8) holds. To show the reverse implication, consider an arbitrary λ belonging to $\mathcal{P}_F(A)$. By relation (14), there exists some $I_0 \in \mathcal{F}(A)$ such that $\lambda \in \mathcal{P}(I_0)$, and so we obtain

$$\inf_{b\in B} f_e(\lambda,\epsilon_b) \leq \sup_{I\in\mathcal{F}(A)}\inf_{b\in B}\sup_{a\in I} f(a,b). \tag{34}$$

This implies

$$\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b) \leq \sup_{I\in\mathcal{F}(A)}\inf_{b\in B}\sup_{a\in I} f(a,b).$$

Also by our minimax result listed in relation (8), we obtain

$$\begin{aligned}\sup_{I\in\mathcal{F}(A)}\inf_{b\in B}\sup_{a\in I} f(a,b) &\leq \inf_{b\in B}\sup_{a\in A} f(a,b)\\ &= \sup_{a\in A}\inf_{b\in B} f(a,b)\end{aligned}$$

and this shows that

$$\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b) \leq \sup_{a\in A}\inf_{b\in B} f(a,b). \tag{35}$$

Because the reverse inequality trivially holds, we can replace the inequality in relation (56) by an equality, and so the function f belongs to $\mathcal{U}_2$. This implies using again the minimax equality in relation (8) that

$$\sup_{\lambda\in\mathcal{P}_F(A)}\inf_{b\in B} f_e(\lambda,\epsilon_b) = \inf_{b\in B}\sup_{a\in A} f(a,b)$$

and by Theorem 5 the function f belongs to $\mathcal{U}_1$. ■

Again using a symmetry argument (replace f by $-f$ and reverse the sets A and B!) in the definition of the sets $\mathcal{U}_1$ and $\mathcal{U}_2$, one can introduce the following class of functions.

Definition 11. *The payoff function $f : A \times B \to R$ belongs to the set $\mathcal{V}_1$ if*

$$\inf_{J\in\mathcal{F}(B)}\sup_{\lambda\in\mathcal{P}_F(A)}\min_{b\in J} f_e(\lambda,\epsilon_b) = \sup_{a\in A}\inf_{b\in B} f(a,b),$$

whereas $f : A \times B \to R$ belongs to the set $\mathcal{V}_2$ if

$$\inf_{\mu\in\mathcal{P}_F(B)}\sup_{a\in A} f_e(\epsilon_a,\mu) = \inf_{b\in B}\sup_{a\in A} f(a,b).$$

By the same symmetry argument, one can easily derive the following corollary from Theorem 6.

Corollary 2. *The minimax result in relation (8), given by*

$$\inf_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\inf_{a\in A} f(a,b),$$

holds if and only if f belongs to the set $\mathcal{V}_1 \cap \mathcal{V}_2$.

Before giving a short overview of some minimax theorems that appeared in the literature, we list some definitions and results for functions defined on a metric space. Observe we also include the definition of a continuous function on a metric space. Another equivalent definition of a continuous function on a topological space was already given in Definition 5.

Definition 12. *Let (X,ρ) be a metric space with metric ρ. The function $k : X \to R$ is said to be continuous at the point $x \in X$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that $|k(x) - k(y)| < \epsilon$ for every $y \in X$ satisfying $\rho(x,y) < \delta$. It is called continuous on X if it is continuous at every point $x \in X$. A function $k : X \to R$ is called uniformly continuous on X if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for any $x, y \in X$ satisfying $\rho(x,y) < \delta$, it holds that $|k(x) - k(y)| < \epsilon$. Finally, a collection of functions $k_\gamma : X \to R, \gamma \in \Gamma$ is called equicontinuous if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for every $x, y \in X$ satisfying $\rho(x,y) < \delta$, it holds that $|k_\gamma(x) - k_\gamma(y)| < \epsilon$ for every $\gamma \in \Gamma$.*

Recall in a metric space (X,ρ) with metric ρ, the open ball $B(x_0,\delta)$ with center x_0 and radius $\delta > 0$ is given by

$$B(x_0,\delta) := \{x \in X : \rho(x,x_0) < \delta\}.$$

We now list the following well-known result ([24,33]).

Lemma 5. *For (X,ρ) a compact metric space with metric ρ, a function $k : X \to R$ continuous on X is uniformly continuous on X.*

Proof. Let $\epsilon > 0$ and consider an arbitrary $x \in X$. Because k is continuous at x, there exists some $\delta_x > 0$ such that $|k(x) - k(y)| < 2^{-1}\epsilon$ for every y belonging to $B(x,\delta_x)$. Clearly the collection of open balls $B(x, 2^{-1}\delta_x), x \in X$ is a covering of X, and this implies by the compactness of X that there exists some finite set $F = \{x_1, \ldots, x_n\} \subseteq X$ satisfying

$$X = \cup_{i=1}^{n} B(x_i, 2^{-1}\delta_{x_i}). \tag{36}$$

Let now $\delta := 4^{-1}\min_{1\le i\le n}\delta_{x_i}$ and consider two points $y, z \in X$ satisfying $\rho(z,y) < \delta$. By relation (36) there exists some $1 \le i^* \le n$ such that $\rho(y, x_{i^*}) < 2^{-1}\delta_{x_{i^*}}$ and so $|f(x_{i^*}) - f(y)| \le 2^{-1}\epsilon$. By the triangle inequality of a metric we also obtain, using $\rho(z,y) < \delta$, that

$$\rho(z, x_{i^*}) \leq \rho(z, y) + \rho(y, x_{i^*}) < \delta + 2^{-1}\delta_{x_{i^*}} \leq \delta_{x_i^*},$$

and so $|f(z) - f(x_{i^*})| < 2^{-1}\epsilon$. This shows that

$$|f(z) - f(y)| \leq |f(z) - f(x_{i^*})| + |f(x_{i^*}) - f(y)| < 2^{-1}\epsilon + 2^{-1}\epsilon = \epsilon$$

and we have shown that the function k is uniformly continuous on X. ■

We now recall the minimax equality listed in relation (6). In 1938, Ville ([39]) proved a generalization of the well-known von Neumann minimax result listed in relation (27). This result is shown in Theorem 7 and serves as an important tool in infinite zero-sum or antagonistic game theory ([40]).

Theorem 7. *If A and B are nonempty compact sets in metric spaces and the payoff function $f : A \times B \to R$ is continuous, then*

$$\sup_{\lambda \in \mathcal{P}_F(A)} \inf_{\mu \in \mathcal{P}_F(B)} f_e(\lambda, \mu) = \inf_{\mu \in \mathcal{P}_F(B)} \sup_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \mu).$$

To prove Theorem 7, we show that the conditions imposed on f and the sets A and B imply that the function f belongs to the set $\mathcal{U}_0$. Applying then Theorem 4 yields a proof of Ville's minimax theorem. Actually we show the following result.

Lemma 6. *If the set A is a compact metric space with metric ρ and the collection of functions $f_b : A \to R$, $b \in B$ given by $f_b(a) := f(a, b)$ is equicontinuous with f the payoff function, then f belongs to $\mathcal{U}_0$. In particular, if f is continuous and the sets A and B are compact metric spaces, then f belongs to $\mathcal{U}_0$.*

Proof. For the proof of the first part, it is obvious that

$$\sup_{I \in \mathcal{F}(A)} \inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) \leq \inf_{\mu \in \mathcal{P}_F(B)} \sup_{a \in A} f_e(\epsilon_a, \mu).$$

To show the result, it is therefore sufficient to verify that for every $\epsilon > 0$ there exists some set $I_\epsilon \in \mathcal{F}(A)$ satisfying

$$\inf_{\mu \in \mathcal{P}_F(B)} \sup_{a \in A} f_e(\epsilon_a, \mu) \leq \inf_{\mu \in \mathcal{P}_F(B)} \sup_{a \in I_\epsilon} f_e(\epsilon_a, \mu) + \epsilon.$$

Let $\epsilon > 0$ be given. Because the collection of functions $f_b, b \in B$ is equicontinuous, one can find some $\delta > 0$ such that for every $a_1, a_2 \in A$ satisfying $\rho(a_1, a_2) < \delta$, it holds that

$$|f(a_1, b) - f(a_2, b)| < \epsilon$$

for every $b \in B$. Clearly the collection of open balls $B(a, \delta), a \in A$ covers A, and by the compactness of A one can find a finite set $I_\epsilon \in \mathcal{F}(A)$ such that

$$A = \cup_{a \in I_\epsilon} B(a, \delta). \tag{37}$$

Consider now an arbitrary $\mu \in \mathcal{P}_F(B)$. By relation (37) and $f_b, b \in B$ equicontinuous, it follows for any $a \in A$ that there exists some $a_0 \in I_\epsilon$ such that

$$|f(a,b) - f(a_0,b)| < \epsilon$$

for every $b \in B$. Hence by the definition of $\mathcal{P}_F(B)$, this implies

$$f_e(\epsilon_a,\mu) \leq f_e(\epsilon_{a_0},\mu) + \epsilon \leq \sup_{a \in I_\epsilon} f_e(\epsilon_a,\mu) + \epsilon. \tag{38}$$

Because $a \in A$ is arbitrary, it follows by relation (38) that

$$\sup_{a \in A} f_e(\epsilon_a,\mu) \leq \sup_{a \in I_\epsilon} f_e(\epsilon_a,\mu) + \epsilon$$

and this implies (using μ is arbitrary) the desired inequality. To verify the second part, it follows by the continuity of the function f on the compact metric space $A \times B$ and Lemma 5 that the function f is uniformly continuous on $A \times B$. This shows that the collection $f_b, b \in B$ is equicontinuous, and by the first part the desired result follows. ■

Actually the conditions imposed by Ville can be improved in the following way ([14]).

Theorem 8. *If the pure strategy sets A and B are compact Hausdorff spaces, and $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$, and $a \longmapsto f(a,b)$ is upper semicontinuous for every $b \in B$, and the payoff function f belongs to the space of Borel measurable functions that are Lebesgue absolutely integrable with respect to any Borel product probability measure $\mu \otimes \lambda$ on $B \times A$, then*

$$\sup_{\lambda \in \mathcal{P}_F(A)} \inf_{\mu \in \mathcal{P}_F(B)} f_e(\lambda,\mu) = \inf_{\mu \in \mathcal{P}_F(B)} \sup_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda,\mu).$$

Again this result (for an alternative proof see [14]) can be verified by showing that the above conditions imply that the function f belongs to $\mathcal{U}_0$. Because its proof involves classic results from the set of Borel measures on a compact Hausdorff space ([7,8]) and these results are beyond the scope of this chapter, we refer the reader to [14] for more details on the used techniques. We also like to mention for $\mathcal{P}(A)$ ($\mathcal{P}(B)$) denoting the set of Borel probability measures on A (B) that under the conditions of the next lemma, one can show by a similar type of proof as in Lemma 1 that

$$\sup_{\lambda \in \mathcal{P}(A)} f_e(\lambda,\mu) = \sup_{a \in A} f_e(\epsilon_a,\mu) \tag{39}$$

for every $\mu \in \mathcal{P}(B)$ and

$$\inf_{\mu \in \mathcal{P}(B)} f_e(\lambda,\mu) = \inf_{b \in B} f_e(\lambda,\epsilon_b) \tag{40}$$

for every $\lambda \in \mathcal{P}(A)$.

Lemma 7. *If the pure strategy set A and B are compact Hausdorff spaces, and the function $b \longmapsto f(a,b)$ is lower semicontinuous for every $b \in B$, and $a \longmapsto f(a,b)$ is upper semicontinuous for every $a \in A$, and f belongs to the space of Borel measurable functions that are Lebesgue absolutely integrable with respect to any Borel product probability measure $\mu \otimes \lambda$ on B, then the function f belongs to $\mathcal{U}_0$.*

Proof. Because the function $b \longmapsto f(a,b)$ is Lebesgue absolutely integrable for any Borel probability measure μ on the set B, one can show (see Corollary 2.2 of [13]) that

$$\inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in I} f_e(\epsilon_a,\mu) = \inf_{\mu\in\mathcal{P}(B)} \sup_{a\in I} f_e(\epsilon_a,\mu)$$

for any $I \in \mathcal{F}(A)$. Hence we obtain

$$\sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in I} f_e(\epsilon_a,\mu) = \sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}(B)} \sup_{a\in I} f_e(\epsilon_a,\mu). \tag{41}$$

In the remainder of the proof, we will now verify that

$$\sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}(B)} \sup_{a\in I} f_e(\epsilon_a,\mu) \geq \inf_{\mu\in\mathcal{P}_F(B)} \sup_{a\in A} f_e(\epsilon_a,\mu). \tag{42}$$

Assuming for the moment that this holds, it follows by (41) that f belongs to $\mathcal{U}_0$. To prove relation (42), we observe for B a compact Hausdorff space that the set $\mathcal{P}(B)$ is compact in the weak* topology ([7, 8]) and the function $\mu \longmapsto f_e(\epsilon_a,\mu)$ is lower semicontinuous with respect to the weak* topology (see Lemma 12 of [14]). Hence by Lemma 3 (replace B by $\mathcal{P}(B)$ and $f(a,b)$ by $f_e(\epsilon_a,\mu)$) and relation (39), it follows that

$$\sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}(B)} \sup_{a\in I} f_e(\epsilon_a,\mu) = \inf_{\mu\in\mathcal{P}(B)} \sup_{\lambda\in\mathcal{P}(A)} f_e(\lambda,\mu). \tag{43}$$

Again by Lemma 12 of [14], the function $\mu \longmapsto f_e(\lambda,\mu)$ is upper semicontinuous, and as $\mathcal{P}(A)$ is also weak* compact, we obtain by Kneser's minimax theorem (Lemma 4) (replace A by $\mathcal{P}(A)$ and $f(a,b)$ by the biaffine function $f_e(\lambda,\mu)$) and relation (40) that

$$\inf_{\mu\in\mathcal{P}(B)} \sup_{\lambda\in\mathcal{P}(A)} f_e(\lambda,\mu) = \sup_{\lambda\in\mathcal{P}(A)} \inf_{b\in B} f_e(\lambda,\epsilon_b). \tag{44}$$

Again by the weak*compactness of $\mathcal{P}(A)$ and relation (40), it follows that

$$\sup_{\lambda\in\mathcal{P}(A)} \inf_{b\in B} f_e(\lambda,\epsilon_b) = \inf_{J\in\mathcal{F}(B)} \sup_{\lambda\in\mathcal{P}(A)} \inf_{\mu\in\mathcal{P}(J)} f_e(\lambda,\mu). \tag{45}$$

It is now obvious that

$$\begin{aligned} &\inf_{J\in\mathcal{F}(B)} \sup_{\lambda\in\mathcal{P}(A)} \inf_{\mu\in\mathcal{P}(J)} f_e(\lambda,\mu) \\ &\geq \inf_{J\in\mathcal{F}(B)} \sup_{\lambda\in\mathcal{P}_F(A)} \inf_{\mu\in\mathcal{P}(J)} f_e(\lambda,\mu) \end{aligned}$$

and by Wald's minimax theorem and Lemma 1

$$\inf_{J\in\mathcal{F}(B)} \sup_{\lambda\in\mathcal{P}_F(A)} \inf_{\mu\in\mathcal{P}(J)} f_e(\lambda,\mu) = \inf_{\mu\in\mathcal{P}_F(B)} \sup_{a\in A} f_e(\epsilon_a,\mu). \tag{46}$$

This implies by relations (43) up to (46) that

$$\sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}(B)} \sup_{a\in I} f_e(\epsilon_a,\mu) \geq \inf_{\mu\in\mathcal{P}_F(B)} \sup_{a\in A} f_e(\epsilon_a,\mu)$$

and so relation (42) is proved. ■

We will now consider the minimax equality listed in relation (7) and introduce the following definition used in ([20]).

Definition 13. *The payoff function $f : A \times B \to R$ is called weakly convexlike on B (or belongs to the set WC_B) if for every finite set $I \subseteq A$*

$$\inf_{\alpha \in \Delta_n, b_i \in B, 1 \leq i \leq n, n \in N} \max_{a \in I} \sum_{i=1}^{n} \alpha_i f(a, b_i) \geq \inf_{b \in B} \max_{a \in I} f(a, b).$$

An alternative representation of the above definition is given by

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) \geq \inf_{b \in B} \max_{a \in I} f(a, b)$$

for every I belonging to $\mathcal{F}(A)$. Because the set B can be identified with the set $(\epsilon_b)_{b \in B}$, it follows for $I \in \mathcal{F}(A)$ that

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) \leq \inf_{b \in B} \max_{a \in I} f(a, b)$$

and this shows that in Definition 13, the inequality for a weakly convexlike function on B can be replaced by an equality. Again a function belonging to WC_B has a clear game theoretical interpretation: for any finite set of pure strategies of player 1, it follows that player 2 using its mixed strategy set $\mathcal{P}_F(B)$ can restrict himself to its set of pure strategies. The next result is proved in ([20]).

Theorem 9. *If B is a compact topological space, the payoff function f is weakly convexlike on B, and $b \longmapsto f(a, b)$ is lower semicontinuous on B, for every $a \in A$, then*

$$\inf_{b \in B} \sup_{\lambda \in \mathcal{P}_F(A)} f_e(\lambda, \epsilon_b) = \sup_{\lambda \in \mathcal{P}_F(A)} \inf_{b \in B} f_e(\lambda, \epsilon_b).$$

As before, we check that any function satisfying the assumptions above belongs to the set $\mathcal{U}_1$, and so by Theorem 5 the minimax result in Theorem 9 is proved.

Lemma 8. *If B is a compact topological space, and the payoff function f is weakly convexlike on B, and $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, then f belongs to $\mathcal{U}_1$.*

Proof. Because the function f is weakly convexlike on B, it follows that

$$\sup_{I \in \mathcal{F}(A)} \inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \sup_{I \in \mathcal{F}(A)} \inf_{b \in B} \max_{a \in I} f(a, b).$$

By the compactness of the set B and $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, we may apply Lemma 3, and this shows by the previous equality that f belongs to $\mathcal{U}_1$. ■

Actually as shown by the following counterexample, the set of weakly convexlike functions on B with B a compact set and $b \longmapsto f(a, b)$ continuous for every $a \in A$ is strictly included in the set $\mathcal{U}_1$. Observe the function 1_S denotes the characteristic function of the set S, i.e., $1_S(s) = 1$ for $s \in S$ and $1_S(s) = 0$ otherwise.

Example 2. Let $B = [0,1]$ and $A = \{1,2,3\}$ and introduce the continuous functions $b \longmapsto f(a,b), a \in A$ given by

$$f(1,b) = 2b1_{\{b \leq 2^{-1}\}} + 1_{\{2^{-1}<b\leq 1\}}, f(2,b) = 1_{\{b\leq 2^{-1}\}} + (2-2b)1_{\{2^{-1}<b\leq 1\}}$$

and $f(3,b) = 1_{\{0\leq b\leq 1\}}$. Because A is a finite set, it follows

$$\sup_{I\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in I} f_e(\epsilon_a,\mu) = \inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in A} f_e(\epsilon_a,\mu).$$

Using $f(3,b) = 1$ for every b, we obtain $f_e(\epsilon_3,\mu) = 1$ for every $\mu \in \mathcal{P}_F(B)$ and so

$$\inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in A} f_e(\epsilon_a,\mu) = 1.$$

At the same time, it is easy to see that $\inf_{b\in B} \max_{a\in A} f(a,b) = 1$, and this shows that the function f belongs to $\mathcal{U}_1$. Introducing now the set $I_0 = \{1,2\} \subseteq A$, it follows that

$$\inf_{b\in B} \max_{a\in I_0} f(a,b) = 1.$$

Moreover, because $\mu_0 = 2^{-1}\epsilon_0 + 2^{-1}\epsilon_1$ belongs to $\mathcal{P}_F(B)$, we obtain

$$\inf_{\mu\in\mathcal{P}_F(B)} \max_{a\in I_0} f_e(\epsilon_a,\mu) \leq \max_{a\in I_0} f_e(\epsilon_a,\mu_0) = 2^{-1}$$

and so f is not weakly convexlike on B.

We will now give an overview of the most important different payoff functions f considered in the literature that were used to verify the minimax equality in relation (8). For a more extensive overview, the reader should consult [15] or [36]. In a paper by Ky Fan in 1953 ([10]), the following definition is introduced. In the literature, these functions are also called convexlike or concavelike.

Definition 14. *The payoff function $f : A \times B \to R$ is called Ky Fan convex on B (or belongs to the set KFC_B) if for every $b_1, b_2 \in B$ and $0 < \alpha < 1$, there exists some $b_0 \in B$ satisfying*

$$f(a,b_0) \leq \alpha f(a,b_1) + (1-\alpha) f(a,b_2)$$

for every $a \in A$. It is called Ky Fan concave on A (or belongs to the set KFC_A) if for every $a_1, a_2 \in A$ and $0 < \alpha < 1$, there exists some $a_0 \in A$ satisfying

$$f(a_0,b) \geq \alpha f(a_1,b) + (1-\alpha) f(a_2,b)$$

for every $b \in B$. The payoff function $f : A \times B \to R$ is called Ky Fan concave-convex on the Cartesian product $A \times B$ if f is Ky Fan concave on A and Ky Fan convex on B.

To rewrite the definition of a Ky Fan convex (concave) function in our notation, we introduce for D some topological space the set $\mathcal{P}_2(D) \subseteq \mathcal{P}_F(D)$ of two-point probability measures on D. This means that the probability measure λ belongs to $\mathcal{P}_2(D)$ if and only if

$$\lambda = s(\lambda_1)\epsilon_{d_1} + s(\lambda_2)\epsilon_{d_2}$$

with $d_i, 1 \leq i \leq 2$ different elements of the pure strategy set D and $s(\lambda)^\top = (s(\lambda_1), s(\lambda_2)) \in \Delta_2$ with $s(\lambda_i) > 0, 1 \leq i \leq 2$. Using this notation, it follows that the payoff function $f : A \times B \to R$ is Ky Fan convex on B if for every μ belonging to $\mathcal{P}_2(B)$ there exists some $b_0 \in B$ satisfying

$$f(a, b_0) \leq f_e(\epsilon_a, \mu)$$

for every $a \in A$. Clearly this property also has a clear game theoretical interpretation. For such a payoff function, every two-point mixed strategy of player 2 is dominated by a pure strategy. Actually by an easy induction argument, one can also show for f Ky Fan convex on B that for any $\mu \in \mathcal{P}_F(B)$ there exists some $b_0 \in B$ satisfying

$$f(a, b_0) \leq f_e(\epsilon_a, \mu)$$

for any $b \in B$. This means that every mixed strategy of player 2 is dominated by a pure strategy. In [10], the following minimax result is shown.

Theorem 10. *If B is compact topological space, the payoff function f is Ky Fan concave-convex on $A \times B$, and $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, then*

$$\inf_{b \in B} \sup_{a \in A} f(a, b) = \sup_{a \in A} \inf_{b \in B} f(a, b)$$

and inf *can be replaced by* min *in the above expression.*

By the well-known symmetry argument (replace f by $-f$ and reverse A and B), one can easily derive from Theorem 10 that the above minimax result holds if A is a compact topological space, the function f is Ky-Fan concave-convex on $A \times B$, and $a \longmapsto f(a, b)$ is upper semicontinuous for every $b \in B$. Another more general class of functions was introduced by König in 1968 ([23]). Actually König only introduced the next class with $\beta = \frac{1}{2}$, but indicates at the the end of his paper that the same results also holds with $0 < \beta < 1$.

Definition 15. *The payoff function $f : A \times B \to R$ is called König convex on B (or belongs to the set KC_B) if there exists some $0 < \beta < 1$ such that for every $b_0, b_1 \in B$, there exists some $b_0 \in B$ satisfying*

$$f(a, b_0) \leq \beta f(a, b_1) + (1 - \beta) f(a, b_0)$$

for every $a \in A$. It is called König concave on A (or belongs to the set KC_A) if there exists some $0 < \beta < 1$ such that for every $a_1, a_2 \in A$, there exists some $a_0 \in A$ satisfying

$$f(a_0, b) \geq \beta f(a_1, b) + (1 - \beta) f(a_2, b)$$

for every $b \in B$. The payoff function $f : A \times B \to R$ is called König concave-convex on $A \times B$, if f is König concave on A and König convex on B.

Although the above definition is rather technical, it has a clear interpretation in game theory. Denoting by $\mathcal{P}_{2,\beta}(D) \subseteq \mathcal{P}_2(D)$ the set of two-point probability measures on the topological space D with probabilities β and $1 - \beta$ (β fixed), it means that any mixed strategy of player 2 belonging to $\mathcal{P}_{2,\beta}(B)$ is dominated by a pure strategy. In [23], the same minimax result is shown as in Theorem 10 under the weaker conditions that B is a compact topological space, $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, and f is König concave-convex on $A \times B$. Another more general class of functions is considered in [12] or [17].

Definition 16. *The payoff function $f : A \times B \to R$ is called closely convex on B (or belongs to the set CC_B) if for every $\epsilon > 0, 0 < \alpha < 1$ and $b_1, b_2 \in B$, there exists some $b_0 \in B$ satisfying*

$$f(a, b_0) \leq \alpha f(a, b_1) + (1 - \alpha) f(a, b_2) + \epsilon$$

for every $a \in A$. It is called closely concave on A (or belongs to the set CC_A) if for every $\epsilon > 0, 0 < \alpha < 1$ and $a_1, a_2 \in A$ there exists some $a_0 \in B$ satisfying

$$f(a_0, b) \geq \alpha f(a_1, b) + (1 - \alpha) f(a_2, b) - \epsilon$$

for every $b \in B$. The payoff function $f : A \times B \to R$ is called closely concave–closely convex on $A \times B$ if f is closely concave on A and closely convex on B.

Again in our notation, it follows that the payoff function f is closely convex on B if for every $\epsilon > 0$ and every $\mu \in \mathcal{P}_2(B)$ there exists some $b_0 \in B$ satisfying

$$f(a, b_0) \leq f_e(\epsilon_a, \mu) + \epsilon$$

for every $a \in A$. This also has an obvious game theoretical interpretation. In [12], one also shows the minimax result in relation (8) under the weaker condition that B is a compact topological space, $b \longmapsto f(a, b)$ is lower semicontinuous for every $a \in A$, and f is closely concave closely–convex on $A \times B$. To show the above results by means of Theorem 6, we need to verify that all the considered payoff functions actually belong to the set $\mathcal{U}_1 \cap \mathcal{U}_2$. In the next result, we say that $0 \leq \beta \leq 1$ is a König concave constant on A if for every $\lambda \in \mathcal{P}_{2,\beta}(A)$ there exists some $a_0 \in A$ satisfying $f(a_0, b) \geq f_e(\lambda, \epsilon_b)$ for every $b \in B$.

Lemma 9. *It holds that $KFC_A \subseteq KC_A \subseteq CC_A \subseteq \mathcal{U}_2$.*

Proof. It is obvious that the inclusion $KFC_A \subseteq KC_A$ holds. To show that $KC_A \subseteq CC_A$, it is sufficient to verify that the set $S \subseteq [0, 1]$ given by

$S := \{0 \leq \beta \leq 1 : \beta$ a a König concave constant$\}$ satisfies $cl(S) = [0,1]$. Clearly the numbers 0 and 1 belong to S. Because the function f is König concave on A, we know that there exists some $0 < \beta < 1$ belonging to S. Moreover, if the numbers $\beta_i, i = 1,2$ belong to S, it follows for every $\lambda_i = \beta_i \epsilon_{a_1} + (1-\beta_i)\epsilon_{a_2} \in \mathcal{P}_{2,\beta_i}(A)$ with $a_i \in A, i = 1,2$ that there exists some elements $a(\beta_i) \in A, i = 1,2$ satisfying

$$f(a(\beta_i), b) \geq f_e(\lambda_i, \epsilon_b) \tag{47}$$

for every $b \in A$ and $i = 1,2$. This implies using β belongs to S that for $\lambda = \beta\epsilon_{a(\beta_1)} + (1-\beta)\epsilon_{a(\beta_2)} \in \mathcal{P}_{2,\beta}(A)$, there exists some $a_0 \in A$ satisfying

$$f(a_0, b) \geq f_e(\lambda, \epsilon_b)$$

for every $b \in B$. Hence by relation (47), we obtain

$$f(a_0, b) \geq (\beta\beta_1 + (1-\beta)\beta_2) f(a_1, b) + (1 - \beta\beta_1 - (1-\beta)\beta_2) f(a_2, b)$$

for every $b \in B$. This means for any $\beta_i \in S, i = 1,2$ that also $\beta\beta_1 + (1-\beta)\beta_2$ belongs to S, and in [16] it is shown that such a set is dense in $[0,1]$. To verify the last inclusion, one can show by induction that for f closely concave on A, it follows for every $\epsilon > 0$ and $\lambda \in \mathcal{P}_F(A)$ that there exists some a_0 satisfying

$$f(a_0, b) \geq f_e(\lambda, \epsilon_b) - \epsilon$$

for every $b \in B$. This implies for every $\epsilon > 0$ and $\lambda \in \mathcal{P}_F(A)$ that

$$\inf_{b \in B} f(a_0, b) \geq \inf_{b \in B} f_e(\lambda, \epsilon_b) - \epsilon \tag{48}$$

and hence

$$\sup_{a \in A} \inf_{b \in B} f(a, b) \geq \inf_{b \in B} f_e(\lambda, \epsilon_b) - \epsilon. \tag{49}$$

Because $\lambda \in \mathcal{P}_F(A)$ and ϵ are arbitrary, we obtain

$$\sup_{a \in A} \inf_{b \in B} f(a, b) \geq \sup_{\lambda \in \mathcal{P}_F(A)} \inf_{b \in B} f_e(\lambda, \epsilon_b), \tag{50}$$

and so f belongs to $\mathcal{U}_2$. ■

Actually one can show that the above inclusions are strict ([12]). Moreover, one can also show the following result

Lemma 10. *If B is a compact topological space and the function $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$, then $KFC_B = KC_B = CC_B \subseteq \mathcal{U}_1$.*

Proof. As in Lemma 9, one can show without any additional conditions that $KFC_B \subseteq KC_B \subseteq CC_B$, and to prove equality it is sufficient to verify for B a compact topological space and $b \longmapsto f(a,b)$ lower semicontinuous for every $a \in A$ that $CC_B \subseteq KFC_B$. We only give a proof of this result for B a compact metric space. (For B a compact topological space, one can apply a similar proof replacing sequences by nets (see section 4 of [7]). If the function f is closely convex on the compact metric space B, then for every $n \in N, 0 < \alpha < 1$ and $b_1, b_2 \in B$, there exists some $b_{0,n} \in B$ satisfying

$$f(a, b_{0,n}) \leq \alpha f(a, b_1) + (1-\alpha) f(a, b_2) + \frac{1}{n}$$

for every $a \in A$. Because B is a compact metric space, there exists some converging subsequence $b_{0,n}, n \in K \subseteq N$ with limit $b_0 \in N$. This implies by the lower semicontinuity of the function $b \longmapsto f(a,b)$ for every $a \in A$ that

$$f(a, b_0) \leq \liminf_{n \in K, n \uparrow \infty} f(a, b_{0,n}) \leq \alpha f(a, b_1) + (1-\alpha) f(a, b_2)$$

and so the function f is Ky Fan convex on B. To show the inclusion $CC_B \subseteq \mathcal{U}_1$, we can verify in a similar way as done in the last part of the proof of Lemma 9 that for f closely convex on B, it follows for every $I \in \mathcal{F}(A)$ that

$$\inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \inf_{b \in B} \max_{a \in I} f(a,b)$$

This implies

$$\sup_{I \in \mathcal{F}(A)} \inf_{\mu \in \mathcal{P}_F(B)} \max_{a \in I} f_e(\epsilon_a, \mu) = \sup_{I \in \mathcal{F}(A)} \inf_{b \in B} \max_{a \in I} f(a,b) \tag{51}$$

and applying Lemma 3 to the last expression in relation (51), we obtain that f belongs to $\mathcal{U}_1$. ■

Using now Lemmas 9 and 10, we obtain for B a compact topological space, $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$, and f closely concave–closely convex on $A \times B$ that f belongs to the set $\mathcal{U}_1 \cap \mathcal{U}_2$, and so by Theorem 6 the classic minimax result in relation (8) holds.

4 Relations Between the Different Minimax Theorems

In this section, we investigate in more detail the relations between the different minimax results discussed in Section 3 and given by relations (6) up to (8). Introducing the notation L_i and R_i for the left-hand and right-hand sides of relation (i) for $i = 6, 7, 8$, we obviously obtain that

$$L_8 = L_7 \geq L_6 \geq R_6 = R_7 \geq R_8. \tag{52}$$

This implies that

$$(8) \Rightarrow (7) \Rightarrow (6). \tag{53}$$

Below we show by means of some counterexamples that none of the arrows in relation (53) can be reversed. In the first counterexample, we show an instance for which (7) holds and (8) does not hold.

Example 3. Let $A = [0,1] \subset R$, $B = \{b_1, b_2, b_3\} \subset R$ and introduce the function $f : A \times B \to R$ given by

$$f(a,b) = \begin{cases} a^2 & if\ b = b_1 \\ (a-1)^2 & if\ b = b_2 \\ 2^{-1} & if\ b = b_3 \end{cases}.$$

For this bifunction, we have

$$L_8 := \min_{b \in B} \sup_{a \in A} f(a,b) = 1/2,$$

whereas

$$R_8 := \sup_{a \in A} \min_{b \in B} f(a,b) = 1/4,$$

and so (8) does not hold. Because $L_8 = L_7 = 2^{-1}$ and it is obvious to check that $R_7 = 2^{-1}$, we obtain that (7) holds.

In the next counterexample, we show an instance for which (6) holds and (7) does not hold.

Example 4. Take $A = [0,1]$, $B = \{b_1, b_2\} \subset R$, and introduce the function $f : A \times B \to R$ given by

$$f(a,b) = \begin{cases} a^2 & if\ b = b_1 \\ (a-1)^2 & if\ b = b_2 \end{cases}.$$

Consider now the mixed strategy $\lambda^* \in \mathcal{P}_F(A)$ given by $\lambda^* = 2^{-1}\epsilon_{a_1} + 2^{-1}\epsilon_{a_2}$ with $a_1 = 0$ and $a_2 = 1$. It is easy to check that

$$\min_{b \in B} f_e(\lambda^*, \epsilon_b) = 2^{-1},$$

and so it follows that $R_7 \geq 2^{-1}$. Moreover, we observe by the definition of the sets A and B that

$$L_6 = \inf_{0 \leq s_1(\mu) \leq 1} \sup_{a \in A} \{s_1(\mu) f(a, b_1) + (1 - s_1(\mu)) f(a, b_2)\}. \tag{54}$$

Using now that the last expression in relation (54) equals

$$\inf_{0 \leq s_1(\mu) \leq 1} \max\{s_1(\mu), 1 - s_1(\mu)\} = 2^{-1}, \tag{55}$$

we obtain that $L_6 = 2^{-1}$. Because we already know that $L_6 \geq R_7 = R_6$ and $R_7 \geq 2^{-1}$, we obtain

$$L_6 = R_7 = R_6 = 2^{-1},$$

It is now easy to check that $L_7 = 1$, and hence we have found an instance for which (6) holds and (7) does not hold.

To conclude these investigations, we give an instance showing that (6) can also fail. Consider the set c_0 of all (real valued) sequences converging to 0. It is well-known that the space c_0 endowed with the norm

$$\|a\|_{c_0} = \sup_{k \in N} |a_k|$$

is a Banach space.

Example 5. Let $A = \{a = (a_k) \in c_0 : a_1 = 0\}$, $B = [0,1] \subset R$ and take the function $f : A \times B \to R$ given by

$$f(a,b) = f((a_k),b) = \begin{cases} 1 \; if \; there \; exist \; some \; k \in N \; such \; that \; b = a_k \\ 0 \; otherwise \end{cases} \tag{56}$$

Consider some $\lambda \in \mathcal{P}_F(A)$. Hence there exists a finite number of sequences $a^i = (a^i_k)_{k\in N}, 1 \leq i \leq m$, belonging to A and some vector $s(\lambda) = (s_1(\lambda), ..., s_m(\lambda))$, $s_i(\lambda) > 0$ and $\sum_{i=1}^m s_i(\lambda) = 1$ such that

$$\lambda = \sum_{i=1}^m s_i(\lambda)\epsilon_{a^i}.$$

Because the set $[0,1]$ contains more than a countable number of elements, one can now choose a number $b \in [0,1]$ such that none of the above sequences $a^i, 1 \leq i \leq m$, contain this number. Using this number and the definition of f, it can be easily seen that

$$\inf_{b\in[0,1]} f_e(\lambda, \epsilon_b) = \inf_{b\in[0,1]} \sum_{i=1}^m s_i(\lambda) f(a^i, b) = 0,$$

and so $R_6 = 0$. On the other hand, consider some $\mu \in \mathcal{P}_F(B)$. By definition, one can find some finite set $\{b_1, \ldots, b_p\} \subseteq [0,1]$ and a vector $s(\mu) = (s_1(\mu), \ldots, s_p(\mu))$, $s_j(\mu) > 0$ with $\sum_{j=1}^p s_j(\mu) = 1$ such that

$$\mu = \sum_{j=1}^p s_j(\mu)\epsilon_{b_j}.$$

Taking the element $a_0 := (0, b_1, \ldots, b_p, 0, 0, ...) \in c_0$, it is obvious by the definition of f that

$$\sup_{a\in A} f_e(\epsilon_a, \mu) \geq \sum_{j=1}^p s_j(\mu) f(a_0, b_j) = 1. \tag{57}$$

Because f is bounded by 1, this shows that

$$L_6 := \inf_{\mu\in\mathcal{P}_F(B)} \sup_{a\in A} f_e(\epsilon_a, \mu) = 1,$$

and so we have verified that (6) does not hold.

5 On Sion's Minimax Theorem

In this section, we give an alternative and elementary proof of Sion's minimax theorem. This famous result is a generalization of von Neumann's minimax theorem ([30]). Its original proof made use of the KKM lemma, which is equivalent to Brouwer's fixed point theorem ([9, 42]). However, as it will turn out, we do not need such a heavy machinery to verify this result. Actually we will give a proof of a slightly more general result by using a less known technique called the level set method originally developed by Joo ([19]). It remains an open question whether it is possible to verify this minimax result by means of Theorem 6.

Definition 17. *A real valued function $k : C \longrightarrow R$ is called quasiconvex on the (convex) set C if all its lower level sets $\{x \in C : k(x) \leq r\}, r \in R$ are convex. It is called quasiconcave on C if $-f$ is quasiconvex on C.*

It is well-known ([3]) that an equivalent description of a quasiconvex function is given by

$$k(\beta x + (1-\beta)y) \leq \max\{k(x), k(y)\}$$

for every $0 < \beta < 1$ and $x, y \in C$. By this representation, it is easy to see that the class of quasiconvex functions strictly contains the class of convex functions. We now list the following result due to Sion ([37]).

Theorem 11. *If the payoff function $f : A \times B \to R$ with B a compact convex subset of a linear topological space and A a convex subset of a linear topological space satisfies $a \longmapsto f(a,b)$ is quasiconcave and upper semicontinuous for every $b \in B$, and $b \longmapsto f(a,b)$ is quasiconvex and lower semicontinuous for every $a \in A$, then the minimax result in relation (8) given by*

$$\inf_{b \in B} \sup_{a \in A} f(a,b) = \sup_{a \in A} \inf_{b \in B} f(a,b)$$

holds, and in the above expressions inf *can be replaced by* max.

The following result is the starting point of the so-called level set method and shown in ([19]). Remember the values r^* and r_* are given in relations (1) and (2). As observed in Section 1, it is always assumed that $r^* > -\infty$. Also for convenience, we denote the lower level set of level r of a function $k : C \to R$ by

$$L(k,r) := \{x \in C : k(x) \leq r\}.$$

Lemma 11. *Let $f : A \times B \to R$ be a given payoff function and introduce the function $f_a : B \to R$ given by $f_a(b) = f(a,b)$. Then $r^* = r_*$ if and only if $\cap_{a \in A} L(f_a, r)$ is nonempty for every $r > r_*$.*

Proof. If $r^* = r_*$, then for every $r > r_* = r^* > -\infty$ there exists by the definition of r^* some $b_0 \in B$ satisfying $\sup_{a \in A} f(a, b_0) < r$. This shows that b_0 belongs to the intersection $\cap_{a \in A} L(f_a, r)$ and so $\cap_{a \in A} L(f_a, r)$ is nonempty. To verify the reverse implication, it is sufficient to check that $r^* \leq r_* + \epsilon$ for every $\epsilon > 0$. Take now $r = r_* + \epsilon$ for some $\epsilon > 0$. By our assumption, we know that $\cap_{a \in A} L(f_a, r)$ is nonempty and so there exists some $b_0 \in B$ satisfying $\sup_{a \in A} f(a, b_0) \leq r$. This implies $r^* = \inf_{b \in B} \sup_{a \in A} f(a,b) \leq r = r_* + \epsilon$, and the proof is completed. ■

For relation (8) to hold, it is necessary and sufficient by Lemma 11 to show that the intersection $\cap_{a \in A} L(f_a, r)$ is nonempty for every $r > r_*$. It can be easily verified that for arbitrary functions f, this result does not hold and so we must impose some conditions on f. Before defining the proper class of functions, we recall some well-known notions within topology. For X a subset of a topological space with topology $\mathcal{F}$, the set $S \subseteq X$ is called open in X

if there exists some set O belonging to $\mathcal{F}$ with $S = X \cap O$. The open sets generated in this way are called the relative topology induced by X, and with this topology the set X is a topological space. Another well-known notion within topology is given in the next definition ([9, 33]).

Definition 18. *For any topological space X, a set $C \subseteq X$ is called connected if for any two disjoint sets C_1 and C_2, both open (closed) in C and satisfying $C = C_1 \cup C_2$, it follows that C_1 or C_2 is empty.*

In [26], the following class of functions is introduced.

Definition 19. *Let X be a topological space. The function $k : X \to R$ is called connected if for every $r \in R$ the lower level set $L(k, r) \subseteq X$ is connected.*

It is well-known that every convex subset of a linear topological space X is connected and so any quasiconvex function $k : X \to R$ is connected. As for quasiconvex functions, one can give an equivalent definition of a connected function.

Lemma 12. *The function $k : X \to R$ is connected if and only if for every $x_1, x_2 \in X$ there exists some connected set $C_{x_1x_2} \subseteq X$ containing x_1, x_2 such that $k(x) \leq \max\{k(x_1), k(x_2)\}$ for every $x \in C_{x_1x_2}$.*

Proof. To show that a connected function satisfies the above property, consider $x_1, x_2 \in X$ and introduce $r := \max\{k(x_1), k(x_2)\}$. Take now the set $C_{x_1x_2}$ equal to the connected set $L(k, r)$. This set satisfies the desired property. To prove the reverse implication that the lower level sets are connected, consider some nonempty lower level set $L(k, r)$ with x_1 belonging to $L(k, r)$ and let x_2 be another arbitrary point belonging to $L(k, r)$. (The empty set is connected by definition.) By assumption, there exists some connected set $C_{x_1x_2} \subseteq X$ containing x_1, x_2 such that

$$k(x) \leq \max\{k(x_1), k(x_2)\}$$

for every x belonging to $C_{x_1x_2}$. This shows $C_{x_1x_2} \subseteq L(k, r)$, and as x_2 is an arbitrary element of $L(k, r)$, we obtain

$$\cup_{x_2 \in L(k,r)} C_{x_1x_2} = L(k, r). \tag{58}$$

By construction, the intersection $\cap_{x_2 \in L(k,r)} C_{x_1x_2}$ contains the vector x_1 and because for every $x_2 \in L(k, r)$ the set $C_{x_1x_2}$ is connected, also $\cup_{x_2 \in L(k,r)} C_{x_1x_2}$ is connected (cf. [9]). Applying now relation (58) shows that the function k is connected. ∎

Using the above representation of a connected function, it can be shown ([11]) that the set of connected functions strictly includes the set of quasiconvex functions. This means that there exists a connected function that is not quasiconvex. To prove our main theorem, we also introduce the following class of functions.

Definition 20. *Let X be a topological space. The collection of functions $k_\gamma : X \to R, \gamma \in \Gamma$ is called equiconnected if for every $x_1, x_2 \in X$ there exists a connected set $C_{x_1,x_2} \subseteq X$ containing x_1, x_2 such that*

$$k_\gamma(x) \leq \max\{k_\gamma(x_1), k_\gamma(x_2)\}$$

for every $x \in C_{x_1 x_2}$ and $\gamma \in \Gamma$.

If X is a convex subset of a linear topological space and for every $\gamma \in \Gamma$ the function k_γ is quasiconvex, then by taking

$$C_{x_1 x_2} = \{\beta x_1 + (1-\beta)x_2 : 0 \leq \beta \leq 1\}$$

it follows immediately that the collection of functions $k_\gamma, \gamma \in \Gamma$ is equiconnected.

Definition 21. *The payoff function $f : A \times B \to R$ belongs to the class $\mathcal{C}_0$ if*

1. *The function $a \longmapsto f(a,b)$ is upper semicontinuous for every $b \in B$;*
2. *The function $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$;*
3. *For every $I \in \mathcal{F}(A)$, the function $b \longmapsto \max_{a \in I} f(a,b)$ is connected;*
4. *The collection of functions $-f_b, b \in B$ with $f_b(a) := f(a,b)$ is equiconnected.*

For any set of quasiconvex functions $k_\gamma, \gamma \in \Gamma$, it follows that the function $x \longmapsto \sup_{\gamma \in \Gamma} k(x)$ is also quasiconvex. Using this observation, it is easy to see for any payoff function f satisfying $a \longmapsto f(a,b)$ is quasiconcave and upper semicontinuous for every $b \in B$ and $b \longmapsto f(a,b)$ is quasiconvex and lower semicontinuous for every $a \in A$ actually belongs to the set $\mathcal{C}_0$. Hence the payoff function f mentioned in Sion's minimax theorem belongs to $\mathcal{C}_0$. One can now show the following important intersection result.

Theorem 12. *If the payoff function f belongs to the class $\mathcal{C}_0$, then for every $r > r_*$ and $I \in \mathcal{F}(A)$ the intersection $\cap_{a \in I} L(f_a, r)$ is nonempty.*

Proof. If $I = \{a_0\} \subseteq A$, then for every $r > r_*$ we obtain by the definition of r_* that $r > \inf_{b \in B} f(a_0, b)$ and so $L(f_{a_0}, r)$ is nonempty. Suppose now for all sets I belonging to $\mathcal{F}(A)$ and consisting of at most k elements that

$$\cap_{a \in I} L(f_a, r) \neq \emptyset \tag{59}$$

for every $r > r_*$. To prove the result for all sets $I \in \mathcal{F}(A)$ consisting of at most $k+1$ elements, we assume by contradiction that there exists some set $I_0 = \{a_0, \ldots, a_k\} \subseteq A$ and some $r_0 > r_*$ satisfying

$$\cap_{i=0}^k L(f_{a_i}, r_0) = \emptyset. \tag{60}$$

Because the collection of functions $-f_b, b \in B$ is equiconnected, one can find some connected set $C_{a_0 a_1} \subseteq A$ containing a_0 and a_1 satisfying

$$f(a,b) \geq \min\{f(a_0,b), f(a_1,b)\} \tag{61}$$

for every $a \in C_{a_0a_1}$ and $b \in B$. We now introduce the set valued mapping $\Phi_r : C_{a_0a_1} \longrightarrow 2^B$, given by

$$\Phi_r(a) = \cap_{\gamma \in \{a_2,a_3,\ldots,a_k,a\}} L(f_\gamma, r). \tag{62}$$

(In case $k = 1$, put $\Phi_r(a) = L(f_a, r)$.) By the definition of $L(f_\gamma, r)$, this yields

$$\Phi_r(a) = \{b \in B : \max_{\gamma \in \{a_2,a_3,\ldots,a_k,a\}} f(\gamma, b) \leq r\}. \tag{63}$$

Because the function

$$b \longmapsto \max_{\gamma \in \{a_2,a_3,\ldots,a_k,a\}} f(\gamma, b)$$

is connected and lower semicontinuous (use $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$), it follows by relation (63) that the sets $\Phi_r(a), a \in C_{a_0a_1}$ are connected and closed for every $r > r_*$. Moreover, by the induction hypothesis in relation (59), the sets $\Phi_{r_0}(a), a \in C_{a_0a_1}$are nonempty and satisfy by relation (61)

$$\Phi_{r_0}(a) \subseteq \Phi_{r_0}(a_0) \cup \Phi_{r_0}(a_1) \tag{64}$$

for every $a \in C_{a_0a_1}$ and by relation (60)

$$\Phi_{r_0}(a_0) \cap \Phi_{r_0}(a_1) = \emptyset. \tag{65}$$

Introducing now the nonempty sets

$$S_i := \{a \in C_{a_0a_1} : \Phi_{r_0}(a) \subseteq \Phi_{r_0}(a_i)\}, i = 0, 1, \tag{66}$$

we obtain by relation (65) that the intersection $S_0 \cap S_1$ is empty. To show that $S_0 \cup S_1 = C_{a_0a_1}$ we first observe that $S_0 \cup S_1 \subseteq C_{a_0a_1}$. For the reverse inclusion, consider for a given $a \in C_{a_0a_1}$ the closed sets

$$A_i(a) := \Phi_{r_0}(a) \cap \Phi_{r_0}(a_i), i = 0, 1.$$

By relation (64), we obtain that

$$A_0(a) \cup A_1(a) = \Phi_{r_0}(a) \tag{67}$$

and because $\Phi_{r_0}(a)$ is connected, it must follow by relation (67) and $A_i(a), i = 0, 1$ closed that $A_0(a)$ or $A_1(a)$ is empty. This means by relation (64) that either $\Phi_{r_0}(a) \subseteq \Phi_{r_0}(a_0)$ or $\Phi_{r_0}(a) \subseteq \Phi_{r_0}(a_1)$ and so the point a belongs to $S_0 \cup S_1$. Hence we have verified that the sets $S_i, i = 0, 1$ satisfy

$$S_0 \cap S_1 = \emptyset,\ S_0 \cup S_1 = C_{a_0a_1}. \tag{68}$$

We will now show that the sets $S_i, i = 0, 1$ are also open in $C_{a_0a_1}$. Let a^* be an arbitrary point belonging to S_0. By our induction hypothesis, we know that the sets $\Phi_r(a^*)$ are nonempty for every $r > r_*$ and this implies by the definition of $\Phi_r(a^*)$ in relation (63) that

$$\inf_{b\in B} \max_{\gamma\in\{a_2,a_3,\ldots,a_k,a^*\}} f(\gamma, b) \leq r$$

for every $r > r_*$. This shows by letting $r \downarrow r_*$ that

$$\inf_{b\in B} \max_{\gamma\in\{a_2,a_3,\ldots,a_k,a^*\}} f(\gamma, b) \leq r_* < r_0$$

and so one can find some $b_0 \in \Phi_{r_0}(a^*) \subseteq B$ ($b_0 \in B$ for $k = 1$) satisfying

$$f(a^*, b_0) < r_0. \tag{69}$$

By the upper semicontinuity of $a \longmapsto f(a, b_0)$ and relation (69), there exists some open neighborhood $\mathcal{U}(a^*)$ of a^* satisfying $f(a, b_0) < r_0$ for every $a \in \mathcal{U}(a^*)$ and because $b_0 \in \Phi_{r_0}(a^*)$, this yields $b_0 \in \Phi_{r_0}(a)$ for every $a \in \mathcal{U}(a^*) \cap C_{a_0a_1}$ or equivalently

$$b_0 \in \Phi_{r_0}(a^*) \cap \Phi_{r_0}(a)$$

for every $a \in \mathcal{U}(a^*) \cap C_{a_0a_1}$. This implies by relation (68) and $a^* \in S_0$ that $\Phi_{r_0}(a) \subseteq \Phi_{r_0}(a_0)$ for every $a \in \mathcal{U}(a^*) \cap C_{a_0a_1}$ or equivalently

$$\mathcal{U}(a^*) \cap C_{a_0a_1} \subseteq S_0.$$

Because $a^* \in S_0$ is arbitrary, we obtain that

$$S_0 = \cup_{a^*\in S_0}(\mathcal{U}(a^*) \cap C_{a_0a_1}) = C_{a_0a_1} \cap (\cup_{a^*\in S_0}\mathcal{U}(a^*))$$

and so S_0 is open in $C_{a_0a_1}$. Similarly, one can verify that the set S_1 is open in $C_{a_0a_1}$ and by relation (68) and $C_{a_0a_1}$ connected we obtain that either S_0 or S_1 is empty. Because by relation (66) the point a_i belongs to $S_i, i = 0, 1$, this yields a contradiction, and the proof is completed. ■

Applying Lemma 11 we immediately deduce from Theorem 12 the following result.

Theorem 13. *Let the payoff function $f : A \times B \to R$ belong to the class $\mathcal{C}_0$. If A is a finite set, then*

$$\inf_{b\in B} \max_{a\in A} f(a, b) = \max_{a\in A} \inf_{b\in B} f(a, b),$$

whereas for A an infinite set

$$\sup_{I\in\mathcal{F}(A)} \inf_{b\in B} \max_{a\in I} f(a, b) = \sup_{a\in A} \inf_{b\in B} f(a, b).$$

Proof. The first formula is an immediate consequence of Lemma 11 and Theorem 12. To verify the second formula, we observe

$$\sup_{a\in A} \inf_{b\in B} f(a, b) = \sup_{I\in\mathcal{F}(A)} \sup_{a\in I} \inf_{b\in B} f(a, b).$$

Applying now the first part yields the desired result. ■

By Theorem 13 and Lemma 3, one can show the following result, which contains as a special case (see observation after Definition 21) Sion's minimax theorem listed in Theorem 11.

Theorem 14. *If B is a compact topological space and the payoff function f belongs to the class $\mathcal{C}_0$, then*

$$\inf_{b\in B}\sup_{a\in A} f(a,b) = \sup_{a\in A}\inf_{b\in B} f(a,b)$$

and $\inf$ *can be replaced by* $\min$ *in the above expressions.*

Proof. Because B is a compact topological space and $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$, we obtain by Lemma 3 and the observation after this lemma that

$$\inf_{b\in B}\sup_{a\in A} f(a,b) = \sup_{I\in\mathcal{F}(A)}\inf_{b\in B}\max_{a\in I} f(a,b).$$

Applying now the second part of Theorem 13 and Lemma 2 yields the desired result. ■

Actually by Lemma 3, one can slightly weaken the condition that A is a compact topological space by replacing the compactness assumption by the condition that there exists some set $I \in \mathcal{F}(A)$ such that for every $r \in R$, the set $\cap_{a\in I}\{b \in B : f(a,b) \leq r\}$ is compact. It is possible ([11]) to construct a payoff function f that satisfies the conditions of Theorem 14 but does not satisfy the conditions of Sion's minimax result.

Definition 22. *The payoff function $f : A \times B \to R$ belongs to the class $\mathcal{C}_1$ if*

1. *The function $a \longmapsto f(a,b)$ is upper semicontinuous for every $b \in B$;*
2. *The function $b \longmapsto f(a,b)$ is lower semicontinuous for every $a \in A$;*
3. *For every $J \in \mathcal{F}(B)$, the function $a \longmapsto \min_{b\in J} f(a,b)$ is connected;*
4. *The collection of functions $f_a, a \in A$ with $f_a(b) := f(a,b)$ is equiconnected.*

By the symmetry argument and Theorem 14, it follows easily that the minimax equality in relation (8) holds if the payoff function f belongs to the class $\mathcal{C}_1$ and A is a compact topological space. Finally, we like to mention that Wald's minimax result is a special case of Sion's minimax result. However, from the proof of Theorem 12, it should be clear that the only properties of convex sets that are important are the observation that any intersection of convex sets is again convex and every convex set is connected. This shows that Sion's minimax result is actually a topological result based on connectedness.

6 On n-Player Nonzero-Sum Noncooperative Games

In this section, we will extend the two-player zero-sum noncooperative games discussed in the previous sections to n-player nonzero-sum noncooperative games, $n \geq 2$. In this framework, there are n players, and each player

$i, 1 \leq i \leq n$ has a pure strategy set X_i and a payoff function $f_i : X \longrightarrow R$ with $X = \Pi_{i=1}^n X_i$ denoting the Cartesian product of the sets X_i. In case each player $i, i = 1, \ldots, n$ selects independently of each other the strategy x_i, the gain given by player i is given by $f_i(x_1, \ldots, x_n)$ (for a complete description of such games and examples see [4, 40] or [38]). In this section, we assume that the sets $X_i, 1 \leq i \leq n$ are subsets of (possibly different) linear topological spaces $\mathcal{X}_i$ ([34]). We also assume in this section that the players only use their pure strategy sets and they do not use their mixed strategy sets. For these n-person noncooperative games, an important concept is given by a Nash equilibrium point. Observe for $n = 2$ (taking $f_2 = -f_1$) this reduces to the minimax concept used within a two-player zero-sum noncooperative game.

Definition 23. *Let the payoff functions $f_i : X \to R$ of each player be given. The point $x^* = (x_1^*, \ldots, x_n^*)$ is called a Nash equilibrium point if*

$$f_i(x_1^*, \ldots, x_i^*, \ldots, x_n^*) \geq f_i(x_1^*, \ldots, x_i, \ldots, x_n^*)$$

for every $x_i \in X_i$ and $1 \leq i \leq n$.

We are now interested in under which conditions a Nash equilibrium point exists for an n-person noncooperative game. To show this, we need the following definition ([5]).

Definition 24. *Let X be a nonempty set and $\varphi : X \times X \to R$ some function. The point x^* is called an equilibrium point of the function φ if $\varphi(x^*, y) \geq 0$ for every $y \in X$.*

Using the above definition of an equilibrium point for the mapping φ, we show the following result.

Lemma 13. *Let $X = \Pi_{i=1}^n X_i$ be the Cartesian product of the sets $X_i, i = 1, \ldots, n$. The point x^* is a Nash equilibrium point if and only if x^* is an equilibrium point of the function $\varphi : X \times X \longrightarrow R$, given by*

$$\varphi(x, y) = \sum\nolimits_{i=1}^{n} f_i(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, y_i, \ldots, x_n) \tag{70}$$

with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Proof. Let x^* be a Nash equilibrium and consider an arbitrary $y = (y_1, \ldots, y_n) \in X$. By definition

$$f_i(x_1^*, \ldots, x_i^*, \ldots, x_n^*) \geq f(x_1^*, \ldots, y_i, \ldots, x_n^*)$$

for every $1 \leq i \leq n$. This shows $\varphi(x^*, y) \geq 0$ and so x^* is an equilibrium point of the function φ. For x^* an equilibrium point of the function φ, consider some $1 \leq i \leq n$ and introduce the vector $y = (x_1^*, \ldots, y_i, \ldots, x_n^*) \in X$. Clearly for this vector y, it follows that

$$0 \leq \varphi(x^*, y) = f_i(x_1^*, \ldots, x_i^*, \ldots, x_n^*) - f_i(x_1^*, \ldots, y_i, \ldots, x_n^*)$$

and as $1 \leq i \leq n$ is arbitrary, we obtain that x^* is a Nash equilibrium point. ∎

Hence by the above lemma, we have reduced the proof of existence of a Nash equilibrium point to the proof of existence of a equilibrium point for the mapping φ listed in relation (70). To show in general the existence of an equilibrium point of a mapping $\varphi : X \times X \longrightarrow R$, we observe that the point x^* is an equilibrium point of the mapping φ if and only if the intersection $\cap_{y \in X}\{x \in X : \varphi(x, y) \geq 0\}$ is nonempty. Unfortunately, it seems not to be possible (in general) to prove the existence of an equilibrium point by means of LP duality or convex analysis techniques as was done for a two-person noncooperative game. To show the existence of a Nash equilibrium under certain conditions on the sets X_i and the payoff functions f_i we need the so-called KKM (Knaster–Kuratowski–Mazurkiewicz) lemma ([9]). Observe the simplex Δ_J for any subset $J \subseteq \{1, \ldots, k\}$ is given by

$$\Delta_J := co(\{e_j : j \in J\})$$

with e_j the jth unit vector in R^k.

Definition 25. *The collection of sets $E_i \subseteq R^k, 1 \leq i \leq k$ satisfy the KKM property if $\Delta_J \subseteq \cup_{i \in J} E_i$ for every set $J \subseteq \{1, \ldots, k\}$.*

The KKM lemma is given by the following result (for its proof see [42]).

Lemma 14. *If the sets $E_i \subseteq R^k, 1 \leq i \leq k$ are closed and satisfy the KKM property, then $\cap_{i=1}^k E_i$ is nonempty.*

The KKM lemma is an easy consequence of Sperner's lemma (see Theorem 2.5.6 of [43] or Lemma 3.5.1 of [27]), and Sperner's lemma can be proved by combinatorial arguments (cf. [1] or Theorem 3.4.3 of [27]). Because our function φ in a so-called equilibrium problem is defined on the set $X \times X$ with X a convex subset of a linear topological space $\mathcal{X}$, we need to discuss the extensions of the KKM lemma to these spaces. This can be done in the following way. Let $\Phi : X \to 2^X$ be a set valued mapping with nonempty values, where X is a convex subset of some (real) linear topological space $\mathcal{X}$ and 2^X the power set of X, and consider for a given collection $\{x_1, \ldots, x_n\} \subseteq X$ and $x \in X$ the (possibly empty) finite dimensional sets

$$E(x) = \{\lambda \in \Delta_N : \sum\nolimits_{j=1}^n \lambda_j x_j \in \Phi(x)\}$$

with $N := \{1, \ldots, n\}$. Denoting by $L := lin(\{x_1, \ldots, x_n\})$ the smallest linear subspace containing the set $\{x_1, x_2, \ldots, x_n\}$, then clearly

$$E(x) = \{\lambda \in \Delta_N : \sum\nolimits_{j=1}^n \lambda_j x_j \in \Phi(x) \cap L\} \subseteq R^n. \tag{71}$$

If we know that the sets $E(x) \subseteq \Delta_N$ are closed for every $x \in X$, and for a given collection $\{x_1, \ldots, x_n\} \subseteq X$, the nonempty sets $E_i := E(x_i), 1 \leq i \leq n$ satisfy the KKM property, then by the KKM lemma we obtain that $\cap_{i=1}^n E(x_i)$ is nonempty. This shows that there exists some $\lambda^* \in \Delta_N$ satisfying $\sum_{j=1}^n \lambda_j^* x_j \in \cap_{i=1}^n \Phi(x_i)$, and so we have verified that $\cap_{i=1}^n \Phi(x_i) \neq \emptyset$. To introduce a topology on $E(x)$, we recall the following definition.

Definition 26. *The set valued mapping $\Phi : X \to 2^X$ with X a convex subset of a linear topological space $\mathcal{X}$ is called finitely closed if for every $x \in X$ and every finite dimensional subspace $L \subseteq \mathcal{X}$ the set $\Phi(x) \cap L$ is closed in the Euclidean topology on L.*

It is obvious that Φ finitely closed implies $E(x)$ is closed for every $x \in X$. In the next lemma, we give a sufficient condition for Φ to be finitely closed.

Lemma 15. *If the set-valued mapping $\Phi : X \to 2^X$ with X a convex subset of a linear topological space $\mathcal{X}$ has closed values $\Phi(x), x \in X$, then the mapping Φ is finitely closed.*

Proof. If $L \subseteq \mathcal{X}$ is a finite dimensional subspace, there exists some finite set $\{z_1, \ldots, z_n\} \subseteq \mathcal{X}$ of linearly independent vectors satisfying

$$L = lin(\{z_1, \ldots, z_n\}).$$

To show that $\Phi(x) \cap L$ is closed in the Euclidean topology on L, we need to verify for any sequence $(x_q)_{q \in N} \subseteq \Phi(x) \cap L$ satisfying $x_q \to x_\infty$ in the Euclidean topology on L that $x_\infty \in \Phi(x) \cap L$. Because every element of L can be uniquely represented as a linear combination of the vectors $z_i, 1 \leq i \leq n$, it follows that $x_q \to x_\infty$ in the Euclidean topology on L if and only if $\lim_{q \uparrow \infty} \beta_q = \beta_\infty$ with $\beta_q^\top = (\beta_{q,1}, \ldots, \beta_{q,n}) \in R^n$, $\beta_\infty^\top = (\beta_{\infty,1}, \ldots, \beta_{\infty,n}) \in R^n$,

$$x_q = \sum_{j=1}^{n} \beta_{q,j} z_j, q \in N, \tag{72}$$

and

$$x_\infty = \sum_{j=1}^{n} \beta_{\infty,j} z_j. \tag{73}$$

Moreover, because $\mathcal{X}$ is a linear topological space, it follows that the mapping $h : R^n \to X$, given by $h(\alpha) = \sum_{j=1}^{n} \alpha_j z_j$, is continuous in this topology. This shows, using $x_q = h(\beta_q) \in \Phi(x)$ for every $q \in N$, that

$$x_\infty = h(\beta_\infty) = \lim_{q \uparrow \infty} h(\beta_q) \in cl(\Phi(x))$$

with the closure taken with respect to the topology on $\mathcal{X}$. Using now that $\Phi(x)$ is closed, we obtain that $x_\infty \in \Phi(x)$ and so x_∞ belongs to $\Phi(x) \cap L$. ■

We next recall the definition of a KKM mapping for set-valued functions $\Phi : X \to 2^X$.

Definition 27. *Let X be a convex subset of a linear topological space $\mathcal{X}$. The set valued mapping $\Phi : X \to 2^X$ is called a KKM mapping if $co(\{x_1, \ldots, x_k\}) \subseteq \cup_{j=1}^{k} \Phi(x_j)$ for every finite subset $\{x_1, \ldots, x_k\} \subseteq X$.*

Clearly by the above definition, it follows for a KKM mapping Φ that x belongs to $\Phi(x)$ for every $x \in X$. In the next lemma, we extend the KKM lemma, to set-valued mappings.

Lemma 16. *If the set valued mapping* $\Phi : X \to 2^X$ *is a KKM mapping with* $\Phi(x)$ *closed for every* $x \in X$*, then* $\cap_{i=1}^k \Phi(x_i)$ *is nonempty for every finite set* $\{x_1, \ldots, x_k\} \subseteq X$*.*

Proof. If Φ is a KKM mapping, then by definition

$$co(\{x_1, \ldots, x_k\}) \subseteq \cup_{j=1}^k \Phi(x_j) \tag{74}$$

for every finite subset $\{x_1, \ldots, x_k\} \subseteq X$. To prove the desired result, we verify by induction that

$$co(\{x_1, \ldots, x_q\}) \cap (\cap_{j=1}^q \Phi(x_j)) \neq \emptyset \tag{75}$$

for every finite subset $\{x_1, \ldots, x_q\} \subseteq X$. By relation (74) it follows that (75) holds for $q = 1$. Suppose now that relation (75) holds for $q \leq k-1$ ($k \geq 2$) and consider a subset $\{x_1, \ldots, x_k\} \subseteq X$. Let $\Delta_k := \{\lambda \in R^k : \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1\}$ and introduce for every $1 \leq i \leq k$ the sets E_i, given by

$$E_i = \{\lambda \in \Delta_k : \sum_{j=1}^k \lambda_j x_j \in \Phi(x_i)\} \subseteq R^k.$$

For L denoting the linear subspace $lin(\{x_1, \ldots, x_k\})$, it is obvious that

$$E_i = \{\lambda \in \Delta_k : \sum_{j=1}^k \lambda_j x_j \in \Phi(x_i) \cap L\},$$

and as by Lemma 15 the set valued mapping Φ is finitely closed, it follows that the sets $E_i, 1 \leq i \leq k$ are closed in the Euclidean topology on L. Moreover, to show that the sets $E_i, 1 \leq i \leq k$, satisfy the KKM property, we observe for every $J \subseteq \{1, .., k\}$ and $\lambda \in co(\{e_j : j \in J\}) \subseteq \Delta_k$ that

$$\lambda = (\lambda_1, \ldots, \lambda_k), \lambda_j = 0, j \notin J, \lambda_j \geq 0, j \in J, \sum_{j \in J} \lambda_j = 1.$$

This implies by relation (74) with k replaced by $|J|$ that

$$\sum_{j=1}^k \lambda_j x_j = \sum_{j \in J} \lambda_j x_j \in co(\{x_j : j \in J\}) \subseteq \cup_{j \in J} \Phi(x_j)$$

and we have verified that λ belongs to $\cup_{j \in J} E_j$. Because $\lambda \in co(\{e_j : j \in J\})$ is arbitrary, this shows that

$$co(\{e_j : j \in J\}) \subseteq \cup_{j \in J} E_j$$

and so the collection $E_i, 1 \leq i \leq k$ satisfies the KKM property. Hence by the KKM lemma, it follows that $\cap_{i=1}^k E_i$ is nonempty and so there exists some $\lambda^* \in \Delta_k$ satisfying $\sum_{j=1}^k \lambda_j^* x_j \in \cap_{i=1}^k \Phi(x_i)$. This proves the induction for k and the proof is completed. ■

We are now able to show that under certain conditions, a Nash equilibrium point exists. To prove this, we first need the following lemma.

Lemma 17. *Let X be a convex subset of a linear topological space $\mathcal{X}$. If the function $\varphi : X \times X \to R$ satisfies $\varphi(x,x) \geq 0$ and $y \longmapsto \varphi(x,y)$ is convex on X for every $x \in X$, then the set valued mapping $\Phi : X \longrightarrow 2^X$ given by*

$$\Phi(y) = \{x \in X : \varphi(x,y) \geq 0\}$$

is a KKM mapping.

Proof. Because $\varphi(x,x) \geq 0$, it follows immediately that y belongs to $\Phi(y)$. Suppose now by contradiction that there exists some finite set $\{y_1^*, \ldots, y_k^*\} \subseteq X, k \geq 2$ such that y^* belonging to $co(\{y_1^*, \ldots, y_k^*\})$ does not belong to $\cup_{j=1}^k \Phi(y_j^*)$. By the first part, it follows that y^* is not equal to y_i^* for some $1 \leq i \leq k$. This means that one can find some $\lambda^* \in \Delta_k$ with at least two positive components smaller than 1 satisfying

$$\max_{1 \leq i \leq k} \varphi(\sum_{j=1}^k \lambda_j^* y_j^*, y_i^*) < 0.$$

By the convexity of the function $y \longmapsto \varphi(\sum_{j=1}^k \lambda_j^* y_j^*, y)$, this implies

$$0 \leq \varphi(\sum_{j=1}^k \lambda_j^* y_j^*, \sum_{i=1}^k \lambda_i^* y_i^*) \leq \sum_{i=1}^k \lambda_i^* \varphi(\sum_{j=1}^k \lambda_j^* y_j, y_i^*) < 0$$

and we obtain a contradiction. ■

Finally, we can give a proof of the following important result.

Theorem 15. *If the pure strategy sets $X_i, 1 \leq i \leq n$ are convex compact subsets of (maybe different) linear topological spaces $\mathcal{X}_i$, the payoff functions $f_i : X \to R, 1 \leq i \leq n$ are continuous on X for every $1 \leq i \leq n$ and satisfy*

$$x_i \longmapsto f_i(x_1, \ldots, x_i, \ldots, x_n)$$

are concave for every $1 \leq i \leq n$ and every fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, then the n-person noncooperative game has a Nash equilibrium point.

Proof. By Lemma 13, we have to show for $X = \Pi_{i=1}^n X_i$ that the function $\varphi : X \times X \longrightarrow R$, given by

$$\varphi(x,y) = \sum_{i=1}^n f_i(x_1, \ldots, x_i, \ldots, x_n) - f_i(x_1, \ldots, y_i, \ldots, x_n) \tag{76}$$

with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ has an equilibrium point, and by the observations after Lemma 13 this means that $\cap_{y \in X} \Phi(y)$ is nonempty with $\Phi(y) := \{x \in X : \varphi(x,y) \geq 0\}$. First observe by the continuity of f_i $(1 \leq i \leq n)$ that the function $x \longmapsto \varphi(x,y)$ listed in relation (76) is continuous on X for every $y \in X$. This shows for every $y \in X$ that the set $\Phi(y)$ is closed and because X is compact, that $\Phi(y)$ is compact as well. Moreover, because $x_i \longmapsto f_i(x_1, \ldots, x_i, \ldots, x_n)$ is concave for every $1 \leq i \leq n$, we obtain that the function $y \longmapsto \varphi(x,y)$ is convex, and together with $\varphi(x,x) = 0$ this implies by Lemma 17 that the set-valued map Φ is a KKM map. Applying now Lemma 16, it follows for every finite subset $F \subseteq X$ that $\cap_{y \in F} \Phi(y)$ is nonempty. This shows by the finite intersection property for compact sets that $\cap_{y \in X} \Phi(y)$ is nonempty, and we have shown the desired result. ■

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Nonlinear Games

Ferenc Szidarovszky

SIE Department, The University of Arizona, Tucson, Arizona, USA
szidar@sie.arizona.edu

Abstract This paper gives an overview of the existence and computation of equilibrium in nonlinear n-person games. After some introductory examples, sufficient existence results are presented in both cases of single-valued and multiple-valued best responses. The uniqueness of the equilibrium is also shown under general conditions. A special iterative method is discussed for the computation of the unique equilibrium based on a variational inequality, and a single-objective optimization model is introduced to provide the equilibria. An example of repeated oligopolies completes the paper.

Key words: Nash equilibria, fixed points, variational inequality, optimization

1 Introduction

An n-person game is defined by specifying the players ($k = 1, 2, \dots, n$), the set S_k of feasible strategies (choices) of each player k, and the payoff function $f_k : S \mapsto R$ for each player k, where S is a subset of the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$. The set S_k contains all feasible decision alternatives for player k, and the payoff function f_k gives the consequence of the decisions of all players for player k. If $x_l \in S_l$ is the selected strategy of player l ($l = 1, 2, \dots, n$), then $f_k(x_1, \dots, x_n)$ is the payoff (meaning profit, savings, etc.) of player k.

In most cases, it is assumed that $S = S_1 \times S_2 \times \cdots \times S_n$, in which case the players may select their strategies independently of each other. However in some applications, such as in production modeling, the resources are limited, which poses an additional condition that the total amount of resources used by all players is limited. We will use the **strategic form representation** of n-person games in this chapter:

$$G = \{ n; S_1, \dots, S_n, S; f_1, \dots, f_n \} .$$

For convenience, we will use the simplifying notation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x}_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, so we may write $\mathbf{x} = (\mathbf{x}_{-k}, x_k)$. The **best response mapping** of player k with given $\mathbf{x}_{-k}$ is defined as

$$g_k(\mathbf{x}_{-k}) = \arg\max_{x_k} \{ f_k(\mathbf{x}_{-k}, x_k) \,|\, (\mathbf{x}_{-k}, x_k) \in S \} \tag{1}$$

assuming that maximum exists. Note that $g_k(\mathbf{x}_{-k})$ is the set of strategies x_k of player k that maximize its payoff with any given $\mathbf{x}_{-k}$, where $\mathbf{x}_{-k}$ shows the choices of all other players. For the sake of convenience, we will write the best response mapping as $g_k(\mathbf{x})$, where we know that g_k does not depend explicitly on x_k. In many cases $g_k(\mathbf{x})$ is single valued, for example, when f_k is strictly concave in x_k, however in most applications $g_k(\mathbf{x})$ is a subset of S_k.

The **Nash equilibrium** ([9]) of game G is a strategy vector $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ such that

(i) $\mathbf{x}^* \in S$;
(ii) $\mathbf{x}_k^* \in g_k(\mathbf{x}^*)$ for all k.

By using the definition of the best response mapping, condition (ii) can be rewritten as

$$f_k\left(\mathbf{x}_{-k}^*, x_k\right) \le f_k\left(\mathbf{x}_{-k}^*, x_k^*\right) \tag{2}$$

for all k and $(\mathbf{x}_{-k}^*, x_k) \in S$. By introducing mapping

$$\mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) = g_1(\mathbf{x}) \times g_2(\mathbf{x}) \times \ldots \times g_n(\mathbf{x}), \tag{3}$$

it is clear that $\mathbf{x}^*$ is a Nash equilibrium if and only if $\mathbf{x}^* \in \mathbf{g}(\mathbf{x}^*)$, that is, $\mathbf{x}^*$ is a fixed point of the point-to-set mapping $\mathbf{g}$. If $\mathbf{g}$ is single valued, then $\mathbf{x}^*$ is a Nash equilibrium if and only if $\mathbf{x}^* = \mathbf{g}(\mathbf{x}^*)$.

The concepts of best response mapping and Nash equilibrium are illustrated in the following examples.

Example 1. Select $n = 2$, $S_1 = S_2 = R_+$, assume $S = S_1 \times S_2$, and $f_1(x_1, x_2) = f_2(x_1, x_2) = x_1 + x_2$. In this case, neither player has best response, because if player k increases the value of x_k (with unchanged strategy of the other player), its payoff increases. Therefore no Nash equilibrium exists.

Example 2. Select again $n = 2$ and assume that both players have two feasible strategies, that is $S_1 = S_2 = \{1; 2\}$. The payoff functions are given below:

$f_1(x_1, x_2)$	$x_2 = 1$	$x_2 = 2$
$x_1 = 1$	1	2
$x_1 = 2$	2	0

$f_2(x_1, x_2)$	$x_2 = 1$	$x_2 = 2$
$x_1 = 1$	2	1
$x_1 = 2$	4	5

These 2×2 matrices are called the **payoff matrices** of players 1 and 2. The best responses are clearly as follows:

$$g_1(1,1) = g_1(2,1) = 2, \qquad g_1(1,2) = g_1(2,2) = 1$$

and

$$g_2(1,1) = g_2(1,2) = 1, \qquad g_2(2,1) = g_2(2,2) = 2.$$

By using mapping (3), we have

$$\begin{aligned} \mathbf{g}(1,1) &= (2,1), \qquad & \mathbf{g}(1,2) &= (1,1) \\ \mathbf{g}(2,1) &= (2,2), \qquad & \mathbf{g}(2,2) &= (1,2) \end{aligned}$$

so mapping $\mathbf{g}$ has no fixed point. Therefore there is no Nash equilibrium.

Example 3. By modifying the payoffs of the previous example as

$f_1(x_1,x_2)$	$x_2 = 1$	$x_2 = 2$
$x_1 = 1$	2	0
$x_1 = 2$	3	1

$f_2(x_1,x_2)$	$x_2 = 1$	$x_2 = 2$
$x_1 = 1$	2	3
$x_1 = 2$	0	1

we have

$$g_1(1,1) = g_1(2,1) = 2, \qquad g_1(1,2) = g_1(2,2) = 2$$

and

$$g_2(1,1) = g_2(1,2) = 2, \qquad g_2(2,1) = g_2(2,2) = 2$$

therefore

$$\begin{aligned} &\mathbf{g}(1,1) = (2,2),\ \mathbf{g}(1,2) = (2,2) \\ &\mathbf{g}(2,1) = (2,2),\ \mathbf{g}(2,2) = (2,2). \end{aligned}$$

Hence we have a unique Nash equilibrium, $x_1 = x_2 = 2$.

Example 4. [15] Assume that n firms produce the same product or offer the same service. Let x_k denote the output of firm k, $C_k(x_k)$ the cost function of firm k, and let $p(s)$ be the price function, where $s = x_1 + x_2 + \cdots + x_n$ is the total output of the industry. If L_k denotes the capacity limit of firm k, then $S_k = [\,0, L_k\,]$ is the set of all feasible strategies of firm k, and

$$f_k(x_1, \ldots, x_n) = x_k \cdot p\left(\sum_{i=1}^{n} x_i\right) - C_k(x_k) \tag{4}$$

is its payoff function. If there is sufficient amount of energy, manpower, etc., for all firms to produce maximum output, then we may assume that $S = S_1 \times S_2 \times \cdots \times S_n$. This n-person game is called **Cournot oligopoly without product differentiation**.

Assume that functions p and $C_k (k = 1, 2, \ldots, n)$ are twice continuously differentiable, furthermore

(A) $p'(s) - C''_k(x_k) < 0$

and

(B) $p'(s) + x_k p''(s) \leq 0$ for all $x_k \in [0, L_k]$ and $s \in [0, \sum_{i=1}^n L_i]$.

Introduce notation $s_k = \sum_{i \neq k} x_i$, then the profit of firm k can be rewritten as

$$f_k(s_k, x_k) = x_k\, p(s_k + x_k) - C_k(x_k),$$

which is strictly concave in x_k. Therefore the best response of firm k is unique and can be obtained as follows. Notice first that

$$\frac{\partial f_k}{\partial x_k}(\mathbf{x}) = p(s) + x_k p'(s) - C'_k(x_k). \tag{5}$$

It is convenient to consider x_k as a function of the total output s, then we have three possibilities. If $p(s) - C'_k(0) \leq 0$, then the best choice of firm k is $x_k(s) = 0$. If $p(s) + L_k p'(s) - C'_k(L_k) \geq 0$, then the best choice of firm k is $x_k(s) = L_k$. Otherwise the best choice is interior and can be obtained as the unique solution of equation

$$p(s) + x_k p'(s) - C'_k(x_k) = 0 \tag{6}$$

in interval $(0, L_k)$. The left-hand side of this equation is continuously differentiable and strictly decreasing in x_k with fixed values of s as a consequence of assumption (A), furthermore its value at $x_k = 0$ is positive and at $x_k = L_k$ is negative. Therefore there is a unique solution $x_k = x_k(s)$. By implicit differentiation of equation (6), we have

$$p'(s) + x'_k(s)p'(s) + x_k(s)p''(s) - C''_k(x_k(s))x'_k(s) = 0$$

implying that

$$x'_k(s) = -\frac{p'(s) + x_k p''(s)}{p'(s) - C''_k(x_k)} \leq 0. \tag{7}$$

By combining the above three cases, we conclude that $x_k(s)$ is unique for all $s \in [0, \sum_{i=1}^n L_i]$ and is nonincreasing in s. The Nash equilibrium therefore is the unique solution s^* of the single variable monotonic equation

$$\sum_{k=1}^n x_k(s) - s = 0 \tag{8}$$

where s^* gives the total equilibrium output of the industry, and the equilibrium output of firm k is obtained as $x_k^* = x_k(s^*)$.

Example 5. By dropping the differentiability of the price function in Cournot oligopolies, we might lose the uniqueness of the Nash equilibrium. As an example with multiple equilibrium, consider the special case of duopoly ($n = 2$) with $S_1 = S_2 = [0, 1.5]$, $C_k(x_k) = 0.5x_k$ ($k = 1, 2$), and

$$p(s) = \begin{cases} 1.75 - 0.5s, & \text{if } 0 \le s \le 1.5; \\ 2.5 - s, & \text{if } 1.5 \le s \le 2.5; \\ 0, & \text{if } s \ge 2.5. \end{cases}$$

Then it can be shown that the set of all Nash equilibria is given as

$$X^* = \{(x_1, x_2) | 0.5 \le x_1 \le 1, 0.5 \le x_2 \le 1, x_1 + x_2 = 1.5\}.$$

2 Existence of Nash Equilibrium

As we have seen in the previous examples, there is no general guarantee for the existence of a Nash equilibrium in n-person games. Our first result gives the probability of the existence of Nash equilibria in special games with randomly selected payoff functions.

Theorem 1. *Let $n = 2$, $S_1 = \{1, 2, \ldots, m\}$, $S_2 = \{1, 2, \ldots, n\}$, all $f_1(i, j)$ values be independent, identically distributed with the same continuous distribution, furthermore $f_2(i, j) = -f_1(i, j)$ for all i and j. Then the probability that this two-person zero-sum game has Nash equilibrium is*

$$\frac{m!n!}{(m + n - 1)!}. \tag{9}$$

Proof. Notice first that:

1. The probability that all values $f_1(i, j)$ are different is one;
2. The probability that (i, j) is an equilibrium is the same for all i and j;
3. The probability that there is an equilibrium equals mn times the probability that $(1, 1)$ is an equilibrium;
4. $(1, 1)$ is an equilibrium, if $f_1(1, 1)$ is the largest in its column and smallest in its row in payoff matrix $(f_1(i, j))_{i,j=1}^{m,n}$. That is, if we order the elements $f_1(m, 1), f_1(m-1, 1), \ldots, f_1(1, 1), f_1(1, 2), \ldots, f_1(1, n)$ in an increasing order, then
$$f_1(m, 1), f_1(m - 1, 1), \ldots, f(2, 1)$$
have to be before $f_1(1, 1)$, and all elements
$$f_1(1, 2), f_1(1, 3), \ldots, f_1(1, n)$$
have to be after $f_1(1, 1)$. Because there are altogether $m + n - 1$ elements in the set, the probability of a such order is
$$\frac{(m - 1)!(n - 1)!}{(m + n - 1)!},$$
therefore the probability that there is a Nash equilibrium is
$$mn\frac{(m - 1)!(n - 1)!}{(m + n - 1)!}$$
where we used item 3 given above. This formula equals the assertion of the theorem. ■

We have also seen before that a vector $\mathbf{x}^* \in S$ is a Nash equilibrium if and only if $\mathbf{x}^*$ is a fixed point of the point-to-set mapping (3). Therefore any existence theorem of fixed points can be applied to find conditions for the existence of Nash equilibria. The most frequently used fixed point theorem is the Kakutani theorem ([7]), which can be directly applied to prove the following very powerful existence result.

Theorem 2. *([10]) Assume that for all k,*

(i) S_k is a nonempty, convex, compact subset of a finite dimensional Euclidean space;
(ii) f_k as an n-variable function of $(x_1, x_2, \ldots, x_n)$ is continuous on $S = S_1 \times S_2 \times \cdots \times S_n$;
(iii) f_k is concave in x_k with any fixed $\mathbf{x}_{-k} \in X_{i \neq k} S_i$.
Then there is at least one Nash equilibrium.

Example 6. Consider again the n-person oligopoly game without product differentiation, which was analyzed earlier in Example 4. All strategy sets $S_k = [\,0, L_k\,]$ are compact in the real line, all payoff functions are twice continuously differentiable, so continuous, furthermore conditions (A) and (B) imply that f_k is concave in x_k. Therefore there is at least one Nash equilibrium. In Example 4, we have also proved the uniqueness of the equilibrium and presented a simple computer procedure to find the equilibrium by solving the single variable, monotonic equation (8).

In case when the best response is single valued, we have a much more simple existence theorem.

Theorem 3. *Assume that conditions (i) and (ii) of Theorem 2 hold, furthermore the best response mapping (3) is single valued. Then the game has at least one Nash equilibrium.*

Proof. Because $\mathbf{g}(\mathbf{x})$ is single valued and f_k is continuous for all k, $g_k(\mathbf{x})$ is also continuous. Hence mapping $\mathbf{g}$ is a continuous mapping of S into itself, therefore the Brouwer fixed point theorem ([3]) implies the existence of at least one fixed point of $\mathbf{g}$, which is a Nash equilibrium. ∎

The Nikaido–Isoda theorem can be proved also by using the Brouwer fixed point theorem (see [4]), and the algorithm introduced in [14] can be used as a practical method to find the equilibrium. There are many generalizations of the Nakaido–Isoda theorem known from the literature. Such a result is the following,

Theorem 4. *Assume that for all k,*

(i) S_k is a nonempty, convex, compact subset of a finite dimensional Euclidean space;
(ii) f_k is upper semicontinuous on $S = S_1 \times \cdots \times S_n$;

(iii) for any fixed $x_k \in S_k$, f_k *is lower semicontinuous in* $\mathbf{x}_{-k}$ *on* $S_{-k} = X_{i \neq k} S_i$;
(iv) for any $\mathbf{x} \in S$, *the best reply* $\mathbf{g}(\mathbf{x})$ *is convex.*
Then there is at least one Nash equilibrium.

Note that condition (iv) holds if f_k is quasiconcave in x_k on S_k.

The existence of equilibrium can be examined without assuming topological structure of the strategy sets based on only monotonicity of the best response. The fixed point theorem of [18] is the theoretical basis for such approach, which was successfully applied in oligopoly models by [19].

Another family of existence results can be obtained by imposing certain continuity and concavity conditions on the "aggregator" function $H : S \times S \mapsto R$ as

$$H(\mathbf{x}, \mathbf{z}) = \sum_{k=1}^{n} f_k(\mathbf{x}_{-k}, z_k). \tag{10}$$

It is easy to show that $\mathbf{x}^* \in S$ is a Nash equilibrium if and only if for all $\mathbf{z} \in S$,

$$H\left(\mathbf{x}^*, \mathbf{z}\right) \leq H\left(\mathbf{x}^*, \mathbf{x}^*\right). \tag{11}$$

By using Fan's inequality (see, for example, [1]), the following result can be shown.

Theorem 5. *([17]) Assume that for all* k, *(i)* S_k *is a nonempty, convex, compact subset of a finite dimensional Eudlidean space; (ii)* $\sum_{i=1}^{n} f_i$ *is upper semicontinuous on* $S = S_1 \times \cdots \times S_n$; *(iii)* f_k *is lower semicontinuous on* $S_{-k} = X_{i \neq k} S_i$ *with any fixed value of* $x_k \in S_k$; *(iv) for any fixed* $\mathbf{x} \in S$, *the function* $H(\mathbf{x}, \mathbf{z})$ *is quasiconcave in* $\mathbf{z}$ *on* S. *Then there is at least one Nash equilibrium.*

The existence of a Nash equilibrium can be proved also based on certain transfer continuity and transfer concavity. Let Z be a subset of a finite dimensional Euclidean space and let $A \subset Z$. A function $U : A \times Z \mapsto R$ is said to be **diagonally transfer continuous** in X if for every $(\mathbf{x}, \mathbf{y}) \in A \times Z$, $U(\mathbf{x}, \mathbf{y}) > U(\mathbf{x}, \mathbf{x})$ implies that there exist some point $\mathbf{y}' \in Z$ and some neighborhood $N(\mathbf{x}) \subset A$ of $\mathbf{x}$ such that $U(\mathbf{y}', \mathbf{z}) > U(\mathbf{z}, \mathbf{z})$ for all $\mathbf{z} \in N(\mathbf{x})$.

Function $U : Z \times B \mapsto R$ is said to be **diagonally transfer quasiconcave** in $\mathbf{y}$ if, for any finite subset $Y^m = \left\{ \mathbf{y}^1, \ldots, \mathbf{y}^m \right\} \subset B$, there exists a corresponding finite subset $X^m = \left\{ \mathbf{x}^1, \ldots, \mathbf{x}^m \right\} \subset Z$ such that for any subset $X^s = \left\{ \mathbf{x}^{k_1}, \ldots, \mathbf{x}^{k_s} \right\} \subset X^m$ $(1 \leq s \leq m)$ and any $\mathbf{x}^{k_0}$ from the convex hull of X^s we have

$$\min_{1 \leq l \leq s} U\left(\mathbf{x}^{k_0}, \mathbf{y}^{k_l}\right) \leq U\left(\mathbf{x}^{k_0}, \mathbf{x}^{k_0}\right). \tag{12}$$

Theorem 6. *([2]) Assume that for all k, (i) S_k is a nonempty, convex, compact subset of a finite dimensional Euclidean space; (ii) The aggregator function $H(\mathbf{x}, \mathbf{y})$ is diagonally transfer continuous in $\mathbf{x}$. Then the n-person game has a Nash equilibrium if and only if H is diagonally transfer quasiconcave in $\mathbf{y}$.*

Example 7. ([2]) Consider a price-setting duopoly in which the firms operate with zero cost. Assume $S_1 = S_2 = [\,0, p^{max}\,]$, where p^{max} is the maximum feasible price to be selected by either firm. Let $c > 0$ be a given constant and assume that

$$f_1(p_1, p_2) = \begin{cases} p_1, & \text{if } p_1 \leq p_2; \\ p_1 - c, & \text{otherwise} \end{cases}$$

and

$$f_2(p_1, p_2) = \begin{cases} p_2, & \text{if } p_2 \leq p_1; \\ p_2 - c, & \text{otherwise.} \end{cases}$$

This game can be interpreted as a duopoly, when each firm has committed to pay brand-loyal customers a fixed amount c if the other firm beats its price. The payoff functions are neither continuous, nor quasiconcave. However the aggregator function H assembled from f_1 and f_2 is diagonally transfer continuous and diagonally transfer quasiconcave. Therefore there is at least one Nash equilibrium.

3 Uniqueness of Nash Equilibrium

Because Nash equilibra are fixed points of the point-to-set mapping $\mathbf{g} : S \mapsto S$ defined in (3), any result on the uniqueness of fixed points can be directly applied to establish the uniqueness of Nash equilibria.

Assume first that $\mathbf{g}(\mathbf{x})$ is single valued. Function $\mathbf{g}$ is called a **contraction** if there is a constant $\varepsilon \in [0, 1)$ such that

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \leq \varepsilon \cdot \|\mathbf{x} - \mathbf{y}\| \tag{13}$$

for all $\mathbf{x}, \mathbf{y} \in S$, where $\| \cdot \|$ is a vector norm. Under this condition, there is at most one fixed point of function $\mathbf{g}$. On the contrary, assume that $\mathbf{x}$ and $\mathbf{y}$ are both fixed points, then $\mathbf{x} = \mathbf{g}(\mathbf{x})$ and $\mathbf{y} = \mathbf{g}(\mathbf{y})$, so

$$\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \leq \varepsilon \cdot \|\mathbf{x} - \mathbf{y}\| \tag{14}$$

which cannot hold for $\mathbf{x} \neq \mathbf{y}$.

Introduce next function $\mathbf{G}(\mathbf{x}) = \mathbf{x} - \mathbf{g}(\mathbf{x})$, then the equilibrium is clearly unique, if G is one-to-one. In the mathematical literature, there are several conditions that guarantee that G is one-to-one. Assuming that $\mathbf{G}$ is continuously differentiable, the most frequently applied conditions are as follows. Let $\mathbf{J}(\mathbf{x})$ denote the Jacobian of $\mathbf{G}(\mathbf{x})$.

(i) All leading principle minors of $\mathbf{J}(\mathbf{x})$ are positive (that is, $\mathbf{J}(\mathbf{x})$ is a P-matrix) (see [5]);
(ii) All leading principle minors of $\mathbf{J}(\mathbf{x})$ are negative (that is, $\mathbf{J}(\mathbf{x})$ is an N-matrix) (see [6]);
(iii) Matrix $\mathbf{J}(\mathbf{x}) + \mathbf{J}(\mathbf{x})^T$ is negative (or positive) semidefinite, and between any pair $\mathbf{x}^1 \neq \mathbf{x}^2$ of points there is a point $\mathbf{x}^0$ such that $\mathbf{J}(\mathbf{x}^0) + \mathbf{J}(\mathbf{x}^0)^T$ is negative (or positive) definite (see [12]).

Assume next that $\mathbf{g}(\mathbf{x})$ is set valued. Mapping $\mathbf{G}$ is called **strictly monotone** if for all $\mathbf{x}^1 \neq \mathbf{x}^2$ and $\mathbf{y}^1 \in \mathbf{G}(\mathbf{x}^1)$ and $\mathbf{y}^2 \in \mathbf{G}(\mathbf{x}^2)$,

$$(\mathbf{x}^1 - \mathbf{x}^2)^T (\mathbf{y}^1 - \mathbf{y}^2) > 0. \tag{15}$$

Under condition (15), we cannot have multiple Nash equilibrium. Assume that $\mathbf{x}^1 \neq \mathbf{x}^2$ are both equilibria, then $\mathbf{x}^1 \in \mathbf{g}(\mathbf{x}^1)$ and $\mathbf{x}^2 \in \mathbf{g}(\mathbf{x}^2)$, so we may select $\mathbf{y}^1 = \mathbf{0}$ and $\mathbf{y}^2 = \mathbf{0}$. Then the left-hand side of (15) is zero, which is a contradiction. Notice that it is sufficient to assume that for $\mathbf{x}^1 \neq \mathbf{x}^2$, $\mathbf{x}^1 - \mathbf{x}^2$ must not be perpendicular to $\mathbf{y}^1 - \mathbf{y}^2$ with any $\mathbf{y}^1 \in \mathbf{G}(\mathbf{x}^1)$ and $\mathbf{y}^2 \in \mathbf{G}(\mathbf{x}^2)$.

More complex uniqueness conditions can be given if the payoff functions are continuously differentiable and the strategy sets S_k are defined by a finite set of continuously differentiable inequalities. Consider game $\{\, n; S_1, \ldots, S_n, S; f_1, \ldots, f_n \,\}$ in which $S = S_1 \times S_2 \times \cdots \times S_n$ and for all k,

(i) $S_k = \{\, x_k \in R^{m_k} \mid q_k(x_k) \geq 0 \,\}$ is nonempty, q_k is continuously differentiable in an open set containing S_k, and all components of q_k are concave;
(ii) There exists an $\bar{x}_k$ such that $q_k(\bar{x}_k) > 0$;
(iii) Payoff function f_k is twice continuously differentiable in an open set containing S.

Let now $\mathbf{x}^*$ be a Nash equilibrium, then for all k,

$$x_k^* = \arg\max \left\{\, f_k \left(\mathbf{x}_{-k}^*, x_k\right) \mid x_k \in S_k \,\right\}$$

so the Kuhn–Tucker necessary conditions (see, for example, [8]) imply that there exists a nonnegative vector u_k^* such that

$$\begin{aligned} \bigtriangledown_k f_k(\mathbf{x}^*) + u_k^{*T} \bigtriangledown_k q_k(x_k^*) &= 0 \\ u_k^{*T} q_k(x_k^*) &= 0 \end{aligned} \tag{16}$$

where $\bigtriangledown_k f_k$ is the gradient (as a row) vector of function f_k with respect to x_k, and $\bigtriangledown_k q_k$ is the Jacobian matrix of q_k. If in addition, f_k is concave in x_k with any fixed $\mathbf{x}_{-k}$, then the Kuhn–Tucker conditions (16) are also sufficient.

Introduce with some nonnegative vector $\mathbf{r} \in R^n$ the following function $h : S \mapsto R^M$

$$\mathbf{h}(\mathbf{x}, \mathbf{r}) = \begin{pmatrix} r_1 \bigtriangledown_1 f_1(\mathbf{x}) \\ r_2 \bigtriangledown_2 f_2(\mathbf{x}) \\ \vdots \\ r_n \bigtriangledown_n f_n(\mathbf{x}) \end{pmatrix} \tag{17}$$

where $M = m_1 + m_2 + \cdots + m_n$, m_k being the dimension of strategy vector x_k for $k = 1, 2, \ldots, n$. The n-person game is said to be **diagonally strictly concave** if with some $\mathbf{r} \geq \mathbf{0}$,

$$\left(\mathbf{x}^1 - \mathbf{x}^0\right)^T \left(\mathbf{h}\left(\mathbf{x}^1, \mathbf{r}\right) - \mathbf{h}\left(\mathbf{x}^0, \mathbf{r}\right)\right) < 0. \tag{18}$$

Notice that condition (18) means that $-\mathbf{h}$ is strictly monotone in the sense of (15).

Theorem 7. *([13]) Assume that conditions (i)–(iii) hold, and the game is diagonally strictly concave. Then the game has at most one Nash equilibrium.*

Proof. Assume that $\mathbf{x}^0 = \left(x_1^0, \ldots, x_n^0\right)$ and $\mathbf{x}^1 = \left(x_1^1, \ldots, x_n^1\right)$ are both equilibria. Then from (16),

$$\begin{aligned} \triangledown_k f_k\left(\mathbf{x}^l\right) + {u_k^l}^T \triangledown_k q_k\left(x_k^l\right) &= 0 \\ {u_k^l}^T q_k\left(x_k^l\right) &= 0 \end{aligned} \tag{19}$$

for $l = 0, 1$. If the dimension of q_k (as well as that of u_k^l) is p_k, then the first equation can be rewritten as

$$\triangledown_k f_k\left(\mathbf{x}^l\right) + \sum_{j=1}^{p_k} u_{kj}^l \triangledown_k q_{kj}\left(x_k^l\right) = 0 \tag{20}$$

where u_{kj}^l and q_{kj} denote the j^{th} component of u_k^l and q_k. Multiplying (20) by $r_k(x_k^1 - x_k^0)^T$ for $l = 0$ and by $r_k(x_k^0 - x_k^1)^T$ for $l = 1$ and adding the resulted equations for $k = 1, 2, \ldots, n$, we get

$$0 = \left\{\left(\mathbf{x}^1 - \mathbf{x}^0\right)^T \mathbf{h}\left(\mathbf{x}^0, \mathbf{r}\right) + \left(\mathbf{x}^0 - \mathbf{x}^1\right)^T \mathbf{h}\left(\mathbf{x}^1, \mathbf{r}\right)\right\}$$

$$+ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{p_k} r_k \left[u_{kj}^0 \left(x_k^1 - x_k^0\right)^T \triangledown_k q_{kj}\left(x_k^0\right) + u_{kj}^1 \left(x_k^0 - x_k^1\right)^T \triangledown_k q_{kj}\left(x_k^1\right) \right] \right\}.$$

The first term is positive as the consequence of assumption (18), therefore the second (summation) term must be negative. By the concavity of functions q_{kj} we have

$$0 > \sum_{k=1}^{n} \left\{ \sum_{j=1}^{p_k} r_k \left[u_{kj}^0 \left(q_{kj}\left(x_k^1\right) - q_{kj}\left(x_k^0\right)\right) + u_{kj}^1 \left(q_{kj}\left(x_k^0\right) - q_{kj}\left(x_k^1\right)\right) \right] \right\}.$$

Using the second equation of (19), we get an obvious contradiction:

$$0 > \sum_{k=1}^{n} \left\{ \sum_{j=1}^{p_k} r_k \left[u_{kj}^0 q_{kj}\left(x_k^1\right) + u_{kj}^1 q_{kj}\left(x_k^0\right) \right] \right\} \geq 0. \tag{21}$$

■

Example 8 ([4]). Consider a quadratic game with $S_k = \left\{ \mathbf{x}_k | \mathbf{x}_k \geq \mathbf{0}, \mathbf{1}^T \mathbf{x}_k = 1 \right\}$ $(k = 1, 2, \ldots, n)$ being a simplex in R^{m_k} and with payoff functions

$$f_k(\mathbf{x}) = \sum_{j=1}^{n} \left[\mathbf{c}_{kj}^T + \mathbf{x}_k^T \mathbf{C}_{kj} \right] \mathbf{x}_j \tag{22}$$

where $\mathbf{c}_{kj}^T$ is a constraint row vector and $\mathbf{C}_{kj}$ is a constant matrix. It is easy to see that the Jacobian of $\mathbf{h}(\mathbf{x}, \mathbf{r})$ has the special form

$$\mathbf{J}(\mathbf{x}, \mathbf{r}) = \mathbf{DC} \tag{23}$$

with

$$\mathbf{C} = \begin{pmatrix} 2\mathbf{C}_{11} & \mathbf{C}_{12} & \ldots & \mathbf{C}_{1n} \\ \mathbf{C}_{21} & 2\mathbf{C}_{22} & \ldots & \mathbf{C}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{n1} & \mathbf{C}_{n2} & \ldots & 2\mathbf{C}_{nn} \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} r_1 \mathbf{I}_{m_1} & 0 & \ldots & 0 \\ 0 & r_2 \mathbf{I}_{m_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_n \mathbf{I}_{m_n} \end{pmatrix}$$

where $\mathbf{I}_{m_k}$ denotes the $m_k \times m_k$ identity matrix. It is known (see, for example, [12]) that condition (18) holds if $\mathbf{DC} + \mathbf{C}^T \mathbf{D}$ is negative definite.

4 Computation of Nash Equilibria

There are several different concepts in computing Nash equilibria. In this section, we will outline the three most frequently used method families: solution for fixed points, reduction to variational inequalities, and transforming the equilibrium problem to an optimization problem.

Let $\mathbf{g}$ denote the best response mapping (3), then $\mathbf{x}^*$ is a Nash equilibrium if and only if $\mathbf{x}^* \in \mathbf{g}(\mathbf{x}^*)$. If $\mathbf{g}(\mathbf{x})$ is single valued, then $\mathbf{x}^*$ is a fixed point if and only if $\mathbf{x}^* = \mathbf{g}(\mathbf{x}^*)$. In this case, we have a (usually nonlinear) system of algebraic equations to solve. The numerical analysis literature offers a large variety of methods (see, for example, [16]) including the Newton method, several variants of the gradient method, fixed point iteration, etc. If $\mathbf{g}(\mathbf{x})$ is a set, then it is usually described by a system of $\mathbf{x}$-dependent inequalities, and we have to find a feasible solution of these inequalities. With surplus and slack variables, we are able to rewrite the inequalities into equations, so any method for solving systems of algebraic equations can be useful again.

Assume next that all conditions of Theorem 7 are satisfied and all payoff functions f_k are concave in x_k with any fixed value of $\mathbf{x}_{-k}$. Introduce the generalized "aggregator" function

$$H_r(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{n} r_k f_k(\mathbf{x}_{-k}, y_k) \tag{24}$$

with $\mathbf{r} = (r_k) > 0$. It is easy to see that $\mathbf{x}^* \in S$ is an equilibrium if and only if

$$H_r(\mathbf{x}^*,\mathbf{x}^*) \geq H_r(\mathbf{x}^*,\mathbf{y}) \tag{25}$$

for all $\mathbf{y} \in S$.

Theorem 8. *([20]) A vector* $\mathbf{x}^*$ *satisfies (25) if and only if*

$$\max_{\mathbf{x}\in S} \left\{ \mathbf{h}(\mathbf{x}^*,\mathbf{r})^T (\mathbf{x}-\mathbf{x}^*) \right\} = 0 \tag{26}$$

where $\mathbf{h}(\mathbf{x},\mathbf{r})$ *is defined by (17).*

Proof. Assume first that $\mathbf{x}^*$ satisfies (25). Because $H_r(\mathbf{x}^*,\mathbf{y})$ has its maximum at $\mathbf{y} = \mathbf{x}^*$, for all $\mathbf{y} \in S$

$$\nabla_y H_r(\mathbf{x}^*,\mathbf{x}^*)(\mathbf{y}-\mathbf{x}^*) \leq 0$$

which is (26).

Assume next that (26) is satisfied. Then the concavity of f_k in x_k and (18) imply that

$$\begin{aligned} &H_r(\mathbf{x}^*,\mathbf{x}^*) - H_r(\mathbf{x}^*,\mathbf{y}) \geq \mathbf{h}(\mathbf{y},\mathbf{r})^T(\mathbf{x}^*-\mathbf{y}) \geq \\ &\mathbf{h}(\mathbf{y},\mathbf{r})^T(\mathbf{x}^*-\mathbf{y}) + \mathbf{h}(\mathbf{x}^*,\mathbf{r})^T(\mathbf{y}-\mathbf{x}^*) > 0. \end{aligned}$$

■

Note that (26) is a **variational inequality**, so by finding its solutions, the Nash equilibria are obtained.

Theorem 9. *([20]) A vector* $\mathbf{x}^*$ *satisfies relation (25) if and only if* $(\mathbf{x}^*,\mathbf{x}^*)$ *is a Nash equilibrium of the two-person zero-sum game with sets* $S_1 = S_2 = S$ *of strategies, and payoff functions* $f_1 = f$, $f_2 = -f$ *with* $f(\mathbf{x},\mathbf{y}) = \mathbf{h}(\mathbf{y},\mathbf{r})^T$ $(\mathbf{x}-\mathbf{y})$.

Proof. Assume first that $\mathbf{x}^*$ satisfies (25). Then by Theorem 8, for all $\mathbf{x} \in S$,

$$\mathbf{h}(\mathbf{x}^*,\mathbf{r})^T (\mathbf{x}-\mathbf{x}^*) \leq 0, \tag{27}$$

that is

$$f(\mathbf{x},\mathbf{x}^*) \leq 0 = f(\mathbf{x}^*,\mathbf{x}^*).$$

We will next prove that $f(\mathbf{x}^*, \mathbf{x}) \geq 0$ for all $\mathbf{x} \in S$. Assume in contrary that there exists an $\mathbf{y} \in S$ such that $f(\mathbf{x}^*, \mathbf{y}) < 0$.

Then by (18),

$$0 > f\left(\mathbf{x}^*, \mathbf{y}\right) = \mathbf{h}\left(\mathbf{y}, \mathbf{r}\right)^T \left(\mathbf{x}^* - \mathbf{y}\right) > \mathbf{h}\left(\mathbf{x}^*, \mathbf{r}\right)^T \left(\mathbf{x}^* - \mathbf{y}\right),$$

that is

$$\mathbf{h}\left(\mathbf{x}^*, \mathbf{r}\right)^T \left(\mathbf{y} - \mathbf{x}^*\right) > 0$$

contradicting (27).

Assume next that $(\mathbf{x}^*, \mathbf{x}^*)$ is an equilibrium of the two-person zero-sum game. Then for all $\mathbf{x}, \mathbf{y} \in S$,

$$f\left(\mathbf{x}, \mathbf{x}^*\right) \leq f\left(\mathbf{x}^*, \mathbf{x}^*\right) \leq f\left(\mathbf{x}^*, \mathbf{y}\right).$$

Notice that the first inequality can be rewritten as

$$\mathbf{h}\left(\mathbf{x}^*, \mathbf{r}\right)^T \left(\mathbf{x} - \mathbf{x}^*\right) \leq 0$$

therefore Theorem 8 implies that $\mathbf{x}^*$ satisfies (25). ■

Consider now the following iteration algorithm to solve the variational inequality (26). Let $\mathbf{x}^{(1)} \in S$ be an arbitrary vector and solve the optimization problem

$$\begin{aligned} &\max f\left(\mathbf{x}, \mathbf{x}^{(1)}\right) \\ &s.t.\ \ \mathbf{x} \in S. \end{aligned} \tag{28}$$

Let $\mathbf{x}^{(2)}$ be a solution and define $\mu_1 = f(\mathbf{x}^{(2)}, \mathbf{x}^{(1)})$. If $\mu_1 = 0$, then $\mathbf{x}^{(1)}$ is an equilibrium, so the procedure terminates. Otherwise $\mu_1 > 0$. The general k^{th} step is the following. We already have $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ and scalars $\mu_1, \ldots, \mu_{k-1} > 0$. Then the new $\mathbf{x}^{(k+1)}$ and μ_k are solutions of the problem

$$\begin{aligned} &\max \mu \\ &s.t.\ f\left(\mathbf{x}, \mathbf{x}^{(i)}\right) \geq \mu, \quad (i = 1, 2, \ldots, k) \\ &\mathbf{x} \in S. \end{aligned} \tag{29}$$

Notice that $f\left(\mathbf{x}^{(k)}, \mathbf{x}^{(i)}\right) \geq \mu_{k-1} \geq 0, \quad (i = 1, 2, \ldots, k-1)$ and $f\left(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}\right) = 0$, therefore $\mu_k \geq 0$.

The convergence of the algorithm is guaranteed by the following result.

Theorem 10. *([20]) There is a subsequence of the iteration sequence* $\left\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots\right\}$ *that converges to the unique Nash equilibrium.*

Proof. The proof consists of several steps.

1. We first show that $\mu_k \to 0$ as $k \to \infty$. Because at each step an additional constraint is added, sequence $\{\mu_k\}$ is monotonic and bounded, therefore it is convergent. Because S is compact, there is a convergent subsequence $\left\{\mathbf{x}^{(k_i)}\right\}$ of the iteration sequence $\left\{\mathbf{x}^{(k)}\right\}$. Clearly

$$\begin{aligned} 0 \le \mu_{k_i-1} &= \max\left\{\min_{1\le k\le k_i-1} \mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}-\mathbf{x}^{(k)}\right) \mid \mathbf{x}\in S\right\} \\ &= \min_{1\le k\le k_i-1} \mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}^{(k_i)}-\mathbf{x}^{(k)}\right) \\ &\le \mathbf{h}\left(\mathbf{x}^{(k_{i-1})},\mathbf{r}\right)^T\left(\mathbf{x}^{(k_i)}-\mathbf{x}^{(k_{i-1})}\right) \to 0 \qquad \text{as } i\to\infty \end{aligned}$$

   implying that $\mu_{k_i-1} \to 0$, and because sequence $\{\mu_k\}$ is monotonic, the entire sequence must converge to zero.

2. Consider function

$$\delta(t) = \min\left\{(\mathbf{h}(\mathbf{x},\mathbf{r})-\mathbf{h}(\mathbf{y},\mathbf{r}))^T(\mathbf{y}-\mathbf{x}) \mid \|\mathbf{x}-\mathbf{y}\|\ge t,\ \mathbf{x},\mathbf{y}\in S\right\} \quad (30)$$

   which exists, because S is compact and $\|\mathbf{x}-\mathbf{y}\| \ge t$ is a closed inequality, furthermore it is positive as the consequence of assumption (18). Let now $\mathbf{x}^*$ be an equilibrium. Define indices k_i according to

$$\delta\left(\left\|\mathbf{x}^{(k_i)}-\mathbf{x}^*\right\|\right) = \min_{1\le k\le i}\delta\left(\left\|\mathbf{x}^{(k)}-\mathbf{x}^*\right\|\right),\ (i=1,2,\dots)$$

   then for $k=1,2,\dots,i$, we have

$$\begin{aligned} \delta\left(\left\|\mathbf{x}^{(k_i)}-\mathbf{x}^*\right\|\right) &\le \left[\mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)-\mathbf{h}\left(\mathbf{x}^*,\mathbf{r}\right)\right]^T\left(\mathbf{x}^*-\mathbf{x}^{(k)}\right) \\ &= \mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}^*-\mathbf{x}^{(k)}\right)-\mathbf{h}\left(\mathbf{x}^*,\mathbf{r}\right)^T\left(\mathbf{x}^*-\mathbf{x}^{(k)}\right) \\ &\le \mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}^*-\mathbf{x}^{(k)}\right) \end{aligned}$$

   because $\mathbf{h}\left(\mathbf{x}^*-\mathbf{r}\right)^T\left(\mathbf{x}^{(k)}-\mathbf{x}^*\right)\le 0$ by Theorem 9. Therefore

$$\begin{aligned} \delta\left(\left\|\mathbf{x}^{(k_i)}-\mathbf{x}^*\right\|\right) &\le \min_{1\le k\le i}\mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}^*-\mathbf{x}^{(k)}\right) \\ &\le \max_{\mathbf{x}\in S}\min_{1\le k\le i}\mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}-\mathbf{x}^{(k)}\right) \\ &= \min_{1\le k\le i}\mathbf{h}\left(\mathbf{x}^{(k)},\mathbf{r}\right)^T\left(\mathbf{x}^{(i+1)}-\mathbf{x}^{(k)}\right) \\ &= \mu_i \to 0 \qquad \text{as } i\to\infty. \end{aligned}$$

   Consequently, $\delta\left(\left\|\mathbf{x}^{(k_i)}-\mathbf{x}^*\right\|\right)\to 0 \quad$ as $i\to\infty$.

3. Function δ clearly satisfies the following properties:
   (i) $\delta(t)$ is continuous in t;
   (ii) $\delta(t) > 0$ as $t > 0$, because the game is diagonally strictly concave.

   Therefore $\delta(t_i) \to 0$ implies that $t_i \to 0$ for any convergent sequence $\{t_i\}$, which implies that $\left\|\mathbf{x}^{(k_i)} - \mathbf{x}^*\right\| \to 0$ as $i \to \infty$, so $\mathbf{x}^{(k_i)} \to \mathbf{x}^*$.

■

Assume next that all conditions of Theorem 7 are satisfied. If $\mathbf{x}^*$ is an equilibrium, then the Kuhn–Tucker conditions show that with some nonnegative vectors $\mathbf{u}_k^*$ $(1 \le k \le n)$, relations (16) hold. Introduce the notation

$$\psi_k(\mathbf{x}, u_k) = \bigtriangledown_k f_k(\mathbf{x}) + u_k^T \bigtriangledown_k q_k(x_k) \tag{31}$$

and consider the following optimization problem

$$\left.\begin{array}{ll} \min \sum_{k=1}^n u_k^T q_k(x_k) & \\ s.t. & u_k \ge 0 \\ & q_k(x_k) \ge 0 \\ & \psi_k(\mathbf{x}, u_k) = 0 \end{array}\right\} \quad k = 1, 2, \ldots, n. \tag{32}$$

Theorem 11. *If $\mathbf{x}^*$ is an equilibrium, then there exist nonnegative vectors $u_k^*(k = 1, 2, \ldots, n)$ such that $(\mathbf{x}^*, u_1^*, \ldots, u_n^*)$ is an optimal solution of problem (32).*

Proof. If $\mathbf{x}^*$ is an equilibrium, then the Kuhn–Tucker necessary conditions (16) are satisfied, so $(\mathbf{x}^*, u_1^*, \ldots, u_n^*)$ is a feasible solution of problem (32) with zero objective function value. Because at any feasible solution the objective function is nonnegative, $(\mathbf{x}^*, u_1^*, \ldots, u_n^*)$ must be optimal. ■

Theorem 12. *If in addition, f_k is concave in x_k with any fixed $\mathbf{x}_{-k}$, then for any optimal solution $(\mathbf{x}, u_1, \ldots, u_n)$ of problem (32), $\mathbf{x}$ is an equilibrium.*

Proof. Under the additional condition, the Kuhn–Tucker conditions are also sufficient. ■

The application of problem (32) will be illustrated in the following examples.

Example 9 (Bimatrix games). Assume $n = 2$, linear strategy sets

$$\begin{aligned} S_1 &= \left\{\mathbf{x}_1 \,|\, \mathbf{x}_1 = \left(x_1^{(i)}\right) \in R^m,\ \mathbf{x}_1 \ge \mathbf{0}, \textstyle\sum_i x_1^{(i)} = 1\right\} \\ S_2 &= \left\{\mathbf{x}_2 \,|\, \mathbf{x}_2 = \left(x_2^{(i)}\right) \in R^n,\ \mathbf{x}_2 \ge \mathbf{0}, \textstyle\sum_i x_2^{(i)} = 1\right\}, \end{aligned} \tag{33}$$

and quadratic payoff functions

$$f_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1^T \mathbf{A} \mathbf{x}_2 \quad \text{and} \quad f_2(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1^T \mathbf{B} \mathbf{x}_2 \tag{34}$$

where $\mathbf{A}$ and $\mathbf{B}$ are given $m \times n$ constant real matrices.

In this case, we may select

$$q_1(\mathbf{x}_1) = \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_1^{(m)} \\ x_1^{(1)} + \cdots + x_1^{(m)} - 1 \\ -x_1^{(1)} - \cdots - x_1^{(m)} + 1 \end{pmatrix}$$

with Jacobian

$$\nabla_1 q_1(\mathbf{x}_1) = \begin{pmatrix} \mathbf{I} \\ \mathbf{1}^T \\ -\mathbf{1}^T \end{pmatrix}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix and $\mathbf{1}$ is the m-element vector, all components of which are equal to one. Similarly,

$$q_2(\mathbf{x}_2) = \begin{pmatrix} x_2^{(1)} \\ \vdots \\ x_2^{(n)} \\ x_2^{(1)} + \cdots + x_2^{(n)} - 1 \\ -x_2^{(1)} - \cdots - x_2^{(n)} + 1 \end{pmatrix}$$

with

$$\nabla_2 q_2(\mathbf{x}_2) = \begin{pmatrix} \mathbf{I} \\ \mathbf{1}^T \\ -\mathbf{1}^T \end{pmatrix}.$$

The objective function of problem (32) has now the special form

$$\sum_{i=1}^{m} u_1^{(i)} x_1^{(i)} + u_1^{(m+1)} \left(\sum_i x_1^{(i)} - 1 \right) + u_1^{(m+2)} \left(- \sum_i x_1^{(i)} + 1 \right)$$

$$+ \sum_{j=1}^{n} u_2^{(j)} x_2^{(j)} + u_2^{(n+1)} \left(\sum_j x_2^{(j)} - 1 \right) + u_2^{(n+2)} \left(- \sum_j x_2^{(j)} + 1 \right).$$

By introducing the notation

$$\alpha = u_1^{(m+2)} - u_1^{(m+1)} \quad \text{and} \quad \beta = u_2^{(n+2)} - u_2^{(n+1)},$$

problem (32) can be further simplified as

$$\begin{aligned} \min\ & \mathbf{u}_1^T\mathbf{x}_1 + \mathbf{u}_2^T\mathbf{x}_2 - \alpha\left(\mathbf{1}^T\mathbf{x}_1 - 1\right) - \beta\left(\mathbf{1}^T\mathbf{x}_2 - 1\right) \\ \text{s.t}\ & \mathbf{u}_1 \geq \mathbf{0},\ \mathbf{u}_2 \geq \mathbf{0} \\ & \mathbf{x}_1 \geq \mathbf{0},\ \mathbf{x}_2 \geq \mathbf{0} \\ & \mathbf{1}^T\mathbf{x}_1 = 1,\ \mathbf{1}^T\mathbf{x}_2 = 1 \\ & \mathbf{x}_2^T\mathbf{A}^T + \mathbf{u}_1^T + \left(u_1^{(m+1)} - u_1^{(m+2)}\right)\mathbf{1}^T = \mathbf{0}^T \\ & \mathbf{x}_1^T\mathbf{B} + \mathbf{u}_2^T + \left(u_2^{(n+1)} - u_2^{(n+2)}\right)\mathbf{1}^T = \mathbf{0}^T. \end{aligned} \tag{35}$$

From the last two constraints

$$\mathbf{u}_1^T = \alpha\mathbf{1}^T - \mathbf{x}_2^T\mathbf{A}^T \quad \text{and} \quad \mathbf{u}_2^T = \beta\mathbf{1}^T - \mathbf{x}_1^T\mathbf{B}$$

so the objective function of (35) is the following:

$$\begin{aligned} & (-\mathbf{x}_2^T\mathbf{A}^T + \alpha\mathbf{1}^T)\mathbf{x}_1 + (-\mathbf{x}_1^T\mathbf{B} + \beta\mathbf{1}^T)\mathbf{x}_2 - \alpha(\mathbf{1}^T\mathbf{x}_1 - 1) - \beta(\mathbf{1}^T\mathbf{x}_2 - 1) \\ & = -\mathbf{x}_1^T\mathbf{A}\mathbf{x}_2 - \mathbf{x}_1^T\mathbf{B}\mathbf{x}_2 + \alpha + \beta. \end{aligned}$$

Hence we have a quadratic optimization problem with linear constraints:

$$\begin{aligned} \max\ & \mathbf{x}_1^T(\mathbf{A} + \mathbf{B})\mathbf{x}_2 - \alpha - \beta \\ \text{s.t.}\ & \mathbf{x}_1 \geq \mathbf{0}, \quad \mathbf{x}_2 \geq \mathbf{0} \\ & \mathbf{1}^T\mathbf{x}_1 = 1, \quad \mathbf{1}^T\mathbf{x}_2 = 1 \\ & \mathbf{A}\mathbf{x}_2 \leq \alpha\mathbf{1}, \quad \mathbf{B}^T\mathbf{x}_1 \leq \beta\mathbf{1}. \end{aligned} \tag{36}$$

As a numerical example, select

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}.$$

Therefore, problem (36) is the following

$$\begin{aligned} \max\ & 3x_1^{(1)}x_2^{(1)} - 2x_1^{(1)}x_2^{(2)} - 2x_1^{(2)}x_2^{(1)} + 3x_1^{(2)}x_2^{(2)} - \alpha - \beta \\ \text{s.t.}\ & x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)} \geq 0 \\ & x_1^{(1)} + x_1^{(2)} = 1, \quad x_2^{(1)} + x_2^{(2)} = 1 \\ & 2x_2^{(1)} - x_2^{(2)} \leq \alpha \\ & -x_2^{(1)} + x_2^{(2)} \leq \alpha \\ & x_1^{(1)} - x_1^{(2)} \leq \beta \\ & -x_1^{(1)} + 2x_1^{(2)} \leq \beta. \end{aligned}$$

A computer program is applied to find three optimal solutions shown in Table 1:

Table 1. Solution of Example 9

$\mathbf{x}_1^T$	(1,0)	(0,1)	(3/5, 2/5)
$\mathbf{x}_2^T$	(1,0)	(0,1)	(2/5, 3/5)
α	2	1	1/5
β	1	2	1/5

Example 10 (Matrix games). Consider a special bimatrix game in which $\mathbf{B} = -\mathbf{A}$, that is, the game is zero sum.

In this case, the optimization problem (36) is linear and it can be broken up to two linear programming problems:

$$\begin{array}{ll} \min & \alpha \\ \text{s.t.} & \mathbf{A}\mathbf{x}_2 \leq \alpha \mathbf{1} \\ & \mathbf{1}^T \mathbf{x}_2 = 1 \\ & \mathbf{x}_2 \geq \mathbf{0}. \end{array} \qquad \begin{array}{ll} \min & \beta \\ \text{s.t.} & \mathbf{A}^T\mathbf{x}_1 \geq -\beta \mathbf{1} \\ & \mathbf{1}^T \mathbf{x}_1 = 1 \\ & \mathbf{x}_1 \geq \mathbf{0}. \end{array} \tag{37}$$

Example 11. Consider now again the single-product oligopoly game without product differentiation (Example 4 gave the definition and notation). We may select

$$S_k = \{x_k | x_k \geq 0, L_k - x_k \geq 0\}$$

so

$$q_k(x_k) = \begin{pmatrix} x_k \\ L_k - x_k \end{pmatrix}.$$

Notice that

$$\nabla_k q_k(x_k) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\nabla_k f_k(\mathbf{x}) = p\left(\sum_i x_i\right) + x_k p'\left(\sum_i x_i\right) - C_k'\left(x_k\right).$$

By introducing $\alpha_k = u_k^{(1)} - u_k^{(2)}$ and $\beta_k = u_k^{(2)}$, the objective function of problem (32) becomes

$$\sum_{k=1}^{n} (\alpha_k x_k + \beta_k L_k),$$

and the last constraint can be written as

$$p\left(\sum_i x_i\right) + x_k p'\left(\sum_i x_i\right) - C_k'\left(x_k\right) + \alpha_k = 0$$

from which we have

$$\alpha_k = -p\left(\sum_i x_i\right) - x_k p'\left(\sum_i x_i\right) + C_k'(x_k)$$

and by substituting this expression into the objective function, we have

$$\sum_{k=1}^{n}\left(-x_k\left[p\left(\sum_i x_i\right) + x_k p'\left(\sum_i x_i\right) - C_k'(x_k)\right] + \beta_k L_k\right).$$

Notice that $u_k^{(1)} = \alpha_k + \beta_k$ must be nonnegative, so we have the following optimization problem to solve

$$\begin{aligned} &\max \textstyle\sum_{k=1}^{n}\{x_k([p(\sum_i x_i) + x_k p'(\sum_i x_i) - C_k'(x_k)]) - \beta_k L_k\} \\ &\text{s.t.} \qquad 0 \le x_k \le L_k \\ &\qquad \beta_k \ge max\{0; p(\textstyle\sum_i x_i) + x_k p'(\sum_i x_i) - C_k'(x_k)\}. \end{aligned} \tag{38}$$

As a numerical example, select $n = 3$, $C_k(x_k) = kx_k^3 + x_k$, $L_k = 1\,(k = 1, 2, 3)$, and $p(s) = 2 - 2s - s^2$, where $s = \sum_{k=1}^{3} x_k$.

In this particular case, problem (38) becomes

$$\begin{aligned} &\max \sum_{k=1}^{3}\{x_k(2 - 2s - s^2 - 2x_k - 2sx_k - 3kx_k^2 - 1) - \beta_k\} \\ &\text{s.t.} \quad 0 \le x_k \le 1 \\ &\qquad x_1 + x_2 + x_3 = s \\ &\qquad \beta_k \ge max\{0; 2 - 2s - s^2 - x_k(2 + 2s) - 3kx_k^2 - 1\}. \end{aligned} \tag{39}$$

A computer program gives the optimal solution:

$$x_1^* = 0.1077$$

$$x_2^* = 0.0986$$

$$x_3^* = 0.0919$$

Theorem 11 and Theorem 12 show how to transform an equilibrium problem into an optimization problem under certain conditions. We will next illustrate that for any optimization problem, we can formulate a two-person zero-sum game such that the equilibria of the game provide optimal solutions. Consider therefore the very general optimization problem

$$\begin{aligned} &\max\ f(\mathbf{x}) \\ &\text{s.t.}\ \ \mathbf{x} \in X \\ &\qquad\ \mathbf{g}(\mathbf{x}) \geq \mathbf{0} \end{aligned} \tag{40}$$

where $f : X \mapsto R$, $\mathbf{g} : X \mapsto R^m$ are arbitrary functions, and $X \subseteq R^n$ is an arbitrary (possibly even discrete) set. The Lagrangian of this problem is

$$L(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) + \mathbf{u}^T \mathbf{g}(\mathbf{x}) \tag{41}$$

for all nonnegative vectors $\mathbf{u} \in R^m$. Define now the zero-sum two-person game with strategy sets $S_1 = X$, $S_2 = R^m_+$, and payoff functions $f_1 = L$, $f_2 = -L$.

Theorem 13. *If* $(\mathbf{x}^*, \mathbf{u}^*)$ *is an equilibrium, then* $\mathbf{x}^*$ *is an optimal solution of problem (40).*

Proof. Because $(\mathbf{x}^*, \mathbf{u}^*)$ is an equilibrium, for all $\mathbf{x}$ and $\mathbf{u}$,

$$f(\mathbf{x}^*) + \mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) \geq f(\mathbf{x}) + \mathbf{u}^{*T}\mathbf{g}(\mathbf{x}) \tag{42}$$

$$-\left[f(\mathbf{x}^*) + \mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*)\right] \geq -\left[f(\mathbf{x}^*) + \mathbf{u}^T\mathbf{g}(\mathbf{x}^*)\right]. \tag{43}$$

From (43), we see that $\mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) \leq 0$ with the choice of $\mathbf{u} = \mathbf{0}$. Next we show that $\mathbf{g}(\mathbf{x}^*) \geq 0$. Assume that for a component, $g_i(\mathbf{x}^*) < 0$. Then select sufficiently large value of u_i, then (43) is violated. Therefore $\mathbf{x}^*$ is a feasible solution of (40).

Because $\mathbf{u}^* \geq \mathbf{0}$ and $\mathbf{g}(\mathbf{x}^*) \geq 0$, $\mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) \geq 0$, and comparing this inequality to $\mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) \leq 0$ (which was shown above), we see that

$$\mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) = 0$$

Finally we show that $\mathbf{x}^*$ is optimal. From (42),

$$f(\mathbf{x}^*) = f(\mathbf{x}^*) + \mathbf{u}^{*T}\mathbf{g}(\mathbf{x}^*) \geq f(\mathbf{x}) + \mathbf{u}^{*T}\mathbf{g}(\mathbf{x}) \geq f(\mathbf{x})$$

for any feasible solution $\mathbf{x}$, which shows the optimality of $\mathbf{x}^*$. ■

5 A Dynamic Extension

Repeated games and dynamic extensions of different classes of games were examined by many authors. In this section, a special dynamic oligopoly game will be briefly discussed.

Consider a single-product oligopoly without differentiation (such as the game introduced earlier in Example 4). The marginal profit of firm k is given as

$$\frac{\partial f}{\partial x_k}(\mathbf{x}) = p\left(\sum_{i=1}^{n} x_i\right) + x_k p'\left(\sum_{i=1}^{n} x_i\right) - C_k'(x_k). \tag{44}$$

Assuming continuous timescale, it is realistic to assume that if the marginal profit is positive, then the firm wants to increase its output. If the marginal profit is negative, then the firm wants to decrease its output, and if the marginal profit is zero, then (assuming concavity of f_k in x_k) the output maximizes the profit, so the firm does not want to change output. This adjustment concept can be naturally modeled as follows:

For $k = 1, 2, \ldots, n$,

$$\dot{x}_k(t) = K_k \cdot \left[p\left(\sum_{i=1}^{n} x_i(t)\right) + x_k(t)p'\left(\sum_{i=1}^{n} x_i(t)\right) - C_k'(x_k(t))\right], \tag{45}$$

where K_k is a positive constant for $k = 1, 2, \ldots, n$.

Clearly any steady state of this system is an interior equilibrium, however corner equilibria (when $x_k = 0$ or $x_k = L_k$) are not always steady states of this dynamic system. Most research on dynamic games is interested in the asymptotic behavior of the trajectory as $t \to \infty$. Local asymptotic stability is usually examined by linearization and based on the locations of the eigenvalues of the Jacobian. The Jacobian of the system has a special structure

$$\mathbf{J} = \mathbf{D} + \mathbf{a} \cdot \mathbf{1}^T \tag{46}$$

where

$$\mathbf{D} = \mathrm{diag}(K_1(p'\left(\sum_i x_i\right) - C_1''(x_1)), \ldots, K_n(p'\left(\sum_i x_i\right) - C_n''(x_n)))$$

$$\mathbf{1}^T = (1, 1, \ldots, 1)$$

and

$$\mathbf{a} = \begin{pmatrix} K_1(p'(\sum_i x_i) + x_1 p''(\sum_i x_i)) \\ K_2(p'(\sum_i x_i) + x_2 p''(\sum_i x_i)) \\ \vdots \\ K_n(p'(\sum_i x_i) + x_n p''(\sum_i x_i)) \end{pmatrix}.$$

Conditions (A) and (B) (introduced in Example 4) imply that the diagonal elements of $\mathbf{D}$ are negative and all elements of $\mathbf{a}$ are nonpositive.

For the sake of simplicity, let d_i and a_i denote the i^{th} diagonal element of $\mathbf{D}$ and the i^{th} element of vector $\mathbf{a}$.

The characteristic polynomial of the Jacobian can be given as follows

$$\varphi(\lambda) = \det(\mathbf{D} + \mathbf{a} \cdot \mathbf{1}^T - \lambda\mathbf{I}) = \det(\mathbf{D} - \lambda\mathbf{I}) \cdot \det(\mathbf{I} + (\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{a} \cdot \mathbf{1}^T).$$

Now we will use the fact that for any n-element vectors $\mathbf{u}$ and $\mathbf{v}$, $\det(\mathbf{I} + \mathbf{u}\mathbf{v}^T) = 1 + \mathbf{v}^T\mathbf{u}$, which can be proved by using mathematical induction with respect to size of the vectors. Then

$$\varphi(\lambda) = \prod_{k=1}^{n}(d_i - \lambda) \cdot \left[1 + \sum_{k=1}^{n} \frac{a_i}{d_i - \lambda}\right] = 0. \tag{47}$$

The roots of the first product are all negative, and we will show that all roots of the bracketed term are also real and negative implying the local asymptotic stability of the steady state. Introduce function

$$g(\lambda) = \sum_{i=1}^{n} \frac{a_i}{d_i - \lambda} \tag{48}$$

where we may assume that the d_i values are different, as terms with identical denominator can be written as one term by adding their numerators. Clearly

$$\lim_{\lambda \to \pm\infty} g(\lambda) = 0,$$
$$\lim_{\lambda \to d_i + 0} g(\lambda) = \infty,$$
$$\lim_{\lambda \to d_i - 0} g(\lambda) = -\infty$$

and

$$g'(\lambda) = \sum_{i=1}^{n} \frac{a_i}{(d_i - \lambda)^2} < 0.$$

The graph of this function is shown in Figure 1. Equation $g(\lambda) = -1$ has a solution before d_1 and one solution between each pair (d_i, d_{i+1})

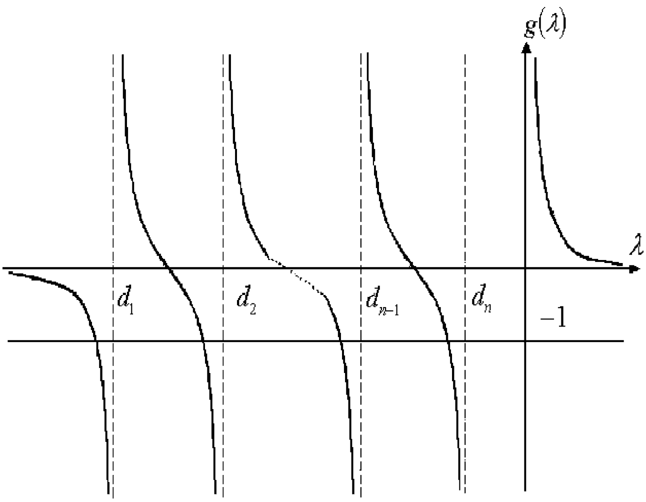

Figure 1. Graph of function $g(\lambda)$

$(i = 1, 2, \ldots, n-1)$. Notice that this equation is equivalent to a polynomial equation of degree n, so there are n real (or complex) roots. We found n real roots, so all roots are real and negative. Hence the steady state is locally asymptotically stable.

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Scalar Asymptotic Contractivity and Fixed Points for Nonexpansive Mappings on Unbounded Sets

George Isac

Department of Mathematics Royal Military College of Canada, STN Forces
Kingston, Ontario, Canada
isac-g@rmc.ca

Abstract Based on the notion of asymptotically contractive mapping due to Penot [16], we propose in this paper a new method for the study of existence of fixed points for nonexpansive mappings defined on unbounded sets.

Key words: fixed-point theory, nonexpansive mappings, scalar asymptotically contractive mapping, scalar asymptotic derivability

1 Introduction

The fixed-point theory is one of the most popular chapters considered in nonlinear functional analysis.

Nonlinear functional analysis is an area of mathematics that has suddenly grown up over the past few decades, influenced by nonlinear problems posed in physics, mechanics, operations research, as well as in economics. In the fixed-point theory, an important chapter is the study of fixed points for nonexpansive mappings.

Nonexpansive mappings are used in many practical problems. Many authors have studied the existence of fixed points for nonexpansive mappings in many papers as for example [1, 2, 4–6, 10–13, 16–21], among others.

The nonexpansivity is related, in some sense, with the contractivity. For comparison of various definitions of contractive mapping, the reader is referred to the classic paper [17].

Generally, in many papers, the existence of fixed points for nonexpansive mappings have been considered with respect to bounded closed convex sets, or with respect to compact convex sets.

In 1992, Luc [13] presented a fixed-point theorem for nonexpansive mappings with respect to unbounded sets using the notion of recessive compactness.

Using the notion of *asymptotically contractive mapping*, Penot [16] generalized to unbounded sets some fixed-point theorems proved some time ago by Browder [1], Göhde [6], Kirk [10], and Luc [13].

Inspired by Penot's results, we present in this paper a new method for the study of existence of fixed points, for nonexpansive mappings, defined on unbounded sets. This method is based on the notion of "scalar asymptotically contractive mapping."

This method, which is somewhat related to the *scalar asymptotic derivability* [8,9], seems to be an interesting method and it opens a new research direction in the study of existence of fixed points for nonexpansive mappings defined on a closed unbounded convex set.

2 Preliminaries

We denote by $(E, \|\cdot\|)$ a Banach space and by $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space. Let $C \subset E$ be a nonempty unbounded closed convex set and $h : C \to E$ be a mapping. We recall some known definitions. We say that h is *nonexpansive* if and only if, for any $x, y \in C$ we have $\|h(x) - h(y)\| \leq \|x - y\|$. The mapping h is said to be *ρ-Lipschitzian*, if there exists a constant $\rho > 0$ such that for any $x, y \in C$ we have $\|h(x) - h(y)\| \leq \rho\|x - y\|$. If $0 < \rho < 1$, then in this case we say that h is a contractive mapping.

We recall that a Banach space $(E, \|\cdot\|)$ is *uniformly convex*, if and only if for every $\epsilon \in [0, 2[$ there is a real number $\delta(\epsilon) \in]0, 1]$ such that whenever $\|x\| \leq r$, $\|y\| \leq r$, $\|x - y\| \geq \epsilon r$, $x, y \in E$, $r > 0$, then it follows that

$$\left\|\frac{x+y}{2}\right\| \leq (1 - \delta(\epsilon))r.$$

Any Hilbert space is uniformly convex and any $L_p(\Omega)$ space with $1 < p < \infty$ and Ω a domain in $\mathbb{R}^n$ is uniformly convex. For more details and results about uniformly convex Banach spaces, the reader is referred to [3,21], and [22].

We say that a mapping $h : C \to E$ is *demi-closed* on C if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ weakly convergent to an element $x^* \in E$ and such that the sequence $\{h(x_n)\}_{n\in\mathbb{N}}$ is convergent in norm to an element y^* we have that $x^* \in C$ and $h(x^*) = y^*$. The demi-closedness is related to the notion of strongly continuous mapping [3,22].

It is known that, if h is nonexpansive and E is uniformly convex, then $I - h$ is demi-closed. (We denoted by I the identity mapping.)

For a proof of this result, see ([2], Theorem 8.4) and ([22], Proposition 10.9). It is remarked in [16] that the boundedness of C used in [2] and [22] is not necessary.

We note that in some papers of Russian mathematicians, the *demi-closed operator* is *called regular operator.*

3 Scalar Asymptotically Contractive Mappings in Hilbert Spaces

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $C \subseteq H$ be a nonempty unbounded closed convex set.

Definition 1. *We say that a mapping $f : C \to H$ is scalar asymptotically contractive on C if and only if there exists an element $x_0 \in C$ such that*

$$\limsup_{x \in C,\ \|x\| \to \infty} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} < 1.$$

We have the following result.

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $C \subset H$ be an unbounded closed convex subset. Let $f : C \to H$ be a mapping such that the following assumptions are satisfied:*

(i) f is nonexpansive,
(ii) $f(C) \subseteq C$,
(iii) f is scalar asymptotically contractive on C. Then f has a fixed point in C.

Proof. Let $x_0 \in C$ be the element defined in assumption (iii) and let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence in $]0, 1[$ such that $\lim\limits_{n \to \infty} \alpha_n = 0$. For any $n \in \mathbb{N}$, we consider the mapping $f_n : C \to H$ defined by $f_n(x) = (1 - \alpha_n) f(x) + \alpha_n x_0$. Because C is a convex set, we have that $f_n(x) \in C$ for any $x \in C$. (We used also assumption (ii)). For any $n \in \mathbb{N}$, the mapping f_n is a contraction with rate $(1 - \alpha_n)$ (because f is nonexpansive). Applying the *Banach contractive principle*, we obtain an element $x_n \in C$ such that $f_n(x_n) = x_n$. The sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded. Indeed, if this is not the case, considering a subsequence (if necessary), we may assume that $\{x_n\}_{n \in \mathbb{N}}$ is convergent to ∞, as $n \to \infty$.

Let $\beta \in]0, 1[$ and $\rho > 0$ such that $\langle f(x) - f(x_0), x - x_0 \rangle \leq \beta \|x - x_0\|^2$, for $x \in C$ satisfying $\|x\| > \rho$. For $n \in \mathbb{N}$, large enough, we have

$$\begin{aligned}
&\|x_n\|^2 - \|x_n\| \|x_0\| \leq \langle x_n, x_n - x_0 \rangle = \langle (1 - \alpha_n) f(x_n) + \alpha_n x_0, x_n - x_0 \rangle \\
&= \langle (1 - \alpha_n) f(x_n) - (1 - \alpha_n) f(x_0) + (1 - \alpha_n) f(x_0) + \alpha_n x_0, x_n - x_0 \rangle \\
&= (1 - \alpha_n) \langle f(x_n) - f(x_0), x_n - x_0 \rangle + (1 - \alpha_n) \langle f(x_0), x_n - x_0 \rangle \\
&\quad + \alpha_n \langle x_0, x_n - x_0 \rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|x_n\|^2 - \|x_n\| \|x_0\| \leq (1 - \alpha_n) \beta \|x_n - x_0\|^2 + (1 - \alpha_n) \|f(x_0)\| \|x_n - x_0\| \\
&+ \alpha_n \|x_0\| \|x_n - x_0\| \leq (1 - \alpha_n) \beta (\|x_n\|^2 + 2\|x_n\| \|x_0\| + \|x_0\|^2) \\
&+ (1 - \alpha_n) \|f(x_0)\| (\|x_n\| + \|x_0\|) + \alpha_n \|x_0\| (\|x_n\| + \|x_0\|).
\end{aligned}$$

Dividing both sides by $\|x_n\|^2$ and taking limits, we obtain $1 \leq \beta$, which is a contradiction. Thus $\{x_n\}_{n\in\mathbb{N}}$ is bounded and we can show that $\{f(x_n)\}_{n\in\mathbb{N}}$ is also bounded (using the fact that f is nonexpansive).

Now, because for any $n \in \mathbb{N}$ we have

$$x_n = (1 - \alpha_n)f(x_n) + \alpha_n x_0,$$

we deduce that

$$\|x_n - f(x_n)\| = \alpha_n \|x_0 - f(x_n)\| \to 0, \text{ as } n \to \infty.$$

The space H being reflexive and $\{x_n\}_{n\in\mathbb{N}}$ a bounded sequence, we may assume (eventually considering a subsequence) that $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to an element $x^* \in C$, (we used also *Eberlein's Theorem*). Because H is uniformly convex and f is nonexpansive, we have that $I - f$ is demi-closed. Therefore, because $\|x_n - f(x_n)\| \to 0$ as $n \to \infty$, we deduce that $f(x^*) = x^*$ and the proof is complete. ■

Corollary 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone, and $f : \mathbb{K} \to \mathbb{K}$ a nonexpansive mapping. If f is scalar asymptotically contractive on $\mathbb{K}$, then f has a fixed-point in K.*

Remark 1. Corollary 1 is an existence theorem for fixed points on a closed convex cone. The theory of fixed point on convex cones has many applications.

Corollary 2. *Let $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone, and $h : K \to K$ a k_0-Lipschitzian mapping ($k_0 > 0$) such that $h(0) \neq 0$. If there exists an element $x_0 \in \mathbb{K}$ such that*

$$\limsup_{\substack{x\in\mathbb{K},\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\langle h(x) - h(x_0), x - x_0\rangle}{\|x - x_0\|^2} < k$$

with $k > k_0$, then k is an eigenvalue of h associated with an eigenvector in $\mathbb{K}$.

Proof. We apply Theorem 1 taking $f = \frac{1}{k}h$. ■

Remark 2. J.P. Penot introduced in [16] the following notion. Let $(E, \|\cdot\|)$ be a Banach space, and let $C \subset E$ be an unbounded set. We say that $f : C \to E$ is *asymptotically contractive* on C if there exists $x_0 \in C$ such that

$$\limsup_{x\in C,\|x\|\to\infty} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1.$$

Several examples of asymptotically contractive mappings are given in [16].

We remark that in the case of Hilbert spaces, any asymptotically contractive mapping is scalar asymptotically contractive but the converse is not true.

The method presented above, on Hilbert spaces, to obtain the existence of fixed points for nonexpansive mappings on unbounded sets, can be extended on Banach spaces. In the next section, we present this extension.

4 G-Scalar Asymptotically Contractive Mappings in Banach Spaces

Let $(E, \|\cdot\|)$ be a reflexive Banach space and $C \subset E$ be an unbounded closed convex set. Let $B : E \times E \to \mathbb{R}$ be a bilinear mapping satisfying the following properties:

(b1) there exists $b > 0$ such that $B(x, y) \leq b\|x\|\|y\|$ for any $x, y \in E$
(b2) there exists $a > 0$ such that $a\|x\|^2 \leq B(x, x)$, for any $x \in E$.

If we denote by $G = \frac{1}{a}B$ and $M = \frac{b}{a}$, then we have that $G(x, y) \leq M\|x\|\|y\|$, for any $x, y \in E$ and $\|x\|^2 \leq G(x, x)$ for any $x \in E$. The function G used in this section will be a such function.

Definition 2. *We say that a mapping $f : C \to E$ is G-scalar asymptotically contractive on C if there exists $x_0 \in C$ such that*

$$\limsup_{x \in C,\ \|x\| \to \infty} \frac{G(f(x) - f(x_0), x - x_0)}{\|x - x_0\|^2} < 1.$$

Remark 3.

1. If E is a Hilbert space and G is the inner product $\langle \cdot, \cdot \rangle$, defined on E, then in this case by Definition 2 we obtain the notion of *scalar asymptotically contractive mapping* introduced by Definition 1.
2. If the mapping G used in Definition 2 satisfies the property
   (b1′) $G(x, y) \leq \|x\|\|y\|$ for any $x, y \in E$,
   then, in this case any asymptotically contractive mapping f (in Penot's sense) is G-scalar asymptotically contractive mapping.

The main result of this section is the following:

Theorem 2. *Let $(E, \|\cdot\|)$ be a reflexive Banach space and $C \subset E$ be an unbounded closed convex set. Let $f : C \to E$ be a mapping such that the following assumptions are satisfied:*

(i) f is nonexpansive,
(ii) $f(C) \subseteq C$,
(iii) $I - f$ is demi-closed,
(iv) f is G-scalar asymptotically contractive on C.

Then f has a fixed point in C.

Proof. The proof follows the same ideas used in the proof of Theorem 1, but we have some particular details.

Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence of $]0, 1[$ such that $\lim_{n \to \infty} \alpha_n = 0$ and let $x_0 \in C$ be the element used in *assumption* (iv). For every $n \in \mathbb{N}$, we consider the mapping $f_n : C \to E$ defined by

$$f_n(x) = (1 - \alpha_n)f(x) + \alpha_n x_0 \text{ for any } x \in C.$$

Obviously, the convexity of C with (ii) implies that $f_n(x) \in C$, for any $x \in C$. Because f is nonexpansive, we have that for any $n \in \mathbb{N}$, f_n is a contraction.

Applying, for any $n \in \mathbb{N}$, the *Banach contraction principle*, we obtain an element $x_n \in C$ such that $f_n(x_n) = x_n$.

The sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ is bounded. Indeed, if this is not the case, considering a subsequence (if necessary) we may assume that $\{\|x_n\|\}_{n\in\mathbb{N}}$ is convergent to $+\infty$. Using assumption (iv), we find $\beta \in]0,1[$ and $\rho > 0$ such that $G(f(x) - f(x_0), x - x_0) \leq \beta\|x - x_0\|^2$ for $x \in C$ satisfying $\rho < \|x\|$. We have,

$$\begin{aligned}
&\|x_n\|^2 - M\|x_n\|\|x_0\| \\
&\leq G(x_n, x_n - x_0) = G((1-\alpha_n)f(x_n) + \alpha_n x_0, x_n - x_0) \\
&= G((1-\alpha_n)f(x_n) + \alpha_n x_0 - (1-\alpha_n)f(x_0) + (1-\alpha_n)f(x_0), x_n - x_0) \\
&= (1-\alpha_n)G(f(x_n) - f(x_0), x_n - x_0) + (1-\alpha_n)G(f(x_0), x_n - x_0) \\
&+ \alpha_n G(x_0, x_n - x_0) \leq (1-\alpha_n)\beta\|x_n - x_0\|^2 + (1-\alpha_n)M\|f(x_0)\|\|x_n - x_0\| \\
&+ \alpha_n M\|x_0\|\|x_n - x_0\| \leq (1-\alpha_n)\beta[\|x_n\|^2 + 2\|x_n\|\|x_0\| + \|x_0\|^2] \\
&+ (1-\alpha_n)M\|f(x_0)\|[\|x_n\| + \|x_0\|] + \alpha_n M\|x_0\|[\|x_n + x_0\|].
\end{aligned}$$

Dividing both sides by $\|x_n\|^2$ and taking limits, we obtain $1 \leq \beta$, which is a contradiction.

Thus $\{x_n\}_{n\in\mathbb{N}}$ is bounded, and because f is nonexpansive, we can show that $\{f(x_n)\}_{n\in\mathbb{N}}$ is also bounded.

Taking into consideration that

$$x_n = (1-\alpha_n)f(x_n) + \alpha_n x_0, \text{ for any } n \in \mathbb{N},$$

we deduce that

$$\|x_n - f(x_n)\| = \alpha_n\|x_0 - f(x_n)\| \to 0 \text{ as } n \to \infty.$$

Because the space E is reflexive and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, we may assume (eventually considering a subsequence) that $\{x_n\}_{n\in\mathbb{N}}$ is weakly convergent to an element $x^* \in C$. By the fact that $I - f$ is supposed to be demi-closed, we obtain that $f(x^*) = x^*$, and the proof is complete. ∎

Considering Remark 3 (2) of this section, we deduce from Theorem 2 the following corollary.

Corollary 3 ([16]). *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space and $C \subset E$ be an unbounded closed convex subset. Let $G : E \times E \to P$ be a bilinear mapping satisfying properties (b1) and (b2) with $a = b = 1$. Let $f : C \to E$ be a mapping such that the following assumptions are satisfied:*

(i) f is nonexpansive
(ii) $f(C) \subseteq C$,
(iii) f is asymptotically contractive in Penot's sense.

Then f has a fixed point in C.

5 (G, A)-Scalar Asymptotically Contractive Mappings in Banach Spaces

In this section, we put in evidence some relations between scalar asymptotically contractive, scalar asymptotically derivable, and asymptotic derivability.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $C \subset H$ be an unbounded closed convex set. Let $x_0 \in C$ be an element and $f : C \to H$ be a mapping. We introduced in [7] the following notion. If C is a closed convex cone and $T : H \to H$ is a continuous linear mapping, we say that T is a *scalar asymptotic derivative* of f along C if

$$\limsup_{x \in C,\ \|x\| \to \infty} \frac{\langle f(x) - T(x), x \rangle}{\|x\|^2} \leq 0.$$

We recall that T is an *asymptotic derivative* of f along C if

$$\limsup_{x \in C,\ \|x\| \to \infty} \frac{\|f(x) - T(x)\|}{\|x\|} \leq 0.$$

If T is an asymptotic derivative of f along C, then T is a scalar asymptotic derivative. M.A. Krasnoselskii introduced the concept of asymptotic derivative, which is much used in nonlinear analysis.

We can generalize the concept of *scalar asymptotic derivative*, considering T a general mapping, not necessarily linear, eventually being an element of a particular class of nonlinear mappings.

We consider the following notations: $U = C - x_0$, $u = x - x_0$, where $x \in C$ and $g(u) = f(u + x_0)$. Obviously, $0 \in U$, $g(0) = f(x_0)$ and for any $u \in U$ we have $f(u + x_0) = f(x)$. If

$$\limsup_{u \in U,\ \|u\| \to \infty} \frac{\langle g(u) - g(0), u \rangle}{\|u\|^2} \leq 0,$$

then we have

$$\limsup_{x \in C, \|x\| \to \infty} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} = \limsup_{u \in U,\ \|u\| \to \infty} \frac{\langle g(u) - g(0), u \rangle}{\|u\|^2} \leq 0.$$

If we consider $g(0)$ as a scalar asymptotic derivative of g along U, then we have that f is scalar asymptotically contractive on C.

This fact implies the following generalization of the notion of G-scalar asymptotically contractivity. To do this, we need to recall some notions and to introduce some conditions.

Let $(E, \|\cdot\|)$ be a Banach space and let $C \subset E$ be an unbounded closed convex set. We recall that a *semi-inner-product* in Lumer's sense [Trans. Amer. Math. Soc. 100, 29–43 (1961)], is a mapping satisfying the following properties:

(s1) $[x + y, z] = [x, z] + [y, z]$, for any $x, y, z \in E$,

(s2) $[\lambda x, y] = \lambda[x, y]$, for any $\lambda \in \mathbb{R}$,
(s3) $[x, x] > 0$ for any $x \in E$, $x \neq 0$,
(s4) $|[x, y]|^2 \leq [x, x][y, y]$, for any $x, y \in E$.

It is known that, for any Banach space we can define a semi-inner-product. Also, it is known that the mapping $x \to [x, x]^{1/2}$ is a norm on E. *If this norm coincides with the norm $\|\cdot\|$ given on E, we say that the semi-inner-product is compatible with the norm $\|\cdot\|$.*

We say that a mapping $A : C \to E$ is ϕ-asymptotically bounded on C if there exist $r, c > 0$ such that:

(α_1) $\|A(x)\| \leq c\phi(\|x\|)$ for all $x \in C$ with $\|x\| > r$,
(α_2) $\lim\limits_{t\to+\infty} \frac{\phi(t)}{t} = 0$.

Now, suppose that a mapping $G : E \times E \to P$ satisfies the following properties:

(β_1) $G(x_1 + x_2, y) = G(x_1, y) + G(x_2, y)$, for any $x_1, x_2, y \in E$,
(β_2) $G(\lambda x, y) = \lambda G(x, y)$, for any $\lambda > 0$ and any $x, y \in E$,
(β_3) $\|x\|^2 \leq G(x, x)$, for any $x \in E$,
(β_4) $G(x, y) \leq M\|x\|\|y\|$, for some $M > 0$ and any $x, y \in E$.

Obviously, any semi-inner-product compatible with the norm $\|\cdot\|$ satisfies the properties (β_1), (β_2), (β_3), and (β_4).

Definition 3. *We say that a mapping $f : C \to E$ is a (G, A)-scalar asymptotically contractive mapping on C, if there exists a mapping $G : E \times E \to P$ satisfying the properties (β_1)–(β_4) and a ρ-contractive mapping $A : C \to E$ such that*

$$\limsup_{x\in C,\ \|x\|\to\infty} \frac{G(f(x) - A(x), x)}{\|x\|^2} < 1.$$

We have the following result.

Theorem 3. *Let $(E, \|\cdot\|)$ be a reflexive Banach space and $C \subset E$ be an unbounded closed convex set. Let $f : C \to E$ be a nonexpansive mapping. If the following assumptions are satisfied:*

(1) $f(C) \subseteq C$,
(2) $I - f$ is demi-closed,
(3) f is (G, A)-scalar asymptotically contractive on C,
(4) A is ϕ-asymptotically bounded and $A(C) \subseteq C$,

then f has a fixed point in C.

Proof. First, we observe that f and A are bounded mappings, i.e., $f(D)$ and $A(D)$ are bounded sets, whenever $D \subset C$ is bounded. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $]0, 1[$ such that $\lim\limits_{x\to\infty} \lambda_n = 0$. For every $n \in \mathbb{N}$, we consider the mapping $f_n : C \to E$ defined by

$$f_n(x) = (1 - \lambda_n)f(x) + \lambda_n A(x).$$

Because C is convex and considering the assumptions (1) and (4), we have that $f_n(C) \subseteq C$.

Using the properties of f and A, we can show that, for any $n \in \mathbb{N}$, f_n is a contractive mapping with the rate $k_n = (1-\lambda_n) + \lambda_n \rho \in]\rho, 1[$. Applying the *Banach contraction principle*, we obtain an element $x_n \in C$ such that $f_n(x) = x_n$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence.

Indeed, if this is not the case considering (if necessary) a subsequence, we may assume that $\{x_n\}_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$.

Because f is (G, A)-scalar asymptotically contractive, there exist $\beta \in]0, 1[$ and $\rho_0 > 0$ such that

$$G(f(x) - A(x), x) \leq \beta \|x\|^2 \text{ for } x \in C \text{ satisfying } \|x\| > \rho_0.$$

For $n \in \mathbb{N}$, large enough we have

$$\begin{aligned}
&\|x_n\|^2 \leq G(x_n, x_n) = G((1-\lambda_n)f(x_n) + \lambda_n A(x_n), x_n) \\
&= G((1-\lambda_n)f(x_n) - (1-\lambda_n)A(x_n) + A(x_n), x_n) \\
&= (1-\lambda_n)G(f(x_n) - A(x_n), x_n) + G(A(x_n), x_n) \\
&\leq (1-\lambda_n)\beta\|x_n\|^2 + cM\phi(\|x_n\|)\|x_n\|.
\end{aligned}$$

Dividing both sides by $\|x_n\|^2$ and taking limits we obtain $1 \leq \beta$, which is a contradiction.

Thus $\{x_n\}_{n\in\mathbb{N}}$ is bounded and consequently $\{f(x_n)\}_{n\in\mathbb{N}}$ and $\{A(x_n)\}_{n\in\mathbb{N}}$ are bounded sequences. Because for any $n \in \mathbb{N}$ we have

$$x_n = (1-\lambda_n)f(x_n) + \lambda_n A(x_n),$$

and we obtain

$$\|x_n - f(x_n)\| = \lambda_n \|A(x_n) - f(x_n)\| \to 0 \text{ as } n \to \infty.$$

The space E being reflexive and $\{x_n\}_{n\in\mathbb{N}}$ being a bounded sequence, we may assume (eventually considering a subsequence and Eberlein's Theorem) that $\{x_n\}_{n\in\mathbb{N}}$ is a weakly convergent sequence to an element $x^* \in C$. Because $I - f$ is demi-closed, we have that $f(x^*) = x^*$ and the proof is complete. ■

Remark 4.

1. If the space $(E, \|\cdot\|)$ is a uniformly convex Banach space, then in this case in Theorem 3 it is not necessary to suppose that $I - f$ is demi-closed.
2. If in Theorem 3, C is a closed convex cone, we have a fixed-point theorem on closed convex cones. The fixed-points theorem on cones have many applications.

Corollary 4. *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space and $C \subset E$ be a closed convex cone. Let $h : C \to E$ be a k_0-Lipschitzian mapping. If the following assumptions are satisfied:*

(1) $h(C) \subseteq C$ and $h(0) \neq 0$,
(2) the mapping $f = \frac{1}{k_0} h$ is (G, A)-scalar asymptotically contractive on C,
(3) A is ϕ-asymptotically bounded and $A(C) \subseteq C$.
Then k_0 is a positive eigenvalue of h associated with an eigenvector in C.

Proof. We apply Theorem 3 to the mapping $f = \frac{1}{k_0} h$. ■

Corollary 5. *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space and $C \subset E$ be a closed convex cone. Let $h : C \to E$ be a k_0-Lipschitzian mapping and $\psi : C \to E$ be a ρ_0-Lipschitzian mapping. If the following assumptions are satisfied:*

(1) $h(C) \subseteq C$ and $h(0) \neq 0$ and $\psi(C) \subseteq C$,
(2) $\rho_0 < k_0$,
(3) ψ is ϕ-asymptotically bounded,
(4) $\limsup\limits_{x \in C,\ \|x\| \to \infty} \frac{G(h(x) - \psi(x), x)}{\|x\|^2} \leq k_0$,

then any $k \geq k_0$ is an eigenvalue of h associated with an eigenvector in C.

Proof. For any k we apply Theorem 3 taking $f = \frac{1}{k} h$ and $A = \frac{1}{k}\psi$. ■

Now, we can generalize the notion of scalar asymptotic derivative.

Definition 4. *We say that a mapping $A : C \to E$ is a G-scalar asymptotic derivative of the mapping $f : C \to E$ along C if*

$$\limsup_{x \in C,\ \|x\| \to \infty} \frac{G(f(x) - A(x), x)}{\|x\|^2} \leq 0.$$

Remark 5. If A is a ρ-contraction and a G-scalar asymptotic derivative for f along C, then f is a (G, A)-scalar asymptotically contractive mapping.

From, Theorem 3, we deduce the following result.

Corollary 6. *Let $(E, \|\cdot\|)$ be a reflexive Banach space and $C \subset E$ be an unbounded closed convex set. Let $f : C \to E$ be a nonexpansive mapping. If the following assumptions are satisfied:*

(1) $f(C) \subseteq C$,
(2) $I - f$ is demi-closed,
(3) f has a G-scalar asymptotic derivative $A : C \to E$ such that A is ρ-contractive, ϕ-asymptotically bounded and $A(C) \subseteq C$,

then f has a fixed point in C.

6 Comments

We presented in this paper some fixed-point theorems for nonexpansive mappings on unbounded closed convex sets of a reflexive Banach space. The results are based on some notions of scalar asymptotic contractivity inspired by the notion of asymptotic contractivity defined recently in [16]. A relation with a notion of scalar asymptotic derivability is established. A few existence results for positive eigenvalues for nonlinear mappings defined on a closed convex cone are also given. Applications of the results presented in this paper may be the subject of another paper.

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Cooperative Combinatorial Games

Imma Curiel

University of the Netherlands Antilles, Curaçao, Netherlands Antilles
romali@interneeds.net

Abstract This chapter is concerned with cooperative combinatorial games that model situations in which the decision makers who agree to cooperate encounter a combinational optimization problem to maximize profit or minimize cost. Eight cooperative combinatorial games that have received most attention in the literature are surveyed and analyzed, and the similarities and differences in their analysis are pointed out.

Key words: cooperative game theory, combinatorial optimization, combinatorial games, assignment games, permutation games, sequencing games, travelling salesman games, routing games, minimum cost spanning tree games, location games, delivery games, core of games, profit or cost allocation

1 Introduction

Cooperative game theory is concerned with situations in which at least two decision makers can increase their profits or decrease their costs by cooperating. One can think, for example, of a case where one person has the resources to make a certain product, another one has the know-how to make it, and yet a third one has the means to transport it to a market where it can be sold. Alone, none of them can generate a profit. By working together they can.

In case a group of decision makers decide to cooperate to increase profits or decrease costs, they will also have to decide how to allocate the total profit or costs. This allocation method should appeal to each member of the group otherwise he will not consent to cooperate.

Solution concepts from cooperative game theory can be used as allocation methods in these situations.

In this chapter, we consider cooperative combinatorial games. These games model situations in which the decision makers who decide to cooperate have to solve a combinatorial optimization problem to maximize profits or minimize costs. Although cooperative games that fit this description appeared earlier in the literature, the bundling of these games into a class was first done in [10]. Seven types of combinatorial games were discussed there with most attention being given to the core of the games. This paper was followed by others that studied other solution concepts from cooperative game theory as well as other cooperative combinatorial games.

Potentially, every combinatorial optimization problem can give rise to a cooperative combinatorial game. In this chapter, we will treat the eight cooperative combinatorial games that have received the most attention in the literature.

In Section 2, we will provide some background of cooperative game theory. In the subsequent sections, we will study assignment and permutation games, sequencing games, travelling salesman and routing games, minimum cost spanning tree games, location games, and delivery games. In the last section, we will briefly point out the similarities and differences in the analysis of these games. We will also mention some work that deals with other cooperative combinatorial games than the eight given above and some topics that are not treated in Sections 2–8.

2 Cooperative Games and Solution Concepts

Formally, a cooperative game in characteristic function form is defined as follows.

Definition 1. *A cooperative game in characteristic function form is an ordered pair $< N, v >$ where N is a finite set, the set of players, and the characteristic function v is a function from 2^N to R with $v(\emptyset) = 0$.*

A subset S of N is called a *coalition*. The number $v(S)$ gives the worth of coalition S in the game. When no confusion about the player set N is possible, the game $< N, v >$ will be identified with the function v. The set N will usually be taken to equal $\{1, 2, \ldots, n\}$.

For it to be worthwhile for coalitions to form, the whole should be at least as profitable as its parts. This property is captured in the following definition of superadditivity.

Definition 2. *A cooperative game v is called superadditive if*

$$v(S \cup T) \geq v(S) + v(T) \text{ for all } S, T \in 2^N \text{ with } S \cap T = \emptyset.$$

If the reverse inequality holds, the game is called subadditive. If equality holds, the game is called additive.

A stronger property than superadditivity is convexity.

Definition 3. *A cooperative game v is called convex if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \text{ for all } S, T \in 2^N.$$

If the reverse inequality holds, the game is called concave.

As mentioned in the introduction, the players will have to decide how to allocate the total profit if they decide to work together. Naturally, a player will compare the amount that he is to receive if he cooperates with the amount that he can generate on his own. If the comparison turns out to be unfavourable for the amount received under cooperation, he will prefer to work alone. Coalitions can do a similar exercise and reach a similar conclusion. A solution concept that takes these aspects of the game into consideration is the *core*. Let $x \in R^N$ be an allocation with x_i being the amount assigned to player $i \in N$, and let $\sum_{i \in N} x_i$ be denoted by $x(N)$.

Definition 4. *The core $C(v)$ of the game v is defined to be the set*

$$C(v) := \{x \in R^n | x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \in 2^N\}.$$

If an allocation that is an element of the core is used to divide profits, then no player or coalition can do better by splitting and working on his/its own. Unfortunately, the core of a game can be empty. The concept of *balancedness* can be used to characterize games with a nonempty core.

Definition 5. *A collection $\mathcal{B}$ of nonempty subsets of N is called a balanced collection if for all $S \in \mathcal{B}$ there exist positive numbers λ_S such that $\sum_{S \in \mathcal{B}} \lambda_S 1_S = 1_N$.*

The numbers λ_S are called the *weights* of the elements of $\mathcal{B}$.

Definition 6. *A cooperative game v is called a balanced game if*

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)$$

for every balanced collection $\mathcal{B}$ with weights $\{\lambda_S\}_{S \in \mathcal{B}}$.

For each coalition S, the *subgame* $< S, v_S >$ of a game v is defined by $v_S(T) = v(T)$ for all $T \subset S$. A game v for which each subgame is balanced is called *totally balanced*. The following theorem is due to Bondareva [5] and Shapley [102].

Theorem 1. *A cooperative game has a nonempty core if and only if it is balanced.*

This theorem is the cooperative game theoretic version of the well-known duality theorem of linear programming.

Convex games are balanced, and the core of a convex game v is the convex hull of the marginal vectors $m^\pi(v)$ of the game v. These are defined as follows.

Definition 7. *Let π be a permutation of N. The marginal vector $m^\pi(v)$ of the game v is defined by*

$$m_i^\pi(v) := v(P(\pi, i) \cup \{i\}) - v(P(\pi, i)) \text{ for all } i \in N,$$

where $P(\pi, i) := \{j \in N | \pi(j) < \pi(i)\}$ is the set of predecessors of i with respect to the permutation π.

Shapley [101] introduced and characterized a solution concept for cooperative games that can be viewed as the average of the marginal vectors. Let Π_N denote the set of all permutations of N.

Definition 8. *The Shapley-value $\phi(v)$ of a cooperative game v is defined by*

$$\phi_i(v) := \frac{1}{n!} \sum_{\pi \in \Pi_N} m_i^\pi(v) \text{ for all } i \in N.$$

It follows that the Shapley-value of a convex game lies in the barycenter of the core of the game. The Shapley-value of a nonconvex game need not be an element of the core of the game.

A solution concept for cooperative games that coincides with the core for convex games is the bargaining set that was introduced in [1]. The bargaining set considers the imputations of a game. An *imputation* of a game v is a vector $x \in R^n$ that satisfies $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. If x satisfies only the first of these conditions, then x is called a *pre-imputation* of v. The set of imputations of v is denoted by $I(v)$ and the set of pre-imputations of v is denoted by $PI(v)$. An *objection* of player i against player j with respect to an imputation x in the game v is a pair $(y; S)$ where S is any coalition that contains i but not j, and $y = (y_k)_{k \in S}$ is an $|S|$-tuple of real numbers satisfying

$$y(S) = v(S) \text{ and } y_k > x_k \text{ for all } k \in S.$$

A *counter objection* to the objection $(y; S)$ is a pair $(z; T)$ with T being a coalition that contains j but not i, and $z = (z_k)_{k \in T}$ being a $|T|$-tuple of real numbers satisfying

$$z(T) = v(T),\ z_k \geq y_k \text{ for } k \in S \cap T \text{ and } z_k \geq x_k \text{ for } k \in T \setminus S.$$

Definition 9. *An imputation $x \in I(v)$ is said to belong to the bargaining set $\mathcal{M}(v)$ of the game v, if for any objection of one player against another with respect to x, there exists a counter objection.*

The bargaining set is always nonempty and contains the core. The fact that for convex games the bargaining set and the core coincide was proven in [76]. In [108], a necessary and sufficient condition for the bargaining set to coincide with the core for superadditive games was given.

To measure the degree of unhappiness of a coalition S with a payoff vector x in a game v, we consider the *excess* $e(S, x)$ of S with respect to x, which is defined by

$$e(S, x) := v(S) - x(S).$$

Player i can compare the payoff he receives according to x with that of player j by taking the maximum of all the excesses $e(S, x)$ over the coalitions S that contain i but not j. Let us denote this maximum by $s_{ij}(x)$ and let T be a coalition with $s_{ij}(x) = e(T, x)$. By splitting off from the grand coalition, forming T, and allocating to the other players in T the payoff given by x, player i remains with $s_{ij}(x)$ and player j will have to see how he will fend for himself. Therefore, $s_{ij}(x)$ can be regarded as the weight of a possible threat of i against j. If x is an imputation with $x_j = v(\{j\})$, then j will have no fear of threats by any player because he can obtain $v(j)$ by working alone. We say that i outweighs j with respect to x if

$$x_j > v(\{j\}) \text{ and } s_{ij}(x) > s_{ji}(x).$$

The kernel that was introduced in [20] consists of those imputations for which no player outweighs another one.

Definition 10. *The kernel $\mathcal{K}(v)$ of a game v is defined by*

$$\mathcal{K}(v) := \{x \in I(v) | s_{ij}(x) = s_{ji}(x) \textit{ or } x_j = v(\{j\}) \textit{ for all } i, j \in N\}.$$

The pre-kernel $\mathcal{PK}(v)$ of v is defined similarly with $I(v)$ replaced by $PI(v)$ and the condition $x_j = v(\{j\})$ left out.

The kernel and the pre-kernel are always nonempty. The kernel is a subset of the bargaining set. For superadditive games, the kernel and the pre-kernel coincide.

The nucleolus of a game, introduced in [100], minimizes the maximum excess in a lexicographical sense. Let $\theta(x)$ be the vector that arranges the excesses of the 2^n subsets of N in decreasing order. If x is lexicographically smaller than y, we denote that by $x <_L y$, and $x \leq_L y$ indicates that either $x <_L y$ or $x = y$.

Definition 11. *The nucleolus $\nu(v)$ of a game v is defined by*

$$\nu(v) := \{x \in I(v) | \theta(x) \leq_L \theta(y) \textit{ for all } y \in I(v)\}.$$

The pre-nucleolus is defined similarly with $I(v)$ replaced by $PI(v)$.

The nucleolus of a game always consists of one point, which lies in the kernel, and which is an element of the core whenever the core is nonempty. If v is a convex game, we have $\mathcal{PK}(v) = \mathcal{K}(v) = \nu(v)$.

In [26], a class of games is introduced for which the nucleolus has a simple expression. This is the class of 1-convex games. Driessen and Tijs show that for these games, the nucleolus coincides with the τ-value, which is easily computed. The τ-value was introduced in [117]. It uses the *upper vector* and *lower vector* of a game. The upper vector M^v of a game v is given by

$$M_i^v := v(N) - V(N \setminus \{i\}) \text{ for all } i \in N.$$

The lower vector μ^v of v is given by

$$\mu_i^v := \max_{S: S \ni i} (v(S) - \sum_{j \in S \setminus \{i\}} M_j^v) \text{ for all } i \in N.$$

The upper vector gives an upper bound on what a player can expect to receive, as if he asks for more the others would be better off by working without him.

The lower vector gives a lower bound on what he will accept, as he can achieve this by forming a coalition that reaches the maximum and giving to the other members of the coalition their upper bounds.

Definition 12. *A game v is called quasi-balanced if*

$$\mu_i^v \leq M_i^v \textit{ for all } i \in N \textit{ and } \mu^v(N) \leq v(N) \leq M^v(N).$$

Every balanced game is quasi-balanced. The τ-value of a quasi-balanced game is defined as a suitable convex combination of the upper and lower vectors.

Definition 13. *Let v be a quasi-balanced game. The τ-value $\tau(v)$ of v is defined by*

$$\tau(v) := \lambda\mu^v + (1 - \lambda)M^v$$

where $\lambda \in [0, 1]$ is uniquely determined by $\sum_{i\in N} \tau_i(v) = v(N)$.

The τ-value need not be an element of the core. In [26], necessary and sufficient conditions for $\tau(v)$ to be an element of $C(v)$ are given.

The τ-value is in general not easily computable but as mentioned above for the class of 1-convex games, both the τ-value as well as the nucleolus can be computed without much work. To define this class, we need the *gap function* g^v of the game v. The gap function is given by

$$g^v(S) := M^v(S) - v(S) \text{ for all } S \in 2^N.$$

Definition 14. *A game v is called 1-convex if*

$$0 \leq g^v(N) \leq g^v(S) \textit{ for all } S \subset N, S \neq \emptyset.$$

If the reverse inequalities hold, then v is called 1-concave.

The following results can be found in [24, 25], and [26].

Theorem 2. *Let v be a 1-convex game. Then*

$$\tau_i(v) = \nu_i(v) = M_i^v - \frac{1}{n}g^v(N) \textit{ for all } i \in N.$$

Furthermore, the extreme points of the core of a 1-convex game are the n vectors $M^v - g^v(N)e^i$ where e^i is the vector with 1 in the i-th place and 0 everywhere else.

From Theorem 2, it follows that for a 1-convex game, the τ-value and the nucleolus lie in the barycenter of the core.

Another class of games given in [26] for which the τ-value has a simple expression is the class of *semiconvex games.*

Definition 15. *A game v is called semiconvex if v is superadditive and $g^v(i) \leq g^v(S)$ for all $i \in N$ and $S \subset N$ with $i \in S$. If v is subadditive and the reverse inequalities hold, then v is called semiconcave.*

Every convex game is semiconvex and every semiconvex game is quasi-balanced.

Theorem 3. *The τ-value of a semiconvex game v is given by $\tau(v) = \lambda \underline{v} + (1-\lambda)M^v$ where $\underline{v} = (v(1), v(2), \ldots, v(n))$ and where $\lambda \in [0,1]$ is such that $\sum_{i \in N} \tau_i(v) = v(N)$.*

This brings us to the end of our short introduction of cooperative games and their solutions concepts. We have limited ourselves to the topics that will be most useful in the subsequent sections. For more extensive and detailed treatments, the reader is referred to [84] and [25].

We have introduced all the game-theoretic concepts from the viewpoint of profit games. With some adaptation (usually the reversal of an inequality), they apply to cost games, too.

3 Assignment Games and Permutation Games

The *assignment problem* is a well-known and well-solved combinatorial optimization problem, cf. [85]. Its mathematical formulation is

$$\begin{array}{lll} \max \sum_{i \in B} \sum_{j \in P} a_{ij} x_{ij} & & \\ \text{s.t.} \quad \sum_{j \in P} x_{ij} \leq 1 & \text{for all } i \in B & \\ \qquad \sum_{i \in B} x_{ij} \leq 1 & \text{for all } j \in P & \\ \qquad x_{ij} \in \{0,1\} & \text{for all } i \in B, j \in P. & \end{array} \tag{1}$$

Here B and P are two disjoint sets and $a_{ij} \geq 0$. This problem is also known as the bipartite weighted matching problem.

The *assignment game* was introduced in [104]. It models a situation in which the player set N can be partitioned into two sets B and P. A player $i \in B$ and a player $j \in P$ can create a profit $a_{ij} \geq 0$. Two players from the same set cannot create a profit. The classic example takes one set to be the set of buyers and the other the set of sellers. The value $v(S)$ of coalition S in an assignment game is the sum of the profits that pairs of players in S can create and is given by (1) with 1 replaced by $1_S(i)$ and $1_S(j)$ in the first and second inequality, respectively. In [104], it is proven that the assignment game has a nonempty core by considering the dual problem of the linear programming relaxation of the 0,1-programming problem that determines $v(N)$. Because the matrix involved in the assignment problem is totally unimodular, the relaxation and the original problem have the same optimal solution(s). In fact, it is shown in [104] that the core of the assignment game corresponds with the set of optimal solutions of the dual problem of the relaxation of (1). Below we give an alternative proof of the balancedness of the assignment game.

A game that is related to the assignment game is the *permutation game* introduced in [118]. The value of a coalition S in the permutation game v is given by

$$v(S) := \max_{\pi \in \Pi_S} \sum_{i \in S} k_{i\pi(i)}. \tag{2}$$

Here Π_S is the subset of Π_N that contains the permutations that do not permute nonmembers of S, and $k_{i\pi(i)}$ is the value of permutation π for player i. Note that this value depends only on $\pi(i)$, so on the position of player i according to the permutation π.

An alternative formulation to (2) is

$$\begin{array}{ll} \max & \sum_{i\in N}\sum_{j\in N} k_{ij} \\ \text{s.t.} & \sum_{j\in N} x_{ij} = 1_S(i) \text{ for all } i \in N \\ & \sum_{i\in N} x_{ij} = 1_S(j) \text{ for all } j \in N \\ & \quad x_{ij} \in \{0,1\} \qquad \text{for all } i,j \in N. \end{array} \tag{3}$$

In [19], it is shown that every assignment game is a permutation game but that the reverse is not true. Because of this, the proof given below of the balancedness of permutation games also implies balancedness of assignment games.

Theorem 4. *Let v be an assignment games or a permutation game. Then v has a nonempty core.*

Proof. In view of the result stated above, it is sufficient to show that a permutation game has a nonempty core. From the Birkhoff–von Neumann theorem, which states that the extreme points of the set of doubly stochastic matrices are the permutation matrices, it follows that the value of (3) is equal to the value of its linear relaxation. The dual problem of the linear problem is

$$\begin{array}{lll} \min \sum_{i\in N} 1_S(i)y_i + \sum_{j\in N} 1_S(j)z_j & & \\ \text{s.t.} \qquad\qquad y_i + z_j \geq k_{ij} & & \text{for all } i,j \in N. \end{array} \tag{4}$$

Let $(\hat{y}, \hat{z})$ be an optimal solution for (4) for $S = N$. Then

$$\sum_{i\in N} (\hat{y}_i + \hat{z}_i) = v(N)$$

and for all $S \in 2^N$

$$\sum_{i\in S} (\hat{y}_i + \hat{z}_i) = \sum_{i\in N} 1_S(i)\hat{y}_i + \sum_{i\in N} 1_S(i)\hat{z}_i \geq v(S).$$

The inequality follows from the fact that $(\hat{y}, \hat{z})$ is a solution of problem (4) for all $S \in 2^N$. So $u \in R^n$ given by $u_i = \hat{y}_i + \hat{z}_i$ is an element of the core of v. ∎

In essence, this proof runs similarly to that in [104]. The linear programming relaxation of the combinatorial optimization problem that determines $v(N)$ is shown to be equivalent to the original 0,1-programming problem, The dual problem of the linear problem provides core elements of the game. This is a rather efficient way of finding core elements as one does not need to compute the value of all $2^n - 1$ nonempty coalitions.

Several results concerning the core of assignment games can be found in the literature. In [8], an iterative process is described to arrive at an optimal assignment and a core allocation in a model of job matching. In [63], a generalization of this model is studied. In [2], Balinski and Gale show that the core of an assignment game has at most $\binom{2r}{r}$ extreme points where $r = \min\{|B|, |P|\}$. They prove that the core has the maximum number of extreme points for square games ($|B| = |P| = r$) when there are r supercompatible pairs. A pair of players $(i, j) \in B \times P$ is called supercompatible if for all $S \subset B \cup P$ with $i, j \in S$, $x_{ij} = 1$ in any optimal solution of the problem that determines $v(S)$. In [91], Rochford introduces *symmetrically pairwise bargained allocations* (SPB allocations) for assignment games and proves that the set of SPB allocations is equal to the intersection of the kernel and the core. Solymosi shows in [108] that the bargaining set and the core of an assignment game coincide. Because the kernel is always a subset of the bargaining set, it follows that the set of SPB allocations is equal to the kernel.

In [110], Solymosi and Raghavan characterize assignment games that are exact in terms of the assignment matrix entries a_{ij}. A cooperative game v is called *exact* if for all $S \subset N$ there is an $x \in C(v)$ with $x(S) = v(S)$. Exact games are semiconvex, so for these games the τ-value has the simple expression given in Theorem 3. Also in [110], assignment games for which the core is a stable set are characterized in terms of the assignment matrix entries. A stable set or von Neumann–Morgenstern solution contains imputations that do not dominate each other. Furthermore, any imputation not in a stable set is dominated by an imputation in the set.

In [54], it is shown that the extreme points of the core of an assignment game are marginal vectors.

In [79], Núñez and Rafels define *buyer-seller exact* assignment games as assignment games in which no a_{ij} can be increased without changing the core of the game. In such a game, each mixed pair coalition attains its value in the core. It is shown that every assignment game has a unique buyer-seller exact representation.

Results on one-point solution concepts also exist. In [109], Solymosi and Raghavan give an algorithm of order $\mathcal{O}(r^3|P|)$ for computing the nucleolus of an assignment game. In [80], it is shown that the τ-value of an assignment game is the midpoint of the buyers-optimal core allocation and the sellers-optimal core allocation. In [78], it is shown that the nucleolus of an assignment game coincides with that of its buyer-seller exact representation. In [111], a $\mathcal{O}(n^4)$ algorithm for computing the nucleolus of a cyclic permutation game is discussed. A cyclic permutation game is a permutation game for which the value $v(N)$ is given by a permutation consisting of a single cycle. In [64], maximum cardinality matching games are discussed. It is shown that the nucleolus of such a game can be computed efficiently.

Modifications and generalization of assignment and permutation games are also found in the literature. In [19], *tridimensional assignment games* and *bipermutation games* are introduced. These are extended to *multiassignment* and *multipermutation* games in [11]. The linear relaxation of the 0, 1-programming problem that determines these games is not equivalent to the original problem and no proof of balancedness based on such a result can be carried over from the assignment and permutation games to these generalizations. In fact, multiassignment and multipermutation games need not be balanced. Subclasses that contain balanced games are given. In [74], assignment games in which one of the sets B,P is an infinite set are studied. In [105], multisided matching games are discussed. In [112], Sotomayor looks at an extension of assignment games in which the players can form more than one partnership. In [53], a class of assignment games called neighbour games are introduced and an $\mathcal{O}(n^2)$ algorithm for computing the nucleolus of a neighbour game is given. In [65], an $\mathcal{O}(n^3)$ algorithm for computing the leximax solution of a neighbour game is given.

The assignment and permutation situations modelled in this section as cooperative games can alternatively be modelled as economies with indivisibilities leading to similar results with respect to existence of core-elements plus results on price equilibria. These models can be found in [11, 19, 40, 57–59, 67, 86, 121].

In [103], Shapley and Scarf model a permutation situation as a *game without side payments* also called a *nontransferable utility game*. They show that the core defined by strong domination is always nonempty but the core defined by weak domination can be empty. In [92, 120, 122], other implications of the difference between weak and strong domination are explored.

An ordinal approach to bipartite as well as nonbipartite matching situations is taken in the literature on matchings. In these models, *preference relations* are used instead of numbers that describe the value of a certain assignment or permutation for a player or coalition. A matching is called *stable* if there do not exist two participants that prefer each other to the partner that the matching assigns to them. In fact, the ordinal approach predates the cardinal approach as it was introduced in 1962 by Gale and Shapley in [41]. They also proved the existence of stable matchings and provided an algorithm to arrive at a stable matching. In [27] and [42], the strategy-proofness of this algorithm is investigated. In [93], the strategy-proofness of matching procedures in general is discussed. Further results on matchings can be found in [11, 66, 75, 94–98, 113].

4 Sequencing Games

Sequencing games resemble permutation games in the sense that in sequencing games, the value of a coalition S is also derived by maximizing a function over the set of admissible permutations for S. However, the set of admissible permutations in a sequencing game has some restrictions. These can be best understood by looking at the way sequencing games were introduced in [16] from *sequencing situations*.

Definition 16. *A sequencing situation consists of a finite set* $N = \{1, 2, \ldots, n\}$ *and an ordered triple* $(\sigma; \alpha; s)$ *where* $\sigma \in \Pi_N$, $\alpha \in R^n$, *and* $s \in R^n_+$.

The classic example of a sequencing situation considers customers standing in a queue in front of a counter waiting to be served. The order in which they are standing is given by σ with $i \in N$ having position $\sigma(i)$ in the queue. The service time of customer i is $s_i > 0$. Depending on his completion time, each customer has costs that are given by a cost function $c_i : R_+ \to R$. The cost functions are taken to be linear, so $c_i(t) = \alpha_i t + \beta_i$. By rearranging, the customers can decrease their total cost. To find an optimal permutation, it is convenient to consider the *urgency index* $u_i = \alpha_i / s_i$ of customer $i \in N$. In [107], it is shown that in order to minimize total cost, the customers should be arranged in order of decreasing urgency indices. The cost savings that are achieved in this way have to be divided among the customers. In [16], the *Equal Gain Splitting rule* or *EGS rule* is introduced as a method to do this. Let i and j be two customers who are standing next to each other with i in front of j. We denote the *gain* that they can achieve by switching positions by g_{ij}. Then

$$g_{ij} := (\alpha_j s_i - \alpha_i s_j)_+ = \max\{\alpha_j s_i - \alpha_i s_j, 0\}.$$

Definition 17. *The EGS rule assigns to each customer* $i \in N$ *in a sequencing situation* $(\sigma; \alpha; s)$ *the amount*

$$EGS_i(\sigma; \alpha; s) = \frac{1}{2} \sum_{k \in P(\sigma,i)} g_{ki} + \frac{1}{2} \sum_{j; i \in P(\sigma,j)} g_{ij}.$$

In [16], the *EGS* rule is characterized by the dummy, equivalence, and switch-properties. In [56], gain splitting rules that divide g_{ij} in some way (not necessarily equally) between i and j are studied. The *split core* is defined as the set containing all allocations generated by gain splitting rules. The split core is shown to be a subset of the core of the sequencing game that was introduced in [16]. The value of a coalition S in a sequencing game is the total amount of cost savings that S can achieve by rearranging its members without jumping over nonmembers. To formalize this the concept of a *connected coalition* comes in handy. A coalition is called connected if there are no nonmembers standing between the members. Let T be a connected coalition. Then the value of T in the sequencing game v corresponding with the sequencing situation $(\sigma; \alpha; s)$ is given by

$$v(T) := \sum_{i \in T} \sum_{k \in P(\sigma,i) \cap T} g_{ki}.$$

For a coalition S that is not connected, we define a *component* of S as a maximal connected subset of S. The components of S form a partition of S, which is denoted by S/σ. The value of S in the sequencing game v is given by

$$v(S) := \sum_{T \in S/\sigma} v(T).$$

In [16], it is shown that sequencing games are convex and that $EGS(\sigma; \alpha; s) \in C(v)$ where v is the sequencing game corresponding with $(\sigma; \alpha; s)$.

In [17], *σ-component additive games* were introduced as a generalization of sequencing games.

Definition 18. *Let $\sigma \in \Pi_N$. Then a cooperative game v is called σ-component additive if*

(a) $v(\{i\}) = 0$ for all $i \in N$,
(b) v is superadditive,
(c) $v(S) = \sum_{T \in S/\sigma} v(T)$.

These games need not be convex. In [17] and [18], the β rule, an extension of the EGS rule to σ-component additive games, is studied, and it is shown that the β rule applied to a σ-component additive game generates an allocation that is in the core of the game. In [14], σ-component additive games with restricted cooperation are studied. It is shown that the allocation generated by the β rule is equal to the nucleolus of the restricted game. This implies that for sequencing games the allocation generated by the EGS rule is equal to the nucleolus of the restricted game. In [89], *Γ-component additive games*, a generalization of σ-component additive games, are studied. It is shown that for these games the bargaining set is equal to the core and the kernel is equal to the nucleolus. In [11], this result is used to derive conditions that guarantee the equality of the nucleolus and the allocation generated by the EGS rule.

In [16], expressions for the Shapley-value and the τ-value of a sequencing game are given in terms of the parameters of the corresponding sequencing situation. So one does not need to compute the $v(S)$'s to obtain these values. Both these values divide the gain generated by two players among them and the players standing between them. The Shapley-value does it equally.

In both sequencing games and permutation games, the value $v(N)$ is obtained by maximizing over the set Π_N. But as shown in [11], the proof of the balancedness of a permutation game cannot be mimicked for a sequencing game because in general it is not possible to extend the function to be maximized to a linear function on the set of doubly stochastic matrices.

Two classes of σ-component additive games are sequencing games with ready times and sequencing games with due dates. The first are studied in [51] and the second in [6]. In [55] and [9], sequencing games with multiple machines are studied. Several classes of balanced multimachine sequencing games are identified.

In [37], a new monotonicity property for sequencing situations is studied. Already in the very first paper on sequencing, [16], a relaxation of the game was considered in which the members of a coalition were allowed to jump over nonmembers. It was shown that such a game can have an empty core. In [17], four relaxations of sequencing games were discussed that permit rearrangements of a coalition that involve jumping over nonmembers as long as this does not cause a delay in the starting time of the nonmembers. The question was posed whether these games were balanced. This question remained open till recently when it was answered by Slikker in [106]. He showed that the games generated by all four relaxations are balanced.

In [119], sequencing games are studied in which a particular player receives a preferential treatment in that he alone in a coalition is allowed to select another player in the coalition and switch places with this player even though this involves jumping over nonmembers. It is shown that these games are balanced.

In [13], a survey of sequencing games is given in which several aspects mentioned in this chapter are treated in more detail.

5 Travelling Salesman Games and Routing Games

The *travelling salesman problem* is a very well-known $\mathcal{NP}$-complete combinatorial optimization problem. It can be stated as follows: Given a directed graph with weights on the arcs, find a minimum weight cycle that visits each vertex exactly once. For our purposes, we may assume without loss of generality that the graph is complete and that the weights satisfy the triangle inequality. The last property implies that going directly from vertex i to vertex j is not more expensive than going from i to k and from k to j.

The travelling salesman problem together with the fixed route cost allocation problem studied in [38] can be viewed as the parents of the *travelling salesman game* introduced in [88]. A travelling salesman game models the following problem. In a given complete directed graph with weights on the arcs, all vertices but one are associated with players. The vertex that does not correspond with a player is denoted by 0. Each coalition S wants to construct a tour that starts in 0, visits each vertex of S exactly once, ends in 0, and has minimum weight. So S wants to find a minimal weight travelling salesman tour on the complete graph with set of vertices equal to $S \cup \{0\}$. Let e be a bijection from $\{1, \ldots, |S|\}$ to S. Such a bijection describes a tour that starts in 0 then visits $e(1)$, then $e(2)$, etc. The last vertex that the tour visits in S is $e(|S|)$ after which it returns to 0. Let $E(S)$ denote the set of bijections from $\{1, \ldots, |S|\}$ to S and let the weight of the arc going from i to j be denoted by w_{ij}. Then the travelling salesman game c is defined by

$$c(S) := \min_{e \in E(S)} (w_{0e(1)} + w_{e(1)e(2)} + \cdots + w_{e(|S|)0}) \text{ for all } S \in 2^N \setminus \emptyset.$$

In [88], it was shown that a travelling salesman game need not be balanced but that it will be balanced if it has three or less players. A 4-player travelling salesman game with empty core was given.

A travelling salesman game is called symmetric if $w_{ij} = w_{ji}$ for all $i, j \in N \cup \{0\}$. In [115], it was shown that a 4-player symmetric travelling salesman game is balanced. An example of a 6-player symmetric travelling salesman game with empty core was given. In [69], it was proven that a 5-player symmetric travelling salesman game is balanced. These proofs are based on a result in [39] implying that for $n \leq 6$, the n-vertex symmetric travelling salesman problem can be formulated as a linear programming problem. So the 0, 1-programming problem that determines the cost of a coalition S is equivalent to its linear programming relaxation. A similar procedure as that in Section 3 yields the nonemptiness of cores of these games.

In general, the integer constraints cannot be dropped and the procedure does not work.

In [11] and [87], classes of (not necessarily symmetric) travelling salesman games that are balanced are studied. In [87], this is done by showing that the travelling salesman games under consideration coincide with *routing games* that are always balanced. Routing games that were introduced in [88] model the same type of situations as travelling salesman games. Only now the assumption is that after an optimal tour has been found for the grand coalition N, any other coalition S does not go through the trouble and expense of computing an optimal tour but simply adopts the tour chosen by N by skipping the vertices that do not belong to S. Let $e \in E(N)$ be such that

$$w_{0e(1)} + w_{e(1)e(2)} + \cdots w_{e(n)0} = \min_{f \in e(N)} (w_{0f(1)} + w_{f(1)f(2)} + \cdots + w_{f(n)0}).$$

In the routing game $< N, c_e >$, the cost $c_e(N)$ of the grand coalition is given by

$$c_e(N) = w_{0e(1)} + w_{e(1)e(2)} + \cdots + w_{e(n)0},$$

and the cost $c_e(S)$ of coalition S is given by

$$c_e(S) = w_{0e_S(1)} + w_{e_S(1)e_S(2)} + \cdots + w_{e_S(|S|)0}$$

where $e_S \in E(S)$ is defined by

$$e_S^{-1}(i) < e_S^{-1}(j) \Leftrightarrow e^{-1}(i) < e^{-1}(j) \text{ for all } i, j \in S.$$

Theorem 5. *Let c_e be a routing game with e being an optimal tour for N. Then $C(c_e) \neq \emptyset$.*

The proof of Theorem 5 relies again on the equivalence of a 0, 1-programming problem and its linear relaxation just as in Section 3.

A routing game defined with respect to a nonoptimal tour for N can have an empty core. In [23], it is shown that a routing game c_e has a nonempty core if and only if $c_e(N) \leq c_e(S) + c_e(N \setminus S)$ for all $S \subset N$. A procedure is described to construct a nearest neighbour tour for which the corresponding routing game satisfies this condition.

Considering the cooperative combinatorial games that we have treated so far, one may be induced to think that if a combinatorial optimization problem is polynomially solvable, then the corresponding cooperative combinatorial game has a nonempty core. However, already in 1988 this conjecture was negated in [31]. There the wallpaper game, a subclass of travelling salesman games arising from the wallpaper problem (that) is polynomially solvable, cf. [73], was introduced. It was shown that a wallpaper game can have an empty core. Actually, the fact that polynomial solvability does not imply balancedness was a kind of hidden knowledge for much longer. In 1962 in [41], it was shown that the roommate problem, which is a nonbipartite version of the marriage problem, does not need to have a stable matching. Translating this ordinal case to a transferable utility nonbipartite weighted matching game with an empty core is straightforward. And since 1965, it was shown in [28] that the weighted matching problem can be solved in polynomial time.

More recently, this problem and the related problem of deciding whether a given travelling salesman game has a nonempty core were discussed in [82].

In [44], graphs that give rise to submodular travelling salesman games are studied. The nucleolus of vehicle routing games is studied in [43].

6 Minimum Cost Spanning Tree Games

The *minimum cost spanning tree problem* is another well-known problem in combinatorial optimization. Contrary to the travelling salesman problem, it is a well-solved problem. The problem can be stated as follows: Given a connected graph with costs on the edges, find a spanning tree (a connected subgraph without cycles with the same set of nodes as the original graph), that has minimum cost among all spanning trees. If n is the number of nodes of the graph, then the minimum cost spanning tree problem can be solved in $\mathcal{O}(n^2)$ time.

By associating each node of the graph except one with a player and assuming that every player wants to be connected with the node that is not a player, we obtain a *minimum cost spanning tree game*. The costs of making the appropriate connections have to be allocated among the players. In [7], this problem was first treated but without explicit use of cooperative game theory. In [4], a game-theoretic approach was first proposed. Let $G = (N_0, E)$ be the complete graph with set of nodes $N_0 = N \cup \{0\}$ and set of edges E. Let $k_{ij} = k_{ji}$ denote the cost of constructing the link $\{i, j\} \in E$.

Definition 19. *The minimum cost spanning tree game (mcst-game) c on G is given by*

$$c(S) = \sum_{\{i,j\} \in E_{T_S}} k_{ij} \text{ for all } S \in 2^N.$$

Here E_{T_S} is the set of edges of a minimum cost spanning tree in the complete graph $G_S = (S_0, E_S)$.

In [4], Bird proposed the following cost allocation scheme. Let T be a minimum cost spanning tree for the graph (N_0, E). For each $I \in N$, the amount that i has to pay is equal to the cost of the edge incident upon i on the unique path from 0 to i in T. Because there can be more than one minimum cost spanning tree, this cost allocation scheme may generate more than one cost allocation in a mcst-game. Any cost allocation generated in this way will be called a Bird tree allocation.

Theorem 6. *Let c be a mcst-game. Let x be a Bird tree allocation for c. Then $x \in C(c)$.*

It follows that mcst-games are balanced. An alternative way to prove this is just as in Section 3 to consider the linear relaxation of the 0, 1-programming problem that determines $c(N)$. In [29], it is shown that the linear relaxation is equivalent to the 0, 1-programming problem. Similar to the approach used in Section 3, this leads to the construction of a core element for c.

In [4], the *irreducible core* $IC(c)$ of a mcst-game c is introduced. The irreducible core of c is the core of a mcst-game $\hat{c}$ defined with the aid of a minimum cost spanning tree of the original mcst-problem. In general, a mcst-game need not be concave but the game $\hat{c}$ is a concave game. In [4], it is shown that the extreme points of the irreducible core of c are precisely the Bird tree allocations.

In [46], mcst-games with efficient coalition structures are defined. The components of this structure induce other mcst-games, and it is shown that the core and nucleolus of the original game are the Cartesian products of the cores and nucleoli of these games. In [47], *permutationally concave games* are defined. A definition of concavity that is equivalent to the one given in Section 2 states that a game c is concave if and only if for all $S^2 \subset S^1 \subset N \setminus R$ we have

$$c(S^1 \cup R) - c(S^1) \leq c(S^2 \cup R) - c(S^2). \tag{5}$$

A game is called a permutationally concave game if property (5) holds for certain coalitions given by a permutation π.

Definition 20. *A game is called permutationally concave if there exists a permutation $\pi \in \Pi_n$ such that for all $1 \leq p_2 \leq p_1 \leq n$ and all $R \subset N \setminus S^\pi(p_1)$ the following is true*

$$c(S^\pi(p_1) \cup R) - c(S^\pi(p_1)) \leq c(S^\pi(p_2) \cup R) - c(S^\pi(p_2)). \tag{6}$$

Here $S^\pi(p) = \{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(p)\}$.

In [47], it is shown that permutationally concave games are balanced and that mcst-games are permutationally concave. If π is a permutation for which (6) holds, then the marginal vector $m^\pi(c)$ is in the core of c. Related to this is the result derived in [4] that the set of restricted weighted Shapley-values is a subset of the irreducible core. In the computation of a restricted weighted Shapley-value, only so-called feasible permutations are considered.

Inequality (6) is satisfied by every feasible permutation. It follows that any convex combination of marginal vectors arising from feasible permutations is in the core. The Shapley-value itself is in general not an element of the core of an mcst-game. In [60], an axiomatization of the Shapley-value of mcst-games is given. In [33], it is shown that deciding whether a given vector is in the core of an mcst-game is $\mathcal{NP}$-complete. In [36], similar results are obtained for Steiner tree games. In [48], it is shown that the nucleolus of an mcst-game is the unique point in the intersection of the core and the kernel. In [49], the nucleolus of a standard tree game is characterized, and an algorithm to compute it is discussed. In [77], population monotonic allocation schemes for mcst-games are discussed, and an algorithm to compute such a scheme for an mcst-game is presented. In [35], it is shown that computing the nucleolus of a minimum cost spanning tree game is in general an $\mathcal{NP}$-hard problem. In [71], an $\mathcal{O}(n^3|\mathcal{B}|)$ algorithm that can be used to compute the nucleolus of a mcst-game is discussed. There the set $\mathcal{B}$ is a subset of 2^N. In [34], an algorithm to compute the nucleolus for certain classes of mcst-games based on the ellipsoid method and Maschler's scheme for approximating the pre-kernel is given. In [3], a noncooperative game is associated with every mcst-problem, and Nash equilibria and subgame perfect Nash equilibria of the game are studied. In [11], a short overview of work that discusses models that are related to mcst-games is given.

7 Location Games

Several *location problems* have been studied in the literature. In general, these problems treat situations in which certain facilities have to be placed in the nodes or along the edges of a given graph. There may be restrictions and/or demands with respect to the location of the facilities. There may be setup costs involved in establishing the facilities and costs that depend on the distance of the facilities from a given set in the graph. The problem is to minimize the costs that arise. Several *location games* arising from the various location problems have been studied. In [116], the following location game was studied. A connected graph $G = (V, E)$ is given. Each edge has a given length. The distance $d(v_1, v_2)$ between two nodes v_1, v_2 of G is defined to be the length of a shortest path from v_1 to v_2. Two subsets N and Q of V are given. N is the set of players. Each player is considered to be located in the corresponding node. $Q = \{q_1, \ldots, q_t\}$ is the set of possible locations for the facilities. The cost of establishing a center at q_j is $c_j \geq 0$. Player $i \in N$ demands that at least one center be located at a distance of at most r_i from him. The problem is to find a location of the facilities that satisfies all demands and minimizes costs. It is assumed that all the demands can be met. In the corresponding location game c, the cost $c(S)$ of coalition S is the minimum cost needed to satisfy the demands of the members of S. Let A be the $n \times t$-matrix defined by

$$a_{ij} = \begin{cases} 1 \text{ if } d(i, q_j) \leq r_i \\ 0 \text{ otherwise.} \end{cases}$$

Then for each $S \in 2^N \setminus \emptyset$, the cost $c(S)$ is given by

$$\begin{aligned} c(S) = \min \quad & cx \\ \text{s.t.} \quad & Ax \geq e^S \\ & x \in \{0, 1\} \end{aligned} \tag{7}$$

where e^S is the vector in R^n with 1 in the i-th place if $i \in S$ and 0 otherwise. In general, the location problem described above is $\mathcal{NP}$-hard, and a location game as defined above can have an empty core. But in [114] and [116], it was shown that if the graph G is a tree, then the corresponding location problem is polynomially solvable and the location game will be balanced. The proof uses the fact established in [114] that the matrix A is balanced if G is a tree. Therefore, problem (7) is equivalent to its linear programming relaxation. A similar approach to that used in Section 3 yields the nonemptiness of the core of the location game.

In [11], the following situation is studied. A connected graph $G = (N, E)$ is given. The set of nodes N corresponds with the set of players. Every edge $e \in E$ has a positive length l_e. The distance $d(x, y)$ between two points x, y anywhere on an edge is defined as the length of a shortest path from x to y. The length of a path is the sum of the length of the edges and parts of edges that belong to the path. For a finite subset A of points anywhere on the edges of G and a node $I \in N$, we define the distance $d(i, A)$ by

$$d(i, A) := \min_{x \in A} d(i, x).$$

Facilities can be constructed on any point along an edge of the graph. For each $i \in N$, a weight w_i is given such that the cost $c(\{i\})$ is equal to $w_i d_i$ where d_i is the distance between i and the set of facilities. Each coalition S is allowed to build p_S facilities. We assume that $p_S < |S|$. For each player $i \in N$, his cost of not having access to any facility is denoted by $L(i)$. This cost is taken to be very high. Two types of games arising from this situation are discussed in [11]. They are the *p-center* and the *p-median game*.

Definition 21. *The p-center game c_p is given by*

$$c_p(S) := \begin{cases} L_S & \text{if } p_S = 0 \\ \min_{A:|A|=p_S} \max_{i \in S} w_i d(i, A) & \text{if } p_S > 0. \end{cases}$$

Here $L_S = \max_{i \in S} L(i)$.

Definition 22. *The p-median game m_p is given by*

$$m_p(S) := \begin{cases} L(S) & \text{if } p_S = 0 \\ \min_{A:|A|=p_S} \sum_{i \in S} w_i d(i, A) & \text{if } p_S > 0. \end{cases}$$

Here $L(S) = \sum_{i \in S} L(i)$.

The p-center and p-median games arise from the p-center and p-median optimization problems. In [61] and [62], it is shown that for $p > 1$, these problems are $\mathcal{NP}$-hard.

In [11], it is shown that under certain conditions, both games are balanced. Also conditions are given that guarantee that c_p and m_p are 1-concave or semiconcave.

Another location game studied in [11] is the *simple plant location game.* In this game, the players correspond with the nodes of a tree. Facilities can be located only in the nodes of the tree. With each node, a certain setup cost is incurred if a facility is built in that node. With each edge of the tree, there is associated a travel cost. The aim of a coalition is to minimize the sum of the setup costs and travel costs of its members. Let o_j denote the setup cost if a facility is built in node $j \in N$.

Definition 23. *The simple plant location game c is defined by*

$$c(S) := \min_{\emptyset \neq A \subset N} \left(\sum_{j \in A} o_j + \sum_{i \in S} w_i d(i, A)\right) \text{ for all } S \subset N \setminus \emptyset.$$

Alternatively, $c(S)$ can be formulated as the value of a set covering problem. For each $i \in N$, let $0 = r_{i1} \leq r_{i2} \leq \cdots \leq r_{in}$ be the ordered sequence of distances between node i and all the nodes, including i. We define r_{in+1} to be a number that is much larger than the sum of all setup costs and travel costs that occur in the problem. The $n^2 \times n$-matrix $H = [h_{i_k j}]$ is defined by

$$h_{i_k j} = \begin{cases} 1 \text{ if } d(i,j) \leq r_{ik} \text{ for } i, j, k \in N \\ 0 \text{ otherwise.} \end{cases}$$

Let $d_{ik} = w_i(r_{ik+1} - r_{ik})$. Then $c(S)$ is also given by

$$\begin{array}{lll} c(S) = \min & \sum_{j=1}^n o_j x_j + \sum_{i \in S} \sum_{k=1}^n d_{ik} z_{ik} & \\ \text{s.t.} & \sum_{j=1}^n h_{i_k j} x_i x_j + z_{ik} \geq 1 & \text{for } i \in S,\ k \in N \\ & x_j \in \{0,1\} & \text{for } j \in N \\ & z_{ik} \in \{0,1\} & \text{for } i \in S,\ k \in N. \end{array} \tag{8}$$

In general, the set-covering problem is $\mathcal{NP}$-hard. In [68], it is shown that the set covering problem (8), which arises from a simple plant location problem, is equivalent to its linear programming relaxation. With the aid of the dual problem of the set-covering problem that determines $c(N)$, we can construct an element of the core of the simple plant location game in a similar fashion as was done in Section 3.

Two types of location games on trees that can have empty cores and that are discussed in [11] are the *median game with budget constraints* and the *center game with budget constraints.* In both games, the number of facilities that a coalition is allowed to build is not prescribed but is limited by the budget of the coalition. Let b_i be the budget of player i and let the budget of coalition S be $b(S) = \sum_{i \in S} b_i$.

Definition 24. *The median game with budget constraints m is given by*

$$m(S) := \begin{cases} \{\min_{A \subset N, o(A) \leq b(S)} \sum_{i \in S} w_i d(i, A) & \text{if } b(S) \geq \min_{j \in N} o_j \\ L(S) & \text{otherwise.} \end{cases}$$

In [11], conditions are given that guarantee the balancedness of a median game with budget constraints.

Definition 25. *The center game with budget constraints c is given by*

$$c(S) := \begin{cases} \min_{A \subset N, o(A) \leq b(S)} \max_{i \in S} w_i d(i, A) & \text{if } b(S) \geq \min_{j \in N} o_j \\ L_S & \text{otherwise.} \end{cases}$$

Conditions for the center game with budget constraints to be balanced are given in [11].

In [90], location games arising from continuous single facility location problems are studied. Sufficient conditions for the nonemptiness of the core of such a game are given.

8 Delivery Games

In a *delivery game* introduced in [52], each edge of a given undirected connected graph corresponds with a player. Each edge $j \in N$ has a travel cost t_j associated with it. A coalition S faces the following problem. Construct a cheapest walk that starts and ends in a specified node v_0 of the graph and that visits each edge in S at least once. The cost of a walk is the sum of the costs of the edges that it traverses. The cost $c(S)$ of coalition S in the delivery game c is the cost of such a walk minus the sum of the costs of the edges in S. The example used in [52] to illustrate this game is that of a post office that has to deliver mail along streets that correspond with edges in the graph and thus players in the game. Each player is responsible for covering the costs of traversing his street once. The other travel costs necessary to complete the walk have to be allocated among the players. Let $D(S)$ denote the set of walks that start and end in v_0 and that visit each edge of S at least once. For each walk $w \in D(S)$, let $t(w)$ be the cost of w.

Definition 26. *The cost $c(S)$ of coalition S in the delivery game c is given by*

$$c(S) := \min_{w \in D(S)} (t(w) - \sum_{i \in S} t_i).$$

The minimization problem above is known as the *Chinese postman problem* and was introduced in [72]. In [30], a polynomial algorithm for solving it was given. In [52], it was shown that delivery games need not be balanced. So just like the wallpaper game and the nonbipartite weighted matching game mentioned in Section 5, this is another example of a class of games that involve a combinatorial optimization problem that can be solved in polynomial time but that do not need to be balanced. In [52], *bridge-connected Euler graphs* are introduced, and it is shown that the delivery game on a bridge-connected Euler graph is balanced. A *bridge* in a graph is an edge that if removed causes the graph to become disconnected. A graph is called a bridge-connected Euler graph if each component that remains after all bridges have been removed contains an Euler cycle. Let b be a bridge in a bridge-connected Euler graph. The cost allocation that divides $t(b)$ among all the players that really need it to obtain service from v_0, i.e., for which b is on every path from v_0, is an element of the core of the delivery game.

In [50], *bridge-connected cyclic graphs*, a subset of bridge-connected Euler graphs, are introduced, and it is shown that the delivery game on a bridge-connected cyclic graph is concave. In [45], graphs that generate balanced, totally balanced, and submodular delivery games are studied. In [44], this is extended to the study of locally balanced, locally totally balanced, and locally submodular delivery games.

9 Conclusion

In this chapter, we have discussed cooperative games arising from combinatorial optimization problems. We have focused on eight classes of games with their variations and generalizations. From these classes, the variety of techniques used to analyze these games becomes clear. To establish balancedness and to find an element of the core, a general approach is possible when the combinatorial optimization problem that determines the game is equivalent to its linear optimization relaxation and the right-hand sides of the constraints satisfy certain conditions discussed in [10] and [12]. The approach used in these cases is similar to the one from [83] for linear production games. In other cases, methods that explore the particular structure of the game under consideration are used to establish balancedness. And in again other cases, the games are not balanced in general and modifications and/or subclasses that contain balanced games are studied. Another approach used for the classes of games that are in general not balanced is the use of *least tax cores* as in [15] and *ϵ-cores* as in [32] and [70]. In all these three articles, other games than the ones considered in this chapter are also studied.

Algorithmic aspects of the core, conditions that characterize totally balancedness, and conditions that characterize concavity for classes of combinatorial games not considered in this chapter are studied in [21, 22], and [81], respectively.

The study of one-point solution concepts reveals a picture that is even more diverse. Algorithms and/or formulae that compute or describe these really depend on the game. For the nucleolus and the τ-value, the situation is simplified in some cases because not all coalitions need to be considered.

Computational complexity, a consideration that has always been present in works on cooperative game theory given that most solution concepts involve computing the value or cost of all coalitions (which makes them exponential to compute unless there are simplifying circumstances), has been receiving more attention in the last ten to fifteen years. Examples of this mentioned in the previous sections are the questions of the core nonemptiness of a given instance of a combinatorial optimization game and the membership of the core of a given allocation x. Another example is the consideration of bounded agents in [99]. These topics may seem to pertain more to computer science than to game theory, but I believe that it is a sign of the maturity of an area of study when it starts raising questions in other areas.

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Algorithmic Cooperative Game Theory

Xiaotie Deng[1] and Qizhi Fang[2]

[1] Department of Computer Science, City University of Hong Kong, Hong Kong, P.R. China
csdeng@cityu.edu.hk

[2] Department of Mathematics, Ocean University of China, Qingdao, P.R. China
qfang@ouc.edu.cn

Abstract In this treatise, we survey some progress in cooperative game theory, in particular those involved with algorithmic and computational complexity issues. Central to these results is the linear program duality characterization of the core for some combinatorial optimization games. We highlight the linear and integer programming techniques and computational complexity approach applied to the core and the Nucleolus for various kinds of games, such as linear production game, flow game, minimum cost spanning tree game, packing and covering games, matching game, and facility location game.

Key words: cooperative game, core, nucleolus, linear program, LP duality, computational complexity, $\mathcal{NP}$-hard

1 Introduction

Cooperative game theory studies the problem of the revenue allocation for a set N of participants, called players, in a joint project where a value function v is defined for each subset of players, representing the revenue achieved by the players in the subset without assistance of other players. Much of cooperative game theory is built around the question how to distribute the collective income in fair and rational manners. Different philosophies result in different solution concepts, *e.g.*, the core, the Shapley value, the Nucleolus, the bargaining set, and the von Neumann–Morgenstern solution set [9, 65]. There are arguments why each such proposal is a reasonable mathematical rendering of the intuitive concept of "fairness."

In general, each solution concept defines, for each cooperative game (N, v), a set $\mathcal{F}$ of allocation vectors. Intuitively, an allocation is considered "fair" if it belongs to this class. Such a set $\mathcal{F}$ could be a singleton such as the case of the Shapley value and the Nucleolus. It could also contain an indefinite number of vectors such as the core. Deciding whether an allocation is in a targeted solution concept set is in general a nontrivial problem and has always been an important issue in the study of cooperative game theory. A particularly interesting theme to the study of such decision problems is that of the bounded rationality, which argues that decisions made by real-life agents may not spend an unbounded amount of resources to evaluate all the possibilities for optimal outcome [66]. Much effort has been made in the study of the bounded rationality in computational resource for solution concepts of cooperative games.

The computational complexity study on cooperative games is especially interesting as the definition of a game involves an exponential number (in the number of players) of values, one for each subset of players. Moreover, the definitions of many solution concepts would involve an exponential number of constraints. Megiddo [50] observed that, for many games, the game value is calculated through succinctly defined structures and for such games, he suggested that finding a solution should be done by a good algorithm (following Edmonds [17]), *i.e.*, within time polynomial in the number of players. Deng and Papadimitriou [14] suggested computational complexities be taken into consideration as another measure of fairness for evaluating and comparing different solution concepts.

An especially fruitful case for the computational complexity approach in cooperative game theory is the class of combinatorial optimization games (see, *e.g.*, [12]). In a cooperative game, when the value of a subset of players is evaluated via a combinatorial optimization problem, subject to constraints of resources controlled by members in the subset, the input size is usually polynomial in the number of players. Therefore, such combinatorial optimization games fit well into the framework of algorithm theory. Indeed, such a line of research has been very active in the past decade. In this treatise, we should focus on those in the study of the core and the Nucleolus.

The organization of the treatise is as follows. In Section 2, formal definitions of cooperative game and solution concepts are given. We also give a sketch of combinatorial optimization game models and related algorithmic and complexity results. In Section 3, we introduce Owen's linear production game [56] and Granot's generalized model [31]. In view of the importance of Owen's model, we focus on the Linear Programming (LP) duality characterization of the core allocation in Owen's work and exemplify its applications with flow game and minimum cost spanning tree game. Additionally, complexity results on these models are also discussed. In Section 4, the packing, covering, and partition games are brought in as a natural extension of Owen's model with integrality condition explicit. The common necessary and sufficient condition on the balancedness of these games is that a corresponding LP relaxation has an integer optimal solution. The sufficiency of this condition follows immediately from Owen's work [56]. Here, we use matching game, vertex covering game, and minimum coloring game to illustrate a variety of computational complexity results for this class of games. In Section 5, we further investigate the linear and integer programming approach and LP duality techniques applied to facility location games. Finally in Section 6, we conclude with some further discussions and remarks.

2 Definitions and Models

A cooperative game with side payments is a pair (N, v), where $N=\{1, 2, \ldots, n\}$ is a finite set and $v : 2^N \to R$ is a function with $v(\emptyset) = 0$. The elements of N are called players, and the subsets S of N are called coalitions. For each coalition $S \subseteq N$, $v(S)$ is the value of S that is interpreted as the profit or cost achieved by the collective action of players in S without any assistance of players in $N \setminus S$. The function v is called the characteristic function. A game is called a profit (cost) game if $v(S)$ measures the profit (cost) achieved by the coalition S. In this section, we present the definitions only for profit games. Symmetric statement holds for cost games.

The focus of cooperative game theory has always been how to fairly distribute the collective income. We denote the income distributed to individual players by a vector $x = (x_1, x_2, \ldots, x_n)$ satisfying $\sum_{i=1}^{n} x_i = v(N)$, called an allocation. Throughout this treatise, we use the shorthand notation $x(S) = \sum_{i \in S} x_i$ for $S \subseteq N$. An allocation vector x is called an *imputation* of the game (N, v) if it also satisfies the individual rationality condition:

$$\forall i \in N : \quad x_i \geq v(\{i\}).$$

The set of imputations of game (N, v) is denoted by $I(v)$. Additional requirements for fairness, stability, and rationality lead to different sets of allocations, which are generally referred to as solution concepts. As limited by space, we shall discuss two of the most important ones: the core and the Nucleolus.

2.1 The Core

The concept of the core was first introduced by Gillies [28] based on the concept of subgroup rationality.

Definition 1. *The core of a game* (N, v) *is defined by*

$$C(v) = \{x \in R^{|N|} : x(N) = v(N) \text{ and } x(S) \geq v(S),\ \forall S \subseteq N\}.$$

The constraints imposed on $C(v)$ ensure that no coalition would have an incentive to split from the grand coalition N and do better on its own. Consider the following allocation linear program (AP):

$$\text{AP}: \qquad \begin{array}{ll} \min & \sum_{i \in N} x_i \\ \text{s.t.} & \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N. \end{array}$$

It is quite obvious that $C(v) \neq \emptyset$ if and only if the optimum value of the linear program (AP) is equal to $v(N)$, in which case any optimal solution to (AP) lies in $C(v)$. Taking the linear program dual to (AP), an equivalent condition for $C(v) \neq \emptyset$ can be obtained based on the concept of balanced sets. A collection $\mathcal{B}$ of nonempty subsets of N is balanced if there exists a set of positive numbers γ_S, $S \in \mathcal{B}$, such that for each $i \in N$, $\sum_{i \in S \in \mathcal{B}} \gamma_S = 1$. A cooperative game (N, v) is *balanced* if $\sum_{S \in \mathcal{B}} \gamma_S v(S) \geq v(N)$ holds for every balanced collection $\mathcal{B}$ with weights $\{\gamma_S : S \in \mathcal{B}\}$. Bondareva [3] and Shapley [62] proved that a game has a nonempty core if and only if it is balanced. For a subset $S \subseteq N$, we define the induced subgame (S, v_S) on S, in which $v_S(T) = v(T)$ for every subset $T \subseteq S$. A cooperative game is *totally balanced* if all its subgames are balanced.

For the core of a cooperative game (as for other solution concepts that form a subset of imputations), we have the following algorithmic and complexity problems:

Testing nonemptiness: Can it be tested in polynomial time whether a given instance of the game has a nonempty core?
Checking membership: Can it be checked in polynomial time whether a given imputation belongs to the core?
Finding a core member: Is it possible to find an imputation in the core in polynomial time?

The three problems are closely related, however, they may in general possess different complexities. It is possible that a core member can be found in polynomial time but it is *co*-$\mathcal{NP}$-complete for the membership checking problem [18, 24]. For a game of sum of edge weight defined on a graph, Deng and Papadimitriou [14] proved that both problems of testing nonemptiness and checking membership of the core are $\mathcal{NP}$-hard.

Some related solution concepts arise from the core. Shapley and Shubik [64] recommended the concepts of *(strong)* ε*-core* and *weak* ε*-core* for a cooperative game. Their main idea is to relax the requirements of subgroup rationality by $x(S) \geq v(S) - \varepsilon$ and $x(S) \geq v(S) - \varepsilon|S|$ for each proper subset S of N, respectively. Later, Tijs and Driesssen [72] introduced the concept of *multiplicative* ε*-tax core* by using $x(S) \geq v(S) - \varepsilon[v(S) - \sum_{i \in S} v(i)]$ instead. Faigle and Kern [22] modified the requirement of Tijs and Driessen's as $x(S) \geq (1 - \varepsilon)v(S)$ to define another approximate core, called *multiplicative* ε*-core*. One explanation of these concepts is that cooperation may not be as hopeless even when the core is empty. Cooperation may be possible with the subsidies of the central authority. Therein, the computational complexity approach is also very promising in order to properly foster the necessary cooperation.

2.2 The Nucleolus

One of the dissatisfactions with the core and some other solution concepts is that there is no definite outcome, though this may allow for flexibility in applications of these concepts to some areas such as economics and political science. The Nucleolus, introduced by Schmeidler [61], trying to capture the intuition of minimizing dissatisfaction of players, is one of the most well-known solution concepts among various attempts to obtain a unique solution and has been made popular especially because of a discovery by Aumann and Maschler [1] in its association with a Talmudic myth.

Let (N, v) be a cooperative game with n players. Given an allocation $x \in R^n$, we define $e(S, x) = x(S) - v(S)$ for each $S \subseteq N$. This number is called the *excess* of S at x and can be interpreted as a measure of satisfaction of the coalition S with the allocation x. Thus, we arrive at the core $C(v)$ as the set of allocations whose excesses are all non-negative. For an allocation $x \in R^n$, let $\theta(x)$ denote the $(2^n - 2)$-dimensional vector whose components are the non-trivial excesses $e(S, x)$, $\emptyset \neq S \neq N$, arranged in a nondecreasing order. That is, $\theta_i(x) \leq \theta_j(x)$, for $1 \leq i < j \leq 2^n - 2$. Denote by $\succeq_l$ the "lexicographically greater than" relationship between vectors of the same dimension.

Definition 2. *The Nucleolus $\eta(v)$ of a game (N, v) is the set of imputations that lexicographically maximize $\theta(x)$ over the set of all imputations $x \in I(v)$. That is,*

$$\eta(v) = \{x \in I(v) : \theta(x) \succeq_l \theta(y) \text{ for all } y \in I(v)\}.$$

Surprisingly, such a complicatedly defined solution, according to Aumann and Maschler [1], was the foundation that dictated a particular schema for the estate division problem set by Rabbi Nathan that baffled Talmudic scholars for two millennia. The problem is one of three wives married to a man who promised them 100, 200, and 300 zuz, respectively, upon his death. The husband died leaving an estate worth less than 600 zuz. According to the Talmud recommendation, the wives would receive an equi-partition of the estate if it is worth 100; but a proportional partition of the promised amount if it is worth 300. And even more unexpectedly, if the estate is worth 200, the wives will received 50, 75, and 75, even though the last two wives' claims were not equal. Such an intricacy has been made clear only after the work of Aumann and Maschler, showing the Tamudic solution's coincidence with the Nucleolus. The Talmud rule has since been credited as anticipation of the modern cooperative game theory.

Even though, by definition, the Nucleolus may contain multiple points, it was proved by Schmeidler [61] that the Nucleolus of a game with the nonempty imputation set contains exactly one element. Kopelowitz [44], with Maschler, Peleg, and Shapley [52] proposed to compute the Nucleolus by recursively solving the following sequential linear programs:

$$\mathrm{LP}_k : \quad \begin{array}{ll} \max \varepsilon & \\ \quad x(S) = v(S) + \varepsilon_r & \forall S \in \mathcal{J}_r \quad r = 0, 1, \ldots, k-1 \\ \quad x(S) \geq v(S) + \varepsilon & \forall S \in 2^N \setminus \displaystyle\bigcup_{r=0}^{k-1} \mathcal{J}_r \\ \quad x \in I(v) & \end{array}$$

Here, we set $\mathcal{J}_0 = \{\emptyset, N\}$ and $\varepsilon_0 = 0$ initially; the number ε_r is the optimum value of the r-th program (LP$_r$), and $\mathcal{J}_r = \{S \in 2^N : x(S) = v(S) + \varepsilon_r$ for every $x \in X_r\}$, where $X_r = \{x \in I(v) : (x, \varepsilon_r)$ is an optimal solution to LP$_r\}$. It can be shown that after at most $n - 1$ iterations, one arrives at a unique optimal solution (x^*, ε_k). The vector x^* is just the Nucleolus of the game.

Because the computation of the Nucleolus requires solutions of sequential linear programs, each with constraints exponential in the number of players, it has been a challenge to obtain polynomial time algorithms [38,50]. Though in some cases the Nucleolus can be calculated in polynomial time, such as the assignment game [67] and the convex game [21, 46], it is in general very hard [11, 19]. There are still some general algorithms for the computation of the Nucleolus [36,59], however, they do not guarantee a polynomially bounded running time except for some special classes of models.

2.3 Combinatorial Optimization Game Models

An important application of cooperative games is that they provide a mathematical formulation for collective decision-making and optimization problems. In such circumstances, very often, the characteristic function value of a coalition can be represented succinctly as the optimum value of a combinatorial optimization problem. Such cooperative games are called combinatorial optimization games.

Combinatorial optimization has been a rich and fruitful research field. The usual consideration in an optimization problem is a single objective function of one agent. Often, however, problems arising from its application involve more than one participant who may have different objectives and control different resources. Cooperative game theory has developed important methodologies to study fairness and rationality in collaborations and deal with conflicting interests. In an example of the facility location model, customers from a given set are in need of certain service that can be provided by connecting them to some facilities. These facilities could be railway stations, libraries, or supermarkets. From a certain authority's point of view, the goal is to minimize the total cost, which is made up of the costs of building facilities and connecting the customers to the open facilities. On the other hand, it is expected to find a fair allocation of the total cost to all customers such that none of the coalitions of customers has any incentive to build their own facility or to ask a competitor to serve them. The cooperative game theoretical approach becomes the natural choice for such problems.

The combinatorial optimization games lead to the applications of a variety of combinatorial optimization techniques, especially the linear and integer programming techniques, which have proved to be powerful in design and analysis of algorithms, as well as establishing complexity results. Here, one of the most interesting results is the LP duality characterization of the core.

In this subsection, we give a sketch of several classic combinatorial optimization game models and related algorithmic results.

An example to formulate a two-sided market as a cooperative game, the assignment game, was given by Shapley and Shubik [63]. The underlying structure is a bipartite graph $(V_1, V_2; E)$, where V_1 is the set of sellers and V_2 is the set of buyers. For the simplest case, each seller has an item to sell and each buyer wants to purchase an item. The i-th seller values his item at c_i dollars and the j-th buyer values the item of the i-th seller at h_{ij} dollars. Between this pair, we may define a value $v(\{i,j\}) = h_{ij} - c_i$ if $h_{ij} \geq c_i$ and set (i,j) an edge in E with weight $v(\{i,j\})$. Otherwise, there is no edge between i and j as no deal is possible if the seller values the item more than the buyer does. Consider a game with side-payment; the value $v(S)$ of a subset S of buyers and sellers is defined as the weight of a maximum weighted matching in the bipartite subgraph $G[S]$ induced by S. Shapley and Shubik [63] established a complete characterization of the core of this model, which says the core is exactly the set of optimal dual solutions to the linear program formulation of the assignment problem. A major factor for this result is that the optimal solution to the corresponding linear program can be achieved at integer points. This approach has been exploited extensively in other game models, such as, location games defined on trees in Tamir [68], partition games in Faigle and Kern [23], and packing/covering games in Deng, Ibaraki, and Nagamochi [12].

The model of Shapley and Shubik is a theoretical formulation for a pure exchange economy. The linear production game in Owen [56] applies their ideas to a production economy. Therein, each player j $(j \in N)$ is in possession of an individual resource vector b^j. For a coalition S of players, the profit obtained by S is the optimum value of the following linear program:

$$\max\ \{c^t y : Ay \leq \sum_{j \in S} b^j, y \geq 0\}.$$

Thus, the characteristic function value is what the coalition can achieve in the linear production model with the resources under their control. Owen showed that one imputation in the core can also be constructed through an optimal dual solution to the linear program that determines the value of N. However, unlike the assignment game, there are in general some imputations in the core that cannot be obtained in this way. The problem of checking membership of the core for linear production games has been proved to be $\mathcal{NP}$-hard [5, 24].

Kalai and Zemel studied games of network flows [40, 41]. In this game, the players are associated with arcs of the network. The value of a coalition is the maximum flow value from the 'source' to the 'sink' in the subnetwork consisting of the original vertex set and those arcs corresponding with the players in the coalition. For a simple network in which arc capacities are all one, the core of the associated flow game coincides with the convex hull of the indicator vectors of minimum cuts in the network. We note that although a flow game can be formulated as a linear production game, the size of the reduction may be exponential in space, and consequently, the complexity results are independent.

Bird [2], and independently, Claus and Granot [6] have formulated a minimum cost spanning tree game for cost allocation problem in a communication network. In this game model, each player corresponds with a vertex of the network, and there is one more external vertex 0 representing a central supplier. The cost of a coalition S is defined as the weight of a minimum spanning tree in the subnetwork induced by the vertex 0 and vertices in S. It was shown that the core of this game is always nonempty and an imputation in the core can be calculated easily from a minimum spanning tree of the network [2, 7, 34]. However, it was proved by Faigle *et al.* [18] that checking membership of the core is *co*-$\mathcal{NP}$-complete in this model.

As extensions to minimum cost spanning tree games, there are some related game models that were investigated, such as minimum cost forest game [45] and minimum base game on matroid [53]. In another direction, Megiddo has formulated a network cost allocation problem differently by defining the cost of a coalition as the weight of a minimum Steiner tree that contains not only vertices corresponding with the coalition but also some switches in the network [51]. This model results in a computationally harder problem because given a subset S of vertices, it is $\mathcal{NP}$-hard to evaluate the weight of a minimum Steiner tree spanning S. By contrast with the minimum cost spanning tree games, the core of this game may be empty. Fang *et al.* [25] further proved that both problems of testing nonemptiness of the core and testing total balancedness are $\mathcal{NP}$-hard for the Steiner tree games.

A facility location game is introduced to formulate the cost allocation problem in a facility location model. In this model, there is a set of customers that needs a certain service from some facilities and a set of possible locations for opening the facilities. For each coalition of customers, its value is defined as the minimum total cost consisting of the costs of opening facilities and connecting each customer in this coalition to an open facility. Goemans and Skutella [29] proved that for this game, it is in general $\mathcal{NP}$-hard to decide whether the core is nonempty and decide whether a given allocation belongs to the core. However, given the information that the core is nonempty, both finding a core member and checking whether a given allocation belongs to the core can be solved efficiently. In a special case where all the customers and facilities are located on the vertices of a tree, it was proved by Kolen [43] that testing nonemptiness and checking membership of the core are both polynomially solvable.

There are still many game models arising from classic combinatorial optimization problems, including dominating set game [73], traveling salesman (TSP) game [15, 58, 70], Chinese postman game [39], and so on.

In the following sections, we will highlight the linear and integer programming techniques and computational complexity approach applied to the cores and the Nucleoli of cooperative games. The focal point of our discussion will be three typical combinatorial optimization games: linear production games, packing/covering/partition games, and facility location games.

3 Linear Production Games

The most interesting connections with combinatorial optimization theory in the study of cooperative games is the characterization of the core. From this aspect, LP duality has proven itself a very powerful tool. Shapley and Shubik [63] proved that for the assignment game associated with a two-sided market, the core is exactly the set of optimal dual solutions to the linear program formulation of the assignment problem. This approach was further exploited in Owen's linear production game [56], where a core allocation can be immediately obtained from an optimal solution to a corresponding dual program. After Owen's work, the linear production game has been fully utilized as a unified model to explain the nonemptiness of the core for many combinatorial optimization games.

In Owen's model, there are n players, and each player possesses a certain amount of m different resources. The resources vector of player i ($i = 1, 2, \ldots, n$) is $b^i = (b^i_1, b^i_2, \ldots, b^i_m)^t$ with $b^i_k \geq 0$ being the amount of the k-th resource possessed by player i. These resources can be used to produce p different products, and each unit of product j ($j = 1, 2, \ldots, p$) can be sold at a given market price c_j, and we denote $c = (c_1, c_2, \ldots, c_p)$. Let $A = [a_{kj}]_{m\times p}$ be the linear production matrix, where a_{kj} is the amount of the k-th resource needed to produce one unit of the j-th product. Then the linear production game $\Gamma_{lp} = (N, v)$ is defined as follows:

(i) The player set is $N = \{1, 2, \ldots, n\}$;
(ii) For each coalition $S \subseteq N$, $v(S)$ is the maximum profit that the coalition S can achieve with the resources under its control, *i.e.*,

$$v(S) = \max\ \{cx:\ Ax \leq \sum_{j\in S} b^j,\ x \geq 0\}.$$

Theorem 1. *([56]) The linear production games are totally balanced.*

A constructive proof presented by Owen [56] obtains an imputation in the core from an arbitrary optimal dual solution to the linear program that determines $v(N)$. Let w be an optimal solution to the linear program:

$$\min\ \{\sum_{j\in N} w^t b^j:\ w^t A \geq c^t,\ w \geq 0\},$$

which is dual to the following linear program *w.r.t.* the grand coalition N,

$$v(N) = \max\ \{cx:\ Ax \leq \sum_{j\in N} b^j,\ x \geq 0\}.$$

Define $u = (u_1, u_2, \ldots, u_n)$ by $u_j = w^t b^j,\ j \in N$. The LP duality theorem implies that $u(N) = v(N)$. On the other hand, let x^*_S be an optimal solution to the linear program that determines $v(S)$, then

$$u(S) = \sum_{i \in S} w^t b^i \geq w^t A x_S^* \geq c x_S^* = v(S).$$

Hence, $u \in C(v)$.

Note that this proof describes a simple way to arrive at a core allocation, implying that for linear production games, the problem of finding a core member can be done in polynomial time. However, unlike the assignment games, there are in general some core members that cannot be obtained in this way. It is natural to ask if we can determine an imputation is in the core or not with an efficient algorithm. A negative answer has been given by Chvátal [5] and Fang *et al.* [24].

Theorem 2. *([5, 24]) For linear production games, the problem of checking membership of the core is co-$\mathcal{NP}$-complete.*

In Owen's linear production game, it is required that for each coalition $S \subseteq N$, the total amount of the k-th resource controlled by S satisfies additivity assumption, *i.e.*, $b_k(S) = \sum_{i \in S} b_k^i$ $(k = 1, 2, \ldots, m)$. A generalized model investigated by Granot [31] retains the main linear program structure of Owen's model but allows right-hand sides of the resource constraints not to satisfy the additivity assumption. That is, in the generalized linear production game, the value of a coalition $S \subseteq N$ is defined by

$$v(S) = \max\{cx : \ Ax \leq b(S), \ x \geq 0\},$$

where $b(S) = (b_1(S), b_2(S), \ldots, b_m(S))^t$ and each $b_k(S)$ $(k = 1, \ldots, m)$ is a general function of S.

Theorem 3. *([31]) If the games consisting of player set N with value function $b_k(S)$, $S \subseteq N$, for all $k \in \{1, 2, \ldots, m\}$ are balanced (resp., totally balanced), then the generalized linear production game is also balanced (resp., totally balanced).*

With the same technique as in Owen's work, a core allocation can also be constructed from an optimal dual solution to the corresponding linear program when the generalized linear production game is balanced.

For certain classes of cooperative games, such as flow games and minimum cost spanning tree games, there is a natural way to formulate them as (generalized) linear production games. Therefore, Owen's model, including Granot's generalized model, has been employed as a unified tool to show the balancedness of these games.

Note that the linear production games are equivalent to the class of nonnegative totally balanced games [9, 16]. However, the reductions in the equivalence proof requires exponential time and space in the number of players. Consequently, complexities results for different totally balanced games should be independent.

3.1 Flow Games

Flow games were first discussed by Kalai and Zemel [40, 41]. Consider a directed network $D = (V, E; \omega)$, where V is the vertex set, E is the arc set, and $\omega : \ E \to R^+$ is the arc capacity function. Let s and t be two distinct vertices by which we denote the 'source' and the 'sink' of the network, respectively. We assume that each player controls one arc in the network. Then the flow game $\Gamma_f = (E, v)$ associated with the network D is defined as follows:

(i) The player set is $E = \{1, 2, \ldots, n\}$;
(ii) For each coalition $S \subseteq E$, $v(S)$ is the value of a maximum flow from s to t in the subnetwork of D consisting only of arcs belonging to S.

Let $\mathcal{P}$ be the set of s-t paths in D and $A = [a_{ij}]$ be the arc-path incidence matrix, where the rows of A correspond with the arcs in E, and the columns correspond with s-t paths in $\mathcal{P}$, $a_{ij} = 1$ if arc i is on the jth s-t path, and $a_{ij} = 0$ otherwise. Also define a vector $h_{_S} \in R^n$ with the j-th component being $\omega(j)$ if $j \in S$ and 0 otherwise. Thus the flow game $\Gamma_f = (E, v)$ can be formulated as a linear production game as follows:

$$\forall\, S \subseteq E, \quad v(S) = \max\{1^t y : \ Ay \leq h_{_S}, \ y \geq 0\}$$

It follows directly from Theorem 1 that the flow game Γ_f is totally balanced and a core allocation can be obtained from a minimum cut of D, which corresponds with an optimal dual solution to the linear program associated with $v(E)$. However, like linear production games, checking membership of the core is still $\mathcal{NP}$-hard for flow games [24].

Theorem 4. *([12, 24]) The flow games are totally balanced, and finding a core member can be done in polynomial time. On the other hand, checking membership of the core is co-$\mathcal{NP}$-complete.*

Because the cardinality of (s, t)-paths is typically huge, the arc-path formulation described above is of little use from an algorithmic point of view. Hence, an alternative arc-flow formulation was exploited to study the core by Kalai and Zemel [41]. A network is called simple if the capacity of each arc is equal to 1. Given a simple network $D = (V, E)$, for $W \subseteq V$, denote by $\delta^+(W)$ and $\delta^-(W)$ the sets of arcs leaving and entering W, respectively. Define a function $c : E \to \{0, 1\}$ with $c(e) = 1$ if $e \in \delta^+(\{s\})$, and $c(e) = 0$ otherwise. Then the maximum flow problem in the network D has the following linear program formulation:

$$(\mathrm{LP}_f) \qquad \begin{array}{lll} \max \displaystyle\sum_{e \in E} c(e) y(e) & & \\ s.t. \displaystyle\sum_{e \in \delta^+(\{v\})} y(e) - \sum_{e \in \delta^-(\{v\})} y(e) = 0 & & \forall\, v \in V \setminus \{s, t\} \\ \quad 0 \leq y(e) \leq 1 & & \forall\, e \in E \end{array}$$

The dual program of (LP_f) is

$$(\mathrm{DLP}_f) \qquad \begin{array}{lll} \min \displaystyle\sum_{e \in E} z(e) & & \\ s.t. \;\; z(e) + \phi(v) - \phi(w) \geq c(e) & & \forall\, e = (v,w) \in E \\ z(e) \geq 0 & & \forall\, e \in E \end{array}$$

Theorem 5. *([41]) Let z be a core member of the flow game (E, v) defined on a simple network $D = (V, E)$. Then there exists $\phi = \{\phi(v) : v \in V\}$ such that (z, ϕ) is an optimal solution to (DLP_f).*

Followed immediately from Theorem 5 and the fact that the minimum cuts of D constitute the extreme dual solutions to the maximum flow problem, it concludes that for a simple network, the core of the corresponding flow game is exactly the convex hull of the indicator vectors of the minimum cuts. Kalai and Zemel [41] also conjectured that Theorem 5 may serve as a practical basis for calculating the Nucleolus. Recently, with an elegant application of LP duality approach in Kalai and Zemel's work [41], Deng, Fang, and Sun [11] proposed an efficient algorithm for computing the Nucleolus of a simple flow game, settling the conjecture. They also gave an $\mathcal{NP}$-hardness proof on both computation and recognition of the Nucleolus for general flow games.

Theorem 6. *([11]) Let (E, v) be the flow game defined on a simple network $D = (V, E)$. Then the Nucleolus $\eta(v)$ can be computed in polynomial time.*

Theorem 7. *([11]) Let $D = (V, E; \omega)$ be a network with general arc capacities and (E, v) be the corresponding flow game. Then both problems of computing the Nucleolus $\eta(v)$ and checking whether a given core member is the Nucleolus are $\mathcal{NP}$-hard.*

3.2 Minimum Cost Spanning Tree Games

The power of Granot's generalized linear production model can be applied to prove the nonemptiness of the core for several games beyond those of Owen's model. In particular, these include some games associated with network optimization problems, such as the minimum cost spanning tree game, the network synthesis game, and the weighted matching game [31].

The minimum cost spanning tree game, MCST game for short, has been studied extensively after its introduction by Bird [2]. Denote by $N = \{1, 2, \ldots, n\}$ a set of customers who all need to be connected to some central supplier denoted by 0. Establishing a direct link between any pair (i, j) $(i, j \in N \cup \{0\})$ is assumed to cost a non-negative weight $\omega(i, j) = \omega(j, i)$. The objective is to create a connected graph on the vertex set $N \cup \{0\}$ and to distribute the resulting total cost among all the customers. This brings out the MCST game $\Gamma_{st} = (N, v)$ in a natural way:

(i) The player set is $N = \{1, 2, \ldots, n\}$;
(ii) For each coalition $S \subseteq N$, $v(S)$ is the weight of a minimum spanning tree in the induced subgraph $G[S \cup \{0\}]$.

Granot [31] formulated the MCST game as a generalized linear production game with exponential number of constraints, and consequently proved its balancedness. A shortcoming of his proof is that it does not provide an efficient scheme to compute a core member. Tamir [69] further presented a linear program formulation with polynomial size, showing that several discrete network synthesis games, including MCST games, satisfy Owen's linear production game model.

Let us describe Tamir's formulation as follows. For an MCST game $\Gamma_{st} = (N, v)$, the value of a coalition $S \subseteq N$ can be represented as the optimum value of the following mixed integer program (MP):

$$
\begin{aligned}
v(S) = \min \quad & \sum_{\forall i,j \in N \cup \{0\}} \omega_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{\forall j \in N \cup \{0\}} f_{ij}^k - \sum_{\forall j \in N \cup \{0\}} f_{ji}^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, 0 \end{cases} \qquad \forall\, k \in S, \forall\, i \in N \cup \{0\} \\
& 0 \leq f_{ij}^k \leq y_{ij} \qquad \forall\, k \in S,\ \forall\, i, j \in N \cup \{0\} \\
& y_{ij} \in \{0, 1\} \qquad \forall\, i, j \in N \cup \{0\}
\end{aligned}
$$

For each $S \subseteq N$, let $\tilde{v}(S)$ denote the optimum value of the LP-relaxation of (MP). Tamir showed that each core member of the game $(N, \tilde{v})$ is also in the core of the original game $\Gamma_{st} = (N, v)$. Thereby, the formulation (MP) casts the MCST game as Owen's linear production game, implying that the core of the MCST game is nonempty and a core member can be generated from an optimal dual solution to the linear program that determines $\tilde{v}(N)$.

It is very interesting that one of those dual optimal solutions corresponds with the particular core member given in a "greedy" allocation scheme. This "greedy" scheme was originally discussed in, *e.g.*, Claus and Kleitman [7] and Bird [2], and has been rigorously proved to yield a core allocation by Granot and Huberman [33]: find a minimum spanning tree T^* on $N \cup \{0\}$ and allocate to each player $i \in N$ the weight of the first edge that i encounters on the unique path from i to 0 in T^*.

Theorem 8. *([2, 7, 33, 69]) The MCST games are balanced, and finding a core member can be done in polynomial time.*

Even though an imputation in the core can be found easily for an MCST game, Faigle *et al.* [18] showed that it is $\mathcal{NP}$-hard to decide whether a given imputation is a core member. Tamir [69] also pointed out that a result of Chvátal's [5] implies $\mathcal{NP}$-hardness of checking membership of the core for the class of network synthesis games, which includes MCST games.

Theorem 9. *([18]) For MCST games, the problem of checking membership of the core is co-$\mathcal{NP}$-complete.*

In regard to the Nucleolus, a variety of algorithmic results have been established for MCST games. Megiddo [50] first described an $O(n^3)$ algorithm for computing the Nucleolus in a special case where the underlying graph is a tree. Galil [27] subsequently reduced the number of operations in the algorithm to $O(nlogn)$. Faigle, Kern, and Kuipers [21] proposed an efficient algorithm of computing the Nucleolus based on ellipsoid method for a class of more general MCST games. An even more special case is obtained when G itself is restricted to a chain. In Littlechild [48], he identified a class of $O(n)$ coalitions that are the only relevant coalitions for the computation of the Nucleolus, and essentially developed an $O(n^2)$ algorithm. Later, this result was improved to a linear time algorithm by Galil [27] and Granot *et al.* [35, 36]. For general cases, however, a negative answer for computing the Nucleolus was given by Faigle, Kern, and Kuipers [19].

Theorem 10. *([19]) For MCST games, computing the Nucleolus is $\mathcal{NP}$-hard.*

4 Packing, Covering, and Partition Games

Another way to extend Owen's model is to explicitly require integer solutions in the definition of a linear production model, yielding some game models with combinatorial nature. Specifically, one may define the game value $v(S)$ as the optimum value of an integer program instead of a linear program:

$$v(S) = \max \{cx : \ Ax \leq \sum_{j \in S} b^j, \ x \geq 0 \text{ and } x \text{ is integral}\}.$$

For the assignment game of Shapley and Shubik [63], the integer program can be solved by its LP-relaxation, as there is always an integer solution for the latter. In Shapley and Shubik's model, b^j is a unit vector and $b(N)$ is a vector of all ones. It is this particular structure of linear constraints that makes the core to be identified with the set of optimal dual solutions to a corresponding linear program. This property is further investigated by Faigle and Kern for partition games [23], and by Deng, Ibaraki, and Nagamochi for packing and covering games [12].

Let A be an $m \times n$ $\{0,1\}$-matrix, $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ be the corresponding index sets of rows and columns, respectively. Let $c = (c_1, \ldots, c_m)^t$ be an m-dimensional vector and $d = (d_1, \ldots, d_n)^t$ be an n-dimensional vector. For a subset $S \subseteq N$, let $\mathbf{1}_S \in R^n$ denote the indicator vector of S, where $\mathbf{1}_S(i) = 1$ if $i \in S$, and $\mathbf{1}_S(i) = 0$ otherwise.

The corresponding packing game $\Gamma_{pac} = (N, v)$ is defined by

(i) The player set is N;
(ii) For each coalition $S \subseteq N$, $v(S)$ is the optimum value of the integer program (IP$_{pac}$):

$$\begin{aligned} v(S) = &\max x^t c \\ &\text{s.t.} \quad x^t A \leq \mathbf{1}_S, \ x \in \{0,1\}^m \end{aligned}$$

The corresponding covering game $\Gamma_{cov} = (M, v)$ is defined by

(i) The player set is M;
(ii) For each coalition $T \subseteq M$, $v(T)$ is the optimum value of the integer program (IP$_{cov}$):

$$v(T) = \min d^t x$$
$$\text{s.t.} \quad A_{T,N} x \geq \mathbf{1}, \; x \in \{0,1\}^n.$$

where $A_{T,N}$ is the submatrix consisting of the rows of A *w.r.t.* the coalition T, and $\mathbf{1} \in R^{|T|}$ is the vector with all components being 1.

The corresponding partition game $\Gamma_{par} = (N, v)$ is defined by

(i) The player set is N;
(ii) For each coalition $S \subseteq N$, $v(S)$ is the optimum value of the integer program (IP$_{par}$):

$$v(S) = \max x^t c$$
$$\text{s.t.} \quad x^t A = \mathbf{1}_S, \; x \in \{0,1\}^m.$$

In the rest of this section, for all game models discussed, we let ILP* denote the corresponding integer program that determines the value of the grand coalition, and LP* and DLP* denote the corresponding LP-relaxation of ILP* and its dual program, respectively. The next theorem provides a description on the combinatorial structure for the corresponding packing, covering, and partition games to be balanced.

Theorem 11. *([12,23]) The core of the packing game Γ_{pac} is nonempty if and only if the LP-relaxation LP* has an integer optimal solution. In such case, the core coincides with the set of optimal solutions to the dual program DLP*. The same conclusion holds for the covering game Γ_{cov} and the partition game Γ_{par}.*

There are many interesting cooperative games defined on graphs, which can be formulated as packing and covering games [12]. For example,

(1) *s*-*t* edge connectivity game, *s*-*t* vertex connectivity game, and maximum *r*-arborescence game;
(2) matching game and vertex covering game;
(3) independent set game and edge covering game;
(4) minimum coloring game.

These game models offer a variety of complexity results on the computational problems related to their cores. For the games in the first category, all of them are always balanced, and both problems of finding a core member and checking membership of the core can be solved in polynomial time. An especially interesting case is the matching game and the vertex covering game defined on general graphs in the second category. For this pair of graph optimization problems, one integer program is polynomially solvable and the other is

$\mathcal{NP}$-hard. The LP-relaxations of this pair are dual to each other, and the condition for the balancedness is polynomially checkable for both games. This is not necessarily true for all $\mathcal{NP}$-hard combinatorial optimization problems. For the minimum coloring games in the fourth category, both problems of testing balancedness and checking membership of the core are $\mathcal{NP}$-hard.

4.1 Matching Games and Assignment Games

The matching game is one of the most important combinatorial optimization games and has attracted much attention from researchers. Let $G = (V, E; \omega)$ be a graph with edge weight function $\omega : E \to R^+$. The matching game $\Gamma_{mt} = (V, v)$ associated with graph G is defined by

(i) The player set is V;
(ii) For each coalition $S \subseteq V$, $v(S)$ is the weight of a maximum weighted matching in the induced subgraph $G[S]$.

Let $A = [a_{ij}]$ be the incidence matrix of graph G in which rows correspond with edges in E, and columns correspond with vertices in V; $a_{ij} = 1$ if edge i and vertex j are incident, and $a_{ij} = 0$ otherwise. Then for each coalition $S \subseteq V$,

$$v(S) = \max\{x^t\omega : \ x^t A \leq \mathbf{1}_S, \ x \in \{0,1\}^{|E|}\},$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_{|E|})^t$. This formulation casts the matching game in the scope of packing games.

It is obvious that the assignment game is a special kind of matching games whose underlying structure is a bipartite graph. Because the LP-relaxation LP* for the maximum matching problem on a bipartite graph always has an integer optimal solution (the incidence matrix A is totally unimodular [37]), the corresponding assignment game is balanced, and the core is precisely the set of optimal solutions to the dual program DLP*.

Theorem 12. *([63]) The core of an assignment game coincides with the set of optimal dual solutions to the linear program of the corresponding assignment problem.*

However, the above nice property breaks down for matching games on general graphs. Deng, Ibaraki, and Nagamochi [12] showed that the core of a matching game is nonempty only for some special classes of graphs. Their constructive proof provides us a polynomial time algorithm to decide whether the game is balanced and to generate a core member when the core is indeed nonempty. In addition, for the problem of checking membership of the core, it suffices to check whether the sum of the values on two endpoints of every edge is at least one.

Later, Deng *et al.* [13] proved that the matching game is totally balanced if and only if the underlying graph is bipartite. That is, a matching game is totally balanced if and only if it can be formulated as an assignment game.

Theorem 13. *([12,13]) For matching games, all problems of testing balancedness and total balancedness, checking membership of the core, and finding a core allocation can be solved in polynomial time.*

Now we consider the algorithmic results on the Nucleolus for matching games. Solymosi and Raghavan [67] constructed an $O(n^4)$ algorithm for computing the Nucleolus in the bipartite case (*i.e.*, assignment games). This was also obtained in Granot, Granot, and Zhu [36]. Faigle *et al.* [20] introduced a new solution concept, the Nucleon, as an alternative of the Nucleolus, and presented an efficient algorithm for its computation. More recently, Kern and Paulusma [42] proposed an efficient algorithm for computing the Nucleolus of a matching game in the unweighted case. However, computing the Nucleolus for general matching games remains unsolved. We guess it is $\mathcal{NP}$-hard.

Theorem 14. *([36, 67]) The Nucleolus of an assignment game can be computed in polynomial time.*

Theorem 15. *([42]) The Nucleolus of an unweighted matching game can be computed in polynomial time.*

Tamir and Mitchell [71], with Granot [30], discussed a kind of generalized matching games: b-matching games (also called roommate games). Given an edge weighted graph $G = (V, E; \omega)$, let each vertex $i \in V$ be associated with a positive integer b_i, and let $\delta(i)$ denote the set of edges incident to vertex $i \in V$. A b-matching of graph G is an $|E|$-dimensional non-negative integer vector x satisfying the degree constraints: $x(\delta(i)) \leq b_i$ for each $i \in V$. The b-matching game on graph G is defined in a similar way as the matching game. It has the players on the vertices of V, and for each coalition $S \subseteq V$, $v(S)$ is defined as the weight of a maximum weighted b-matching in the induced subgraph $G[S]$.

This game generalizes the original matching game by substituting $\mathbf{1}_S$ with the integer vector $\mathbf{b}_S$ on the right-hand sides of the constraints in the integer program that determines $v(S)$. Still, with a similar approach applied to Owen's linear production game, Tamir and Mitchell [71] proved that if the LP-relaxation LP* for the maximum b-matching problem on G has an integer optimal solution, then the core of this game is nonempty, and a core member can be constructed from an optimal solution to the corresponding dual program DLP*. However, the substitution of $\mathbf{b}_S$ makes the necessary and sufficient condition on the balancedness for packing games (Theorem 11) fail for general b-matching games.

4.2 Vertex Covering Game

In this subsection, we consider the vertex covering game to exemplify the result of Theorem 11 on general covering games. Given a graph $G = (V, E)$, the associated vertex covering game $\Gamma_{vc} = (E, v)$ is defined by

(i) The player set is E;

(ii) For each coalition $S \subseteq E$, $v(S)$ is the cardinality of a minimum vertex cover in the edge induced subgraph $G[S]$, *i.e.*,

$$v(S) = \min\{\mathbf{1}^t y : \; A_{S,V}\, y \geq \mathbf{1}, \; y \in \{0,1\}^{|V|}\},$$

where the matrix A is the incidence matrix of G as described in the matching game.

For the pair of matching game and vertex covering game defined on a common unweighted graph, their corresponding LP-relaxations LP* are dual to each other. However, the maximum matching problem can be solved in polynomial time, whereas the problem of finding the minimum vertex cover is in general $\mathcal{NP}$-hard.

We remark that when we think of a computational task for a cooperative game (N, v), we have the characteristic function v as an oracle that outputs the value $v(S)$ for a queried set $S \subseteq N$ and consider that one oracle call can be done in a constant time. In the vertex covering game, we take graph G as the input, and the running time of algorithms is measured by the encoding length of G, not by the oracle complexity model. Surprisingly, for this vertex covering game that is associated with an $\mathcal{NP}$-hard problem, Deng, Ibaraki, and Nagamochi [12] showed that all the questions about the core can be answered in polynomial time.

In detail, the core of a vertex covering game on graph G is nonempty if and only if the size of a maximum matching is equal to the size of a minimum vertex cover in the graph G [12]. It is very interesting that testing this condition can be transformed into an instance of 2-Satisfiability problem, yielding a polynomial time algorithm. Moreover, when the game is balanced, an imputation is in the core if and only if it is a convex combination of the indicator vectors of maximum matchings.

Theorem 16. *([12]) For vertex covering games, all problems of testing core nonemptiness, checking membership of the core, and finding a core member can be solved in polynomial time.*

4.3 Minimum Coloring Game

The minimum coloring games arise in applications if the smallest number of conflict-free groups are sought in a system where vertices represent members and edges represent conflicts between members. Such conflict graphs can be found in many resource sharing problems, for example, the channel assignment problem for mobil communication. Let $\chi(G)$ denote the chromatic number of an undirected graph G, *i.e.*, the minimum number of maximal independent sets that together cover all vertices of G. The minimum coloring game $\Gamma_{col} = (V, v)$ on a graph $G = (V, E)$ is defined by

(i) The player set is V;
(ii) For each coalition $S \subseteq V$, $v(S)$ equals $\chi(G[S])$, *i.e.*, the chromatic number of the induced subgraph $G[S]$.

This game can be formulated as a covering game as follows: for the constraint matrix A in the covering game formulation, the rows of A correspond with the vertices in V, and the columns correspond with all the maximal independent sets of G.

It is well-known that the computation of $\chi(G)$ is generally $\mathcal{NP}$-hard. Unlike the vertex covering games discussed above, all the algorithmic problems related to the core are $\mathcal{NP}$-hard for this game [12].

Theorem 17. *([12]) For minimum coloring games, all problems of testing core nonemptiness, checking membership of the core, and finding a core member are $\mathcal{NP}$-hard.*

Let $\omega(G)$ denote the size of a maximum clique in G that satisfies $\omega(G) \leq \chi(G)$, as widely known in graph coloring theory. A graph G is called perfect if $\omega(G[S]) = \chi(G[S])$ holds for all subset $S \subseteq V$. The minimum coloring game defined on a perfect graph possesses nice algorithmic properties on its core. Given a perfect graph G, it was proved by Deng, Ibaraki, and Nagamochi [12] that the associated minimum coloring game is balanced, and finding a core member and checking membership of the core can both be solved in polynomial time [12, 55]. Subsequently, it was also proved that a minimum coloring game is totally balanced if and only if the associated graph is perfect by Deng *et al.* [13]. It follows that the decision problem on the total balancedness of a minimum coloring game is as hard as recognizing perfect graphs, which is recently shown to be done in polynomial time [8].

Theorem 18. *([13]) The minimum coloring game on a graph $G = (V, E)$ is totally balanced if and only if graph G is perfect.*

In Okamoto [55], the algorithmic issues were further investigated for other solution concepts of minimum coloring games. A characterization of the Nucleolus for some special classes of graphs, including complete multipartite graphs and chordal graphs, leads to an efficient algorithm for its computation.

5 Facility Location Games

In a facility location problem, customers from a given set N are in need of a certain service from some facilities. For a given set F of possible locations for the facilities, opening facility $i \in F$ causes a predefined cost $f_i \geq 0$, and connecting customer $j \in N$ to this facility requires cost $c_{ij} \geq 0$. The collective goal is to minimize the sum of total cost, which is made up of the costs to open facilities and to connect each customer to an open facility. This is referred to as the unconstrained facility location problem. For the constrained case, some further requests have to be taken into consideration, such as some facilities can only handle a limit number of customers, or customers from different groups cannot be assigned to the same facility.

The corresponding facility location game $\Gamma_{fl} = (N, v)$ is defined as follows:

(i) The player set is $N = \{1, 2, \ldots, n\}$;
(ii) For each coalition $S \subseteq N$, $v(S)$ is the minimum total cost of providing the service only to the players in S.

A systematic study on the core of the facility location game was carried out by Goemans and Skutella [29]. Linear and integer programming approach and LP duality technique are also crucial in their work.

To give the formulation of the facility location problem, let us first define two kinds of variables. For each $i \in F$, the variable y_i is set to 1 if facility i is opened, and 0 otherwise; for each $i \in F$ and $j \in N$, the variable x_{ij} is set to 1 if customer j is connected to facility i, and 0 otherwise. In order to model the constrained case, for each facility $i \in F$, a collection of feasible subsets $\mathcal{F}_i$ is introduced to represent all the possibilities of the set of customers that can be connected to this facility, and accordingly, define $P_i \subseteq \{0,1\}^{n+1}$ by

$$P_i := \{(0, \ldots, 0)\} \cup \{(1, \mathbf{1}_S) | \, S \in \mathcal{F}_i\},$$

where $\mathbf{1}_S \in \{0,1\}^n$ is the indicator vector of the subset S. Then the general facility location problem can be formulated as the following integer linear program:

$$\mathrm{IP}^*: \quad \begin{array}{ll} \min & \displaystyle\sum_{i \in F} f_i y_i + \sum_{i \in F, j \in N} c_{ij} x_{ij} \\ \text{s.t.} & \begin{cases} \displaystyle\sum_{i \in F} x_{ij} = 1 & \text{for all } \; j \in N \\ (y_i, x_{i1}, x_{i2}, \ldots, x_{in}) \in P_i & \text{for all } \; i \in F \\ x_{ij}, y_i = 0, 1 & \text{for all } \; i \in F, \; j \in N \end{cases} \end{array}$$

Replacing each discrete vector set P_i by its conic hull $\mathrm{cone}(P_i) = \{\sum_{u \in P_i} \lambda_u u : \lambda_u \geq 0\}$, rather than the most nature convex hull $\mathrm{conv}(P_i)$, we obtain the following LP-relaxation (LP*):

$$\mathrm{LP}^*: \quad \begin{array}{ll} \min & \displaystyle\sum_{i \in F} f_i y_i + \sum_{i \in F, j \in N} c_{ij} x_{ij} \\ \text{s.t.} & \begin{cases} \displaystyle\sum_{i \in F} x_{ij} = 1 & \text{for all } \; j \in N \\ (y_i, x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathrm{cone}(P_i) & \text{for all } \; i \in F \\ x_{ij}, y_i \geq 0 & \text{for all } \; i \in F, \; j \in N \end{cases} \end{array}$$

By making use of the technique of Lagrange dual, Goemans and Skutella [29] proved that this LP-relaxation (LP*) is equivalent to the cost allocation problem (CAP) of the associated facility location game:

$$\mathrm{CAP}: \quad \max\{\sum_{j \in N} x_j : \sum_{j \in S} x_i \leq v(S), \;\; \forall S \subseteq N\}.$$

Theorem 19. *([29]) For a facility location problem and its corresponding game model, the cost allocation problem (CAP) is equivalent to the dual of the LP-relaxation LP^*. In particular, their values are equal and the core of the facility location game is nonempty if and only if there is no integrality gap for LP^*.*

Note that for the unconstrained case,

$$\mathrm{conv}(P_i) = \mathrm{cone}(P_i) = \{(y_i, x_{i1}, x_{i2}, \ldots, x_{in}) : 0 \le x_{ij} \le y_i,\ \forall j \in N\}.$$

Theorem 19 implies that both checking membership of the core and finding a core member may reduce to solving the dual program of LP* for a balanced unconstrained facility location game. However, the problem of testing the balancedness is $\mathcal{NP}$-hard.

Theorem 20. *([29]) For the unconstrained facility location games, it is $\mathcal{NP}$-complete to decide whether the core is nonempty. If the unconstrained facility location game is balanced, both problems of finding a core member and checking membership of the core can be done in polynomial time.*

In the rest of this section, we will restrict our attention to a special case of unconstrained facility location games in which all players and facilities are located on the vertices of a tree. For each player $j \in N$, the cost for connecting j to facility $i \in F$ is equal to the distance between them in the underlying tree. This kind of game is referred to as simple facility location games. Goemans and Skutella [29] proved that in this case, the core is always nonempty. This result was also obtained by Kolen [43] and Curiel [9]. Let us consider the integer program formulations given by Kolen [43] to exploit how the LP duality approach is applied to simple facility location games.

Let $T = (N, E)$ be a tree with $|N| = n$, the cost connecting i and j be equal to the distance between them, denoted by $d(i, j)$ for each pair $i, j \in N$. Assume that the players correspond with vertices in N. For each vertex $i \in N$, let $r_{i1} \le r_{i2} \le \cdots \le r_{in}$ be the ordered sequence of distances between vertex i and all the vertices, including i. Also define r_{in+1} to be a number that is larger than the sum of all opening costs and connecting costs involved. Define an $n^2 \times n$-matrix $H = [h_{i_k j}]$ by

$$h_{i_k j} = \begin{cases} 1 & \text{if } d(i, j) \le r_{ik} \text{ for } i, j, k \in N \\ 0 & \text{otherwise.} \end{cases}$$

Then the coalition values can be formulated as follows. For each $j \in N$, set variable $x_j = 1$ if and only if a facility is built in vertex j; for each $i, k \in N$, set variable $z_{ik} = 1$ if and only if there is no facility within distance r_{ik} from vertex i. The number of built facilities that are within distance r_{ik} from vertex i are given by $\sum_{j=1}^{n} h_{i_k j} x_j$. For each coalition $S \subseteq N$, the game value $v(S)$ is then the optimum value of the following integer program:

$$\min \quad \sum_{j=1}^{n} f_j x_j + \sum_{i\in S}\sum_{k=1}^{n}(r_{ik+1} - r_{ik})z_{ik}$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} h_{i_k j}x_j + z_{ik} \geq 1 & \text{for } i \in S,\ k \in N \\ z_{ik} \in \{0,1\} & \text{for } i \in S,\ k \in N \\ x_j \in \{0,1\} & \text{for } j \in N. \end{cases}$$

In this formulation, the total balancedness of the constraint matrix implies the integrality of the LP-relaxation of the integer program (IP*) that determines $v(N)$ [43]. Accordingly, the nonemptiness of the core can be shown in a similar way as that for Owen's linear production game: let y^* be an optimal dual solution of the LP-relaxation (LP*) that determines $v(N)$, then $u = (u_1, u_2, \ldots, u_n) \in R^n$ with $u_i = \sum_{k=1}^{n} y^*_{ik}$ is in the core of the game.

Theorem 21. *([9, 29, 43]) For a simple facility location game, the core is nonempty, and finding a core member can be done in polynomial time.*

Some different game models arising from facility location problems were discussed in the literature. Tamir [68] considered a facility location model on a tree where each customer has to be connected to a facility within a given distance. This can be formulated as a special case of the games discussed above. Another game model, called fixed cost spanning forest game, was introduced by Granot *et al.* [32]. In this model, there are no proximity constraints on the distances between the customers and the facilities. Each customer must be connected to some designated central facility, not necessarily the closest one to the customer. In general, the core of this game may be empty. However, it was shown in Granot *et al.* [32] that when the underlying network is a tree, the game is balanced and a core allocation can be found with a strongly polynomial time algorithm.

6 Further Discussions and Remarks

In the study of cooperative games, it is suggested to have polynomial time algorithms for finding and checking solutions to game models [50]. It is further suggested that computational complexity be taken as one extra factor in considering rationality and fairness of a solution concept and comparing different solution concepts [10,14] in a way derived from the concept of bounded rationality [54,57,66]. In this line of approach, one may be lured to try to classify solution concepts by their complexities. However, very often, they may display different orders in the complexity hierarchy from game to game. Some concepts may be easier to compute in one game but more difficult in others. But, still, we may ask this question: what is the worst-case complexity of a solution concept? With all algorithms we know of, the concepts of the core, the bargaining set, and the von Neumann–Morgenstern solution should be in an order of increasing complexity [10, 14]. However, it would be nice to have a definite proof in terms of lower bound.

To make the study of complexity and algorithmic issues for cooperative games meaningful in the associated application areas, it is vital to have computational complexity as an integrated part of theoretical consideration for solution concepts. Even in the case in which the solutions of a game model do not exist or are difficult to compute, it may not be easy to simply dismiss the problem as hopeless especially when the game arises from important applications. Various conceptual approaches, in particular the approximation in fair allocations, are proposed to resolve this problem.

The core, the set of feasible outcomes of a social or economic situation that cannot be improved upon by any coalition of players, is a fundamental equilibrium concept. When the core is empty, it motivates conditions ensuring nonemptiness of approximate cores in economies and game models. A natural way to approximate the core is the *least core*, which was introduced by Maschler, Peleg, and Shaplsy [52]. Let (N, v) be a profit cooperative game. Given a real number ε, the ε-core is defined to contain the allocations such that $x(S) \geq v(S) - \varepsilon$ for each nonempty proper subset S of N. The *least core*, denoted by $LC(v)$, is the intersection of all nonempty ε-core. Let ε^* be the minimum value of ε such that the ε-core is empty, then the least core is the same as the ε^*-core. It is not hard to see that the least core is always nonempty.

The concept of least core poses new challenges on algorithmic issues. The most natural problem is how to efficiently compute the value ε^* for a given cooperative game. The catch is that the computation of ε^* requires one to solve a linear program with exponential number of constrains. Though there are cases the value ε^* can be computed in polynomial time, it is in general very hard. If we consider the value of ε^* as some subsidies given by the central authority to ensure the existence of the cooperation, then it is significant to give the approximate value of it even when the computation problem is $\mathcal{NP}$-hard. This needs new techniques in design and analysis of algorithms.

Another possible approach we are interested in is to interpret approximation as bounded rationality. For example, it would be interesting to know if there are some of those games with the property that, for any $\varepsilon > 0$, checking membership in the ε-core can be done in polynomial time but it is $\mathcal{NP}$-hard to tell if an imputation is in the core or not. In such cases, the restoration of cooperation would be a result of bounded rationality. That is to say, the players would not care an extra gain or loss of ε at the expense of another order of degree of computational resources.

As an important solution concept in economics and game theory, the Nucleolus and related solution concepts have been applied to study insurance policies by Lemaire [47], to real estate by Raghavan and Solymosi [60], to study peer group by Brânzei, Solymosi, and Tijs [4], and to bankruptcy by Aumann and Maschler [1] as well as Malkevitch [49]. However, it has been a challenge to obtain polynomial time algorithms for computing the Nucleolus.

For some special game models, the Nucleolus can be computed in polynomial time, such as, assignment games, simple flow games, and unweighted matching games. On the other hand, it is $\mathcal{NP}$-hard for more general game models including minimum cost spanning tree games [18], flow games, and linear production games [11].

There are still many unsolved complexity questions concerning the Nucleolus. Kern and Paulusma [42] conjectured that for general matching games, computation of the Nucleolus is $\mathcal{NP}$-hard. In addition, it is very interesting for us to know the complexity of computing the Nucleolus for other games in the class of packing and covering games, such as vertex covering games and minimum coloring games. In Deng, Fang, and Sun [11], an elegant application of LP duality approach yields a polynomial time algorithm for computing the Nucleolus of a simple flow game. Because flow games fall into the scope of packing games, we believe that LP duality technique will be useful in design of algorithms for this class of games.

For cooperative games arising from $\mathcal{NP}$-hard combinatorial optimization problems, computation of the Nucleolus may in general be a hard task. But till now, no such $\mathcal{NP}$-hardness result is known. For example, facility location problem is a classic $\mathcal{NP}$-hard combinatorial optimization problem. We guess that for facility location games, the Nucleolus is difficult to compute in general, though there may be efficient algorithms for some special cases. Moreover, when computation of the Nucleolus is difficult, we are also interested in seeking meaningful approximation concepts of the Nucleolus especially from the political and economic background.

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A Survey of Bicooperative Games

Jesús M. Bilbao, Julio R. Fernández, Nieves Jiménez, and Jorge J. López

Department of Applied Mathematics II, University of Seville, Escuela Superior de Ingenieros, Sevilla, Spain
{mbilbao,julio,niejimjim,jorlopvaz}@us.es

Abstract The aim of the current chapter is to study several solution concepts for bicooperative games. For these games introduced by Bilbao [1], we define a one-point solution called the Shapley value, as this value can be interpreted in a similar way to the classic Shapley value for cooperative games. The first result is an axiomatic characterization of this value. Next, we define the core and the Weber set of a bicooperative game and prove that the core of a bicooperative game is always contained in the Weber set. Finally, we introduce a special class of bicooperative games, the so-called bisupermodular games, and show that these games are the only ones in which the core and the Weber set coincide.

Key words: bicooperative game, bisupermodular game, Bore, Shapley value, Weber set

1 Introduction

The theory of cooperative games studies situations where a group of people/players are associated to obtain a profit as a result of their cooperation. Thus, a cooperative game is defined as a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function satisfying that $v(\emptyset) = 0$. For each $S \in 2^N$, the worth $v(S)$ can be interpreted as the maximal gain or minimal cost that the players forming coalition S can achieve by themselves against the best offensive threat by the complementary coalition $N \setminus S$. Hence, we can say that a cooperative game has orthogonal coalitions (see Myerson [12]). Classic market games for economies with private goods are examples of cooperative games.

Games with nonorthogonal coalitions are games in which the worth of coalition S depends on the actions of coalition $N \setminus S$. For instance, the joint owners of a building are considering hiring a gardener to work in the common areas of their residence. The garden is a public good. Each owner can decide to support the proposal or to veto it. However, some of them may decide not to take part in the decision making and would thus not necessarily be *defenders* or *detractors* of the project. This is the case with multicriteria decision making when underlying scales are bipolar, i.e., a central value exists on each scale and it is considered a neutral value. Thus, social situations involving externalities and public goods are such cases.

These situations may be modeled in the following manner. We consider pairs (S,T), with $S,T \subseteq N$ and $S \cap T = \emptyset$. Thus, (S,T) is a partition of N in three groups. Players in S are defenders of modifying the *status quo* and they want to accept a proposal; players in T do not agree with modifying the situation and they will take action against any change. Finally, the members of $N \setminus (S \cup T)$ are not convinced of the profits derived from the proposal and they vote abstention.

Thus, in our model we consider the set of all ordered pairs of disjoint coalitions $3^N = \{(S,T) : S,T \subseteq N, \;\; S \cap T = \emptyset\}$, and define a worth function $b : 3^N \to \mathbb{R}$. For each $(S,T) \in 3^N$, the worth $b\,(S,T)$ can be interpreted as the maximal gain (whenever $b\,(S,T) > 0$) or minimal loss (whenever $b\,(S,T) < 0$) that the players of the coalition S can achieve when they decide to play together against the players of T and the players of $N \setminus (S \cup T)$ not taking part. This leads us in a natural way into the concept of bicooperative game introduced by Bilbao [1].

Definition 1. *A bicooperative game is a pair (N,b) with N a finite set and b is a function $b : 3^N \to \mathbb{R}$ with $b\,(\emptyset,\emptyset) = 0$.*

Similar to the cooperative case in which each coalition $S \in 2^N$ can be identified with a $\{0,1\}$-vector $\mathbf{1}_S$, each pair $(S,T) \in 3^N$ can be identified with the $\{-1,0,1\}$-vector $\mathbf{1}_{(S,T)}$ defined, for all $i \in N$, by

$$\mathbf{1}_{(S,T)}\,(i) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

A special kind of bicooperative games has been studied by Felsenthal and Machover [5] who consider *ternary voting games*. This concept is a generalization of voting games that recognizes abstention as an option alongside *yes* and *no* votes. These games are given by mappings $u : 3^N \to \{-1,1\}$ satisfying the following three conditions: $u\,(N,\emptyset) = 1$, $u\,(\emptyset,N) = -1$, and $\mathbf{1}_{(S,T)}\,(i) \leq \mathbf{1}_{(S',T')}\,(i)$ for all $i \in N$, implies $u\,(S,T) \leq u\,(S',T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of a bill.

In Chua and Huang [3], the Shapley–Shubik index for ternary voting games is considered. More recently, several works by Freixas [6, 7] and Freixas and Zwicker [8] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes. A new approach to bicooperative games is presented by Grabisch and Lange [11] by using the product of finite distributive lattices. They consider a set of players $N = \{1,\ldots,n\}$ and the product $L_1 \times \cdots \times L_n$ of the lattices

$L_i = (\{-1, 0, 1\}, \leq)$, $i \in N$, equipped with the pointwise order. Here, 1 means voting or playing in favor, -1 means voting or playing against, and 0 means abstention.

A one-point solution concept for cooperative games is a function that assigns to every cooperative game a n-dimensional real vector that represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. The most important solution concept is the *Shapley value* as proposed by Shapley [14]. The Shapley value assumes that every player is equally likely to join to any coalition of the same size, and all coalitions with the same size, are equally likely. Each component of the Shapley value $\Phi(v) \in \mathbb{R}^n$ is a weighted average of the marginal contributions $v(S \cup \{i\}) - v(S)$ of player $i \in N$, and it is given by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-1-s)!}{n!} \left[v(S \cup \{i\}) - v(S)\right],$$

where $s = |S|$ and $n = |N|$.

Another way to introduce the Shapley value is based on the marginal worth vectors and corresponds with the following interpretation. Each permutation $\pi = (i_1, i_2, \ldots, i_n)$ of the elements of N can be interpreted as a sequential process of formation of the grand coalition N. Beginning from the empty set, first the player i_1 is included, next the player i_2 and so until the inclusion of the player i_n gives rise to the coalition N. In each one of these processes, the corresponding *marginal worth vector* $a^\pi(v) \in \mathbb{R}^n$ evaluates the marginal contribution of every player to the coalition formed by his predecessors, that is,

$$a_i^\pi(v) = v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right) \text{ for all } i \in N,$$

where π^i is the set of the predecessors of player i in the order π. The Shapley value $\Phi(v)$ assigns the expected amount received by each player $i \in N$, that is,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \left[v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right)\right].$$

where Π_n is the set of all permutations of N and π^i is the set of the predecessors of player i in the order π.

A solution concept for cooperative games is a function that assigns a subset of n-dimensional real vectors to every cooperative game (N, v). These vectors represent the payoff distribution over the players.

The core [9] is one of the most studied solution concepts. The core of a cooperative game (N, v) consists of all payoff vectors that distribute the total savings $v(N)$ among players and secure every coalition $S \in 2^N$ at least the amount it can obtain by operating on its own, that is,

$$C(N, v) = \left\{x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N\right\},$$

where $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$.

Although the core of a cooperative game is considered as a very natural solution concept, most of the time it is empty. The core is nonempty for the class of convex games [15]. This leads us to consider other solution concepts. In 1978, Weber [17] proposed as a solution concept for a cooperative game: a set that contains the core, which is always nonempty and easy to compute. Its definition is based on the marginal worth vectors. The *Weber set* of game v is the convex hull of all marginal worth vectors, that is,

$$W(N, v) = conv\{a^{\pi}(v) : \pi \in \Pi_n\}.$$

Let us outline the contents of our work. In the next section, we study some properties and characteristics of the distributive lattice 3^N. The aim of the third section is to introduce the Shapley value for a bicooperative game. We obtain an axiomatization of the Shapley value in this context as well as a nice formula to compute it. This value is the only one that satisfies our five axioms. Four of them are extensions of the classic axioms for the Shapley value: linearity, symmetry, dummy, and efficiency. The fifth axiom is refereed to the structure of the family of coalitions in 3^N. In the fourth section, we define the above solutions concepts for bicooperative games and prove that the core is always contained in the Weber set. The bisupermodular games, which are introduced in the fifth section, play an important role in the relationship between the Weber set and the core. We see that the bisupermodular games are the only ones for which their Weber set and the core coincide, establishing a characterization of these games. Throughout this chapter, we will write $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$, respectively.

2 The Lattice 3^N

Let $N = \{1, \ldots, n\}$ be a set and let $3^N = \{(A, B) : A, B \subseteq N,\ A \cap B = \emptyset\}$. Grabisch and Labreuche [10] proposed a relation in 3^N given by

$$(A, B) \sqsubseteq (C, D) \Longleftrightarrow A \subseteq C,\ B \supseteq D.$$

The set $\left(3^N, \sqsubseteq\right)$ is a partially ordered set (or poset) with the following properties:

1. $(\emptyset, N)$ is the first element: $(\emptyset, N) \sqsubseteq (A, B)$ for all $(A, B) \in 3^N$.
2. $(N, \emptyset)$ is the last element: $(A, B) \sqsubseteq (N, \emptyset)$ for all $(A, B) \in 3^N$.
3. Each pair $\{(A, B), (C, D)\}$ of elements of 3^N has a join

$$(A, B) \vee (C, D) = (A \cup C, B \cap D),$$

and a meet

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

Moreover, $\left(3^N, \sqsubseteq\right)$ is a finite distributive lattice. Two pairs (A, B) and (C, D) are comparable if $(A, B) \sqsubseteq (C, D)$ or $(C, D) \sqsubseteq (A, B)$; otherwise, (A, B)

and (C, D) are noncomparable. A chain of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $\left(3^N, \sqsubseteq\right)$, all maximal chains have the same number of elements, and this number is $2n+1$. Thus, we can consider the rank function $\rho : 3^N \to \{0, 1, \ldots, 2n\}$ such that $\rho\left[(\emptyset, N)\right] = 0$ and $\rho\left[(S,T)\right] = \rho\left[(A,B)\right] + 1$ if (S,T) covers (A,B), that is, if $(A,B) \sqsubset (S,T)$ and there exists no $(H,J) \in 3^N$ such that $(A,B) \sqsubset (H,J) \sqsubset (S,T)$. An element of a lattice is $\vee$-irreducible if it covers only one element.

For the distributive lattice 3^N, let P denote the set of all nonzero $\vee$-irreducible elements. Then P is the disjoint union $C_1 + C_2 + \cdots + C_n$ of the chains

$$C_i = \{(\emptyset, N \setminus i),\ (i, N \setminus i)\}, \quad 1 \le i \le n = |N|.$$

An order ideal of P is a subset I of P such that if $x \in I$ and $y \le x$, then $y \in I$. The set of all order ideals of P, ordered by inclusion, is the distributive lattice $J(P)$, where the lattice operations $\vee$ and $\wedge$ are just ordinary union and intersection. The fundamental theorem for finite distributive lattices (see [16, Theorem 3.4.1]) states that the map $\varphi : 3^N \to J(P)$ given by $(A,B) \mapsto \{(X,Y) \in P : (X,Y) \sqsubseteq (A,B)\}$ is an isomorphism (see Figure 1).

Example. Let $N = \{1,2\}$. Then $P = \{(\emptyset,\{1\}), (\emptyset,\{2\}), (\{2\},\{1\}), (\{1\},\{2\})\}$ is the disjoint union of the chains $(\emptyset,\{1\}) \sqsubset (\{2\},\{1\})$ and $(\emptyset,\{2\}) \sqsubset (\{1\},\{2\})$. We will denote $a = (\emptyset,\{1\})$, $b = (\{2\},\{1\})$, $c = (\emptyset,\{2\})$, and $d = (\{1\},\{2\})\}$. Thus we obtain the lattice

$$J(P) = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\}\}.$$

In the following, we will denote by $c\left(3^N\right)$ the number of maximal chains in 3^N and by $c\left(\left[(A,B), (C,D)\right]\right)$ the number of maximal chains in the sublattice $\left[(A,B), (C,D)\right]$.

Proposition 1. *The number of maximal chains of* 3^N *is* $(2n)!/2^n$, *where* $n = |N|$.

Proof. The number of maximal chains of 3^N is equal to the number of maximal chains of $J(P)$, and this number is the number of extensions $e(P)$ of P to a total order (see Stanley [16, Section 3.5]). Because $P = C_1 + \cdots + C_n$, where

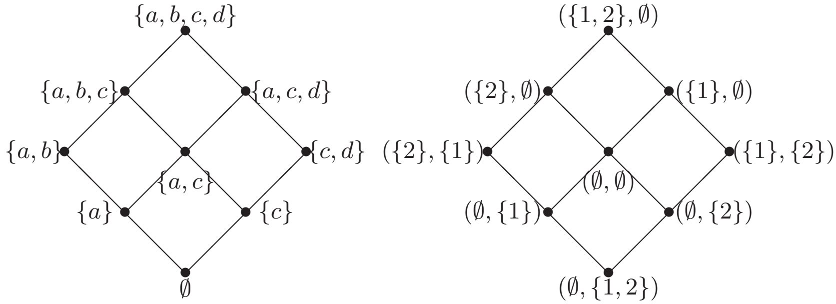

Figure 1. $J(P)$ and 3^N for two players

the chain C_i satisfies $|C_i| = 2$ for $1 \leq i \leq n$, we can apply the enumeration of lattice paths method from Stanley [16, Example 3.5.4], and obtain

$$c\left(3^N\right) = e(P) = \binom{2n}{2, \dots, 2} = \frac{(2n)!}{2^n}.$$

■

Proposition 2. *For all $(A, B) \in 3^N$, the number of maximal chains of the sublattice $[(\emptyset, N), (A, B)]$ is $(n + a - b)!/2^a$, where $a = |A|$ and $b = |B|$.*

Proof. Given the sublattice $[(\emptyset, N), (A, B)]$, we take $N \setminus B = \{i_1, \dots, i_{n-b}\}$ and hence there are $n - b$ elements $(\emptyset, N \setminus i)$ with $i \notin B$ (see Figure 2).

Because $A \subseteq N \setminus B$, then $a \leq n - b$, and thus the set of the irreducible elements of the sublattice can be written as

$$P_{[(\emptyset,N),(A,B)]} = C_1 + \cdots + C_a + C_{a+1} + \cdots + C_{a+(n-b-a)}$$

where for all $i_j \in A$, $1 \leq j \leq a$ and $i_{a+k} \notin A \cup B$, $1 \leq k \leq n - b - a$, we obtain

$$\begin{aligned} C_j &= \{(\emptyset, N \setminus i_j), (i_j, N \setminus i_j)\}, \\ C_{a+k} &= \{(\emptyset, N \setminus i_{a+k})\}. \end{aligned}$$

That is, there are a chains such that $|C_j| = 2$ and there are $n - b - a$ chains such that $|C_{a+k}| = 1$. Because

$$|C_1| + \cdots + |C_a| + |C_{a+1}| + \cdots + \left|C_{a+(n-b-a)}\right| = 2a + (n - b - a),$$

we can apply the enumeration of lattice paths method from Stanley [16, Section 3.5] and we obtain

$$c\left([(\emptyset, N), (A, B)]\right) = \binom{2a + (n - b - a)}{2, \dots, 2, 1, \dots, 1} = \frac{(n + a - b)!}{2^a}.$$

■

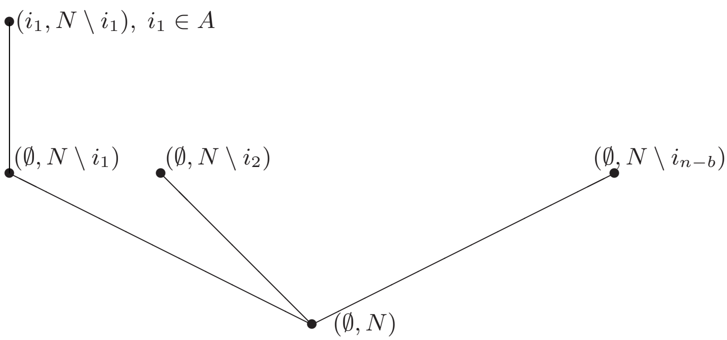

Figure 2. Irreducible elements of the sublattice

Proposition 3. *Let it be* $(A,B),(C,D) \in 3^N$ *with* $(A,B) \sqsubseteq (C,D)$. *The number of maximal chains of the sublattice* $[(A,B),(C,D)]$ *is equal to the number of maximal chains of the sublattice* $[(D,C),(B,A)]$.

Proof. First of all, note that if $(A,B) \sqsubseteq (C,D)$, then $A \subseteq C,\ B \supseteq D$ and hence $(D,C) \sqsubseteq (B,A)$. Therefore, $[(D,C),(B,A)]$ is a sublattice of 3^N.

Let $\varphi : (3^N, \sqsubseteq) \to (3^N, \sqsubseteq)$ be the map defined by $\varphi(A,B) = (B,A)$. This map is one to one as

$$\varphi(A,B) = \varphi(C,D) \Longleftrightarrow (B,A) = (D,C) \Longleftrightarrow (A,B) = (C,D).$$

Clearly, $(A,B) \sqsubset (A_1,B_1) \sqsubset \cdots \sqsubset (A_k,B_k) \sqsubset (C,D)$ is a maximal chain in the sublattice $[(A,B),(C,D)]$ if and only if

$$(D,C) \sqsubset (B_k,A_k) \sqsubset \cdots \sqsubset (B_1,A_1) \sqsubset (B,A)$$

is a maximal chain in the sublattice $[(D,C),(B,A)]$. Finally, it follows that

$$(X,Y) \in [(A,B),(C,D)] \Longleftrightarrow (Y,X) \in [(D,C),(B,A)].$$

■

3 The Shapley Value for Bicooperative Games

We denote by $\mathcal{BG}^N$ the real vector space of all bicooperative games on N, that is

$$\mathcal{BG}^N = \left\{ b : 3^N \to \mathbb{R},\ b(\emptyset,\emptyset) = 0 \right\}.$$

We consider the *identity* games $\left\{\delta_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$, the *superior unanimity* games $\left\{\overline{u}_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$, and the *inferior unanimity* games $\left\{\underline{u}_{(S,T)} : (S,T) \in 3^N,\ (S,T) \neq (\emptyset,\emptyset)\right\}$, which are defined, for any $(S,T) \in 3^N$ such that $(S,T) \neq (\emptyset,\emptyset)$ as follows.

The identity game $\delta_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (A,B) = (S,T), \\ 0 & \text{otherwise.} \end{cases}$$

The superior unanimity game $\overline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is given by

$$\overline{u}_{(S,T)}(A,B) = \begin{cases} 1 & \text{if } (S,T) \sqsubseteq (A,B),\ \ (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

The inferior unanimity game $\underline{u}_{(S,T)} : 3^N \to \mathbb{R}$ is defined by

$$\underline{u}_{(S,T)}(A,B) = \begin{cases} -1 & \text{if } (A,B) \sqsubseteq (S,T),\ \ (A,B) \neq (\emptyset,\emptyset), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove (see [2]) that all the above collections are bases of $\mathcal{BG}^N$.

A *value* on $\mathcal{BG}^N$ is a function $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$, which associates to each bicooperative game b a vector $(\Phi_1(b), \ldots, \Phi_n(b))$ representing the value that every player has in the game b. In order to define a reasonable value for a bicooperative game, we use the following interpretation of the Shapley value in the bicooperative case. We consider that a player i estimates his participation in game b, evaluating his marginal contributions $b(S \cup i, T) - b(S, T)$ in those coalitions $(S \cup i, T)$ that are formed from others (S, T) when i joins S and his marginal contributions $b(S, T) - b(S, T \cup i)$ in those (S, T) that are formed when i leaves $T \cup i$.

Thus, a value for player i can be written as

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \Big[\overline{p}^i_{(S,T)} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S, T \cup i\right) \right) \Big],$$

where for every (S, T), the coefficient $\overline{p}^i_{(S,T)}$ can be interpreted as the subjective probability that the player i has of joining the coalition S and $\underline{p}^i_{(S,T)}$ as the subjective probability that the player i has of leaving the coalition $T \cup i$. Thus, $\Phi_i(b)$ is the value that the player i can expect in the game b.

Figure 3 shows the different sequential orders corresponding with the different chains from $(\emptyset, N)$ to $(N, \emptyset)$ that contain (S, T) and $(S \cup i, T)$ and all chains that contain the coalitions $(S, T \cup i)$ and (S, T).

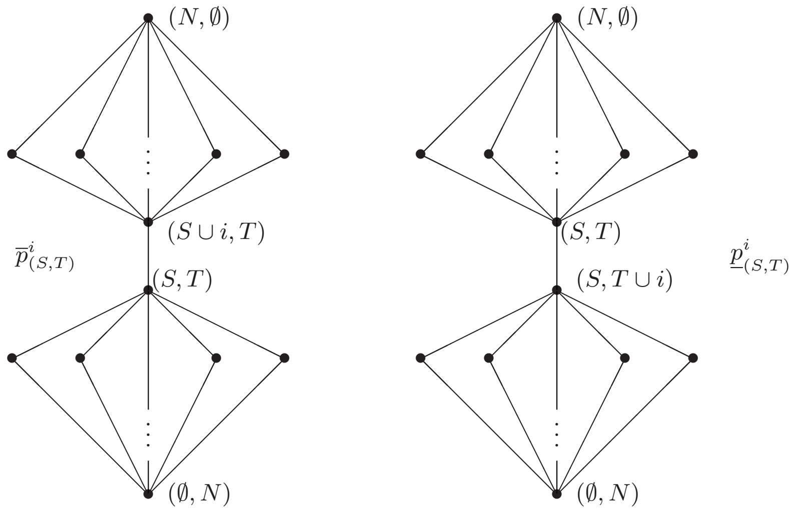

Figure 3. Chains that contain coalitions

If we assume that all sequential orders or chains have the same probability, we can deduce formulas for these probabilities $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ in terms of the number of chains that contain these coalitions.

Applying Propositions 2 and 3, we obtain

$$\begin{aligned}
\overline{p}^i_{(S,T)} &= \frac{c\left([(\emptyset,N),(S,T)]\right)\; c\left([(S\cup i,T),(N,\emptyset)]\right)}{c\left(3^N\right)} \\
&= \frac{\dfrac{(n+s-t)!}{2^s}\cdot\dfrac{(n+t-s-1)!}{2^t}}{\dfrac{(2n)!}{2^n}} \\
&= \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t},
\end{aligned}$$

$$\begin{aligned}
\underline{p}^i_{(S,T)} &= \frac{c\left([(\emptyset,N),(S,T\cup i)]\right)\; c\left([(S,T)\,(N,\emptyset)]\right)}{c\left(3^N\right)} \\
&= \frac{\dfrac{(n+t-s)!}{2^t}\cdot\dfrac{(n+s-t-1)!}{2^s}}{\dfrac{(2n)!}{2^n}} \\
&= \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.
\end{aligned}$$

Taking into account that $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ are independent of player i, and only depend of $s=|S|$ and $t=|T|$, we can establish the following definition.

Definition 2. *The Shapley value for the bicooperative game $b\in\mathcal{BG}^N$ is given, for each $i\in N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \Big[\overline{p}_{s,t}\left(b(S\cup i,T)-b(S,T)\right)+\underline{p}_{s,t}\left(b(S,T)-b\left(S,T\cup i\right)\right)\Big],$$

where, for all $(S,T)\in 3^{N\setminus i}$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t},$$

and

$$\underline{p}_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.$$

With the aim to characterize the Shapley value for bicooperative games, we consider a set of reasonable axioms and we prove that the Shapley value is the unique value on $\mathcal{BG}^N$ that satisfies these axioms.

Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$, and $b, w \in \mathcal{BG}^N$,

$$\Phi_i(\alpha b + \beta w) = \alpha \Phi_i(b) + \beta \Phi_i(w).$$

We now introduce the dummy axiom, understanding that a player is a *dummy player* when his contributions to coalitions $(S \cup i, T)$ formed with his incorporation to S and his contributions to coalitions (S, T) formed with his desertion of $T \cup i$ coincide exactly with his individual contributions. Thus, a player $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ if, for every $(S, T) \in 3^{N \setminus i}$, it holds

$$\begin{aligned} b(S \cup i, T) - b(S, T)) &= b(\{i\}, \emptyset), \\ b(S, T) - b(S, T \cup i) &= -b(\emptyset, \{i\}). \end{aligned}$$

Note that if $i \in N$ is a dummy in $b \in \mathcal{BG}^N$, then for all $(S, T) \in 3^{N \setminus i}$,

$$b(S \cup i, T) - b(S, T \cup i) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}).$$

Because a dummy player i in a game b has no meaningful strategic role in the game, the value that this player should expect in the game b must exactly be the sum up of his contributions.

Dummy axiom. If player $i \in N$ is dummy in $b \in \mathcal{BG}^N$, then

$$\Phi_i(b) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}).$$

In a similar way as in the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.

Symmetry axiom. For all $b \in \mathcal{BG}^N$ and for any permutation π over N, it holds that $\Phi_{\pi i}(\pi b) = \Phi_i(b)$ for all $i \in N$, where $\pi b(\pi S, \pi T) = b(S, T)$ and $\pi S = \{\pi i : i \in S\}$.

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them. Taking into account different situations that can be modeled by a bicooperative game b, we suppose that the amount $b(N, \emptyset)$ is the maximal gain and $b(\emptyset, N)$ is the minimal loss obtained by the players when they decide full cooperation. Then the maximal global gain is given by $b(N, \emptyset) - b(\emptyset, N)$. From this perspective, the value Φ must satisfy the following axiom.

Efficiency axiom. For every $b \in \mathcal{BG}^N$, it holds

$$\sum_{i \in N} \Phi_i(b) = b(N, \emptyset) - b(\emptyset, N).$$

It is easy to check that our Shapley value for bicooperative games verifies the above axioms. But this value is not the unique value that satisfies these four axioms. For instance, the value $\Phi(b)$ defined, for $b \in \mathcal{BG}^N$ and $i \in N$, by

$$\Phi_i(b) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[b\left(S \cup i, N \setminus (S \cup i)\right) - b\left(S, N \setminus S\right) \right],$$

also verifies these axioms. However, note that, for any bicooperative game $b \in \mathcal{BG}^N$, this value is the Shapley value corresponding with the cooperative game (N, v), where $v : 2^N \to \mathbb{R}$ is defined by $v(A) = b(A, N \setminus A)$ if $A \neq \emptyset$, and $v(\emptyset) = 0$. This value is not satisfactory for any bicooperative game because it considers the contributions to pairs of coalitions, in which all players take part. Moreover, there is an infinite number of bicooperative games that give rise to the same cooperative game.

For these reasons, if we want to obtain an axiomatic characterization of our Shapley value for bicooperative games, we need to introduce an additional axiom. Previously, we showed that a value on $\mathcal{BG}^N$ that satisfies the above four axioms is given by the expression

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}_{s,t} \left(b(S,T) - b\left(S, T \cup i\right) \right) \right],$$

where $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ satisfy some conditions. We prove this result in several steps. First of all, we show that a value for player i satisfying the linearity and dummy axioms can be expressed as a linear combination of his contributions.

Theorem 1. *Let Φ_i be a value for player $i \in N$ that satisfies linearity and dummy axioms. Then, for every $b \in \mathcal{BG}^N$,*

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \Big[\overline{p}^i_{(S,T)} \left(b(S \cup i, T) - b(S,T) \right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S, T \cup i\right) \right) \Big],$$

where $\sum_{(S,T) \in 3^{N \setminus i}} \overline{p}^i_{(S,T)} = 1$, *and* $\sum_{(S,T) \in 3^{N \setminus i}} \underline{p}^i_{(S,T)} = 1$.

Proof. The set of identity games is a basis of $\mathcal{BG}^N$, and each game $b \in \mathcal{BG}^N$ can be written as

$$b = \sum_{\{(S,T) \in 3^N : (S,T) \neq (\emptyset,\emptyset)\}} b(S,T) \delta_{(S,T)}.$$

By the linearity axiom,

$$\Phi_i(b) = \sum_{\{(S,T) \in 3^N : (S,T) \neq (\emptyset,\emptyset)\}} \Phi_i(\delta_{(S,T)}) b(S,T).$$

We denote by $a^i_{(S,T)} = \Phi_i\left(\delta_{(S,T)}\right)$ for all $(S,T) \neq (\emptyset, \emptyset)$ and thus, the value $\Phi_i(b)$ is given by

$$\begin{aligned}
&\sum_{(S,T)\in 3^N} a^i_{(S,T)} b(S,T) \\
&= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T)} b(S,T) + \sum_{\{(S,T)\in 3^N : i\in S\}} a^i_{(S,T)} b(S,T) \\
&\quad + \sum_{\{(S,T)\in 3^N : i\in T\}} a^i_{(S,T)} b(S,T) \\
&= \sum_{\left\{(S,T)\in 3^{N\setminus i} : (S,T)\neq(\emptyset,\emptyset)\right\}} a^i_{(S,T)} b(S,T) + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} b(S\cup i,T) \\
&\quad + \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S,T\cup i)} b(S,T\cup i) \\
&= \sum_{(\emptyset,\emptyset)\neq(S,T)\in 3^{N\setminus i}} \left(a^i_{(S,T)} b(S,T) + a^i_{(S\cup i,T)} b(S\cup i,T) + a^i_{(S,T\cup i)} b(S,T\cup i) \right) \\
&\quad + a^i_{(\{i\},\emptyset)} b(\{i\},\emptyset) + a^i_{(\emptyset,\{i\})} b(\emptyset,\{i\}).
\end{aligned}$$

Let us consider the games $w^i_{(A,B)} : 3^N \to \mathbb{R}$ where, for each $(A,B)\in 3^{N\setminus i}$, the game $w^i_{(A,B)}$ is defined by

$$w^i_{(A,B)}(S,T) = \begin{cases} w^i_{(A,B)}(S\setminus i,T) & \text{if } i\in S, \\ w^i_{(A,B)}(S,T\setminus i) & \text{if } i\in T, \\ 1 & \text{if } i\notin S\cup T, (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, player i is a dummy in $w^i_{(A,B)}$ for each $(A,B)\in 3^{N\setminus i}$ and hence $\Phi_i(w^i_{(A,B)}) = 0$ by the dummy axiom. If we apply the above equality to the game $w^i_{(A,B)}$, we get

$$\sum_{\left\{(S,T)\in 3^{N\setminus i} : (\emptyset,\emptyset)\neq(S,T)\sqsubseteq(A,B)\right\}} \left(a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} \right) = 0.$$

We show, by induction on $\rho\left[(S,T)\right]$, the rank of the coalitions, that for all $(S,T)\in 3^{N\setminus i}$, $(S,T)\neq(\emptyset,\emptyset)$, it holds that $a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0$. Note that the first element in $\left(3^{N\setminus i},\sqsubseteq\right)$ is $(\emptyset, N\setminus i)$, and so $\rho\left[(\emptyset,N\setminus i)\right] = 0$. Thus, we obtain

$$\begin{aligned}
&\sum_{\left\{(S,T)\in 3^{N\setminus i} : (S,T)\sqsubseteq(\emptyset,N\setminus i)\right\}} \left(a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} \right) \\
&= a^i_{(\emptyset,N\setminus i)} + a^i_{(\{i\},N\setminus i)} + a^i_{(\emptyset,N)} = 0.
\end{aligned}$$

Now assume the property for $(H,J)\in 3^{N\setminus i}$ with $\rho\left[(H,J)\right] \leq k-1$ and suppose that $(S,T)\in 3^{N\setminus i}$ has $\rho\left[(S,T)\right] = k$. Then

$$\begin{aligned}
\Phi_i(w^i_{(S,T)}) &= \sum_{\left\{(H,J)\in 3^{N\setminus i}:(\emptyset,\emptyset)\neq(H,J)\sqsubseteq(S,T)\right\}} \left(a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}\right) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} \\
&\quad + \sum_{\left\{(H,J)\in 3^{N\setminus i}:(\emptyset,\emptyset)\neq(H,J)\sqsubset(S,T)\right\}} \left(a^i_{(H,J)} + a^i_{(H\cup i,J)} + a^i_{(H,J\cup i)}\right) \\
&= a^i_{(S,T)} + a^i_{(S\cup i,T)} + a^i_{(S,T\cup i)} = 0,
\end{aligned}$$

where the last but one equality follows from the induction hypothesis, and the last one follows from the dummy axiom. Now for each $(S,T) \in 3^{N\setminus i}$, define

$$\overline{p}^i_{(\emptyset,\emptyset)} = a^i_{(\{i\},\emptyset)}, \quad \underline{p}^i_{(\emptyset,\emptyset)} = -a^i_{(\emptyset,\{i\})}, \quad \overline{p}^i_{(S,T)} = a^i_{(S\cup i,T)}, \quad \underline{p}^i_{(S,T)} = -a^i_{(S,T\cup i)},$$

and we compute

$$\begin{aligned}
&\Phi_i(b) \\
&= \sum_{(S,T)\in 3^{N\setminus i}} \Big[\left(\underline{p}^i_{(S,T)} - \overline{p}^i_{(S,T)}\right) b(S,T) \\
&\qquad\qquad + \overline{p}^i_{(S,T)} b(S\cup i,T) - \underline{p}^i_{(S,T)} b\left(S,T\cup i\right)\Big] \\
&= \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}^i_{(S,T)}\left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)}\left(b(S,T) - b\left(S,T\cup i\right)\right)\right].
\end{aligned}$$

Finally, it is easy to check that player i is a dummy in the games $\overline{u}_{(\{i\},N\setminus i)}$ and $\underline{u}_{(N\setminus i,\{i\})}$, and hence

$$\begin{aligned}
\sum_{(S,T)\in 3^{N\setminus i}} \overline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} a^i_{(S\cup i,T)} = \sum_{\{(S,T)\in 3^N:i\in S\}} a^i_{(S,T)} \\
&= \sum_{\{(S,T)\in 3^N:i\in S\}} \Phi_i\left(\delta_{(S,T)}\right) = \Phi_i\left(\sum_{\{(S,T)\in 3^N:i\in S\}} \delta_{(S,T)}\right) \\
&= \Phi_i\left(\overline{u}_{(\{i\},N\setminus i)}\right) = \overline{u}_{(\{i\},N\setminus i)}\left(\{i\},\emptyset\right) - \overline{u}_{(\{i\},N\setminus i)}\left(\emptyset,\{i\}\right) = 1.
\end{aligned}$$

$$\begin{aligned}
\sum_{(S,T)\in 3^{N\setminus i}} \underline{p}^i_{(S,T)} &= \sum_{(S,T)\in 3^{N\setminus i}} -a^i_{(S,T\cup i)} = \sum_{\{(S,T)\in 3^N:i\in T\}} -a^i_{(S,T)} \\
&= \sum_{\{(S,T)\in 3^N:i\in T\}} -\Phi_i\left(\delta_{(S,T)}\right) = \Phi_i\left(\sum_{\{(S,T)\in 3^N:i\in T\}} -\delta_{(S,T)}\right) \\
&= \Phi_i\left(\underline{u}_{(N\setminus i,\{i\})}\right) = \underline{u}_{(N\setminus i,\{i\})}\left(\{i\},\emptyset\right) - \underline{u}_{(N\setminus i,\{i\})}\left(\emptyset,\{i\}\right) \\
&= 1.
\end{aligned}$$

■

Now, we show that if we add the symmetry axiom to the linearity and dummy axioms, the coefficients $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ only depend on the cardinality of S and T.

Theorem 2. *Let Φ_i be a value for player $i \in N$ defined, for every game $b \in \mathcal{BG}^N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \Big[\overline{p}^i_{(S,T)} \left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S,T\cup i\right)\right)\Big].$$

If Φ_i satisfies the symmetry axiom, then $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ and $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ for all $(S,T) \in 3^{N\setminus i}$ with $s = |S|$ and $t = |T|$.

Proof. Let Φ_i be a value for player i given by

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \Big[\overline{p}^i_{(S,T)} \left(b(S\cup i,T) - b(S,T)\right) + \underline{p}^i_{(S,T)} \left(b(S,T) - b\left(S,T\cup i\right)\right)\Big].$$

Let (S_1,T_1) and (S_2,T_2) be coalitions in $3^{N\setminus i}$ such that $(S_1,T_1) \neq (\emptyset,\emptyset) \neq (S_2,T_2)$ satisfying that $|S_1| = |S_2| < n-1$ and $|T_1| = |T_2| < n-1$. Consider a permutation π of N that takes $\pi S_1 = S_2$ and $\pi T_1 = T_2$ while leaving i fixed. Then $\pi\delta_{(S_1,T_1)} = \delta_{(S_2,T_2)}$ and

$$\begin{aligned}
\overline{p}^i_{(S_1,T_1)} &= \Phi_i(\delta_{(S_1\cup i,T_1)}) = \Phi_i(\delta_{(S_2\cup i,T_2)}) = \overline{p}^i_{(S_2,T_2)},\\
\underline{p}^i_{(S_1,T_1)} &= -\Phi_i(\delta_{(S_1,T_1\cup i)}) = -\Phi_i(\delta_{(S_2,T_2\cup i)}) = \underline{p}^i_{(S_2,T_2)},
\end{aligned}$$

where the second equality follows from the symmetry axiom.

Now, let $i,j \in N, i \neq j$ and let $(S,T) \in 3^{N\setminus\{i,j\}}$. Let us consider the permutation π of N that interchanges i and j while leaving the remaining players fixed. Then $\pi\delta_{(S,T)} = \delta_{(S,T)}$ and

$$\begin{aligned}
\overline{p}^i_{(S,T)} &= \Phi_i(\delta_{(S\cup i,T)}) = \Phi_j(\delta_{(S\cup j,T)}) = \overline{p}^j_{(S,T)},\\
\underline{p}^i_{(S,T)} &= -\Phi_i(\delta_{(S,T\cup i)}) = -\Phi_j(\delta_{(S,T\cup j)}) = \underline{p}^j_{(S,T)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\overline{p}^i_{(N\setminus i,\emptyset)} &= \Phi_i(\delta_{(N,\emptyset)}) = \Phi_j(\delta_{(N,\emptyset)}) = \overline{p}^j_{(N\setminus j,\emptyset)},\\
\underline{p}^i_{(\emptyset,N\setminus i)} &= -\Phi_i(\delta_{(\emptyset,N)}) = -\Phi_j(\delta_{(\emptyset,N)}) = \underline{p}^j_{(\emptyset,N\setminus j)}.
\end{aligned}$$

Hence, for every $(S,T) \in 3^{N\setminus i}$ there exist $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ such that $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}$ and $\underline{p}^i_{(S,T)} = \underline{p}_{s,t}$ for all $i \in N$. ∎

The following theorem characterizes the values $\Phi = (\Phi_1, \dots, \Phi_n)$ that satisfy the above axioms and are efficient.

Theorem 3. *Let $\Phi = (\Phi_1, \dots, \Phi_n)$ be a value on $\mathcal{BG}^N$ defined, for every game b and for all $i \in N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}_{s,t}\left(b(S\cup i,T) - b(S,T)\right) + \underline{p}_{s,t}\left(b(S,T) - b(S,T\cup i)\right)\right].$$

Then, the value Φ satisfies the efficiency axiom if and only if it is satisfied

$$\overline{p}_{n-1,0} = \frac{1}{n}, \quad \underline{p}_{0,n-1} = \frac{1}{n},$$

and

$$(n-s-t)\,\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\,\underline{p}_{s,t} + s\overline{p}_{s-1,t}$$

for all $0 \le s,\ t \le n-1$ and $0 < s+t \le n-1$.

Proof. For every $b \in \mathcal{BG}^N$, we have that $\sum_{i\in N}\Phi_i(b)$ is equal to

$$\begin{aligned}
&\sum_{i\in N}\sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}_{s,t}\left(b(S\cup i,T) - b(S,T)\right) + \underline{p}_{s,t}\left(b(S,T) - b(S,T\cup i)\right)\right]\\
&=\sum_{i\in N}\sum_{(S,T)\in 3^{N\setminus i}} \left[\overline{p}_{s,t}b(S\cup i,T) - \underline{p}_{s,t}b(S,T\cup i) + \left(-\overline{p}_{s,t} + \underline{p}_{s,t}\right)b\left(S,T\right)\right]\\
&=\sum_{(S,T)\in 3^N} b\left(S,T\right)\left[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)\left(-\overline{p}_{s,t} + \underline{p}_{s,t}\right)\right]\\
&= b\left(N,\emptyset\right)n\overline{p}_{n-1,0} - b\left(\emptyset,N\right)n\underline{p}_{0,n-1}\\
&\quad + \sum_{\substack{(\emptyset,\emptyset)\neq(S,T)\in 3^N\\ (S,T)\notin\{(\emptyset,N),(N,\emptyset)\}}} b\left(S,T\right)\left[s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)\left(-\overline{p}_{s,t} + \underline{p}_{s,t}\right)\right].
\end{aligned}$$

If the coefficients satisfy the relations for the coefficients, then Φ satisfies the efficiency axiom.

Conversely, fix $(S,T) \in 3^N, (S,T) \neq (\emptyset,\emptyset)$, and applying the preceding equality to the identity game $\delta_{(S,T)}$, we have that $\sum_{i\in N}\Phi_i(\delta_{(S,T)})$ is equal to

$$\begin{cases}
n\overline{p}_{n-1,0} & \text{if } (S,T) = (N,\emptyset),\\
-n\underline{p}_{0,n-1} & \text{if } (S,T) = (\emptyset,N),\\
s\overline{p}_{s-1,t} - t\underline{p}_{s,t-1} + (n-s-t)\left(\underline{p}_{s,t} - \overline{p}_{s,t}\right) & \text{otherwise.}
\end{cases}$$

Thus, if Φ satisfies the efficiency axiom, the relations for the coefficients are true. ■

As we have already indicated, these four axioms are not sufficient to characterize the Shapley value for bicooperative games. Now, we introduce an additional axiom and prove that our Shapley value is the unique value on $\mathcal{BG}^N$ that verifies the five axioms. This new axiom will take into account the structure of the set of the coalitions in 3^N.

First of all, note that the coalitions $(S \setminus j, T)$ and $(S, T \cup i)$ where $j \in S$ and $i \notin S \cup T$ have the same rank

$$\rho\left[(S \setminus j, T)\right] = \rho\left[(S, T \cup i)\right] = n + s - t - 1.$$

However, the number of maximal chains in the sublattice $\left[(\emptyset, N), (S \setminus j, T)\right]$ is not the same as the number of maximal chains in $\left[(\emptyset, N), (S, T \cup i)\right]$ as, by Proposition 2,

$$c\left(\left[(\emptyset, N), (S \setminus j, T)\right]\right) = \frac{(n + s - 1 - t)!}{2^{s-1}},$$
$$c\left(\left[(\emptyset, N), (S, T \cup i)\right]\right) = \frac{(n + s - t - 1)!}{2^{s}}.$$

Hence, beginning from the coalition $(\emptyset, N)$, the probability of formation of the coalition (S, T) with the incorporation of one player j to $(S \setminus j, T)$ must be distinct from the probability of formation (S, T) with the desertion of one player i in $(S, T \cup i)$.

In analogous form, if we consider $(S, T \setminus k)$ with $k \in T$ and $(S \cup i, T)$, which have the same rank, the number of maximal chains in $\left[(S, T \setminus k), (N, \emptyset)\right]$ is not equal to the number of maximal chains in $\left[(S \cup i, T), (N, \emptyset)\right]$. Therefore the probability of formation of $(N, \emptyset)$ beginning from $(S, T \setminus k)$ when one player k leaves the coalition T must be distinct from the probability of formation of $(N, \emptyset)$ when one player i forms the coalition $(S \cup i, T)$.

Taking into account these considerations, the values that one player must obtain in the identity games must be proportional to the number of maximal chains in the corresponding sublattices. It must be also considered that one value verifying the above four axioms assigns a non-negative real number to one player i in the identity game $\delta_{(S,T)}$ if this player belongs to S and a non-positive real number if the player i belongs to T. From this point of view, our value must be satisfied by the following axiom (see Figure 4).

Structural axiom. For every $(S, T) \in 3^{N \setminus i}$, $j \in S$ and $k \in T$, it holds

$$\frac{c\left(\left[(\emptyset, N), (S \setminus j, T)\right]\right)}{c\left(\left[(\emptyset, N), (S, T \cup i)\right]\right)} = -\frac{\Phi_j(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S,T \cup i)}\right)},$$
$$\frac{c\left(\left[(S, T \setminus k), (N, \emptyset)\right]\right)}{c\left(\left[(S \cup i, T), (N, \emptyset)\right]\right)} = -\frac{\Phi_k(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S \cup i,T)}\right)}.$$

Theorem 4. *Let Φ be a value on $\mathcal{BG}^N$. The value Φ is the Shapley value if and only if Φ satisfies the efficiency axiom and each component satisfies linearity, dummy, symmetry, and structural axioms.*

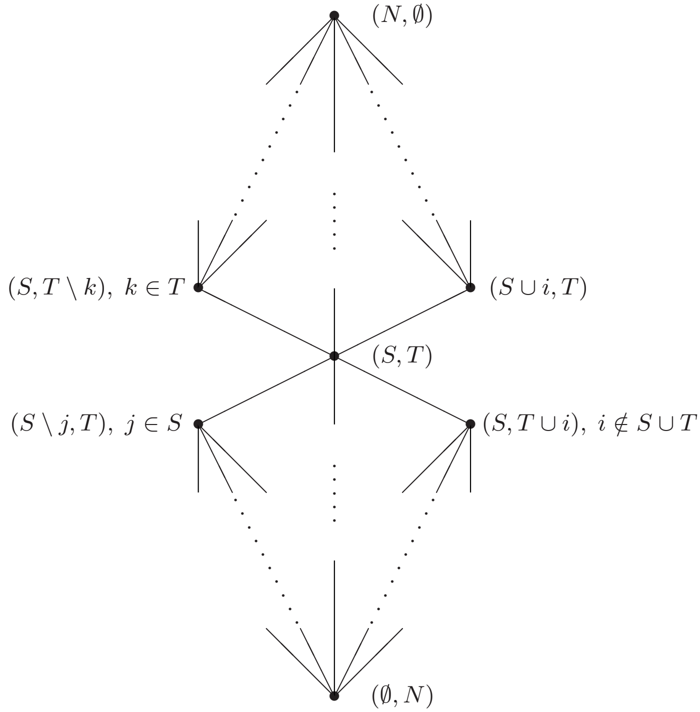

Figure 4. Structural axiom

Proof. If Φ is a value that satisfies linearity, dummy, symmetry, and efficiency, then

$$\Phi_i(b)=\sum_{(S,T)\in 3^{N\setminus i}}\Big[\overline{p}_{s,t}\left(b(S\cup i,T)-b(S,T)\right)+\underline{p}_{s,t}\left(b(S,T)-b\left(S,T\cup i\right)\right)\Big]$$

and the coefficients $\overline{p}_{s,t}$ and $\underline{p}_{s,t}$ satisfy

$$\overline{p}_{n-1,0}=\frac{1}{n},\quad \underline{p}_{0,n-1}=\frac{1}{n},$$

and

$$(n-s-t)\,\overline{p}_{s,t}+t\underline{p}_{s,t-1}=(n-s-t)\,\underline{p}_{s,t}+s\overline{p}_{s-1,t}. \tag{1}$$

Taking into account that the value Φ verifies the structural axiom, then

$$\overline{p}_{s-1,t}=2\underline{p}_{s,t}, \tag{2}$$

$$\underline{p}_{s,t-1}=2\overline{p}_{s,t}. \tag{3}$$

We prove that these coefficients, verifying all above conditions, are determined in unique form. Indeed, consider a coalition (S,T) with $|S|=n-1$ and $|T|=0$. If we apply equation (1) to this coalition, we obtain

$$\overline{p}_{n-1,0} = \underline{p}_{n-1,0} + (n-1)\,\overline{p}_{n-2,0}$$

and by (2), $\overline{p}_{n-2,0} = 2\underline{p}_{n-1,0}$. Taking into account that $\overline{p}_{n-1,0} = \frac{1}{n}$ and combining the above equalities, we have that

$$\frac{1}{n} = (1 + 2\,(n-1))\,\underline{p}_{n-1,0}$$

and hence

$$\underline{p}_{n-1,0} = \frac{1}{n\,(2n-1)} = \frac{1!\,(2n-2)!}{2^{n-1}\,(2n)!}2^n, \quad \overline{p}_{n-2,0} = \frac{2}{n\,(2n-1)} = \frac{1!\,(2n-2)!}{2^{n-2}\,(2n)!}2^n.$$

In similar way, if we apply (1) and (2) to a coalition (S,T) with $|S| = n-2$ and $|T| = 0$, we get

$$\begin{aligned} 2\overline{p}_{n-2,0} &= 2\underline{p}_{n-2,0} + (n-2)\,\overline{p}_{n-3,0}, \\ \overline{p}_{n-3,0} &= 2\underline{p}_{n-2,0}, \end{aligned}$$

and hence

$$\underline{p}_{n-2,0} = \frac{2!\,(2n-3)!}{2^{n-2}\,(2n)!}2^n, \quad \overline{p}_{n-3,0} = \frac{2!\,(2n-3)!}{2^{n-3}\,(2n)!}2^n.$$

If we assume that

$$\underline{p}_{s+1,0} = \frac{(n-s-1)!\,(n+s)!}{2^{s+1}\,(2n)!}2^n, \quad \overline{p}_{s,0} = \frac{(n-s-1)!\,(n+s)!}{2^{s}\,(2n)!}2^n$$

then, for $|S| = s$ and $|T| = 0$, applying (1) and (2),

$$\begin{aligned} (n-s)\,\overline{p}_{s,0} &= (n-s)\,\underline{p}_{s,0} + s\overline{p}_{s-1,0}, \\ \overline{p}_{s-1,0} &= 2\underline{p}_{s,0}, \end{aligned}$$

and combining both expressions, we obtain, for $1 \le s \le n-1$,

$$\underline{p}_{s,0} = \frac{(n-s)!\,(n+s-1)!}{2^{s}\,(2n)!}2^n, \quad \overline{p}_{s-1,0} = \frac{(n-s)!\,(n+s-1)!}{2^{s-1}\,(2n)!}2^n.$$

If we apply the same reasoning with the equalities (1) and (3) beginning with a coalition (S,T) with $|S| = 0$ and $|T| = n-1$, we obtain, for $1 \le t \le n-1$,

$$\overline{p}_{0,t} = \frac{(n-t)!\,(n+t-1)!}{2^{t}\,(2n)!}2^n, \quad \underline{p}_{0,t-1} = \frac{(n-t)!\,(n+t-1)!}{2^{t-1}\,(2n)!}2^n.$$

If we now consider (S,T) with $|S| = s$ and $|T| = 1$, we apply (1) and (3),

$$(n-s-1)\,\overline{p}_{s,1} + \underline{p}_{s,0} = (n-s-1)\,\underline{p}_{s,1} + s\overline{p}_{s-1,1},$$

$$\overline{p}_{s,1} = \frac{1}{2}\underline{p}_{s,0}, \quad \overline{p}_{s-1,1} = \frac{1}{2}\underline{p}_{s-1,0},$$

and substitute the values already obtained, then

$$\overline{p}_{s-1,1} = \frac{(n-s+1)!\,(n+s-2)!}{2^s\,(2n)!}2^n, \quad \underline{p}_{s,1} = \frac{(n-s+1)!\,(n+s-2)!}{2^{s+1}\,(2n)!}2^n.$$

If we assume that

$$\overline{p}_{s-1,t-1} = \frac{(n-s+t-1)!\,(n+s-t)!}{2^{s+t-2}\,(2n)!}2^n,$$
$$\underline{p}_{s,t-1} = \frac{(n-s+t-1)!\,(n+s-t)!}{2^{s+t-1}\,(2n)!}2^n,$$

then applying $\underline{p}_{s,t-1} = 2\overline{p}_{s,t}$ (3) we obtain, for all $0 \leq s, \quad t \leq n-1$ and $s+t \leq n-1$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{2^{s+t}\,(2n)!}2^n.$$

Finally, applying (1) and (2),

$$(n-s-t)\,\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\,\underline{p}_{s,t} + s\overline{p}_{s-1,t},$$
$$\overline{p}_{s-1,t} = 2\underline{p}_{s,t},$$

it holds that

$$\underline{p}_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{2^{s+t}\,(2n)!}2^n$$

for all $0 \leq s, \ \ t \leq n-1$ and $s+t \leq n-1$. ■

4 The Core and the Weber Set

Now, some solution concepts for bicooperative games are introduced, where a solution concept is a rule that assigns to every bicooperative game a set of payoff vectors that distribute the total saving among the players. Taking into account different situations that can be modeled by a bicooperative game (N, b), the amount $b\,(N, \emptyset)$ is the maximal gain and $b\,(\emptyset, N)$ is the minimal loss obtained by the players when they decide full cooperation and so, the maximal global gain is given by $b\,(N, \emptyset) - b\,(\emptyset, N)$. A vector $x \in \mathbb{R}^n$ that satisfies $\sum_{i \in N} x_i = b\,(N, \emptyset) - b\,(\emptyset, N)$ is an *efficient vector*, and the set of all efficient vectors is called *preimputation set*, which is defined by

$$I^*(N, b) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = b\,(N, \emptyset) - b\,(\emptyset, N) \right\}.$$

The *imputations* for game b are the preimputations that satisfy the *individual rationality principle* for all players, that is, each player gets at least the difference between the amount that he can attain by himself taking the rest of players against and the value of the coalition $(\emptyset, N)$,

$$I(N,b) = \{x \in I^*(N,b) : x_i \geq b(i, N \setminus i) - b(\emptyset, N) \text{ for all } i \in N\}.$$

A satisfactory distribution criterion could be that every coalition $(S,T) \in 3^N$ receives at least the amount it can contribute to the coalition $(\emptyset, N)$, that is, the amount $b(S,T) - b(\emptyset, N)$. This leads us to the following definition of the core of a bicooperative game.

Definition 3. *Let $b \in \mathcal{BG}^N$. The core of b is the set*

$$C(N,b) = \left\{ \begin{array}{l} x \in I^*(N,b) : \text{ there exist } y,z \in \mathbb{R}^n \text{ such that } x = y+z, \text{ and} \\ y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N), \text{ for all } (S,T) \in 3^N \end{array} \right\}.$$

Let $x \in I^*(N,b)$ be such that $x = y + z$. Then

$$y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N) \Longleftrightarrow y(N \setminus S) + z(T) \leq b(N, \emptyset) - b(S,T).$$

Therefore, $C(N,b)$ is also the set of vectors $x \in I^*(N,b)$ such that there exist $y, z \in \mathbb{R}^n$ with $x = y + z$ and $y(N \setminus S) + z(T) \leq b(N, \emptyset) - b(S,T)$ for all $(S,T) \in 3^N$. Thus, for each $(S,T) \in 3^N$, the payoff $y(N \setminus S)$ plus the payoff $z(T)$ must not exceed $b(N, \emptyset) - b(S,T)$, which is the amount that is foregone by forming the coalition (S,T) instead of the coalition $(N, \emptyset)$.

Notice also that $x \in C(N,b)$ if and only if there exist $y, z \in \mathbb{R}^n$ such that $x = y + z$, and

$$\begin{aligned} y(S) + z(N \setminus T) &\geq b(S,T) - b(\emptyset, N), \\ y(N \setminus S) + z(T) &\leq b(N, \emptyset) - b(S,T), \end{aligned}$$

for all $(S,T) \in 3^N$. These inequalities are similar in the bicooperative context to the inequalities characterizing the core in a cooperative game $v : 2^N \to \mathbb{R}$,

$$C(v) = \left\{ x \in \mathbb{R}^n : x(S) \geq v(S) - v(\emptyset), \ x(N \setminus S) \leq v(N) - v(S), \ \forall S \in 2^N \right\}.$$

In order to extend the idea of the Weber set to a bicooperative game (N,b), it is assumed that all players estimate that $(N, \emptyset)$ is formed as a sequential process where at each step a player joins the defender coalition or a player leaves the detractor coalition. These sequential processes are obtained for each chain from $(\emptyset, N)$ to $(N, \emptyset)$. For each chain, a player can evaluate his contribution when he joins the defenders or when he leaves the detractors. This can be reflected in the vectors of $\mathbb{R}^n$ called *superior marginal worth vectors* and *inferior marginal worth vectors*. Thus, we introduce the following notation.

For $N = \{1, \ldots, n\}$, let $\overline{N} = \{-n, \ldots, -1, 1, \ldots, n\}$. Let $\Lambda : 3^N \to 2^{\overline{N}}$ be the isomorphism defined by $\Lambda(S,T) = S \cup \{-i : i \in N \setminus T\} \in 2^{\overline{N}}$, for each $(S,T) \in 3^N$. For instance, $\Lambda(\emptyset,N) = \emptyset$ and $\Lambda(N,\emptyset) = \overline{N}$. Because $S \cap T = \emptyset \Leftrightarrow S \subseteq N \backslash T$, we see that $i \in \Lambda(S,T)$ and $i > 0$ imply $-i \in \Lambda(S,T)$.

In the lattice $\left(3^N, \sqsubseteq\right)$, we consider the set of all maximal chains going from $(\emptyset, N)$ to $(N,\emptyset)$ and denote this set by $\Theta\left(3^N\right)$. If $\theta \in \Theta\left(3^N\right)$ is the maximal chain

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

then we can write the following associated chain of sets in $2^{\overline{N}}$,

$$\emptyset \subset \{i_1\} \subset \cdots \subset \{i_1, \ldots, i_j\} \subset \cdots \subset \{i_1, \ldots, i_{2n-1}\} \subset \overline{N},$$

where $\{i_1, \ldots, i_j\} = \Lambda(S_j, T_j)$ for $j = 1, \ldots, 2n$. We define the vector $\theta(i_j) = (i_1, \ldots, i_j)$, where the last component $i_j \in \overline{N}$ satisfies the following property: if $i_j > 0$, then the player $i_j \in S_j$ and $i_j \notin S_{j-1}$, that is, i_j is the last player who joins S_j, and if $i_j < 0$, then the player $-i_j \notin T_j$ and $-i_j \in T_{j-1}$, that is, $-i_j$ is the last player who leaves T_{j-1}. Equivalently, the elements in $\theta(i_j) = (i_1, \ldots, i_j)$ are written following the order of incorporation in the defenders coalitions or desertion from the detractors coalition (depending on the sign of each i_k) in the coalitions in chain θ . Moreover, we write

$$\theta(i_j) \setminus i_j = (i_1, i_2, \ldots, i_{j-1}) = \theta(i_{j-1})$$

and $i_k \in \theta(i_j)$ when i_k is one component of the vector $\theta(i_j)$, that is $1 \leq k \leq j$. Note that an equivalence between maximal chains and vectors $\theta = (i_1, \ldots, i_{2n})$ is obtained. Fix an order $\theta = (i_1, \ldots, i_{2n})$, we also define $\alpha[\theta(i_j)] = (S_j, T_j)$ such that $\Lambda(S_j, T_j) = \{i_1, \ldots, i_j\}$. Moreover, $\alpha[\theta(i_j) \setminus i_j] = \alpha[\theta(i_{j-1})] = (S_{j-1}, T_{j-1})$. In particular, $\alpha[\theta(i_{2n})] = (N, \emptyset)$ and $\alpha[\theta(i_1) \setminus i_1] = (\emptyset, N)$.

For example, let it be $N = \{1,2,3\}$ and let $\theta \in \Theta\left(3^N\right)$ be given by

$$(\emptyset, N) \sqsubset (\emptyset, \{1,3\}) \sqsubset (\{2\}, \{1,3\}) \sqsubset (\{2\}, \{1\}) \sqsubset (\{2\}, \emptyset) \sqsubset (\{2,3\}, \emptyset) \sqsubset (N, \emptyset).$$

Its associated chain of sets in $2^{\overline{N}}$ is given by

$$\emptyset \subset \{-2\} \subset \{-2,2\} \subset \{-2,2,-3\} \subset \{-2,2,-3,-1\} \subset \{-2,2,-3,-1,3\} \subset \overline{N}.$$

and the maximal chain can be represented by the order $\theta = (-2,2,-3,-1,3,1)$. A coalition, for instance $(\{2\}, \emptyset)$, can be also represented by $\alpha[\theta(-1)]$ and by $\Lambda^{-1}(\{-2,2,-3,-1\})$.

Definition 4. *Let $\theta \in \Theta\left(3^N\right)$ and $b \in \mathcal{BG}^N$. The inferior and superior marginal worth vectors with respect to θ are $m^\theta(b), M^\theta(b) \in \mathbb{R}^n$ given by*

$$\begin{aligned} m_i^\theta(b) &= b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]), \\ M_i^\theta(b) &= b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i]), \end{aligned}$$

for all $i \in N$. The vector $a^\theta(b) = m^\theta(b) + M^\theta(b)$ is called the marginal worth vector with respect to θ.

We show that the marginal worth vectors are preimputations.

Proposition 4. *For any* $b \in \mathcal{BG}^N$ *and* $\theta \in \Theta\left(3^N\right)$, *we have*

$$\sum_{i \in N} a_i^\theta (b) = b(N, \emptyset) - b(\emptyset, N).$$

Proof. Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$. It holds that

$$\begin{aligned}
\sum_{i \in N} a_i^\theta (b) &= \sum_{i \in N} \left[m_i^\theta (b) + M_i^\theta (b)\right] \\
&= \sum_{i \in N} \left[b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]) + b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i])\right] \\
&= \sum_{j=1}^{2n} \left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j) \setminus i_j])\right] \\
&= b(\alpha[\theta(i_1)]) - b(\alpha[\theta(i_1) \setminus i_1]) + \sum_{j=2}^{2n} \left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_{j-1})])\right] \\
&= b(N, \emptyset) - b(\emptyset, N).
\end{aligned}$$

■

Proposition 5. *Let* $b \in \mathcal{BG}^N$ *and* $\theta \in \Theta\left(3^N\right)$. *Then,*

$$\sum_{j \in S} M_j^\theta (b) + \sum_{j \in N \setminus T} m_j^\theta (b) = b(S, T) - b(\emptyset, N),$$

for every (S, T) *in the chain* θ.

Proof. Let $\theta \in \Theta\left(3^N\right)$ and (S, T) in the chain θ with $|S| = s, |T| = t, s + t \leq n$ and such that $\Lambda(S, T) = \{i_1, i_2, \ldots, i_{n+s-t}\}$ where the i_j are written following the order of incorporation in θ, that is, $\theta(i_j) = (i_1, i_2, \ldots, i_j)$ for all $1 \leq j \leq n + s - t$. Then,

$$\begin{aligned}
\sum_{j \in S} M_j^\theta (b) + \sum_{j \in N \setminus T} m_j^\theta (b) &= \sum_{\{i_j \in \Lambda(S,T) : i_j > 0\}} M_{i_j}^\theta (b) + \sum_{\{i_j \in \Lambda(S,T) : i_j < 0\}} m_{-i_j}^\theta (b) \\
&= \sum_{i_j \in \Lambda(S,T)} \left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j) \setminus i_j])\right] \\
&= \sum_{j=1}^{n+s-t} \left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j) \setminus i_j])\right] \\
&= b(S, T) - b(\emptyset, N).
\end{aligned}$$

Note that for $(S, T) = (N, \emptyset)$, we have

$$\sum_{j \in N} \left[m_j^\theta (b) + M_j^\theta (b)\right] = b(N, \emptyset) - b(\emptyset, N).$$

■

Definition 5. *Let $b \in \mathcal{BG}^N$. The Weber set of b is the convex hull of the marginal worth vectors, that is, $W(N,b) = conv\left\{a^\theta(b) : \theta \in \Theta\left(3^N\right)\right\}$.*

As the preimputation set is a convex set, $W(N,b) \subseteq I^*(N,b)$. However, in general, the vectors of the Weber set are not imputations. For example, let (N,b) with $N = \{1,2\}$ and $b : 3^N \to \mathbb{R}$ defined as $b(\emptyset, N) = -5$, $b(\emptyset, i) = -4$, $b(i,j) = -1$, $b(i,\emptyset) = 1$, $b(N,\emptyset) = 2$, for all $i,j \in N$. If we consider $\theta = (-2,2,-1,1)$, then $a_1^\theta(b) = m_1^\theta(b) + M_1^\theta(b) = 3$. As $b(1,2) - b(\emptyset,N) = 4$, then $a_1^\theta(b) < b(1, N\setminus 1) - b(\emptyset, N)$ and $a^\theta(b) \notin I(N,b)$.

Because $I(N,b)$ is a convex set, then $W(N,b) \subseteq I(N,b)$ if all marginal worth vectors are imputations. For this, a sufficient condition is the *zero-monotonicity* of the game b.

Definition 6. *A bicooperative game $b \in \mathcal{BG}^N$ is monotonic when for all coalitions $(S_1,T_1),(S_2,T_2)$ with $(S_1,T_1) \sqsubseteq (S_2,T_2)$, it holds that $b(S_1,T_1) \le b(S_2,T_2)$.*

Definition 7. *The zero-normalization of a bicooperative game $b \in \mathcal{BG}^N$ is the game $b_0 \in \mathcal{BG}^N$ defined by*

$$b_0(S,T) = b(S,T) - \sum_{j\in S}\left[b(j, N\setminus j) - b(\emptyset, N)\right], \quad \textit{for all } (S,T) \in 3^N.$$

Definition 8. *A bicooperative game $b \in \mathcal{BG}^N$ is called zero-monotonic if its zero-normalization is monotonic.*

Proposition 6. *Let $b \in \mathcal{BG}^N$ be a zero-monotonic bicooperative game. Then, for every $\theta \in \Theta\left(3^N\right)$, the marginal worth vector associated to θ is an imputation for the game b.*

Proof. Let $\theta \in \Theta\left(3^N\right)$. Because the vector $a^\theta(b)$ is efficient, we prove that

$$\begin{aligned}
a_i^\theta(b) &= b(\alpha[\theta(i)]) - b(\alpha[\theta(i)\setminus i]) + b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i)\setminus -i]) \\
&= b_0(\alpha[\theta(i)]) + \sum_{\{i_j\in\theta(i):i_j>0\}}\left[b(i_j, N\setminus i_j) - b(\emptyset,N)\right] \\
&\quad - b_0(\alpha[\theta(i)\setminus i]) - \sum_{\{i_j\in\theta(i)\setminus i:i_j>0\}}\left[b(i_j, N\setminus i_j) - b(\emptyset,N)\right] \\
&\quad + b_0(\alpha[\theta(-i)]) + \sum_{\{i_j\in\theta(-i):i_j>0\}}\left[b(i_j, N\setminus i_j) - b(\emptyset,N)\right] \\
&\quad - b_0(\alpha[\theta(-i)\setminus -i]) - \sum_{\{i_j\in\theta(-i)\setminus -i:i_j>0\}}\left[b(i_j, N\setminus i_j) - b(\emptyset,N)\right] \\
&= b_0(\alpha[\theta(i)]) - b_0(\alpha[\theta(i)\setminus i]) + b_0(\alpha[\theta(-i)]) \\
&\quad - b_0(\alpha[\theta(-i)\setminus -i]) + b(i, N\setminus i) - b(\emptyset, N) \ge b(i, N\setminus i) - b(\emptyset,N),
\end{aligned}$$

where the inequality follows the zero-monotonicity of b. ∎

Now we prove that the core of a bicooperative game is always included in its Weber set. The proof is closely related to the proof given by Derks [4] of the parallel result for cooperative games.

Theorem 5. *If* $b \in \mathcal{BG}^N$, *then* $C(N,b) \subseteq W(N,b)$.

Proof. Assume that there exists $x \in C(N,b)$ such that $x \notin W(N,b)$. Because $x \in C(N,b)$, then $\sum_{i \in N} x_i = b(N,\emptyset) - b(\emptyset,N)$, and there exist $y,z \in \mathbb{R}^n$ such that $x = y+z$ and $y(S)+z(N\setminus T) \geq b(S,T) - b(\emptyset,N)$ for all $(S,T) \in 3^N$. Because $W(N,b)$ is convex and closed, by the Separation Theorem (see Rockafellar [13]), there exists $u \in \mathbb{R}^n$ such that

$$w \cdot u > x \cdot u \text{ for all } w \in W(N,b). \tag{4}$$

In particular, the above inequality holds for all marginal worth vectors $w = a^\theta(b)$ with $\theta \in \Theta\left(3^N\right)$. If the components of vector u are ordered in nonincreasing order $u_{i_1} \geq u_{i_2} \geq \cdots \geq u_{i_{n-1}} \geq u_{i_n}$, let $\theta \in \Theta\left(3^N\right)$ be the maximal chain given by $\theta = (-i_1, i_1, -i_2, i_2, \ldots, -i_n, i_n)$. Note that $\theta(i_j) \setminus i_j = \theta(-i_j)$ for all $1 \leq j \leq n$, $\theta(-i_j) \setminus -i_j = \theta(i_{j-1})$ for all $2 \leq j \leq n$ and $\alpha[\theta(-i_1) \setminus -i_1] = (\emptyset, N)$. Then

$$
\begin{aligned}
a^\theta(b) \cdot u &= \sum_{j=1}^n a^\theta_{i_j}(b)\, u_{i_j} = \sum_{j=1}^n \left[M^\theta_{i_j}(b) + m^\theta_{i_j}(b)\right] u_{i_j} \\
&= \sum_{j=1}^n u_{i_j}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_j)\setminus i_j]) + b(\alpha[\theta(-i_j)]) - b(\alpha[\theta(-i_j)\setminus -i_j])\right] \\
&= \sum_{j=1}^n u_{i_j}\left[b(\alpha[\theta(i_j)]) - b(\alpha[\theta(i_{j-1})])\right] \\
&= u_{i_n} b(N,\emptyset) + \sum_{j=1}^{n-1} u_{i_j} b(\alpha[\theta(i_j)]) - u_{i_1} b(\emptyset,N) - \sum_{j=2}^{n} u_{i_j} b(\alpha[\theta(i_{j-1})]) \\
&= u_{i_n} b(N,\emptyset) - u_{i_1} b(\emptyset,N) + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right) b(\alpha[\theta(i_j)]) \\
&\leq u_{i_n} b(N,\emptyset) - u_{i_1} b(\emptyset,N) + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right)\left[\sum_{k=1}^j y_{i_k} + \sum_{k=1}^j z_{i_k} + b(\emptyset,N)\right] \\
&= u_{i_n}\left[\sum_{k=1}^n y_{i_k} + \sum_{k=1}^n z_{i_k} + b(\emptyset,N)\right] - u_{i_1} b(\emptyset,N) \\
&\quad + \sum_{j=1}^{n-1}\left(u_{i_j} - u_{i_{j+1}}\right)\left[\sum_{k=1}^j y_{i_k} + \sum_{k=1}^j z_{i_k} + b(\emptyset,N)\right] \\
&= \sum_{j=1}^n u_{i_j}\left(y_{i_j} + z_{i_j}\right) = \sum_{j=1}^n u_{i_j} x_{i_j} = x \cdot u
\end{aligned}
$$

which is in contradiction with the inequality (4). We conclude that $C(N,b) \subseteq W(N,b)$. ■

5 Bisupermodular Games

We now introduce a special class of bicooperative games.

Definition 9. *A bicooperative game $b \in \mathcal{BG}^N$ is called bisupermodular if, for all (S_1, T_1) and (S_2, T_2), it holds*

$$b((S_1, T_1) \vee (S_2, T_2)) + b((S_1, T_1) \wedge (S_2, T_2)) \geq b(S_1, T_1) + b(S_2, T_2),$$

or equivalently

$$b(S_1 \cup S_2, T_1 \cap T_2) + b(S_1 \cap S_2, T_1 \cup T_2) \geq b(S_1, T_1) + b(S_2, T_2).$$

The next proposition characterizes the bisupermodular games as those bicooperative games for which the marginal contributions of a player to one coalition in 3^N is never less than the marginal contribution of this player to any coalition contained in it. This characterization will be used in the proofs of the following results.

Proposition 7. *Let $b \in \mathcal{BG}^N$. The bicooperative game b is bisupermodular if and only if for all $i \in N$ and (S_1, T_1), $(S_2, T_2) \in 3^{N \backslash i}$ such that $(S_1, T_1) \sqsubseteq (S_2, T_2)$, it holds $b(S_2 \cup i, T_2) - b(S_2, T_2) \geq b(S_1 \cup i, T_1) - b(S_1, T_1)$, and $b(S_2, T_2) - b(S_2, T_2 \cup i) \geq b(S_1, T_1) - b(S_1, T_1 \cup i)$.*

Proof. Necessary condition. Let (S_1, T_1), $(S_2, T_2) \in 3^{N \backslash i}$ with $(S_1, T_1) \sqsubseteq (S_2, T_2)$. If $S_1' = S_1 \cup i$ and we apply the definition of bisupermodularity to (S_1', T_1) and (S_2, T_2), it follows

$$b(S_1' \cup S_2, T_1 \cap T_2) + b(S_1' \cap S_2, T_1 \cup T_2) \geq b(S_1 \cup i, T_1) + b(S_2, T_2),$$

and hence $b(S_2 \cup i, T_2) + b(S_1, T_1) \geq b(S_1 \cup i, T_1) + b(S_2, T_2)$.

In an analogous form, taking $T_2' = T_2 \cup i$ and applying the definition of supermodularity to (S_1, T_1) and (S_2, T_2'), it follows

$$b(S_1, T_1 \cup i) + b(S_2, T_2) \geq b(S_1, T_1) + b(S_2, T_2 \cup i).$$

Sufficient condition. Let (S_1, T_1), $(S_2, T_2) \in 3^N$. If $(S_1, T_1) \sqsubseteq (S_2, T_2)$ or $(S_2, T_2) \sqsubseteq (S_1, T_1)$, the equality trivially holds. So, we consider the case $(S_1, T_1) \wedge (S_2, T_2) \neq (S_1, T_1)$ and $(S_1, T_1) \wedge (S_2, T_2) \neq (S_2, T_2)$.

Let $\theta \in \Theta\left(3^N\right)$ be a maximal chain that contains the coalitions (S_2, T_2) and $(S_1, T_1) \vee (S_2, T_2)$. As $\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2) \neq \emptyset$, we assume that $|\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2)| = k$ and $\Lambda(S_1, T_1) \setminus \Lambda(S_2, T_2) = \{i_1, i_2, \ldots, i_k\}$, where the i_j are in the same order as they appear in the order θ, i.e.,

$$\alpha[\theta(i_1)] \sqsubset \alpha[\theta(i_2)] \sqsubset \cdots \sqsubset \alpha[\theta(i_k)].$$

Then, the chain θ is given by

$$\emptyset \subset \cdots \subset \Lambda(S_2, T_2) \subset \Lambda(S_2, T_2) \cup \{i_1\} \subset \cdots \subset \Lambda(S_2, T_2) \cup \{i_1, \ldots, i_k\} \subset \cdots \subset \overline{N}$$

or equivalently

$$(\emptyset, N) \sqsubset \cdots \sqsubset (S_2, T_2) \sqsubset \cdots \sqsubset (S_1, T_1) \vee (S_2, T_2) \sqsubset \cdots \sqsubset (N, \emptyset).$$

If we denote $A_j = \{i_1, i_2, \ldots, i_j\}$, for all $1 \leq j \leq k$, $A_0 = \emptyset$ and $(P, Q) = (S_1, T_1) \wedge (S_2, T_2)$, it holds that $\Lambda^{-1}[\Lambda(P, Q) \cup A_j] \sqsubset \Lambda^{-1}[\Lambda(S_2, T_2) \cup A_j]$ for all $1 \leq j \leq k$. We can apply the hypothesis to $\Lambda^{-1}[\Lambda(P, Q) \cup A_j]$ and $\Lambda^{-1}[\Lambda(S_2, T_2) \cup A_j]$, and we obtain

$$\begin{aligned} & b\left(\Lambda^{-1}(\Lambda(P, Q) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(P, Q) \cup A_{j-1})\right) \\ & \leq b\left(\Lambda^{-1}(\Lambda(S_2, T_2) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(S_2, T_2) \cup A_{j-1})\right) \end{aligned}$$

for all $1 \leq j \leq k$. Hence,

$$\begin{aligned} b((S_1, T_1)) - b((S_1, T_1) \wedge (S_2, T_2)) &= b\left(\Lambda^{-1}(\Lambda(P, Q) \cup A_k)\right) - b(P, Q) \\ &= \sum_{j=1}^{k} \left[b\left(\Lambda^{-1}(\Lambda(P, Q) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(P, Q) \cup A_{j-1})\right)\right] \\ &\leq \sum_{j=1}^{k} \left[b\left(\Lambda^{-1}(\Lambda(S_2, T_2) \cup A_j)\right) - b\left(\Lambda^{-1}(\Lambda(S_2, T_2) \cup A_{j-1})\right)\right] \\ &= b((S_1, T_1) \vee (S_2, T_2)) - b(S_2, T_2). \end{aligned}$$

■

The following result allows the identification of the games for which the marginal worth vectors are in the core.

Theorem 6. *A necessary and sufficient condition so that all marginal worth vectors of a bicooperative game $b \in \mathcal{BG}^N$ are vectors of the core is that the game b is bisupermodular*

Proof. Sufficient condition. Let $\theta \in \Theta\left(3^N\right)$. We know that the marginal worth vectors are efficient, and we prove that the marginal worth vector $a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b)$ satisfies

$$\sum_{j \in S} M_j^\theta(b) + \sum_{j \in N \setminus T} m_j^\theta(b) \geq b(S, T) - b(\emptyset, N), \quad \text{for all } (S, T) \in 3^N.$$

By Proposition 5, for every (S, T) in the chain θ, it holds

$$\sum_{j \in S} M_j^\theta(b) + \sum_{j \in N \setminus T} m_j^\theta(b) = b(S, T) - b(\emptyset, N).$$

We prove that, for every coalition (S, T), not in the chain θ,

$$\sum_{j \in S} M_j^\theta(b) + \sum_{j \in N \setminus T} m_j^\theta(b) \geq b(S, T) - b(\emptyset, N).$$

Indeed, let (S,T) be a coalition that does not belong to the chain θ, such that $\Lambda(S,T)=\{i_1,i_2,\ldots,i_k\}$, $k=n+s-t$, where the elements are written following the order of θ; that is, $\alpha[\theta(i_1)]\sqsubset\alpha[\theta(i_2)]\sqsubset\cdots\sqsubset\alpha[\theta(i_k)]$.

If we denote $A_j=\{i_1,i_2,\ldots,i_j\}$, for all $1\leq j\leq k$, and $A_0=\emptyset$, note that, for all $1\leq j\leq k$, we have that $A_j=\Lambda(S,T)\cap\Lambda(\alpha[\theta(i_j)])$, that is, $\Lambda^{-1}(A_j)=(S,T)\wedge\alpha[\theta(i_j)]$. As b is a bisupermodular game, Proposition 7 implies that, for all $1\leq j\leq k$,

$$b(\alpha[\theta(i_j)])-b(\alpha[\theta(i_j)\setminus i_j])\geq b\left(\Lambda^{-1}(A_j)\right)-b\left(\Lambda^{-1}(A_{j-1})\right),$$

and we obtain

$$\begin{aligned}\sum_{j\in S}M_j^{\theta}(b)+\sum_{j\in N\setminus T}m_j^{\theta}(b)&=\sum_{\{i_j\in\Lambda(S,T):i_j>0\}}M_{i_j}^{\theta}(b)+\sum_{\{i_j\in\Lambda(S,T):i_j<0\}}m_j^{\theta}(b)\\&=\sum_{i_j\in\Lambda(S,T)}[b(\alpha[\theta(i_j)])-b(\alpha[\theta(i_j)\setminus i_j])]\\&=\sum_{j=1}^{n+s-t}[b(\alpha[\theta(i_j)])-b(\alpha[\theta(i_j)\setminus i_j])]\\&\geq\sum_{j=1}^{n+s-t}\left[b\left(\Lambda^{-1}(A_j)\right)-b\left(\Lambda^{-1}(A_{j-1})\right)\right]\\&=b(S,T)-b(\emptyset,N).\end{aligned}$$

Necessary condition. For all $(S_1,T_1),(S_2,T_2)\in 3^N$, consider a maximal chain $\theta\in\Theta\left(3^N\right)$ that contains $(S_1,T_1)\wedge(S_2,T_2)=(S_1\cap S_2,T_1\cup T_2)$ and $(S_1,T_1)\vee(S_2,T_2)=(S_1\cup S_2,T_1\cap T_2)$. As the marginal worth vectors are elements of $C(N,b)$, we have that

$$\sum_{j\in S_1}M_j^{\theta}(b)+\sum_{j\in N\setminus T_1}m_j^{\theta}(b)\geq b(S_1,T_1)-b(\emptyset,N),$$

$$\sum_{j\in S_2}M_j^{\theta}(b)+\sum_{j\in N\setminus T_2}m_j^{\theta}(b)\geq b(S_2,T_2)-b(\emptyset,N),$$

By the election of the maximal chain θ and Proposition 5, it is also satisfied

$$\sum_{j\in S_1\cap S_2}M_j^{\theta}(b)+\sum_{j\in N\setminus(T_1\cup T_2)}m_j^{\theta}(b)=b((S_1,T_1)\wedge(S_2,T_2))-b(\emptyset,N).$$

$$\sum_{j\in S_1\cup S_2}M_j^{\theta}(b)+\sum_{j\in N\setminus(T_1\cap T_2)}m_j^{\theta}(b)=b((S_1,T_1)\vee(S_2,T_2))-b(\emptyset,N).$$

Therefore,

$$
\begin{aligned}
& b\left(S_1, T_1\right)+b\left(S_2, T_2\right)-2 b(\emptyset, N) \\
\leq & \sum_{j \in S_1} M_j^\theta(b)+\sum_{j \in N \backslash T_1} m_j^\theta(b)+\sum_{j \in S_2} M_j^\theta(b)+\sum_{j \in N \backslash T_2} m_j^\theta(b) \\
= & \sum_{j \in S_1 \cup S_2} M_j^\theta(b)+\sum_{j \in S_1 \cap S_2} M_j^\theta(b)+\sum_{j \in N \backslash\left(T_1 \cup T_2\right)} m_j^\theta(b)+\sum_{j \in N \backslash\left(T_1 \cap T_2\right)} m_j^\theta(b) \\
= & b\left(\left(S_1, T_1\right) \wedge\left(S_2, T_2\right)\right)+b\left(\left(S_1, T_1\right) \vee\left(S_2, T_2\right)\right)-2 b(\emptyset, N) .
\end{aligned}
$$

Hence

$$
b\left(S_1, T_1\right)+b\left(S_2, T_2\right) \leq b\left(\left(S_1, T_1\right) \wedge\left(S_2, T_2\right)\right)+b\left(\left(S_1, T_1\right) \vee\left(S_2, T_2\right)\right) .
$$

■

As the core of a bicooperative game $b \in \mathcal{BG}^N$ is a convex set, an immediate consequence of this theorem is the following result.

Corollary 1. *Let $b \in \mathcal{BG}^N$. A necessary and sufficient condition so that $W(N, b)=C(N, b)$ is that the bicooperative game b is bisupermodular.*

Let $b \in \mathcal{BG}^N$. A special element of $W(N, b)=conv\left\{a^\theta(b): \theta \in \Theta\left(3^N\right)\right\}$ is the value that assigns the same probability to all maximal chains. In the next theorem, we prove that this value is the Shapley value of b.

Theorem 7. *The Shapley value for $b \in \mathcal{BG}^N$ is given, for each $i \in N$, by*

$$
\Phi_i(b)=\sum_{\theta \in \Theta\left(3^N\right)} \frac{1}{c\left(3^N\right)} a_i^\theta(b) .
$$

Proof. Let us consider $b \in \mathcal{BG}^N$ and compute

$$
\begin{aligned}
\Psi_i(b) & =\sum_{\theta \in \Theta\left(3^N\right)} \frac{1}{c\left(3^N\right)} a_i^\theta(b) \\
& =\sum_{\theta \in \Theta\left(3^N\right)} \frac{1}{c\left(3^N\right)}\left[m_i^\theta(b)+M_i^\theta(b)\right] \\
& =\sum_{\theta \in \Theta\left(3^N\right)} \frac{1}{c\left(3^N\right)}[b(\alpha[\theta(-i)])-b(\alpha[\theta(-i) \backslash-i])] \\
& \quad+\sum_{\theta \in \Theta\left(3^N\right)} \frac{1}{c\left(3^N\right)}[b(\alpha[\theta(i)])-b(\alpha[\theta(i) \backslash i])],
\end{aligned}
$$

If θ runs over all orders in $\Theta\left(3^N\right)$, the sets $\alpha[\theta(i) \setminus i]$ determine all coalitions $(S, T) \in 3^{N \backslash i}$ in which i is incorporated in the order, and the sets $\alpha[\theta(-i)]$

determine all coalitions $(S,T) \in 3^{N\setminus i}$ in which player i has just left the preceding coalition in the order. Thus, the above expression can be written as

$$\Psi_i(b) = \sum_{(S,T)\in 3^{N\setminus i}} \left[\left(\sum_{\substack{\theta\in\Theta(3^N):\\ \alpha[\theta(i)\setminus i]=(S,T)}} \frac{1}{c(3^N)} \right) [b(\alpha[\theta(i)]) - b(\alpha[\theta(i)\setminus i])] \right.$$
$$\left. + \left(\sum_{\substack{\theta\in\Theta(3^N):\\ \alpha[\theta(-i)]=(S,T)}} \frac{1}{c(3^N)} \right) [b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i)\setminus -i])] \right].$$

Now for each $(S,T) \in 3^{N\setminus i}$, we define

$$\overline{p}^i_{(S,T)} = \sum_{\substack{\theta\in\Theta(3^N):\\ \alpha[\theta(i)\setminus i]=(S,T)}} \frac{1}{c(3^N)}, \quad \underline{p}^i_{(S,T)} = \sum_{\substack{\theta\in\Theta(3^N):\\ \alpha[\theta(-i)]=(S,T)}} \frac{1}{c(3^N)}.$$

The number $\overline{p}^i_{(S,T)}$ represents the quotient between the number of chains from $(\emptyset,N)$ to $(N,\emptyset)$ that contain (S,T) and $(S\cup i,T)$ and the total number of maximal chains, and the number $\underline{p}^i_{(S,T)}$ represents the quotient between the chains that contain the coalitions $(S,T\cup i)$ and (S,T) and the total number of maximal chains (see Figure 3). Applying Propositions 2 and 3, we obtain

$$\begin{aligned}
\overline{p}^i_{(S,T)} &= \frac{c([(\emptyset,N),(S,T)])\ c([(S\cup i,T),(N,\emptyset)])}{c(3^N)} \\
&= \frac{\dfrac{(n+s-t)!}{2^s}\cdot\dfrac{(n+t-s-1)!}{2^t}}{\dfrac{(2n)!}{2^n}} \\
&= \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t},
\end{aligned}$$

$$\begin{aligned}
\underline{p}^i_{(S,T)} &= \frac{c([(\emptyset,N),(S,T\cup i)])\ c([(S,T)(N,\emptyset)])}{c(3^N)} \\
&= \frac{\dfrac{(n+t-s)!}{2^t}\cdot\dfrac{(n+s-t-1)!}{2^s}}{\dfrac{(2n)!}{2^n}} \\
&= \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.
\end{aligned}$$

Therefore, $\Psi_i(b) = \Phi_i(b)$ for all $i\in N$ and $b\in\mathcal{BG}^N$. ■

As a consequence of Theorem 7, the Shapley value of a bisupermodular game b is in $C(N,b)$ and hence, the core of b is nonempty.

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Cost Allocation in Combinatorial Optimization Games

Yannis Marinakis,[1] Athanasios Migdalas,[1] and Panos M. Pardalos[2]

[1] Decision Support Systems Laboratory, Department of Production Engineering and Management, Technical University of Crete, Chania, Greece
marinakis@ergasya.tuc.gr, sakis@verenike.ergasya.tuc.gr

[2] Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida, USA
pardalos@cao.ise.ufl.edu

Abstract Cooperative game theory is concerned primarily with groups of players who coordinate their actions and pool their winnings. One of the main concerns is how to divide the extra earnings (or cost savings) among the members of the coalitions. Thus a number of solution concepts for cooperative games have been proposed. In this chapter, a selection of basic notions and solution concepts for cooperative games are presented and analyzed in detail. The paper is particularly concerned with cost allocation methods in problems that arise from the field of combinatorial (discrete) optimization.

Key words: cost allocation, combinatorial optimization games

1 Introduction

Game theory deals with decisions in which two or more players, possibly with conflicting interests, interact. Each of these players tries to optimize his own objective function. A game can be classified as a cooperative or a non-cooperative game. The difference between the two is that in a cooperative game, the players can make agreements in order to minimize their common cost or to maximize their common payoff, while this is not possible in a non-cooperative game. Even if all players in a cooperative game agree that it is beneficial to minimize their total common cost (or to maximize their common total payoff), each player might want to minimize his individual cost (or to maximize his individual payoff). In this type of a situation, one may need a cost allocation method (payoff allocation method) that reflects the common objectives as well as each player's objectives.

The main purpose of this paper is to demonstrate how cooperative game theory can be applied to combinatorial optimization and supply chain management problems. It deals with cost allocation methods in problems that arise from the field of combinatorial (discrete) optimization. In the next section, the solution concepts of the cooperative game theory, like the core of the game, the Shapley value, the Bargaining set, the Nucleolus of the game, and the Kernel of the game, are given and analyzed. In Section 3, the corresponding combinatorial optimization game for the most important problems of combinatorial optimization and of supply chain management, like Scheduling problems, Assignment problems, and Routing problems, is presented. For these games, the core, the nucleolus, and the other solution concepts were calculated. Finally, the concluding remarks are given in the last section.

2 Cooperative Game Theory and Cost Allocation

2.1 Basic Concepts in Cooperative Game Theory

A *cooperative* n-person game [24] is defined by a pair $(N; u)$ where $N = \{1, 2, \ldots, n\}$ is the set of players and u is a real valued function, called the *characteristic function*, defined on $S \subseteq N$, with $u(\emptyset) = 0$. Each subset $S \subseteq N$ is a *coalition*, and N is called the *grand coalition*. In cooperative *cost games*, the characteristic function is often denoted by $c(S)$ instead of $u(S)$. The *cardinality* or the *size* of a coalition, $|S|$, is equal to the number of players in S. The empty subset of N is called *empty coalition*. When the game involves monetary or physical units that can be transferred between the players, then the game is called *Transferable Utility Game*. The characteristic function in a cost game refers to the cost that arises when a coalition chooses to cooperate. The set of all cooperative games with player set N will be denoted G^N.

A *pre-imputation* y is a vector in R^n such that the cost y_i is allocated to player i and such that $\sum_{i \in N} y_i = c(N)$. An *imputation* is a pre-imputation that satisfies the requirement $y_i \leq c(\{i\})$ for $i \in N$. For simplicity we write $y(S)$ for $\sum_{i \in S} y_i$ and $c(i)$ for $c(\{i\})$.

The *excess* of a nonempty coalition S with respect to a (cost allocation) vector y is $e(S, y) = c(S) - y(S)$.

The *marginal cost* of a player, m_i is the marginal cost of that player in the grand coalition, i.e., $m_i = c(N) - c(N\{i\})$. Note that for a monotone game $m_i \geq 0$ for all i.

A game can satisfy a number of properties:

- A game $(N; c)$ is *monotone*, if the characteristic function c is monotone, i.e., $c(S) \leq c(T)$ for $S \subset T \subset N$.
- A game $(N; c)$ is proper if the characteristic function is *subadditive*, i.e., $c(S) + c(T) \geq c(S \bigcup T)$ for all $S, T \subset N, S \bigcap T = \emptyset$. In a proper game it is always profitable to form large coalitions, which is an incentive to cooperate.
- The weakest form of *subadditivity* occurs if the characteristic function is *additive*, i.e., $c(S) + c(T) = c(S \bigcup T)$ for all $S, T \subset N, S \bigcap T = \emptyset$.

- A game with an additive characteristic function is called an *inessential* game. All others games are called *essential*.
- A cost game is *convex* if its characteristic function is *concave* (or *submodular*).

2.2 Solution Concepts

For the characterization of a solution concept, there is a number of properties or axioms that a solution concept must satisfy. A solution may satisfy some of the properties. The most important of them are:

- *Group rationality* or *Pareto optimality* or *Pareto efficiency*: $\sum_{i \in N} y_i = c(N)$. The total cost allocated to the players must be equal to the total cost of the game.
- *Individual rationality*: $y_i \geq c(\{i\}), \forall i \in N$. The cost allocated in each player should not be higher than the cost the player would have to pay if he acted without the others.
- *Kick-back*: $y_i \geq 0$. The cost allocated to a player must always be non-negative.
- *Dummy player*: If player i contributes nothing to any coalition, $c(S) = c(S \setminus \{i\}) + c(i)$ for all $S \subseteq N, i \in S$, then the cost allocated to i, y_i, is equal to $c(i)$.
- *Anonymity* (or *neutrality* or *symmetry*): The order in which the players are numbered should not affect the cost allocated to the players of the game.
- *Monotonicity*: If the overall cost increases, the allocation to a player should not be lower than before the cost increase.
- *Additivity*: If the cost matrix $C = \{c_{ij}\}$ is divided into two independent cost matrices, $C^1 = \{c^1_{ij}\}$ and $C^2 = \{c^2_{ij}\}$, where $c_{ij} = c^1_{ij} + c^2_{ij}$, for all i, j then $y_i = y^1_i + y^2_i$ for all i.

The Core of a Game

If all players in a game decide to work together, then a question arises of how to divide the total profit. If one or more players believe that a proposed allocation is disadvantageous to them, they can decide to leave. The *core* is the most significant solution concept of a cooperative game that easily can be perceived as fair. In a game $(N; c)$, the core is defined as those imputations, y, that satisfy:

$$y(S) \leq c(S),\ S \subseteq N \tag{1}$$

$$y(N) = c(N) \tag{2}$$

Constraint (1) means that the total cost allocated to the players in a coalition should not exceed the cost of a system dedicated to that coalition. This constraint expresses the group and individual rationality constraints.

Constraint (2) means that the total cost of the game is to be divided among the players. This is the efficiency constraint. Because in a core solution there is no incentive for any coalition to leave the grand coalition, the core solutions are in some sense stable. Constraints (1) and (2) do not necessarily define a unique point. Further, it is possible that the core is empty. Therefore, the core can be seen as a description of candidate allocations, rather than a concept that can be used to find a particular cost allocation. A solution belonging to the core is a cost allocation in which the total cost is allocated to the players in the game in such a way that no subset of players pays more than it would have to do if it acted alone. Empty core means that there was always a coalition that could do better by separating from the grand coalition.

It is often interesting to investigate whether a game can be guaranteed to have a nonempty core. A sufficient condition for nonemptiness is that the game is convex. However, the core may be nonempty even if the game is not convex.

The fact that the core may be empty has led to the introduction of *ϵ-cores.* The *strong ϵ-core* are those solutions y that satisfy the following:

$$\sum_{i \in S} y_i \leq c(S) + \epsilon,\ S \subseteq N \tag{3}$$

$$\sum_{i \in N} y_i = c(N). \tag{4}$$

The *weak ϵ-core* are those solutions y that satisfy the following:

$$\sum_{i \in S} y_i \leq c(S) + |S|\epsilon,\ S \subseteq N \tag{5}$$

$$\sum_{i \in N} y_i = c(N). \tag{6}$$

If ϵ is large enough, the strong and the weak ϵ-cores are always non-empty. The minimal ϵ-value that produces a nonempty ϵ-core in a game with an empty core could, for example, be seen as a measure of the distance from a nonempty core. The minimal ϵ-value that makes the strong ϵ-core nonempty is computed in the procedure for computing the nucleolus.

A Transferable Utility game (N, c) is called *balanced* [6] if it has a non-empty core and *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding with some coalition $T \subset N$, $T \neq \emptyset$ is the game (T, c^T) with $c^T(S) = c(S)$ for all $S \subset T$.

Shapley Value

The rationale behind the *Shapley value* [24,61] is that the marginal cost of each player, when successively forming the grand coalition, is reflected. Each way of forming the grand coalition is considered to be equally probable. Suppose that the grand coalition N, of a game (N, c) is formed by succesively adding players in the order $p_1, p_2, \ldots, p_{|N|}$. There are $|S| - 1)!(|N| - |S|)!$ ways of adding players, such that player $i = p_s$. Furthermore, let a particular coalition S be the coalition defined by $\{p_1, p_2, \ldots, p_s\}$. The marginal cost of player i in coalition S is $(c(S) - c(S \setminus \{i\})$. The Shapley value for player i is computed as the sum over all the coalitions S, of the marginal cost of player i in the coalition S, multiplied by the probability that the grand coalition is formed that way, and is given by

$$\phi_i = \sum_{S \subseteq N | i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} (c(S) - c(S \setminus \{i\})). \tag{7}$$

The Shapley value is a unique solution to a game. It is the only value that satisfies the three properties of additivity, symmetry, and the dummy player property. Furthermore, the Shapley value is efficient and satisfies the anonymity property. But even if the core is nonempty, the Shapley value may not be included in the core.

Bargaining Set

The concept of an objection of a player is formalized and used in the definition of the bargaining set in [2, 13, 20]. Let

$$\Gamma_{ij} = \{S \in 2^N | i \in S, j \notin S\}. \tag{8}$$

An objection of player i against player j with respect to an imputation y in the game $c \in G^N$ is a pair (x, S) where $S \in \Gamma_{ij}$ and $x = (x_k)_{k \in S}$ is a $|S|$-tuple of real numbers satisfying

$$x(S) = c(S) \textit{ and } x_k > z_k \ \forall k \in S. \tag{9}$$

A counter objection to the objection (x, S) is a pair (z, T) where $T \in \Gamma_{ji}$ and $z = (z_k)_{k \in T}$ is a $|T|$-tuple of real numbers such that

$$z(T) = c(T), z_k \geq x_k \text{ for } k \in S \cap T \text{ and } z_k \geq y_k \text{ for } k \in T \setminus S. \tag{10}$$

An imputation y is said to belong to the *bargaining set* $M(c)$ of the game c, if for any objection of one player against another with respect to y, there exists a counter objection.

The Nucleolus of a Game

The nucleolus of a game minimizes maximal discontent for the coalitions. In a game $(N; c)$, it is defined for each imputation y an *excess vector* $\theta(y)$ of dimension $2^{|N|} - 2$. Let the excess vector contain the excesses $e(S, y)$ of each nonempty subset of the grand coalition, with respect to y, in a nondecreasing order. This implies that if $i < j$, $\theta_i(y) \leq \theta_j(y)$ for all $1 \leq i \leq j \leq 2^{|N|-2}$. If there exists a positive integer q, such that $\theta_i(y) = \theta_i(\bar{y})$ whenever $i < q$ and $\theta_i(y) > \theta_i(\bar{y})$ for $i = q$, we say that $\theta(y)$ is lexicographically greater than $\theta(\bar{y})$, and denote this by $\theta(y) >_L \theta(\bar{y})$. With $\theta(y) \geq_L \theta(\bar{y})$ we mean that either $\theta(y) >_L \theta(\bar{y})$ or $\theta(y) = \theta(\bar{y})$. The nucleolus is defined as those imputations y that have the lexicographically greatest associated vector. Schmeidler showed [60] that for those games where the nucleolus, $\sum_{i \in N} c(i) \geq c(N)$, is nonempty, the nucleolus is a unique point. He showed that if the core is nonempty, the nucleolus is included in the core, and also that the nucleolus is a continuous function of the characteristic function.

The nucleolus is efficient, individual rational, anonymous, and possesses the dummy player property. The nucleolus is neither additive nor monotonic. All coalitions are equal in the computation of the nucleolus. If all constraints in an explicit formulation of the core are known, the nucleolus of a game $(N; c)$ can be found by solving successive linear programs [24]. The nucleolus is the cost allocation in which the total cost is allocated among the players in such a way that the least satisfied subset of players is as satisfied as possible, and the second least satisfied subset of players is as satisfied as possible, etc.

The prenucleolus $n(c)$ [26] is defined to be the (unique) allocation $y \in R^n$ that lexicographically maximizes θ over the set of all allocations. The nucleolus is obtained when it is computed the lexicographically maximum over the set of all imputations. The prenucleolus and the nucleolus coincide whenever core(c) is nonempty.

The (pre)nucleolus can be computed [26] by solving a sequence of linear programs as follows. Let $S_0 = \{\emptyset, N\}$ and first solve:

$$(LP_1) \max \epsilon \tag{11}$$

$$\text{s.t.}$$

$$\sum_{i \in N} y_i = c(N) \tag{12}$$

$$\sum_{i \in S} y_i \leq c(S) - \epsilon \ \forall S \notin S_0. \tag{13}$$

If ϵ_1 is the optimal value of (LP_1), let S_1 be the collection of all coalitions that become tight at $\epsilon = \epsilon_1$ and solve

$$(LP_2) \max \epsilon \tag{14}$$

$$\text{s.t.}$$

$$\sum_{i \in N} y_i = c(N) \tag{15}$$

$$\sum_{i \in S} y_i \leq c(S) - \epsilon_1 \quad \forall S \in S_1 \tag{16}$$

$$\sum_{i \in S} y_i \leq c(S) - \epsilon \ \text{ otherwise} \tag{17}$$

Continuing this way, a sequence $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_k$ is calculated until, finally, the optimal solution of (LP_k), namely the prenucleolus $n(c)$ of the game, is unique. This procedure requires the solution of at most $|N|$ linear programming problems, and for that reason usually the nucleolus is calculated indirectly.

The Kernel of a Game

If a cost vector y has been proposed in the game c, player i can compare his position with that of player j by considering the minimum cost $c_{ij}(y)$ of i against j with respect to y, defined by

$$c_{ij} = \max_{S \in \Gamma_{ij}} e(S, y). \tag{18}$$

The minimum cost of i against j with respect to y can be regarded as the lowest cost that player i can pay without the cooperation of j. Player i can do this by forming a coalition without j but with other players who are satisfied with their cost according to y. Therefore, $c_{ij}(y)$ can be regarded as the weight of a possible threat of i against j. If y is an imputation, then player j cannot be threatened by i or any other player when $y_j = c(j)$ because j can be obtained by operating alone. We say that i outweighs j if

$$y_j < c(j) \text{ and } c_{ij}(y) < c_{ji}(y). \tag{19}$$

The *kernel* consists of those imputations for which no player outweighs another one.

3 Combinatorial Optimization Games

3.1 Sequencing/Scheduling Games

The main characteristic of a sequencing situation is that a number of jobs have to be processed in some order on a number of machines in such a way that some cost criterion is minimized. Sequencing situations can be classified on the number of machines, on the specific properties of machines (parallel, serial), on restrictions on the jobs, and on the order in which the jobs have to be processed on the machines (job-shop, flow-shop). A review of scheduling theory is given in [51].

Example 1. Consider a sequencing situation where there is one single machine and 6 different players have a job that must be processed on this machine. The initial order of the jobs is $1, 2, 3, 4, 5, 6$, where the duration of each job is $1, 4, 5, 2, 3, 4$, respectively, and the corresponding cost for each job is $1, 7, 16, 3, 4, 16$. The total cost of the initial order is $1 * 1 + 5 * 7 + 10 * 16 + 12 * 3 + 15 * 4 + 19 * 16 = 596$. After some analysis, it is calculated that the optimal order of the jobs is $6, 3, 2, 4, 5, 1$ with total cost equal to $4 * 16 + 9 * 16 + 13 * 7 + 15 * 3 + 18 * 4 + 19 * 1 = 435$, namely, we have a cost saving equal to 161. This cost saving can be allocated to the players as follows: If players 1 and 2 switch the order of performing their jobs on the machine, then a cost saving of 3 units is generated, which is divided equally among them. So, if we have the initial order $1, 2, 3, 4, 5, 6$ with cost $1 * 1 + 5 * 7 + 10 * 16 + 12 * 3 + 15 * 4 + 19 * 16 = 596$, after the exchange of job 1 with 2 the solution is $2, 1, 3, 4, 5, 6$ with cost $4 * 7 + 5 * 1 + 10 * 16 + 12 * 3 + 15 * 4 + 19 * 16 = 593$. The cost saving is divided equally among the two players and the initial cost allocation is $(1.5, 1.5, 0, 0, 0, 0)$. The following table presents all the exchanges and the corresponding cost allocation.

Exchange	Job order	Total cost	Cost allocation
	1 2 3 4 5 6	$1 * 1 + 5 * 7 + 10 * 16 +$ $12 * 3 + 15 * 4 + 19 * 16 = 596$	
(1,2)	2 1 3 4 5 6	$4 * 7 + 5 * 1 + 10 * 16 +$ $12 * 3 + 15 * 4 + 19 * 16 = 593$	$(1.5, 1.5, 0, 0, 0, 0)$
(1,3)	2 3 1 4 5 6	$4 * 7 + 9 * 16 + 10 * 1 +$ $12 * 3 + 15 * 4 + 19 * 16 = 582$	$(7, 1.5, 5.5, 0, 0, 0)$
(1,4)	2 3 4 1 5 6	$4 * 7 + 9 * 16 + 11 * 3 +$ $12 * 1 + 15 * 4 + 19 * 16 = 581$	$(7.5, 1.5, 5.5, 0.5, 0, 0)$
(1,5)	2 3 4 5 1 6	$4 * 7 + 9 * 16 + 11 * 3 +$ $14 * 4 + 15 * 1 + 19 * 16 = 580$	$(8, 1.5, 5.5, 0.5, 0.5, 0)$
(1,6)	2 3 4 5 6 1	$4 * 7 + 9 * 16 + 11 * 3 +$ $+14 * 4 + 18 * 16 + 19 * 1 = 568$	$(14, 1.5, 5.5, 0.5, 0.5, 6)$
(2,3)	3 2 4 5 6 1	$5 * 16 + 9 * 7 + 11 * 3 +$ $14 * 4 + 18 * 16 + 19 * 1 = 539$	$(14, 16, 20, 0.5, 0.5, 6)$
(5,6)	3 2 4 6 5 1	$5 * 16 + 9 * 7 + 11 * 3 +$ $15 * 16 + 18 * 4 + 19 * 1 = 507$	$(14, 16, 20, 0.5, 16.5, 22)$
(4,6)	3 2 6 4 5 1	$5 * 16 + 9 * 7 + 13 * 16 +$ $15 * 3 + 18 * 4 + 19 * 1 = 487$	$(14, 16, 20, 10.5, 16.5, 32)$
(2,6)	3 6 2 4 5 1	$5 * 16 + 9 * 16 + 13 * 7 +$ $15 * 3 + 18 * 4 + 19 * 1 = 451$	$(14, 34, 20, 10.5, 16.5, 50)$
(3,6)	6 3 2 4 5 1	$4 * 16 + 9 * 16 + 13 * 7 +$ $15 * 3 + 18 * 4 + 19 * 1 = 451$	$(14, 34, 28, 10.5, 16.5, 58)$

In a one-machine sequencing situation [12], there is a queue of players, each with one job, in front of a machine. Each player must have his job processed on this machine. The finite set of players is denoted by $N = \{1, \ldots, n\}$. The positions of the players in the queue are described by the bijection $\sigma \in \Pi_N$. We assume that there is an initial order $\sigma_0 \in \Pi_N$ on the jobs before the processing of the machine starts. The processing time p_i of the job of player i is the time the machine takes to handle this job. For each player $i \in N$, the cost of spending time in the system can be described by a linear cost function $c_i : R_+ \to R$ defined by $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. A sequencing situation as described above is denoted by (N, σ_0, p, α) with $p, \alpha \in R^N_{++}$.

The completion time $C(\sigma, i)$ of the job of player i if the jobs are processed according to the order $\sigma \in \Pi_N$ is given by

$$C(\sigma, i) = \sum_{\{j \in N | \sigma(j) \leq \sigma(i)\}} p_j. \tag{20}$$

By rearranging from the initial order to an optimal order, an allocation problem arises: how should the maximal total cost savings of the players that can obtained be divided among the players? By defining the value of a coalition S as the maximum cost savings, the coalition S can be achieved by rearrangement and, so, we obtain the corresponding sequencing game (N, u), which is defined by

$$u(S) = \max_{\sigma \in A(S)} \{\sum_{i \in S} \alpha_i [C(\sigma_0, i) - C(\sigma, i)]\}, \forall S \subset N \tag{21}$$

where $A(S)$ is the set of admissible orders for a coalition S. If the players decide to save money by rearranging their position, they will need to divide the cost savings that they generate. A division rule is the equal division rule [13], which divides the cost savings equal to the players, but this method does not distinguish between players who actually contribute to the savings and those who do not. Curiel [13] proposed a rule, called Equal Gain Splitting Rule (EGS), which does not have this disadvantage. In this rule, first the gain g_{ij} that players i and j, who are standing next to each other with i in front of j, can be achieved by switching positions. This gain is equal to the difference of the sums of the costs of i and j before and after they change places. If $u_i \geq u_j$, then both players cannot gain anything by switching places. On the other hand, if $u_j > u_i$, then the players can gain $\alpha_j s_i - \alpha_i s_j$. So, $g_{ij} = max\{\alpha_j s_i - \alpha_i s_j, 0\}$. Finally the rule that Curiel proposed is

$$EGS = \frac{1}{2} \sum_{k \in P(\sigma, i)} g_{ki} + \frac{1}{2} \sum_{j : i \in P(\sigma, j)} g_{ij} \text{ for each } i \in N. \tag{22}$$

Curiel, also, proved that a sequencing game is convex, and as a result that is totally balanced. Hammers et al. [42] give a generalization of the EGS rule, which they call split core. The split core contains all allocations generated by gain splitting rules. They also gave a monotonicity property for solutions concepts, which may contain more than one element, and use it together with efficiency and the dummy property to characterize the split core. They showed that all solution concepts that satisfy efficiency, the dummy property, and monotonicity are contained in the split core. The split core is a subset of the core.

In the literature [6], many other classes of sequencing game are studied. Hamers [38] extends the class of one-machine sequencing situations by imposing ready times on the jobs. Borm et al. [5] consider some classes of sequencing situations in which due dates are imposed on the jobs and different cost criteria are used. Hamers et al. [40] consider sequencing situations with m parallel and identical machines in which no restrictions on the jobs are imposed. Van den Nouweland et al. [56] consider multiple machine flow-shop sequencing situation with a dominant machine.

3.2 Permutation and Assignment Games

Permutation games [67] arise from situations in which every player has one job and one machine. Every job has to be processed on a machine and each machine can process every job, but no machine is allowed to process more than one job. If player i processes his job on the machine of player j, the processing costs are α_{ij}. Let $N = \{1, \dots, n\}$ be the set of players. The corresponding permutation game (N, u) is the cooperative game defined by

$$u(S) = \sum_{i \in S} \alpha_{ij} - \min_{\pi \in P_S} \sum_{i \in S} \alpha_{i\pi(i)}, \forall S \subset N, S \neq \emptyset, u(\emptyset) = 0. \tag{23}$$

where the number $u(S)$ denotes the maximal cost savings a coalition S can obtain by processing their jobs according to an optimal schedule compared with the situation in which every player processes his job on his own machine.

Example 2. Let $N = \{1, \dots, 4\}$ be the player set with cost matrix

	1	2	3	4
1	10	1	3	5
2	1	7	6	8
3	7	9	9	2
4	6	7	2	10

The permutation game based on the equation (23) is

S	$u(S)$	S	$u(S)$	S	$u(S)$
1	0	1,3	$19 - 10 = 9$	1,2,3	$26 - 13 = 13$
2	0	1,4	$20 - 11 = 9$	1.2.4	$27 - 13 = 14$
3	0	2,3	$16 - 15 = 1$	1,3,4	$29 - 11 = 18$
4	0	2,4	$17 - 15 = 2$	2,3,4	$26 - 15 = 11$
1,2	$17 - 2 = 15$	3,4	$19 - 4 = 15$	1,2,3,4	$36 - 6 = 30$

So, the optimal schedule for this game is to process player 1 to job 2, player 2 to job 1, player 3 to job 4, and player 4 to job 3 with cost saving $10 + 7 + 9 + 10 - 1 - 1 - 2 - 2 = 36 - 6 = 30$.

An alternative way to calculate $u(S)$ is as the value of the following integer programming problem [13].

$$u(S) = \max \sum_{i \in N} \sum_{j \in N} \alpha_{ij} x_{ij} \tag{24}$$

s.t.

$$\sum_{j \in N} x_{ij} \leq 1_s(i), \ \ i \in N \tag{25}$$

$$\sum_{i \in N} x_{ij} \leq 1_s(j), \ \ j \in N \tag{26}$$

$$x_{ij} \in \{0, 1\}, \ \ \forall i \in N, j \in N \tag{27}$$

A subclass of permutation games is the class of assignment games. A game associated with markets is the assignment game introduced by Shapley and Shubik [62]. They modeled a two-side market with buyers and sellers and showed that the core is exactly the set of optimal solutions to a linear program dual to the optimal assignment problem [15]. In the assignment game, a bipartite graph is used to represent M customers and N merchants in a market. An edge (i, j) with weight α_{ij} represents the joint profit if customer i buys from merchant j. Every customer buys from one merchant and every merchant sells to one customer. Define x_{ij} to be one if customer i buys from merchant j, and zero otherwise. A formulation of an assignment game is the following [13]:

$$u(S) = \max \sum_{i \in M} \sum_{j \in N} \alpha_{ij} x_{ij} \tag{28}$$

s.t.

$$\sum_{j \in N} x_{ij} \leq 1_s(i), \ \ i \in M \tag{29}$$

$$\sum_{i \in M} x_{ij} \leq 1_s(j), \ \ j \in N \tag{30}$$

$$x_{ij} \in \{0, 1\}, \ \ \forall i \in M, j \in N \tag{31}$$

Shapley and Shubik [62] showed that the core of an assignment game corresponds with the set of optimal solutions of the dual problem of the previous formulation. Balinski and Gale [3] showed that the core of an assignment game can have at most $\binom{2m}{m}$ extreme points where m is the minimum of $|M|$ and $|N|$. They also proved that in each extreme core point of an assignment game, there is a player who receives a zero payoff. Nunez and Rafels [57] provided a characterization of extreme points of the core, which is also valid for the class of nonconvex games.

Solymosi and Raghanvan [63] gave an algorithm of order $O(|M|^3|N|)$ to find the nucleolus of an assignment game, where $|M|$ is assumed to be the minimum of $|M|$ and $|N|$. Hamers et al. [43] proposed an algorithm of order p^2, where p is the number of players for calculating the nucleolus of neighbor games, where neighbor games are games that are the intersection of assignment games and the class of component additive games. The core of neighbor games is nonempty and coincides with the bargaining set, and the nucleolus coincides with the kernel.

3.3 Matching Game

In the matching game [28,45], let the complete graph K_m be where the players N correspond with the nodes of the graph. A matching is a set M of edges such that no two edges in M have a node in common. Each edge e in K_n is assigned a weight $w(e)$, and the value $u(S)$ of a coalition is equal to the weight of a maximal matching in the subgraph induced by S. Here each individual player $i \in N$ has value $u(i) = 0$ while value $u(N) > 0$ may be possible. The characteristic function of the game is equal with the value of a maximal weighted matching in K_n. The matching game on K_3 with unit edge weights has an empty core. Solymosi and Raghavan [63] showed that the nucleolus of a matching game can be computed in polynomial time in the bipartite case, in the case where the edges of positive weight in the underlying graph do not contain a circuit of odd length. Because the matching games deal with allocating savings instead of costs, the inequalities of the basic solution concepts (like core, nucleolus, etc.) are reversed.

Faigle et al. [28] introduced the nucleon as the multiplicative analogue of the nucleolus and calculated the nucleon for the matching game. The nucleon of the non-negative game (N, u) is the set of all allocation vectors $x \in R_+^N$ that lexicographically maximize the satisfaction vector $\alpha(x)$, where for every coalition $S \notin S_0$ the satisfaction vector is

$$\alpha(x, S) = \begin{cases} \frac{x(S)}{u(S)}, & \text{if } u(S) > 0 \\ \infty, & \text{if } u(S) = 0. \end{cases} \tag{32}$$

They proved that the nucleon of a non-negative additive game equals the nucleolus.

3.4 Network Flows and Multicommodity Flow Games

Kalai and Zemel studied games of flows [15, 46]. In this game, the players are associated with arcs of the networks. The value of a subgroup is the maximum flow from s to t (source and sink, respectively) for the subgraph consisting of the original node set and those edges corresponding with the subgroup of players. On each of the arcs, there is a capacity restriction and an associated simple control game that describes which coalitions of players are allowed to use the arc. For a simple network game for which arc capacities are all one, they also showed that the core is exactly the same as the set of solutions to a linear program dual to a linear program formulation of the network flow problem.

In the mulicommodity flow game [52], there is a graph with a multicommodity flow between each pair of nodes, satisfying node capacity and demand constraints, and the payoff of a node is the total flow originated or terminated at each node. A payoff allocation is in the core if and only if there is no subset of nodes that can increase their payoff by deleting from the graph. Markakis et al. proved that the core is nonempty in both the transrable utility case and the nontransferable utility case.

3.5 Minimum Cost Spanning Tree

A spanning tree is a tree (i.e., a connected acyclic graph) that spans all the nodes of an undirected network. The cost of the spanning tree is the sum of the costs (lengths) of its arcs. The minimum spanning tree problem is concerned with the identification of a spanning tree of minimum cost. In the simple case, no topological or capacity restrictions are imposed on the tree. The minimum spanning tree problem in terms of graph theory can be presented as follows: Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = 1, \ldots, n$ is the set of nodes and $\mathcal{E}$ is the set of arcs, $(i, j) \in \mathcal{E}$, which connects those nodes. The cost of connecting node i to node j is C_{ij}, where $C_{ij} = C_{ji}, \forall(i, j)$. It is assumed that the cost of the connection matrix satisfies the triangle inequality.

Bird [4] and Claus and Kleitman [11] have formulated a minimum cost spanning tree game, MCST game, for cost allocation of communication networks to its users and introduced several cost allocation criteria. In this game, one player corresponds with a node of the graph. There is one more external node 0. The cost for a subset of players is the weight of minimum spanning tree of the subgraph induced by their corresponding nodes and node 0. The characteristic function of this game, the weight of the minimum spanning tree in a graph, can be calculated in polynomial time. More precisely, a *minimum cost spanning tree game* is a cooperative game (N, c) where the characteristic function $c(S)$ is defined as the optimal objective function value to minimum cost spanning tree problem over the vertices in $S \cup 0$ where 0 is the root vertex. Bird, also, proposed a cost allocation rule [4,6,13]. Let c be a MCST game and let $E_t \subset E$ be the set of arcs of a minimum cost spanning tree T for the graph (N_0, E). For each $i \in N$, let the amount that i has to pay be equal to the cost of the edge incident upon i on the unique path from 0 to i in T. Let $c_{ij} = c_{ji}$ the cost of constructing the link (i, j). It is easy to see that in this way, the total costs are distibuted among the players. So, the cost of a coalition $S \subset N$ in MCST game is $\sum_{(i,j) \in E_{T_S}} c_{ij}$. Because there can be more than one minimum cost spanning tree for a graph, this way of dividing the costs need not lead to a unique cost allocation in a MCST game. A pseudocode of Bird's rule is presented in [6]. Curiel [13] proved that Bird tree allocation rule is an extreme point of the core. Bird [4] proposed the irreducible core of a MCST game as a means of generating more core allocations over those given in the set of Bird tree allocations. He proved that the irreducible core is a subset of the core for all MCST games that have a minimum cost spanning tree with fixed costs on the common edges. MCST games are the types of games that have received the most attention in cooperative theory as the determination of a minimum spanning tree in graph is the "easy" problem whereas the determination of a traveling salesman tour in a graph, which will be studied in Section 3.7, is the "hard" problem.

Example 3. In a complete graph, let the player set be denoted by $\{2, \ldots, 6\}$ and the root node denoted by 1. The cost of the arcs is presented in the following table,

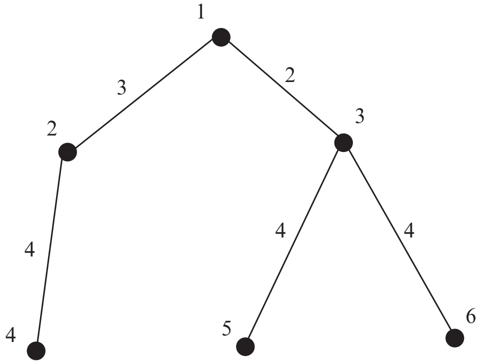

Figure 1. Optimal solution

	1	2	3	4	5	6
1	0	3	2	5	6	7
2	3	0	5	4	8	7
3	2	5	0	5	4	4
4	5	4	5	0	7	7
5	6	8	4	7	0	6
6	7	7	4	7	6	0

By applying the Bird rule in this problem we take the optimal solution, presented in Figure 1, with cost 17. This gives a cost allocation $(3, 2, 4, 4, 4)$ that is an element of the core. It is, in fact, an extreme point of the core of this game.

An overview of MCST problem is given in Aarts [1] and the core, nucleolus, and the Shapley value are studied in Granot and Huberman [36]. The core of the MCST game consists of all vectors y that are fair in the sense that the vector y should be considered fair if the amount $y(S)$ of any coalition S has to pay cumulatively never exceeds the cost $c(S)$ of a minimum spanning tree on $S \cup \{0\}$, which is what S would have to invest in order to connect itself to 0 without any outside considerations [27]. Faigle et al. [27] proved that it is an NP-hard problem to decide whether a given member is not a member of the core. The core of a MCST game is a polyhedron in R^N. Granot and Huberman [36] showed that a solution in the core of a MCST game can be read from an associated MCST graph. Thus, the core of a MCST game is never empty. They also discussed and calculated the core, the nucleolus, and the Shapley value for a minimum spanning tree game with more than one node incident to the root. They proved that the intersection of core and prekernel of a MCST game consists of precisely the nucleolus. Megiddo [55] presented a polynomial algorithm to find the nucleolus and the Shapley value of the game. Tamir [66] presented network synthesis games that include MCST games. Granot and Granot [35] study fixed cost spanning forest problems in which the players form a subset of the set of nodes of an undirected graph, and Aarts [1] studies chain games that are games that have a minimum spanning tree that is a chain.

Fernandez et al. [32] introduced the multicriteria version of MCST game. The characteristic function associates to each coalition S a set $V(S)$ that represents the Pareto minimum cost of constructing a distribution system among the users in S from the source 0. A Pareto minimum cost spanning tree for a given connected graph, with costs on the edges, is a spanning tree that has Pareto-minimum costs among all spanning trees. They proved that an extension of Bird's rule provides dominance core elements in this game, but also gave a family of core solutions that are different from the previous ones, which are based on proportional allocations obtained using scalar solutions of the multicriteria spanning tree problem. They also proved that the preference core of this game is not empty.

Suijs [64] analyzed spanning network problems that feature random connection cost. It is assumed that the agents who need to be connected to the supplier are constant absolute risk averse expected utility maximizers. Because preferences may differ between agents, minimum cost spanning trees have no meaning in this context. To tackle this problem of network formation and cost allocation, the author applied stochastic cooperative game theory. Stochastic cooperative games were designed to explicitely take into account random payoffs and the individuals preferences over these random payoffs. For stochastic spanning tree games, Suijs focused on core allocations and proved that the core is nonempty and which graphs may give rise to core allocations. Furthermore, he pointed toward a specific core allocation called the two-stage Bird allocation. The first stage works just like the standard Bird allocation, but in the second stage, agents are allowed to mutually insure (part of) their random cost.

3.6 Steiner Tree Problem

In this problem, there are costs associated with connecting the nodes of a network to a tree. In addition, there is a potential revenue to collect at each node if it is connected. The problem is to decide which node to connect, and how, so as to maximize the revenue collected minus the connecting costs. Megiddo [54] has formulated this problem defining the cost of a minimum Steiner tree game that contains all corresponding nodes in the original graph. In a cost allocation setting, the Steiner Tree Problem could be solved in order to identify a coalition that is most unsatisfied with a proposed cost allocation in a minimum spanning tree game. Kuipers et al. [48] proposed a cost allocation rule for a variant of Steiner Tree Game, called Vertex Weighted Steiner Tree Game. The Vertex Weighted Steiner Tree Game is similar to the Steiner Tree Game except that each vertex of the game has a reward if and only if all are connected in the tree. They proved that every 5-persons Vertex Weighted Steiner Tree Game has a nonempty core.

3.7 Traveling Salesman Problem Games

Consider a salesman who has to visit n cities. The *Traveling Salesman Problem (TSP)* asks for the shortest tour through all the cities such that no city is visited twice and the salesman returns at the end of the tour back to the starting city. We speak of a symmetric TSP, if for all pairs i, j the distance c_{ij} is equal to the distance c_{ji}. Otherwise, we speak of the assymetric traveling salesman problem. If the cities can be represented as points in the plain such that c_{ij} is the Euclidean distance between point i and point j, then the corresponding TSP is called the Euclidean TSP. Euclidean TSP obeys in particular the triangle inequality $c_{ij} \leq c_{ik} + c_{kj}$ for all i, j, k. The Traveling Salesman Problem (TSP) is one of the most famous hard combinatorial optimization problems. For a review on the traveling salesman problem, we refer to Lawer et al. [50] and to Gutin et al. [37].

The Traveling Salesman Game (TSG) deals with the question of how to allocate the total cost of a tour to the customers served on that tour. In the traveling salesman cost allocation game (N, c), the players correspond with the nodes of the graph. Further, the characteristic function in a TSG is defined as the total cost of the minimum Hamiltonian cycle, meaning the minimum tour of visiting all the nodes in $S \cup \{0\}$. In a TSG with a home city, the home city (which is not a player and corresponds with the depot) must be included in the minimum cost cycle of each $S \subset N$. The value of a subgroup S of players that have been visited is minimum Hamiltonian tour in the subgraph induced by $S \cup 0$ [59]. Let K be the set of the available truck types, V_k the capacity of truck type K, D_i the demand of customer i, c_{ij}^k the cost of transportation between customers i and j, using truck type k, and x_{ij} equal to 1 if customer j is visited immediately after customer i in the tour and is equal to 0 otherwise. A formulation of the Traveling Salesman Game is the following [24]:

$$c^{TSP}(S, k) = \min \sum_{i \in S^0} \sum_{j \in S_0, j \neq i} c_{ij}^k x_{ij} \tag{33}$$

s.t.

$$\sum_{i \in S^0} x_{ij} = 1, \ \ j \in S^0 \tag{34}$$

$$\sum_{j \in S^0} x_{ij} = 1, \ \ i \in S^0 \tag{35}$$

$$\sum_{i \in Q} \sum_{j \in Q, j \neq i} x_{ij} \leq |Q| - 1 \ \ \begin{cases} Q \subset S^0 \\ |Q| \geq 2 \end{cases} \tag{36}$$

$$x_{ij} \in \{0, 1\}, \ \ i, j \in S^0 \tag{37}$$

Conditions (34) and (35) state that exactly one edge should be used entering node j and leaving node i, respectively. Conditions (36) are the subtour elimination inequalities.

Engenval [24] presents two different TSG, the Standard Traveling Salesman Game in which the truck type, k, is given in advance, such that the requirement $V_k \geq \sum_{i\in S} D_i$, holds. Then, the characteristic function for this game is $C^{(\alpha)}S = c^{TSP}(S,k)$, for a given truck type k. The second game is the Variable Cost Traveling Salesman Game in which the truck type is not given in advance. Instead it is defined as the lowest capacity truck type that can be used to serve a given coalition S. The characteristic function, now, is $C^{(\beta)}S = c^{TSP}(S,k)$, where $k = argmin_{k\in K}\{V_k | V_k \geq \sum_{i\in N} D_i\}$.

The Core of Traveling Salesman Problem Games

Example 4. A salesman is invited to travel among 3 different cities to present the products of his company and return back to his company. One way to do this is to go to each city and, then, to return to his city, but this is very expensive for the companies, because they have to pay for a two-way ticket from his city to their city and back. They decide to find an order for him to visit the companies that minimizes the total costs. The costs of the travels in Euro are presented in the following table

	1	2	3	4
1	0	300	450	350
2	300	0	150	200
3	450	150	0	100
4	350	200	100	0

The optimal tour of this problem is $1,2,3,4,1$ with cost 900 Euro.

The problem is how to divide the cost among the companies. They decide to perform a game theoretic analysis of this problem. A core element is, for example, $(366.6, 316.6, 216.6)$. This cost allocation is obtained by divided the travel costs from the company equally among the three companies, and making company i pay all of the travel costs from company i to company j. This is a core allocation element because if player 4 acts alone, he will have to pay 700 Euro, if player 3 acts alone, he will have to pay 900 Euro, and, finally, if player 2 acts alone, he will have to pay 600 Euro. Also, if players 2 and 3 make a coalition, they will have to pay together 900 Euro, if players 2 and 4 make a coalition, they will have to pay together 850 Euro, and, finally, if players 4 and 3 make a coalition, they will have to pay together 900 Euro.

An example with empty core is the following

Example 5. Consider the Traveling Salesman Game with player set $N = \{2,\ldots,7\}$ and the home depot denoted by 1. The cost of the edges are presented in the following table

	1	2	3	4	5	6	7
1	0	10	10	10	20	20	20
2	10	0	20	20	20	10	20
3	10	20	0	20	10	20	20
4	10	20	20	0	20	20	10
5	20	20	10	20	0	10	10
6	20	10	20	20	10	0	10
7	20	20	20	10	10	10	0

The optimal tour is $1, 3, 5, 7, 6, 2, 4, 1$ with cost equal to 80. If all players form coalitions of size 4 (passing through the depot), we take that the possible coalitions are $1, 3, 5, 6, 2, 1$ with cost $y_3 + y_5 + y_6 + y_2 \leq 50$, $1, 3, 5, 7, 4, 1$ with cost $y_3 + y_5 + y_7 + y_4 \leq 50$ and $1, 4, 7, 6, 2, 1$ with cost $y_4 + y_7 + y_6 + y_2 \leq 50$. A linear combination of these three inequalities is $2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 \leq 150$, meaning that $y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \leq 75$, but we know that $y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \geq 80$, so, the core of the problem is empty.

Dror [21] showed that the core of a TSG without a home city is empty. Potters et al. [59] showed that three-person TS games have a nonempty core, and, simultaneously, gave an example of an asymmetric traveling salesman with four players that has an empty core, and provided some conditions for an asymmetric traveling salesman game to have a nonempty core. Tamir [65] showed that each four-person symmetric Traveling Salesman game has a non-empty core and a five-person TS game can have an empty core. Kuipers [47] proved that five-person TS games are balanced. Also Kuipers extended the result of Tamir in a six-person game.

Faigle et al. [29] and Fekete [30] proposed a method for allocating the cost in a TSP tour based on the concept of moat packing. A moat, in a given graph in a plane, is a simple closed strip of constant width that separates two nonempty complementary subsets of the nodes. The inside of the moat is the region containing the depot, the other region is called the outside. A moat packing is a collection of moats with pairwise disjoint interior. The cost of a moat packing is twice the sum of all widths. They proved that if the cost of any moat is distributed twice among the nodes on the outside, the resulting distribution is such that no coalition pays more than its TSP cost. Faigle et al. [29] proved that the core of TSP games may be empty, even for the case of Euclidean distances and, simultaneously, provided an instance of a traveling salesman game in the two-dimensional Euclidean space with six players such that the core is empty. They proved that TSP games whose weights satisfy the triangle inequality always have ϵ-approximately fair (core) allocations for $\epsilon = \frac{1}{2}$. ϵ-approximation means that a coalition S should be charged with an allocation that does not exceed the cost $c(S)$ by more than a fraction ϵ. With use of the above observations, Faigle developed an LP-based allocation rule guaranteeing that no coalitions pay more than α times their own cost, where α is the ratio between the optimal TSP-tour and the optimal value of its Held–Karp relaxation, which is also known as the solution over the subtour polytope.

One of the problems of the computation of the core and the nucleolus to a TSG is that the number of characteristic function evaluations may be very large when the number of customers is large. Engevall proposed [23,24] a constraint generation approach in order to compute a solution in the core or to conclude that the core is empty, and to compute the nucleolus. Note that in a constraint generation approach, the subproblem that must be solved in order to identify a constraint that is needed but not yet included is a traveling salesman subtour problem. Engevall proved that in the special case of the TSG, called standard Euclidean TSG, in which the cost matrix is proportional to the Euclidean distance of the customers, the core might be empty.

Okamoto [58] showed that in general to test the core nonemptiness of a given traveling salesman game is NP-hard. He proved that the core of a traveling salesman game is always nonempty if the distance matrix is a symmetric Monge matrix. The Monge property is known as a polynomially solvable case of TSP. An $N_0 \times N_0$ matrix D is a Monge matrix if D satisfies $d_{ik} + d_{jl} \leq d_{il} + d_{jk}$ for all $i < j$ and $k < l$. If a matrix D is a Monge matrix, then it is also said to have the Monge property. Note that a Monge matrix does not need to satisfy the triangle inequality. The testing of non-emptiness of the core is an NP-hard problem, and only for some special classes of traveling salesman is there a possibility not to be an NP-hard problem. Okamoto proved that the core of the traveling salesman with a Monge matrix is nonempty and can be found in $O(N^2)$.

The Nucleolus of Traveling Salesman Problem Games

For the computation of the nucleolus of a TSG, Göthe-Lundgren et al. [34] proposed a constraint generation approach. Engevall [24] proposed the demand nucleolus, which is the solution that has a lexicographically greatest modified excess vector. To define the demand nucleolus, the elements of the excess vector are modified in such a way that the excess $e(S, y)$ are multiplied with the total demand of the coalition. The effect that the demand nucleolus has on the cost allocation is that the importance of coalitions with a large number is reduced, compared with the nucleolus.

Fixed Routing Games

Potters et al. [59] also introduced the class of fixed routing games. The idea of a fixed routing game is that the salesman decides about the Hamiltonian circuit he will use to visit, meaning that the order in which the players are served is defined beforehand, and remains the same for all coalitions. Then the value of a coalition S in a fixed routing game is defined as the costs of the restricted tour that the salesman visits the players in S in the same order as described by the original Hamiltonian circuit and skips all other players.

They showed that fixed routing games have a nonempty core if the chosen Hamiltonian circuit is an optimal route for the related TS problem and the cost matrix satisfies the triangle inequality. Derks and Kuipers [19] gave a number of procedures to construct tours that guarantee the nonemptiness of the core of the game.

The Traveling Preacher Problem

This game [30, 31] can be considered as a variant of the Traveling Salesman Game, with the difference that there is not a specified central root node for the salesman. They proved that this problem can be solved in polynomial time, showing that the difficulty of finding a core allocation for a combinatorial optimization problem may be caused by the existence of the special node called depot, rather than being a consequence of the hardness of the optimization problem itself.

3.8 Chinese Postman Games

In the Chinese Postman Problem [6, 22], one considers a situation in which a postman has to deliver mail to each street of a certain city. He has to start and finish at the post office. For each street, costs are involved each time the postman visits the street. The postman should choose a route to visit all streets in such a way that costs are minimized. The main differences between several classes of Chinese Postman Problems can be found in the underlying graph that describes the street plan of the city.

A cost allocation problem arises if in the underlying graph each edge corresponds with a different player. Because all the players need the mail delivery service and the nature of this service requires the server to travel from the post office and visit all edges (players) before returning to the post office, the cost allocation problem is concerned with a fair allocation of the cost of a cheapest Chinese Postman Problem tour in the graph. That is, the cost of a cheapest tour, which starts at the post office, visits each edge at least once and returns to the post office.

A Chinese Postman Problem is a tuple $\Gamma = (N, G, u_0, g, t)$ where $N = \{1, \ldots, n\}$ is the set of players, $G = (V, E)$ is a connected undirected graph with vertex set V and edge set E, $u_0 \in V$ represents the post office, $g : E \to N$ is a bijection relating the players to the edges, and $t : E \to R_+$ is a nonnegative cost function assigning costs to the edges. An S-tour [41] with respect to u_0 associated with coalition $S \subset N$ is a closed walk $(u_0, e_1, \ldots, e_k, u_0)$ that starts at the post office u_0, visits each player in S at least once, and returns to u_0. The set of all S-tours is denoted by $D(S)$.

Suppose a coalition S is served according to the S-tour $(u_0, e_1, \ldots, e_k, u_0)$ $\in D(S)$, then the total costs of this tour are $\sum_{j=1}^{k} t(e_j)$. We will assume that each player $i \in S$ pays the costs $t(g^{-1}(i))$ himself. In this way the separable costs are already allocated $\sum_{i\in S} t(g_i^{-1})$ of an S-tour. The remaining nonseparable costs for coalition S, $\sum_{j=1}^{k} t(e_j) - \sum_{i\in S} t(g_i^{-1})$, have to be allocated to its members in some way. This gives rise to the Chinese Postman Game (N, c) defined by

$$c(S) = \min_{(u_0, e_1, \ldots, e_k, .u_0) \in D(S)} [\sum_{j=1}^{k} t(e_j) - \sum_{i\in S} t(g_i^{-1})] \tag{38}$$

for all $S \subset N$.

Hamers et al. [41] introduced and characterized a specific cost allocation rule γ that divides the nonseparable costs of a minimal N-tour among all players. They proved that in delivery games, the core may be empty and also proved that for bridge-connected Euler graphs, the outcome of γ is always a core element. Hamers [39] focused on the concavity property of delivery games, that is for games arising from a delivery model corresponding with a bridge-connected Euler graph.

3.9 Vehicle Routing Problem Games

The *distribution* or *vehicle routing problem* (VRP) is often described as the problem in which vehicles based on a central depot are required to visit geographically dispersed customers in order to fulfill known customer demands. The problem is to construct a low cost, feasible set of routes — one for each vehicle. A route is a sequence of locations that a vehicle must visit along with the indication of the service it provides [7]. The vehicle must start and finish its tour at the depot.

Example 6. Consider a Vehicle Routing Problem, let the depot be denoted with 1 and the set of customers denoted with $(2, \ldots, 8)$. The demand of each customer is 10 units and the fleet of the vehicle is homogeneous with capacity equal to 20 units. The problem is Euclidean and the distances of the customers are according the following table (the meaning of value *cost* will be explained in the following example):

	1	2	3	4	5	6	7	8
1	0	10	10	10	10	10	100	100
2	10	0	*cost*	*cost*	*cost*	*cost*	400	400
3	10	*cost*	0	*cost*	*cost*	*cost*	400	400
4	10	*cost*	*cost*	0	*cost*	*cost*	400	400
5	10	*cost*	*cost*	*cost*	0	*cost*	400	400
6	10	*cost*	*cost*	*cost*	*cost*	0	400	400
7	100	400	400	400	400	400	0	5
8	100	400	400	400	400	400	5	0

The Vehicle Routing Game (VRG) is a game (N, c^v) where the total cost of a VRP is to be divided among the players [24]. The players N of the game are the customers, and the characteristic function $c^v(S), S \subseteq N$ is the optimal cost of a VRP over the customers in S. It is assumed that the cost matrix for each vehicle satisfies the triangular inequality and that it is always at least as expensive to use a higher capacity truck as it is to use a lower capacity one. Furthermore, it is assumed a sufficient supply of each truck type, so that the least costly truck type for a route is always chosen. Göthe-Lundgren et al. [34] presented models for the VRG; they discussed the case that the characteristic function is defined as the optimal objective function to a basic VRP. Let D_i be the demand of customer $i \in N$, K the set of truck types in the fleet, V_k the capacity of truck type $k \in K$, and q the highest capacity truck type that is equal to $\arg\max_{k \in K}\{V_k\}$. The characteristic function value $c^v(S)$ of the VRG can be obtained by solving a Set Partitioning Problem (SPP) formulation as follows [24]:

Assume that for each feasible coalition $S \in R$, a minimal cost route is known. The cost of such a route is denoted by $c^T(S)$, and is given by a solution of a Traveling Salesman Problem (TSP) over the customers in S:

$$\alpha_{ir} = \begin{cases} 1, & \text{if customer } i \text{ belongs to} \\ & \text{(feasible) coalition } S_r \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

$$x_r = \begin{cases} 1, & \text{if the minimum cost route covering the customers} \\ & \text{in coalition } S_r \in R \text{ is used} \\ 0, & \text{otherwise} \end{cases} \tag{40}$$

$$(VRP - SPP)c^v(S) = \min \sum_{r|S_r \in R} c^T(S_r)x_r \tag{41}$$

$$\text{s.t.}$$

$$\sum_{r|S_r \in R} \alpha_{ir}x_r = 1, \quad i \in S \tag{42}$$

$$x_r \geq 0, \quad r|S_r \in R \tag{43}$$

$$x_r \text{ integer} \quad r|S_r \in R$$

The Core of Vehicle Routing Problem Games

Example 7. This example is a continuation of the previous example. If $cost = 15$, then an optimal solution of the problem has a total cost 295, and the routes are $1-2-3-1$, $1-4-5-1$, $1-6-1$, and $1-7-8-1$. Because one of the core constraints expresses that customers 7 and 8 will not pay more than $100 + 5 + 100 = 205$, customers $2, 3, 4, 5, 6$ would have to pay at least 90 together. The customers $2, 3, 4, 5, 6$ form coalitions of size 2, for example $y_2 + y_3 \leq 10 + 15 + 10 = 35$. For all i and j $\{(i, j \in 2, \ldots, 8)\}$, $y_i + y_j \leq 10+15+10 = 35$. A linear combination of these ten inequalities yields $4y_2 + 4y_3 + 4y_4 + 4y_5 + 4y_6 \leq 350$, meaning that $y_2 + y_3 + y_4 + y_5 + y_6 \leq 87.5$, which means that the core of the problem is empty.

On the other hand, if $cost = 25$, then we take the same optimal routes but with cost 315. By making the same analysis as previously, the customers $2, 3, 4, 5, 6$ would have to pay at least 110 together. The customers $2, 3, 4, 5, 6$ form coalitions of size 2, for example $y_2 + y_3 \leq 10 + 25 + 10 = 45$. For all i and j $\{(i, j \in 2, \ldots, 8)\}$, $y_i + y_j \leq 10 + 25 + 10 = 45$. A linear combination of these ten inequalities yields $4y_2 + 4y_3 + 4y_4 + 4y_5 + 4y_6 \leq 450$, meaning that $y_2 + y_3 + y_4 + y_5 + y_6 \leq 112.5$, which means that, now, the core of the problem is not empty.

The core is defined by all solutions that fulfill:

$$y(S) \leq c^v(S), \; S \in R_A, \tag{44}$$

$$y(N) = c^v(N) \tag{45}$$

where $R_A = \{S | S \subset N, S \neq \emptyset\}$.

If all constraints in the core formulation of a VRG are explicitly formulated, it is necessary to solve $2^{|N|} - 2$ VRPs in order to evaluate $c^v(S)$. This is computationally complicated for any nontrivial size of N. However, for the VRG, it is possible to reduce the number of inequalities significantly by only considering the feasible coalitions [24]:

$$y(S) \leq c^v(S), \; S \in R, \tag{46}$$

$$y(N) = c^v(N). \tag{47}$$

Engevall [23] and Göthe-Lundgren et al. [34] observed that in any core solution to the VRG, the customers that are covered by a route in any optimal solution to the VRP over the grand coalition have to carry the full cost of that route.

Göthe-Lundgren et al. [34] proved that the number of inequalities that defines the core in the basic VRG can be reduced significantly by only considering coalitions that can be served by a single vehicle. They also proved that the core of the basic VRG is empty if, and only if, there is an integrality gap between the optimal solution to the Set Partitioning Problem formulation and the optimal solution to the linear relaxation of the SPP formulation. They gave an example of a basic VRG with an empty core. Engevall [24] proposed a solution procedure in order to either find a solution in the core or to conclude that the core is empty based on constraint generation (46). He also proved that the core of the VRG is nonempty if and only if $c^v(N)$ is equal to the optimal value of the linear relaxation of the SPP formulation of a VRP over N.

The Nucleolus of Vehicle Routing Problem Games

The computation of nucleolus in a VRG requires considerable computational effort, because it leads to the need for solving complex combinatorial optimization problems. A method that uses a constraint generation approach and can be applied to compute the nucleolus in a basic VRPG with a nonempty core is presented in [34]. Engevall [24] proposed that if the core of the game is found to be empty, and a branch and price procedure were used to investigate the existence of alternative optimal dual solutions to the relaxed VRP, the branch and price procedure can be continued until an optimal solution to the VRP and thus $c^v(N)$ is found.

3.10 Packing and Covering Games

A packing game (c, A, max) is associated with an integer program [16]. The row of A is indexed by M, and the column of A is indexed by N. N is the set of players. $\forall S \subseteq N, u(S)$ is the value of the following integer program:

$$\max \qquad x^t c \tag{48}$$

$$s.t.\ x^t A_{M,S} \leq 1^t_{|S|},\ x^t A_{M,\bar{S}} \leq 0^t_{n-|S|}, \tag{49}$$

$$x \in \{0,1\}^m \tag{50}$$

where $A_{M,S}$ is the submatrix of A with row set M and column set S, and $u(\emptyset)$ is defined to be 0.

Deng et al. [17] gave a necessary and sufficient condition for maximum packing games to have nonempty cores. They proved that the linear programming relaxation of a maximum packing problem has an integral optimal solution if and only if the associated game has a nonempty core, and if so the core is characterized by the set of optimal solutions of the dual of the linear programming relaxation.

The bin packing game can be stated as follows [53]: given n items of sizes $\alpha_1, \ldots, \alpha_n$ and m bins each of size U, we denote the bin packing game by $G = [m, U; \alpha_1, \ldots, \alpha_n]$. Let us assume that $\alpha_1, \ldots, \alpha_n$ and U are nonnegative integers satisfying $\alpha_i \leq U$ for all $i \in \{1, 2, \ldots, n\}$. The set of items $\{1, 2, \ldots, n\}$ is denoted by I, the set of bins by B, and the vector $\alpha_1, \ldots, \alpha_n$ by α. For any subset of items $I' \subseteq I$, the value $\sum_{i \in I'} \alpha_i$ is calculated. The set N of players consists of all items and all bins, and so $|N| = n + m$. The characteristic function of the game, denoted by $u_G : 2^N \to R$, is defined as follows. When S is a coalition containing $m' = |S \cap B|$ bins and items $S \cap I$, the value $u_G(S)$ is equal to the weight of optimal bin packing with respect to

$$u(S) = max\{\sum_{j=1}^{m'} |\exists I_1, \ldots, \exists I_{m'} \subseteq S \cap I, I_j \cap I_{j'} = \emptyset (j \neq j'),$$

$$\sum_{i \in I_j} \alpha_i \leq U (j = 1, \ldots, m')\}. \tag{51}$$

Faigle and Kern [25] proved that every bin packing game has a nonempty ϵ-core with $\epsilon = \frac{1}{2}$ and constructed a class of bin packing games with empty ϵ-core and $\epsilon = \frac{1}{7}$. Furthermore, Woeginger [70] proved that every bin packing game has a nonempty ϵ-core with $\epsilon = \frac{1}{3}$. Matsui [53] proposed an algorithm for finding an allocation x in the ϵ-core with minimum tax rate ϵ.

Let N be the set of players. For each coalition $S \subseteq N$, the cost of providing a service to the players in S is $C(S)$. The set covering problem can be stated as follows [14]: given a universal set U, and a collection of subsets of U, $T = \{S_1, S_2, \ldots, S_k\}$, and a cost function $c : T \to Q^+$, find a minimum cost subcollection of T that covers all the elements of U. Given an instance of the set cover problem over the set N, the cost of providing the service to a coalition S is the cost of the optimal subcollection of T that covers all the elements in S. Denavur et al. [14] proposed a greedy algorithm for the computation of the set of players that will be served. In the vertex covering game, the players are edges in the graph and the game value is the minimum set covering all the edges.

3.11 Facility Location Games

The location of facilities in order to provide service for customers is a well known problem in operation research. In the basic model, there is a number of places that the facilities can be opened, a cost for opening each facility, and a number of customers assigned to each facility with a predefined cost. The goal of the problem is the minimization of the total cost. Let $G = (N, E)$ be a graph, where N is the set of nodes, which is the same as the set of players in the game. Every edge $e \in E$ has a positive length l_e. The distance $d(x, y)$ between two points x, y anywhere on the edges of the graph is defined as the length of a shortest path from x to y. The length of a path is the sum of the lengths of the edges and parts of edges that belong to the path. Let A be a finite subset of points anywhere on the edges of G and let $i \in N$. The distance $d(i, A)$ between i and A is defined by [13]

$$d(i, A) = \min_{x \in A} d(i, x). \tag{52}$$

The players can construct service facilities at any point on the graph, that is, at any point along an edge of the graph and not only at the nodes of the graph. The cost of a player $i \in N$ is a linear function of the distance between i and a facility that is closest to i. For each $i \in N$, a weight w_i is given such that if this distance is d_i, the cost for i is $w_i d_i$.

Tamir [65] has considered a cost allocation game for a location problem for which he was able to reduce the exponential number of constraints for core ($x(S) \leq c(S)$) to a linear number. Curiel [13] denoted the number of facilities that coalition S is allowed to build by p_s and assumed that $p_s < |S|$. Each player has a cost $L(i)$ associated with not having access to any facility. Curiel studied two classes of games arising from such a situation. In the first, each coalition wants to minimize the maximum cost of its member and is called p-center game, and in the second each coalition wants to minimize the sum of the costs of its members and is called p-median game. The formulation of these games is given in the following [13]:

For the p-center game:

$$c_p(S) = \begin{cases} \max\limits_{i \in S} L(i), & \text{if } p_S = 0 \\ \min\limits_{A:|A|=p_S} \max\limits_{i \in S} w_i d(i, A), & \text{if } p_S > 0 \end{cases} \tag{53}$$

For the p-median game:

$$c_p(S) = \begin{cases} \sum\limits_{i \in S} L(i), & \text{if } p_S = 0 \\ \min\limits_{A:|A|=p_S} \sum\limits_{i \in S} w_i d(i, A), & \text{if } p_S > 0 \end{cases} \tag{54}$$

In a p-center game, each coalition S with $p_S > 0$ has to solve a p-center problem, whereas in a p-median game each coalition S with $p_S > 0$ has to solve a p-median problem. P-center and p-median games have a nonempty core under certain conditions and are balanced [13].

Curiel [13] also presented a simple plant location game that can be formulated as follows. Let N be a set of players. The players correspond with nodes of a tree and need to build facilities that can be located in the nodes of the tree only. These are setup costs depending on the node where the facility is located. There are also travel costs associated with edges of the tree. Each coalition wants to minimize the sum of the setup costs and the weighted travel costs of its members.

Denavur et al. [14] proposed a greedy algorithm for the computation of the set of facilities that will be open, the set of cities to be connected to each facility, and the amount to be charged to each city that has been connected in an uncapacitated facility game. In this game, the opening of a facility causes a fixed cost $f_i \geq 0$ and the cost of assigning customer $j \in N$ to facility i is denoted by $c_{ij} \geq 0$. Goemans and Skutella [33] derived for any kind of constrained facility location an equivalent relaxation in the natural space of variables that contains a variable y_i denoting whether facility i is open and a variable x_{ij} denoting whether customer j is assigned to facility i. For the unconstrained facility location problem, this canonical relaxation turns out to simply be a classic LP relaxation of the problem. Kolen [49] proved that the core is nonempty if and only if this canonical LP relaxation has no integrality gap for the objective function being considered. Chardaire [10] generalized Kolen's result to some sorts of capacitated facility location games. Geomans and Skutella [33] showed that testing the core nonemptiness is NP-complete. Finally, Goemans and Skutella [33] proved that the cost allocation problem is equivalent to the dual of the LP relaxation of the facility location problem.

Chardaire in his PhD thesis [9] calculated the core and the nucleolus of two location games, the uncapacitated facility location game and the capacitated facility location game. He proved that for the first game, a necessary and sufficient condition for the nonemptiness of the core is for the integer problem associated with the grand coalition and the straightforward LP relaxation of that integer problem to have the same optimal values. Moreover, when the core is not empty, he gave a compact reformulation of the core, polynomial in the number of players, based on the dual of the LP relaxation associated with the grand coalition. For the second problem, namely the Capacitated case, the results that were obtained from Chardaire were not so strong as in the first problem, but in the case that the distances are Euclidean the results are similar as in the first case. Chardaire, also, studied the nucleolus of this two games and he proved that when the core of the uncapacitated game is non-empty, the nucleolus of the game is equal with the nucleolus of the relaxed game. He proposed a constraint generation method to compute the nucleolus of the uncapacitated game. He extended his approach in the capacitated case and proved that if the distances of the problem are Euclidean and the core is nonempty, then the constrained generation method for the computation of the nucleolus can also be applied.

3.12 Supply Chain Management and Cooperative Games

The design and management of supply chain are nowadays one of the most active research fields in the area of optimization. In the literature, few papers have been published that use Cooperative Game Theory to study applications of Supply Chain Management. Most of the works published until now concern theoretical results and solution concepts in some combinatorial problems (routing problems, location problem) as it was presented in the previous sections, and not practical applications. In Wang and Parlar [69], a newsvendor game with three players is analyzed, first in noncooperative setting and then under cooperation with and without Transferable Utility. Hartman et al. [44] considered the newsvendor centralization game, a game in which multiple retailers decide to centralize their inventory and split profits resulting from the benefits of risk pooling and showed that this game has a nonempty core under certain restrictions on the demand distribution.

Engevall [24] studied in his PhD thesis a distribution problem in Norsk Hydro Olje AB that markets and sells gas-oil in Sweden. Norsk Hydro is responsible for the transportation of different qualities of gas and gas-oil to the customers of Norsk Hydro. After the transportation of the goods to the customers has taken place, he considered how to allocate the total transportation cost for a tour, or for a set of tours, to the customers served.

Vidal and Goetschalckx [68] presented a model for the optimization of a global supply chain that maximizes the after-tax profits of a multinational corporation and that includes transfer prices and the allocation of transportation costs as explicit decision variables. They formed the problem as a nonconvex optimization problem with a linear objective function, and both linear constraints and bilinear constraints. They also proposed a heuristic algorithm for the solution of the problem.

4 Conclusion

In this paper, the most important cost allocation methods in problems that arise from the field of combinatorial (discrete) optimization were presented. Initially, the solution concepts of the cooperative game theory, like the core of the game, the Shapley value, the Bargaining set, the Nucleolus of the game, and the Kernel of the game, were given and analyzed. Then, for the most important problems of combinatorial optimization and of supply chain management, the corresponding combinatorial optimization game was presented. For these games, the core, the nucleolus, and the other solution concepts were calculated.

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Time-Dependent Equilibrium Problems

Antonino Maugeri[1] and Carmela Vitanza[2]

[1] Department of Mathematics and Computer Sciences, University of Catania, Catania, Italy
maugeri@dmi.unict.it

[2] Department of Mathematics, University of Messina, Messina, Italy
vitanzac@unime.it

Abstract The paper presents variational models for dynamic traffic, dynamic market, and evolutionary financial equilibrium problems taking into account that the equilibria are not fixed and move with time. The authors provide a review of the history of the variational inequality approach to problems in physics, traffic networks, and others, then they model the dynamic equilibrium problems as time-dependent variational inequalities and give existence results. Moreover, they present an infinite dimensional Lagrangean duality and apply this theory to the above time-dependent variational inequalities.

Key words: time-dependent variational inequalities, dynamic traffic, dynamic market and evolutionary financial equilibrium problems, Lagrangean duality.

1 Introduction

The scientific life of the theory of Variational Inequalities has revealed itself full of events and surprises. This theory arose in the 1970s as an innovative and effective method to solve a group of equilibrium problems originated from mathematical physics as the Signorini problem, the obstacle problem, and the elastic-plastic torsion problem, and it is still an open question to decide who must be considered the founder between G. Fichera and G. Stampacchia, who first dealt with Variational Inequalities (see [10] and [15]).

The critical point for which the other theories, available in the literature, have revealed themselves unable to solve the above-mentioned problems is that these problems request a condition of complementarity type on the boundary or on a part of the set where the problems are defined, and, in general, it is not possible to express them as an optimization problem.

After an intense period of successes and of fundamental results obtained by means of the Variational Inequality theory, which someone defines as the Italian way of mathematics, maybe in consequence of the untimely death of

G. Stampacchia in 1979, the interest for Variational Inequalities declined and it seemed that the theory had no more to say.

On the contrary, in the beginning of the 1980s, it was proved by M.J. Smith (see [22]) and S. Dafermos (see [2]) that the traffic network equilibrium problem can be formulated in terms of a finite-dimensional Variational Inequality and, hence, it is possible to study in this way existence, uniqueness, stability of traffic equilibria, and to compute the solutions. In consequence of this fact, the past decades have witnessed an exceptional interest for Variational Inequalities, and an enormous amount of papers and books have been devoted to this topic. As a relentless river, more and more problems arising from the economic world, as the spatial price equilibrium problem, the oligopolistic market equilibrium problem, the migration problem, and many others (see [19]), are formulated in terms of a finite dimensional Variational Inequality and, by means of this theory, solved.

The last event goes back to the end of the 1990s: the traffic network equilibrium problem with feasible path flows that have to satisfy time-dependent capacity constraints and demands has been formulated in [3] and [4] (see also [11]) as an evolutionary Variational Inequality, for which existence theorems and computational procedures are given. Starting from this first result, many other problems with time-dependent data have been formulated in the same terms. In [5] and [6], the authors consider the spatial price equilibrium problem when the prices and the commodity shipment bounds vary over the time. [8] addresses the time-dependent spatial price equilibrium problem in which the variables are commodity shipments. In [7] and [9], the authors consider a time-depending financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time-dependent supplies of the financial holdings.

Although in the theory of Variational Inequalities an important chapter is constituted by parabolic or hyperbolic Variational Inequalities, the models that formulate the above problem are different from the previous ones and then they request an appropriate study and an improvement of some aspects of Variational Analysis. All these problems have a common element: their equilibrium conditions can be handled as generalized complementarity problems and moreover the evolutionary Variational Inequality formulation can be expressed in a unified way (see [1]).

The aim of this paper is to present the essential aspects of the problems considered and to focus on the new questions that the evolutionary framework provides.

2 Time-Dependent Equilibrium Conditions and Evolutionary Variational Inequalities

The driving forces of the problems that we examine are considered time-dependent on a fixed time interval $[0, T]$. Consequently, the response of the system is time-dependent, too. Here the system is assumed to respond to changes of the driving forces so gradually in the considered timescale that, at each instant, equilibrium conditions prevail. However, we can consider models with presence of delay effects on the response (see [21]), but in this paper we will just mention this subject.

We start considering a model of a traffic network on a finite directed graph (see [3] and [4]). There is given a set $\mathcal{W}$ of origin-destination pairs and a set $\mathcal{R}$ of routes. Each route $r \in \mathcal{R}$ links some origin-destination pair $w \in \mathcal{R}$. This leads to the set $\mathcal{R}(w)$ of all $w \in \mathcal{W}$. The topology of the network is described by the pair-route incidence matrix $\Phi = \{\Phi_{w,r}\}$ with $w \in \mathcal{W}$, $r \in \mathcal{R}$, where

$$\Phi_{w,r} = \begin{cases} 1 & \text{if the route } r \text{ connects the pair } w \\ 0 & \text{otherwise.} \end{cases}$$

Because the feasible flows have to satisfy time-dependent capacity constraints and demand requirements, the flow vectors are time-dependent flow vectors $f(t) \in \mathbb{R}^{\mathcal{R}}$, where t varies in the fixed time interval $\mathcal{T} = [0, T]$, while the topology remains fixed. Each component $f_r(t)$ of $f(t)$ gives the flow trajectory $f : \mathcal{T} \to \mathbb{R}^{\mathcal{R}}$, which have to satisfy almost everywhere on $\mathcal{T}$ the capacity constraints

$$\lambda(t) \leq f(t) \leq \mu(t)$$

and the so-called "traffic conservation law":

$$\Phi\, f(t) = \rho(t),$$

where the bounds $\lambda \leq \mu$ and the demand $\rho = (\rho_w)_{w \in \mathcal{W}} \geq 0$ are given. Considering a L^p setting with $p \in (1, \infty)$, we assume that λ and $\mu \in L^p(\mathcal{T}, \mathbb{R}^{\mathcal{R}})$ and that ρ lies in $L^p(\mathcal{T}, \mathbb{R}^{\mathcal{W}})$. Assuming in addition that

$$\Phi\, \lambda(t) \leq \rho(t) \leq \Phi\, \mu(t) \text{ a.e. on } \mathcal{T},$$

we obtain that the set of feasible flows

$$\mathbb{K} = \{f \in E : \lambda(t) \leq f(t) \leq \mu(t),\ \Phi\, f(t) = \rho(t) \text{ a.e. on } \mathcal{T}\} \tag{1}$$

is nonempty (see [13]). Clearly $\mathbb{K}$ is convex and weakly compact.

The cost trajectory C, which assigns to each flow trajectory $f \in \mathbb{K}$ the cost trajectory $C(f)$, is a mapping $C : \mathbb{K} \to E^* = L^q(\mathcal{T}, \mathbb{R}^{\mathcal{R}})$ $\left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$ and it results

$$\ll C(f), g \gg = \int_{\mathcal{T}} \langle C(f(t)), g(t) \rangle \, dt = \int_{\mathcal{T}} \sum_{s \in \mathcal{R}} C_s(f)\, g_s(t)\, dt.$$

The equilibrium condition is given by a generalized version of Wardrop's condition, namely:

Definition 1. *$h \in \mathbb{K}$ is an equilibrium flow if and only if, for all $w \in \mathcal{W}$ and r, $s \in \mathcal{R}(w)$ and a.e. on $\mathcal{T}$ there holds:*

$$C_r(h)(t) < C_s(h)(t) \Longrightarrow h_r(t) = \mu_r(t) \text{ or } h_s(t) = \lambda_s(t). \tag{2}$$

We remark that the kind of equilibrium defined by condition (2) is different from the one obtained considering a minimization of an objective, like total cost set by society or some authority.

The equilibrium approach defined by (2) is called user-oriented traffic equilibrium and has the meaning that every agent in traffic strives for his individual cost and it, when abandoning artificial assumptions of symmetry and thus abandoning the existence of a potential, cannot be formulated as simple optimization problems. The overall flow pattern obtained according to condition (2) fits very well in the framework of the theory of Variational Inequality. In fact in [3] and [4], the following result is shown:

Theorem 1. *$h \in \mathbb{K}$ is an equilibrium solution according to Definition 1 if and only if h is a solution to the following Variational Inequality*

$$\text{“}\textit{Find } h \in \mathbb{K}:$$
$$\ll C(h), f-h \gg = \int_{\mathcal{T}} \langle C(h(t)), f(t)-h(t)\rangle\, dt \geq 0 \quad \forall f \in \mathbb{K}.\text{”} \tag{3}$$

The next equilibrium conditions that we present are those of the spatial price equilibrium problem in the case of the price formulation. In this case, we have n supply markets $P_1, P_2, \ldots, P_n$ and m demand markets $Q_1, Q_2, \ldots, Q_m$ of a commodity m, whose geometry remains fixed during the interval of time $\mathcal{T} = [0, T]$. For each $t \in \mathcal{T}$ we have:

the supply price vector $p(t) \in \mathbb{R}^n$;
the total supply vector $g(t) \in \mathbb{R}^n$;
the demand price vector $q(t) \in \mathbb{R}^m$;
the total demand vector $f(t) \in \mathbb{R}^m$;
the flow vector $x(t) \in \mathbb{R}^{nm}$;
the unit cost vector $c(t) \in \mathbb{R}^{nm}$.

The feasible vectors $u(t) = (p(t), q(t), x(t))$ have to satisfy the time-dependent constraints on prices and transportation flows, namely

$$u(t) \in \prod_{i=1}^{n} \left[\underline{p}_i(t), \overline{p}_i(t)\right] \times \prod_{j=1}^{n} \left[\underline{q}_j(t), \overline{q}_j(t)\right] \times \prod_{i=1}^{n}\prod_{j=1}^{m} \left[\underline{x}_{ij}(t), \overline{x}_{ij}(t)\right]$$

where $\underline{p}_i(t)$, $\overline{p}_i(t)$, $\underline{q}_j(t)$, $\overline{q}_j(t)$, $\underline{x}_{ij}(t)$, $\overline{x}_{ij}(t)$ are given.

The functional setting for the trajectories $u(t)$ is the Hilbert space

$$L = L^2(\mathcal{T}, \mathbb{R}^n) \times L^2(\mathcal{T}, \mathbb{R}^m) \times L^2(\mathcal{T}, \mathbb{R}^{nm})$$

and, hence, the set of feasible vectors $u(t)$ is given by

$$\begin{aligned}\mathbb{K} &= \mathbb{K}_1 \times \mathbb{K}_2 \times \mathbb{K}_3 \\ &= \{p \in L^2(\mathcal{T}, \mathbb{R}^n) : 0 \leq \underline{p}(t) \leq p(t) \leq \overline{p}(t) \ \text{ a.e. on } \mathcal{T}\} \\ &\times \{q \in L^2(\mathcal{T}, \mathbb{R}^m) : 0 \leq \underline{q}(t) \leq q(t) \leq \overline{q}(t) \ \text{ a.e. on } \mathcal{T}\} \\ &\times \{x \in L^2(\mathcal{T}, \mathbb{R}^{nm}) : 0 \leq \underline{x}(t) \leq x(t) \leq \overline{x}(t) \ \text{ a.e. on } \mathcal{T}\},\end{aligned}$$

where $\underline{p}(t)$, $\overline{p}(t) \in L^(\mathcal{T}, \mathbb{R}^n)$, $\underline{q}(t)$, $\overline{q}(t) \in L^(\mathcal{T}, \mathbb{R}^m)$, $\underline{x}(t)$, $\overline{x}(t) \in L^(\mathcal{T}, \mathbb{R}^{nm})$. $\mathbb{K}$ is a convex, closed, weakly compact set. Furthermore, we are giving the mappings:

$$g = g(t, p(t)) : \mathcal{T} \times \mathbb{K}_1 \to L^2(\mathcal{T}, \mathbb{R}^n)$$
$$f = f(t, q(t)) : \mathcal{T} \times \mathbb{K}_2 \to L^2(\mathcal{T}, \mathbb{R}^m)$$
$$c = c(t, x(t)) : \mathcal{T} \times \mathbb{K}_3 \to L^2(\mathcal{T}, \mathbb{R}^{nm})$$

which, at time t, assign to each price trajectory $p \in \mathbb{K}_1$ and $q \in \mathbb{K}_2$ the supply $g \in L^2(\mathcal{T}, \mathbb{R}^n)$ and the demand $f \in L^2(\mathcal{T}, \mathbb{R}^m)$, respectively, and to the flow trajectory $x \in \mathbb{K}_3$ the cost $c \in L^2(\mathcal{T}, \mathbb{R}^{nm})$. Now, we allow that, during the activities of the market in the time interval $[0, T]$, supply and demand excesses can occur, namely that there exists n non-negative functions $s_i(t)$ $i = 1, 2 \ldots, n$ and m non-negative functions $t_j(t)$ $j = 1, \ldots, m$ such that

$$g_i(t, p(t)) = \sum_{j=1}^{m} x_{ij}(t) + s_i(t) \quad i = 1, 2, \ldots, n \tag{4}$$

$$f_j(t, q(t)) = \sum_{i=1}^{n} x_{ij}(t) + t_j(t) \quad j = 1, 2, \ldots, m. \tag{5}$$

The equilibrium conditions of this evolutionary market take the following form:

Definition 2. *$u(t) = (p(t), q(t), x(t)) \in L$ is a dynamic market equilibrium if and only if for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ and a.e. in $\mathcal{T}$ there hold:*

$$\begin{array}{rl} s_i(t) > 0 & \Longrightarrow p_i(t) = \underline{p}_i(t) \\ \underline{p}_i(t) < p_i(t) < \overline{p}_i(t) & \Longrightarrow s_i(t) = 0 \end{array} \quad i = 1, 2, \ldots, n; \tag{6}$$

$$\begin{array}{rl} t_j(t) > 0 & \Longrightarrow q_j(t) = \overline{q}_j(t) \\ \underline{q}_j(t) < q_j(t) < \overline{q}_j(t) & \Longrightarrow t_j(t) = 0 \end{array} \quad j = 1, 2, \ldots, m; \tag{7}$$

$$p_i(t) + c_{ij}(t, x(t)) \begin{cases} > q_j(t) & \text{if } \; x_{ij}(t) = \underline{x}_{ij}(t) \\ = q_j(t) & \text{if } \; \underline{x}_{ij}(t) \le x_{ij}(t) \le \overline{x}_{ij}(t) \\ < q_j(t) & \text{if } \; x_{ij}(t) = \overline{x}_{ij}(t). \end{cases} \tag{8}$$

Conditions (6) and (7), in a reasonable way, are satisfied when the excesses vanish in dependence of the prices; conditions (8) control the amounts of commodity shipments between the supply and the demand markets according to the equilibrium condition that the supply price plus the transportation cost is greater, equal, or less than the demand price.

Denoting by $v : \mathcal{T} \times \mathbb{K} \to L$ the operator defined setting

$$
\begin{aligned}
& v = v(t, u(t)) \\
= \Bigg(& \left(g_i(t, p(t)) - \sum_{j=1}^{m} x_{ij}(t) \right)_{i=1,\dots,n}, \left(f_j(t, q(t)) - \sum_{i=1}^{n} x_{ij}(t) \right)_{j=1,\dots,m}, \\
& (p_i(t) + c_{ij}(t, x(t)) - q_j(t))_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \Bigg),
\end{aligned} \tag{9}
$$

also here the following characterization in terms of Variational Inequalities holds (see [5] and [6]):

Theorem 2. $u(t) = (p(t), q(t), x(t)) \in \mathbb{K}$ *is a dynamic market equilibrium if and only if* $u(t)$ *is a solution to*

$$
\begin{aligned}
& \ll v(u), \tilde{u} - u \gg = \int_0^T \langle v(t, u(t)), \tilde{u}(t) - u(t) \rangle \, dt \\
= & \int_0^T \Bigg\{ \sum_{i=1}^{n} \left(g_i(t, p(t)) - \sum_{j=1}^{m} x_{ij}(t) \right) (\tilde{p}_i(t) - p_i(t)) \\
& - \sum_{j=1}^{m} \left(f_j(t, q(t)) - \sum_{i=1}^{n} x_{ij}(t) \right) (\tilde{q}_j(t) - q_j(t)) \\
& + \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i(t) + c_{ij}(t, x(t)) - q_j(t)) (\tilde{x}_{ij}(t) - x_{ij}(t)) \Bigg\} dt \geq 0 \\
& \forall \tilde{u} = (\tilde{p}, \tilde{q}, \tilde{x}) \in \mathbb{K}.
\end{aligned} \tag{10}
$$

For what concerns the quantity formulation of the spatial price equilibrium problem, in this case the only change is that the supply prices p_i and the demand prices q_j are considered as functions of the supply g and the demand f and the equilibrium conditions are related to a vector $w(t) = (g(t), f(t), x(t), s(t), t(t))$, which represents the variables of the model. More precisely, we are giving two mappings $p = p(t, g(t)) : \mathcal{T} \times L^2([0,T], \mathbb{R}^n_+) \to L^2([0,T], \mathbb{R}^n_+)$ and $q = q(t, f(t)) : \mathcal{T} \times L^2([0,T], \mathbb{R}^m_+) \to L^2([0,T], \mathbb{R}^m_+)$, which assign to each supply $g(t)$ the supply price $p(t, g(t))$ and to each demand $f(t)$ the demand price $q(t, f(t))$. We assume that capacity constraints on p, q and the transportation cost $c(t, x(t))$ are fixed in such a way that:

$$
\underline{p}(t) \leq p(t, g(t)) \leq \overline{p}(t), \quad \underline{q}(t) \leq q(t, f(t)) \leq \overline{q}(t),
$$

$$
\underline{c}(t) \leq c(t, x(t)) \leq \overline{c}(t).
$$

The set of feasible vectors $w(t)$ is given by

$$\mathbb{K} = \Bigg\{ w(t) = (g(t), f(t), x(t), s(t), t(t)) \in$$

$$L^2([0,T],\mathbb{R}^n) \times L^2([0,T],\mathbb{R}^m) \times L^2([0,T],\mathbb{R}^n) \times L^2([0,T],\mathbb{R}^m):$$

$$\left. \begin{aligned} w(t) \geq 0 \text{ a.e. in } [0,T]; \\ g_i(t) = \sum_{j=1}^{m} x_{ij}(t) + s_i(t), \quad i = 1,\dots,n; \\ f_j(t) = \sum_{j=1}^{n} x_{ij}(t) + t_j(t), \quad j = 1,\dots,m \text{ a.e. in } [0,T] \end{aligned} \right\} \tag{11}$$

and the dynamic market equilibrium conditions in the case of the quantity formulation take the following form:

Definition 3. *$w^*(t) \in \mathbb{K}$ is a dynamic market equilibrium if and only if for each $i = 1,\dots,n$ and $j = 1,\dots,m$ and a.e. in $[0,T]$ there hold:*

$$\begin{aligned} &\textit{if } s_i^*(t) > 0, && \textit{then } p_i(t, g^*(t)) = \underline{p}_i(t); \\ &\textit{if } \underline{p}_i(t) < p_i(t, g^*(t)), && \textit{then } s_i^*(t) = 0; \end{aligned} \tag{12}$$

$$\begin{aligned} &\textit{if } t_j(t) > 0, && \textit{then } q_j(t, f^*(t)) = \overline{q}_j(t); \\ &\textit{if } q_j(t, f^*(t)) < \overline{q}_j(t), && \textit{then } t_j^*(t) = 0; \end{aligned} \tag{13}$$

$$\begin{aligned} &\textit{if } x_{ij}^*(t) > 0, \ \textit{then } p_i(t, g^*(t)) + c_{ij}(t, x^*(t)) = q_j(t, f^*(t)); \\ &\textit{if } p_i(t, g^*(t)) + c_{ij}(t, x^*(t)) > q_j(t, f^*(t)), \ \textit{then } x_{ij}^*(t) = 0. \end{aligned} \tag{14}$$

Then in [8], [16], [17] the following result is shown:

Theorem 3. *$w^* \in \mathbb{K}$ is a dynamic market equilibrium if and only if w^* is a solution to the Variational Inequality*

$$\textit{Find } w^* \in \mathbb{K} \textit{ such that}$$

$$\begin{aligned} \ll v(w^*), w - w^* \gg &= \int_0^T \langle v(t, w^*(t)), w(t) - w^*(t) \rangle \, dt \\ = \int_0^T &\big\{ \langle p(t, g^*(t)), g(t) - g^*(t) \rangle - \langle q(t, f^*(t)), f(t) - f^*(t) \rangle \\ &+ \langle c^*(t, x^*(t)), x(t) - x^*(t) \rangle - \langle \underline{p}(t), s(t) - s^*(t) \rangle \\ &+ \langle \overline{q}(t), t(t) - t^*(t) \rangle \big\} \, dt \geq 0 \quad \forall w \in \mathbb{K}. \end{aligned} \tag{15}$$

Here v denotes the operator $v(t,w) = (p(t, g(t)), \ -q(t, f(t)), \ c(t, x(t)), -\underline{p}(t), \ \overline{q}(t))$.

Now we pass to present the evolutionary financial equilibrium conditions and the equivalent variational inequality formulation. We consider a multi-sector, multiinstrument financial equilibrium problem with a general utility function and including policy interventions in form of taxes and price controls.

Then we have m sectors, with a typical sector denoted by i, and n instruments, with a typical financial instrument denoted by j, in the period $[0, T]$. Let $s_i(t)$ be the total financial volume held by sector i at the time t. x_{ij} denotes the amount of instrument j held as an asset in sector i's portfolio, y_{ij} the amount of instrument j held as liability in sector i's portfolio. The assets x_{ij} in sector i's portfolio are grouped into the column vector $x_i(t)$ and the sector asset vectors into the matrix $x(t)$; similarly, $y_i(t)$ denotes the column vector of the liabilities in sector i's portfolio and $y(t)$ the matrix of the sector liability vectors. The instrument prices $r_j(t)$ are variables of the problem but are fixed instrument floor prices $\underline{r}_j(t)$ and instrument ceiling prices $\overline{r}_j(t)$, which represent the form of policy interventions; $r(t)$, $\underline{r}(t)$, $\overline{r}(t)$ denote the column vectors of the prices, of the floor prices, and of the ceiling prices, respectively. Moreover, the policy interventions act by imposing a tax rate $\tau_{ij}(t)$ on sector i's net yield on financial instrument j. We assume that the tax rates in this model have the flexibility of adjusting the tax rate following the evolution of the system. Then, assuming as the functional setting the Lebesgue space $L^2([0, T], \mathbb{R}^p)$, the set of feasible assets and liabilities for the sector i becomes

$$P_i = \left\{ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \in L^2([0, T], \mathbb{R}^{2n}) \colon \right.$$

$$\sum_{j=1}^{n} x_{ij}(t) = s_i(t), \sum_{j=1}^{n} y_{ij}(t) = s_i(t) \text{ a.e. in } [0, T],$$

$$\left. x_{ij}(t) \geq 0, y_{ij}(t) \geq 0 \text{ a.e. in } [0, T] \right\}$$

and the set of feasible instrument prices is

$$\mathcal{R} = \big\{ r(t) \in L^2([0, T], \mathbb{R}^n) \colon$$

$$\underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), j = 1, \dots, n \text{ a.e. in } [0, T] \big\},$$

where $\underline{r}(t)$ and $\overline{r}(t)$ are assumed to belong to $L^2([0, T], \mathbb{R}^n)$. We introduce for each sector i a utility function $U_i(t, x_i(t), y_i(t), r(t))$, which is constituted by two terms (see [9] and [19]):

$$\begin{aligned} & U_i(t, x_i(t), y_i(t), r(t)) \\ &= u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} \big(r_j(t) - \underline{r}_j(t) \big) \left(1 - \tau_{ij}(t) \right) \left(x_{ij}(t) - y_{ij}(t) \right) \end{aligned} \tag{16}$$

The first term is connected with the opposite of the risk-aversion and an example of this type of function is the well-known one used in the quadratic model (see [19] and [7]):

$$u_i(t, x_i(t), y_i(t)) = -\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}$$

where $Q^i(t)$ is a $2n \times 2n$ matrix, which, following the concept that assessment of risk is based on a variance-covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument, represents a measure of this aversion.

In the general case, we require a lot of qualitative assumptions on $u_i(t, x_i(t),\ y_i(t))$. Precisely, we require that $u_i(t, x_i(t), y_i(t))$ is defined and concave on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, is measurable in t, and continuous with respect to x_i and y_i. Moreover, we assume that $\dfrac{\partial u_i}{\partial x_{ij}}$ and $\dfrac{\partial u_i}{\partial y_{ij}}$ exist and they are measurable in t and continuous with respect to x_i and y_i. Further, we require that the following growth conditions hold:

$$\begin{gathered} |u_i(t, x, y)| \leq \alpha_i(t)\, \|x\|\, \|y\|, \quad \forall x, y \in \mathbb{R}^n, \\ \text{a.e. in } [0, T],\ i = 1, \ldots, m; \end{gathered} \tag{17}$$

$$\begin{gathered} \left| \frac{\partial u_i(t, x, y)}{\partial x_{ij}} \right| \leq \beta_{ij}\, \|y\|, \left| \frac{\partial u_i(t, x, y)}{\partial y_{ij}} \right| \leq \gamma_{ij}\, \|x\|, \\ i = 1, \ldots, m;\ j = 1, \ldots, n, \end{gathered} \tag{18}$$

where α_i, β_{ij}, γ_{ij} are non-negative functions of $L^\infty([0, T])$. The second term expresses the request to maximize the value of the asset holding and to minimize the value of the liabilities. Moreover, the second term incorporates the tax rate through the presence of the $(1 - \tau_{ij}(t))$ term premultiplying the $\left(r_j(t) - \underline{r}_j(t)\right)(x_{ij}(t) - y_{ij}(t))$.

We can provide the following definition of an evolutionary financial equilibrium.

Definition 4. *A vector of sector assets, liabilities, and instrument prices* $(x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^{m} P_i \times \mathcal{R}$ *is an equilibrium of the evolutionary financial model if and only if it satisfies the system of inequalities:*

$$\begin{aligned} &-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t))\left(r_j^*(t) - \underline{r}_j(t)\right) - \mu_i^{(1)}(t) \geq 0 \\ &-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} - (1 - \tau_{ij}(t))\left(r_j^*(t) - \underline{r}_j(t)\right) - \mu_i^{(2)}(t) \geq 0, \end{aligned} \tag{19}$$

and equalities:

$$x^*_{ij}(t)\left[-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))\left(r^*_j(t)-\underline{r}_j(t)\right)-\mu^{(1)}_i(t)\right]=0$$

$$y^*_{ij}(t)\left[-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial y_{ij}}-(1-\tau_{ij}(t))\left(r^*_j(t)-\underline{r}_j(t)\right)-\mu^{(2)}_i(t)\right]=0, \tag{20}$$

where $\mu^{(1)}_i(t)$, $\mu^{(2)}_i(t)\in L^2([0,T])$ *are Lagrangean functions, for all sectors* i : $i=1,\ldots,m$ *and for all instruments* $j : j=1,\ldots,n$ *and verifies the condition:*

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left(x^*_{ij}(t)-y^*_{ij}(t)\right)\begin{cases}\leq 0 & \text{if } r^*_j(t)=\overline{r}_j(t)\\ =0 & \text{if } \underline{r}_j(t)<r^*_j(t)<\overline{r}_j(t)\\ \geq 0 & \text{if } r^*_j(t)=\underline{r}_j(t).\end{cases} \tag{21}$$

The meaning of Definition 4 is that to each financial volume $s_i(t)$ held by the sector i, we associate the functions $\mu^{(1)}_i(t)$, $\mu^{(2)}_i(t)$, related, respectively, to the assets and to the liabilities and that represent the "equilibrium disutilities" for unit of the sector i. The financial volume invested in the instrument j as assets $x^*_{ij}(t)$ is greater or equal to zero if the j-th component

$$-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))\left(r^*_j(t)-\underline{r}_j(t)\right)$$

of the disutility is equal to $\mu^{(1)}_i(t)$, whereas if

$$-\frac{\partial u_i(t,x^*_i(t),y^*_i(t))}{\partial x_{ij}}-(1-\tau_{ij}(t))\left(r^*_j(t)-\underline{r}_j(t)\right)>\mu^{(1)}_i(t),$$

then $x^*_{ij}(t)=0$. The same occurs for the liabilities.

The functions $\mu^{(1)}_i(t)$ and $\mu^{(2)}_i(t)$ are Lagrangean functions associated, respectively, with the constraints

$$\sum_{j=1}^{n}(x_{ij}(t)-s_i(t))=0 \quad \text{and} \quad \sum_{j=1}^{n}(y_{ij}(t)-s_i(t))=0.$$

They are not known *a priori*, but this has not influence, as Definition 4 is equivalent to a Variationsl Inequality in which $\mu^{(1)}_i(t)$ and $\mu^{(2)}_i(t)$ do not appear, as the following theorem shows:

Theorem 4. *A vector* $(x^*(t),y^*(t),r^*(t))\in\prod_{i=1}^{m}P_i\times\mathcal{R}$ *is an evolutionary financial equilibrium if and only if it satisfies the following Variational Inequality:*

$$\textit{Find } (x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^{m} P_i \times \mathcal{R} \textit{ such that}$$

$$\sum_{i=1}^{m} \int_0^T \left\{ \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t)) \left(r_j^*(t) - \underline{r}_j(t) \right) \right] \right.$$

$$\times \left[x_{ij}(t) - x_{ij}^*(t) \right]$$

$$+ \sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} - (1 - \tau_{ij}(t)) \left(r_j^*(t) - \underline{r}_j(t) \right) \right]$$

$$\times \left[y_{ij}(t) - y_{ij}^*(t) \right]$$

$$\left. + \sum_{j=1}^{n} \left(x_{ij}^*(t) - y_{ij}^*(t) \right) \times \left[r_j(t) - r_j^*(t) \right] \right\} dt \geq 0$$

$$\forall (x, y, r) \in \prod_{i=1}^{m} P_i \times \mathcal{R}. \tag{22}$$

Now, if we give a look to the variational inequalities and to the underlying constraint sets that express the above equilibrium problems, we are led to conclude that all these problems can be formulated in a unified way. In fact, let us consider the nonempty, convex, closed, bounded subset of $L^2([0,T], \mathbb{R}^q)$ given by:

$$\mathbb{K} = \left\{ u \in L^2([0,T], \mathbb{R}^q) : \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T]; \right.$$

$$\left. \sum_{i=1}^{q} \xi_i u_i(t) = \rho(t) \text{ a.e. in } [0,T],\ \xi_i \in \{-1, 0, 1\},\ i \in \{1, \ldots, q\} \right\}. \tag{23}$$

For chosen values of the scalars ξ_i, of the dimension q, and of the boundaries λ, μ, we obtain each of the previous above cited constraint sets (see [1] for details). Therefore, we obtain the following standard form for the above cited problems:

$$\text{Find } u \in \mathbb{K} \text{ such that}$$

$$\ll F(u), v - u \gg = \int_0^T \langle F(t, u(t)), v(t) - u(t) \rangle \, dt \geq 0 \quad \forall v \in \mathbb{K}, \tag{24}$$

where $\mathbb{K}$ is given by (23) and F is a mapping from $[0,T] \times \mathbb{K}$ onto $L^2([0,T], \mathbb{R}^q)$.

Further, it directly derives from the proofs of Theroems 1–4 that problem (24) is also equivalent to the following one:

$$\text{Find } u \in \mathbb{K} \text{ such that}$$

$$\langle F(t, u(t)), v(t) - u(t) \rangle \quad \forall v \in \mathbb{K}, \text{ a.e. in } [0,T], \tag{25}$$

which can be useful for computational purpose.

3 Qualitative Results

As observed in [3] and [4], there are two standard approaches to the existence of equilibria, namely, with and without a monotonicity requirement. We shall employ the following definitions. $F : [0,T] \times \mathbb{K} \to L^2([0,T],\mathbb{R}^q)$ is said to be

- *pseudomonotone* if and only if for all $u, v \in \mathbb{K}$
$$\ll F(u), v-u \gg \geq 0 \Longrightarrow \ll F(v), v-u \gg \geq 0;$$
- *hemicontinuous* if and only if for all $v \in \mathbb{K}$, the function $u \to \ll F(u), v-u \gg$ is upper semicontinuous on $\mathbb{K}$;
- *hemicontinuous along line segments* if and only if for all $u, v \in \mathbb{K}$, the function $w \to \ll F(w), v-u \gg$ is upper semicontinuous on the line segment $[x, y]$.

The following general result holds:

Theorem 5. *Let* $F : [0,T]\times\mathbb{K} \to L^2([0,T],\mathbb{R}^q)$ *and* $\mathbb{K} \subseteq L^2([0,T],\mathbb{R}^q)$ *convex and nonempty. Assume that*

(a) there exists $A \subseteq \mathbb{K}$ *nonempty, compact and* $B \subseteq \mathbb{K}$ *compact, convex such that, for every* $u \in \mathbb{K} \setminus A$*, there exists* $v \in B$ *with*
$$\ll F(u), v-u \gg < 0;$$

and that either (b) or (c) below holds:

(b) F *is hemicontinuous;*
(c) F *is pseudomonotone and hemicontinuous along line segments.*

Then there exists $u \in \mathbb{K}$ *such that* $\ll F(u), v-u \gg \geq 0$, $\forall v \in \mathbb{K}$.

We may apply this result with $\mathbb{K}$ given by (23). Then $\mathbb{K}$ is convex, closed, and bounded, hence weakly compact. So, if we endow $L^2([0,T],\mathbb{R}^q)$ with the weak topology, then $\mathbb{K}$ is compact and condition (a) in Theorem 5 is automatically satisfied by choosing $A = \mathbb{K}$ and $B = \emptyset$.

If we endow the space $L^2([0,T],\mathbb{R}^q)$ with the strong topology, condition (a) must be used (we can avoid the request of convexity of $\mathbb{K}$, as observed in [4]) as well as (b). Finally, because weak and strong topology coincide on line segments, condition (c) is enough to ensure the existence of a solution.

Now let us suppose that F is a Carathéodory function, namely that $F(t,u)$ is measurable in t and continuous with respect to u and that the following condition holds:
$$\|F(t,u)\|_{\mathbb{R}^q} \leq f(t) + \alpha(t)\,\|u\|_{\mathbb{R}^q} \tag{26}$$
with $f(t) \in L^2([0,T])$, $\alpha(t) \in L^\infty([0,T])$.

Then it is possible to show (see [6], Theorem 3) that F is hemicontinuous. In consequence of this fact, we get the following existence result.

Theorem 6. *Assume that condition (26) holds. Then each of the following conditions is sufficient for the existence of a solution to the Variational Inequality (24):*

1. *There exist $A \subseteq \mathbb{K}$ nonempty, compact and $B \subseteq \mathbb{K}$ compact such that, for every $u \in \mathbb{K} \setminus A$, there exists $v \in B$ with $\ll F(u), v - u \gg < 0$;*
2. *F is pseudomonotone;*
3. *F is hemicontinuous with respect to the weak topology.*

Interesting problems concerning the qualitative study of solutions to the Variational Inequality (24) are the stability and sensitivity analysis and the so-called regularization theory of solutions. The sensitivity analysis tries to clarify the behavior of solutions when some changes in the data occur, and the aim of the stability analysis is to check if a small change in the mapping F produces a small change in the solution. Some results in these fields can be found in [8, 17, 21].

The regularization theory deals with the problem to see if, imposing that the data fulfill some regularity assumptions, as Hölder-continuity, differentiability, and so on, the solutions to (24) verify in turn these major properties. For example, in [13], Section 2.1, the author asks whether the solution to (24) (or (25)) can be in $C([0,T],\mathbb{R}^q)$. Even if some partial results are available, the question is still open.

4 Lagrangean and Duality Theory

It is worth remarking that the Lagrangean theory provides interesting contributions, absolutely necessary for the better understanding and handling of the equilibrium problems considered. In fact, not only do the Lagrangean variables have a meaning intrinsic to the nature of the problems considered, but also the Lagrangean theory is essential in order to obtain the equivalence between the equilibrium conditions and a Variational Inequality. However, in our infinite dimensional setting, new problems arise with respect to the finite dimensional Lagrangean theory. The crucial difference with respect to the finite dimensional setting is that the interior of the cone

$$C = \left\{ v \in L^2([0,T],\mathbb{R}^q) : v(t) \geq 0 \text{ a.e. in } [0,T] \right\} \tag{27}$$

is empty and, as a consequence, the separation theorems as well as the so-called Slater regularity assumption do not hold. Then one can try to overcome this difficulty either introducing the new concept of quasi-relative interior and proving separation theorems by means of this new concept (see [13] for details and applications) or using a more general regularity assumption that does not require any condition on the interior of C. We will follow this second way and, to this end, let us consider the Variational Inequality (24) and let us introduce the following function:

$$\mathcal{L}(v, l_1, l_2, m) = \Psi(v) - \int_{\Omega} \langle l_1(t), v(t) - \lambda(t) \rangle \, dt$$
$$+ \int_{\Omega} \langle l_2(t), v(t) - \mu(t) \rangle \, dt + \int_{\Omega} \langle m(t), \Phi v(t) - \rho(t) \rangle \, dt$$
$$\forall v \in L^2([0,T], \mathbb{R}^q), \quad \forall l_1, l_2 \in C, \ \forall m \in L^2([0,T], \mathbb{R}^l) \tag{28}$$

which is called *Lagrangean functional.* In (28) we denote by

$$\Psi : L^2([0,T], \mathbb{R}^q) \to \mathbb{R}$$

the mapping

$$\Psi(v) = \langle F(u), v - u \rangle$$

with u solution to the Variaional Inequality (24). It results

$$\min_{v \in \mathbb{K}} \Psi(v) = \Psi(u) = 0.$$

By the term $\Phi\, v(t) - \rho(t)$, we denote the term $\sum_{i=1}^{q} \xi_i v_i = \rho(t)$, which appears in the convex set $\mathbb{K}$ given by (23); here $\rho \in L^2([0,T], \mathbb{R}^l)$ and Φ is a $l \times q$ matrix whose entries are -1, 0, 1.

Our aim is to prove the following characterization:

Theorem 7. *$u \in \mathbb{K}$ is a solution to Variational Inequality (24) if and only if there exist $\bar{l}_1$, $\bar{l}_2 \in C$ and $\overline{m} \in L^2([0,T], \mathbb{R}^l)$ such that $(u, \bar{l}_1, \bar{l}_2, \overline{m})$ is a saddle point of the Lagrange functional (28), namely*

$$\begin{gathered} \mathcal{L}(u, l_1, l_2, m) \leq \mathcal{L}(u, \bar{l}_1, \bar{l}_2, \overline{m}) \leq \mathcal{L}(v, \bar{l}_1, \bar{l}_2, \overline{m}) \\ \forall v \in L^2([0,T], \mathbb{R}^q), \quad \forall l_1, \ l_2 \in C \ \textit{and} \ \forall m \in L^2([0,T], \mathbb{R}^l) \end{gathered} \tag{29}$$

and in addition

$$\int_0^T \langle \bar{l}_1(t), u(t) - \lambda(t) \rangle \, dt = 0, \qquad \int_0^T \langle \bar{l}_2(t), u(t) - \mu(t) \rangle \, dt = 0. \tag{30}$$

Proof. Let $(u, \bar{l}_1, \bar{l}_2, \overline{m})$ be a saddle point of the Lagrange functional $\mathcal{L}$. Taking into account that $\mathcal{L}(u, \bar{l}_1, \bar{l}_2, \overline{m}) = 0$, from the right-hand part of (29) we get:

$$\mathcal{L}(v, \bar{l}_1, \bar{l}_2, \overline{m}) \geq 0, \quad \forall v \in L^([0,T], \mathbb{R}^q). \tag{31}$$

Considering (31) for each $v \in \mathbb{K}$, namely for $\lambda(t) \leq v(t) \leq m(t)$ and $\Phi v(t) = \rho(t)$, we obtain:

$$\Phi(v) = \langle F(u), v - u \rangle \geq \mathcal{L}(v, \bar{l}_1, \bar{l}_2, \overline{m}) \geq 0, \forall v \in \mathbb{K} \tag{32}$$

and therefore u is a solution to (29).

Vice versa, let u be a solution to (24) and, first, let us prove that there exist $\bar{l}_1$, $\bar{l}_2 \in C$ and $\overline{m} \in L([0,T],\mathbb{R}^l)$ such that

$$\mathcal{L}'(u,\bar{l}_1,\bar{l}_2,\overline{m})(v-u) \geq 0, \quad \forall v \in L^2([0,T],\mathbb{R}^q) \tag{33}$$

and

$$\int_0^T \langle \bar{l}_1(t), u(t)-\lambda(t)\rangle\, dt = 0, \quad \int_0^T \langle \bar{l}_2(t), u(t)-m(t)\rangle\, dt = 0,$$

where $\mathcal{L}'(u,\bar{l}_1,\bar{l}_2,\overline{m})$ denotes the Fréchet derivative of $\mathcal{L}(u,\bar{l}_1,\bar{l}_2,\overline{m})$ at u. We derive the estimate (33) using Theorem 5.3 of [14], provided that the Kurcyusz–Robinson–Zowe condition (5.2) of [14] (see also [23] and [20]):

$$\begin{aligned} &\begin{pmatrix} g'(u) \\ \\ h'(u) \end{pmatrix} \text{ cone } \left(L^2([0,T],\mathbb{R}^q)-\{u\}\right) \\ &+ \text{ cone } \begin{pmatrix} c+\{g(u)\} \\ \\ 0 \end{pmatrix} = \begin{pmatrix} L^2([0,T],\mathbb{R}^q) \\ \\ L^2([0,T],\mathbb{R}^l) \end{pmatrix} \end{aligned} \tag{34}$$

is fulfilled (see also the remark at the end of page 120 of [14]). In order to verify this condition, let us set $g(v) = (\lambda - v, v - m)$ and $h(v) = \Phi v - \rho$. It results $g'(v)(w) = (-w, w)$, $h'(v)(w) = \Phi w$, and the condition (34) is fulfilled because it results:

$$\begin{aligned} &- \text{ cone } \left(L^2([0,T],\mathbb{R}^q)-\{u\}\right) + \text{ cone } (C+\{\lambda-u\}) \\ &= -L^2([0,T],\mathbb{R}^q) + \text{ cone } \{u\} + C + \text{ cone } \{\lambda-u\} = L^2([0,T],\mathbb{R}^q), \end{aligned} \tag{35}$$

$$\text{cone } \left(L^2([0,T],\mathbb{R}^q)-\{u\}\right) + \text{ cone } (C+\{u-m\}) = L^2([0,T],\mathbb{R}^q), \tag{36}$$

and

$$\begin{aligned} \Phi \text{ cone } \left(L^2([0,T],\mathbb{R}^q)-\{u\}\right) &= \Phi\left(L^2([0,T],\mathbb{R}^q) - \text{ cone } \{u\}\right) \\ &= \Phi\, L^2([0,T],\mathbb{R}^q) = L^2([0,T],\mathbb{R}^l). \end{aligned} \tag{37}$$

Then, the other assumption of Theorem 5.3 of [14] being fulfilled, (33) holds, and, in virtue of the linearity of $\mathcal{L}(v,\bar{l}_1,\bar{l}_2,\overline{m})$ with respect to v, it follows that u is a minimal point for $\mathcal{L}$, namely

$$0 = \mathcal{L}(u,\bar{l}_1,\bar{l}_2,\overline{m}) \leq \mathcal{L}(v,\bar{l}_1,\bar{l}_2,\overline{m}), \quad \forall v \in L^2([0,T]).$$

So the right-hand part of (29) is proved. Now, taking into account that $\mathcal{L}(u,l_1,l_2,m)$ is reduced to

$$\mathcal{L}(u,l_1,l_2,m) = -\int_0^T \langle l_1(t), \lambda(t)-u(t)\rangle\, dt + \int_0^T \langle l_2(t), u(t)-\mu(t)\rangle\, dt \leq 0 \tag{38}$$

for each l_1, $l_2 \in C$, $\forall m \in L^2([0,T],\mathbb{R}^l)$, our result is achieved. ■

Some interesting consequences can be derived from Theorem 7 and from the estimate (29). The first consequence concerns the meaning of the Lagrangean variables. Taking into account that

$$\int_0^T \langle \bar{l}_1(t), u(t) \rangle \, dt = \int_0^T \langle \bar{l}_1(t), \lambda(t) \rangle \, dt = 0,$$

$$\int_0^T \langle \bar{l}_2(t), u(t) \rangle \, dt = \int_0^T \langle \bar{l}_2(t), \mu(t) \rangle \, dt = 0,$$

and that $\Phi\, u(t) = \rho(t)$ form the right-hand part of (29), we get:

$$\int_0^T \langle F(t, u(t)), v(t) - u(t) \rangle \, dt - \int_0^T \langle \bar{l}_1(t), v(t) - u(t) \rangle \, dt$$
$$+ \int_0^T \langle \bar{l}_2(t), v(t) - u(t) \rangle \, dt + \int_0^T \langle \overline{m}, \Phi(v - u) \rangle \, dt \geq 0,$$
$$\forall v \in L^2([0, T], \mathbb{R}^q)$$

and hence

$$F(t, u(t)) - \bar{l}_1(t) + \bar{l}_2(t) + \Phi^T \, \overline{m}(t) = 0. \tag{39}$$

It is possible to derive from (30) and (39) interesting information about the meaning of the Lagrangean variables $\bar{l}_1$, $\bar{l}_2$, and $\overline{m}$. In fact, taking into account that (30) can be rewritten as

$$\bar{l}_1^i(t) \, (u_i(t) - \lambda_i(t)) = 0, \quad \bar{l}_2^i(t) \, (u_i(t) - m_i(t)) = 0 \ \text{ a.e. in } [0, T],$$

it follows that when $\bar{l}_i^i(t) > 0$, then $u_i(t) = \lambda_i(t)$, namely the variables $\bar{l}_i^i(t)$ give information about the point for which the vector attains the minimal value; a similar remark holds also for $\bar{l}_2^i(t)$.

Moreover, from (39) we deduce that $\overline{m}$ gives information about the equilibrium value of the functional $F - \bar{l}_i + \bar{l}_2$, which represents a generalized "cost" functional. Many other consequences about the meaning of the Lagrangean variables could be derived (we refer for this to [3–5, 7, 9, 16]).

Another group of consequences concerns the duality theory. In fact, from estimate (29), we immediately deduce that the so-called duality gap cannot arise, namely that it results

$$\max_{\substack{l_1,\, l_2 \in C \\ m \in L^2([0,T],\mathbb{R}^l)}} \inf_{v \in L^2([0,T],\mathbb{R}^q)} \mathcal{L}(v, l_1, l_2, m)$$
$$= \min_{v \in L^2([0,T],\mathbb{R}^q)} \sup_{\substack{l_1,\, l_2 \in C \\ m \in L^2([0,T],\mathbb{R}^l)}} \mathcal{L}(v, l_1, l_2, m) = \mathcal{L}(u, \bar{l}_1, \bar{l}_2, \overline{m}). \tag{40}$$

Moreover, taking into account (38) and (30), we can introduce a Dual Variational Inequality in the following way:

$$\text{find } \left(u, \bar{l}_1, \bar{l}_2, \overline{m}\right) \in L^2([0,T], \mathbb{R}^q) \times \tilde{\mathbb{K}}:$$

$$\int_0^T \langle \lambda(t) - u(t), l_1(t) - \bar{l}_1(t) \rangle \, dt + \int_0^T \langle u(t) - \mu(t), l_2(t) - \bar{l}_2(t) \rangle \, dt \geq 0$$

$$\forall (u, l_1, l_2, m) \in \tilde{\mathbb{K}}, \tag{41}$$

where

$$\tilde{\mathbb{K}} = \Big\{ (u, l_1, l_2, m) \in \left(L^2([0,T], \mathbb{R}^q)\right)^3 \times L^2([0,T], \mathbb{R}^l) :$$

$$l_1(t),\ l_2(t) \geq 0, \text{ a.e. in } [0,T];\ \Phi\, u(t) - \rho(t) = 0, \text{ a.e. in } [0,T];$$

$$F(t, u(t)) - l_1(t) + l_2(t) + \Phi^T m(t) = 0, \text{ a.e. in } [0,T] \Big\}. \tag{42}$$

So the Dual Variational Inequality associated with our problem is a Quasi-Variational Inequality.

5 Conclusion

We conclude the current paper remarking that the time-dependent theory of equilibrium problems have received from the related Variational Inequality formulation a very fruitful setting and that Variational Inequalities seem to be the key to solve some of the principal challenges of our time. In fact, they allow us to manage the market and financial equilibria, following their evolution in time and achieving a light on the next future.

New future research directions deal with the study of evolutionary equilibria by means of projected dynamic systems theory, the introduction to the elastic model for which the data depend also on the expected equilibrium solutions.

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Differential Games of Multiple Agents and Geometric Structures

Panos M. Pardalos, Vitaliy A. Yatsenko, Altannar Chinchuluun, and Artyom G. Nahapetyan

Center for Applied Optimization, Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida, USA
{pardalos,yatsenko,altannar,artyom}@ufl.edu

Abstract This chapter deals with problems of differential games of multiple agents moving in a region. We describe such a game by a hierarchical structure, which can be simplified using a fiber bundle. Then, using geometric techniques, we study controllability, observability, and optimality problems. In addition, we also consider a cooperative problem when the agent's motions must satisfy a separation constraint throughout the encounter to be conflict-free. A classification of maneuvers based on different commutative diagrams is introduced using their fiber bundles representation. In the case of two agents, these optimality conditions allow us to construct the optimal maneuvers geometrically.

Key words: cooperative game, differential games, multiple agents, hierarchical structure, Yang–Mills field, controllability, observability

1 Introduction

The modern game theory basically deals with dynamical systems on smooth manifolds. However, many practical systems like multiple agents do not have such structures. The axiomatic control theories should adequately reflect in terms of their internal language of notions and control problems (Cressman, 2003 [5]). In terms of these theories, the control structures can make up various hierarchies. According to Kalman, for example, the most general structure is represented by a controllability-reachability structure over which the optimal control structure is built. This approach regarding the structure of optimal control and Yang–Mills Fields was discussed in (Yatsenko, 1985 [27]; Butkovskiy, 1990 [4]).

In this chapter, the geometric description problem of multiple agents is studied. We discuss mathematical aspects of the "Unified game (UGT)" and "Theory of the control structures (TCS)." We consider a game as a hierarchical structure. It is assumed that each agent can be described by a fiber bundle.

A joint maneuver has to be chosen to guide each agent from its starting position to its target position while avoiding conflicts. Among all the conflict-free joint maneuvers, we aim to determine the one with the least overall cost. The cost of an agent's maneuver is its energy, and the overall cost is a weighted sum of the maneuver energies of all individual agents, where the weights represent priorities of the agents.

As an example, we consider the hierarchical structure of such multiagent system on Figure 1. Each agent of the system can be described by stochastic or deterministic differential equation with a control. In the paper, we first reduce the model to a hierarchical geometric representation using fiber bundles. Then we consider an integrated geometric model where the separated model of agents is integrated into single model. For example, the interaction between six robots on Figure 2 can be described by a hierarchical structure. The integrated model allows solving controllability, observability, and cooperative control problems.

In Section 2, we consider geometric aspects of the nonlinear control systems. The section constructs a formal model, where the optimal control structure appears independently from the controllability-reachability structure and that of the space of local system states. The efficiency of this

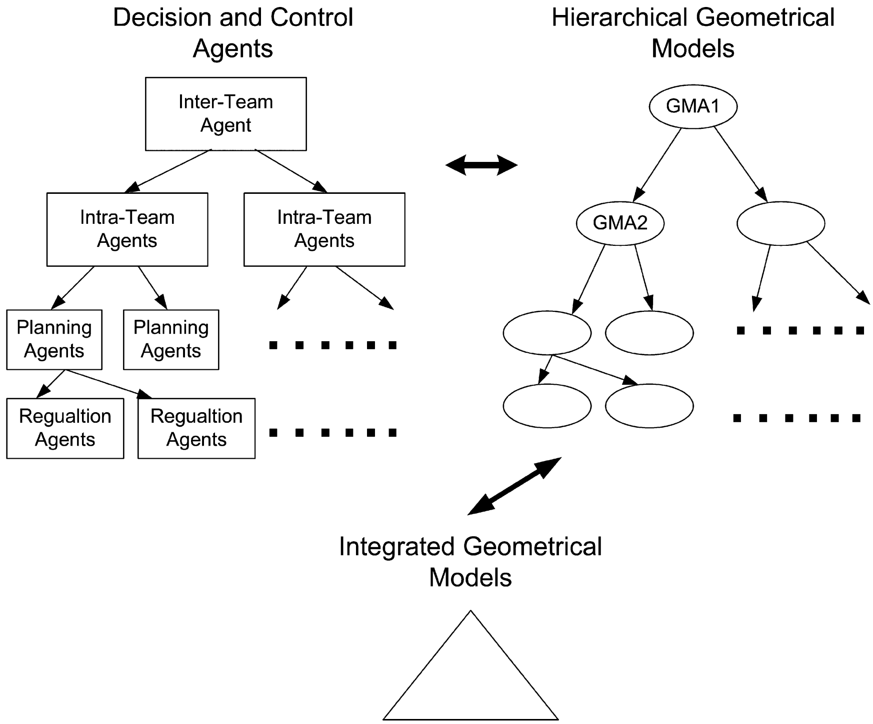

Figure 1. Hierarchical structure of multiple agents

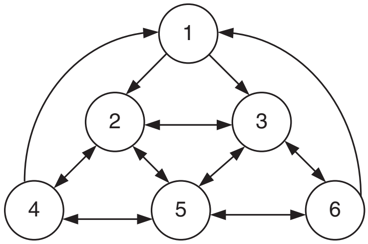

Figure 2. Hierarchical structure of multiple robot

axiomatic approach is illustrated using structural analysis of a general problem of the optimal control. In Section 3, we analyze in detail the relationship between gauge fields, identification problems, and control systems. The result of the analysis is an estimation algebra of a nonlinear estimation problem. The estimation algebra turns out to be a useful concept to explore finite-dimensional nonlinear filters. In Section 4, we consider a Lie group related to Yang–Mills gauge groups. We show that the estimation algebra of the identification problem is a subalgebra of the current algebra. Section 5 focuses on nonlinear control systems and Yang–Mills fields. Section 6 is devoted to geometric models of multiagent systems as controlled dynamical-information objects. It is shown that these systems can be described by commutative diagrams, which allow one to analyze a symmetry.

2 Geometric Structures

We briefly describe the role of topological, metric, and orderness structures. Note that each standard ordinary differential control system or inclusion $x' \in I(x)$, $x \in X$, generates two independent topological structures on X. One of them is generated by a family of inclusions of $x \in X$, i.e., the family of reachability sets $O(x_0, \varepsilon)$ from x_0 for time $\varepsilon \geq 0$, and another one by a family of a controllability area $O(x_0, \varepsilon)$ to x_0 for time $\varepsilon \geq 0$ (observability topology).

Let (X, τ) be a topological space, where X is an abstract nonempty set and τ is a topology on X.

Definition 1. *Control (or admissible control) $\gamma(a, b)$ in (X, τ) is an image of the continuous (in sense of topology τ) map φ: $[0, 1] \to X$,*

$$x = \varphi(t), \quad 0 \leq t \leq T, \quad x \in X, \tag{1}$$

$$a = \varphi(0), \quad b = \varphi(T). \tag{2}$$

$a \in X$ is an initial point and $b \in X$ is a final point of the control $\gamma(a, b)$.

Thus, the control $\gamma(a, b)$ is pathwise connected and linearly ordered subset (sequence) of X where $a \in \gamma(a, b)$ and $b \in \gamma(a, b)$ are the smallest and the largest of its elements, respectively. As results, maps (1) and (2) are admissible parameterizations of the control $\gamma(a, b)$.

The verification of Definition 1 consists of a validation of the controllability and finding optimal control of systems without using any differential or difference structure. Furthermore, we shall consider that there is a metric in topological spaces, which allows one to analyze control problems at various levels of generality. We shall be looking for the "minimal" but not trivial structures, which can be responsible for controls.

2.1 Metric Spaces

The concepts of a metric and a metric space are introduced by the following definitions.

Definition 2. *Metric space (X, ρ) is a pair (X, ρ) where X is an arbitrary nonempty set and p is a metric structure of X, i.e., ρ is a real valued function $\rho = \rho(x, y)$, $(x, y) \in X^2 = X \times X$, or map*

$$\rho : X^2 \to \mathbb{R} \tag{3}$$

with the metric axioms:

$$\rho(a, b) \geq 0 \quad for \ \ \forall \ (a, b) \in X^2, \tag{4}$$

$$\rho(a, a) = 0 \quad for \ \ \forall \ a \in X, \tag{5}$$

$$\rho(a, b) < \rho(a, c) + \rho(c, b) \tag{6}$$

for any $a \in X$, $b \in X$, $c \in X$, and (6) is called "triangle inequality." Sometimes ρ is also called a global metric on X or distance in X.

The metric introduced by Definition 2 differs from the usual concept of metric: there is neither the symmetry axiom ($\rho(a, b) = \rho(b, a)$ for $\forall \ a \in X$, $\forall \ b \in X$) nor the requirement: $\rho(a, b) > 0$ if $a \neq b$. So, the given concept of metric is more adequate to the situation in typical control problems. As is known, the metric space (X, ρ) can also be considered as a topological space (X, τ), where topology T is induced by metric ρ.

But the metric can measure control $\gamma(a, b)$ introduced by Definition 1. This can be done by the following definition.

Definition 3. *The length $l[\gamma(a, b)]$ of the control $\gamma(a, b)$ is a real valued function*

$$l\left[\gamma(a, b)\right] = \lim_{N \to \infty} l_N \left[\gamma(a, b)\right], \tag{7}$$

where

$$l_N \left[\gamma(a, b)\right] = \sum_{i=0}^{N} \rho(x_i, x_{i+1}), \tag{8}$$

where $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ *is the N-th partition* T_N *of* $\gamma(a,b)$*, and the partition* T_N *becomes finer with* $N \to \infty$*. Of course, it is necessary to prove or admit the existence and uniqueness of (7). If so,* $\gamma(a,b)$ *is called measurable (in metric* ρ*). The set* $\gamma(a,b)$ *of all measurable* $\gamma(a,b)$ *is denoted by* $\Gamma(a,b)$*:*

$$\Gamma(a,b) = \{\gamma(a,b)\}. \tag{9}$$

So, in the metric space (X,ρ) the *admissible control* $\gamma(a,b)$ is just a measurable (in sense of metric ρ) sequence and vice versa.

If we have several sequences in X:

$$\gamma_1(x_0,x_1), \gamma_2(x_1,x_2), \ldots, \gamma_n(x_n,x_{n+1}) \tag{10}$$

then we can define their sum

$$\gamma(x_0,x_{n+1}) = \sum_{i=1}^{n} \gamma_i(x_{i-1},x_i), \tag{11}$$

which is also a sequence.

Inversely, if $\gamma(a,b)$ is a sequence and $x_i \in \gamma(a,b)$, $i = 1,\ldots,n$, $x_1 < \cdots < x_n$, then $\gamma(a,b)$ can be represented as the sum of sequences:

$$\gamma(a,b) = \gamma_1(a,x_1) + \gamma_2(x_1,x_2) + \cdots + \gamma_n(x_n,b). \tag{12}$$

Definition 4. *The sequence* $\gamma_i(x_{i-1},x_i)$ *in (11) is called a piece of the sequence* $\gamma(a,b)$*. We accept that functional (7) is additive one:*

$$\rho(a,b) \geq 0 \quad \textit{for} \;\; \forall\, (a,b) \in X^2, \tag{13}$$

$$\rho(a,a) = 0 \quad \textit{for} \;\; \forall\, a \in X. \tag{14}$$

2.2 Optimal Control

Consider the following problem of optimal control in (X,ρ).

1. Determine
$$\bar{l}(a,b) = \{\inf l[\gamma(a,b)] : \; \gamma(a,b) \in \Gamma(a,b)\}. \tag{15}$$
2. Determine $\bar{\gamma} = \bar{\gamma}(a,b)$, if exists, such that
$$l[\bar{\gamma}(a,b)] = \bar{l}(a,b). \tag{16}$$
This admissible $\bar{\gamma}(a,b)$ will be called the minimal of the optimal control problem.
3. Describe all set $\{\bar{\gamma}(a,b)\}$ for fixed $(a,b) \in X^2$ and for all $(a,b) \in X^2$.

A simple but an important property of the minimal $\bar{\gamma}(a,b)$ is given by the following theorem.

Theorem 1. *If the admissible $\gamma(c,d)$ is the minimal of the optimal control problem, then the sequence $\bar{\gamma}(a,b)$ is also minimal.*

This is a consequence of the additivity property of (12). If any admissible sequence $\gamma(a,b)$ is minimal, it does not mean that $\gamma(a,b)$ is also minimal.

It is easy to prove the inequality

$$\rho(a,b) \leq \bar{l}(a,b). \tag{17}$$

Definition 5. *The metric space (X,ρ) is an obstacleness metric space if there exists at least one point $(a,b) \in X^2$ such that*

$$\rho(a,b) \leq \bar{l}(a,b).$$

The metric space (X,ρ) is a generalized metric space iff

$$\rho(a,b) = \bar{l}(a,b) \quad \textit{for} \ \ \forall\ (a,b) \in X^2.$$

An example of a generalized space is the Euclidean space $\mathbb{R}^n$.

The following theorem is valid:

Theorem 2. *$\bar{l} = \bar{l}(a,b)$ is also metric on X, and $(X,\bar{l})$ is also a metric space.*

Definition 6. *Metric $\bar{l} = \bar{l}(a,b)$ is called a secondary metric.*

Generally, $\bar{l}(a,b)$ is distinguished from the initial metric $\rho(a,b)$ on X.

Definition 7. *If the secondary metric $\bar{l}$ coincides with the metric ρ, then ρ is called a self-secondary metric.*

The following theorems are valid:

Theorem 3. *The secondary metric is a self-secondary metric.*

This is similar to the property of projection operator $P : P^2 = P$.

Theorem 4. *The metric space (X,ρ) is a generalized space if the metric ρ is the self-secondary metric.*

We illustrate the application of the above introduced concepts by the following:

Theorem 5. *(Sufficient condition for minimal). The sequence $\gamma(a,b)$ in (X,ρ) is minimal if for any of its admissible $\gamma(c,d)$, the next relation is true:*

$$\bar{l}(c,d) = \bar{l}(c,x) + \bar{l}(x,d) \quad \textit{for} \ \ \forall\ x \in \gamma(c,d), \tag{18}$$

where $\bar{l}$ is a secondary metric of ρ.

It might seem that for $\gamma(a,b)$ to be minimal, just one identity is sufficient:

$$\bar{l}(a,b) = \bar{l}(a,x) + \bar{l}(x,b) \quad \text{for} \ \ \forall\ x \in \gamma(a,b). \tag{19}$$

But it is not true, there exists a contrary example.

From topology standpoint, the secondary metric $\bar{l}$ generally is weaker (rougher) than the "initial" or "first" metric ρ. In other words, topology (X,ρ) is stronger (thinner) than secondary topology $(X,\bar{l})$.

3 Identification of Agents and Yang–Mills Fields

In this section, we consider the models where each agent of the hierarchical system is described by a stochastic differential equation.

3.1 Stochastic Agents

Consider the stochastic differential system:

$$d\theta = 0, \tag{20}$$

$$dx_t = A(\theta)x_t dt + b(\theta)dw_t, \tag{21}$$

$$dy_t = \langle(\theta), x_t\rangle dt + dv_t. \tag{22}$$

Here $\{w_t\}$ and $\{v_t\}$ are independent, scalar, and standard Wiener processes, and $\{x_t\}$ is an $\mathbb{R}^n$-valued process. Assume that θ takes values in a smooth manifold $\Theta \to \mathbb{R}^N$, and the map $\theta \to \Sigma(\theta) := (A(\theta), b(\theta), c(\theta))$ in a smooth map taking values in minimal triples. By the identification problem we shall mean the nonlinear filtering problem associated with equation (21); i.e., the problem of recursively computing conditional expectations of the form $\pi_t(\phi) \,\underline{\Delta}\, E[\phi(x_t, \theta)|Y_t]$, where Y_t is the σ-algebra generated by the observations $\{y_s : 0 \leq s \leq t\}$ and ϕ belongs to a suitable class of functions on $\mathbb{R}^n \times \Theta$.

For given y_t, the joint unnormalized conditional density $\rho \,\underline{\Delta}\, \rho(t, x, \theta)$ of x_t and θ satisfy the stochastic partial differential Stratonovitch equation

$$d\rho = A_0 \rho dt + B_0 \rho dy_t, \tag{23}$$

where the operators A_0 and B_0 are given by

$$A_0 := \frac{1}{2}\left\langle b(\theta), \frac{\partial}{\partial x}^2\right\rangle - \left\langle \frac{\partial}{\partial x}, A(\theta)x\right\rangle - \langle c(\theta), x\rangle^2/2, \tag{24}$$

$$B_0 := \langle c(\theta), x\rangle. \tag{25}$$

From the Bayes formula, it follows that

$$\pi_t(\phi) = \sigma_t(\phi)/\sigma_t(l), \tag{26}$$

where

$$\sigma_t(\phi) = \int_\Theta \int_{\mathbb{R}^n} \phi(x, \theta)\rho(t, x, \theta)|dx||d\theta|, \tag{27}$$

where $|dx|$ and $|d\theta|$ are fixed volume elements on $\mathbb{R}^n$ and Θ, respectively. Further, if $Q(t, \theta)$ denotes the unnormalized posterior density of θ given t, then it satisfies the equation:

$$dQ = E[\langle c(\theta), x_t|\theta\rangle, Y_t]Q(t, \theta)dy_t. \tag{28}$$

The paper on nonlinear filtering theory (Hazewinkel, 1982 [10]) shows that it is natural to look at equation (23) formally as a deterministic partial differential equation,

$$\frac{\partial \rho}{\partial t} = A_0\rho + \dot{y}B_0\rho. \tag{29}$$

By the Lie algebra of the identification problem, we shall mean the operator Lie algebra $\bar{\mathrm{G}}$ generated by A_0 and B_0. For more general nonlinear filtering problems, estimation algebras analogous to $\bar{\mathrm{G}}$ have been emphasized by Brockett (Mitter, 1990 [19]) and others as being objects of central interest. In the papers (Krishnaprasad and Marcus, 1981 [14]), the Lie algebra $\bar{\mathrm{G}}$ is used to classify identification problems and to understand the role of certain sufficient statistics.

3.2 The Estimation Algebra of Nonlinear Filtering Systems

To understand the structure of the estimation algebra, it is well-worth considering an example.

Example 1. Let $dx_t = \theta dw_t$; $d\theta = 0$; $dy_t = x_t dt + dv_t$. Then $A_0 = \frac{\theta^2}{2}\frac{\partial^2}{\partial t^2} - \frac{x^2}{2}$ and $B_0 = x$, and $\bar{\mathrm{G}} = \{A_0, B_0\}_{I..A}$ is spanned by the set of operators $\left(\frac{\theta^2}{2} - \frac{x^2}{2}\right)$, $\left(\theta^{2n}x\right)_{n=0}^{\infty}$, $\left(\theta^{2n}\frac{\partial}{\partial x}\right)_{n=1}^{\infty}$ and $\{\theta^{2n}1\}_{n=1}^{\infty}$. We then notice that,

$$\tilde{G} \subseteq \mathbb{R}\left[\theta^2\right] \otimes \left\{\frac{\partial^2}{\partial x^2}, x\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, x^2, x, 1\right\} L.A.$$

is a subalgebra of the Lie algebra obtained by tensoring the polynomial ring $\mathbb{R}\left[\theta^2\right]$ with a 6-dimensional Lie algebra. Here, L.A. stands for the Lie algebra generated by the elements in the brackets.

The general situation is very much as in this example. Consider the vector space (over the reals) of operators spanned by the set,

$$S := \left\{\frac{\partial^2}{\partial x_i \partial x_j}, x_i\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}, x_i x_i, x_j, 1\right\}, \\ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n. \tag{30}$$

This space of operators has the structure of a Lie algebra henceforth denoted as $\tilde{\mathrm{G}}_0$ (of dimension $3n^2 + 2n + l$) under operator commutation (the commutation rules being $\frac{\partial^2}{\partial x_i \partial x_j}$, $x_k = \delta jk\frac{\partial}{\partial x_i} + \delta ik\frac{\partial}{\partial x_j}$ etc., where δjk denotes the Kronecker symbol). For each choice Θ, A_0 and B_0 take values in $\tilde{\mathrm{G}}_0$. It follows that in general A_0 and B_0 are smooth maps from Θ into $\tilde{\mathrm{G}}_0$. Thus, let us consider the space of smooth maps $C^{\infty}(\Theta; \tilde{\mathrm{G}}_0)$. This space can be given by the structure of a Lie algebra (over the reals) in the following way:

$$\text{given } \varphi,\ \phi\varepsilon C^{\infty}(\Theta; \tilde{\mathrm{G}}_0),$$

define the Lie bracket $[\cdot,\cdot]_C$ on $C^\infty(\Theta;\tilde{\mathrm{G}}_0)$ by

$$[\phi,\psi]_C(P) = [\phi(P),\psi(P)] \quad \text{for every} \quad P\in\Theta. \tag{31}$$

Here the bracket on the right-hand side of equation (31) is in $\tilde{\mathrm{G}}_0$. We denote as $\tilde{\mathrm{G}}_0$ the Lie algebra $(C^\infty(\Theta;\tilde{\mathrm{G}}_0);[.,.]_C)$. Whenever the dimension of Θ is greater than zero, $\tilde{\mathrm{G}}_0$ is infinite dimensional and is an example of a *current algebra*. Current algebras play a fundamental role in the physics of Yang–Mills fields where they occur as Lie algebras of gauge transformations. Elsewhere in mathematics they are studied under the guise of local Lie algebras. The following is immediate.

Proposition 1. *The Lie algebra* $\tilde{\mathrm{G}}$ *of operators generated by*

$$A_0 := \frac{1}{2}\left\langle b(\theta),\frac{\partial}{\partial x}\right\rangle^2 - \left\langle \frac{\partial}{\partial x}, A(\theta)x\right\rangle - \langle c(\theta),x\rangle^2/2 \tag{32}$$

and $B_0 := \langle c(\theta),x\rangle$, *is a subalgebra of the current algebra* $C^\infty(\Theta;\tilde{\mathrm{G}}_0)$.

3.3 Estimation Algebra and Identification Problem

It is known (Marcus, 1984 [17]) that $\tilde{\mathrm{G}}$ admits a faithful representation as a Lie algebra of vector fields on a finite dimensional manifold. Specifically, consider the system of equations,

$$\begin{aligned} d\theta &= 0,\\ dz &= \left[A(\theta) - Pc(\theta)c^T(\theta)\right] z dt + Pc(\theta)dy_t,\\ \frac{dP}{dt} &= A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta) - Pc(\theta)c^T(\theta)P,\\ ds &= \frac{1}{2}\langle c(\theta),z\rangle^2 dt - \langle c(\theta),z\rangle dy_t. \end{aligned} \tag{33}$$

The system of equations (33) evolves on the product manifold $\Theta\times\mathbb{R}^{n(n+3)/2+1}$. Associated with equations (33) there are the pair of vector fields (first-order differential operators),

$$\begin{aligned} a_0^* &= \langle (A(\theta) - Pc(\theta)c^T(\theta))z, \partial/\partial z\rangle\\ &+ \mathrm{tr}\,((A(\theta)P + PA^T(\theta) + b(\theta)b^T(\theta) - Pc(\theta)c^T(\theta)P), \partial/\partial P)\\ &+ 1/2\langle c(\theta),z\rangle^2\partial/\partial s \end{aligned}$$

and

$$b_0^* = \langle P(\theta),\partial/\partial z\rangle - \langle c(\theta),z\rangle\partial/\partial z.$$

Here $\partial/\partial P = [\partial/\partial P_{ij}] = (\partial/\partial P)^T = n\times n$ symmetric matrix of differential operators. Consider the Lie algebra of vector fields generated by a_0^* and b_0^*. Because a_0^* and b_0^* are vertical vector fields with respect to the fibering $\Theta\times\mathbb{R}^{n(n+3)/2+1}\to\Theta$, then every vector field is in this Lie algebra. One of the main results is the following (Lee and Marcus, 1980 [15]):

Theorem 6. *The map*

$$\Phi_k : \tilde{\mathrm{G}}_0 \to \bigcup \Theta \times \mathbb{R}^{n(n+3)/2+1}$$

defined by

$$b_0^* = \langle P(\theta), \partial/\partial z \rangle 1/2 \langle c(\theta), z \rangle \partial/\partial s$$

is a faithful representation of the Lie algebra of the identification problem as a Lie algebra of (vertical) vector fields on a finite dimensional manifold fibered over Θ.

Example 2. To illustrate Theorem 5, consider the Lie algebra of Example 1. The embedding equations (33) take the form

$$\begin{aligned} d\theta &= 0, \\ dp &= \left(\theta^2 - p^2\right) dt, \\ dz &= -pzdt + pdy_t, \\ ds &= z^2/2dt - zdy_t. \end{aligned}$$

Then

$$\Phi_k(B_0) = \Phi_k(x) = b_0^* = p\frac{\partial}{\partial z} + (-z)\frac{\partial}{\partial s}.$$

The induced maps on Lie brackets are given by

$$\begin{aligned} \Phi_k(\theta^{2k}\partial/\partial z) &= \theta^{2k}\partial/\partial z, \quad k = 0, 1, 2, \dots, \\ \Phi_k(\theta^{2k}x) &= \theta^{2k}(p\partial/\partial z - z\partial/\partial s), \quad k = 1, 2, \dots, \\ \Phi_k(\theta^{2k}l) &= \theta^{2k}\partial/\partial s, \quad k = 1, 2, \dots. \end{aligned}$$

The embedding equations have the following statistical interpretation. Assume that the initial condition for (12) is of the form

$$\begin{aligned} \rho_0(x, \theta) &= \left(2\pi \det \sum(\theta)\right)^{-n/2} \\ &\times \exp\left(-\left\langle x - \mu(\theta), \sum^{-1}(\theta)(x - \mu(\theta))\right\rangle\right) \cdot Q_\theta, \end{aligned}$$

where $\theta \to (\mu(\theta), \Sigma(\theta), Q_0(\theta))$ is a smooth map, $\sum(\theta) > 0$, $\theta \in \Theta$ and $Q_0 > 0$ for $\theta \in \Theta$. Suppose equation (11) is initialized at,

$$(\theta_0, z_0, P_0, s_0) = \left(\theta_0, \mu(\theta_0), \sum(\theta_0), -\log(Q_0(\theta))\right) \tag{34}$$

Append to the system (11) an output equation,

$$\bar{Q}_t = e^{-s_t}. \tag{35}$$

Now if (33) is solved with initial condition (34), one can show by differentiating $\bar{Q}_t$ that $\bar{Q}_t$ satisfies the equation (7). In other words, the system (31)–(35) with initial condition (14) is a finite dimensional recursive estimation for the posterior density $Q(t, \theta_0)$. We have thus verified the homomorphism principle of Brockett (Brockett, 1979 [3]): that finite dimensional recursive estimators must involve Lie algebras of vector fields that are homomorphic images of the Lie algebra of operators associated with the unnormalized conditional density equation.

4 Sobolev Lie Group and Yang–Mills Fields

It has been remarked elsewhere that the Cauchy problem associated with (29) may be viewed as a problem of integrating a Lie algebra representation. In this connection, one should be interested whether there is an appropriate topological group associated with $\tilde{G}$. We have the following general procedure.

Let M be a compact Riemannian manifold of dimension d. Let L be a Lie algebra of dimension $n < \infty$. We can always view L as a subalgebra of the general linear Lie algebra $g\ell(m;\mathbb{R}), m > n$ (Ado's theorem).

Assumption 1 *Let $G = \{\exp(L)\}_G \subset g\ell(m;\mathbb{R})$ be the smallest Lie group containing the exponentials of elements of L. We assume that G is a closed subset of $g\ell(m;\mathbb{R})$.*

Define,

$$\begin{aligned}\mathcal{R} &= \mathrm{C}^\infty(\mathrm{M}; g\ell(\mathrm{m};\mathbb{R})),\\ \mathcal{L} &= \mathrm{C}^\infty(\mathrm{M};\mathrm{L}),\\ \mathcal{D} &= \mathrm{C}^\infty(\mathrm{M};\mathrm{G}).\end{aligned}$$

Clearly $\mathcal{R}$ is an algebra under pointwise multiplication and

$$\mathcal{L} \subset \mathcal{R}, \quad \mathcal{D} \subset \mathcal{R}.$$

Let $(U\alpha, \varphi_\alpha)$ be a C^∞ atlas for M. Then for a $f_1, f_2 \in \mathcal{R}$, define

$$\| f_1 - f_2 \| = \left[\int_{\varphi_\alpha(U_\alpha)} \mathrm{dvol} \sum_{\ell=0}^{k} \left| D^\ell (f_1 - f_2) \varphi_\alpha^{-1} \right|^2 \right]^{1/2}, \tag{36}$$

where

$$|f|^2 = \mathrm{tr}\,(f' f). \tag{37}$$

(Here $k = d/2 + s$, $s > 0$). Let $\mathcal{R}_k$ be the completion of $\mathcal{R}$ and $\mathcal{D}_k$, the completion of $\mathcal{D}$ in the norm $\| \cdots \|_k$ ($\mathcal{D}_k$ is closed in $\mathcal{R}_k$). By the Sobolev theorem, $\mathcal{R}_k$ is a Banach algebra and the group operation

$$\begin{aligned}\mathcal{D}_k \times \mathcal{D}_k &\to \mathcal{D}_k, \\ (f_1, f_2) &\to f_1 f_2\end{aligned} \tag{38}$$

when $(f_1 f_2)(m) = f_1(m) f_2(m)$ is continuous. Thus $\mathcal{D}_k$, is a topological group.

By proceeding as before, one can give a Sobolev completion of $\mathcal{L}$ to obtain $\mathcal{L}_k$, an infinite dimensional Lie algebra, where once again by the Sobolev theorem the bracket operation

$$\begin{aligned}[.,.]\mathcal{L}_{\mathrm{k}} \times C_{\mathrm{k}} &\to \mathcal{L}_{\mathrm{k}}, \\ (f_1, f_2) &\to [f_1, f_2]\end{aligned}$$

with $[f_1, f_2](m) = [f_1(m), f_2(m)]$ is continuous. Now, for a small enough neighborhood $V(0)$ of $0 \in \mathcal{L}$, one can define

$$\begin{aligned}\exp : V(0) &\to \mathcal{D}_{\mathrm{k}}, \\ \xi &\to \exp(\xi)\end{aligned}$$

by pointwise exponentiation. This permits us to provide a Lie group structure on $\mathcal{D}_{\mathrm{k}}$ with $\mathcal{L}_{\mathrm{k}}$ canonically identified as the Lie algebra of $\mathcal{D}_{\mathrm{k}}$.

The procedure outlined above appears to play a significant role in several contexts (the index theorem Yang–Mills fields (Milter, 1980, 1981 [19, 20])).

For our purposes, $\mathcal{L}$ will be identified with a faithful matrix representation of $\tilde{G}_0$. Thus we associate with the identification problem a Sobolev Lie group, which is a subgroup of $\mathcal{D}_{\mathrm{k}}$ corresponding with $\tilde{G}_0$

Remark 1. One of the important differences between the problem of filtering and the problems of Yang–Mills theories is that in the latter case there are natural norms for Sobolev completion. This follows from the fact that in Yang–Mills theories, the algebra $\mathcal{L}$ is compact (semisimple) and one has the Killing form to work with. In filtering problems, $\tilde{G}_0$ is never compact.

We use a representation of the form

$$\rho(t, x, \theta) = \exp(g_1(t, \theta)\mathrm{A}^1) \ldots \exp(g_n(t, \theta) A^n) \rho_0 \tag{39}$$

for the solution to the equation (8). In the case of Example 1, this takes the form

$$\begin{aligned}\rho(t, x\theta) = \exp&\left(g_1(t, \theta)\left(\frac{\theta^2}{2}\frac{\theta^2}{\theta_x} - \frac{x^2}{2}\right)\right)\exp\left(g_2(t, \theta)\theta^2 \frac{\partial}{\partial x}\right) \\ &\times \exp\left(g_3(t, \theta)x\right)\exp\left(g_4(t, \theta)l\right)\rho_0.\end{aligned}$$

Differentiating and substituting in (29), we can obtain

$$
\begin{aligned}
\frac{\partial g}{\partial t}(t,\theta) &= 1, \\
\frac{\partial g_2}{\partial t}(t,\theta) &= \cosh(g_1,\theta)\dot{y}, \\
\frac{\partial g_3}{\partial t} &= -\frac{1}{\theta}\sinh(g_1,\theta)\dot{y}, \\
\frac{\partial g_4}{\partial t} &= \frac{\partial q_3}{\partial t}(t,\theta)g_2(t,\theta)
\end{aligned} \tag{40}
$$

and $g_i(0,\theta) = 0$ for $i = 1,2,3,4$, $\theta \in \Theta$. The above first-order partial differential equations may be easily solved by quadrature and one has the representation

$$
\begin{aligned}
\rho(t,x,\theta) = \int_{-\infty}^{\infty} &\sqrt{\frac{1}{2\pi\sinh(|\theta|t)}}\exp\left(-\frac{1}{2}\coth^2\left(\frac{|x|}{|\theta|}^2 + z\right)t|\theta|\right) \\
&\times \exp\left(\frac{xz}{\sqrt{|\theta|\sinh(|\theta|t)}}\right)\exp\left(g_4\left(t,\theta\right)\theta^2\right) \\
&\times \exp\left(g_2(t,\theta)\sqrt{|\theta|z}\right)\rho_0\left(g_3(t,\theta)\theta^2\sqrt{|\theta|z},\theta\right)dz,
\end{aligned} \tag{41}
$$

where $\rho_0(,\theta) \in L_2(\mathbb{R})$ for every $\theta \in \Theta$ and is smooth in θ. Further, $\Theta\mathbb{R}$ is a bounded set and 0 closure Θ.

In equation (39), g_1 should be viewed as canonical coordinates of the second kind on the corresponding Sobolev Lie group. Now expand g_2 and g_3 to obtain

$$
\begin{aligned}
g_2(t,\theta) &= \sum_{k=0}^{\infty}\theta^2\int_0^t \frac{\sigma^{2k}}{(2k)!}\dot{y}_\sigma d\sigma, \quad k = 1,2,\ldots, \\
g_3(t,\theta) &= -\sum_{k=0}^{\infty}\theta^{2k}\int_0^t \frac{\sigma^{2k+1}}{(2k+1)!}\dot{y}_\sigma d\sigma, \quad k = 1,2,\ldots.
\end{aligned} \tag{42}
$$

It follows that all the "information" contained by the observations $\{y_\sigma : 0 \leq \sigma \leq t\}$ about the joint unnormalized conditional density is contained in the sequence

$$
\mathrm{T}\Delta\left\{\int \frac{\sigma^k}{k!}\dot{y}_\sigma d\sigma; \quad k = 0,1,2,\ldots\right\}. \tag{43}
$$

Thus T is nothing but a joint sufficient statistic for the identification problem.

5 Control Agents and Yang–Mills Fields

Consider an object, the motion equation for which can be represented as

$$
\dot{x} = r(x,u), \tag{44}
$$

where $x = (x_1, x_2, x_3) \in Q \subset \mathbb{R}^3$; a function $r(x, u)$ is derived when an equation for dynamics of a particle in a field is reduced to Cauchy form, and the field is characterized by a variable u. The equations similar to (44) are widely used in physics and its applications. The equations of the concrete particle dynamics are considered in (Daniel and Viallet, 1980 [6]) and in many other papers. At present, control dynamics equation construction problems deserve a great attention. For instance, these problems include controllable models of dynamics of particles in scalar, vector, and spinor fields.

This section builds up a controllable model for dynamics of a particle in electromagnetic and charged fields. The model is based on the gauge field concept (Daniel, 1980 [6]), which allows us to formulate different principles for an automatic control of the dynamics of the particles.

Constructing a controllable model means creating a transformation from a field u to *Yang–Mills field*. The essence of this transition is as follows (Mitter, 1979 [18]). Instead of u, consider an n-component vector field $\widehat{f}(\widehat{x})$, $\widehat{x} \in T^1$ in a 4-dimensional space-time T^1. Let $M(\widehat{x})$ be local gauge transformations such that

$$\widehat{f}(\widehat{x}) = M(\widehat{x})\widehat{f}'(\widehat{x}) \tag{45}$$

and, for a fixed x, $M(x)$ form a group $G_1 \in GL(n)$. Introduce an operator ∇_α, i.e.,

$$\nabla_\alpha \widehat{f} = \Big[\partial_\alpha + K_\alpha(\widehat{x})\widehat{f}(\widehat{x})\Big], \tag{46}$$

which satisfies the conditions

$$M(\widehat{x})\nabla'_\alpha \widehat{f}'(\widehat{x}) = \nabla_\alpha \widehat{f}(\widehat{x}), \quad \nabla'_\alpha = \partial_\alpha + K'_\alpha, \tag{47}$$

where $K_\alpha = -Q_b C^b_\alpha$; $\{Q_b\}$ is a basis of Lie algebra $\widehat{g}$ for a group G_1, $[G_a, Q_b] = g^c_{ab} Q_c$; G^c_{ab} are structural constants of the Lie algebra $\widehat{g}$. The equations for the values C^b_α are derived from the Lagrangian $Y^a_{\alpha\beta} Y^{\alpha\beta}_a$, where

$$Y^a_{\alpha\beta} = \frac{\partial C^a_\beta}{\partial \widehat{x}^a} - \frac{\partial C^a_\alpha}{\partial \widehat{x}^\beta} - \frac{1}{2} g^a_{bc}\left(C^b_\alpha C^c_\beta - C^b_\beta C^c_\alpha\right), \tag{48}$$

and the Lagrangian has the following form:

$$\partial_\beta Y^{\alpha\beta} = Y^{\alpha\beta}_b g^b_{ac} C^c_\beta .$$

Relation (47) yields the law of transformation for a field of matrices K_α:

$$K'_\alpha(\widehat{x}) = M^{-1}(\widehat{x}) K_\alpha(\widehat{x}) M(\widehat{x}) + M(x)^{-1} \frac{\partial M(\widehat{x})}{\partial \widehat{x}^\alpha}.$$

Such transformation satisfies the group law g. A set of these transformations forms a gauge group, formally denoted as

$$\widetilde{g} = \prod_{x} g.$$

It is shown in (Yatsenko, 1985 [27]) that the values C_{α}^{b} are Yang–Mills fields. The Yang–Mills field describes a parallel transfer in a charge field and states its curvature. Such field can be brought in correspondence with the notion of connectedness in some main fiber bundle $(P, T^1, \widetilde{g})$, $\pi\colon P \to T^1$, where T^1 is a base and $\widetilde{g}$ is a structure group.

A control in $(P, T^1, \widetilde{g})$, $\pi\colon P \to T^1$ is understood as a connectedness C_{α}^{b}. Notice that one can consider a projection π as a control. Thus, it is possible to deal with a "controllable" fiber bundle $(P, T^1, \widetilde{g})$, $\pi\colon P \to T^1$ and a vector field $r(\widehat{x}, u(C_{\alpha}^{b}))$ on P instead of the initial object described by equation (44).

To solve control problems, it is necessary to construct equivalent and aggregated models. We construct an equivalent model of a controllable object $(P, T^1, \widetilde{g})$, $\pi\colon P \to T^1$ as follows. Let $\pi\colon P \to T^1$ be a main $\widetilde{g}$-fiber-bundle and let $l\colon Z \to T^1$ be some m-dimensional $\widetilde{g}$-vector fiber bundle with a trivial action, exerted by $\widetilde{g}$ onto Z. Assume also that a structure of a $k > 1$-dimensional cellular set can be introduced on T^1. An equivariant embedding of π into 1 is understood as an embedding $h\colon P \to Z$, commutating with projections. If $K > m$, i.e., an action, exerted by $\widetilde{h}$ onto Z is free outside a zero section for 1, then the main $\widetilde{g}$-fiber-bundle $\pi\colon P \to T^1$ can be equivariantly embedded into $l\colon Z \to T^1$. An equivalent model of a controlled process is understood as a ternary $(Z, T^1, \widetilde{g})$. In its turn, an equivalent model admits an exact aggregation, performed by means of a factorization of an induced *vector fiber bundle*. In this case, it is possible to assume that a vector fiber bundle is specified by an interrelation system ω on some set X^1. Introduce an equivalence relation S on X^1. This relation generates an object of the same nature, as the initial object X^1, and a factor-object (F-object) is obtained, which possesses a factorizing equivalence relation S. If (X_1, ω) generates an object of the same nature, possessed by an initial object, and this generation is carried out on a subset X_1 of X_1, then a subobject $X_1(\widetilde{\omega})$ (P-object) is derived. By using the language of mathematical structure theory, it is possible to create a general theory of aggregation of invariant models for nonlinear systems.

Consider the main automatic particle dynamics control principles, with electromechanical systems with distributed parameters as an example. It is shown in (Samoilenko, 1970 [21]) that a closed distributed automatic control system can be represented by two subsystems S_1 and S_2, interrelated by the electromagnetic field

$$S = S_1 \cup S_2.$$

Represent B of fields of a whole control system state by a field of internal states of each subsystem B_1 and B_2, of an interaction field B_0, and a by-side field B_3. In addition, represent B_0 by two components X and U, i.e., $B_0 = X + U$, where X is an information carrier and U is a control field. Consider U as the result of an influence exerted by X onto the control medium and simulated by an operator dependence $U = \widehat{B}(X, E)$, where $\widehat{B}$ is a control operator and E is the control medium power supply field. An external field B is also divided into two components, viz. into V and N, where V is a field of control that is carried out according to a fixed space-time program, and N is a field of disturbing effects. The control object is described by a fiber bundle $(P, T^1, \widetilde{g})$ with a control $C_\alpha^b(X, Y, U, V, N)$, where Y is a field of an internal field state. It is clear that a section is only one in $(P, T^1, \widetilde{g})$, if U, V, and N are physically implementable and uniquely specified. The general problem, concerning calculation of an electromagnetic field of control system, consists of finding such physically implementable operator $\widehat{B}$ and programmed controlling influence V, under which the particle dynamics would meet certain previously formulated requirements.

6 Multiagent Systems and Fiber Bundes

There is active research of controlled multiagent objects as information-transforming systems during the last several years. Despite the achievements that have been made in this area, effective mathematical methods for investigating such systems have not yet been developed. One possible approach is based on the differential geometry methods of system theory (Van der Shaft, is 1982, 1987 [24, 25]). This section is devoted to one of the problems of this area of research, that of developing a method for analyzing a class of mathematical models of symmetric controlled processes. Assuming that the process is described by a commutative diagram (Van der Shaft, 1982, 1987 [24, 25]), which is based on the lamination concept, we propose a geometric method for "identifying" its hidden structure.

Investigation of the information-transformation laws in various systems is one of the most essential stages in the creation of new agents. The goal of the experimental and theoretical research is the implementation of optimal strategy using complex structure nonequilibrium processes in such systems. To investigate these processes, it is required to develop the corresponding mathematical methods. In this context, we propose an approach, which is based on the assumption that one can use models from the mathematical system theory to adequately describe informational processes. The essence of this approach is in the following.

Some dynamic system, S, which implements a transformation, F, or an input informational action, U, into an output one, X, is considered. It is assumed that one can affect the information-transforming process by a reconfiguring action that changes the dynamic behavior, structure, symmetry, etc., of the process. We refer to the objects described in the preceding S as dynamic information-transforming agents (DITA).

The connection between the input and output actions is necessary for obtaining answers to questions about the method of programming the entire system, optimizing the flow of informational signals, and the interconnections among the global system properties (stability, controllability, etc.) and the corresponding local properties of the various subsystems. One has to answer those questions also when solving pattern-recognition problems, constructing an associative memory. A generalized description of DITA that contains a large number of subsystems (for example, a neural network) is postulated in this section: the controlled process in the DITA is described adequately by a commutative diagram that generalizes the concept of a nonlinear controlled dynamic system on a manifold. Taking into account the symmetry concept, which is characteristic of classic mechanics (Arnold, 1983 [1]), one has to transfer it to the DITA, "identify" the hidden structure of the informational process, and demonstrate that the proposed model admits local and/or global decompositions into smaller-dimensionality feedback subsystems.

We note that the decomposition idea was first applied to discretely symmetric automatic control systems by Yu. Samoylenko (the elementary cell method) (Samoilenko, 1970 [22]). Continuous symmetry group dynamic systems were considered by Van der Shaft (Van der Shaft, 1987 [25]). Substantive results on the decomposability of systems with symmetries have been obtained by A.Y. Krener (Krener, 1973 [13]) and others. However, this question remains open for DITAs.

Necessary concepts and definitions. Some definitions and concepts that are necessary for describing the DITA structure and the conditions for its decomposability are presented in this section. The necessary notions about manifolds, connectivities, and distributions are given in (Griffiths, 1983 [8]). We introduce the definition of a nonlinear DITA.

Definition 8. *Consider a triple, $F(B, M, \psi)$, where B is a smooth fiber over M with the projection $\pi : B \to M$; π_M is the natural projection of TM on M; and ψ is a smooth mapping such that the diagram presented in Figure 3 is commutative, by a "geometrical model of the agent."*

We interpret the manifold M as the DITA state space and the $\pi^{-1}(x) \in B$ layer as the space of input action values that depends in the general case on the current system state. If one chooses the coordinates (x, u), which correspond with the B_x layer, then this definition of the DITA, F, corresponds locally with the nonlinear transformation $\psi : (x, u) \to (x, \psi(x, u))$ and the dynamic system

$$\dot{x}(t) = \psi(x(t), u(t)), \quad u(t) \in U. \tag{49}$$

$$\begin{tikzcd} B \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & TM \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd}$$

Figure 3. Diagram of a nonlinear controlled DITA

where x is the DITA state vector, $u = (u^1, u^2)$ are the control actions, $u^1(\cdot, \cdot)$ is the vector of the coded input informational action that depends in general on time and on the current state, and $u^2(\cdot, \cdot)$ is the action used to reconfigure the dynamic properties of the DITA and to train it.

The control algorithm, u^2, inputs to the system the capability of transforming the set of input actions into a set of output signals that allows one to identify the input images uniquely. In essence, it realizes the decoding process, which identifies the input images. In the simplest case, it can be realized on the basis of the successive input action segmentation method. Such a method facilitates a unique separation of the input images by the use of the simplest binary decoding rule.

Definition 9. *Let M be a smooth manifold. We say that the smooth mapping $Q : G \times M$ such that:*

1. *$Q(e, x) = x$ for all $x \in M$, and*
2. *$Q(g, Q(h, x)) = Q(gh, x)$ for any g and $h \in G$, and all $x \in M$, is the left action (or G-action) of the G Lie group on M.*

We fix one of the variables for various time instants and examine the Q action as a function of the remaining variables. Let $Q_g : M \to M$ denote the function $x \ \mapsto Q(g, x)$ and $Q_x : G \to M$ the function $g| \to Q(g, x)$. We note that as $(Q_g)^{-1} = Q_g^{-1}$, Q_g is a diffemorphism.

We introduce the definition of group action on a manifold.

Definition 10. *Let Q be the action of G on M. We say that the set $G \cdot x = \{Q_g(x) | g \in G\}$ is the orbit (Q-orbit) of the point $x \in M$. The action is free at x if $g| \to Q_g(x)$ is one-to-one. It is free on M if and only if it is free at all $x \in M$.*

We now introduce the concept of global symmetry of a controlled DITA.

Definition 11. *Let $\hat{F}(B, M, \psi)$ be a nonlinear controlled DITA, and θ and Q be actions of G on B and M, respectively. Then, F has symmetry (G, θ, Q) if the diagram presented in Figure 4 is cummutative for all $g \in G$.*

We consider, within the framework of the presented definition, the special case in which the symmetry lies "entirely within the state space."

$$
\begin{tikzcd}
B \arrow[rrr, "B_g"] \arrow[dr, "\psi"] \arrow[dd, "\pi"'] & & & B \arrow[dl, "\psi"'] \arrow[dd, "\pi"] \\
& TM \arrow[r, "TQ_g"'] \arrow[dl, "\pi_M"] & TM \arrow[dr, "\pi_M"'] & \\
M \arrow[rrr, "Q_g"'] & & & M
\end{tikzcd}
$$

Figure 4. A commutative diagram of DITA with symmetries

Definition 12. *Let $B = M \times U$, where U is some manifold. Then, (G, Q) is a symmetry of the state space of system $\hat{F}(B, M, \psi)$ if (G, θ, Q) is a symmetry of $\hat{F}$ for $\theta_g = (Q_g, Id_U) : (x, u) \rightarrow (Q_g(x), u)$.*

Global state space symmetry can be defined only for a DITA B_x of which is a trivial lamination as otherwise the input spaces would depend on the state and the problem is made substantially more complicated.

We introduce now the definition of local symmetry.

Definition 13. *We assume that $Q : G \times M \rightarrow M$ is an action and that $\varepsilon \in T_eG$. Then, $Q^{\xi}(R \times M \rightarrow M) : (t, x)| \rightarrow Q(\exp t\xi, x)$, where $\exp : T_eG \rightarrow G$ is the usual exponential mapping, is the $\mathbb{R}$-action on M, and Q^{ξ} is the complete flow on M. We say that the corresponding vector field on M, which is defined by the expression*

$$\xi_m(x) = \frac{d}{dt} Q(\exp t\xi, x)\Big|_{t=0}, \tag{50}$$

is the infinitesimal action generator, which corresponds with ξ.

Let X_t denote the flow of the vector field X, that is, $X_t = F_t(X_0)$. It is obvious from the definition of the infinitesimal generator that if (G, θ, Q) is a symmetry of the $\hat{F}(B, M, \psi)$ system, then the diagram presented in Figure 5 is commutative for all $t \in \mathbb{R}$ and $\xi \in T_eG$.

On the basis of the local commutativity property, we present the following definition of infinitesimal DITA symmetry.

Definition 14. *Let $\hat{F}(B, M, \psi)$ be a nonlinear DITA. Then, (G, θ, Q) is an infinitesimal symmetry of F if, for each $x_0 \in M$, there exists an open neighborhood $\hat{O}$ of the point x_O and $\xi > 0$ such that*

$$(\xi_M)_t * \psi(\xi) = \psi((\xi_b)_t(b)), \tag{51}$$

for all $b \in \pi^{-1}(\hat{O})$, $|t| < \xi$, and $\parallel \xi \parallel < 1$, $\xi \in T_eG$, where $\parallel \cdot \parallel$ is an arbitrary fixed norm on T_eG.

One can define an infinitely small state space symmetry for nontrivial laminations of the input actions manifold when one can introduce integratable connectivity. For this, we introduce Definition 15.

$$\begin{tikzcd}
B \arrow[rrr, "\xi_B t"] \arrow[dr, "\psi"] \arrow[dd, "\pi"'] & & & B \arrow[dl, "\psi"'] \arrow[dd, "\pi"] \\
& TM \arrow[r, "T(\xi_M)t"'] \arrow[dl, "\pi_M"] & TM \arrow[dr, "\pi_M"'] & \\
M \arrow[rrr, "(\xi_M)t"'] & & & M
\end{tikzcd}$$

Figure 5. Diagram of a symmetric DITA

Definition 15. *Let $H(\cdot)$ be an integratable connectivity on B and (G, θ, Q) be a symmetry of F. Then, (G, θ, Q) is an infinitesimal state space symmetry if $\xi_B(b) \in H(b)$ for all $\xi \in T_e G$, that is, the infinitesimal generators θ are horizontal.*

We introduce a definition of feedback equivalence of two DITAs in analogy with (Van der Shaft, 1982 [24]).

Definition 16. *A system, $F(B, M, \psi)$, is feedback equivalent to a system, $F'(B, M, \widetilde{\psi})$, if there exists an isomorphism, $\gamma : B \to B$, such that the diagram presented in Figure 6 is commutative.*

Isomorphism means that, for $x \in M$, γ_x is a mapping from the layer over x' into the layer over x', and it is a diffeomorphism. Consequently, this corresponds with a "control feedback."

The local structure of DITAs with symmetries. Because we are interested in the local structure of a DITA, we have to assume that the system has an infinitesimal symmetry, which satisfies some nonsingularity condition. For this, we set the dimensionality of M to n and that of G to k, where $k < n$. We note that the action $Q : G \times M \to M$ is free at the point $m \in M$ if $Q_m : G \to M$ is one-to-one. This is equivalent to saying that the tangent mapping Q is of full rank, that is, rank $Q = \dim G$. Hence, Q is free on M if and only if it is free in some neighborhood of m. We say that an action that satisfies this condition is nonsingular at the point m.

The basic result of this section is that the existence of an infinitesimal symmetry in a neighborhood of a singular point in a DITA makes it possible to decompose the system into a cascade union of simpler subsystems. The structure of these subsystems depends, in general, on the symmetry group G. If, for example, G has a nontrivial center, then one of the subsystems is in fact a quadrature subsystem.

Let, in addition, $C = h \in G | jg = gh$ for all $g \in G$ be the center of the G group to which the kernel, C_+, of the Lie semialgebra $T_e G$, which has the same dimensionality as C, corresponds. Hence, if G has an l-dimensional center, there exist linearly independent vectors $\xi^1, \ldots, \xi^k \in T_e G$ such that $[\xi^i, \xi^j] = 0$ for all $1 \le i \le l$ and $1 \le j \le k$.

$$\begin{tikzcd}
B \arrow[r, "\psi"] \arrow[dr, "\pi"'] \arrow[rr, bend left, "\gamma"] & TM \arrow[d, "\pi_M"] & B \arrow[l, "\psi"'] \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd}$$

Figure 6. Diagram of feedback-equivalent DITAs

Using the results of Van der Shaft, Markus, and Grizzle's investigations (Van der Shaft, 1982, 1987 [24, 25]; Marcus, 1973, 1984 [16, 17]; Grizzle, 1983 [9]) that deal with the properties of systems with symmetries as applied to DITAs, one can formulate the following theorems.

Theorem 7. *Let us assume that $\hat{F}(B, M, \xi)$ is a controlled DITA with an infinitesimal state space symmetry, (G, θ, Q), that G has an l-dimensional center, and that Q is nonsingular at the point $m \in M$. Then, the B coordinates $(x_1, \ldots, x_n, u)$ in a neighborhood of m exist such that $\hat{F}$ is given in these coordinates by the expression.*

Using the obtained results for systems for infinitesimal state space symmetries, one can propose the structure of the decomposed system. It suffices to demonstrate for this that the decomposed system with infinitesimal symmetry is locally feedback-equivalent to the original system with infinitesimal state space symmetry.

Definition 17. *Let $\hat{F}(B, M, \psi)$ be a controlled DITA and $\hat{O}$ be an open subset of M. Then, we say that a system of the form $\hat{F}(\pi^{-1}(\hat{O}), \hat{O}, \psi)|\pi^{-1}(O)$ is $\hat{F}|\hat{O}$ ($\hat{F}$ bounded on $\hat{O}$).*

Theorem 8. *Let $\hat{F}(B, M, \psi)$ have an infinitesimal symmetry (G, θ, Q) and Q be nonsingular at the point m. There exist a neighborhood of m and a system F with infinitesimal symmetry (G, θ, Q) such that $\hat{F}|\hat{O}$ is feedback equivalent to the $\hat{F}$ system.*

Let $\hat{F}(B, M, \psi)$ be a controlled DITA with symmetry (G, θ, Q) and Q be nonsingular at the point m. Then, in a neighborhood of m, $\hat{F}$ is feedback-equivalent to $\hat{F}$ with infinitesimal symmetry and has the structure shown in Figure 7, where γ is the feedback function, the L^i are nonlinear subsystems of dimensions $n-k$ and $k-l$, respectively, and Q is an l-dimensional "quadrature" system

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_{n-k}, u), \quad i = 1, \ldots, n-k, \\ \dot{x}_j &= f_j(x_1, \ldots, x_{n-1}, u), \quad i = n-k+1, \ldots, k. \end{aligned} \tag{52}$$

The global structure of DITA. The decomposability of a DITA with global symmetries is the result of factoring the DITA state space, which follows from the properties of a symmetry.

We introduce the definition of proper action.

Definition 18. *Let Q be a G-action on M. We say that Q acts properly if $(g, m) \to m$ is a proper mapping, that is, if the pre-images of compact sets are compact.*

This definition is equivalent to the following assertion: whenever x_n converges on M and $Q_{g_n}(x_n)$ converges on M, g_n includes a subsequence,

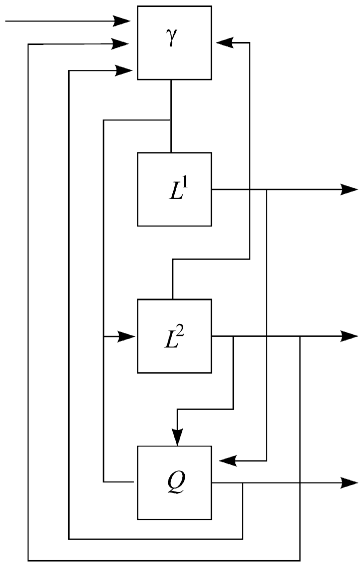

Figure 7. Local structure of DITA with infinitesimal symmetries

which converges in G. Hence, if G is compact, this condition is satisfied automatically. Membership in the same Q-orbit is an equivalence relation on M. Let M/G be the set of equivalence classes and $p : M \to M/G$ be specified by the relation $p(m) = Gm$. We introduce on M/G a relations topology, that is, $V \subset M/G$ is open if and only if $p^{-1}(V)$ is open on M. In general, M/G can be a rather poor space.

If G acts freely and properly on M, then M/G is a smooth manifold and $p : M \to M/G$ is the principal lamination with Lie group G.

We introduce the following constraints on the principal lamination:

(1) p is a smooth full-rank function;
(2) $p : M \to M/G$ has a cross section (that is, a smooth mapping $\sigma : M/G \to M$ such that $p \cdot \sigma$ a is the identity mapping on M/G if and only if M is equivalent to $M/G \times G$;
(3) the topological conditions that guarantee the existence of a section, that is, if M/G or G is a contraction mapping, a cross section must exist, are specified.

We formulate a theorem, which is necessary for obtaining a global factorization of the DITA state space.

Let $Q_m : G \to G \cdot m$ be specified by $g \to Q(G, m)$. The following result about the global structure of a DITA with symmetries holds.

Theorem 9. *We assume that $\hat{F}(M \times U, M, \psi)$ is a controlled DITA with a state space symmetry (C, Q). Then, if Q is free and proper, and $p : M \to M/G$ has a cross section σ, then $\hat{F}$ is isomorphic to the system*

$$\dot{y} = \Psi(y, u),$$
$$\dot{g} = (T_e L_g)(T_e Q_{\sigma(y)})^{-1} \left[\Psi(\sigma(y), u) - (T_y \sigma)\Psi(y, u)\right], \tag{53}$$

defined on $M/G \times G$.

We formulate an assertion on feedback equivalence of DITAs with symmetries.

Assertion 1 *Let the DITA $F(M \times U, M, \psi)$ have a symmetry (G, θ, Q) such that Q is free and proper. Then, there exists a system F with symmetry (G, Q) to which F is feedback equivalent under the condition that $p : M \to M/G$ has a cross section σ.*

Combining Theorem 9 and Assertion 1, we obtain the following corollary

Corollary 1. *Let DITA $\hat{F}(M \times U, M, \psi)$ have a symmetry (G, θ, Q), Q be free and proper, and $p : M \to M/G$ have a cross section. Then, there exists a model of DITA F with state space symmetry (G, Q) to which $\hat{F}$ is feedback-equivalent. Consequently, F has a global structure.*

The feasibility of applying the results to the investigation of agents. It is of interest to investigate the decomposability of DITAs composed of neural-like agents that are described by the system of equations

$$\dot{x}(t) = \psi(x(t), u(t)). \tag{54}$$

One can define for (54) a decomposed system L as a nontrivial cascade of subsystem L^1 and L^2. If the Lie algebra $\hat{L}(L)$ is the semidirect sum of finite-dimensional subalgebra L^1 and the ideal of L^2, it has a nontrivial cascade decomposition into subsystems L^1 and L^2 such that $\hat{L}(L^1) = L^1$, and $\hat{L}(L^2) = L^2$. Using this fact and Levy's theorem, one can demonstrate that if $\hat{L}(L)$ is finite-dimensional, the DITA admits a nontrivial decomposition into a parallel cascade of L^i systems with simple Lie algebras followed by a cascade of one-dimensional systems, L^j. As a result, the basic informational transformation is done in subsystems with simple Lie algebras. The state space, M, of the original system, L, is adopted here as the state space of these systems. Therefore, despite the fact that the system has been partitioned into simpler parts, the overall dimensionality of these parts is, in general, larger than that of the original system. (One can reduce at the local level this dimensionality by replacing the L^i system by matrix equivalents defined on the exponential functions of the Lie algebras that correspond with them.) These results can be compared with the conditions for decomposability obtained by analyzing the DITA symmetries described in this section for which the subsystem dimensionality equals that of the original system. No assumptions about the finite dimensionality of the Lie algebra are required here. We consider a class of neural nets described by the linear-analytic equations

$$\dot{x}(t) = f(x) + \sum_{i=1}^{k} u_i g_i(x). \tag{55}$$

One can formulate for it the necessary and sufficient conditions for parallel-cascade decomposability by Lie algebras. In doing so, one can pose the condition that each component of the input action be applied to only one of the subsystems, that is, the decomposition procedure partitions the inputs into disjoint subsets. However, such an approach cannot be applied to the decomposition of a DITA with scalar input.

If DITA $\hat{F}(B, M, \psi)$ has an infinitesimal symmetry (G, θ, Q), local commutativity of the diagram means that $\psi * \varepsilon_B = \varepsilon_m$ and $\pi * \varepsilon_B = \varepsilon_n$. Let $\Delta_B = \text{span}\{\varepsilon \| \varepsilon_B \in T_e G\}$ and the same hold for Δ_m. Then, $\psi * \Delta B \subset \Delta_m$ and $\pi * \Delta_B = \Delta$, and Δ_m, is a controlled invariant distribution. Models of neural networks, including affine ones, have invariant distributions that induce decompositions of the system into simpler subsystems. However, because the symmetry conditions are constraints, the decompositions are obtained as more detailed and structured.

A class of dynamic information-transforming systems that are described by a commutative diagram is examined in this section. Constraints on systems with symmetry under which one can expose, explicitly the hidden structure of the controlled process are formulated. We show that the effect of the DITA on the information-transforming process depends substantially on the type of system symmetry. The informational process is subject here to the action of cascade group, transformations, or the action of a dynamic-transformation operator with feedback. The obtained results can be expanded to adaptive learning systems by introducing the corresponding optimization models. When doing so, one can expect that a DITA of which the quality functional is invariant in symmetry-conserving transformations will be described adequately by a nonlinear system with optimal feedback and will have a differential-geometric structure, which is of interest from the point of view of applications. We plan to use the results of the investigations presented here in the study of a synergetic model of a neural network on the basis of potential-dependent ion channels in biomembranes.

7 Fiber Bundles and Observability

In the past decade, an important work has been done on a differential geometric approach to nonlinear input state-output systems, which in local coordinates have the form

$$\dot{x} = g(x, u), \quad y = h(x), \tag{56}$$

where x is the *state* of the system, u is the *input*, and y is the *output*. Most of the attention has been directed to the formulation in this context of fundamental system theoretic concepts like controllability, observability, minimality, and realization theory.

In spite of some very natural formulations and elegant results, which have been achieved, there are certain disadvantages in the whole approach, from which we summarize the following points,

(a) Normally the equations

$$\dot{x} = g(x, u) \tag{57}$$

are interpreted as a family of vector fields on a manifold parameterized by u; i.e., for every fixed $\overline{u}$, $g(\cdot, \overline{u})$ is a globally defined vector field. We propose another framework by looking at (57) as a coordinatization of

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd}$$

where B is a *fiber bundle* above the state space manifold X, and the fibers of B are the *state-dependent* input spaces, and TX is as usual the tangent bundle of X (the possible velocities at every point of X).

(b) The "usual" definition of *observability* has some drawbacks. In fact, observability is defined as *distinguishability*; i.e., for every x_1 and x_2 (elements of X) there exists a *certain* input function (in principle dependent on x_1 and x_2) such that the output function of the system starting from x_1 under the influence of this input function is different from the output function of the system starting from x_2 under the influence of the same input function. Of course, from a practical point of view this notion of observability is not very useful, and also is not in accord with the usual definition of observability or reconstructibility for general systems.

Hence, despite the work of Sussmann (Sussmann, 1983 [23]) on *universal* inputs, i.e., input functions, which distinguish between every two states x_1 and x_2, this approach remains unsatisfactory.

(c) In the class of nonlinear systems (56), *memoryless* systems

$$y = h(u) \tag{58}$$

are not included. Of course, one could extend the system (56) to the form

$$\dot{x} = g(x, u), \quad y = h(x, u), \tag{59}$$

but this gives, if one wants to regard observability as distinguishability, the following rather complicated notion of observability. As can be seen, distinguishability of (59) with $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ is equivalent to distinguishability of

$$\dot{x} = g(x, u), \quad \overline{y} = \overline{h}(x), \tag{60}$$

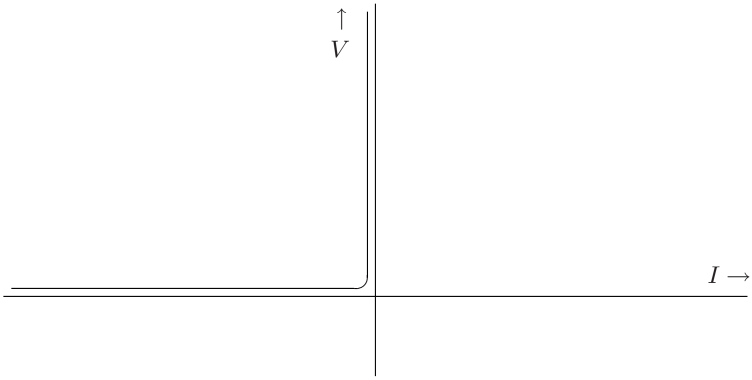

Figure 8. Input State-Output System for Ideal Diode

where $\overline{h} : \mathbb{R}^n \to (\mathbb{R}^p)^{\mathbb{R}^m}$ is defined by $\overline{h}(x)(u) = h(x, u)$.
Checking the Lie algebra conditions for distinguishability for the system (60) is not very easy.

(d) It is often not clear how to distinguish *a priori* between inputs and outputs. Especially in the case of a nonlinear system, it could be possible that a separation of what we shall call *external variables* in input variables and output variables should be interpreted only *locally*. An example is the (nearly) ideal diode given by the $I - V$ characteristic in Figure 8. For $I < 0$, it is natural to regard I as the input and V as the output, while for $V > 0$ it is natural to see V as the input and I as the output. Around an input-output description should be given in the scattering variables $(I - V, I + V)$. Moreover, in the case of nonlinear systems, it can happen that a global separation of the external variables in inputs and outputs is simply not possible! This results in a definition of a system, which is a generalization of the usual input-output framework. It appears that various notions like the definitions of autonomous (i.e., without inputs), memoryless, time-reversible, Hamiltonian, and gradient systems are very natural in this framework.

7.1 Nonlinear Model of Agents

The (say C^∞) agents can be represented in the commutative diagram

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\pi"'] & & TX \times W \arrow[dl, "\pi_X"] \\ & X & \end{tikzcd} \tag{61}$$

where (all spaces are smooth manifolds) B is a fiber bundle above X with projection π, TX is the tangent bundle of X, π_x the natural projection of TX on X, and f is a smooth map. W is the space of external variables (think of the inputs *and* the outputs). X is the state space, and the fiber $\pi_{-1}(x)$ in B above $\in X$ represents the space of inputs (to be seen initially as *dummy* variables), which is state dependent (think of forces acting at different points of a curved surface).

This definition formalizes the idea that at every point $x \in X$ we have a set of possible velocities (elements of TX) and possible values of the external variables (elements of W), namely the space

$$f(\pi^{-1}(x)) \subset T_x X \times W.$$

We denote the system (61) by $\Sigma(X, W, B, f)$. It is easily seen that in local coordinates x for X, v for the fibers of B, w for W, and with f factored in $f = (g, h)$, the system is given by

$$\dot{x} = g(x, v), \quad w = h(x, v). \tag{62}$$

Of course one should ask oneself how this kind of system formulation is connected with the usual input-output setting. In fact, by adding more and more assumptions successively to the very general formulation (61), we shall distinguish among three important situations, of which the last is equivalent to the "usual" interpretation of system (56).

(i) Suppose the map h restricted to the fibers of B is an *immersive* map into W (this is equivalent to assuming that the matrix $\partial h/\partial v$ is injective). Then:

Lemma 1. *Let h restricted to the fibers of 4 be an immersion into W. Let $(\overline{x}, \overline{u})$ and $\overline{w}$ be points in B and W, respectively, such that $h(\overline{x}, \overline{v}) = \overline{w}$. Then locally around $(\overline{x}, \overline{v})$ and $\overline{w}$ there are coordinates (x, v) for B (such that v are coordinates for the fibers of), coordinates (w_1, w_2) for W, and a map $\overline{h}$ such that h has the form*

$$(x, v) \gg h > (w_1, w_2) = (\overline{h}(x, v), v). \tag{63}$$

Proof. The lemma follows from the implicit function theorem.

Hence *locally* we can interpret a part of the external variables, i.e., w_1, as the outputs, and a complementary part, i.e., w_2, as the inputs! If we denote w_1 by y and w_2 by u, then system (62) has the form (of course only locally)

$$\dot{x} = y(x, u), \quad y = \overline{h}(x, u). \tag{64}$$

■

(ii) Now we not only assume that $\partial h/\partial v$ is injective, which results in a *local* input-output parameterization (64), but we also assume that the output set denoted by Y is *globally* defined. Moreover, we assume that W is a fiber bundle above Y, which we call $p : W \to Y$, and that h is a bundle morphism (i.e., maps fibers of B into fibers of W). Then:

Lemma 2. *Let $h : B \to W$ be a bundle morphism, which is a diffeomorphism restricted to the fibers. Let $\overline{x} \in X$ and $\overline{y} \in Y$ be such that $h(\pi^{-1}(\overline{x})) = p^{-1}(\overline{y})$. Take coordinates x around $\overline{x}$ for X and coordinates y around $\overline{y}$ for Y. Let $(\overline{x}, \overline{v})$ be a point in the fiber above $\overline{x}$ and let $(\overline{y}, \overline{u})$ be a point in the fiber above $\overline{y}$ such that $h(\overline{x}, \overline{v}) = (\overline{y}, \overline{u})$. Then there are local coordinates v around $\overline{v}$ for the fibers of B, coordinates u around $\overline{u}$ for the fibers of W, and a map $\overline{h} : X \to Y$ such that h has the form*

$$(x, v) \gg h > (y, u) = (\overline{h}(x), v). \tag{65}$$

Proof. Choose a locally trivializing chart $(0, \phi)$ of W around $\overline{y}$. Then $\phi : p^{-1}(0) \to 0 \times U$, with U the standard fiber of W. Take local coordinates u around $\overline{u} \in U$. Then (y, u) forms a coordinate system for W around $(\overline{y}, \overline{u})$. Because h is a bundle morphism, it has the form

$$(x, \overline{v}) \gg h > (y, u) = (\overline{h}(x), h'(x, \overline{v})),$$

where $(x, \overline{v})$ is a coordinate system for B around $(\overline{x}, \overline{v})$. Now adapt this last coordinate system by defining

$$v = (h')^{-1}(x, u) \quad \text{with } x \text{ fixed.}$$

Because h restricted to the fibers is a diffeomorphism, v is well defined and (x, v) forms a coordinate system for B in which h has the form

$$(x, v) \gg h > (y, u) = (\overline{h}(x), u).$$

Hence under the conditions of Lemma 2, our system is locally (around $\overline{x} \in X$ and $\overline{y} \in Y$) described by

$$\dot{x} = g(x, u), \quad y = \overline{h}(x). \tag{66}$$

■

This input-output formulation is essentially the same as the one proposed by Brockett and Takens, who take the input spaces as the fibers of a bundle above a globally defined output space Y. In fact, this situation should be regarded as the normal setting for nonlinear control systems.

(iii) Take the same assumptions as in (ii) and assume moreover that W is a *trivial* bundle, i.e., $W = Y \times U$, and that B is a trivial bundle, i.e., $B = X \times V$. Because h is a diffeomorphism on the fibers, we can identify U and V. In this case, the output set Y and the input set U are *globally* defined, and the system is described by

$$\dot{x} = g(x, u), \quad y = \overline{h}(x), \tag{67}$$

where for each fixed $\overline{u}, g(\cdot, \overline{u})$ is a globally defined vector field on X. This is the "usual" interpretation of (56).

Remark 2.

1. When h restricted to the fibers of B is *not* an immersion, we have a situation where we could speak of "hidden inputs." In fact, in this case there are variables in the fibers of B that can affect the internal state behavior via the equation $\dot{x} = g(x, v)$ but that cannot be directly identified with some of the external variables.
2. The splitting of the external variables into inputs and outputs as described in Lemma 1 is of course by no means unique! This fact has interesting implications, even in the linear case, which we shall not pursue further here.
3. From Lemma 2, it is clear that the coordinatization of the fibers of the bundle W uniquely determines, via h, the coordinatization of the fibers of B. It should be remarked that a coordinatization of the fibers of W is locally equivalent to the existence of an (integrable) *connection* on the bundle W, and that one coordinatization is linked with another by what is essentially an output feedback transformation, i.e., a bundle isomorphism from W into itself. Again we do not comment further on this point.
4. A beautiful example of this kind of system is the Lagrangian system. Here the output space is equal to the configuration space Q of a mechanical system. The state space X is the configuration space with the velocity space, so $X = TQ$. The space W is equal to T^*Q (the cotangent bundle of Q), with the fibers of T^*Q representing the external forces. When we denote the natural projection of TQ on Q by ρ, then B is just ρ^*T^*Q (the pullback bundle via ρ). Now given a function $L : TQ \to \mathbb{R}$ (called the Lagrangian), we can construct a symplectic form $d(\partial L/\partial \dot{q}) \wedge dq$ (with $(q, \dot{q})$ coordinates for TQ) on TQ, which uniquely determines a map $g : B \to TTQ$. Finally, in coordinates the system is given by

$$\ddot{q} = F(q, \dot{q}) + \sum_j u_j Z_j(q, \dot{q}), \quad y = q, \tag{68}$$

   with the vector fields $F(q, \dot{q})$ and $Z_j(q, \dot{q})$ satisfying certain conditions. Moreover the vector fields Z_j commute, i.e., $[Z_i, Z_j] = 0$ for all i, j, a fact that has a very interesting interpretation.
5. Most cases where B can be taken as trivial are generated by a space X such that TX is a trivial bundle. For instance, when X is a Lie group, TX is automatically trivial.

7.2 Minimality and Observability

Minimality. We want to give a definition of minimality for a general nonlinear agent.

Definition 19. *Let $\Sigma(X, W, B, f)$ and $\Sigma'(X', W, B', f')$ be two smooth systems. Then we say $\Sigma' \leqq \Sigma$ if there exist surjective submersions $\phi : X \to X', \Phi : B \to B'$ such that the diagram*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & TX \times W \arrow[dl] \\ & X & \end{tikzcd} \tag{69}$$

commutes.

Σ is called *equivalent to* Σ' (denoted $\Sigma \sim \Sigma'$) if ϕ and Φ are diffeomorphisms.

We call Σ *minimal* if $\Sigma' \leqq \Sigma \Rightarrow \Sigma' \sim \Sigma$.

$$\begin{tikzcd} B \arrow[rrr, "\Phi"] \arrow[dd, "\pi"'] \arrow[dr, "f"'] & & & B' \arrow[dd, "\pi'"] \arrow[dl, "f'"] \\ & W \times TX \arrow[r, "id", "\phi_*"'] \arrow[dl, "\pi_X"] & W \times TX' \arrow[dr, "\pi_{X'}"'] & \\ X \arrow[rrr, "\phi"'] & & & X' \end{tikzcd}$$

Remark 3. This definition formalizes the idea that we call Σ' *less complicated* than $\Sigma (\Sigma' \leqq \Sigma)$ if Σ' consists of a set of trajectories in the state space, smaller than the set of trajectories of Σ, but which generates the same *external behavior*. (The external behavior Σ_e of $\Sigma(X, W, B, f)$ consists of the possible functions $w : \mathbb{R} \to W$ generated by $\Sigma(X, W, B, f)$. Hence, when we define $\Sigma := \{(x, w) : \mathbb{R} \to X \times W|_x$ absolutely continuous and $(\dot{x}(t), w(t)) \inf(\pi^{-1}(x(t)))a.e.\}$, then Σ_e, is just the projection of Σ on $W^{\mathbb{R}}$).

Remark 4. Notice that we only formalize the *regular* case by asking that Φ and ϕ be surjective as well as submersive. In fact we could, for instance, allow that at isolated points ϕ or Φ are not submersive. However, we do not discuss this problem here, and treat only the regular case as described in Definition 19.

Remark 5. Notice that $\Sigma_1 \leqq \Sigma_2$ and $\Sigma_2 \leqq \Sigma_2$ need not imply $\Sigma_1 \sim \Sigma_1$. This fact leads to very interesting problems, which we do not pursue further at this time.

Of course, Definition 19 is an elegant but rather abstract definition of minimality. From a differential geometric point of view, it is very natural to see what these conditions of commutativity mean *locally*. In fact, we will see in Theorem 11 that locally these conditions of commutativity do have a very direct interpretation. But first we have to state some preparatory lemmas and theorems.

Let us look at (69). Because Φ is a submersion, it induces an involutive distribution D on B given by

$$D := \{Z \in TB | \Phi_* \dot{Z} = 0\}$$

(the foliation generated by D is of the form $\Phi^{-1}(c)$ with c constant). In the same way, ϕ induces an involutive distribution E on X. Now the information in the diagram (69) is contained in three subdiagrams (we assume $f = (g, h)$ and $f' = (g', h')$):

$$\begin{tikzcd}
B \arrow[r, "\Phi"] \arrow[d, "h"'] & B' \arrow[d, "h'"] \\
W \arrow[r, leftrightarrow, "id"'] & W
\end{tikzcd} \qquad \mathrm{I}$$

$$\begin{tikzcd}
B \arrow[r, "\Phi"] \arrow[d, "\pi"'] & B' \arrow[d, "\pi'"] \\
X \arrow[r, "\phi"'] & X'
\end{tikzcd} \qquad \mathrm{II}$$

$$\begin{tikzcd}
B \arrow[r, "\Phi"] \arrow[d, "g"] & B' \arrow[d, "g'"'] \\
TX \arrow[r, "\phi_*"'] & TX'
\end{tikzcd} \qquad \mathrm{III}$$

Lemma 3. *Locally the diagrams I, II, III are equivalent, respectively, to*

$$\begin{aligned} I' &: \quad D \subset \ker dh, \\ II' &: \quad \pi_* D = E, \\ III' &: \quad g_* D \subset TE = T\pi_*(D). \end{aligned} \tag{70}$$

Proof. I' and II' are trivial. For III' observe that, when ϕ induces a distribution E on X, then ϕ_* induces the distribution TE on TX. ■

Now we want to relate conditions I', II', III' with the theory of nonlinear disturbance decoupling. Consider in local coordinates the system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \quad \text{on a manifold } X.$$

We can interpret this as an affine distribution on manifold.

Theorem 10. *Let $D \in A(\Delta_0)$. Then the condition*

$$[\Delta, D] \subseteq D + \Delta_0 \tag{71}$$

(we call such a $D \in A(\Delta_0)\Delta(\operatorname{mod} \Delta_0)$ invariant) is equivalent to the two conditions (a) there exists a vector field $F \in \Delta$ such that $[F, D] \subseteq D$; (b) there exist vector fields $B_i \in \Delta_0$ such that span $\{B_i\} = \Delta_0$ and $[B_i, D] \subset D$.

With the aid of this theorem, the disturbance decoupling problem is readily solved. The key to connecting our situation with this theory is given by the concept of the *extended system*, which is of interest in itself.

Definition 20. (Extended system). *Let*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr, "\pi"'] & & TX \times W \arrow[dl, "\pi_X"] \\ & X & \end{tikzcd}$$

Then we define the extended system of $\Sigma(X, W, B, f)$ as follows: We define Δ_0 as the vertical tangent space of B, i.e.,

$$\Delta_0 := \{Z \in TB | \pi_* Z = 0\}.$$

Note that Δ_0 is *automatically involutive.*

Now take a point $(\overline{x}, \overline{v}) \in B$. Then $g(\overline{x}, \overline{v})$ is an element of $T_{\overline{x}}X$. Now define

$$\Delta(\overline{x}, \overline{v}) := \{Z \in T_{(\overline{x}, \overline{v})} | \pi_* Z = g(\overline{x}, \overline{v})\}.$$

So $\Delta(\overline{x}, \overline{v})$ consists of the possible lifts of $g(\overline{x}, \overline{v})$ in $(\overline{x}, \overline{v})$. Then it is easy to see that Δ is an affine distribution on B, and that $\Delta - \Delta = \Delta_0$. We call the affine system (Δ, Δ_0) on B constructed in this way, together with the output function $h : B \to W$, the extended system $\Sigma^e(X, W, B, f)$.

We have the following:

Lemma 4.

(a) Let D be an involutive distribution on B such that $D \cap \Delta_0$ has constant dimension. Then $\pi_* D$ is a well-defined and involutive distribution on X if and only if $D + \Delta_0$ is an involutive distribution.

(b) Let D be an involutive distribution on B and let $D \cap \Delta_0$ have constant dimension. Then the following two conditions are equivalent: (i) $\pi_* D$ is a well-defined and involutive distribution on X, and $g_* D \subset T\pi_* D$. (ii) $[\Delta, D] \subset D + \Delta_0$.

Proof.

(a) Let $D + \Delta_0$ be involutive. Because D and Δ_0 are involutive, this is equivalent to $[D, \Delta_0] \subset D + \Delta_0$. Applying Theorem 10 to this case gives a basis $\{Z_1, \ldots, Z_k\}$ of D such that $[Z_i, \Delta_0] \subset= \Delta_0$. In coordinates (x, u) for B, the last expression is equivalent to $Z_i(x, u) = (Z_{ix}, Z_{iu}(x, u))$, where Z_{ix} and Z_{iu} are the components of Z_i in the x- and u-directions, respectively. Hence $\pi_* D = \mathrm{span}\,\{Z_{1x}, \ldots, Z_{kx}\}$ and is easily seen to be involutive. The converse statement is trivial.

(b) Assume (i); then there exist coordinates (x, u) for B such that $D = \{\partial/\partial x_1, \ldots, \partial/\partial x_x\}$ (the integral manifolds of D are contained in the sections $u = \text{const}$). Then $g_* D \subset T\pi_* D$ is equivalent to

$$\left(\frac{\partial g}{\partial x_i}\right)_{j^e \text{comp}} = 0$$

with $i = 1, \ldots, k$ and $j = k+l, \ldots, n$ (n is the dimension of X). From these expressions, $[\Delta, D] \subset D + \Delta_0$ readily follows. The converse statement is based on the same argument.

■

Now we are prepared to state the main theorem of this section. First we have to give another definition.

Definition 21. (Local minimality). *Let $\Sigma(X, W, B, f)$ be a smooth system. Let $\overline{x} \in X$. Then $\Sigma(X, W, B, f)$ is called locally minimal (around $\overline{x}$) if when D and E are distributions (around $\overline{x}$) that satisfy conditions I', II', III' of Lemma 3, then D and E must be the zero distributions.*

It is readily seen from Definition 19 that minimality of $\Sigma(X, W, B, f)$ *locally* implies local minimality (locally every involutive distribution can be factored out).

Combining Lemma 3, Definition 20, and Lemma 4 we can state:

Theorem 11. *$\Sigma(X, W, B, f = (g, h))$ is locally minimal if and only if the extended system $\Sigma^e(X, W, B, f = (g, h))$ satisfies the condition that there exist no nonzero involutive distribution D on B such that*

$$\begin{aligned} (i) \quad & [\Delta, D] \subset D + \Delta_0, \\ (ii) \quad & D \subset \ker dh. \end{aligned} \tag{72}$$

Remark 6. It is very surprising that the condition of minimality locally comes down to a condition on the extended system, which is in some sense an infinitesimal version of the original system.

Remark 7. Actually there is a conceptual algorithm to check local minimality. Define

$$\Delta^{-1}(\Delta_0 + D) := \{\text{vector fields } Z \text{ on } B \mid [\Delta, Z] \subseteq \Delta_0 + D\}.$$

Then we can define the sequence $\{D^{mu}\}, \mu = 0, 1, 2, \ldots$ as follows:

$$D^0 = \ker dh,$$
$$D^\mu = D^{\mu-1} \cap \Delta^{-1}(\Delta_0 + D^{\mu-1}), \quad \mu = 1, 2, \ldots.$$

Then $\{D^\mu\}$, $\mu = 0, 1, 2, \ldots$, is a decreasing sequence of involutive distributions, and for some $k \geqq \dim(\ker dh) D^k = D^\mu$ for all $\mu \geqq k$. Then D^k is the *maximal* involutive distribution that satisfies

$$\begin{aligned} (i) \quad & [\Delta, D^k] \subset D^k + \Delta_0, \\ (ii) \quad & D^k \subset \ker dh. \end{aligned}$$

From Theorem 11, it follows that $\Sigma(X, W, B, f)$ is locally minimal if and only if $D^k = O$.

Observability. It is natural to suppose that our definition of minimality has something to do with controllability and observability. However, because the definition of a nonlinear system (61) also includes autonomous systems, (i.e., no inputs), minimality cannot be expected to imply, in general, some kind of controllability. In fact, an autonomous linear system

$$\dot{x} = Ax, \quad y = Cx$$

is easily seen to be minimal if and only if (A, C) is observable. Moreover, it seems natural to define a notion of *observability* only in the case that the system (61) has at least a local input-output representation; i.e., we make the standing assumption that $(\partial h/\partial v)$ is injective (see Lemma 1). Therefore, *locally* we have as our system

$$\dot{x} = g(x, u), \quad y = \overline{h}(x, u) \tag{73}$$

for every possible input-output coordinatization (y, u) of W. For such an input-output system local minimality implies the following notion of observability, which we call *local distinguishability.*

Proposition 2. *Choose a local input-output parameterization as in (73). Then local minimality implies that the only involutive distribution E on X that satisfies*

(i) $[g(\cdot,u),E] \subset E$ *for all* u *(E is invariant under $g(\cdot,u)$),*
(ii) $E \subset \ker d_x h(\cdot,u)$ *for all* u *($d_x\overline{h}$ means differentiation with respect to x) is the zero distribution.*

Proof. Let E be a distribution on X that satisfies (i) and (ii). Then we can lift E in a trivial way to a distribution D on B by requiring that the integral manifolds of D be contained in the sections $u = \text{const}$. Then one can see that D satisfies $[\Delta, D] \subset D + \Delta_0$ and $D \subset \ker dh$. Hence $D = 0$ and $E = 0$. ■

Remark 8. It is easily seen that, under the condition $(\partial h/\partial v)$ injective local minimality. We can state the following Corollary 2.

Corollary 2. *Suppose there exists an input-output coordinatization*

$$\dot{x} = g(x,u), \quad y = \overline{h}(x). \tag{74}$$

Then heal minimality implies local weak observability.

Proof. As can be seen from Proposition 2, local minimality in this more restricted case implies that the only involutive distribution E on X that satisfies

$$\begin{aligned} &(i) \quad [g(\cdot,u),E] \subset E \quad \text{for all} \quad u,\\ &(ii) \quad E \subset \ker d\overline{h} \end{aligned}$$

is the zero distribution. It can be seen that the biggest distribution that satisfies (i) and (ii) is given by the null space of the codistribution P generated by elements of the form

$$L_{g(\cdot,u^1)} L_{g(\cdot,u^2)} \cdots L_{g(\cdot,u^k)} d\overline{h}, \quad \text{with} \quad u^j \text{ arbitrary}.$$

Because this distribution has to be zero, the codistribution P equals $T^*_x X$, in every $\in X$. This is, apart from singularities (which we don't want to consider), equivalent to local weak observability. ■

Moreover, let (74) be locally weakly observable. Then all feedback transformations $u \mapsto v = \alpha(x,u)$ that leave the form (74) invariant (i.e., y is only the function x) are exactly the output feedback transformations $u \mapsto v = \alpha(y,u)$. It can be easily seen in local coordinates that after such output feedback is applied, the modified system is still locally weakly observable.

In Proposition 2 and its corollary, we have shown that local minimality implies a notion of observability, which generalizes the usual notion of local weak observability. Now we will define a much stronger notion. Let us denote the (defined only locally) vector field $\dot{x} = g(x,\overline{u})$ for fixed $\overline{u}$ by $g^{\overline{u}}$ and the function $\overline{h}(x,\overline{u})$ by $h^{\overline{u}}$ (with g and $\overline{h}$ as in (73)).

Definition 22. *Let $\Sigma(X, W, B, f) = (g, h)$ be a smooth nonlinear system. It is called strongly observable if for every possible input-output coordinatization (73) the autonomous system*

$$\dot{x} = g^{\overline{u}}(x), \quad y = h^{\overline{u}}(x) \tag{75}$$

with $\overline{u}$ constant is locally weakly observable, for all $\overline{u}$.

Remark 9. Let $\Sigma(X, W, B, f = (g, h))$ be strongly observable. Take one input-output coordinatization (y, u). The system has the form (in these coordinates)

$$\dot{x} = g(x, u), \quad y = \overline{h}(x, u).$$

Because the system is strongly observable, every *constant* input-function (constant in *this* coordinatization) distinguishes between two nearby states. However, in every other input-output coordinatization, every constant (i.e., in *this* coordinatization) input function also distinguishes. This implies that in the first coordinatization, every C^∞ input function distinguishes. Because the C^∞ input functions are dense in a reasonable set of input functions, every input function in this coordinatization distinguishes.

Proposition 3. *Consider the Pfaffian system constructed as follows:*

$$P = dh^{\overline{u}} + L_{g^{\overline{u}}} dh^{\overline{u}} + L_{g^{\overline{u}}}(L_{g^{\overline{u}}} dh^{\overline{u}}) + \cdots + L_{g^{\overline{u}}}^{n-1} dh^{\overline{u}},$$

with n the dimension of X and $L_{g^{\overline{u}}}$ the Lie derivative with respect to $g^{\overline{u}}$. As is well-known, the condition that the Pfaffian system P as defined above satisfies the condition $P_x = T_x^* X$ for all $x \in X$ (the so-called observability rank condition) implies that the system

$$\dot{x} = g^{\overline{u}}(x), \quad y = h^{\overline{u}}(x)$$

is locally weakly observable. Hence, when the observability rank condition is satisfied for all u, the system is strongly observable.

We will call the Pfaffian system P the *observability codistribution.*

Remark 10. As is known, local weak observability of the system

$$\dot{x} = g^{\overline{u}}(x), \quad y = h^{\overline{u}}(x)$$

implies that the observability rank condition (i.e., $\dim P_x = T_x^* X$) is satisfied almost everywhere (in fact, in the analytic case everywhere). Because we don't want to go into singularity problems, for us local weak observability and the observability rank condition are the same.

Remark 11. It is easily seen that when for one input-output coordinatization the observability rank condition for all u is satisfied, then for *every* input-output coordinatization the observability rank condition for all u is satisfied. This follows from the fact that the observability rank condition is an open condition.

Controllability. The aim of this section is to define a kind of controllability that is "dual" to the definition of local distinguishability (Proposition 2) and that we shall use in the following section. The notion of controllability we shall use is the so-called "strong accessibility."

Definition 23. *Let $\dot{x} = g(x,u)$ be a nonlinear system in local coordinates. Define $R(T,x_0)$ as the set of points reachable from x_0 in exactly time T; in other words,*

$$R(T,x_0) := \{x_1 \in X \mid \exists \text{ state trajectory } x(t) \text{ generated by } g \\ \text{such that } \; x(0) = x_0 \;\text{ and }\; x(T) = x_1\}.$$

We call the system *strongly accessible* if for all $x_0 \in X$ and for all $T > 0$ the set $R(T,x_0)$ has a nonempty interior.

For systems of the form (in local coordinates)

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{76}$$

(i.e., affine systems) we can define A as the smallest Lie algebra that contains $\{g_1,\ldots,g_m\}$ and that is invariant under f (i.e., $[f,A] \subset A$). It is known that $A_x = T_x X$ for every $x \in X$ implies that the system (76) is strongly accessible. In fact, when the system is analytic, strong accessibility and the rank condition $A_x = T_x X$ for every $x \in X$, are equivalent. We call A the *controllability distribution* and the rank condition the controllability rank condition. Now it is clear that for affine systems (76) this kind of controllability is an elegant "dual" of local weak observability.

It is well-known that the extended system (see Definition 20) is an affine system. Hence for this system we can apply the rank condition described above. This makes sense because the strong accessibility of $\Sigma(X,W,B,f)$ is very much related to the strong accessibility of $\Sigma^e(X,W,\,B,f)$, which can be seen from the following two propositions.

Proposition 4. *If $\Sigma^e(X,W,B,f = (g,h))$ is strongly accessible, then $\Sigma(X,W,\,B,f = (g,h))$ is strongly accessible as well.*

Proof. In local coordinates, the dynamics of Σ^e and Σ are given by

$$\begin{array}{rll} I & \dot{x} = g(x,u) & (\Sigma), \\ II & \dot{x} = g(x,v) & (\Sigma^e), \\ & \dot{v} = u. & \end{array}$$

It is easy to show that if for Σ^e one can steer to a point x_1, then the same is possible for Σ (even with an input that is smoother). ■

The converse is harder:

Proposition 5. *Let $\Sigma(X, W, B, f = (g, h))$ be strongly accessible. In addition, if the fibers of B are connected, then $\Sigma^e(X, W, B,\ f = (g, h))$ is strongly accessible.*

Proof. Consider the same representation of Σ and Σ^e as in the proof of Proposition 4. Let $x_0 \in X$ and x_1 be in the (nonempty) interior of $R_\Sigma(x_0, T)$ (the reachable set of system Σ). Then it is possible to reach x_1 from x_0 by an input function $v(t)$ that cannot be generated by the differential equation $\dot{v} = u$. However, we know that the set of the v generated in this way is dense in L^2. (For this we certainly need that the fibers of B are connected.) Because we only have to prove that the interior of a set is nonempty, this makes no difference. Now it is obvious from the equations

$$\dot{x} = g(x, v), \quad \dot{v} = u$$

that if we can reach an open set in the x-part of the (extended) state, then it is surely possible in the hole (x, v)-state. ■

8 Conclusion

In this chapter, the problem of geometric description of multiple agents is studied. The connection of the optimal game and Yang–Mills fields has been established. A geometric model of a controlled agent as dynamic information-transforming system is examined. A description of the information-transforming system within the framework of the geometric formalism is also proposed. After a classification of the fiber bundle types of conflict and conflict-free maneuvers, a weighted energy can be proposed as the cost function to select the optimal one. Various local and global controllability and observability conditions are derived. For the general multiagent case, a convex optimization algorithm is proposed to find the optimal multilegged maneuvers. To completely characterize the optimal conflict-free maneuvers, many issues remain to be addressed. Possible directions of future research include the analysis of the proposed mathematical models in terms of its performance and its robustness with respect to uncertainty of the agents' positions and velocities, and a more realistic study for the agent dynamics.

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Convexity in Differential Games

Valentin Ostapenko

Training-Scientific Complex "Institute of Applied System Analysis," National Technical University of Ukraine "Kyiv Polytechnical Institute," Ukraine
nmo@mmsa.ntu-kpi.kiev.ua

Abstract The current chapter is devoted to the development of convex analysis concepts in the context of solving pursuit-evasion problems in differential games. Classic convex analysis is generalized; new concepts such as matrix-convex sets and H-convex sets are introduced and studied. With the help of these, it is shown possible to describe a rather wide class of differential games where players' strategies are produced in a comparatively constructive manner. The main attention is on studying those properties of matrix convexity that are required for the theory of differential games.

Operational constructions for the initial positions sets, favorable to each player, for the derivation of the players' strategies are also described.

Key words: differential games, matrix-convexity, H-convexity, operational constructions

1 Introduction

Many important results in differential games have been derived during the past 40 years. The book "Differential Games" by R. Isaacs [1] initiated the research on this subject. The book introduced a wide range of applied plan problems with inherent game characteristics, which, however, were not entirely contained within the borders of the formed theory of optimum control. The ideas described in [1] received precise mathematical formalization in subsequent works by other authors [3, 7–9, 11, 13, 14]. The theory of differential games has since matured and developed into an independent scientific discipline that has its own field of problems and methods.

The large number of approaches, methods, and algorithms for decisions in various classes of problems have been developed within the borders of the differential games theory. These are problems of pursuit, pursuit-evasion, keeping, escaping problem, and game problems of dynamic search. Firstly, the general approaches for decisions in differential games of pursuit-evasion were founded. In these approaches, the structure of the game is described by stable bridges [8, 9], one-parametrical semigroups (operational constructions) [13, 14], or alternative integral of L.S. Pontryagin [11]. The theorems of alternative break the phase space of the game into sets of initial positions that are favorable for this or that player. Thus, the theorems of existence of optimum strategies for the players are proved.

The next group of methods is also devoted to decisions in general differential games, but these are more constructive approaches, including approximate methods [14] to them. However, numerical realization of these methods meets difficulties due to the extensive calculations and the necessity for the derivation of special theories that would effectively describe the sets and operations needed. Comprehensive numerical results have only been obtained for two- and three-dimensional spaces.

A third group of methods are not applied to all differential games as a whole, rather they are directed toward decisions in certain classes of games. Such methods are the first method by L.S. Pontryagin [11], the method of resolving functions [5], and a method based on H-convex sets [10, 14]. As a rule, these methods are solving certain classes of linear differential games. Thus, the convexity of the terminal set or of the areas of players' controls plays the important role. At a more detailed level of studying linear differential games, generalized concepts of convexity are used in order to expand the field of application of the methods of pursuit. Such concepts are the H-convexity [4] and the matrix convexity, which constitutes a further development of the H-convexity.

Issues of the application of elements from convex analysis and generalized convex analysis to decisions in linear games of pursuit-evasion are addressed in the current chapter.

2 Auxiliary Results

2.1 Notation

We use the following notations throughout the chapter.

E^n - n-dimensional Euclidean space with the scalar product $\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i$, where $x, y \in E^n$;

Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} (x^i)^2}$;

E^1 - one-dimensional space that coincides with the set of real numbers;

E - identity operator in space E^n or the identity matrix of dimensions $n \times n$;

$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ - norm of linear operator in E^n;

A^* - adjoint operator to the operator A that operates in E^n, or the transposed matrix of matrix A.
$\mathrm{cl}\, A$ - closure of the set A;
$\mathrm{int}\, A$ - interior of the set A;
∂A - boundary of the set A;
$\rho(A,B) = \max\left\{\sup\limits_{x\in A}\inf\limits_{y\in B}\|x-y\|, \sup\limits_{y\in B}\inf\limits_{x\in A}\|x-y\|\right\}$ - Hausdorff distance between the sets A and B;
$A+B = \bigcup\{x+y : x\in A, y\in B\}$ - sum of two sets $A, B \subset E^n$;
$\lambda A = \bigcup\{\lambda x : x \in A\}$ - multiplication of the scalar λ with the set $A\subset E^n$;
$A \overset{*}{-} B = \{z\in E^n : B+z\subset A\}$ - geometric remainder of the two sets A and B.
$\mathrm{co}\, A$ - convex hull of the set A (smallest convex set containing A);
$\overline{\mathrm{co}} A$ - closed convex hull of the set A;
$\mathrm{con}\, A$ - conical hull of the set A (smallest cone containing A);
$S(x,r) = \{y\in E^n : \|y-x\|\le r\}$ - sphere with the center at x and radius r;
$S = S(0,1)$, $\partial S = \{x\in E^n : \|x\| = 1\}$.
$W_A(x^*) = \sup\limits_{x\in A}\langle x, x^*\rangle$ - support function of the set $A\subset E^n$.

2.2 Multivalued Mappings and Philippov's Lemma

Let us introduce the notation:

2^Y - set of all subsets of the set Y;

$F : X\to 2^Y$ - multivalued mapping that assigns a subset $F(x)$ of Y to the point $x\in X$;

$\int\limits_a^b F(t)\,dt$ - integral of the multivalued mapping $F:[a,b]\to 2^{E^n}$, that is, the set of all integrals $\int\limits_a^b f(t)\,dt$, where $f(t):[a,b]\to E^n$ is a measurable, bounded, single-valued mapping such that $f(t)\in F(t)$.

If $f(t,u)$, $f: E^1\times E^r\to E^n$, is a continuous function and $U[a,b]$ is a set of measurable functions $u(t)$ with values in the compact set $U\subset E^r$, then, by definition,

$$\int\limits_a^b f(t,U)\,dt = \bigcup\left\{\int\limits_a^b f(t,u(t))\,dt : u(\cdot)\in U[a,b]\right\}.$$

Theorem 1. *[2] Under the made assumptions,* $\int\limits_a^b f(t,U)\,dt$ *is a convex set.*

Let us consider the linear multivalued mapping $F(t) = A(t)M$, $t\in[a,b]$, where $A(t)$ is a linear operator in the space E^n for each $t\in[a,b]$, and $M\subset E^n$

is a closed subset. We assume that the family of operators $A\,(t)$, $t \in [a,b]$, is bounded and measurable. We shall understand the integral $\int_a^b A\,(t)\,M dt$ as the set of all integrals of the form $\int_a^b A\,(t)\,m\,(t)\,dt$, where $m\,(t) \in M$, $t \in [a,b]$, is a measurable bounded function. We do not impose any condition of boundedness on the set M. However, from Theorem 1, it is not difficult to obtain the following result:

Corollary 1. *The set* $\int\limits_a^b A\,(t)\,M dt$ *is convex.*

Let us recall the concept of the lexicographic order for vectors in E^n. Let $x = \left(x^1, \ldots, x^n\right) \in E^n$, $y = \left(y^1, \ldots, y^n\right) \in E^n$. We say that x is less than y in lexicographic sense if for some $k = 1, \ldots, n$, $x^i = y^i$, $i < k$, $x^k < y^k$. It is easy to see that if $K \subset E^n$ is a compact set, then there exists the unique point $x_* \in K$, which is maximal in lexicographic sense for all $x \in K$.

Let $f\,(u, v, t)$ be a continuous function with values in E^n, $u \in U$, $v \in V$, $U \subset E^r$, $V \subset E^s$, U and V compact sets. Let $\xi\,(t) \in E^n$ be a measurable function on $[a,b]$ and let the equation $f\,(u,v,t) = \xi\,(t)$ be solvable for any $v \in V$, $t \in [a,b]$. Denote by $u_*\,(v,t)$ its lexicographic maximal solution.

Theorem 2. *[6] If* $v\,(t) \in V$ *is a measurable function on* $[a,b]$*, then* $u_*\,(v\,(t)\,,t)$ *is also a measurable function of* t *on* $[a,b]$*.*

This statement can be made more exact if we note that the constructed function $u_*\,(v,t)$ is Borel measurable w.r.t. v.

Theorem 3. *If* $x\,(t) \in E^n$ *is a fully continuous function on the segment* $[a,b]$ *that fulfills almost everywhere the inclusion*

$$\dot{x} \in f\,(x, t, U),$$

where $f : E^{n+1} \times U \to E^n$ *is a continuous mapping, and* $U \subset E^r$ *is a compact set, then there exists a function* $u\,(\cdot) \in U\,[a,b]$ *such that nearly everywhere*

$$\dot{x}\,(t) = f\,(x\,(t)\,, t, u\,(t)).$$

3 Matrix Convexity

The concept of convex sets is generalized in the current section. The sums of products of numbers by vectors are used in the definition of common convexity. In the given generalization, the role of numbers is played by matrices leading, thus, to the term “matrix convexity.” By analogy, common convexity can be named “scalar.”

Necessity for studying matrix convexity has emerged in control theory and differential games theory. With the help of matrix convexity, it was possible to describe a rather wide class of differential games where players' strategies are produced comparatively constructively. However, the main attention has been given to the study of those properties of matrix convexity that were important for the theory of differential games.

Matrix-convex sets are not necessarily convex in common scalar sense generally. However, a class of matrix-convex sets is a subclass of scalar-convex sets under certain assumptions. Moreover, it has been shown that matrix-convex sets are H-convex in a general enough case. It is necessary to note that H-convex sets are well studied and described comparatively constructively in a number of concrete examples. Thus, the subclass of convex sets is constructed for each set of the matrices determining convexity.

3.1 Scalar-Convex Sets

Let us consider certain properties of common convex sets. The following is the traditional definition of convexity.

Definition 1. *A set $M \subset E^n$ is called a convex set, if for any $x, y \in M$ and any numbers $\lambda_i \geq 0$ such that $\lambda_1 + \lambda_2 = 1$, the inclusion*

$$\lambda_1 x + \lambda_2 y \in M \tag{1}$$

is fulfilled.

We can write the inclusion (1) in a slightly different manner using the concept of the sum of two sets.

Definition 2. *A set $M \subset E^n$ is called a convex set, if for any $x, y \in M$ and any numbers $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$, the equality*

$$\lambda_1 M + \lambda_2 M = M \tag{2}$$

is fulfilled.

In order to show that a set M is a convex set, it is enough to show that (2) is fulfilled for any λ_1, λ_2. On the other hand, what is possible to say about the set M if λ_1 and λ_2 are fixed? It turns out that equality (2) gives convexity in case M is a closed set. Let us formulate this statement as a lemma.

Lemma 1. *Let λ_1 and λ_2 be fixed positive numbers such that $\lambda_1 + \lambda_2 = 1$. A closed set M is a convex set if and only if equality (2) is fulfilled.*

If M is not a closed set, then Lemma 1 is not correct. One can consider, as an example, the set of rational points from the segment $[0, 1]$ that have the degrees of two in the denominator. Then the equality (2) is fulfilled for $\lambda_1 = \lambda_2 = \frac{1}{2}$.

3.2 Matrix Convexity for Two Operators

Henceforth we shall assume that M is a convex set.

Let A, B be linear operators that operate in E^n and such that $A + B = E$. We shall denote the family of these operators by $\mathbf{R}$, $\mathbf{R} = \{A, B\}$.

Definition 3. *A set $M \subset E^n$ is called an $\mathbf{R}$- convex set if*

$$AM + BM = M. \tag{3}$$

Definition 3 differs from Definition 2 with respect to the linear operators that are represented by matrices in concrete bases and have replaced the scalars λ_1 and λ_2. It is natural to call this type convexity "matrix convexity" and the common convexity "scalar convexity." If $A = \lambda_1 E$, $B = \lambda_2 E$, where $\lambda_1, \lambda_2 \in (0, 1)$, then Definition 3 is the definition of the common, scalar convexity by virtue of (1).

Let us note that (3) is satisfied not only by scalar-convex sets but by other sets as well.

Example 1. Consider the two-dimensional space

$$E^2 = \left\{x = \left(x^1, x^2\right), x^i \in E^1\right\}.$$

Let the linear operators A and B be defined by the following matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $M = \{(0, 0)\,; (1, 0)\,; (0, 1)\,; (1, 1)\}$.

The given set consists of four points, which are the vertices of a square.

Matrix A transfers the points $(0, 1)$ and $(1, 1)$ to the points $(0, 0)$ and $(1, 0)$, respectively, while leaving the other two points in place. From this, $AM = \{(0, 0); (1, 0)\}$. Similarly, $BM = \{(0, 0)\,; (0, 1)\}$. It is not difficult to see that $AM + BM = \{(0, 0)\,; (1, 0)\,;\ (0, 1)\,;\ (1, 1)\} = M$. Thus, an example of a nonconvex set that satisfies (3) for certain A and B has been constructed.

Because Definition 3 is a generalization of the common convexity, the question under what conditions equation (3) guarantees scalar convexity arises naturally.

Theorem 4. *Let $\|A - B\| < 1$. Then scalar convexity of the set M follows from (3).*

Remark 1. Note that in case $A = \lambda_1 E B = \lambda_2 E \lambda_1, \lambda_2 \in [0, 1]$, the condition $\|A - B\| < 1$ turns into the condition $|\lambda_1 - \lambda_2| < 1$, which implies $\lambda_1 > 0$, $\lambda_2 > 0$. The last condition coordinates to that of Lemma 1.

Proof. Let us show that for any points $x, y \in M$, the midpoint $\bar{x} = \frac{1}{2}(x+y)$ of the segment $[x, y]$ belongs to the set M. The convexity of M will follow from this by virtue of Lemma 1 for $\lambda_1 = \lambda_2 = \frac{1}{2}$.

Let $x_1 = Ax + By$, $y_1 = Ay + Bx$. It follows from (3) that $x_1, y_1 \in M$. We note that

$$\frac{1}{2}(x_1 + y_1) = \frac{1}{2}(A(x+y) + B(x+y)) = \frac{1}{2}(A+B)(x+y) = \frac{1}{2}(x+y) = \bar{x},$$

that is, a midpoint of the segment $[x_1, y_1]$ is a midpoint of the segment $[x, y]$. Let us estimate the distance between x_1 and y_1:

$$\|x_1 - y_1\| = \|A(x-y) + B(y-x)\| = \|(A-B)(x-y)\| \leq \|A-B\| \cdot \|x-y\|.$$

This distance is already less than the distance between x and y by virtue of the hypothesis of the theorem.

Construct next the sequences

$$x_k = Ax_{k-1} + By_{k-1},\ y_k = Ay_{k-1} + Bx_{k-1};\ k = 1, 2, \ldots,\ x_0 = x,\ y_0 = y.$$

Using induction on k, one can show, as above, that

$$\frac{1}{2}(x_k + y_k) = \bar{x},$$

$$\|x_k - y_k\| \leq \|(A-B)\| \cdot \|x_{k-1} - y_{k-1}\| = \|(A-B)\|^k \cdot \|x - y\|$$

Because $\|A - B\| < 1$, the sequences $\{x_k\}$, $\{y_k\}$ converge to $\bar{x}$. From equality (3) it follows that $x_k, y_k \in M$ for all $k = 0, 1, 2, \ldots$. This relation and the fact that M is closed imply $\bar{x} \in M$. The theorem has been proved. ■

Let us study a class of **R**-convex sets. We will cite necessary and sufficient conditions for **R**-convexity. Set

$$M(x^*) = \{x \in M : \langle x, x^* \rangle = W_M(x^*)\}.$$

By definition, $W_M(x^*) = \sup\limits_{x \in M} \langle x, x^* \rangle$. Therefore either the set $M(x^*)$ is empty, or the supremum is attained on its points. Let us note that it is natural to assume that $M(0) = M$ in the case $x^* = 0$.

Theorem 5. *For the realization of equality (3) it is necessary that, for all $x^* \neq 0$, the following inclusions are fulfilled:*

$$M(x^*) \subset M(A^* x^*),\ M(x^*) \subset M(B^* x^*). \tag{4}$$

Proof. Let us consider the support function of the left part in (3)

$$\begin{aligned} W_{AM+BM}(x^*) &= \sup\{\langle x, x^* \rangle : x \in AM + BM\} \\ &= \sup\{\langle Ax + By, x^* \rangle : x, y \in M\} \\ &= \sup\{\langle x, A^* x^* \rangle + \langle y, B^* x^* \rangle : x, y \in M\} \\ &= \sup_{x \in M} \langle x, A^* x^* \rangle + \sup_{x \in M} \langle x, B^* x^* \rangle \\ &= W_M(A^* x^*) + W_M(B^* x^*). \end{aligned}$$

It follows from (3) that $W_M(A^*x^*) + W_M(B^*x^*) = W_M(x^*)$. From this relation and from the obvious equality $x^* = (A^* + B^*)x^*$ it follows that, for all $x \in M(x^*)$,

$$\sup_{y\in M} \langle y, A^*x^*\rangle + \sup_{y\in M} \langle y, B^*x^*\rangle - \langle x, (A^* + B^*)x^*\rangle = 0$$

or

$$\left[\sup_{y\in M} \langle y, A^*x^*\rangle - \langle x, A^*x^*\rangle\right] + \left[\sup_{y\in M} \langle y, B^*x^*\rangle - \langle x, B^*x^*\rangle\right] = 0. \qquad (5)$$

Each of the expressions in the square brackets is non-negative. Therefore, it follows from (3) that

$$\sup_{y\in M} \langle y, A^*x^*\rangle = \langle x, A^*x^*\rangle, \ \sup_{y\in M} \langle y, B^*x^*\rangle = \langle x, B^*x^*\rangle$$

and, hence, $x \in M(A^*x^*)$, and $x \in M(B^*x^*)$, which proves the theorem. ■

The set $\{x \in E^n : \langle x, x^*\rangle \le c\}$ is understood as a half-space. The half-space is determined by the vector x^* and the number c.

It is known that every convex set can be represented as the intersection of half-spaces:

$$M = \bigcap_{x^*\in H(M)} \{x \in E^n : \langle x, x^*\rangle \le c(x^*)\}, \qquad (6)$$

where $H(M)$ is a some set of nonzero vectors from E^n, $c(x^*)$ is a number, probably equal to $+\infty$.

Lemma 2. *Let a set M be represented in the form*

$$M = \bigcap_{\alpha\in a} \{x \in E^n : f_\alpha(x) \le 0\},$$

where $\{f_\alpha\}$ is a set of convex functions, a is an arbitrary set of indices; and there exists some number α_0 such that the inequality $f_{\alpha_0}(x) < 0$ is correct for any $x \in M$. Let $a_0 = a\{\alpha_0\}$ and $M_0 = \bigcap_{\alpha\in a_0} \{x : f_\alpha(x) \le 0\}$. Then $M = M_0$.

Proof. Let us assume the opposite. Then, there exists $x_0 \in M_0$ such that $x_0 \notin M$. Because $f_\alpha(x_0) \le 0$ is fulfilled for any $\alpha \in a_0$, then the relation $x_0 \notin M$ implies $f_{\alpha_0}(x_0) > 0$. Because the function f_α is continuous by virtue of convexity, the set M is closed. Therefore, there exists $x_1 \in M$, which is the point nearest to x_0. Let us consider the point $x_\lambda = \lambda x_1 + (1-\lambda)x_0$. Because x_1 is the point nearest to x_0, then $x_\lambda \notin M$ for all $\lambda \in [0,1)$. From the convexity of M_0 it follows that $x_\lambda \in M_0$ for $\lambda \in [0,1]$. The function $g(\lambda) = f_{\alpha_0}(x_\lambda)$ is continuous on the segment $[0,1]$. Moreover, $g(0) > 0$, but $g(1) < 0$ from the hypothesis of the lemma. This implies the existence of a $\lambda_0 \in (0,1)$ such that $g(\lambda_0) = 0$, that is, $f_{\alpha_o}(x_{\lambda_0}) = 0$. Comparing the given equality to the inclusion $x_{\lambda_0} \in M_0$ we obtain that, for any $\alpha \in a$, $f_\alpha(x_{\lambda_0}) \le 0$, that is, $x_{\lambda_0} \in M$. But as $\lambda_o < 1$, we obtain a contradiction. The lemma has been proved. ■

Theorem 6. *Let a set M be represented in the form (6). Then for the fulfillment of the equality (3) it is enough to show that, for all $x^* \in H(M)$, inclusions (4) are fulfilled.*

Proof. Let $x^* \in H(M)$. From Lemma 2, it follows that it is enough to consider the case $M(x^*) \neq \emptyset$. Then there exists $x \in M(x^*)$ such that

$$\sup_{y \in M} \langle y, A^* x^* \rangle = \langle x, A^* x^* \rangle, \ \sup_{y \in M} \langle y, B^* x^* \rangle = \langle x, B^* x^* \rangle.$$

It follows from this that equality (5) holds, and, consequently,

$$\begin{aligned} W_{AM+BM}(x^*) &= \sup \{ \langle y, x^* \rangle : y \in AM + BM \} \\ &= \langle x, (A^* + B^*) x^* \rangle \leq \sup_{x \in M} \langle x, x^* \rangle \leq c(x^*). \end{aligned}$$

The last inequality means that the left part of (3) is included in the right part. The reverse inclusion is obvious. The theorem has been proved. ■

Theorems 5 and 6 give a description of the class of **R**-convex sets. However, this description is obviously not sufficient. The results deduced below give a more constructive description of **R**-convexity in terms of H-convexity. Let us give the definition of H-convexity.

Definition 4. *Let H be a subset of the unit sphere in E^n, that is, $H \subset \{x^* \in E^n : \|x^*\| = 1\}$. The set M is called H-convex if it can be written in the form*

$$M = \bigcap_{x^* \in H} \{x \in E^n : \langle x, x^* \rangle \leq c(x^*)\}, \tag{7}$$

where the scalar $c(x^*)$ can accept any value (even $+\infty$).

The representation (7) means that the H-convex set M is defined by an intersection of half-spaces that are described only by vectors $x^* \in H$. Other half-spaces do not participate in the construction of M.

Note that a set of the form (6) is a $H(M)$-convex set. In case H coincides with the entire unit sphere, H-convexity becomes common convexity.

Let us consider another example of H-convex sets.

Example 2. Let $H = \{\pm e_i, i = 1, \ldots, n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots 0)$ is the i-th unit vector. Then, the parallelepipeds with edges parallel to the axes of the coordinates are H-convex sets. The set M is an H-convex set if and only if it is represented in form

$$M = \left\{ x = \left(x^1, \ldots, x^n\right) : a_i \leq x^i \leq b_i, \ i = 1, \ldots, n \right\},$$

where the numbers a_i can accept the value $-\infty$, and b_i can accept the value $+\infty$.

Let us connect the set H to the operators A and B. We shall denote by H a set of unit vectors $x^* \in E^n$ that satisfy the conditions:

a) $A^*x^* = \lambda_A(x^*)\,x^*, \; B^*x^* = \lambda_B(x^*)\,x^*$;
b) the numbers $\lambda_A(x^*)$ and $\lambda_B(x^*)$ are non-negative.

Thus, vectors $x^* \in H$ are eigenvectors of the operators A^* and B^* with non-negative eigenvalues.

We shall assume subsequently that H is the set determined as above.

Theorem 7. *Let the set M be an* **R**-*convex and scalar-convex set, let* $\operatorname{int} M \neq \emptyset$. *Then M is an H-convex set.*

Proof. It is known [15] that, for a convex set M, if $x_0 \in \partial M$ then, in any neighborhood of x_0, there exists a point $x_1 \in \partial M$ such that the cone of normal directions at it is spanned by one vector. Let us denote this vector by $n(x_1)$ and assume that $\|n(x_1)\| = 1$.

Let us suppose that M is not a H-convex set and consider the set

$$M_1 = \bigcap_{x^* \in H} \{x \in E^n : \langle x, x^* \rangle \leq W_M(x^*)\}.$$

The set M_1 is the H-convex hull of M. Because $M \neq M_1$, there exists a point $x_0 \in \partial M$ such that $x_0 \in \operatorname{int} M_1$. Indeed, if such a point does not exist, then $\operatorname{int} M = \operatorname{int} M_1$ and these sets would coincide by virtue of the closure of M and M_1.

The existence of a point $x_1 \in \partial M \bigcap \operatorname{int} M_1$ for which the cone of normals is spanned by one vector has been noticed above. It means that if $x_1 \in M(x^*)$, then $x^* = \lambda_0 n(x_1)$ for some $\lambda_0 > 0$. It is clear that $n(x_1) \notin H$. From the definition of H it follows that $n(x_1)$ cannot be eigenvector of the operators A^* and B^* with non-negative eigenvalues. Let $A^*n(x_1) \neq \lambda n(x_1)$ for $\lambda \geq 0$. But this means that $A^*n(x_1)$ does not belong to the cone of normals at the point x_1 and, therefore, $x_1 \notin M(A^*n(x_1))$. Thus, the necessary conditions of Theorem 5 are violated. The obtained contradiction proves the theorem. ■

Theorem 8. *Let M be an H-convex set. Then M is an* **R**-*convex set.*

Proof. The proof follows from Theorem 6 for $H(M) = H$. ■

Let us discuss the hypothesis of Theorem 7. The necessity for requiring the scalar convexity of M has been verified by Example 1 in which M is not only an H-convex set but also a scalar-convex set. Moreover, the set M is an **R**-convex set. The requirement $\operatorname{int} M \neq \emptyset$ is also essential.

Example 3. Let $M = \{m\}$. In other words, M consists of one point. Then the equality (3) is obviously fulfilled. However, it is not difficult to select A and B for which a point is not a H-convex set. Thus, the operators A^* and B^* may have only one-dimensional subspace of eigenvectors. In this case, some half-spaces and hyperplanes of dimension $n-1$ will be H-convex sets.

Let us consider various H-convex sets, where H is connected to A, B.

Example 4. Let

$$A = \operatorname{diag}\{\alpha_1, \ldots, \alpha_n\}, \; B = \operatorname{diag}\{\beta_1, \ldots, \beta_n\}, \; \alpha_i, \beta_i \geq 0.$$

Then the set H will be the set from Example 2 and the H-convex sets will be the corresponding ones.

Example 5. Let A, B be represented by diagonal matrices in the same way as in Example 4 but such that

$$\alpha_1 = \cdots = \alpha_{m_1} = \alpha^1, \alpha_{m_1+1} = \cdots = \alpha_{m_2} = \alpha^2, \ldots, \alpha_{m_{k-1}+1} = \cdots = \alpha_n = \alpha^k,$$

$$\beta_1 = \cdots = \beta_{m_1} = \beta^1, \beta_{m_1+1} = \cdots = \beta_{m_2} = \beta^2, \ldots, \beta_{m_{k-1}+1} = \cdots = \beta_n = \beta^k.$$

Set $n_i = m_i - m_{i-1}$, $i = 1, \ldots, k$, $m_0 = 0$. In this case, the H-convex sets have the form

$$M = \left\{x = \left(x^1, \ldots, x^k\right) : x^i \in M_i\right\},$$

where x^i is a vector from the space E^{n_i}, and M_i is a convex subset in E^{n_i}.

Example 6. Let us consider the case

$$A = \begin{pmatrix} \alpha & \gamma \\ 0 & \alpha \end{pmatrix}, B = \begin{pmatrix} \beta & -\gamma \\ 0 & \beta \end{pmatrix},$$

where $\alpha, \beta > 0$, and γ are arbitrary numbers. In this case, the vectors $\pm(0, 1)$ are eigenvectors of A^* and B^*.

The sets of the form $M = \left\{x = \left(x^1, x^2\right) : a \leq x^2 \leq b\right\}$ are H-convex sets, where a may take the value $-\infty$, and b the value $+\infty$.

3.3 Matrix Convexity for Various Families of Operators

We generalize the results of the previous sections to the case of several operators. Let $\mathbf{R}_k = \{A_1, \ldots, A_k\}$ be a family of linear operators operating in E^n such that $A_1 + \cdots + A_k = E$.

Definition 5. *A set $M \subset E^n$ is called $\mathbf{R}_k$-convex if*

$$A_1 M + A_2 M + \cdots + A_k M = M. \tag{8}$$

Let $A_{i_1}, \ldots, A_{i_m}$ $(1 \leq m < k)$ be operators from $\mathbf{R}_k$, and define $A = A_{i_1} + \cdots + A_{i_m}$ and $B = E - A$.

Lemma 3. *Let M be an $\mathbf{R}_k$-convex set. Then M is an $\mathbf{R}$-convex set, $\mathbf{R} = \{A, B\}$.*

Proof. Let $A_{i_{m+1}}, \dots, A_{i_k}$ be operators from $\mathbf{R}_k$.

Then $B = A_{i_{m+1}} + \dots + A_{i_k}$. Consider arbitrary $x, y \in M$. Because M is an $\mathbf{R}_k$-convex set, then

$$Ax + By = A_{i_1}x + \dots + A_{i_m}x + A_{i_{m+1}}y + A_{i_k}y \in M$$

which proves the $\mathbf{R}$-convexity of M. ∎

Theorem 9. *Suppose that, for some set of operators $A_{i_1}, \dots, A_{i_m}$ from $\mathbf{R}_k$, the inequality $\|E - 2(A_{i_1} + \dots + A_{i_m})\| < 1$ is fulfilled. Then an $\mathbf{R}_k$-convex set is also scalar-convex.*

Proof. Let us set $A = A_{i_1} + \dots + A_{i_m}$, $B = E - A$. It follows from Lemma 3 that if M is an $\mathbf{R}_k$-convex set, then M is an $\mathbf{R}$-convex set. From the hypothesis of the theorem, it follows that $\|A - B\| = \|E - 2A\| < 1$. Applying Theorem 4, we obtain the scalar convexity of M. ∎

We shall denote by H_k a set of unit vectors $x^* \in E^n$ that satisfy the following conditions:

(a) $A_i^* x^* = \lambda_i(x^*)\, x^*$, $i = 1, \dots, k$;
(b) the numbers $\lambda_i(x^*) \geq 0$, $i = 1, \dots, k$.

Theorem 10. *Let M be an $\mathbf{R}_k$-convex and scalar-convex set with* $\operatorname{int} M \neq \emptyset$. *Then M is an H_k-convex set.*

Proof. Denote by H^j the set of unit eigenvectors of A_j^* with corresponding non-negative eigenvalues, that is, $x^* \in H^j$ if $A_j^* x^* = \lambda x^*$, where $\lambda \geq 0$.

From the definitions of H_k and H^j, it follows that

$$H_k = \bigcap_{j=1}^{k} H^j. \tag{9}$$

Set $A = A_j$, $B = E - A$. By Lemma 3, the set M is an $\mathbf{R}$-convex set. By Theorem 7, it is an H^j-convex set. This implies that M may be represented as the intersection of half-spaces of the form $\{x \in E^n : \langle x, x^* \rangle \leq c(x^*)\}$, $x^* \in H^j$.

Because this holds for any j, then it follows from (9) that $x^* \in H_k$ and, therefore, M is a an H_k-convex set. ∎

Theorem 11. *Let M be an H_k-convex set. Then M is an $\mathbf{R}_k$-convex set.*

Proof. We shall validate this statement using mathematical induction on k. For $k = 2$, the conclusion of the theorem follows from Theorem 8.

Assume that the inductive hypothesis is correct for $k - 1$ and prove it for k.

Set $A = A_1 + \dots + A_{k-1}$, $B = A_k$, and consider the case $detA = 0$. Denote by ε_0 the minimum modulus from all nonzero eigenvalues of operator A (real and complex values). Then, for all $\varepsilon \in (0, \varepsilon_0)$, the operator $A + \varepsilon E$ has an inverse operator.

Set
$$A_i^\varepsilon = A_i + \frac{\varepsilon}{k-1}E, A^\varepsilon = A + \varepsilon E.$$

Then $\sum\limits_{i=1}^{k-1} A_i^\varepsilon = A^\varepsilon$.

Set $\bar{A}_i^\varepsilon = (A^\varepsilon)^{-1} A_i^\varepsilon$,

$$H_{k-1}^\varepsilon = \left\{x^* \in \partial S : \bar{A}_i^\varepsilon x^* = \lambda_i^\varepsilon (x^*) x^*,\ \lambda_i^\varepsilon (x^*) \geq 0,\ i = 1, \ldots, k-1\right\}.$$

Because, for any $x^* \in H_k$, the relation

$$\lambda_i^\varepsilon (x^*) = \left(\lambda_i (x^*) + \frac{\varepsilon}{k-1}\right) \Big/ \left(\sum_{i=1}^{k-1} \lambda_i (x^*) + \varepsilon\right) \geq 0;$$

is fulfilled, then $H_k \subset H_{k-1}^\varepsilon$. From this, it follows that M is an H_{k-1}^ε-convex set. By the inductive hypothesis $\sum\limits_{i=1}^{k-1} \bar{A}_i^\varepsilon M = M$. Multiplying both sides of the equality by A^ε we obtain $\sum\limits_{i=1}^{k-1} A_i^\varepsilon M = A^\varepsilon M$. From this, we have

$$\sum_{i=1}^{k-1} A_i^\varepsilon M + A_k M = A^\varepsilon M + A_k M.$$

Because the operator $A^\varepsilon + A_k = A + \varepsilon E + A_k = (1+\varepsilon) E$ has an inverse operator, we obtain, by analogous reasoning, $A^\varepsilon M + A_k M = (1+\varepsilon) EM$ from which

$$(1+\varepsilon)^{-1} \left(\sum_{i=1}^{k-1} A_i^\varepsilon M + A_k M\right) = M.$$

Let $x_i \in M$ be arbitrary points and set

$$\bar{x} = \sum_{i=1}^{k} A_i x_i, x_\varepsilon = (1+\varepsilon)^{-1} \left(\sum_{i=1}^{k-1} A_i^\varepsilon x_i + A_k x_k\right).$$

Because $\lim\limits_{\varepsilon \to 0} A_i^\varepsilon = A_i$, then $\lim\limits_{\varepsilon \to 0} x_\varepsilon = \bar{x}$. Whereas for any $\varepsilon \in (0, \varepsilon_0)$ the inclusion $x_\varepsilon \in M$ is fulfilled and M is a closed set, then $\bar{x} \in M$. It follows that M is an $\mathbf{R}_k$-convex set.

The case $det A \neq 0$ is more simple and it is easy to reduce it to the previous reasoning with $\varepsilon = 0$. The theorem has been proved. ■

Next, we generalize the results to the case of infinite number of operators.

Set $\mathbf{R}_\infty = \{A(t), t \in [0,1]\}$, where $A(t)$ is a linear operator that operates in E^n. We assume that $\mathbf{R}_\infty$ is a bounded and measurable family of operators and that $\int\limits_0^1 A(t)\, dt = E$.

Definition 6. *A set M is called* $\mathbf{R}_\infty$*-convex if*

$$\int_0^1 A(t)\, M dt = M. \tag{10}$$

Let Ω_i, $i = 1, \ldots, k$, be a collection of measurable subsets of the segment $[0,1]$ such that $\Omega_i \bigcap \Omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{k} \Omega_i = [0,1]$.

Define A_i as

$$A_i = \int_{\Omega_i} A(t)\, dt, \mathbf{R}_k = \{A_1, \ldots, A_k\}.$$

Lemma 4. *Let M be an* $\mathbf{R}_\infty$*-convex set. Then M is an* $\mathbf{R}_k$*-convex set.*

Proof. Let $x_i \in M$ be arbitrary points, $i = 1, \ldots, k$. Let us set $x(t) = x_i$, $t \in \Omega_i$. From the $\mathbf{R}_\infty$-convexity it follows that

$$\sum_{i=1}^{k} A_i x_i = \int_0^1 A(t)\, x(t)\, dt \in M,$$

which implies the $\mathbf{R}_k$-convexity of M. The lemma is proved. ■

We shall denote by H_∞ the set of all unit vectors $x^* \in E^n$ such that, for nearly all $t \in [0,1]$, the following conditions are fulfilled:

(a) $A^*(t) = \lambda(t|x^*)\, x^*$,
(b) the numbers $\lambda(t|x^*) \geq 0$.

Lemma 5. *Let M be an* $\mathbf{R}_\infty$*-convex set. Then it is a scalar-convex set.*

Proof. From Corollary 1, it follows that $\int_0^1 A(t)\, M dt$ is a convex set. From this and from (10), the convexity of M is implied. ■

Theorem 12. *Let M be an* $\mathbf{R}_\infty$*-convex set and* $\operatorname{int} M \neq \emptyset$*. Then M is an* H_∞*-convex set.*

Proof. Let us set

$$A_{t_1,t_2} = \int_{t_1}^{t_2} A(t)\, dt, B_{t_1,t_2} = E - A_{t_1,t_2}, \mathbf{R}_{t_1,t_2} = \{A_{t_1,t_2}, B_{t_1,t_2}\},$$

where $t_1, t_2 \in [0,1]$, $t_1 < t_2$.

From Lemma 4, it follows that M is an $\mathbf{R}_{t_1,t_2}$-convex set.

Let

$$H_{t_1,t_2} = \left\{x^* \in \partial S : A^*_{t_1,t_2} x^* = \lambda_{t_1,t_2}(x^*)\, x^*, \lambda_{t_1,t_2}(x^*) \geq 0\right\}.$$

Because for $x^* \in H_\infty$, then $\lambda_{t_1,t_2}(x^*) = \int\limits_{t_1}^{t_2} \lambda(t|x^*)\, dt \geq 0$, and $H_\infty \subset H_{t_1,t_2}$.

Whereas M is $\mathbf{R}_{t_1,t_2}$-convex set then, from Theorem 7, it follows that M is an H_{t_1,t_2}-convex set. Let us show that $H_\infty = \bigcap\limits_{t_1,t_2 \in [0,1]} H_{t_1,t_2}$. It will imply the H_∞-convexity of M.

Let $t \in [0,1)$ be an arbitrary number, and $\Delta t > 0$ such that $t + \Delta t \in [0,1]$. Assume that, for any $\Delta t > 0$, $x^* \in H_{t,t+\Delta t}$ and, hence, the following equality is fulfilled:

$$\frac{1}{\Delta t} \int\limits_{t}^{t+\Delta t} A^*(\tau)\, d\tau x^* = \lambda_{t,t+\Delta t}(x^*)\, x^*,$$

where $\lambda_{t,t+\Delta t}(x^*) \geq 0$.

The limit, as $\Delta t \to 0$, of the left side of the equality exists for nearly all $t \in [0,1]$. Therefore, passing to the limit, we obtain that, for nearly all t,

$$A^*(t)\, x^* = \lambda(t|x^*)\, x^*, \lambda(t|x^*) \geq 0.$$

From this, it follows that $x^* \in H_\infty$. The theorem has been proved. ■

Theorem 13. *Let M be an H_∞-convex set. Then M is an $\mathbf{R}_\infty$-convex set.*

Proof. Set $C = \sup\{\|A(t)\|, t \in [0,1]\}$. Let $x(t)$ be an arbitrary bounded measurable function such that $x(t) \in M$ for all $t \in [0,1]$. For any $\varepsilon > 0$ there exists a collection of measurable subsets Ω_i, $i = 1, \ldots, k$, of the segment $[0,1]$ such that $\Omega_i \bigcap \Omega_j = \emptyset$, $i \neq j$ and $\bigcup\limits_{i=1}^{k} \Omega_i = [0,1]$, and a collection of points $x_i \in M$ such that for the function $x_\varepsilon(t) = x_i$, $t \in \Omega_i$, the following inequality is fulfilled:

$$\int\limits_0^1 \|x_\varepsilon(t) - x(t)\|\, dt \leq \frac{\varepsilon}{C}.$$

It follows from this that

$$\left\| \int\limits_0^1 A(t)\, x(t)\, dt - \int\limits_0^1 A(t)\, x_\varepsilon(t)\, dt \right\| \leq \int\limits_0^1 \|A(t)\| \cdot \|x(t) - x_\varepsilon(t)\|\, dt \leq \varepsilon.$$

Let H_k be the same set as previously. Because, for any $x^* \in H_\infty$,

$$\lambda_i(x^*) = \int\limits_{\Omega_i} \lambda(t|x^*)\, dt \geq 0,$$

then $H_\infty \subset H_k$. Therefore, M is an H_k-convex set. Theorem 11 implies that M is an $\mathbf{R}_k$-convex set. From this it follows that

$$\int_0^1 A(t)\, x_\varepsilon(t)\, dt = \sum_{i=1}^k A_i x_i \in M.$$

From the arbitrariness of $\varepsilon > 0$ and the closure of M we receive

$$\int_0^1 A(t)\, x(t)\, dt \in M,$$

which implies the $\mathbf{R}_\infty$-convexity of M. ■

3.4 *H*-convex Sets and Integration of Linear Multivalued Mappings

We state and prove some lemmas that are needed later on.

Let H be an arbitrary subset of unit vectors from E^n, and let $A : E^n \to E^n$ be a linear operator.

Lemma 6. *Let the operator A have an inverse operator, and assume that $A^* x^* \in \operatorname{con} H$ is fulfilled for any $x^* \in H$. If M is H-convex, then $A^{-1}M$ is also H-convex.*

Proof. The set M has the form

$$M = \bigcap_{x^* \in H} \{x \in E^n : \langle x, x^* \rangle \le c(x^*)\}, \tag{11}$$

where $c(x^*)$ may take the value $+\infty$.

From this

$$\begin{aligned} A^{-1}M &= \bigcap_{x^* \in H} \{x \in E^n : \langle Ax, x^* \rangle \le c(x^*)\} \\ &= \bigcap_{x^* \in H} \{x \in E^n : \langle x, A^* x^* \rangle \le c(x^*)\}. \end{aligned}$$

From the hypothesis of the lemma $\dfrac{A^* x^*}{\|A^* x^*\|} \in H$. Therefore,

$$A^{-1}M = \bigcap_{x^* \in H} \left\{x \in E^n : \left\langle x, \frac{A^* x^*}{\|A^* x^*\|} \right\rangle \le \frac{c(x^*)}{\|A^* x^*\|} \right\}.$$

Thus, $A^{-1}M$ is an H-convex set and the lemma is proved. ■

Lemma 7. *Let the operator A have an inverse operator, and assume that $A^* x^* = \lambda(x^*)\, x^*$ is fulfilled for all $x^* \in H$, where $\lambda(x^*)$ is a number. Moreover, if $x^* \in H$, then $(-x^*) \in H$. Then, if M is an H-convex, the set AM is also an H-convex set.*

Proof. Let M have the form (11), and assume that $\lambda(x^*) > 0$. Then

$$\begin{aligned} AM &= \bigcap_{x^* \in H} \left\{x \in E^n : \left\langle A^{-1}x, x^*\right\rangle \le c(x^*)\right\} \\ &= \bigcap_{x^* \in H} \left\{x \in E^n : \left\langle x, \left(A^{-1}\right)^* x^*\right\rangle \le c(x^*)\right\} \\ &= \bigcap_{x^* \in H} \left\{x \in E^n : \frac{1}{\lambda(x^*)} \langle x, x^*\rangle \le c(x^*)\right\} \\ &= \bigcap_{x^* \in H} \left\{x \in E^n : \langle x, x^*\rangle \le \lambda(x^*)\, c(x^*)\right\} \end{aligned}$$

that is, AM is an H-convex set.

If $\lambda(x^*) < 0$, then

$$AM = \bigcap_{x^* \in H} \{x \in E^n : \langle x, -x^*\rangle \le -\lambda(x^*)\, c(x^*)\}.$$

The lemma has been proved. ■

Remark 2. The hypothesis of Lemma 6 follows from the hypothesis of Lemma 7.

Let us consider a family of bounded and measurable on t operators $\{A(t), t \in [0, \theta]\}$ that operate in the space E^n, where θ is a fixed number.

In Subsection 3.3, a family of operators is determined on the interval $[0, 1]$ $(\theta = 1)$ that satisfies the condition

$$\int_0^1 A(t)\, dt = E. \tag{12}$$

The first condition $(\theta = 1)$ is not essential, and it may be removed by a change of the variable $t = \theta\tau$, $\tau \in [0, 1]$. The condition (12), or a more general condition that is expressed in terms of existence of the inverse operator to $\int_0^1 A(t)\, dt$ is more essential. In the current section, the case when this condition is not fulfilled will be studied.

In the current subsection, we shall understand as H a set of unit vectors $x^* \in E^n$ that satisfy the conditions:

(a) $A^*(t)\, x^* = \lambda(t|x^*)\, x^*$ for any $t \in [0, \theta]$, where $\lambda(t|x^*)$ is a number;
(b) the numerical function $\lambda(\,\cdot\mid x^*)$ does not change sign on the interval $[0, \theta]$ for fixed x^*.

Let $A = \int_0^\theta A(t)\, dt$.

Note that, for any x^*,

$$A^* x^* = \int_0^\theta \lambda\,(t|x^*)\,dt x^*.$$

Theorem 14. *Let M be an H-convex set, and let $x\,(s)$, $s \in [0, \theta]$, be a measurable and bounded function with values in E^n. If, for each $s \in [0, \theta]$, the inclusion*

$$Ax\,(s) \in M \tag{13}$$

is fulfilled, then

$$\int_0^\theta A\,(t)\,x\,(t)\,dt \in M. \tag{14}$$

Proof. Without loss of the generality, it is possible to assume that $\theta = 1$. Let us consider the case when there exists an inverse operator A^{-1}. From (13), it follows that for all, $s \in [0, \theta]$,

$$x\,(s) \in A^{-1}M.$$

From Lemma 7 and from Remark 2, it follows that the set $A^{-1}M$ is an H-convex set. From the definition of operator A

$$\int_0^1 A^{-1}A\,(t)\,dt = E. \tag{15}$$

Let us take advantage of Theorem 13. Then, from (15), it follows that

$$\int_0^1 A^{-1}A\,(t)\,x\,(t)\,dt \in A^{-1}M.$$

From this the inclusion (14) is implied.

Consider the case when A may not have an inverse operator. Let

$$A_\varepsilon\,(t) = A\,(t) + \varepsilon E, \quad A_\varepsilon = \int_0^1 A_\varepsilon\,(t)\,dt = A + \varepsilon E.$$

For small enough positive ε, the operator A_ε has an inverse operator. Whereas $Ax\,(s) \in M$, then $A_\varepsilon x\,(s) = Ax\,(s) + \varepsilon x\,(s) \in M + \varepsilon DS$, where D is a constant such that $\|x\,(s)\| \le D$ for $s \in [0, 1]$. Let us assume that M has the form (11). Then $A_\varepsilon x\,(s) \in M + \varepsilon DS \subset M_\varepsilon$, where

$$M_\varepsilon = \bigcap_{x^* \in H} \{x \in E^n : \langle x, x^* \rangle \le c\,(x^*) + \varepsilon D\}.$$

It follows from the first part of proof that

$$\int_0^1 A_\varepsilon(t)\, x(t)\, dt \in M_\varepsilon.$$

From this, whereas $\left\| \int_0^1 x(t)\, dt \right\| \le D$, then

$$\int_0^1 A(t)\, x(t)\, dt \in M_\varepsilon - \varepsilon \int_0^1 x(t)\, dt \subset M_\varepsilon + \varepsilon DS \subset M_{2\varepsilon}.$$

By virtue of the arbitrariness of $\varepsilon > 0$ we obtain the inclusion (14) and the theorem is proved. ∎

Theorem 15. *Let the hypothesis of Theorem 14 be fulfilled; let $\tau \in [0, \theta]$ be an arbitrary fixed number, and assume that $0 \in M$. Then*

$$\int_0^\tau A(t)\, x(t)\, dt \in M.$$

Proof. Let us introduce the function

$$x_1(s) = \begin{cases} x(s), & s \in [0, \tau], \\ 0, & s \in (\tau, \theta]. \end{cases}$$

Because $0 \in M$, then $Ax_1(s) \in M$ for $s \in [0, \theta]$. From this and from Theorem 14 it follows that

$$\int_0^\tau A(t)\, x(t)\, dt = \int_0^\theta A(t)\, x_1(t)\, dt \in M.$$

The theorem has been proved. ∎

Corollaries 2 and 3 are implied from Theorems 14 and 15.

Corollary 2. *Let M be an H-convex set and assume that, for the set $W \subset E^n$, the inclusion $AW \subset M$ is fulfilled. Then*

$$\int_0^\theta A(t)\, W dt \subset M.$$

Corollary 3. *Let the hypothesis of Corollary 2 be fulfilled; and assume that $0 \in M$. Then, for any $\tau \in [0, \theta]$,*

$$\int_0^\tau A(t)\, W dt \subset M.$$

From Lemma 7 and Corollary 2, we have the following:

Corollary 4. *If M is an H-convex set and the operator A has an inverse operator, then*

$$\int_0^\theta A(t)\,M dt = AM. \tag{16}$$

Theorem 16. *Let M be an H-convex and compact set. Then (16) is fulfilled.*

Proof. Set $B_\varepsilon = \varepsilon A + \varepsilon^2 E$. The vectors $x^* \in H$ are the eigenvectors of the operator A^*, and, therefore, of the operator B_ε^*. For small enough $\varepsilon > 0$, the corresponding eigenvalues of the operator B_ε^* have the same sign with the eigenvalues of the operator A^*, and, therefore, of the operators $A^*(t)$. (We assume that zero has a sign identical to the sign of any number.)

Let $m \in M$. Then, as $0 \in M - m$,

$$\begin{aligned}\int_0^\theta A(t)\,M dt &= \int_0^\theta A(t)\,(M-m)\,dt + Am\\ &\subset \int_0^\theta A(t)\,(M-m)\,dt + B_\varepsilon\,(M-m) + Am.\end{aligned}$$

For small enough $\varepsilon > 0$, the operator $A + B_\varepsilon$ has an inverse operator. From this and from Corollary 4

$$\int_0^\theta A(t)\,(M-m)\,dt + B_\varepsilon\,(M-m) = (A + B_\varepsilon)\,(M-m).$$

By virtue of the arbitrariness of ε and the compactness of M we obtain

$$\int_0^\theta A(t)\,M dt \subset A(M-m) + Am = AM.$$

The reverse inclusion is obvious. The theorem has been proved. ■

4 Operational Constructions in Differential Games

4.1 Dynamics of Game Problems

Consider a dynamic system described by the differential equation

$$\dot{z} = f(z, u, v), \tag{17}$$

where $z \in E^n$, $u \in U$, $v \in V$, U and V are compact in Euclidean spaces.

The players P (pursuer) and E (evader) dispose the parameters u and v respectively. By admissible controls for players P and E we shall understand the functions $u(t)$ and $v(t)$ with values in U and V respectively. The sets of all admissible controls of players P and E determined on a segment $[a, b]$ (half-interval $[a, b)$) will be denoted by $U[a, b]$, $V[a, b]$ ($U[a, b)$, $V[a, b)$) respectively.

We shall assume further that the function f and the sets U and V satisfy the following assumptions:

Assumption 1 *The function $f(z, u, v)$ is continuous and locally Lipschitz w.r.t. z (i.e., the function satisfies a Lipschitz condition w.r.t. z on every compact set $K \subset E^n$ with the Lipschitz constant L_K depending on K).*

Assumption 2 *There exists a constant $C \geq 0$ such that, for all $z \in E^n$, $u \in U$, and $v \in V$,*

$$|\langle z, f(z, u, v)\rangle| \leq C\left(1 + \|z\|^2\right).$$

Assumption 3 *The set $f(z, U, v)$ is a convex set for all $z \in E^n$, and $v \in V$.*

Assumptions 1 and 2 guarantee the existence, uniqueness, and continuity of the solution $z(t)$ to equation (17) on all semi-axis $[0, +\infty)$ for arbitrary initial condition $z(0) = z_0$ and any admissible controls $u(t)$ and $v(t)$ for players P and E in place of the parameters u and v in (17).

Denote by $z(t|u(\cdot), v(\cdot), z_0)$ the solution $z(t)$ to equation (17) corresponding with $u(t)$, $v(t)$, and the initial condition $z(0) = z_0$.

Consider an arbitrary interval $[0, \theta]$, $\theta < +\infty$. Assumption 3 guarantees, in the topology of uniform convergence on segment $[0, \theta]$, compactness of the solutions set corresponding with various admissible controls $u(\cdot)$ for player P and the initial position z_0. This fact remains valid even if the initial position z_0 is not fixed and runs instead over some compact set $K \subset E^n$.

From the described property, it follows that if $u_k(\cdot) \in U[0, \theta]$, $x_k \in K$, $k = 1, 2, \ldots$, are some sequences, and

$$z_k(t) = z(t|u_k(\cdot), v(\cdot), x_k)$$

is the sequence of corresponding solutions to equation (17), then there exists a subsequence $\{z_{k_m}(\cdot)\}$ of the sequence $\{z_k(\cdot)\}$ that converges uniformly on $[0, \theta]$ to the function $z_0(\cdot)$. Moreover, there exist $u(\cdot) \in U[0, \theta]$ and $x \in K$ such that

$$z_0(t) = z(t|u(\cdot), v(\cdot), x).$$

The same statement is valid if we consider directedness instead of sequences. This question will be addressed below.

The aims of the players are described with the help of a terminal set $M \subset E^n$ and a set of phase constraints $N \subset E^n$. The sets M and N are assumed to be closed, moreover, $M \subset N$.

Let us fix a moment $\theta > 0$. The aim of player P is to achieve the inclusions $z(\theta) \in M$, $z(t) \in N$, for all $t \in [0, \theta]$, i.e., to draw a trajectory $z(t)$ on M at moment θ, while keeping it in the set N. The aim of player E is opposite and is to achieve either $z(\theta) \notin M$ or, for some $t < \theta$, $z(t) \notin N$.

Various strategies for player P may be used. There are ε-strategies in which the greatest informational discrimination of the opponent is assumed: player E informs player P of his control for some time $\varepsilon > 0$ in advance. Moreover, player P uses the information about the current position. Because player E disposes the parameter ε, ε-strategies are equivalent to the strategies in which player P chooses his current control knowing the initial position and the entire prehistory of the opponent's actions. These strategies are constructed on the basis of some Volterra mappings. Strategies in which player P chooses his current control knowing the initial position and the current control of the opponent are particular cases of the latter strategies. Such a strategy will be called counter-strategy.

4.2 Convergence in the Set of Closed Subsets

In order to describe the initial positions sets that are favorable for any player under the construction of strategies for the players, it is necessary to use various operations on sets, namely addition, union, and intersection. It is also necessary to consider sequences and directedness of sets and to study questions of their convergence.

In a space of closed subsets, it is possible to introduce the concept of convergence in various ways. The S-convergence [12], which describes convergence of unbounded sets, is used here. In the case of bounded sets, S-convergence coincides with convergence in Hausdorff metric

Let us recall the following definition.

Definition 7. *Let X be an arbitrary set, and I an ordered set, i.e., such that for any $\alpha_1, \alpha_2 \in I$ we can find $\alpha \in I$ such that $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$. A mapping $\alpha \to x_\alpha$ of the set I into X is said to be a directedness and is denoted by $\{x_\alpha\}$, $\alpha \in I$.*

Definition 8. *A directedness $\{y_\beta\}$, $\beta \in J$, is said to be a subdirectedness of a directedness $\{x_\alpha\}$, $\alpha \in I$, if, for any $\alpha \in I$, there exists an index $\beta(\alpha) \in J$ such that, for any $\beta' \in J$ with $\beta' \geq \beta(\alpha)$, we can find $\alpha' \in I$, $\alpha' \geq \alpha$, such that $x_{\alpha'} = y_{\beta'}$.*

The notions of directedness and subdirectedness are natural generalizations of the notions of sequence and subsequence. If I is a set of natural numbers then $\{x_\alpha\}$, $\alpha \in I$, is the usual sequence, and its subsequences satisfy the conditions of Definition 8.

Next, we introduce with an example, where the set I has its elements arranged in increasing order, the notion of sets directed to increase.

Example 7. Let $[0, \theta]$ be some interval. A partition ω is a finite sequence of numbers $\{\tau_0, \tau_1, \ldots, \tau_k\}$, where k is an arbitrary number, such that $\tau_0 = 0 \le \tau_1 \le \cdots \le \tau_k = \theta$. Assume that $|\omega| = \theta$, and note that ω is a partition of the interval $[\,0,\ \theta]$. On the set $I = \{\omega : |\omega| = \theta\}$ of all partitions of the interval $[0, \theta]$ we introduce a partial order. Let $\omega_j = \left\{\tau_0^j = 0 \le \tau_1^j \le \cdots \le \tau_{k_j}^j = \theta\right\}$. We write $\omega_1 \ge \omega_2$ if all numbers τ_i^2 coincide with some numbers τ_i^1, i.e., the partition ω_1 includes the partition ω_2 and perhaps some other points, too. For two arbitrary partitions ω_1 and ω_2, we shall understand $\omega = \omega_1 \cup \omega_2$ as the partition formed by all numbers τ_i^1 and τ_i^2 in increasing order.

We say that the constructed set I is directed to increase as $\omega_1 \bigcup \omega_2 \ge \omega_1$ and the relation $\omega_1 \bigcup \omega_2 \ge \omega_2$ holds true for any ω_1, ω_2. We shall also say that t belongs to the partition $\omega = \{\,\tau_0 \le \tau_1 \le \cdots \le \tau_k = \theta\,\}$ $(t \in \omega)$ if $t = \tau_i$ for some i.

Definition 9. *The directedness* $\{\,x_\alpha\}$, $\alpha \in I$, *of elements of the topological space* X, *is converging to an element* $x \in X$ *if, for any neighborhood* W *of point* x, *there exists* $\alpha_W \in I$ *such that* $x_\alpha \in W$ *for all* $\alpha \ge \alpha_W$.

Let us note that a topological space X is a compact space if, and only if, each directedness in X contains a subdirectedness converging to some point of X.

Let X be a topological space, and let $\mathbf{R}\,(X)$ be the family of all its closed subsets.

Definition 10. *We shall say that the directedness* $\{\,F_\alpha\}$, $\alpha \in I$, *of closed subsets from* X, *is* S*-convergent to the closed set* $F \subset X$ *if:*

1. *for any* $x \in F$ *and any neighborhood* W_x *of the element* x *we can find a* $\beta \in I$ *such that* $F_\alpha \bigcap W_x \ne \emptyset$ *for all* $\alpha \ge \beta$;
2. *for any* $y \notin F$ *we can find a neighborhood* W_y *of the element* $y \in X$ *and a* $\gamma \in I$ *such that* $F_\alpha \bigcap W_y = \emptyset$ *for all* $\alpha \ge \gamma$.

Example 8. Let X be a two-dimensional space of vectors (x, y). Assume that $F_t = \{\,(x, y) :\ x - yt = 0\,\}$. The set $F_0 = \{(x, y) :\ x = 0\,\}$ is the coordinate axis $o'W$ffl and the set $F_t,\ t \ne 0$, is the straight line $y = (1/t)\,x$. It is not difficult to see that, for $t \to 0$, $\{F_t\}$ S-converges to the set F_0. The set F_t is not included in an ε-neighborhood of the set F_0 for any $\varepsilon > 0$ and $t \ne 0$.

The set F_t is a simple and a characteristic example of sets motion according to some differential equations. Indeed, let us consider the system $\dot{x} = -y,\ \dot{y} = 0$. If $x\,(t) = x\,(0) - y\,(0)\ t$, then the set F_t is a set of initial values $(x(0), y(0))$ from which the trajectory reaches the set F_0 precisely at moment t.

The following results describe the basic properties of S-convergence.

Theorem 17. *It is possible to choose a converging subdirectedness from any directedness* $\{\,F_\alpha\}$, $\alpha \in I$, $F_\alpha \in \mathbf{R}\,(X)$.

Definition 11. *The directedness* $\{F_\alpha\}$, $\alpha \in I$, $F_\alpha \in \mathbf{R}(X)$, *is called nondecreasing if* $F_\alpha \supset F_\beta$ *for* $\alpha \geq \beta$, *and it is called nonincreasing if* $F_\alpha \subset F_\beta$ *for* $\alpha \geq \beta$.

Theorem 18. *The nondecreasing directedness* $\{F_\alpha\}$, $\alpha \in I$, *S-converges to* $cl \bigcup_{\alpha \in I} F_\alpha$. *The nonincreasing directedness* $\{F_\alpha\}$, $\alpha \in I$, *S-converges to* $\bigcap_{\alpha \in I} F_\alpha$.

Definition 12. *The subset* M *of a set* $\mathbf{R}(X)$ *is called S-closed if the S-convergence of some directedness of elements from* M *to the closed subset* $F \subset X$ *implies* $F \in M$.

Theorem 19. *The family of all S-closed sets from* $\mathbf{R}(X)$ *is the set of the closed sets of some topology from* $\mathbf{R}(X)$.

From this, the correctness of the next definition follows.

Definition 13. *We call S-topology on* $\mathbf{R}(X)$ *a topology in which only S-closed sets are closed.*

Furthermore, if X is a Euclidean space or its closed subset (i.e., locally compact Hausdorff space), then the following statement is of interest.

Theorem 20. *Let* X *be a locally compact Hausdorff space. Then a convergence of directednesses from* $\mathbf{R}(X)$ *in S-topology is equivalent to S-convergence.*

For a space X with metric $d(\cdot,\cdot)$ it is possible to introduce the Hausdorff distance between two compact sets $A, B \subset X$ if we assume that

$$\rho\,(A,B) = \max\left\{\max_{x\in A}\min_{y\in B} d\,(x,y)\,,\ \max_{y\in B}\min_{x\in A} d\,(x,y)\right\}.$$

Theorem 21. *Let* X *be a compact metric space. Then the S-topology coincides with the topology induced by the Hausdorff metric* $\rho(\cdot,\cdot)$.

4.3 Operators over Sets

We consider the dynamic system that is described by equation (17) and satisfies the Assumptions 1–3.

Definition 14. *Let* P_ε, $\varepsilon \geq 0$, *be an operator that to each closed set* $M \subset E^n$ *corresponds a set* $P_\varepsilon M$ *of all points* $z_0 \in E^n$ *that satisfy the condition: For any admissible control* $v\,(t)$, $t \in [0,\varepsilon]$, *of player E there exists an admissible control* $u\,(t)$, $t \in [0,\varepsilon]$, *of player P such that for the appropriate solution* $z\,(t) = z\,(t \mid u\,(\cdot), v\,(\cdot), z_0)$ *to equation (17) with start at* z_0 *the inclusion* $z\,(\varepsilon) \in M$ *is fulfilled, i.e., the trajectory* $z\,(t)$ *reaches* M *at the moment* ε.

Using the operations of union and intersection, we can describe the operators P_ε as follows:

$$P_\varepsilon M = \bigcap_{v(\cdot)\, \in V[0,\varepsilon]} \ \bigcup_{u(\cdot)\, \in U[0,\varepsilon]} \{ z_0 \in E^n : z(\varepsilon \mid u(\cdot), v(\cdot), z_0) \in M \}. \quad (18)$$

Remark 3. In Definition 14, one can consider controls $u(\cdot)$ and $v(t)$ determined only on the half-open interval $[0, \varepsilon)$ as a change of the control values $u(t)$ and $v(t)$ at one point does not change the trajectory. Here, it is always possible to uniquely and continuously prolong on $[0, \varepsilon)$ the solution $z(t)$ determined on $[0, \varepsilon)$, assuming $z(\varepsilon) = \lim\limits_{t \to \varepsilon} z(t)$. This fact will be used later on.

Remark 4. We can interpret the set $P_\varepsilon M$ as a set of initial positions z_0 from which player P may hit the trajectory $z(t)$ on M at moment ε if he knows the control $v(t)$ of player E over the entire interval $[0, \varepsilon]$ in advance. If $z_0 \notin P_\varepsilon M$ then there exists a control for player E such that, for all admissible controls of player P, the relation $z(\varepsilon) \notin M$ holds. In this case, the players' strategies are preassigned, i.e., they choose their controls over the entire interval $[0, \varepsilon]$. Moreover, player E knows z_0, and player P uses the information about z_0 and the chosen control $v(t)$, $t \in [0, \varepsilon]$.

Lemma 8. *The set* $P_\varepsilon M$ *is closed.*

Proof. Let $\{z_k\}$ be a sequence from $P_\varepsilon M$ converging to the point z_0. For the proof of the closure of $P_\varepsilon M$ it should be shown that $z_0 \in P_\varepsilon M$.

Because $\{z_k\}$ converges, one can assume that $z_k \in K$, where K is a compact subset from E^n. As $z_k \in P_\varepsilon M$ then, for any admissible control $v(\cdot) \in V[0, \varepsilon]$, there exists an admissible control $u_k(\cdot) \in U[0, \varepsilon]$ and a moment $\delta_k \in [0, \varepsilon]$ such that $z_k(\varepsilon) \in M$, where $z_k(t) = z(t \mid u_k(\cdot), v(\cdot), z_k)$.

By virtue of Assumption 3, a set of trajectories corresponding with various admissible controls of player P and initial positions from K is a compact set in topology of uniform convergence. Therefore, there exists a subsequence of the sequence $\{z_k(\cdot)\}$ converging to a solution $z(t)$ of equation (17) that corresponds with some admissible control $u(\cdot) \in U[0, \varepsilon]$ and control $v(\cdot)$. Without loss of generality, one can assume that the sequence $\{z_k(\cdot)\}$ converges to $z(\cdot)$. Note that z_0 is initial point of the solution $z(t)$. By virtue of the closure of M, we get $z(\varepsilon) = \lim\limits_{k \to \infty} z_k(\varepsilon) \in M$. This implies $z_0 \in T_\varepsilon M$. The lemma has been proved. ■

Assume $P_{N,\varepsilon} M = (P_\varepsilon M) \cap N$. It is clear that $P_{N,\varepsilon} M$ is a closed set if M and N are closed sets. Thus, Lemma 8 allows us to apply the operators $P_\varepsilon M$ and $P_{N,\varepsilon} M$ repeatedly.

Lemma 9. *The following properties hold:*

(1) $P_{N,0} M = M$;
(2) $P_{N,\varepsilon} M_1 \subset P_{N,\varepsilon} M_2$, *if* $M_1 \subset M_2$;
(3) for any family of closed sets $\{M_\alpha\}, \alpha \in I$, $\bigcap\limits_{\alpha \in I} P_{N,\varepsilon} M_\alpha \supset P_{N,\varepsilon} \bigcap\limits_{\alpha \in I} M_\alpha$.

Proofs of properties 1 and 2 follow directly from the definitions. Property 3 follows from property 2.

Lemma 10. *The inclusion* $P_{N,\,\varepsilon_1}P_{N,\,\varepsilon_2}M \subset P_{N,\,\varepsilon_1+\varepsilon_2}M$ *holds.*

Proof. It follows from Lemma 9 (property 2) that

$$P_{N,\,\varepsilon_1}P_{N,\,\varepsilon_2}M = \left(P_{\varepsilon_1}\left(P_{\varepsilon_2}M\right)\cap N\right)\cap N \subset \left(P_{\varepsilon_1}P_{\varepsilon_2}M\right)\cap N.$$

Therefore, it is sufficient to prove the lemma for the operator P_ε. It is not difficult to do so by applying the representation (18). Informally, player P outputs from points $z_0 \in P_{\varepsilon_1+\varepsilon_2}M$ the trajectory $z\,(t)$ on M at moment $\varepsilon_1+\varepsilon_2$ as he knows a preassigned control $v\,(t)$ on the entire interval $[0,\,\varepsilon_1+\varepsilon_2]$. If $z_0 \in P_{\varepsilon_1}P_{\varepsilon_2}M$, then player P outputs at first the trajectory on $P_{\varepsilon_2}M$ at moment ε_1 as he knows a control $v\,(t)$, $t \in [0, \varepsilon_1)$, and subsequently he finds out a control $v\,(t)$ on $[\varepsilon_1, \varepsilon_1+\varepsilon_2]$ and outputs the trajectory $z\,(t)$ on M at moment $\varepsilon_1+\varepsilon_2$. Thus, in the second case, player P is less informed in advance about the opponent's actions than in the first case. ∎

The inclusion considered in Lemma 9 (property 3) is in general strict, i.e., the inverse inclusion may not be fulfilled. We next consider a case where this inclusion becomes equality.

Lemma 11. *Let* $\{M_\alpha\}, \alpha \in I, M_\alpha \subset N$, *be a nonincreasing directedness of closed sets. Then*

$$\bigcap_{\alpha\in I} P_{N,\varepsilon}M_\alpha = P_{N,\varepsilon}\bigcap_{\alpha\in I} M_\alpha.$$

Proof. By virtue of Lemma 9 (property 3), it is sufficient to show inclusion

$$\bigcap_{\alpha\in I} P_{N,\varepsilon}M_\alpha \subset P_{N,\varepsilon}\bigcap_{\alpha\in I} M_\alpha.$$

Let $z_0 \in P_{N,\varepsilon}M_\alpha$ for any $\alpha \in I$. It means that $z_0 \in N$ and, for any control $v\,(\cdot) \in V\,[0,\varepsilon)$, there exists a control $u_\alpha\,(\cdot) \in U\,[0,\varepsilon)$ such that for an appropriate solution $z_\alpha\,(t) = z\,(t\,|, u_\alpha\,(\cdot)\,, v\,(\cdot)\,, z_0\,)$ the relation $z_\alpha\,(\varepsilon) \in M$ is fulfilled. By virtue of Assumption 3, a set of solutions to equations (17), appropriate to various admissible controls of player P, represents a compact set in topology of uniform convergence. Therefore, there exists a subdirectedness of the directedness $\{z_\alpha\,(\cdot)\}$ converging to some solution $z\,(\cdot)$ that corresponds with some control $u\,(\cdot)$ and to control $v\,(\cdot)$, i.e., $z\,(t) = z\,(t\,|u\,(\cdot)\,, v\,(\cdot)\,, z_0\,)$. Without loss of generality, it is possible to assume that the directedness $\{z_\alpha\,(\cdot)\}$ converges to $z\,(\cdot)$. As $M_\beta \subset M_\alpha$ and as $\beta \geq \alpha$, then $z_\beta\,(\varepsilon) \in M_\alpha$ for all $\beta \geq \alpha$. From this $z\,(\varepsilon) = \lim_\beta z_\beta\,(\varepsilon) \in M_\alpha$. By virtue of the arbitrariness of α we have $z\,(\varepsilon) \in \bigcap_{\alpha\,\in I} M_\alpha$ implying $z_0 \in P_{N,\,\varepsilon}\bigcap_{\alpha\,\in I} M_\alpha$. The lemma has been proved. ∎

The next lemma follows from Theorem 18 and Lemma 11.

Lemma 12. *Let* $\{M_\alpha\}$, $\alpha \in I$, $M_\alpha \subset N$, *be a nonincreasing directedness of closed sets. Then the directednesses* $\{P_{N,\varepsilon} M_\alpha\}$, $\alpha \in I$, S*-converges to the set* $P_{N,\varepsilon} \bigcap\limits_{\alpha \in I} M_\alpha$.

Let $\omega = \{\tau_0 = 0 \le \tau_1 \le \cdots \le \tau_k = t\}$ be a finite partition of the interval $[0,\, t]$ (see Example 7), and assume that

$$P_N^\omega M = P_{N,\delta_1} P_{N,\delta_2} \ldots P_{N,\delta_k} M,$$

where $\delta_i = \tau_i - \tau_{i-1},\ \ i = 1, \ldots, k$.

Remark 5. If $N = E^n$ we write $P^\omega M = P_{\delta_1} P_{\delta_2} \ldots P_{\delta_k} M$. Let $z_0 \in P^\omega M$ and assume that at the initial moment of time, player P knows in advance the control of player E at time δ_1. Then, P may aim and hit on $P_{\delta_2} \ldots P_{\delta_k} M$ at moment $\delta_1 = \tau_1$. If he has a hit on $P_{\delta_2} \ldots P_{\delta_k} M$, then he will know the control of player E at time δ_2, and therefore player P may aim and hit on $P_{\delta_3} \ldots P_{\delta_k} M$ at time moment $\tau_2 = \delta_1 + \delta_2$. Continuing the process further, player P attains an inclusion $z\,(t) \in M$. In addition, player P chooses his control at the points $\tau_{i-1},\ \ i = 1, \ldots, k$, on an interval $[\tau_{i-1}, \tau_i)$ knowing $z(\tau_{i-1})$ and the future control of player E on interval $[\tau_{i-1}, \tau_i)$. Analogously, if $z_0 \notin P^\omega M$, then player E at moment $\tau_0 = 0$ can choose a control such that, for any control of player P, a corresponding trajectory does not hit on the set $P_{\delta_2} \ldots P_{\delta_k} M$ at moment $\delta_1 = \tau_1$. Continuing the process further, we get $z\,(t) \notin M$. In addition, having $z\,(\tau_{i-1})$, player E chooses his control at the point τ_{i-1} on the next interval $[\tau_{i-1}, \tau_i)$.

The next proposition follows from Lemma 11.

Lemma 13. *Let* $|\omega_1| = |\omega_2| = t$ *and* $\omega_1 \ge \omega_2$. *Then* $P_N^{\omega_1} M \subset P_N^{\omega_2} M$.

Definition 15. $\tilde{P}_{N,t} M = \bigcap\limits_{|\omega| = t} P_N^\omega M$, $\tilde{P}_t M = \bigcap\limits_{|\omega| = t} P^\omega M$.

Because of Lemma 13, the families of sets $\{P_N^\omega M\}, |\omega| = t$, is a nonincreasing directedness of closed sets. Consequently, from Theorem 18 we get the following results:

Lemma 14. *The directedness* $\{P_N^\omega M\}$, $|\omega| = t$, S*-converges to the set* $\tilde{P}_{N,\,t} M$.

Theorem 22. *The following equality holds:*

$$\tilde{P}_{N,\,t_1 + t_2} M = \tilde{P}_{N,\,t_1} \tilde{P}_{N,\,t_2} M.$$

Proof. Let $\omega(|\omega| = t_1 + t_2)$ be an arbitrary partition formed by numbers $\tau_i, i = 0, \ldots, k$, where $\tau_i \le t_1, i = 1, \ldots, k_1$, and $\tau_i > t_1, i = k_1 + 1, \ldots, k$. Let us denote by ω_1 an interval partition $[0, t_1]$ formed by numbers $\tau_i, i = 0, \ldots, k_1$, and t_1, and by an interval partition ω_2 of $[0, t_2]$, formed by the zero and the numbers $\tau_i - t_1, i = k_1 + 1, \ldots, k$.

Because of Lemma 13 and the definition of P_N^{ω} , it follows that $P_N^{\omega_1} P_N^{\omega_2} M \subset P_N^{\omega} M$. From this, $\tilde{P}_{N,t_1} \tilde{P}_{N,t_2} M \subset P_N^{\omega_1} \tilde{P}_{N,t_2} M \subset P_N^{\omega_1} P_N^{\omega_2} M \subset P_N^{\omega} M$, because of Lemma 9 (property 2), and by the arbitrariness of ω, we get

$$\tilde{P}_{N,\, t_1} \tilde{P}_{N,\, t_2} M \subset \tilde{P}_{N,\ t_1 + t_2} M.$$

To prove the inverse inclusion, let ω_1, $|\omega_1| = t_1$ and ω_2, $|\omega_2| = t_2$, be arbitrary partitions formed by numbers $\tau_i^1,\ i = 0, \ldots, k_1$, and $\tau_i^2,\ i = 0, \ldots, k_2$, respectively. Let us form a new partition ω from the numbers τ_i^1 and $\tau_i^2 + t_1$. It is clear that $|\omega| = t_1 + t_2$ and $P_N^{\omega_1} P_N^{\omega_2} M = P_N^{\omega} M \supset \tilde{P}_{N,\ t_1 + t_2} M$. From this, because of Lemma 11, we get

$$\tilde{P}_{N,t_1+t_2} M \subset \bigcap_{|\omega_1| = t_1} \ \bigcap_{|\omega_2| = t_2} P_N^{\omega_1} P_N^{\omega_2} M$$

$$= \bigcap_{|\omega_1| = t_1} P_N^{\omega_1} \Big(\bigcap_{|\omega_2| = t_2} P_N^{\omega_2} M \Big) = \tilde{P}_{N,t_1} \tilde{P}_{N,t_2} M.$$

The theorem has been proved. ■

Remark 6. The family $\left\{ \tilde{P}_{N,t} \right\}, t \geq 0$, is a family of operators acting on spaces of closed subsets in $N \subset E^n$. Because of Theorem 22, this family is a one-parameter semigroup (the parameter is t). Let us compare Theorem 22 with a known result from differential equations theory. Consider the equation $\dot{x} = f(x),\ x \in E^n$. It is known that there exists a one-parameter group of operators G_t acting in E^n such that a solution $x(t)$ to the considered differential equation may be represented in the form $x(t) = G_t x_0$, where x_0 is an initial position. In case x varies in a Banach space, it is possible to guarantee only a semigroup property for G_t. Thus, the semigroup $\tilde{P}_{N,\, t}$ describes the dynamic of the set M motion.

4.4 ε-Strategies Description and Game Move

The characteristic feature of an ε-strategy is that player P uses the information about the future control of player P on some time interval with length determined by player E.

It is possible to introduce different definitions of ε-strategies, however, they are all equivalent. We will below turn our attention to one of them. Assume that the game will take place on a finite interval $[0, \theta]$. Player E chooses at the initial moment of time a finite partition of the interval $[0, \theta]$, say $\omega = \{ \tau_0 = 0 \leq \tau_1 \leq \cdots \leq \tau_k = \theta \}$. At moment τ_{i-1}, let the dynamic system be at the point $z(\tau_{i-1})$, $i = 1, 2, \ldots$. Using this information, player E chooses his control $v_i(t),\ t \in [\tau_{i-1}, \tau_i)$. We assume that player P knows $\tau_{i-1},\ z(\tau_{i-1})$

and $v_i(t), t \in [\tau_{i-1}, \tau_i)$, and that he chooses his control $u_i(t), t \in [\tau_{i-1}, \tau_i)$. If we substitute $v_i(t)$ and $u_i(t)$ in (17), we can find a solution $z(t)$ to equation (17) with the beginning at $z(\tau_{i-1})$. Because $z(t)$ is Lipschitz, this solution can be prolonged on the interval $[\tau_{i-1}, \tau_i]$. Thus, at moment τ_i, the dynamic system is at the point $z(\tau_i)$ and we can repeat the process further. Because the number of points is τ_i or finite, we construct a solution to equation (17) on the entire interval $[0, \theta]$ using the described process.

Consider the game from Subsection 4.3.

Theorem 23.

1. *Let $z_0 \notin \tilde{P}_\theta M$. Then there exists an ε-strategy for player E such that, for any ε-strategy of player P for the corresponding trajectory $z(t)$ with the beginning at z_0, the inclusion $z(\theta) \in M$ is fulfilled.*
2. *Let $z_0 \notin \tilde{P}_\theta M$. Then there exists an ε-strategy for player E such that, for any ε-strategy of player P for the corresponding trajectory $z(t)$ with the beginning at z_0, the inclusion $z(\theta) \notin M$ is fulfilled.*

Proof.

1. Let $z_0 \in \tilde{P}_\theta M$ and assume that player E chooses a partition
$$\omega = \{\tau_0 = 0 \leq \tau_1 \leq \cdots \leq \tau_k = \theta\}.$$
At the moment $\tau_0 = 0$, player E constructs a control $v(\cdot) \in V[0, \tau_1)$. Player P knows this control. From Theorem 22, it follows that
$$z_0 \in \tilde{P}_\theta M = \tilde{P}_{\tau_1} \tilde{P}_{\theta-\tau_1} M \subset P_{\tau_1} \tilde{P}_{\theta-\tau_1} M.$$
The definition P_ε implies the existence of a $u(\cdot) \in U[0, \tau_1)$ such that
$$z(\tau_1) = z(\tau_1 \mid u(\cdot), v(\cdot),\ z_0) \in \tilde{P}_{\theta-\tau_1} M.$$
We take the position $z(\tau_1)$ as the initial position and repeat a process. As a result, at the i-th step, a control $u(\cdot) \in U[0, \tau_i)$ will be constructed such that, for the corresponding trajectory, the inclusion $z(\tau_i) \in P_{\theta-\tau_i} M$ is fulfilled. We have $z(\theta) \in M$ at k-th step.
2. Let $z_0 \notin \tilde{P}_\theta M$. The definition of $\tilde{P}_\theta M$ implies the existence of a partition $\omega = \{\tau_0 = 0 \leq \tau_1 \leq \cdots \leq \tau_k = \theta\}$, $\delta_i = \tau_i - \tau_{i-1}$, such that $z_0 \notin P^\omega M$. Player E chooses a partition ω at the initial moment. Because $z_0 \notin P_{\delta_1} \ldots P_{\delta_k} M$, there exists a $v(\cdot) \in V[0, \delta_1)$ such that, for any control $u(\cdot) \in V[0, \delta_1)$, the relation
$$z(\tau_1) = z(\delta_1) = z(\delta_1 \mid u(\cdot),\ v(\cdot),\ z_0) \notin P_{\delta}, \ldots P_{\delta_k} M$$
is fulfilled.
Let us take the point $z(\tau_1)$ as the initial point and repeat the described process. Then, at the i-th step, a control $v(\cdot) \in V[0, \tau_i)$ will be constructed such that, for the corresponding trajectory, the relation

$$z(\tau_i) = z(\delta_1 + \cdots + \delta_i) \notin P_{\delta_{i+1}} \dots P_{\delta_k} M.$$

is fulfilled. We get $z(\theta) \notin M$ at the k-th step. ■

Theorem 23 is a theorem of alternatives. Because of this theorem, the entire game space E^n is divided in two subsets $\tilde{P}_\theta M$ and $E^n / \tilde{P}_\theta M$. The first subset describes all initial positions favorable to player P, and the second subset describes all initial positions favorable to player E.

Theorem 23 describes the game structure without phase constraints, i.e., $N = E^n$. Let us consider the case when N is subset of the space E^n.

Lemma 15. *Let $z_0 \in \tilde{P}_{N,\varepsilon} M$. Then for any $v(\cdot) \in V[0,\varepsilon)$ there exists a $u(\cdot) \in U[0,\varepsilon)$ such that $z(\varepsilon\,|u(\cdot), \nu(\cdot), \varepsilon) \in M$ and $z(t\,|u(\cdot), v(\cdot), z_0) \in N$ for all $t \in [0,\varepsilon)$.*

Proof. Let $\omega = \left\{\tau^\omega = 0 \le \tau_1^\omega \le \cdots \le \tau_{k(\omega)}^\omega = \varepsilon\right\}$ be a partition of the interval $[0,\varepsilon]$. Fix $v(\cdot) \in V[0,\varepsilon)$. The construction of $P_N^\omega M$ and the definition of $P_{N,\delta} M$ imply that if $z_0 \in P_N^\omega M$ then, for $v(\cdot) \in V[0,\varepsilon)$, there exists a $u^\omega(\cdot) \in U[0,\varepsilon)$ such that, for the corresponding trajectory $z^\omega(t)$, the inclusions $z^\omega(\varepsilon) \in M$ and $z^\omega(\tau_i^\omega) \in N$ are fulfilled for all $i = 1, \dots, k(\omega)$.

Assumption 3 implies that one can chose a converging subdirectedness from the directednesses of the trajectories $\{z^\omega(\cdot)\}$. Without loss of generality, it is possible to assume that $\{z^\omega(\cdot)\}$ converges, and that $z(t)$ is its limit. The function $z(\cdot)$ is a solution to (17) corresponding with some control $u(\cdot) \in U[0, \varepsilon)$ and the fixed control $v(\cdot)$. A solution $z(\cdot)$ has the properties $z(\varepsilon) \in M$ and $z(t) \in N$ for all $t \in [0,\varepsilon]$. The first property is obvious. We shall prove the validity of the second property. We fix an arbitrary moment $t \in [0,\varepsilon]$. For this t there exists a partition ω_t such that $t \in \omega_t$. For all $\omega \ge \omega_t$, the relation $t \in \omega$ will also be fulfilled. Therefore, $z^\omega(t) \in N$ for all $\omega \ge \omega_t$. This implies $z(t) \in N$. The lemma has been proved. ■

Using Lemma 15 we can prove the next theorem.

Theorem 24.

1. *Let $z_0 \in \tilde{P}_{N,\theta} M$. Then there exists an ε-strategy for player P such that, for any ε-strategy of player E for the corresponding trajectory $z(t)$ with the beginning at z_0, the inclusions $z(\theta) \in M$ and $z(t) \in N$ are fulfilled for all $t \in [0,\theta]$.*
2. *Let $z_0 \notin \tilde{P}_{N,\theta} M$. Then there exists an ε-strategy for player E such that, for any ε-strategy of player P for the corresponding trajectory with the beginning at z_0, either $z(\theta) \notin M$ is fulfilled or there exists $t \in [0,\theta]$ such that $z(t) \notin N$.*

5 Complex Analysis in Linear Differential Games

5.1 Linear Games with Scalar Matrix

Let us consider the game dynamics described by the equation

$$\dot{z} = Az + B(u, v), \tag{19}$$

where $A : E^n \to E^n$ is a linear operator and $B : U \times V \to E^n$ is a continuous mapping. Let us study the cases $A = 0$ and $A = aE$, where a is a number and E is a unit identity matrix. For such games we can describe, under certain convexity condition, the operators $\tilde{P}_t$, illustrate further and exemplify the essence of the methods used for the solution of wider game classes. The special cases of Volterra mappings (disstrategies) will be considered.

The case $A = 0$ is called a simple moving. The equation (19) takes the simple form

$$\dot{z} = B\,(u,\ v).$$

According to Assumption 3, the set $B\,(U, v)$ is convex for all $v\ \in V$. Assume that M and N are convex sets. Let us set

$$P_t^*\, M = \bigcap_{v \in V}\ \bigcup_{u \in U} \{\, z_0 \in E^n :\ \ z_0 + tB\,(u, v) \in M\,\},\ P^*_{N,t}\, M = N \bigcap P_t^*\, M \tag{20}$$

We the set $P^*_{N,\theta}\, M$ represents. Analogous arguments can be considered for any $t \in [0,\ \theta]$.

Let $z_0 \in P_\theta^*\, M$. Then for any $v\ \in V$ there exists $u_{z_0}\,(v) \in U$ such that

$$z_0 + \theta\, B\,(\, u_{z_0}\,(v)\,, v) \in M. \tag{21}$$

It follows from (21) that the mapping $u_{z_0} : V \to U$ depends on the initial position z_0. The value $u_*(v)$ can be found by solving the inclusion $z_0 + \theta B(u, v) \in M$ with respect to $u\ \in U$. There can be many such solutions. Let us assume that $u_{z_0}(v)$ is the least of the solutions in lexicographic sense. In this case, it follows from Philipov's lemma (Theorem 2) that if $v\,(\cdot) \in V\,[0,\ \theta]$ then the function $u\ (t) = u_{z_0}\,(v\ (t))$ is a measurable function and therefore $u\,(\cdot) \in U\,[0,\ \theta]$.

If player E has selected the control $v\,(\cdot) \in V\,[0,\ \theta]$, then it follows from (21) that for any $t \in [0,\ \theta]$

$$z_0 + \theta B\,(u_{z_0}\,(v(t))\,, v(t)) \in M. \tag{22}$$

If we divide both parts of inclusion (22) by θ and integrate w.r.t. t from 0 to θ, we get

$$z(\theta) = z_0 + \int_0^\theta B\,(u_{z_0}\,(v\,(t))\,, v(t))\,dt \in \frac{1}{\theta}\int_0^\theta M\,dt = M.$$

Thus $z(\theta) \in M$. It means that $z_0 \in \tilde{P}_\theta M$. This implies that $P_\theta^* M \subset \tilde{P}_\theta M$.

Let $z_0 \in P^*_{N,\,\theta}\, M$ and u_* be constructed by the above mapping. Because $M \subset N$, it follows from (22) that

$$\theta B\,(u_{z_0}\,(v(t))\,, v(t)) \in N - z_0 \tag{23}$$

Because $0 \in N - z_0$, then, for any $\tau \in [0,\ \theta]$,

$$\int_0^{\tau} (N - z_0)\ dt \subset \int_0^{\theta} (N - z_0)\ dt.$$

From this and from (23)

$$\int_0^{\tau} B\left(u_{z_0}\left(v\left(t\right)\right), v\left(t\right)\right) dt \in \frac{1}{\theta}\int_0^{\tau} (N - z_0)\ dt \subset \frac{1}{\theta}\int_0^{\theta} (N - z_0)\ dt = N - z_0.$$

In result, we get $z\left(\tau\right) \in N$. This implies that

$$P^*_{N,\theta}M \subset \tilde{P}_{N,\theta}M.$$

Remark 7. The mapping u_{z_0} was originally constructed in order to output the trajectory on the set M at the moment θ. However, the same mapping allows a trajectory on the set N to contact with M. Why does this occur? Let $m_{z_0}\left(v\right) = z_0 + \theta B\left(u_{z_0}\left(v\right), v\right) \in M$ and consider the convex hulls

$$M_* = \overline{\mathrm{co}}\ \{m_{z_0}(v),\ v \in V\},\ \ N_* = \overline{\mathrm{co}}\,\{z_0,\ M_*\}\,.$$

It follows from the construction that the end of the trajectory is $z(\theta) \in M_*$. Thus, the entire trajectory lays in the set N_*. Therefore, if it "targets" on the set M, as in inclusion (21), the trajectory will not automatically abandon the set $N_* \subset N$.

Let us now consider the case $z_0 \notin P^*_{N,\theta}M$. Then either $z_0 \notin N$ or there exists $v_{z_0} \in V$, depending on z_0, such that for any $u \in U$

$$z_0 + \theta B\left(u, v_{z_0}\right) \notin M. \tag{24}$$

Because $B\left(U, v_{z_0}\right)$ is convex, then, for any $u(\cdot) \in U\left[0, \theta\right]$, there exists a $u \in U$ such that

$$\int_0^{\theta} B\left(u\left(t\right), v_{z_0}\right) dt = \theta B\left(u, v_{z_0}\right).$$

It follows from this and from (24) that, for any $u\left(\cdot\right) \in U\left[0,\ \theta\right]$,

$$z\ (\theta) = z_0 + \int_0^{\theta} B\left(u\left(t\right), v_{z_0}\right) dt = z_0 + \theta B\left(u, v_{z_0}\right) \notin M.$$

It is possible to consider the described process of the control construction $v(t) \equiv v_{z_0}$ as a special case of an ε-strategy for player E in which the trivial partition $\omega = \{0, \theta\}$ is selected at the initial moment and the constant control z_0 on the interval $[0, \theta]$ is constructed. Therefore $z_0 \notin P^*_{N,\theta}M$and so $\tilde{P}_{N,\theta}M \subset P^*_{N,\theta}M$.

Thus the following theorem has been proved.

Theorem 25.

1. *Let $z_0 \in P^*_{N,\theta} M$. Then there exists a mapping $u_{z_0} : V \to U$ such that for any $v\,(\cdot) \in V\,[0,\,\theta]$:*
   (a) $u_{z_0}\,(v(t))$ is an admissible control of player P,
   (b) for the trajectory $z(t)$ with the beginning at z_0, corresponding with the controls $u_{z_0}\,(v(t))$ and $v(t)$, the inclusions $z(\theta) \in M$ and $z(t) \in N$ are fulfilled for all $t \in [0, \theta]$.
2. *Let $z_0 \notin P^*_{N,\theta} M$. Then either $z_0 \notin N$ or there exists $v_{z_0} \in V$ such that, for the trajectory $z(t)$ with the beginning at z_0, corresponding with the arbitrary control $u(\cdot) \in U\,[0, \theta]$ and to the control $v(t) \equiv v_{z_0}$, the relation $z(\theta) \notin M$ is fulfilled.*
3. *$\tilde{P}_{N,\theta} M = P^*_{N,\theta} M$.*

Let us consider the case of scalar matrix. The previous arguments, without any additional constructions, are transferred to the case of games with dynamics $\dot{z} = az + B\,(u, v)$, where a is a number.

Let us generalize the formula (20). Assume that

$$P^*_\theta M = \bigcap_{v \in V} \bigcup_{u \in U} \left\{ z_0 \in E^n : e^{a\theta} z_0 + \int_0^t e^{a(\theta - t)} dt\, B(u, v) \in M \right\}, \tag{25}$$

$$P^*_{N,\theta} M = N \bigcap P^*_\theta M.$$

Theorem 26. *The statements of Theorem 25 hold if the operators $P^*_{N,\theta}$ are replaced by the operators defined in (25).*

Let $z_0 \in P^*_\theta M$. Then we construct a mapping $u_{z_0} : V \to U$ using, instead of the inclusion (24), the inclusion

$$e^{a\theta} z_0 + \int_0^\theta e^{a(\theta - s)} ds\ B\,(u_{z_0}\,(v)\,, v) \in M. \tag{26}$$

Let us assume that player E realizes the control $v\,(\cdot) \in V\,[0,\,\theta]$. Then it follows from (26) that, for any $t \in [0,\ \theta]\,$,

$$e^{a\theta} z_0 + \int_0^\theta e^{a(\theta - s)} ds\ B\,(u_{z_0}\,(v(t))\,, v(t)) \in M. \tag{27}$$

Let us divide (27) by $\int_0^\theta e^{a(\theta - s)} ds$ and then multiply by $e^{a(\theta - t)}$ and integrate w.r.t. t from 0 to θ. We get

$$z\,(\theta) = e^{a\theta} z_0 + \int_0^\theta e^{a(\theta - s)}\ B\,(u_{z_0}\,(v\,(t))\,, v\,(t))\, dt$$

$$\in \int_0^\theta e^{a(\theta-t)} \left(\int_0^\theta e^{a(\theta-s)} ds \right)^{-1} M dt = M.$$

This implies that $z_0 \in \tilde{P}_\theta M$ and $P_\theta^* M \subset \tilde{P}_\theta M$.

Let now $z_0 \in P_{N,\theta}^* M$. Using the fact that $e^{a(\cdot)} z_0 - z_0 = \int_0^\theta e^{a(\theta-s)} ds a z_0$, we get from (27):

$$\int_0^\theta e^{a(\theta-s)} ds \left[B \left(u_{z_0} \left(v \left(t \right) \right), v(t) \right) + a z_0 \right] \in N - z_0 \tag{28}$$

for any $t \in [0,\ \theta]$. Fix $\tau \in [0,\ \theta]$. If we divide (28) by $\int_0^\theta e^{a(\theta-s)} ds$ and then multiply by $e^{a(\tau-t)}$ and integrate w.r.t. t from 0 to τ we get

$$\int_0^\tau e^{a(\tau-t)} \left[B \left(u_{z_0} \left(v \left(t \right) \right), v \left(t \right) \right) + a z_0 \right] \, dt$$

$$\in \int_0^\tau e^{a(\tau-t)} \left(\int_0^\theta e^{a(\theta-s)} ds \right)^{-1} (N - z_0) \, dt$$

$$= \int_0^\tau e^{at} \left(\int_0^\theta e^{as} ds \right)^{-1} (N - z_0) \, dt \subset N - z_0.$$

The last inclusion follows from the fact $0 \in N - z_0$ and $e^{at} > 0$. From this

$$z\left(\tau\right) = e^{a\tau} z_0 + \int_0^\tau e^{a(\tau-t)} B \left(u_{z_0} \left(v \left(t \right) \right), v \left(t \right) \right) \, dt \in N.$$

Thus, $z_0 \in \tilde{P}_{N,\,\theta} M$ and $P_{N,\theta}^* M \subset \tilde{P}_{N,\theta} M$.

Let now $z_0 \notin P_{N,\,\theta}^* M$. Then either $z_0 \notin N$ or there exists $v_{z_0} \in V$ such that, for any $u \in U$,

$$e^{a\theta} z_0 + \int_0^\theta e^{a(\theta-s)} ds \, B \left(u, v_{z_0} \right) \, dt \notin M.$$

As $B\left(U, v_*\right)$ is convex then, for any $u(\cdot) \in U\left[0, \theta\right]$, there exists $u \in U$ such that

$$\int_0^\theta e^{a(\theta-t)} B \left(u(t), v_{z_0} \right) dt = \int_0^\theta e^{a(\theta-s)} ds \, B \left(u, v_{z_0} \right).$$

This implies that for any $u(\cdot) \in U[0,\ \theta]$

$$z(\theta) = e^{a\theta} z_0 + \int_0^{\theta} e^{a(\theta - t)} B(u(t),\ v_{z_0})\ dt\ \ \notin M.$$

Thus, $z_0 \notin \tilde{P}_{N,\theta} M$ and $\tilde{P}_{N,\theta} M \subset P^*_{N,\theta} M$.

Let us construct two examples showing that the convexity conditions on the sets M and N are essential. We show that if M is not convex or a is a matrix (not a scalar matrix), then there are cases for which $P^*_t M \neq \tilde{P}_t M$.

Example 9. We consider the game with dynamics $\dot{z} = v$, where $n = 1$, $z \in E^1$, $V = \{-1, +1\}$, $M = \{-1, +1\}$, $\theta = 1$. It is not difficult to see that $z_0 = 0 \in P^*_1 M$ as for $z_0 = 0$ $z(1) = \pm 1 \in M$ is fulfilled. We show that $0 \notin \tilde{P}_1 M$. Suppose that player E has chosen the control

$$v(t) = \begin{cases} -1; & t \in [0,\ 1/2) \\ +1; & t \in [1/2;\ 1]. \end{cases}$$

It is not difficult to see that $z(1) = 0 \notin M$ and therefore $P^*_1 M \neq \tilde{P}_1 M$.

Example 10. We consider the game with dynamics $\dot{z} = Az + v$, where $n = 2$, $z \in E^2$, $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \left\{(1,0),\left(0,(e-1)^{-1}\right)\right\}$, $M = \mathrm{co}\,\{(1,0),\ (0,1)\}$, and $\theta = 1$. It is not difficult to see that $z_0 = (0,\ 0) \in P^*_1 M$. We show that $z_0 \notin \tilde{P}_1 M$. Suppose, that player E has chosen the control

$$v(t) = \begin{cases} (1,0); & t \in [0,\ 1/2) \\ \left(0,\ (e-1)^{-1}\right); & t \in [1/2;\ 1]. \end{cases}$$

Because $e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$, it is not difficult to see that

$$z(1) = \left(1/2,\ \int_{\frac{1}{2}}^{t} e^t dt\,(e-1)^{-1}\right).$$

The set M has the form $M = \{(\lambda_1, \lambda_2),\ \lambda_i \geq 0,\ \lambda_1 + \lambda_2 = 1\}$. As $\frac{1}{2} + \frac{e - e^{1/2}}{e-1} \neq 1$ then $z(1) \notin M$. This implies that $z_0 \notin M$ and therefore $P^*_1 M \neq \tilde{P}_1 M$.

5.2 *H*-convexity in Linear Games

Games with scalar matrices were considered in the previous subsection and sufficiently effective methods for their solution were produced. It was shown that the convexity conditions for the sets M and N are essential. Example 10 showed that the ordinary convexity of M and N is not sufficient for an arbitrary matrix A. However, for each matrix, it is possible to choose an appropriate class of convexity such that if M and N belong to this class, then it is again possible to construct effective solution methods.

Let us consider in addition to the space E^n, in which z changes, the space L. Assume that $\dim L \leq n$. Let $\varphi : L \to E^n$ be a linear inclusion operator, $\pi : E^n \to L$ a linear mapping; $A : E^n \times E^n$ a linear operator; and $B : U \times V \to L$ a continuous mapping, where U and V are as previously compact sets in Euclidean spaces.

The dynamics of the games to be studied are described by the equation

$$\dot{z} = Az + \varphi\, B\,(u, v). \tag{29}$$

The terminal set M and the set of the phase constraints N are in the form

$$M = \{\, z \in E^n : \ \pi\, z \in M_L \}\,, \quad N = \{\, z \in E^n \ : \ \pi z \in N_L \},$$

where $M_L \subset N_L$ are closed sets in the space L.

Set $C(t) = \pi e^{A(\theta - t)}\varphi$, $t \in [0,\ \theta]$, and assume that

$$P^*_{N,\theta} M = \bigcap_{v \in V} \bigcup_{u \in U} \left\{ z \in N : \ \pi e^{A\theta} z + \int\limits_0^\theta C\,(t)\ dt\, B\,(u,\ v) \in M_L \right\}.$$

Denote by H the set of the unit vectors $x^* \in L$ such that

(a) $C^*\,(t)\, x^* = \lambda\,(t \,|\, x^*\,)\, x^*$ for all $t \in [0,\ \theta]$, where $\lambda\,(t \,|\, x^*\,)$ is a number;
(b) for fixed x^* the numerical function $\lambda\,(\cdot \,|\, x^*\,)$ does not change sign on the interval $[0,\ \theta]$.

Note that the given definition of the set M coincides with the definition in Subsection 3.4 for the operator family $\{A\,(t)\,,\ t \in [0, \theta]\}$ acting in E^n.

Theorem 27. *Let M_L be a H-convex set. If $z_0 \in P^*_{N,\theta}M$ then there exists a mapping $u_* : V \to U$ such that for any $v\,(\cdot) \in V\,[0,\ \theta]$:*

(a) $u_{z_0}\,(v\,(t))$ is an admissible control of player P;
(b) for the solution $z\,(t)$ to equation (29) with the beginning at z_0 and corresponding controls $u_{z_0}\,(v\,(t))$ and $v\,(t)$, the inclusion $\pi z\,(\theta) \in M_L$ is fulfilled.
(c) if, furthermore, N_L is an H-convex set and $Az_0 \in \varphi L$ then $\pi z\,(\tau) \in N_L$ for all $\tau \in [0,\ \theta]$.

Proof. Let $z_0 \in P^*_{N,\theta}\ M$. Then for any $v \in V$ there exists $u_{z_0}(v) \in U$ such that

$$\pi e^{A\theta} z_0 + \int\limits_0^\theta C(t)\ dt\, B(u,\ v) \in M_L \tag{30}$$

By Philipov's lemma (Theorem 2), a mapping $u_{z_0}(v)$ can be chosen such that the function $u_{z_0}(v(t))$ is measurable if $v(t\ )$ is an admissible control of player E.

Let us assume that player E realizes some control $v(t)$, $t \in [0,\ \theta]$. Then, for any $s \in [0,\ \theta]$,

$$\int\limits_0^\theta C(t)\ dt\, B(u_{z_0}(v),\ v(s)) \in M_L - \pi e^{A\theta} z_0 \tag{31}$$

The set $M_L - \pi e^{A\theta} z_0$ is H-convex. Therefore, applying Theorem 14, we get from (31)

$$\pi z(\theta) = \pi e^{A\theta} z_0 + \int\limits_0^\theta C(t)\ \ B(u_{z_0}(v(t)),\ v(t))\, dt \in M_L.$$

Thus, claims "a" and "b" of the theorem are proved. We shall now show that the disstrategy constructed in (30) keeps the trajectory in N_L under the made assumptions. It follows from (30) and from $M_L \subset N_L$ that

$$\pi e^{A\theta} z_0 + \int\limits_0^\theta \pi e^{A(\theta - t)} \varphi\ dt\, B(u_{z_0}(v),\ v) \in N_L$$

or

$$\pi e^{A\theta} z_0 + \int\limits_0^\theta \pi e^{At} \varphi\ dt\, B(u_{z_0}(v),\ v) \in N_L \tag{32}$$

As

$$\int\limits_0^t e^{As} A\ ds\ = e^{At} - E, \tag{33}$$

it follows from (32) and (33) that

$$\int\limits_0^\theta \pi e^{At} \varphi\ dt\ [B(u_{z_0}(v),\ v) + x_0] \in N_L - \pi z_0, \tag{34}$$

where x_0 is an element from L such that $A z_0 = \varphi x_0$. Fix $\tau \in [0,\ \theta]$. Then, it follows from (34) that for any $s \in [0,\ \tau]$:

$$\int_0^\theta \pi e^{At}\varphi\, dt\, \left[B\left(u_{z_0}\left(v\left(\tau-s\right)\right),\, v\left(\tau-s\right)\right)+x_0\right] \in N_L - \pi z_0 \tag{35}$$

Because $0 \in N_L - \pi z_0$, applying Theorem 15 we get from (35):

$$\int_0^\tau \pi e^{At}\varphi\ \left[B\left(u_{z_0}\left(v\left(\tau-t\right)\right),\, v\left(\tau-t\right)\right)+x_0\right] dt \in N_L - \pi z_0$$

Taking into account (34) and replacing variables t by $\tau - t$, we have

$$\pi z\left(\tau\right) = \pi e^{A\tau} z_0 + \int_0^\tau \pi e^{A(\tau-t)}\varphi\ B\left(u_{z_0}\left(v\left(t\right)\right),\, v\left(t\right)\right) dt\ \in N_L.$$

This finishes the proof of the theorem. ■

Theorem 28. *Let $B\left(U, v\right)$ be a H-convex set for all $v \in V$. If $z_0 \notin P^*_{N,\theta}M$ then either $\pi z_0 \notin N_L$ or there exists $v_{z_0} \in V$ such that, for the trajectory $z\left(t\right)$ corresponding with the arbitrary control $u\left(\cdot\right) \in U\left[0,\ \theta\right]$ and control $v\left(t\right) \equiv v_{z_0}$ with the beginning at z_0 the relation $z\left(\theta\right) \notin M_L$ is fulfilled.*

Proof. Let $z_0 \notin P^*_{N,\theta}M$ and $\pi z_0 \in N_L$. Then there exists $v_{z_0} \in V$ such that for any $u \in U$:

$$\pi e^{A\theta} z_0 + \int_0^\theta C\left(t\right)\ dt\, B\left(u,\, v_{z_0}\right) \notin M_L. \tag{36}$$

Because $B\left(U,\, v_{z_0}\right)$ is a H-convex set, it follows from Theorem 16 that

$$\int_0^\theta C\left(t\right)\ B\left(U,\, v_{z_0}\right)\, dt = \int_0^\theta C\left(t\right)\ dt\, B\left(U,\, v_{z_0}\right).$$

It follows from this and from (36) that for any $u\left(\cdot\right) \in U\left[0,\ \theta\right]$:

$$\pi\, z\left(\theta\right) = \pi e^{A\theta} z_0 + \int_0^\theta C\left(t\right)\ B\left(u\left(t\right),\, v_{z_0}\right)\, dt\ \notin M_L.$$

The theorem is proved. ■

It follows from Theorem 27 that under certain constraints on the sets M and N, the inclusion $P^*_{N,\theta}M \subset \tilde{P}_{N,\theta}M$ is fulfilled, i.e., $P^*_{N,\theta}M$ is an estimation from below for the set $\tilde{P}_{N,\theta}M$. Under the conditions of Theorem 25 $\tilde{P}_{N,\theta}M \subset P^*_{N,\theta}M$, that is, in this case $P^*_{N,\theta}M$ is an estimation from above for $\tilde{P}_{N,\theta}M$. Thus, the following results:

Corollary 5. *Let M_L, N_L, and $B\,(U,\,v)$ for any $v \in V$, be H-convex sets and either $N = L$ or $AE^n \subset \varphi L$. Then*

$$\tilde{P}_{N,\,\theta} M = P^*_{N,\,\theta} M.$$

Let us make a few remarks and give some examples. At first we turn our attention to the fulfillment conditions of item (a) in the definition of the set H. We will find that the following statement is useful later on.

Theorem 29. *Let*

$$\left(\pi e^{At} \varphi\right)^* x^* = \lambda\,(t)\; x^*,$$

for any $t \in [0,\;\theta]$, $\theta > 0$, where $\lambda\,(t)$ is a numerical function. Then x^* is an eigenvector of the operators $\left(\pi A^k \varphi\right)^*, k = 0, 1, 2, \ldots$

Proof. The analyticity of mapping $\left(\pi e^{At} \varphi\right)^*$ implies the analyticity of the function $\lambda\,(t)$. Let $\lambda\,(t) = \sum\limits_{k=0}^{\infty} \lambda_k \frac{t^k}{k!}$. Then it follows from the condition of the theorem that

$$\sum_{k=0}^{\infty} \left[\, \left(\pi A^k \varphi\right)^* x^* - \lambda_k x^* \right]\; \frac{t^k}{k!} \equiv 0.$$

From this, because of the arbitrariness of $t \in [0,\;\theta]$, we get $\left(\pi A^k \varphi\right)^* x^* = \lambda_k x^*$ and this proves the theorem. ■

Corollary 6. *Let the conditions of Theorem 26 be fulfilled, $E^n = L$, and π, φ be identity operators. Then, $\lambda\,(t) = e^{\lambda t}$, where λ is an eigenvalue of A^* associated with an eigenvector $x*$.*

The opportunity for the application of the method of H-convex sets is illustrated by the examples below. The set H for concrete classes of games will be described there as well as the introduction of the operator π and φ.

Example 11. Let $E^n = L$, and let π and φ be the identity operators. Then equation (29) has the form

$$\dot{z} = Az + B\,(u, v).$$

Let us denote by H_A the set of the unit length eigenvectors of the operator A^*, including a value associated with zero.

We will show that, in this case, one can take H_A as H. Indeed, if x^* is an eigenvector of A^* and $\lambda\,(x^*)$ is an appropriate eigenvalue of A^*, then

$$e^{A^*(\theta - t)} x^* = e^{\lambda(x^*)(\theta - t)} x^*$$

whence $\lambda\,(\,t \mid x^*\,) = e^{\lambda(x^*)(\theta - t)} > 0$. Thus, the conditions "a" and "b" in the definition of H are fulfilled. It follows from Theorem 26 that if x^* is an eigenvector of the operator $e^{A^* t}$, then x^* is an eigenvector of the operator A^* implying, thus that H_A is the maximal set satisfying the conditions "a" and "b" in the definition of H.

Because $E^n = L$, the conditions $A\,z_0 \in \varphi\,L$ are fulfilled automatically.

Example 12. We shall consider a game with dynamics

$$\ddot{x} = D\dot{x} + B\,(u, v), \tag{37}$$

where $x \in L$, and D is a linear operator acting in the space L. Equation (37) describes a control moving according to the Newton's law accounting friction force. Rewrite the last equation in the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = Dy + B\,(u, v)\,. \end{cases}$$

In this case, the space $E^n = L \times L$ and the operators A, π, and φ may be represented by the matrices

$$A = \begin{bmatrix} 0 & E_L \\ 0 & 0 \end{bmatrix}, \quad \pi\,\left[E_l\;0\right], \quad \text{and} \quad \phi = \begin{bmatrix} 0 \\ E_L \end{bmatrix},$$

where E_L is the unit operator acting in space L. Using the Cauchy formula, it is not difficult to find that

$$e^{At} = \begin{bmatrix} E_L & \int_0^t e^{Ds} ds \\ 0 & e^{Dt} \end{bmatrix},$$

$$\pi\, e^{At} = \begin{bmatrix} E_L & \int_0^t e^{Ds} ds \end{bmatrix},$$

and

$$\pi\, e^{At}\varphi = \int_0^t e^{Ds} ds.$$

This implies that, if $x\,(0) = x_0,$ and $\dot{x}\,(0) = y_0$, the solution to equation (37) is of the form:

$$x\,(t) = \pi\, z\,(t) = x_0 + \int_0^t e^{Ds} ds\, y_0 + \int_0^t \int_0^{t-s} e^{D\tau} d\tau\; B\,(\,u\,(s)\,,\, v\,(s))\; ds\,.$$

Let H_D be, as above, the set of the unit eigenvectors of D^*. We show that it is possible to take H_D as the set H. Indeed, if x^* is an eigenvector of D^* and $\lambda\,(x^*)$ the corresponding eigenvalue, then

$$\left(\pi\, e^{A(\theta-t)}\varphi\right)^*\; x^* = [\lambda\,(x^*)]^{-1} \left(e^{\lambda(x^*)(\theta-t)} - 1\right)\; x^*.$$

Therefore, $\lambda\,(\,t\mid x^*\,) = [\lambda\,(x^*)]^{-1}\left(e^{\lambda(x^*)(\theta-t)} - 1\right) \geq 0$ for $t \in [0,\ \theta]$. If $\lambda(x^*) = 0$, then $\lambda\,(\,t\mid x^*\,) = \theta - t \geq 0$. Thus, the conditions “a” and “b” in the definition of H are fulfilled.

Let us show that H_D is the maximal set satisfying the conditions "a" and "b" in the definition of H. For this purpose, by virtue of Theorem 29, it is sufficient to show that $\left(\pi A^2 \varphi\right) = D$. Because

$$\pi e^{At}\varphi = \int_0^t e^{Ds} ds,$$

we have $\dfrac{d^2}{dt^2}\left(\pi e^{At}\varphi\right) = \pi A^2 e^{At}\varphi = De^{Dt}$.

The required equality follows from this for $t = 0$.

In the given example, the condition $Az_0 \in \varphi L$ means

$$A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ Dy_0 \end{pmatrix} \in \varphi L = \begin{pmatrix} 0 \\ L \end{pmatrix},$$

whence $x(0) = y_0 = 0$.

Example 13. In the previous examples, the set H did not depend on θ. We consider here an example where such a dependence takes place. Let the game dynamics be described by the equation

$$\ddot{x} = -Dx + B(u, v), \tag{38}$$

where $x \in L$. Rewrite (38) in the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -Dx + B(u, v). \end{cases}$$

In this case $E^n = L \times L$,

$$A = \begin{bmatrix} 0 & E_L \\ -D & 0 \end{bmatrix}, \quad \pi\left[E_l\ 0\right], \quad \text{and} \quad \varphi = \begin{bmatrix} 0 \\ E_L \end{bmatrix}.$$

It is not difficult to show that

$$\pi e^{At} = \left[\cos\left(\sqrt{D}\,t\right),\ \left(\sqrt{D}\right)^{-1}\sin\left(\sqrt{D}\,t\right)\right],\ \pi e^{At}\varphi\ = \left(\sqrt{D}\right)^{-1}\sin\left(\sqrt{D}\,t\right),$$

where the notations $\cos\left(\sqrt{D}\,t\right)$ and $\left(\sqrt{D}\right)^{-1}\sin\left(\sqrt{D}\,t\right)$ are introduced for the series:

$$\cos\left(\sqrt{D}\,t\right) = E_L - \frac{1}{2!}D\,t^2 + \frac{1}{4!}D^2t^4 - \cdots,$$

$$\left(\sqrt{D}\right)^{-1}\sin\left(\sqrt{D}\,t\right) = E_L t - \frac{1}{3!}D\,t^2 + \frac{1}{5!}D^2t^5 - \cdots$$

From this, if $x(0) = x_0$ and $\dot{x}(0) = y_0$, the solution to equation (38) has the form

$$x(t) = \pi\,z(t) = \cos\left(\sqrt{D}\,t\right)x_0 + \left(\sqrt{D}\right)^{-1}\sin\left(\sqrt{D}\,t\right)y_0$$

$$+\left(\sqrt{D}\,\right)^{-1}\int\limits_{0}^{t}\sin\!\left(\sqrt{D}\,(t-s)\right)B\big(u(s),v(s)\big)\,ds.$$

Let $x^* \in L$ be a unit eigenvector of D^* with a corresponding eigenvalue $\lambda\,(x^*)$. It is possible to take as H those unit eigenvectors of D^* for which the function $\left(\sqrt{\lambda\ (x^*)}\,\right)^{-1}\ \sin\!\left(\sqrt{\lambda\ (x^*)}\ t\,\right)$ does not change sign on $[0,\ \theta]$. Thus,

$$\begin{aligned}\left(\pi\, e^{A(\theta-t)}\varphi\right)^* x^* &= \left(\sqrt{D^*}\right)^{-1}\sin\!\left(\sqrt{D^*}\,(\theta-t)\right)\ \ x^*\\ &= \left(\sqrt{\lambda\ (x^*)}\right)^{-1}\ \sin\!\left(\sqrt{\lambda\ (x^*)}\,(\theta-t)\ \right)\ x^*.\end{aligned}$$

As in Example 12, the condition $Az_0 \in \varphi L$ has also the form $x\,(0)=0$. The constructed set H depends essentially on θ and may be empty.

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Game Dynamic Problems for Systems with Fractional Derivatives

Arkadii A. Chikrii

Glushkov Cybernetics Institute NAS Ukraine, Kiev Ukraine
chik@insyg.kiev.ua

Abstract A general approach to solving game approach problems for systems with Volterra evolution is outlined. It is based on the method of resolving functions [11] (the latter is also referred to as the method of Minkowski inverse functionals [12]) and employs the apparatus of the theory of set-valued mappings. Suggested scheme encompasses a wide range of functional-differential systems, in particular integral, integro-differential, and difference-differential systems of equations that specify dynamics of the conflict-controlled process.

In this chapter, we examine in great detail the case when dynamics of the conflict-controlled process is described by a system with fractional derivatives. Note that we deal with both the Riemann–Liouville and Dzhrbashyan–Nersesyan–Caputo fractional derivatives.

Here solutions to such systems are given in the form of Cauchy formula analog. Sufficient conditions for terminating the game of approach in some guaranteed time are obtained. These conditions are based on the Pontryagin condition analog [48], expressed in terms of the Mittag–Leffler matrix functions [13,14]. Using the asymptotic expansions of these functions allows one to develop conditions for solvability of the game problems.

Key words: fractional derivative, set-valued mapping, Minkowski functional, Mittag–Leffler function, Pontryagin's condition

1 Introduction

This investigation is concerned with the processes with fractional derivatives. It should be emphasized that systems of fractional order go back to the Abel integral equation, to be specific, to representation of its solution [18,53]. The key to understanding the operation of fractional integration lies also with the Cauchy formula for multiple integration of a function. Extensive literature is devoted to investigating the operators of fractional integration and differentiation. Monographs [24,32,38,40,46,53] are worthy of notice as they give the reader a comprehensive idea of the subject. The studies [3,33,47] are devoted to the physical and geometrical interpretations of the fractional integration and differentiation. In the past 20 years, fractional calculus has found applications in physics [7], hydrology [4], finance [51], seismic analysis [26], viscous damping [31], electrochemical problems involving diffusion [40], fractional-order sinusoidal oscillators [41], robotics, as well as in the theory of control of finite-dimensional [2, 22, 34, 35, 42–44] and infinite-dimensional systems [36, 37] and in solving the Cauchy problem for systems of fractional order [17, 25].

It should be noted that to solve a system of differential equations with Riemann–Liouville fractional derivative, a fractional integral of appropriate order should be given, instead of a conventional Cauchy data at the initial time $t = 0$. This is explained by the fact that the solution to such equation has singularity at $t = 0$ and only generalized initial conditions are meaningful in such case. Yet, from the physical standpoint, it is desirable to deal with conventional Cauchy problem for a system of equations with fractional derivatives.

M. M. Dzhrbashyan and A. B. Nersesyan in their joint paper [17] suggested to consider an equation with fractional derivatives, in which instead of Riemann–Liouville derivative, its regularized value is used and conventional Cauchy data stands for the initial condition. Simultaneously, M. Caputo performed the same expedient in his study [6]. That is why in the sequel the regularized fractional derivative will be referred to as the Dzhrbashyan–Nersesyan–Caputo fractional derivative.

In this paper, to study conflict-controlled processes described by fractional-order systems, we use the method of resolving functions [11]. This method was initiated by the paper of Pshenichnyi [49], devoted to the group pursuit problem in the case of "simple motions." Later on, B. N. Pshenichnyi, A. A. Chikrii, and J. S. Rappoport developed a general method for solving the linear problems of group pursuit, in their number those under the state constraints. Important results for the group pursuit problems were obtained by N. L. Grigorenko [20], with the help of analogous techniques, and by N. N. Petrov [45], who examined such problems under the state constraints. The gist of the method of resolving functions consists in constructing, on the basis of known process parameters, certain numerical functions. These function integrally characterize the course of a conflict-controlled process, namely the trajectory proximity to the terminal set, and play a key role in solving specific problems. On the one hand, this method appears as a general approach to investigation of conflict-controlled processes, closely related with the Pontryagin first direct method [48]. On the other hand, it substantiates the rule of parallel pursuit, well-known to engineers engaged in design of rocket and space technology.

After publication of the monograph [11], the method of resolving functions has been extended to systems of variable structure [29] and to the integral and integro-differential games for linear systems possessing polar singularity [19]. Game problems with a terminal functional were a subject of research in [15], where some ideas of Fenchel–Moreau and Minkowski were used.

Some of the results concerning systems with fractional derivatives, provided below, were obtained jointly with S. D. Eidelman.

2 Formulation of the Problem. Auxiliary Results. Scheme of the Method

Let us denote by R^n the real n-dimensional Euclidean space and by $R_+ = \{t : t \geq 0\}$ the positive semi-axis. Consider the process evolving according to the equation

$$z(t) = g(t) + \int_0^t \Omega(t,\tau)\varphi\left(u(\tau), v(\tau)\right) d\tau, \qquad t \geq 0. \tag{1}$$

Function $g(t)$, $g : R_+ \to R^n$, is Lebesgue measurable and bounded for $t > 0$, matrix function $\Omega(t,\tau)$, $t \geq \tau \geq 0$, is measurable in τ and also summable in τ for any $t \in R_+$. The control block is given by function $\varphi\left(u, v\right), \varphi : U \times V \to R^n$, which is assumed to be jointly continuous in its variables on the direct product of nonempty compacts U and V, i.e., $U, V \in K(R^n)$. Control actions of the players, $u\left(\tau\right)$, $u : R_+ \to U$, and $v(\tau)$, $v : R_+ \to V$, are measurable functions.

In addition to the process (1), a terminal set is given having a cylindrical form

$$M^* = M_0 + M, \tag{2}$$

where M_0 is a linear subspace from R^n and $M \in K(L)$, where L is an orthogonal complement to M_0 in R^n.

The goals of the first (u) and the second (v) player are opposite. The first one strives in the shortest time to drive a trajectory of the process (1) to the set (2), the second one strives to maximally postpone the instant of time when the process trajectory hits the set M^*.

Let us take the side of the first player and assume that his opponent chooses as controls arbitrary measurable functions with values from V. We also assume that the game (1), (2) takes place on the interval $[0, T]$ and that the first player chooses as controls measurable functions of the form:

$$u(t) = u\left(g\left(T\right), v_t\left(\cdot\right)\right), \quad t \in [0, T], \quad u(t) \in U, \tag{3}$$

where $v_t\left(\cdot\right) = \{v\left(s\right) : \ 0 \leq s \leq t\}$ is a pre-history of the second player's control up to the instant t. If, for example, $g(t) = A(t)z_0$, where $A(t)$ is a matrix function such that $A(0) = E$ (E is a unit matrix) and $z(0) = z_0$, then we may consider that $u(t) = u\left(z_0, v_t\left(\cdot\right)\right)$, i.e., control of the first player appears as a special type quasistrategy [11, 28].

The goal of the paper is, under the information condition (3), to develop sufficient conditions for solvability of the problem in favor to the first player in some guaranteed time, as well as to estimate this time and to find the control of first player that allows for the realization of this result.

Now let us describe the method of solving this problem. Original assumptions about functions $g(t)$, $\Omega(t,\tau)$, $\varphi(u,v)$ and sets U, V, M^* allow us to realize constructions already known from the theory of differential games [9–12]. Let us briefly outline them.

Define by π the orthoprojector acting from R^n onto L.

Setting

$$\varphi(U,v) = \{\varphi(u,v) : \ u \in U\}$$

let us consider the following set-valued mappings

$$W(t,\tau,v) = \pi\Omega(t,\tau)\,\varphi(U,v)\,,$$

$$W(t,\tau) = \bigcap_{v\in V} W(t,\tau,v),$$

defined on sets $\Delta \times V$ and Δ respectively, where

$$\Delta = \{(t,\tau) : \ 0 \le \tau \le t < \infty\}\,.$$

Condition 1 (Pontryagin's condition). Set-valued mapping $W(t,\tau)$ takes nonempty values on set Δ.

By virtue of continuity of the function $\varphi(u,v)$ and the condition $U \in K(R^n)$, the mapping $\varphi(U,v)$ is continuous in v in Hausdorff metric.

Taking into account the assumptions concerning matrix function $\Omega(t,\tau)$, one can infer that the set-valued mappings $W(t,\tau,v)$ and $W(t,\tau)$ are measurable in τ [23]. Recall that a set-valued mapping $F(t)$, $F:[0,T] \to 2^{R^n}$ is called measurable if for any open set Y, $Y \subset R^n$, the set $\{t \in [0,T] : \ F(t) \cap Y \ne \emptyset\}$ is measurable.

Let us denote by $P(R^n)$ a set of all nonempty closed sets from space R^n. Then, obviously,

$$W(t,\tau,v) : \ \Delta \times V \to P(R^n),$$
$$W(t,\tau) : \ \Delta \to P(R^n).$$

In this case, the set-valued mappings $W(t,\tau,v)$ and $W(t,\tau)$ are usually referred to as normal in τ [23].

It follows from Pontryagin's condition and some results of the papers [1, 16, 23] that for any $t \ge 0$ there exists at least one τ-measurable selection $\gamma(t,\tau) \in W(t,\tau)$. By assumptions concerning the parameters of process (1) such selection $\gamma(t,\tau)$ is a function that is summable in τ for any fixed $t \ge 0$, $\tau \in [0,t]$. Denote

$$\xi(t, g(t), \gamma(t,\cdot)) = \pi g(t) + \int_0^t \gamma(t,\tau)\,d\tau.$$

Now let us define a function

$$\begin{aligned}\alpha(t,\tau,v) = \sup\{\alpha \geq 0 : [W(t,\tau,v) - \gamma(t,\tau)] \\ \cap \alpha[M - \xi(t,g(t),\gamma(t,\cdot))] \neq \emptyset\}\end{aligned} \tag{4}$$

and call it the resolving function. This function will play a key role in the sequel.

By virtue of assumptions concerning the parameters of process (1) and some known results from [11], one can infer that function (4) is measurable in τ and upper semicontinuous in v.

In what follows, our prime concern will be with the joint dependence of function $\alpha\,(t,\tau,v)$ in variables τ and v. Let us fix some t and set $\alpha\,(\tau,v) = \alpha\,(t,\tau,v)$. We will say that function $\alpha\,(\tau,v)$, $\alpha : [0,T] \times V \to R_+$, is superpositionally measurable if for any measurable function $v(\tau)$, $v : [0,T] \to V$, the superposition $\alpha\,(\tau, v\,(\tau))$, $\alpha : [0,T] \to R_+$, is a τ-measurable function. Sufficiently general assumption ensuring function $\alpha\,(\tau,v)$ to be superpositionally measurable is that of its $L \times B$ measurability [1,16], i.e., of measurability with respect to σ-algebra being a product of σ-algebras $L\,([0,T])$ and $B\,(R^n)$. This σ-algebra consists of subsets of the set $[0,T] \times R^n$ generated by sets of the form $X \times Y$, where X is Lebesgue measurable subset of the interval $[0,T]$ and Y is a Borel measurable subset of R^n.

Denote $W(t,\tau,v) - \gamma\,(T,\tau) = H\,(\tau,v)$, $M - \xi\,(T, g\,(T)\,, \gamma\,(T,\cdot)) = M_1$ and introduce a set-valued mapping

$$\Xi\,(\tau,v) = \{\alpha \in R_+ :\ H\,(\tau,v) \cap \alpha M_1 \neq \emptyset\}\,. \tag{5}$$

Then

$$\alpha\,(\tau,v) = \sup\{\alpha \in R_+ :\ \alpha \in \Xi\,(\tau,v)\}\,.$$

Let us study properties of the set-valued mapping (5). The following general result is true, which generalizes the known statement from [23] and follows, in particular, from the work [30].

Lemma 1. *Let $X \in P\,(R^n)$ and $F(w)$, $F : X \to P\left(R^k\right)$, and $H\,(w)$, $H : X \to P\,(R^n)$, be normal set-valued mappings and let $M\,(w,x)$, $M : X \times R^k \to P\,(R^n)$, be a Caratheodory mapping (measurable in w and continuous in x).*

Then the mapping

$$\Xi\,(w) = \{x \in F\,(w) :\ H\,(w) \cap M\,(w,x) \neq \emptyset\}$$

is normal.

Setting in the statement of Lemma 1 $w = (\tau,v)$, $x = \alpha$ and respectively $F\,(w) = R_+$ and $M\,(w,x) = \alpha M_1$, we infer that the mapping $\Xi\,(\tau,v)$ is $L \times B$ measurable, as the mapping $H\,(\tau,v)$ is $L \times B$ measurable by virtue of its Lebesgue measurability in τ and continuity in v [16].

Now let us show that the function $\alpha(\tau, v)$ is $L \times B$ measurable. Indeed, because the formula is true:

$$\alpha(\tau, v) = \sup_{\alpha \in \Xi(\tau, v)} \alpha = C(\Xi(\tau, v); 1),$$

where $C(X, p)$ is a support function of set X in direction p [52], its $L \times B$ measurability follows from the $L \times B$ measurability of the set-valued mapping $\Xi(\tau, v)$ [23].

Thus, the function $\alpha(\tau, v)$ is $L \times B$ measurable, bounded below by zero and semicontinuous in v [11].

Let us show that the function $\inf_{v \in V} \alpha(\tau, v)$ is measurable. To do this we will treat V as a constant set-valued mapping. It is a measurable mapping [23]. The approximation set in V can be formed, for instance, by functions $v_m(\tau) = v_m$, where $V_* = \{v_1, v_2, \ldots\}$ is a countable dense subset of set V. Then, by virtue of $L \times B$ measurability of the considered function, it is superpositionally measurable whence follows that functions $\alpha(\tau, v_m)$ are measurable in τ. Let us now show that

$$\inf_{v \in V} \alpha(\tau, v) = \inf_{v_m} \alpha(\tau, v_m).$$

For this purpose, we set $\alpha(\tau) = \inf_{v \in V} \alpha(\tau, v)$ and fix τ. By definition of the greatest lower bound, for any $\varepsilon > 0$ there exists an element $v_\varepsilon \in V$ such that

$$\alpha(\tau, v_\varepsilon) \leq \alpha(\tau) + \varepsilon.$$

On the other hand, from the upper semicontinuity in v of the function $\alpha(\tau, v)$, it follows that a neighborhood $O(v_\varepsilon)$ of element v_ε exists such that for any $v \in O(v_\varepsilon)$

$$\alpha(\tau, v) \leq \alpha(\tau, v_\varepsilon) + \varepsilon.$$

In its turn, from here and from the definition of set V_*, it follows that an element $v_m \in V_* \cap O(v_\varepsilon)$ exists such that

$$\alpha(\tau, v_m) \leq \alpha(\tau, v_\varepsilon) + \varepsilon \leq \alpha(\tau) + 2\varepsilon.$$

Then

$$\inf_{v_m} \alpha(\tau, v_m) \leq \alpha(\tau).$$

What is more, because the inverse inequality is always true in view of the inclusion $V_* \subset V$, then

$$\alpha(\tau) = \inf_{v \in V} \alpha(\tau, v) = \inf_{v_m} \alpha(\tau, v_m)$$

and therefore function $\alpha(\tau)$ is measurable as the greatest lower bound of countable set of measurable functions [23].

The following statement is a consequence of formula (4). If for some t the inclusion $\xi(t, g(t), \gamma(t, \cdot)) \in M$ is satisfied, then function $\alpha(t, \tau, v)$ turns into infinity for all $\tau \in [0, t]$, $v \in V$.

Let us introduce a mapping

$$T\left(g\left(\cdot\right),\gamma\left(\cdot,\cdot\right)\right)=\left\{t\geq 0:\ \int\limits_0^t \inf_{v\in V}\alpha\left(t,\tau,v\right)d\tau\geq 1\right\}. \qquad (6)$$

If for some t the integral in expression (6) turns into infinity, then the inequality in braces is readily satisfied. If, on the other hand, the inequality in (6) fails for any t, then we set $T\left(g\left(\cdot\right),\gamma\left(\cdot,\cdot\right)\right)=\emptyset$.

We can now formulate the main result of the paper.

Theorem 1. *Let in the game (1), (2) Pontryagin's condition hold, $M = coM$ and let for some bounded function $g\left(t\right)$, $t > 0$, and some measurable in τ selection $\gamma\left(t,\tau\right)$, $t\geq\tau\geq 0$, of the set-valued mapping $W\left(t,\tau\right)$ the following relations be true:*

$$T\left(g\left(\cdot\right),\gamma\left(\cdot,\cdot\right)\right)\neq\emptyset \text{ and } T\in T\left(g\left(\cdot\right),\gamma\left(\cdot,\cdot\right)\right),\ T<+\infty.$$

Then a trajectory of process (1) can be brought from the initial state $g(T)$ to the terminal set in time T, using control of the form (3).

Proof. Consider the case $\xi\left(T,g\left(T\right),\gamma\left(T,\cdot\right)\right)\overline{\in}M$. Let $v_T\left(\cdot\right)$ be an arbitrary measurable function with values in V. Analogously to [10, 11], we introduce a test function

$$h\left(t\right)=1-\int\limits_0^t\alpha\left(T,\tau,v\left(\tau\right)\right)d\tau,\quad t\in\left[0,T\right].$$

Because the function $\alpha\left(T,\tau,v\right)$ is $L\times B$ measurable, it is superpositionally measurable as well, i.e., function $\alpha\left(T,\tau,v\left(\tau\right)\right)$ is measurable. On the other hand, by assumptions concerning the parameters of process (1), (2) the latter is bounded for almost all $\tau<T$ and therefore integrable on any finite interval of time. From this it follows that the function $h\left(t\right)$ is continuous, nonincreasing, and $h\left(0\right)=1$. Therefore, there exists an instant $t_*=t\left(v\left(\cdot\right)\right)$, $t_*\in\left(0,T\right]$, such that $h\left(t_*\right)=0$.

In the sequel the segments $[0,t_*)$ and $[t_*,T]$ will be referred to as "active" and "passive" respectively. Let us describe how the first player chooses his control on each of them. For this purpose consider a set-valued mapping

$$\begin{aligned}U\left(\tau,v\right)=\{u\in U:\ &\pi\Omega\left(T,\tau\right)\varphi\left(u,v\right)-\gamma\left(T,\tau\right)\\ &\in\alpha\left(T,\tau,v\right)\left[M-\xi\left(T,g\left(T\right),\gamma\left(T,\cdot\right)\right)\right]\}.\end{aligned} \qquad (7)$$

Because the function $\alpha\left(T,\tau,v\right)$ is $L\times B$ measurable, $M\in K\left(R^n\right)$, and the vector $\xi\left(T,g\left(T\right),\gamma\left(T,\cdot\right)\right)$ is bounded, then the mapping

$$\alpha\left(T,\tau,v\right)\left[M-\xi\left(T,g\left(T\right),\gamma\left(T,\cdot\right)\right)\right]$$

is $L \times B$ measurable. In addition, it is obvious that the left side of inclusion in (7) is jointly $L \times B$ measurable function in τ and v and continuous in u. From here, in view of the known statement from [23], it follows that the mapping $U(\tau, v)$ is $L \times B$ measurable. Therefore its selection

$$u(\tau, v) = \operatorname{lex\,min} U(\tau, v) \tag{8}$$

is $L \times B$ measurable function. Let us set control of the first player on the active segment $[0, t_*)$ equal to

$$u(\tau) = u(\tau, v(\tau)). \tag{9}$$

By virtue of the function $u(\tau, v)$ $L \times B$ measurability, it is superpositionally measurable, which implies the measurability of function $u(\tau)$.

Let us analyze the "passive" segment $[t_*, T]$. We set in expression (7) $\alpha(T, \tau, v) \equiv 0$ for $\tau \in [t_*, T]$, $v \in V$, and choose control of the first player in accordance with the above-outlined scheme using expressions (7)–(9).

In the case $\xi(T, g(T), \gamma(T, \cdot)) \in M$, control of the first player on the interval $[0, T]$ is chosen from the same relations as on the passive segment, i.e., by the scheme (7)–(9) with $\alpha(T, \tau, v) \equiv 0$, $\tau \in [0, T]$, $v \in V$.

Let us show that if the control of the first player is chosen in the form (9), then in both cases, in view of relations (7), (8), a trajectory of process (1) will be brought to set M at instant T for any control of the second player.

From expression (1) we have

$$\pi z(T) = \pi g(T) + \int_0^T \pi \Omega(T, \tau) \varphi(u(\tau), v(\tau)) \, d\tau. \tag{10}$$

Let us first analyze the case $\xi(T, g(T), \gamma(T, \cdot)) \overline{\in} M$. To do this, we add and subtract from the right side of equality (10) the term $\int_0^T \gamma(T, \tau) \, d\tau$. Using the above-outlined rule for control choice of the first player, we obtain from (10) the inclusion

$$\pi z(T) \in \xi(T, g(T), \gamma(T, \cdot)) \left[1 - \int_0^{t_*} \alpha(T, \tau, v(\tau)) \, d\tau \right] + \int_0^{t_*} \alpha(T, \tau, v(\tau)) M d\tau.$$

Because M is a convex compact, $\alpha(T, \tau, v(\tau))$ is a nonnegative function for $\tau \in [0, t_*)$, and

$$\int_0^{t_*} \alpha(T, \tau, v(\tau)) \, d\tau = 1,$$

then $\int_0^{t_*} \alpha(T, \tau, v(\tau)) M d\tau = M$ and, consequently, $\pi z(T) \in M$ and $z(T) \in M^*$.

Suppose that $\xi(T, g(T), \gamma(T, \cdot)) \in M$. Then, taking into account the control law of the first player, from equality (10) one can immediately deduce the inclusion $\pi z(T) \in M$. ■

3 Some General Properties of the Resolving Function. Explicit Formulas

As seen from the method scheme, in order to evaluate the instant of the game termination and to construct the control law of the first player in the form (3), we need an explicit form of the resolving function. The following statements provide solution to this problem under some specific assumptions, namely when some parameters of the process (1), (2) appear as convex sets (for example, polyhedral, elliptic, or spherical).

Lemma 2. *Let in the game problem (1), (2) Pontryagin's condition be satisfied, $M = coM$, and let the mapping $W(t, \tau, v)$ be convex-valued. Then*

$$\alpha(t, \tau, v) = \inf_{p \in P(t)} \{C(W(t, \tau, v); p) - (p, \gamma(t, \tau))\}, \tag{11}$$

where

$$P(t) = \{p \in L : \ C(M; p) + (p, \xi(t, g(t), \gamma(t, \cdot))) = -1\}. \tag{12}$$

Proof. The nonemptiness of the intersection in definition of function $\alpha(t, \tau, v)$, in view of both sets' closureness and convexity, in the terms of support functions [52], is equivalent to the inequality

$$C(W(t, \tau, v); p) - (p, \gamma(t, \tau)) + \alpha[C(M; -p) + (p, \xi(t, g(t), \gamma(t, \cdot)))] \geq 0$$

for all $p \in L$, or put it otherwise, to the inequality

$$\begin{aligned} &-\alpha[C(M; -p) + (p, \xi(t, g(t), \gamma(t, \cdot)))] \\ &\leq C(W(t, \tau, v); p) - (p, \gamma(t, \tau)). \end{aligned} \tag{13}$$

The right-hand side of the last inequality is non-negative for all p, in view of Pontryagin's condition and the choice of selection $\gamma(t, \tau)$. Therefore, if $C(M; -p) + (p, \xi(t, g(t), \gamma(t, \cdot))) \geq 0$, then for any $\alpha \geq 0$, inequality (13) is readily satisfied.

Normalizing p with the help of expression (12), one can infer formula (11) from inequality (13). ■

Lemma 3. *Let the process (1), (2) be linear ($\varphi(u, v) = u - v$), for this process Pontryagin's condition holds, and let, in addition,*

$$\pi \Omega(t, \tau) U = \{x \in L : \ (p_i, x) \leq a_i(t, \tau), \ i = 1, \ldots, k\},$$

where $a_i : \Delta \to R^1$ *is a function that is summable in* τ *for all* $t \geq 0$*,* $p_i \in L$*. Then, if* $\xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right) \overline{\in} M$ *then*

$$\alpha\left(t, \tau, v\right) = \max_{m \in M} \min_{i \in I(m)} \left\{ \frac{a_i\left(t, \tau\right) - \left(p_i, \pi \Omega\left(t, \tau\right) v + \gamma\left(t, \tau\right)\right)}{\left(p_i, m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right)} \right\}, \tag{14}$$

where

$$I\left(m\right) = \left\{i \in \{1, \ldots, k\} : \left(p_i, m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right) > 0\right\}.$$

Proof. It follows from formula (4) for the resolving function [11] that

$$\alpha\left(t, \tau, v\right) = \max_{m \in M} \alpha\left(t, \tau, v, m\right),$$

where

$$\begin{aligned} \alpha\left(t, \tau, v, m\right) = \sup\{\alpha \geq 0 : \alpha\left(m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right) \\ \in W\left(t, \tau, v\right) - \gamma\left(t, \tau\right)\}, \end{aligned} \tag{15}$$

and that function $\alpha\left(t, \tau, v, m\right)$ is upper semicontinuous in m.

From the inclusion in relation (15), with account of assumptions of Lemma 3, we have

$$\alpha\left(m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right) \in \pi \Omega\left(t, \tau\right) U - \pi \Omega\left(t, \tau\right) v - \gamma\left(t, \tau\right)$$

or, to take it differently,

$$\begin{aligned} \alpha\left(p_i, m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right) \leq a_i\left(t, \tau\right) \\ - \left(p_i, \pi \Omega\left(t, \tau\right) v + \gamma\left(t, \tau\right)\right), \ i = 1, \ldots, k. \end{aligned} \tag{16}$$

It follows from Pontryagin's condition that the right-hand side of inequality (16) is non-negative for all $(t, \tau) \in \Delta$, $v \in V$, $i = 1, \ldots, k$, and therefore for all i such that $\left(p_i, m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right) \leq 0$, inequality (16) is readily satisfied for all $\alpha \geq 0$. Therefore,

$$\alpha\left(t, \tau, v, m\right) = \min_{i \in I(m)} \left\{ \frac{a_i\left(t, \tau\right) - \left(p_i, \pi \Omega\left(t, \tau\right) v + \gamma\left(t, \tau\right)\right)}{\left(p_i, m - \xi\left(t, g\left(t\right), \gamma\left(t, \cdot\right)\right)\right)} \right\}$$

and, thus, function $\alpha\left(t, \tau, v\right)$ is given by expression (14). ■

Corollary 1. *If in the conditions of Lemma 3*

$$U = \left\{u : \left(p_i, u\right) \leq a_i, \ i = 1, \ldots, k\right\}$$

and $\pi \Omega\left(t, \tau\right) U = r\left(t, \tau\right) U$*, where* $r\left(t, \tau\right)$ *is a function that is summable in* τ *for any* t*, then* $a_i\left(t, \tau\right) \equiv a_i r\left(t, \tau\right)$*.*

Corollary 2. *Let the assumptions of Lemma 3 be satisfied, let set U be a symmetric polyhedron about the origin:*

$$U = -U = \{u : (p_i, u) \leq a_i,\ i = 1, \dots, k\},$$

and $\pi\Omega(t,\tau)U = r(t,\tau)U$, where $r(t,\tau)$ is a function, summable in τ for any t, and let set M be homothetic to U ($M = \lambda U$, $\lambda \geq 0$).

Then if $\xi(t, g(t), \gamma(t,\cdot)) \overline{\in} M$, the following formula is true

$$\alpha(t,\tau,v) = \min_{i \in I_*} \left\{ \frac{a_i r(t,\tau) - (p_i, \pi\Omega(t,\tau)v + \gamma(t,\tau))}{\lambda a_i - (p_i, \xi(t, g(t), \gamma(t,\cdot)))} \right\},$$

where

$$I_* = \{i \in \{1, \dots, k\} : \lambda a_i - (p_i, \xi(t, g(t), \gamma(t,\cdot))) > 0\}.$$

Lemma 4. *Let in the game problem (1), (2) Pontryagin's condition be satisfied, $\varphi(u,v) = u - v$, and let*

$$\pi\Omega(t,\tau)U = \{x \in L : (x - x_0, F(t,\tau)(x - x_0)) \leq 1\},$$

where $F(t,\tau)$ is a matrix function, which is summable in τ for any t and appears as a positively defined, symmetric square matrix for all $(t,\tau) \in \Delta$.

Then if $\xi(t, g(t), \gamma(t,\cdot)) \overline{\in} M$

$$\alpha(t,\tau,v) = \max_{m \in M} \alpha(t,\tau,v,m),$$

where $\alpha(t,\tau,v,m)$ is the greatest positive root of the quadratic equation for α:

$$\begin{gathered}(\alpha(m - \xi(t, g(t), \gamma(t,\cdot))) + \pi\Omega(t,\tau)v + \gamma(t,\tau) - x_0, F(t,\tau) \\ \times (\alpha(m - \xi(t, g(t), \gamma(t,\cdot))) + \pi\Omega(t,\tau)v + \gamma(t,\tau) - x_0)) = 0.\end{gathered}$$

Proof of Lemma 4 is analogous to that of Lemma 3.

Lemma 5. *Let for the process (1), (2) Pontryagin's condition hold, $\varphi(u,v) = u - v$, and let set U be an ellipsoid of the form*

$$U = \{u : (u - u_0, F(u - u_0)) \leq 1\}, \tag{17}$$

and $\pi\Omega(t,\tau)U = r(t,\tau)U$, where F is a symmetric square matrix, $r(t,\tau)$ is a function, summable in τ for any t, and set M is homothetic to ellipsoid U, $M = \lambda U$, $\lambda \geq 0$.

Then, if $\xi(t, g(t), \gamma(t,\cdot)) \overline{\in} M$, then the resolving function $\alpha(t,\tau,v)$ appears as the greatest positive root of the quadratic equation for α:

$$\begin{gathered}(\pi\Omega(t,\tau)v + \gamma(t,\tau) - \alpha\xi(t, g(t), \gamma(t,\cdot)) - u_0, \\ F(\pi\Omega(t,\tau)v + \gamma(t,\tau) - \alpha\xi(t, g(t), \gamma(t,\cdot)) - u_0)) = r(t,\tau) + \lambda.\end{gathered} \tag{18}$$

The statement of Lemma 5 follows from the equality (4), by virtue of the symmetry property of ellipsoid U (17).

Remark 1. If in the conditions of Lemma 5 F is a unit matrix, then in the case of spherical U and M, quadratic equation (18) determines the resolving function.

This result can be found in monograph [11].

In view of formula (6), the following statement is important.

Lemma 6. *Let the process (1), (2) be linear ($\varphi(u,v) = u - v$), Pontryagin's condition be satisfied, and let U and M be convex sets.*

If $\xi(t, g(t), \gamma(t,\cdot)) \overline{\in} M$, then function $\alpha(t,\tau,v)$ is concave in v and attains its minimum in v.

Proof. Let us make use of the representation of resolving function in the form (11), (12). In our case

$$\alpha(t,\tau,v) = \inf_{p \in P(t)} Q(t,\tau,v,p),$$

where function

$$Q(t,\tau,v,p) = C(\pi\Omega(t,\tau)U; p) - (\pi\Omega(t,\tau)v, p) - (p, \gamma(t,\tau))$$

is linear in v. Note that the lower bound in p of a set of linear functions is a concave function. The corresponding inequality can be easily obtained. Then, because of the compactness of set U, the concave function attains on this set its minimum [20]. ■

Various types of sufficient conditions for the continuity in v of function $\alpha(t,\tau,v)$ can be found in the book [11]. They ensure the attainability of the lower bound in v in the definition of the game termination time.

4 Finiteness of the Game Termination Time

When solving specific problems, there is a need to have explicit formulas for the functions under study that would allow one to draw a conclusion whether (or not) the time of game termination is finite.

Denote

$$\Phi(t) = \int_0^t \inf_{v \in V} \alpha(t,\tau,v)\, d\tau.$$

Then the shortest time for the game termination (in the framework of advanced scheme) is defined by the formula

$$T(g(\cdot), \gamma(\cdot,\cdot)) = \inf\{t \geq 0 : \Phi(t) \geq 1\}.$$

Now we find an explicit form of function $\Phi(t)$ for some specific values of the parameters of process (1), (2).

Let us consider the ellipsoids

$$Q = \{x \in R^n : (x, Fx) \leq 1\}, \quad Q_L = \{x \in L : (x, F_L x) \leq 1\},$$

where F and F_L are positive symmetric matrices.

Lemma 7. *Let for conflict-controlled process (1), (2) $\varphi(u,v) = u - v$, $U = aQ$ ($a > 1$), $V = Q$, $M^* = \{0\}$ and let an inverse matrix to $\Omega(t,\tau)$ exist for all $t \geq \tau \geq 0$.*

Then, if $\gamma(t,\tau) \equiv 0$, then

$$\Phi(t) = \int_0^t \frac{a-1}{\sqrt{(q, Fq)}} d\tau,$$

where $q = \Omega^{-1}(t,\tau)\, g(t)$.

Proof. From formula (4) with account of assumptions of the lemma, it follows that

$$\alpha(t,\tau,v) = \sup\{\alpha \geq 0 : \Omega(t,\tau)\, v - \alpha g(t) \in a\Omega(t,\tau)\, Q\}. \tag{19}$$

Here selection $\gamma(t,\tau)$ may be chosen identically equal to zero because the mapping $W(t,\tau)$ always contains zero. Note that as $M^* = \{0\}$, then $M = M_0 = \{0\}$, $L = R^n$, and therefore π is an operator of identity transformation, defined by a unit matrix E.

The inclusion in (19) can be rewritten in the form $v - \alpha q \in aQ$, where $q = \Omega^{-1}(t,\tau)\, g(t)$. Because the vector $v - \alpha q$ linearly depends on α, then the least upper bound in α in expression (19) is furnished by number α, such that vector $v - \alpha q$ lies on the boundary of ellipsoid aQ. The last statement means that

$$(v - \alpha q, F(v - \alpha q)) = a^2$$

and therefore the resolving function appears as the greatest root of the quadratic equation for α:

$$\alpha^2 (q, Fq) - 2\alpha (v, Fq) + (v, Fv) - a^2 = 0.$$

Then

$$\alpha(t,\tau,v) = \frac{(v, Fq) + \sqrt{(v, Fq)^2 + (q, Fq)\left[a^2 - (v, Fv)\right]}}{(q, Fq)}$$

and

$$\min_{v \in Q} \alpha(t,\tau,v) = \frac{a-1}{\sqrt{(q, Fq)}},$$

where minimum is furnished by the element $v = -\frac{q}{\sqrt{(q,Fq)}}$.

The last equality implies the required result. ∎

Lemma 8. *Let the parameters of process (1), (2) satisfy conditions:* $\varphi(u,v) = u - v$, $\pi\Omega(t,\tau) = w(t,\tau)E$, *where* $w(t,\tau)$ *is a numerical function,* $U = aQ_L$ $(a > 1)$, $V = Q_L$, $M = lQ_L$ $(l \geq 0)$.

Then if $\gamma(t,\tau) \equiv 0$ *then*

$$\Phi(t) = \int_0^t \frac{|w(t,\tau)|(a-1)}{\sqrt{(g,Fg)} - l} d\tau, \tag{20}$$

where $g = \pi g(t)$.

Proof. It is easily seen that in this case, Pontryagin's condition is satisfied and the set-valued mapping $W(t,\tau)$ contains zero. Set $\gamma(t,\tau) \equiv 0$. According to representation (4), the resolving function is the greatest number, satisfying the inclusion

$$w(t,\tau)v - \alpha\pi g(t) \in (a|w(t,\tau)| + \alpha l) Q_L.$$

Setting $w(t,\tau) = w$, $\pi g(t) = g$, we rewrite the above inclusion, taking into account the symmetry of the ellipsoid $(Q_L = -Q_L)$:

$$wv - \alpha g \in (a|w| + \alpha l) Q_L. \tag{21}$$

The value of α in (21) is maximal in the case, when vector from the left part of inclusion lies on the boundary of the ellipsoid $(a|w| + \alpha l) Q_L$. To put it otherwise,

$$(wv - \alpha g, F_L(wv - \alpha g)) = (a|w| + \alpha l)^2.$$

As a result, we obtain the quadratic equation for α:

$$\left(\|g\|^2 - l^2\right)\alpha^2 - 2\alpha[w(g,F_Lv) - |w|al] + w^2\left[(v,F_Lv) - a^2\right] = 0.$$

Setting $\overline{w} = w(g,F_Lv) - |w|al$, we have

$$\alpha(\cdot) = \frac{\overline{w} + \sqrt{\overline{w}^2 + [(g,F_Lg) - l^2]w^2[a^2 - (v,F_Lv)]}}{(g,F_Lg) - l^2},$$

whence follows that

$$\min_{v \in Q_L} \alpha(\cdot) = \frac{|w|(a-1)}{\sqrt{(g,F_Lg)} - l}. \tag{22}$$

Here the minimum is furnished by the element $v = -\operatorname{sign} w \frac{g}{\sqrt{(g,F_Lg)}}$.

Now, taking into account the notations, made above, one can deduce from (22) formula (20). ■

In the case of spherical parameters of the conflict-controlled process, the following statements are true.

Corollary 3. *Under the conditions of Lemma 7, if* $F = E$ *and* $\gamma(t,\tau) = 0$, *then*

$$\Phi(t) = \int_0^t \frac{a-1}{\|\Omega^{-1}(t,\tau)g(t)\|} d\tau.$$

Corollary 4. *Under the conditions of Lemma 8, if Q_L is a sphere in L $(F_L = E)$ and $\gamma(t,\tau) \equiv 0$, then*

$$\Phi(t) = \int_0^t \frac{|w(t,\tau)|(a-1)}{\|\pi g(t)\| - l} d\tau.$$

5 Comparison with the First Direct Method of L. S. Pontryagin

The suggested method gives sufficient conditions for termination of the approach game (1), (2) in finite time $T(g(\cdot), \gamma(\cdot,\cdot))$, where $\gamma(t,\tau)$, $t \geq \tau \geq 0$ is some fixed selection. Under the same assumptions, the first direct method of L. S. Pontryagin [11, 39, 48] ensures that the game can be terminated at the instant of time

$$P(g(\cdot)) = \min\left\{ t \geq 0 : \pi g(t) \in M - \int_0^t W(t,\tau)\, d\tau \right\}. \tag{23}$$

In so doing, the first player applies the counter-control

$$u(t) = u(g(\cdot), v(t)). \tag{24}$$

Let us compare the times $T(g(\cdot), \gamma(\cdot,\cdot))$ and $P(g(\cdot))$. It is easy to see that

$$\inf_{\gamma(\cdot,\cdot)} T(g(\cdot), \gamma(\cdot,\cdot)) \leq P(g(\cdot)). \tag{25}$$

The last inequality follows from the following statement.

Proposition 1. *Let Pontryagin's condition be satisfied for the conflict-controlled process (1), (2). Then in order that the inclusion*

$$\pi g(t) \in M - \int_0^t W(t,\tau)\, d\tau \tag{26}$$

holds, it is necessary and sufficient that there exists a selection $\gamma(t,\tau)$ of set-valued mapping $W(t,\tau)$, which is summable in τ, $0 \leq \tau \leq t$, and such that

$$\xi(t, g(t), \gamma(t,\cdot)) \in M. \tag{27}$$

Proof of this statement immediately follows from the definitions of function $\xi(t, g(t), \gamma(t,\cdot))$ and the integral of set-valued mapping.

Thus, inclusion (26) implies the inclusion (27) whence follows that the function $\alpha(t,\tau,v)$ turns into $+\infty$. This clearly demonstrates the fact that the case when the resolving function turns into infinity corresponds with the first direct method of L. S. Pontryagin.

On the other hand, of interest is the case of equality in (25), i.e., when the time of the game termination is unaffected by information on the prehistory of second player's control.

Using methods presented in [10, 11], one may obtain the following result.

Proposition 2. *Let for the conflict-controlled process (1), (2) Pontryagin's condition be satisfied, the mapping $W(t,\tau;v)$ be convex-valued, and the terminal set be an affine manifold ($M=\{m\}$ is a point). Then for any $g(\cdot)$*

$$\min_{\gamma(\cdot,\cdot)} T\left(g\left(\cdot\right),\gamma\left(\cdot,\cdot\right)\right)=P\left(g\left(\cdot\right)\right).$$

Examples of game problems, demonstrating that each of the two conditions of Proposition 2 is essential in the case of differential games, are given in [11]. In these examples, if one of mentioned conditions fails, relation (25) may turn into a strict inequality. Also, local conditions for the equality and the strict inequality in (25) in terms of cones are contained in [11].

6 Game Problems for Fractional Systems

In this section, we introduce in a standard way the classic notions of Riemann–Liouville fractional integral and fractional derivative. To them corresponds the equation with fractional derivative in which instead of standard Cauchy condition at the initial instant $t=0$, the fractional integral of appropriate fractional order is given. The reason is that, generally speaking, the solution of such equation has singularity at $t=0$ and therefore only generalized initial conditions have sense here. However, from the physical point of view, it is desirable to have a standard Cauchy problem for equations with fractional derivatives.

In [17], Dzhrbashyan and Nersesyan introduced an equation with fractional derivative, in which instead of Riemann–Liouville derivative its regularized value is used and a standard Cauchy condition stands for the initial condition. Later on the new notion of fractional derivative was called the Dzhrbashyan–Nersesyan–Caputo regularized derivative.

Let us define the fractional Riemann–Liouville integral of order β, $\beta\in(0,1)$, of a function $z\left(t\right)$, $t>0$, by the formula [53]

$$\left(I^{\beta}_{0+}z\right)(t)=\frac{1}{\Gamma\left(\beta\right)}\int\limits_0^t\frac{z\left(s\right)}{\left(t-s\right)^{1-\beta}}ds,$$

where $\Gamma(\beta)$ is the Euler γ-function. Then, the fractional Riemann–Liouville derivative of order β has the form

$$\left(D^{\beta}_{0+}z\right)(t)=\frac{d}{dt}\left(I^{1-\beta}_{0+}\right)(t),$$

and the regularized Dzhrbashyan–Nersesyan–Caputo fractional derivative of order β [17,25] has the form

$$\left(D^{(\beta)}_{0+}z\right)(t)=\left(D^{\beta}_{0+}z\right)(t)-\frac{t^{-\beta}}{\Gamma\left(1-\beta\right)}z\left(+0\right).$$

We will associate each of the fractional derivatives with appropriate game problem.

Thus, let in the first problem the evolution of a conflict-controlled process be described by the system of differential equations

$$D^{\beta}\hat{z} = A\hat{z} + \varphi\left(u, v\right), \quad \hat{z} \in R^n, \quad u \in U, \quad v \in V, \tag{28}$$

under the initial condition

$$I^{1-\beta}\hat{z}|_{t=0} = \hat{z}_0, \tag{29}$$

and in the second problem by the system

$$D^{(\beta)}z = Az + \varphi\left(u, v\right), \quad z \in R^n, \quad u \in U, \quad v \in V, \tag{30}$$

under the initial condition

$$z|_{t=0} = z_0. \tag{31}$$

In the notations of fractional derivatives in (28), (30), some symbols are omitted for the simplicity of exposition.

In addition to the dynamics of processes (28), (30), the terminal set of the form (2) is given. The goals of the players in each of the cases are the same as in the general problem statement. Note that in the problems (28), (29) and (30), (31), the first player (u) chooses his control in the form of measurable functions $u\left(t\right) = u\left(\hat{z}_0, v_t\left(\cdot\right)\right)$ and $u\left(t\right) = u\left(z_0, v_t\left(\cdot\right)\right)$, respectively, with values in the domain U.

Let us proceed to the deduction of integral representations for functions $\hat{z}\left(t\right)$ and $z\left(t\right)$. For this purpose, for any arbitrary positive number ρ and complex number μ, we define a generalized matrix function of Mittag–Leffler

$$E_{\rho}\left(B; \mu\right) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma\left(k\rho^{-1} + \mu\right)},$$

where B is an arbitrary square matrix of order n with complex-valued elements. Matrix function $E_{\rho}\left(B; \mu\right)$ is an integer function of argument B.

Theorem 2. *Under the players' controls chosen, the solution $\hat{z}\left(t\right)$ to the system (28), (29) is defined by the formula*

$$\begin{gathered}\hat{z}\left(t\right) = t^{\beta-1}E_{1/\beta}\left(At^{\beta}; \beta\right)\hat{z}_0 \\ + \int_0^t \left(t-\tau\right)^{\beta-1} E_{1/\beta}\left(A\left(t-\tau\right)^{\beta}; \beta\right)\varphi\left(u\left(\tau\right), v\left(\tau\right)\right)d\tau,\end{gathered} \tag{32}$$

and the solution $z\left(t\right)$ to the system (30), (31) by the formula

$$\begin{gathered}z\left(t\right) = E_{1/\beta}\left(At^{\beta}; 1\right)z_0 \\ + \int_0^t \left(t-\tau\right)^{\beta-1} E_{1/\beta}\left(A\left(t-\tau\right)^{\beta}; \beta\right)\varphi\left(u\left(\tau\right), v\left(\tau\right)\right)d\tau.\end{gathered} \tag{33}$$

Proof. Let us first note that the function $F(\tau) = \varphi(u(\tau), v(\tau))$, $\tau > 0$, is measurable and essentially bounded. This implies that the integrals in formulas (32), (33) converge absolutely. The proof consists of two parts. In the first one, we will prove that the first terms in formulas (32), (33) are solutions of the homogeneous equations, satisfying the initial conditions (29), (31), respectively. In the second part we will show that the second term in formulas (32), (33)

$$z_2(t) = \int_0^t (t-\tau)^{\beta-1} E_{1/\beta}\left(A(t-\tau)^\beta; \beta\right) F(\tau)\, d\tau \tag{34}$$

is a solution to nonhomogeneous equations (32), (33).

The fact that $z_2(t)$ satisfies the zero initial condition immediately follows from the boundedness of the functions $E_{1/\beta}\left(A(t-\tau)^\beta; \beta\right)$ and $F(\tau)$ and that $\beta > 0$.

Denoting

$$\hat{z}_1(t) = t^{\beta-1} E_{1/\beta}\left(At^\beta; \beta\right) \hat{z}_0$$

we proceed to calculations:

$$\begin{aligned}
\left(D^\beta \hat{z}_1\right)(t) &\equiv D^\beta\left[t^{\beta-1} E_{1/\beta}\left(At^\beta; \beta\right) \hat{z}_0\right] \\
&= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt}\left(\int_0^t (t-\tau)^{-\beta} \tau^{\beta-1} \sum_{k=0}^{\infty} \frac{A^k \tau^{\beta k}}{\Gamma(\beta(k+1))} d\tau\right) \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{\infty} \frac{A^k \hat{z}_0}{\Gamma(\beta k+\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} \tau^{\beta(k+1)-1} d\tau \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{\infty} \frac{A^k \hat{z}_0 B(1-\beta, \beta k+\beta)}{\Gamma(\beta k+\beta)} \frac{d}{dt} t^{\beta k}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\beta)} \sum_{k=1}^{\infty} \frac{A^k \Gamma(1-\beta) \Gamma(\beta k+\beta)}{\Gamma(\beta k+\beta) \Gamma(\beta k+1)} \beta k t^{\beta k-1} \hat{z}_0 \\
&= \sum_{k=1}^{\infty} \frac{\beta k A^k t^{\beta k-1}}{\Gamma(\beta k+1)} \hat{z}_0 = \sum_{k=1}^{\infty} \frac{A^k t^{\beta k-1}}{\Gamma(\beta k)} \hat{z}_0 \\
&\overset{k=k'+1}{=} A t^{\beta-1} \sum_{k'=0}^{\infty} \frac{A^{k'} t^{\beta k'}}{\Gamma(\beta k'+\beta)} \hat{z}_0 = A \hat{z}_1(t).
\end{aligned}$$

Here $B(z, w) = \int\limits_0^1 x^{z-1}(1-x)^{w-1} dx = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ is Euler β-function.

Let us now show that function $\hat{z}_1(t)$ satisfies the initial condition (29).

$$
\begin{aligned}
\left(I^{1-\beta}\hat{z}_1\right)(t) &= \frac{1}{\Gamma(1-\beta)}\int\limits_0^t \frac{\hat{z}_1(\tau)}{(t-\tau)^{\beta}}d\tau \\
&= \frac{1}{\Gamma(1-\beta)}\int\limits_0^t \frac{\tau^{\beta-1}}{(t-\tau)^{\beta}}\sum_{k=0}^{\infty}\frac{A^k\tau^{\beta k}}{\Gamma(\beta k+\beta)}\hat{z}_0 d\tau \\
&= \frac{1}{\Gamma(1-\beta)}\sum_{k=0}^{\infty}\frac{A^k\hat{z}_0}{\Gamma(\beta k+\beta)}\int\limits_0^t \tau^{\beta(k+1)-1}(t-\tau)^{-\beta}d\tau \\
&= \frac{1}{\Gamma(1-\beta)}\sum_{k=0}^{\infty}\frac{A^k t^{\beta k}}{\Gamma(\beta k+\beta)}\frac{\Gamma(\beta(k+1))\Gamma(1-\beta)}{\Gamma(\beta k+1)}\hat{z}_0 \\
&= \sum_{k=0}^{\infty}\frac{A^k t^{\beta k}}{\Gamma(\beta k+1)}\hat{z}_0 \overset{t\to 0}{\to} \hat{z}_0.
\end{aligned}
$$

Consider the function

$$
z_1(t) = E_{1/\beta}\left(At^{\beta};1\right)z_0 \equiv E_{1/\beta}\left(At^{\beta}\right)z_0,
$$

where $E_{1/\beta}\left(At^{\beta}\right)$ is the matrix function of Mittag–Leffler.

Then

$$
\begin{aligned}
\left(D^{(\beta)}z_1\right)(t) &= \frac{1}{\Gamma(1-\beta)}\left[\frac{d}{dt}\left(\int\limits_0^t (t-\tau)^{-\beta}\sum_{k=0}^{\infty}\frac{A^k\tau^{\beta k}}{\Gamma(\beta k+1)}d\tau\right) - t^{-\beta}\right]z_0 \\
&= \frac{1}{\Gamma(1-\beta)}\left[\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\beta k+1)}\frac{d}{dt}\int\limits_0^t (t-\tau)^{-\beta}\tau^{\beta k}d\tau - t^{-\beta}\right]z_0 \\
&= \frac{1}{\Gamma(1-\beta)}\left[\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\beta k+1)}\frac{d}{dt}B(1-\beta,\beta k+1)\,t^{1-\beta+\beta k} - t^{-\beta}\right]z_0 \\
&= \frac{1}{\Gamma(1-\beta)}\left[\sum_{k=0}^{\infty}A^k t^{\beta k-\beta}\frac{\Gamma(1-\beta)\Gamma(\beta k+1)}{\Gamma(\beta k+1)\Gamma(2+\beta k-\beta)}(1-\beta+\beta k) - t^{-\beta}\right]z_0 \\
&= \frac{1}{\Gamma(1-\beta)}\left[\Gamma(1-\beta)\sum_{k=0}^{\infty}\frac{A^k t^{\beta(k-1)}}{\Gamma(1+\beta(k-1))} - t^{-\beta}\right]z_0 \\
&= A\sum_{k=1}^{\infty}\frac{A^{k-1}t^{\beta(k-1)}}{\Gamma(1+\beta(k-1))}z_0 = Az_1(t).
\end{aligned}
$$

Moreover, $z_1(t)$ satisfies the initial condition (31) as

$$
\lim_{t\to 0} z_1(t) = \lim_{t\to 0}\sum_{k=0}^{\infty}\frac{A^k t^{\beta k}}{\Gamma(\beta k+1)}z_0 = z_0.
$$

Let us analyze function $z_2(t)$, defined by formula (34), and show that it satisfies equations (28), (30), under the zero initial conditions.

We have

$$\left(D^{\beta} z_2\right)(t) = \left(D^{(\beta)} z_2\right)(t) = \frac{1}{\Gamma(1-\beta)} \\ \times \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} \left(\int_0^{\tau} (\tau - s)^{\beta-1} E_{1/\beta}\left(A(\tau-s)^{\beta}; \beta\right) F(s)\, ds \right) d\tau. \tag{35}$$

Separately we will study the function

$$\psi(t) = \int_0^t (t-\tau)^{-\beta} \left(\int_0^{\tau} (\tau - s)^{\beta-1} \sum_{k=0}^{\infty} \frac{A^k (\tau-s)^{\beta k}}{\Gamma(k\beta+\beta)} F(s)\, ds \right) d\tau \\ = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\beta+\beta)} \int_0^t (t-\tau)^{-\beta} \left(\int_0^{\tau} (\tau - s)^{\beta(k+1)-1} F(s)\, ds \right) d\tau. \tag{36}$$

For this purpose, we consider the following integrals

$$\dot{I}_k = \int_0^t \int_0^{\tau} (t-\tau)^{-\beta} (\tau - s)^{\beta(k+1)-1} F(s)\, ds d\tau \\ = \iint_{(\Delta_t)} (t-\tau)^{-\beta} (\tau - s)^{\beta(k+1)-1} F(s)\, d\tau ds,$$

$$\Delta_t = \{(s, \tau) : 0 \le s \le \tau \le t\}.$$

The latter integral converges absolutely, which allows, by virtue of Fubini theorem, to change the order of integration using Dirichlet formula.

Then

$$\dot{I}_k = \int_0^t \left(\int_s^t (t-\tau)^{-\beta} (\tau - s)^{\beta(k+1)-1} d\tau \right) F(s)\, ds \\ = B(1-\beta, \beta k + \beta) \int_0^t (t-s)^{\beta k} F(s)\, ds \\ = \frac{\Gamma(1-\beta)\Gamma(k\beta+\beta)}{\Gamma(k\beta+1)} \int_0^t (t-s)^{\beta k} F(s)\, ds. \tag{37}$$

From equalities (36), (37) it follows that

$$\psi(t) = \Gamma(1-\beta) \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\beta+1)} \int_0^t (t-s)^{\beta k} F(s)\, ds.$$

Because the function $F(t)$ is measurable and bounded, then $\psi(t)$ has the derivative almost everywhere:

$$\begin{aligned} \frac{d\psi}{dt} &= \Gamma(1-\beta) \left\{ F(t) + \sum_{k=1}^{\infty} \frac{A^k \beta k}{\Gamma(\beta k+1)} \int_0^t (t-s)^{k\beta-1} F(s)\, ds \right\} \\ &= \Gamma(1-\beta) \left\{ F(t) + \int_0^t \sum_{k=1}^{\infty} \frac{A^k (t-s)^{k\beta-1}}{\Gamma(\beta k)} F(s)\, ds \right\} \\ &= \Gamma(1-\beta) \left\{ F(t) + A \int_0^t (t-s)^{\beta-1} E_{1/\beta}\left(A(t-s)^{\beta}; \beta\right) F(s)\, ds \right\}. \end{aligned} \tag{38}$$

Substituting (38) into (35), we obtain the equalities

$$D^{\beta} z_2 = D^{(\beta)} z_2 = A z_2 + \varphi(u, v). \qquad \blacksquare$$

7 Fractional Conflict-Controlled Processes with Integral Block of Control

Alongside the conflict-controlled processes (28), (29) and (30), (31), we will analyze the processes differing from them by the block of controls appearing in the integral form. To be specific, in the case of Riemann–Liouville derivative, we will study the process

$$\begin{gathered} D^{\beta} \hat{y} = A\hat{y} + \int_0^t (t-\tau)^{\gamma-1} \varphi(u(\tau), v(\tau))\, d\tau, \\ 0 < \gamma < 1, \quad 0 < \beta < 1, \end{gathered} \tag{39}$$

under the initial condition

$$I^{1-\beta} \hat{y}|_{t=0} = \hat{y}_0 \tag{40}$$

and in the case of regularized Dzhrbashyan–Nersesyan–Caputo derivative the process

$$D^{(\beta)} y = Ay + \int_0^t (t-\tau)^{\gamma-1} \varphi(u(\tau), v(\tau))\, d\tau \tag{41}$$

under the initial condition

$$y|_{t=0} = y_0. \tag{42}$$

Theorem 3. *Under controls of the players chosen, the solution $\hat{y}(t)$ to the problem (39), (40) is given by the formula*

$$\hat{y}(t) = t^{\beta-1} E_{1/\beta}\left(At^{\beta};\beta\right)\hat{y}_0$$
$$+\int_0^t \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\gamma+\beta\right)\varphi(u(\tau),v(\tau))\,d\tau \tag{43}$$

and the solution $y(t)$ to the problem (41), (42) by the formula

$$y(t) = E_{1/\beta}\left(At^{\beta};1\right)y_0$$
$$+\int_0^t \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\gamma+\beta\right)\varphi(u(\tau),v(\tau))\,d\tau. \tag{44}$$

Proof. Taking into account the reasoning, presented in the proof of Theorem 2, it suffices to show that the function

$$y_2(t) = \int_0^t \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\gamma+\beta\right)\varphi(u(\tau),v(\tau))\,d\tau$$

is a solution of equations (43), (44) under the zero initial condition.

After application of formulas (32), (33) to systems (39), (41), under the zero initial conditions we have

$$\hat{y}(t) \equiv y(t) = \int_0^t (t-\tau)^{\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\beta\right)\int_0^{\tau}(\tau-s)^{\gamma-1}F(s)\,ds d\tau$$
$$= \int_0^t\int_s^t (t-\tau)^{\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\beta\right)(\tau-s)^{\gamma-1}\,d\tau F(s)\,ds.$$

Let us calculate the integral

$$I(t-s) = \int_s^t (t-\tau)^{\beta-1} E_{1/\beta}\left(A(t-\tau)^{\beta};\beta\right)(\tau-s)^{\gamma-1}\,d\tau$$
$$\overset{\tau-s=\hat{\tau}}{=} \int_0^{t-s} (t-s-\hat{\tau})^{\beta-1} E_{1/\beta}\left(A(t-s-\hat{\tau})^{\beta};\beta\right)\hat{\tau}^{\gamma-1}d\hat{\tau}$$
$$= \int_0^{t-s} \hat{\tau}^{\beta-1}E_{1/\beta}\left(A\hat{\tau}^{\beta};\beta\right)(t-s-\hat{\tau})^{\gamma-1}\,d\hat{\tau}.$$

In view of the matrix analog to formula (1.16) ([18], p. 120), we eventually obtain

$$I\left(t-s\right)=\Gamma\left(\gamma\right)\left(t-s\right)^{\gamma+\beta-1}E_{1/\beta}\left(A\left(t-s\right)^{\beta};\gamma+\beta\right),$$

whence follows that

$$\hat{y}\left(t\right)=y\left(t\right)=\Gamma\left(\gamma\right)\int\limits_{0}^{t}\left(t-s\right)^{\gamma+\beta-1}E_{1/\beta}\left(A\left(t-s\right)^{\beta};\gamma+\beta\right)F\left(s\right)ds. \quad \blacksquare$$

Remark 2. If $\gamma+\beta\geq 1$, then the solutions (43), (44) appear as absolutely continuous functions [53], having bounded derivatives almost everywhere.

Remark 3. Integrals in equations (39), (41) may have arbitrary τ-summable kernels.

Thus, for the game problems with the fractional derivatives of Riemann–Liouville and Dzhrbashyan–Nersesyan–Caputo of the types (28)–(29), (30)–(31), (39)–(40), (41)–(42), the solutions can be presented by formulas (32), (33), (43), (44), which are specific cases of representation (1).

The above-outlined general method can be applied for solution to each of the mentioned game problems.

8 Specific Case of Simple Matrix, the Origin as a Terminal Set and Spherical Control Domains

For illustration of the method, we now analyze various specific cases in which solution can be obtained in analytic form.

In a sequel, for the brevity of exposition and the unification of notions, we will distinguish the four above-outlined problems by assigning to their parameters the values of indices i, j: $i=1,2$, $j=1,2$. Then a trajectory $z_{11}\left(t\right)$ corresponds with the process with Riemann–Liouville derivative and conventional block of control (28) and $z_{12}\left(t\right)$ to that with the integral block of control (39). In the turn, a trajectory $z_{21}\left(t\right)$ corresponds with the process with the regularized Dzhrbashyan–Nersesyan–Caputo derivative and the block of control in conventional form, and $z_{22}\left(t\right)$ to that with the integral block of control (41).

Thus, we have the four processes

$$z_{ij}\left(t\right)=g_{ij}\left(t\right)+\int\limits_{0}^{t}\Omega_{ij}\left(t,\tau\right)\varphi\left(u\left(\tau\right),v\left(\tau\right)\right)d\tau,\ i=1,2,\ j=1,2, \tag{45}$$

where

$$\begin{aligned}
&g_{11}(t) = G_{11}(t)\,\hat{z}_0,\ \ G_{11}(t) = t^{\beta-1}E_{1/\beta}\left(At^{\beta};\beta\right),\\
&\Omega_{11}(t,\tau) = (t-\tau)^{\beta-1}E_{1/\beta}\left(A(t-\tau)^{\beta};\beta\right),\\
&g_{12}(t) = G_{12}(t)\,\hat{y}_0,\ \ G_{12}(t) = t^{\beta-1}E_{1/\beta}\left(At^{\beta};\beta\right),\\
&\Omega_{12}(t,\tau) = \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1}E_{1/\beta}\left(A(t-\tau)^{\beta};\gamma+\beta\right),\\
&g_{21}(t) = G_{21}(t)\,z_0,\ \ G_{21}(t) = E_{1/\beta}\left(At^{\beta};1\right),\\
&\Omega_{21}(t,\tau) = (t-\tau)^{\beta-1}E_{1/\beta}\left(A(t-\tau)^{\beta};\beta\right),\\
&g_{22}(t) = G_{22}(t)\,y_0,\ \ G_{22}(t) = E_{1/\beta}\left(At^{\beta};1\right),\\
&\Omega_{22}(t,\tau) = \Gamma(\gamma)(t-\tau)^{\gamma+\beta-1}E_{1/\beta}\left(A(t-\tau)^{\beta};\gamma+\beta\right).
\end{aligned} \tag{46}$$

Let

$$A = \lambda E,\ \ \varphi(u,v) = u - v,\ \ M^* = \{0\},\ \ U = aS,\ \ a > 1,\ \ V = S, \tag{47}$$

where λ is a number and S is the unit ball centered at the origin. Then $L = R^n$ and the orthoprojector π appears as the operator of identical transformation, defined by the unit matrix. All the matrix functions $G_{ij}(t)$ and $\Omega_{ij}(t,\tau)$ have the forms

$$G_{ij}(t) = \hat{g}_{ij}(t)\,E, \quad \Omega_{ij}(t,\tau) = w_{ij}(t,\tau)\,E, \quad i = 1,2, \quad j = 1,2,$$

where $\hat{g}_{ij}(t)$ and $w_{ij}(t,\tau)$ are scalar functions. In addition, note that for matrix $B = \lambda E$, the following equality is true

$$E_{\rho}(B;\mu) = E_{\rho}(\lambda;\mu)\,E,$$

where $E_{\rho}(\lambda;\mu)$ is the generalized scalar function of Mittag–Leffler [18, 53].

Then

$$\begin{aligned}
W_{ij}(t,\tau,v) &= w_{ij}(t,\tau)(aS - v),\\
W_{ij}(t,\tau) &= |w_{ij}(t,\tau)|\,(a-1)\,S.
\end{aligned}$$

Consequently, Pontryagin condition holds if $a \geq 1$.

Set $\gamma_{ij}(t,\tau) \equiv 0$. Then

$$\xi_{ij}(t, g_{ij}(t), \gamma_{ij}(t,\tau)) = g_{ij}(t) = \hat{g}_{ij}(t)\,z_{ij}^0, \quad z_{ij}^0 \neq 0,$$

and

$$\alpha_{ij}(t,\tau,v) = \sup\left\{\alpha \geq 0 :\ \alpha\hat{g}_{ij}(t)\,z_{ij}^0 \in w_{ij}(t,\tau)(aS - v)\right\}$$

is the greatest root of the square equation for α:

$$\left\| w_{ij}\left(t,\tau\right)v-\alpha\hat{g}_{ij}\left(t\right)z_{ij}^{0}\right\| =\left|w_{ij}\left(t,\tau\right)\right|a.$$

Therefore

$$\alpha_{ij}\left(t,\tau,v\right)=\frac{\left(v_{0},q\right)+\sqrt{\left(v_{0},q\right)^{2}+\left\|q\right\|^{2}\left(a_{0}^{2}-\left\|v_{0}\right\|^{2}\right)}}{\left\|q\right\|^{2}},\tag{48}$$

where $v_0=w_{ij}\left(t,\tau\right)v$, $q=\hat{g}_{ij}\left(t\right)z_{ij}^{0}$, $a_0=\left|w_{ij}\left(t,\tau\right)\right|a$.

It should be noted that $\hat{g}_{ij}(t)\neq 0$ up to the instant of the game termination. The game can be terminated at the moments when this function vanishes, with the help of the first direct method.

It is evident that

$$\min_{\|v\|\le 1}\alpha_{ij}\left(t,\tau,v\right)=\frac{\left|w_{ij}\left(t,\tau\right)\right|\left(a-1\right)}{\left\|\hat{g}_{ij}\left(t\right)z_{ij}^{0}\right\|},$$

where the minimum is furnished by the element

$$v_{ij}(t,\tau)=-\mathrm{sign}\left\{\hat{g}_{ij}\left(t\right)w_{ij}(t,\tau)\right\}\frac{z_{ij}^{0}}{\left\|z_{ij}^{0}\right\|}.$$

Then the time of the game termination appears as the least root of the equation

$$\int\limits_{0}^{t}\frac{\left(a-1\right)\left|w_{ij}\left(t,\tau\right)\right|}{\left|\hat{g}_{ij}\left(t\right)\right|\left\|z_{ij}^{0}\right\|}d\tau=1,$$

as the functions $w_{ij}\left(t,\tau\right)$ are continuous in t.

Let us introduce the functions

$$\Phi_{ij}\left(t\right)=\int\limits_{0}^{t}\frac{\left|w_{ij}\left(t,\tau\right)\right|}{\left|\hat{g}_{ij}\left(t\right)\right|}d\tau.$$

Then the time of the game termination can be given by the formula

$$T_{ij}\left(z_{ij}^{0},0\right)=\min\left\{t\ge 0:\ \Phi_{ij}\left(t\right)\ge\frac{\left\|z_{ij}^{0}\right\|}{a-1}\right\},\tag{49}$$

where the functions $\Phi_{ij}\left(t\right)$ have the forms

$$\Phi_{11}\left(t\right)=\int\limits_{0}^{t}\left|\tau^{\beta-1}E_{1/\beta}\left(\lambda\tau^{\beta};\beta\right)\right|d\tau/\left|t^{\beta-1}E_{1/\beta}\left(\lambda t^{\beta};\beta\right)\right|,$$

$$\Phi_{12}\left(t\right)=\Gamma\left(\gamma\right)\int\limits_{0}^{t}\left|\tau^{\gamma+\beta-1}E_{1/\beta}\left(\lambda\tau^{\beta};\beta+\gamma\right)\right|d\tau/\left|t^{\beta-1}E_{1/\beta}\left(\lambda t^{\beta};\beta\right)\right|,$$

$$\begin{aligned} \Phi_{21}(t) &= \int_0^t \left|\tau^{\beta-1} E_{1/\beta}\left(\lambda\tau^\beta;\beta\right)\right| d\tau / E_{1/\beta}\left(\lambda t^\beta;1\right), \\ \Phi_{22}(t) &= \Gamma(\gamma) \int_0^t \left|\tau^{\gamma+\beta-1} E_{1/\beta}\left(\lambda\tau^\beta;\beta+\gamma\right)\right| d\tau / E_{1/\beta}\left(\lambda t^\beta;1\right). \end{aligned} \tag{50}$$

To determine whether (or not) the time of the game termination $T_{ij}\left(z_{ij}^0, 0\right)$ is finite, an asymptotic representation of the generalized scalar function of Mittag–Leffler plays a key role. We take interest in specification of formulas (2.23), (2.24) from [18], p. 134, giving such representation for function $E_\rho(x;\mu)$ for real x, $\rho > \frac{1}{2}$ and arbitrary μ.

From these formulas, it follows that for positive x

$$E_\rho(x;\mu) = \rho x^{\rho(1-\mu)} e^{x^\rho} - \sum_{k=1}^{p} \frac{x^{-k}}{\Gamma\left(\mu - k\rho^{-1}\right)} + O\left(|x|^{-1-p}\right) \tag{51}$$

and for negative x

$$E_\rho(x;\mu) = -\sum_{k=1}^{p} \frac{x^{-k}}{\Gamma\left(\mu - k\rho^{-1}\right)} + O\left(|x|^{-1-p}\right). \tag{52}$$

As seen from asymptotic representations (51), (52), in our example it is reasonable to analyze two cases: $\lambda > 0$ and $\lambda < 0$.

Let $\lambda > 0$. Then the generalized functions of Mittag–Leffler, appearing in the formulas for $\Phi_{ij}(t)$, are positive. From this and from formula (1.15) ([18], p. 120) here having the form

$$\int_0^x E_\rho\left(\lambda x^{1/\beta};\mu\right) \tau^{\mu-1} d\tau = x^\mu E_\rho\left(\lambda x^{1/\beta};\mu+1\right), \ (\mu > 0), \ \lambda \in R, \tag{53}$$

we infer formulas for functions $\Phi_{ij}(t)$

$$\begin{aligned} &\Phi_{11}(t) = t^\beta E_{1/\beta}\left(\lambda t^\beta;\beta+1\right) / t^{\beta-1} E_{1/\beta}\left(\lambda t^\beta;\beta\right), \\ &\Phi_{12}(t) = \Gamma(\gamma)\, t^{\gamma+\beta} E_{1/\beta}\left(\lambda t^\beta;\beta+\gamma+1\right) / t^{\beta-1} E_{1/\beta}\left(\lambda t^\beta;\beta\right), \\ &\Phi_{21}(t) = t^\beta E_{1/\beta}\left(\lambda t^\beta;\beta+1\right) / E_{1/\beta}\left(\lambda t^\beta;1\right), \\ &\Phi_{22}(t) = \Gamma(\gamma)\, t^{\gamma+\beta} E_{1/\beta}\left(\lambda t^\beta;\beta+\gamma+1\right) / E_{1/\beta}\left(\lambda t^\beta;1\right). \end{aligned} \tag{54}$$

Set $\rho = \frac{1}{\beta}$, $x = \lambda t^\beta$ in formula (48). It should be noted that as $\beta \in (0,1)$, then $\rho \in (1,\infty)$ and therefore $\rho > \frac{1}{2}$. From this it follows the asymptotic representation

$$\begin{aligned} E_{1/\beta}\left(\lambda t^\beta;\mu\right) &= \frac{1}{\beta}\left(\lambda t^\beta\right)^{\frac{1}{\beta}(1-\mu)} e^{\left(\lambda t^\beta\right)^{1/\beta}} - \sum_{k=1}^{p} \frac{\left(\lambda t^\beta\right)^{-k}}{\Gamma(\mu - k\beta)} \\ &+ O\left(\left(t^\beta\right)^{-1-p}\right) = \frac{1}{\beta} \lambda^{\frac{1}{\beta}(1-\mu)} t^{1-\mu} e^{\lambda^{1/\beta} t} + \cdots. \end{aligned} \tag{55}$$

Using this representation, the following relations can be deduced

$$t^{\beta}E_{1/\beta}\left(\lambda t^{\beta};\beta+1\right)=\frac{1}{\beta}\lambda^{\frac{1}{\beta}(1-(\beta+1))}t^{\beta}t^{-\beta}e^{\lambda^{1/\beta}t}+\cdots=\frac{1}{\beta}\lambda^{-1}e^{\lambda^{1/\beta}t}+\cdots$$

$$t^{\beta-1}E_{1/\beta}\left(\lambda t^{\beta};\beta\right)=\frac{1}{\beta}t^{\beta-1}\lambda^{\frac{1}{\beta}(1-\beta)}t^{1-\beta}e^{\lambda^{1/\beta}t}+\cdots=\frac{1}{\beta}\lambda^{\frac{1}{\beta}-1}e^{\lambda^{1/\beta}t}+\cdots$$

$$\begin{aligned}&\Gamma\left(\gamma\right)t^{\gamma+\beta}E_{1/\beta}\left(\lambda t^{\beta};\beta+\gamma+1\right)\\&=\Gamma\left(\gamma\right)\frac{1}{\beta}t^{\gamma+\beta}\lambda^{\frac{1}{\beta}(1-(\gamma+\beta+1))}t^{1-(\gamma+\beta+1)}e^{\lambda^{1/\beta}t}+\cdots\\&=\frac{\Gamma\left(\gamma\right)}{\beta}\lambda^{-\frac{\gamma+\beta}{\beta}}e^{\lambda^{1/\beta}t}+\cdots\end{aligned}\tag{56}$$

$$E_{1/\beta}\left(\lambda t^{\beta};1\right)=\frac{1}{\beta}\lambda^{\frac{1}{\beta}(1-1)}t^{(1-1)}e^{\lambda^{1/\beta}t}+\cdots=\frac{1}{\beta}e^{\lambda^{1/\beta}t}+\cdots.$$

From formulas (54) and asymptotic representations (56), we obtain the following equalities

$$\lim_{t\to\infty}\Phi_{11}\left(t\right)=\frac{\frac{1}{\beta}\lambda^{-1}}{\frac{1}{\beta}\lambda^{\frac{1}{\beta}-1}}=\lambda^{-\frac{1}{\beta}},$$

$$\lim_{t\to\infty}\Phi_{12}\left(t\right)=\frac{\frac{\Gamma(\gamma)}{\beta}\lambda^{-\frac{\gamma+\beta}{\beta}}}{\frac{1}{\beta}\lambda^{\frac{1}{\beta}-1}}=\Gamma\left(\gamma\right)\lambda^{-\frac{\gamma+1}{\beta}},$$

$$\lim_{t\to\infty}\Phi_{21}\left(t\right)=\frac{\frac{1}{\beta}\lambda^{-1}}{\frac{1}{\beta}}=\lambda^{-1},$$

$$\lim_{t\to\infty}\Phi_{22}\left(t\right)=\frac{\frac{\Gamma(\gamma)}{\beta}\lambda^{-\frac{\gamma+\beta}{\beta}}}{\frac{1}{\beta}}=\Gamma\left(\gamma\right)\lambda^{-\frac{\gamma+\beta}{\beta}}.$$

Thus, when $\lambda>0$, the times under study are finite if the inequalities hold, respectively:

$$T_{11}\left(z_{11}^{0},0\right)\ \text{ if } \lambda^{-\frac{1}{\beta}}>\frac{\left\|z_{11}^{0}\right\|}{a-1},$$

$$T_{12}\left(z_{12}^{0},0\right)\ \text{ if } \Gamma\left(\gamma\right)\lambda^{-\frac{\gamma+1}{\beta}}>\frac{\left\|z_{12}^{0}\right\|}{a-1},$$

$$T_{21}\left(z_{21}^{0},0\right)\ \text{ if } \lambda^{-1}>\frac{\left\|z_{21}^{0}\right\|}{a-1},$$

$$T_{22}\left(z_{22}^{0},0\right)\ \text{ if } \Gamma\left(\gamma\right)\lambda^{-\frac{\gamma+\beta}{\beta}}>\frac{\left\|z_{22}^{0}\right\|}{a-1}.$$

Let us consider the case when $\lambda<0$. Set in formula (52) $\rho=\frac{1}{\beta},\ x=\lambda t^{\beta}$. Then

$$E_{1/\beta}\left(\lambda t^{\beta};\mu\right)=-\sum_{k=1}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(\mu-k\beta\right)}+O\left(t^{-(1+p)\beta}\right).$$

Using this asymptotic representation we obtain

$$t^{\beta}E_{1/\beta}\left(\lambda t^{\beta};\beta+1\right)=t^{\beta}\left[-\sum_{k=1}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(\beta+1-k\beta\right)}+O\left(t^{-\beta(1+p)}\right)\right]$$

$$=t^{\beta}\left[-\frac{\lambda^{-1}t^{-\beta}}{\Gamma\left(1\right)}-\frac{\lambda^{-2}t^{-2\beta}}{\Gamma\left(1-\beta\right)}-\cdots\right]=-\lambda^{-1}-\frac{\lambda^{-2}t^{-\beta}}{\Gamma\left(1-\beta\right)}-\cdots,$$

$$t^{\beta-1}E_{1/\beta}\left(t\lambda^{\beta};\beta\right)=t^{\beta-1}\left[-\sum_{k=1}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(\beta-k\beta\right)}+\cdots\right]$$

$$=t^{\beta-1}\left[-\sum_{k=2}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(\beta-k\beta\right)}+\cdots\right]=t^{\beta-1}\left[-\frac{\lambda^{-2}t^{-2\beta}}{\Gamma\left(-\beta\right)}-\cdots\right]$$

$$=-\frac{\lambda^{-2}t^{-2\beta}}{\Gamma\left(-\beta\right)}-\cdots, \tag{57}$$

$$\Gamma\left(\gamma\right)t^{\gamma+\beta}E_{1/\beta}\left(t\lambda^{\beta};\gamma+\beta+1\right)$$

$$=\Gamma\left(\gamma\right)t^{\gamma+\beta}\left[-\sum_{k=1}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(\gamma+\beta+1-k\beta\right)}+\cdots\right]=-\Gamma\left(\gamma\right)\frac{\lambda^{-1}t^{-\gamma}}{\Gamma\left(\gamma+1\right)}-\cdots,$$

$$E_{1/\beta}\left(\lambda t^{\beta};1\right)=-\sum_{k=1}^{p}\frac{\lambda^{-k}t^{-k\beta}}{\Gamma\left(1-k\beta\right)}+O\left(t^{-\beta(1+p)}\right)=-\frac{\lambda^{-1}t^{-\beta}}{\Gamma\left(1-\beta\right)}-\cdots$$

Let us analyze an asymptotic behavior of functions $\Phi_{ij}\left(t\right)$ given by formulas (54) in the case when $\lambda<0$. Note that functions (57) are not of necessity positive. However, using the inequality

$$\left|\int_{0}^{t}f\left(\tau\right)d\tau\right|\leq\int_{0}^{t}\left|f\left(\tau\right)\right|d\tau$$

for arbitrary summable function $f\left(\tau\right)$, asymptotic representations (57) and formulas (54), one can easily infer that

$$\Phi_{ij}\left(t\right)\underset{t\to\infty}{\to}\infty,\quad\forall i,j=1,2.$$

Thus, the times $T_{ij}\left(z_{ij}^{0},0\right)$ given by formula (49) are finite for any $z_{ij}^{0},\ i,j=1,2$. This means that in the case when $\lambda<0$, the process under study is completely conflict-controllable [11] in each of the problems (28)–(29), (30)–(31), (39)–(40), and (41)–(42).

Let $\lambda=0$. Then, taking into account formulas (54) for the functions $\Phi_{ij}(t)$, $i,j=1,2$, together with expression (49), one can calculate the precise values of the termination times for the games under study, namely:

$$T_{11}\left(z_{11}^{0},0\right)=\beta\ \frac{\left\|z_{11}^{0}\right\|}{a-1},$$

$$T_{21}\left(z_{21}^{0},0\right)=\left[\Gamma\left(\beta+1\right)\ \frac{\left\|z_{21}^{0}\right\|}{a-1}\right]^{\frac{1}{\beta}},$$

$$T_{12}\left(z_{12}^{0},0\right)=\left[\frac{\beta+\gamma}{B(\gamma+\beta)}\ \frac{\left\|z_{12}^{0}\right\|}{a-1}\right]^{\frac{1}{\gamma+1}}, \tag{58}$$

$$T_{22}\left(z_{22}^{0},0\right)=\left[\frac{\Gamma\left(\beta+\gamma+1\right)}{\Gamma(\gamma)}\ \frac{\left\|z_{22}^{0}\right\|}{a-1}\right]^{\frac{1}{\gamma+\beta}}.$$

9 "Parallel Approach" and the Method of Resolving Functions

The rule of parallel approach (pursuit) for moving objects is well-known in engineering practice. It is geometric-descriptive in the case of simple motions in the plane and provides for the lines of sight being parallel in the course of pursuit (the line of sight is a straight line connecting current states of the players). This geometric phenomenon was rigorously justified with the help of the method of resolving functions [11]. In the case when by the meeting is meant an exact capture of the evader by the pursuer, this rule provides the optimal time of pursuit.

Below is given a formal definition of the "parallel approach" for a wider class of problems, including, in particular, the above-mentioned case of simple motions in the plane.

Definition 1. *In the game (1), (2), let $g(t) = A(t)z^0$, $A(0) = E$, where $A(t)$ is a square matrix, z_0 is the initial state of process (1), and let the terminal set be a linear subspace: $M^* = M_0$. We say that the parallel approach has place in the course of the game if the first player, employs a strategy, defining the control $u(t) = u\left(z_0, v_t(\cdot)\right)$, such that for any control of the second player, the projection of a trajectory of system (1) onto the subspace L, $L = M_0^{\perp}$, has the form*

$$\pi z(t) = \rho(t)\pi z_0,\ t \geq 0, \tag{59}$$

where $\rho(t)$ is a scalar function, vanishing at some finite instant of time.

Let us address the notion of "parallel approach," defined by condition (59), as applied to the problem treated in Section 8.

Preserving the notations for each of the four processes (45), (46), we will study separately the case of (47). Note that each of the indices i and j corresponds with some of the problems (28)–(29), (30)–(31) or (39)–(40), (41)–(42).

It can be shown that in the differential games with a simple matrix, spherical control domains, and a linear subspace as the terminal set, the method of resolving functions provides the "parallel approach."

From formula (4), with the account of assumptions (47), one can deduce an expression for the resolving function

$$\alpha_{ij}(t,\tau,v) = \sup\left\{\alpha \geq 0 : -\alpha \hat{g}_{ij}(t)\, z_{ij}^0 \in \omega_{ij}(t,\tau)(aS - v)\right\}.$$

Set

$$\alpha(z,v) = \frac{(z,v) + \sqrt{(z,v)^2 + \|z\|^2\left(a^2 - \|v\|^2\right)}}{\|z\|^2}, \|z\| \neq 0.$$

Then

$$\alpha_{ij}(t,\tau,v) = \frac{\omega_{ij}(t,\tau)}{\hat{g}_{ij}(t)}\alpha\left(z_{ij}^0, v\right). \tag{60}$$

If $\hat{g}_{ij}(t) = 0$ for some $t > 0$, then function $\alpha_{ij}(t,\tau,v)$ turns into infinity, which, as was mentioned in Section 5, corresponds with the first direct method of L.S. Pontryagin.

From the proof of Theorem 1 for the case (47), one can infer an expression for the set-valued mapping $U_{ij}(\tau,v)$, defined by formula (7), which determines a strategy of the pursuer

$$U_{ij}(\tau,v) = \left\{u : u \in aS, -\alpha_{ij}^*(T_{ij},\tau,v)\,\hat{g}_{ij}(T_{ij})\, z_{ij}^0 \in \omega_{ij}(T_{ij},\tau)(u - v)\right\}, \tag{61}$$

where

$$T_{ij} = T_{ij}\left(z_{ij}^0, 0\right), \alpha_{ij}^*(t,\tau,v) = \begin{cases} \alpha_{ij}(t,\tau,v), \tau \in [0, t_{ij}^*), \\ 0, \qquad \tau \in \left[t_{ij}^*, t\right]. \end{cases} \tag{62}$$

In each of the cases, the moment of switching t_{ij}^* can be found from the equation

$$1 - \int_0^{t_{ij}^*} \alpha_{ij}(T_{ij},\tau,v(\tau))\, d\tau = 0. \tag{63}$$

It follows from formulas (60), (61) that the set-valued mapping $U_{ij}(\tau,v)$ consists of a single element $u_{ij}(\tau,v)$ of the form

$$u_{ij}(\tau,v) = v - \alpha^*\left(z_{ij}^0, v\right) z_{ij}^0,$$

where

$$\alpha^*\left(z_{ij}^0, v(\tau)\right) = \begin{cases} \alpha\left(z_{ij}^0, v(\tau)\right), \tau \in [0, t_{ij}^*), \\ 0, \qquad \tau \in \left[t_{ij}^*, T_{ij}\right]. \end{cases}$$

Substituting $u(\tau) = u_{ij}(\tau, v(\tau))$ into formula (45), in view of conditions (47) and the fact that π is an operator of identical transformation, we obtain

$$z_{ij}(t) = \hat{g}_{ij}(t)\, z_{ij}^0 + \int_0^t \omega_{ij}(t,\tau)(u(\tau) - v(\tau))\, d\tau$$

$$= \left[\hat{g}_{ij}(t) - \int_0^t \omega_{ij}(t,\tau)\,\alpha^*\left(z_{ij}^0, v(\tau)\right) d\tau\right] z_{ij}^0 = \rho_{ij}(t) z_{ij}^0.$$

From formulas (60)–(63), it follows that at $t = T_{ij}$, $\rho_{ij}(t) = 0$.

Thus, when applied to the problems (45)–(47), the method of resolving functions realizes the "parallel approach" as relation (59) holds true in the course of approach.

Let us dwell upon on the case of simple motions, i.e., when $\lambda = 0$ in condition (47).

Then, if the conflict-controlled process is described by a system of equations with Riemann–Liouville fractional derivatives (28) under the initial condition in the form of the fractional integral (29), then, by virtue of formula (60), the resolving function has the forma

$$\alpha_{11}(t,\tau,v) = \sup\left\{\alpha \geq 0 : -\alpha\frac{t^{\beta-1}}{\Gamma(\beta)}\hat{z}_0 \in \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}(aS - v)\right\}$$

$$= \frac{(t-\tau)^{\beta-1}}{t^{\beta-1}}\alpha\left(\hat{z}_0, v\right)$$

The bringing of a trajectory of the system $D^\beta \hat{z} = u - v$ from the initial state $\hat{z}_0$ into the origin is realized with the help of control

$$u(\tau) = v(\tau) - \alpha^*\left(\hat{z}_0, v(\tau)\right)\hat{z}_0,$$

where

$$\alpha^*\left(z^0, v(\tau)\right) = \begin{cases} \alpha\left(\hat{z}_0, v(\tau)\right), \tau \in [0, t^*_{11}), \\ 0, \qquad\qquad \tau \in \left[t^*_{11}, \beta\frac{\|\hat{z}_0\|}{a-1}\right], \end{cases}$$

and the instant t^*_{11} can be found from the equation

$$1 - \int\limits_0^{t^*_{11}} \alpha_{11}\left(T_{11}, \tau, v(\tau)\right) d\tau = 0, \text{ where } T_{11} = \beta\frac{\|\hat{z}_0\|}{a-1}. \tag{64}$$

The instant $\beta\frac{\|\hat{z}_0\|}{a-1}$ is just the instant of time when a trajectory hits the origin. In so doing, presentation (59), where

$$\rho_{11}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}\left[1 - \int\limits_0^t \frac{(t-\tau)^{\beta-1}}{t^{\beta-1}}\alpha^*\left(\hat{z}_0, v(\tau)\right) d\tau\right],$$

is true.

From the above formulas, it follows that at $t = \beta\frac{\|\hat{z}_0\|}{a-1}$ $\rho_{11}(t) = 0$.

Under the same conditions ($A = 0$, $\varphi(u,v) = u - v$, $U = aS$, $V = S$, $M^* = \{0\}$), in the case when the process under consideration is described by the equations with fractional derivatives of Riemann–Liouville having the integral block of control, and the fractional integral as the initial condition, the resolving function takes the form

$$\alpha_{12}(t,\tau,v) = \sup\left\{\alpha \geq 0 : -\alpha\frac{t^{\beta-1}}{\Gamma(\beta)}\hat{y}_0 \in \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)}(t-\tau)^{\gamma+\beta-1}(aS-v)\right\}$$
$$= B(\gamma+\beta)\frac{(t-\tau)^{\gamma+\beta-1}}{t^{\beta-1}}\alpha(\hat{y}_0,v).$$

The bringing of a trajectory of the system

$$D^{\beta}\hat{y} = \int_0^t (t-\tau)^{\gamma-1}(u(\tau)-v(\tau))\,d\tau, I^{1-\beta}\hat{y}|_{t=0} = \hat{y}_0,$$

into the origin at the instant $T_{12} = \left[\frac{\beta+\gamma}{B(\gamma+\beta)}\frac{\|\hat{y}_0\|}{a-1}\right]^{1/\gamma+1}$ is provided by the control

$$u(\tau) = v(\tau) - \alpha^*(\hat{y}_0, v(\tau))\,\hat{y}_0,$$

where

$$\alpha^*(\hat{y}_0,v(\tau)) = \begin{cases}\alpha(\hat{y}_0,v(\tau)), \tau \in [0,t_{12}^*),\\ 0, \qquad\qquad \tau \in [t_{12}^*,T_{12}],\end{cases}$$

and the instant t_{12}^* can be found from the equation

$$1 - \int_0^{t_{12}^*} \alpha_{12}(T_{12},\tau,v(\tau))\,d\tau = 0.$$

The function $\rho_{12}(t)$, appearing in definition of the "parallel approach," is given by the formula

$$\rho_{12}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}\left[1 - \int_0^t B(\gamma+\beta)\frac{(t-\tau)^{\gamma+\beta-1}}{t^{\beta-1}}\alpha^*(\hat{y}_0,v(\tau))\,d\tau\right]$$

and vanishes at $t = T_{12}$.

For simple motions with the regularized fractional derivatives of Dzhrbashyan–Nersesyan–Caputo (30) and Cauchy initial conditions, the resolving function is defined by the formula

$$\alpha_{21}(t,\tau,v) = \sup\left\{\alpha \geq 0 : -\alpha z_0 \in \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}(aS-v)\right\} = \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\alpha(z_0,v).$$

The control of the pursuer

$$u(\tau) = v(\tau) - \alpha^*(z_0,v(\tau))\,z_0,$$

where

$$\alpha^*(z_0,v(\tau)) = \begin{cases}\alpha(z_0,v(\tau)), \tau \in [0,t_{21}^*)\\ 0, \qquad\qquad \tau \in [t_{21}^*,T_{21}],\end{cases}$$

$$\int_0^{t_{21}^*} \alpha_{21}\left(T_{21}, \tau, v(\tau)\right) d\tau = 1, T_{21} = \left[\Gamma\left(\beta+1\right) \frac{\|z_0\|}{a-1}\right]^{1/\beta},$$

guarantees the bringing of a trajectory into the origin at $t = T_{21}$. In this case, the function $\rho_{21}(t)$ is given by the expression

$$\rho_{21}(t) = 1 - \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} \alpha^*\left(z_0, v(\tau)\right) d\tau,$$

and vanishes at $t = T_{21}$.

For the same problem but with the integral block of control (41)–(42), the following formula is true:

$$\alpha_{22}(t,\tau,v) = \sup\left\{\alpha \geq 0 : -\alpha y_0 \in \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)}\left(t-\tau\right)^{\gamma+\beta-1}\left(aS - v\right)\right\}$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)}\left(t-\tau\right)^{\gamma+\beta-1} \alpha\left(y_0, v\right).$$

The control of the pursuer

$$u(\tau) = v(\tau) - \alpha^*\left(y_0, v(\tau)\right) y_0,$$

where

$$\alpha^*\left(y_0, v(\tau)\right) = \begin{cases} \alpha\left(y_0, v(\tau)\right), \tau \in [0, t_{22}^*), \\ 0, \qquad \tau \in [t_{22}^*, T_{22}], \end{cases} \qquad T_{22} = \left[\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)} \frac{\|y_0\|}{a-1}\right]^{1/\gamma+\beta},$$

and

$$\int_0^{t_{22}^*} \alpha_{22}\left(T_{22}, \tau, v(\tau)\right) d\tau = 1,$$

brings a trajectory of the process into the origin at the time T_{22}.

In so doing, the function $\rho_{22}(t)$ has the form

$$\rho_{22}(t) = 1 - \int_0^t \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)} (t-\tau)^{\gamma+\beta-1} \alpha^*\left(y_0, v(\tau)\right) d\tau.$$

This function specifies the rate of the "parallel approach."

10 Conflict-Controlled Functional-Differential Processes

Conflict-controlled process (1), (2), encompasses a much wider scope of game problems than the games, described by the systems with fractional derivatives, presented in Sections 6–8. In particular, in view of Cauchy formula, the quasilinear differential games can be presented in the form (1).

Note that the differential games with the integral block of control were not studied earlier. However, even in case of ordinary differential game with simple motions, an origin as the terminal set, and spheres as the control domains, the time of transition into the origin (under an advantage of the first player) differs in essence from that for the game with the integral block of control.

One can verify this, performing simple calculations in accordance with the suggested scheme.

Now we will touch upon the game problems for the systems of second-order Volterra integral equations, the systems of integral-differential equations, and also the systems of difference-differential equations.

Let the dynamic of a process be described by the system of integral equations [19]

$$z(t) = f(t) + \int_0^t K(t,s)\, z(s)\, ds + \int_0^t Q(t,s)\, \varphi(u(s), v(s))\, ds, \qquad (65)$$

where the parameters of the process, namely functions $f(\cdot)$, $\varphi(\cdot)$ and matrix functions $K(\cdot)$, $Q(\cdot)$, enjoy rather "good" properties. Then a solution of system (65) can be presented by formula (1), where

$$g(t) = f(t) + \int_0^t R(t,s)\, f(s)\, ds,$$

$$\Omega(t,\tau) = Q(t,\tau) + \int_\tau^t R(t,s)\, Q(s,\tau)\, ds,$$

and a resolvent $R(t,s)$ has the form $R(t,s) = \sum\limits_{m=1}^{\infty} K_m(t,s)$, $K_m(t,s) = \int\limits_s^t K_1(t,\tau)\, K_{m-1}(\tau,s)\, d\tau$, $K_1(t,s) = K(t,s)$.

In the case when kernels $K(t,s)$, $Q(t,s)$ have the polar singularity, $g(t)$ and $\Omega(t,\tau)$ can be expressed in convenient form in terms of Mittag–Leffler function.

Let the dynamic system be described by the integral-differential equation

$$\dot{z}(t) = Az(t) + \int_0^t K(t,s)\, z(s)\, ds + \varphi(u,v), \quad z(0) = z_0, \qquad (66)$$

where A is a square matrix of order n, $u \in U$, $v \in V$, $U, V \in K(R^n)$, and both kernel $K(t,s)$ and function $\varphi(u,v)$ are continuous functions.

Then, under chosen controls, the solution to system (66) can be presented by formula (1), where

$$g(t) = \Omega(t,0)\, z_0,$$
$$\Omega(t,\tau) = e^{A(t-\tau)} + \int_\tau^t \hat{R}(t,s)\, e^{A(s-\tau)} ds$$

and

$$\hat{R}(t,s) = \sum_{m=1}^{\infty} \hat{K}_m(t,s)\,,\ \ \hat{K}_m(t,s) = \int_s^t \hat{K}_1(t,\tau)\, \hat{K}_{m-1}(\tau,s)\, d\tau,$$
$$\hat{K}_1(t,s) = e^{A(t-\tau)} K(t,s)\,.$$

It goes without saying that kernel $K(t,s)$ can enjoy polar singularity, and the control block $\varphi(u,v)$ in (66) can be presented in the integral form analogously to system (65).

If the dynamics of a process is given by the difference-differential equation

$$\dot{z}(t) = Az(t) + Bz(t-\tau) + \varphi(u,v) \tag{67}$$

or even by a more general equation (maybe with the integral control block), then, in view of Cauchy formula, using Dirichlet formula and Fubini theorem one can deduce representation (1).

Alongside the systems (28), (30) with fractional derivatives, the advanced method allows one to study, in the frames of suggested scheme, the processes of aftereffect type

$$D^\beta z = Az + \int_0^t K(t,\tau)\, z(\tau)\, d\tau + \int_0^t Q(t,\tau)\, \varphi(u(\tau), v(\tau))\, d\tau. \tag{68}$$

Performing calculations, one can obtain the solution of system (68) in the form (1).

Also, the method of resolving functions can be successfully applied for investigation of the game problems for systems (65)–(68). In so doing, sufficient conditions for solvability of the approach problem in a finite time can be obtained.

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Projected Dynamical Systems, Evolutionary Variational Inequalities, Applications, and a Computational Procedure

Monica–Gabriela Cojocaru,[1] Patrizia Daniele,[2] and Anna Nagurney[3]

[1] Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, Canada
mcojocar@uoguelph.ca
[2] Department of Mathematics and Computer Sciences, University of Catania, Catania, Italy
daniele@dmi.unict.it
[3] Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts, USA
nagurney@gbfin.umass.edu

Abstract In this paper, we establish the equivalence between the solutions to an evolutionary variational inequality and the critical points of a projected dynamical system in infinite-dimensional spaces. We then present an algorithm, with convergence results, for the computation of solutions to evolutionary variational inequalities based on a discretization method and with the aid of projected dynamical systems theory. A numerical traffic network example is given for illustrative purposes.

Key words: projected dynamical systems, evolutionary variational inequalities, critical points, regularization procedure, discretization

1 Introduction

Numerous problems in engineering, in operations research and the management sciences, as well as in economics and finance involve interactions among decision-makers and the competition for resources. In such problems, the concept of equilibrium plays a central role and provides a valuable benchmark against which an existing state of such complex systems can be compared. Examples, par excellence, of such equilibrium problems include: congested urban transportation networks, the Internet, multisector, multi-instrument financial equilibrium problems, as well as a variety of decentralized supply chain networks (see, e.g., [27, 35, 36]).

Various methodologies have been developed to formulate and solve such problems, which are often large-scale. For example, [13] showed that the traffic network equilibrium conditions as formulated by [41] were a finite-dimensional variational inequality and then utilized the theory to establish both existence and uniqueness results of the equilibrium traffic flow pattern as well as to propose an algorithm with convergence results (see also [14]). Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above, as well as to game theoretic problems, such as oligopolistic market equilibrium problems (see, e.g., [15, 26, 36], and the references therein).

As important as the study of the equilibrium state is that of the study of the underlying dynamics or disequilibrium behavior of such systems. Note that because such problems typically involve more than a single decision-maker who is faced with constraints (such as, for example, budgetary, conservation of flow, non-negativity assumptions on the variables, among others), classic dynamical systems theory is no longer sufficient for the formulation and solution of such problems. Toward that end, Dupuis and Nagurney [25] introduced a new class of dynamical system with a discontinuous right-hand side and provided the foundational theory for such *projected dynamical systems*. Moreover, they established, under suitable conditions, that the set of stationary points of a projected dynamical system coincided with the set of solutions of the associated finite-dimensional variational inequality. This connection allowed for the investigation of the disequilibrium behavior preceding the attainment of the equilibrium. Zhang and Nagurney [44] (see also [38]), subsequently, developed the stability theory for finite-dimensional projected dynamical systems. Such results are relevant because without such a theory, the concept of equilibrium may not be valid.

Isac and Cojocaru [30, 31] initiated the systematic study of projected dynamical systems on infinite-dimensional Hilbert spaces in 2002 with the fundamental issue of existence of solutions to such problems answered by Cojocaru [8] in her thesis (see also Cojocaru and Jonker [9]).

Evolutionary variational inequalities, which are also infinite-dimensional, were originally introduced by Lions and Stampacchia [33] and by Brezis [5] in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems. Steinbach [42], on the other hand, studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity. Daniele, Maugeri, and Oettli [21, 22] motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others as we shall highlight later.

As noted by Cojocaru, Daniele, and Nagurney [10], the theory and application of evolutionary variational inequalities was developing in parallel to that of projected dynamical systems. That reference reviews the theoretical foundations of both of these methodologies and surveys the historical developments.

Moreover, it makes explicit for the first time the connection between projected dynamical systems on Hilbert spaces and evolutionary variational inequalities. Finally, the authors provide an illustrative dynamic traffic network example. In [11], the same authors established further results on the unified theory of projected dynamical systems and evolutionary variational inequalities in the context of double-layered dynamics. Moreover, stability analysis results were provided for the curve of equilibria.

This paper expands upon the theme of that first and second joint paper of ours – that of the synthesis and expansion of the theories of projected dynamical systems and evolutionary variational inequalities to enable the richer modeling and rigorous analysis of a plethora of complex dynamic problems subject to constraints. In particular, here we provide a new proof of the equivalence between solutions to an evolutionary variational inequality and the critical points of a projected dynamical system in infinite dimensions. In addition, we propose a new algorithm for the computation of solutions to evolutionary variational inequalities that exploits the equivalence. Convergence results are also provided.

We now recall some fundamentals and results of our prior work, which, along with the preliminary results in Section 2, will allow us to establish the main contributions of this paper.

Let $\mathbb{K}$ be a convex polyhedral set in $\mathbb{R}^n$, $F : \mathbb{K} \to \mathbb{R}^n$ and let us introduce the operator

$$\Pi_{\mathbb{K}} : \mathbb{R} \times \mathbb{K} \to \mathbb{R}^n$$

defined by means of the directional derivative in the sense of Gâteaux

$$\Pi_{\mathbb{K}}(x, -F(x)) = \lim_{t \to 0^+} \frac{P_{\mathbb{K}}(x - tF(x)) - x}{t}$$

of the projection operator $P_{\mathbb{K}} : \mathbb{R}^n \to \mathbb{K}$ given by

$$\|P_{\mathbb{K}}(z) - z\| = \inf_{y \in \mathbb{K}} \|y - z\|.$$

In [25], Dupuis and Nagurney considered the differential equation with a discontinuous right-hand side

$$\frac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t)))$$

and the associated Cauchy problem

$$\begin{cases} \dfrac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t))) \\ \\ x(0) = x_0 \in \mathbb{K}, \end{cases} \tag{1}$$

whose solutions (see also [44]) they called *projected dynamical systems* (PDS). A similar idea, in different contexts, can be found in the papers [1, 12, 29] and in the book [2], as we shall see in Remark 1. In [24] and [25], existence theorems of an absolutely continuous solution are shown, provided that F is assumed to be Lipschitz continuous and with linear growth.

The key trait of a projected dynamical system was first found by Dupuis and Nagurney [25]. In particular, the authors proved the following theorem.

Theorem 1. *The critical points of equation*

$$\frac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t))), \tag{2}$$

namely, the solutions such that $\dfrac{d\,x(t)}{d\,t} \equiv 0$, *are the same as the solutions to the variational inequality*

$$\textit{Find } x \in \mathbb{K} : \ \langle F(x), y - x\rangle \geq 0, \quad \forall y \in \mathbb{K}.$$

As noted above, variational inequalities in the finite-dimensional case have been used to formulate a spectrum of problems arising in engineering, operations research and the management sciences, transportation science, economics, and finance, as, for example, in the case of the traffic network equilibrium, spatial price equilibrium, oligopolistic market equilibrium, and financial equilibrium problems. All these applications have also benefited from the theory of projected dynamical systems in terms of analysis and computation (see [10, 35], and the references therein).

As also noted above, projected dynamical systems have been considered in the framework of Hilbert spaces (see [8–10, 28] and [39]). We now provide a definition of a projected dynamical system.

Definition 1. *A projected dynamical system is given by a mapping* $\Psi : \mathbb{R}_+ \times \mathbb{K} \to \mathbb{K}$, *which solves the initial value problem:*

$$\dot{\Psi}(t, x) = \Pi_{\mathbb{K}}(\Psi(t, x), -F(\Psi(t, x))), \quad \Psi(0, x) = x \in \mathbb{K}.$$

In [8] and [9], the following theorem has been proved.

Theorem 2. *Let H be a Hilbert space and let $\mathbb{K} \subset H$ be a nonempty, closed, and convex subset. Let $F : \mathbb{K} \to H$ be a Lipschitz continuous vector field with Lipschitz constant b. Let $x_0 \in \mathbb{K}$ and $L > 0$ such that $\|x_0\| \leq L$. Then the initial value problem (1) admits a unique solution in the class of the absolutely continuous functions on the interval $[0, l]$ where* $l = \dfrac{L}{\|F(x_0)\| + b\,L}$.

In fact, in [8], the author shows that solutions to problem (1) on Hilbert spaces can be extended to $\mathbb{R}_+$, so Definition 1 also holds in the context of Hilbert spaces. The important consequence of such a theory in the Hilbert space is that we can establish a connection between the solutions to an evolutionary variational inequality and the stationary solutions to projected dynamical equations in Hilbert spaces (see [8] and [9]).

For completeness and definiteness, we now provide some additional citations to evolutionary variational inequalities and applications. In [21] and [22], Daniele, Maugeri, and Oettli formulated time-dependent traffic equilibria as evolutionary variational inequalities. In [19], Daniele and Maugeri developed a time-dependent spatial equilibrium model (price formulation) in which bounds over the time on the supply and demand market prices and on the commodity shipments between supply and demand market pairs were imposed. Moreover, the authors addressed the time-dependent spatial price equilibrium problem in which the variables were commodity shipments. In [18], Daniele introduced a time-dependent financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time-depending supplies of the financial holding.

Cojocaru, Daniele, and Nagurney [10] showed that all the above considered problems can be formulated into a unified definition as we recall below. We consider the nonempty, convex, closed, bounded subset of the Hilbert space $L^2([0,T],\mathbb{R}^q)$ given by

$$\mathbb{K} = \Big\{ u \in L^2([0,T],\mathbb{R}^q) : \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \\ \sum_{i=1}^{q} \xi_{ji}\, u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \qquad (3) \\ \xi_{ji} \in \{-1,0,1\}, \quad i \in \{1,\ldots,q\} \;\; j \in \{1,\ldots,l\} \Big\}.$$

Let $\lambda, \mu \in L^2([0,T],\mathbb{R}^q)$, $\rho \in L^2([0,T],\mathbb{R}^l)$ be convex functions. For chosen values of the scalars ξ_{ji}, of the dimensions q and l, and of the constraints λ, μ, we obtain each of the previous above-cited model constraint set formulations (see [10]), as follows:

- for the traffic network problem (see [21, 22]), we let $\xi_{ji} \in \{0,1\}$, $i \in \{1,\ldots,q\}$, $j \in \{1,\ldots,l\}$, and $\lambda(t) \ge 0$ for all $t \in [0,T]$;
- for the quantity formulation of spatial price equilibrium (see [16]), we let $q = n+m+nm$, $l = n+m$, $\xi_{ji} \in \{-1,0,1\}$, $i \in \{1,\ldots,q\}$, $j \in \{1,\ldots,l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0,T]$;
- for the price formulation of spatial price equilibrium (see [17] and [19]), we let $q = n+m+mn$, $l = 1$, $\xi_{ji} = 0$, $i \in \{1,\ldots,q\}$, $j \in \{1,\ldots,l\}$, $\lambda(t) \ge 0$ for all $t \in [0,T]$, and $\rho_j(t) = 0$ for all $t \in [0,T]$ and $j \in \{1,\ldots,l\}$;
- for the financial equilibrium problem (see [18]), we let $q = 2mn+n$, $l = 2m$, $\xi_{ji} = \{0,1\}$ for $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0,T]$.

Then, setting

$$\ll \Phi, u \gg = \int_0^T \langle \Phi(t), u(t) \rangle \, dt$$

where $\Phi \in L^2([0,T],\mathbb{R}^q)^*$ and $u \in L^2([0,T],\mathbb{R}^q)$, if F is given such that $F : \mathbb{K} \to L^2([0,T],\mathbb{R}^q)$, we have the following standard form of the evolutionary variational inequality:

$$\text{find } u \in \mathbb{K} : \ll F(u), v-u \gg \geq 0, \quad \forall v \in \mathbb{K}. \tag{4}$$

In [22], sufficient conditions that ensure the existence of a solution to (4) are given.

Now the following general result holds in Hilbert spaces (see [8,9,28] and [39]), as we shall prove in Section 4.

Theorem 3. *Assume that the hypotheses of Theorem 2 hold. Then the solutions to the variational inequality (4) are the same as the critical points of the projected differential equation (PrDE) (2), that is, the points $x \in \mathbb{K}$ such that*

$$\Pi_{\mathbb{K}}(x(t), -F(x(t))) = 0,$$

and vice versa.

As a consequence, and by choosing the Hilbert space H to be $L^2([0,T],\mathbb{R}^p)$, we find that the solutions to the evolutionary variational inequality:

$$\text{find } u \in \mathbb{K} : \int_0^T \langle F(u(t)), v(t) - u(t) \rangle \, dt \geq 0, \quad \forall v \in \mathbb{K} \tag{5}$$

are the same as the critical points of the equation:

$$\frac{d\,u(t,\tau)}{d\,\tau} = \Pi_{\mathbb{K}}(u(t,\tau), -F(u(t,\tau))), \tag{6}$$

that is, the points such that

$$\Pi_{\mathbb{K}}(u(t,\tau), -F(u(t,\tau))) \equiv 0 \text{ a.e. in } [0,T],$$

which are obviously stationary with respect to τ.

As noted in [10], the meaning of the two "times" in (6) needs to be well understood. Intuitively, at each instant $t \in [0,T]$, the solution of the evolutionary variational inequality (5) represents a static state of the underlying system. As t varies over $[0,T]$, the static states describe one (or more) curves of the equilibria. In contrast, τ here is the time that describes the dynamics of the system until it reaches one of the equilibria of the curve.

Section 2 is dedicated to the presentation of additional definitions and preliminary results that we need in the subsequent sections. In Section 3, we present a self-contained proof of Theorem 3 and we reference similar existing results. In Section 4, we show how a solution to the evolutionary variational inequality (5) can be computed with the aid of the projected dynamical systems theory. In Section 5, we present a proof of the convergence of the algorithm. In Section 6, we present a numerical dynamic traffic network example that is distinct from the one in [10].

2 Definitions and Preliminary Results

Following Gwinner [28], let us recall some well-known objects of convex analysis that we need in what follows.

Let H be a real Hilbert space, whose inner product we denote by $\langle \cdot, \cdot \rangle$.

Definition 2. *For a subset $M \subset H$, the polar M^0 is defined by*

$$M^0 = \{\xi \in H : \langle \xi, x\rangle \leq 1, \quad \forall x \in M\}.$$

For a cone C, Definition 2 simplifies into

$$C^0 = C^- = \{\xi \in H : \langle \xi, x\rangle \leq 0, \ \forall x \in C\}.$$

Definition 3. *Let $\mathbb{K}$ be a nonempty, closed, convex subset of H. For all $z \in \mathbb{K}$, we define the support cone (or tangent cone, or contingent cone) to $\mathbb{K}$ at x as the set*

$$T_{\mathbb{K}}(x) = \overline{\bigcup_{\lambda > 0} \lambda\,(\mathbb{K} - x)}.$$

Definition 4. *We define the normal cone to $\mathbb{K}$ at x as the set*

$$N_{\mathbb{K}}(x) = \{\xi \in H : \ \langle \xi, z - x\rangle \leq 0, \ \forall z \in \mathbb{K}\}.$$

Proposition 1. *We then have the following result:*

$$\left(T_{\mathbb{K}}(x)\right)^0 = N_{\mathbb{K}}(x) = \left(T_{\mathbb{K}}(x)\right)^-.$$

Proof. It is clear (see [2], Proposition 2, page 220) that

$$\left(T_{\mathbb{K}}(x)\right)^0 \subseteq N_{\mathbb{K}}(x) = \{\xi \in H : \ \langle \xi, z - x\rangle \leq 0, \ \forall z \in \mathbb{K}\},$$

because $z - x \in T_{\mathbb{K}}(x)$, $\forall z \in \mathbb{K}$. Vice versa, $N_{\mathbb{K}}(x) \subseteq \left(T_{\mathbb{K}}(x)\right)^0$, because if $y = \lim_n \lambda_n(z_n - x)$, $z_n \in \mathbb{K}$, $\lambda_n \geq 0$ $\forall n \in \mathbb{N}$, for each $\xi \in N_{\mathbb{K}}(x)$:

$$\langle \xi, \lambda_n(z_n - x)\rangle \leq 0, \ \forall n \in \mathbb{N}$$

and, hence,

$$\langle \xi, y\rangle \leq 0, \ \forall y \in T_{\mathbb{K}}(x),$$

and the assertion is proved. ■

The set $T_{\mathbb{K}}(x)$ is clearly a closed convex cone with vertex 0 and it is the smallest cone C whose translate $x+C$ has vertex x and contains $\mathbb{K}$. The utility of the support cone derives from the following result:

Theorem 4. *If we denote by $P_{\mathbb{K}} = \mathrm{Proj}\ (\mathbb{K}, \cdot)$ the projection onto $\mathbb{K}$ of an element of H, then:*

$$P_{\mathbb{K}}(x + \lambda h) = x + \lambda P_{T_{\mathbb{K}}(x)} h + o(\lambda)$$

for any x, h, and $\lambda > 0$.

Proof. See [43] Lemma 4.6 page 300. ■

Corollary 1. *If we define the projection of h at x with respect to $\mathbb{K}$ as the directional derivative in the sense of Gâteaux*

$$\Pi_{\mathbb{K}}(x,h) = \lim_{\lambda \to 0^+} \frac{P_{\mathbb{K}}(x+\lambda h) - x}{\lambda},$$

then

$$\Pi_{\mathbb{K}}(x,h) = P_{T_{\mathbb{K}}(x)} h,$$

namely, $\Pi_{\mathbb{K}}(x,h)$ is the projection of h on the support cone $T_{\mathbb{K}}(x)$.

Definition 5. *The set of unit inward normals to $\mathbb{K}$ at x is defined by*

$$n_{\mathbb{K}}(x) = \{v : \|v\| = 1 \text{ and } \langle v, x-y\rangle \leq 0,\ \forall y \in \mathbb{K}\}.$$

Then, using Proposition 1, we have that

Proposition 2. *The set of unit normals to $\mathbb{K}$ at x satisfies:*

$$n_{\mathbb{K}}(x) = \partial B(0,1) \cap -\left(T_{\mathbb{K}}(x)\right)^0,$$

where $\partial B(0,1) = \{z : \|z\| = 1\}$.

Now, because in infinite dimensions the interior as well as the relative algebraic interior of a convex set can be empty, we introduce the concepts of quasi interior of $\mathbb{K}$, which may be nonempty.

Definition 6. *We call the quasi interior of $\mathbb{K}$ (denoted by* qi $\mathbb{K}$*) the set of those $x \in \mathbb{K}$ for which $T_{\mathbb{K}}(x) = H$.*

Definition 7. *We define the quasi boundary of a closed convex set $\mathbb{K}$ (denoted by* qbdry $\mathbb{K}$*) as the set $\mathbb{K} \setminus$* qi $\mathbb{K}$.

Then the following proposition holds.

Proposition 3. $x \in$ qbdry $\mathbb{K}$ *if and only if $n_{\mathbb{K}}(x) \neq \emptyset$.*

Proof. Let $x \in$ qbdry $\mathbb{K}$. Then, by virtue of Proposition 2.1 in [6], there exists a $\xi \neq 0$ such that $\langle \xi, x\rangle \leq \langle \xi, y\rangle$, $\forall y \in \mathbb{K}$, and, hence:

$$\left\langle \frac{\xi}{\|\xi\|}, x-y \right\rangle \leq 0 \quad \forall y \in \mathbb{K}.$$

vice versa, if $n_{\mathbb{K}}(x)$ is nonempty, then there exists a ξ with $\|\xi\| = 1$ such that $\langle \xi, x-y\rangle \leq 0$, $\forall y \in \mathbb{K}$. Then $x \notin$ qi $\mathbb{K}$, because: if $x \in$ qi $\mathbb{K}$, then $\langle \xi, x-y\rangle \leq 0$, $\forall y \in \mathbb{K}$ implies $\langle \xi, \lambda(x-y)\rangle \leq 0$, $\forall \lambda > 0$ and $\forall y \in \mathbb{K}$. If $y \in T_{\mathbb{K}}(x)$, then we can write $y = \lim_n \lambda_n(z_n - x)$ and so $\langle \xi, \lambda_n(z_n - x)\rangle \leq 0$, $\forall n \in \mathbb{N}$. When $n \to \infty$, then we get $\langle \xi, y\rangle \leq 0$, $\forall y \in T_{\mathbb{K}}(x)$. Therefore, if $x \in$ qi $\mathbb{K}$, then $T_{\mathbb{K}}(x) = H$ and, hence, $\langle \xi, y\rangle \leq 0$ $\forall y \in H$. Choosing $-y \in H$, we get $\langle \xi, -y\rangle \leq 0$, that is, $\langle \xi, y\rangle = 0$ $\forall y \in H$. Choosing $y = \xi$, we obtain $\|\xi\| = 0$, and then $\xi = 0$, which is an absurdity as $\|\xi\| = 1$. ■

Following an idea of Dupuis [23] on Euclidean space, later used in [25] for the theory of finite-dimensional PDS, we present next a generalization of the geometric interpretation of the operator $\Pi_{\mathbb{K}}$ on infinite-dimensional H-spaces. A similar result, also in infinite-dimensional spaces, can be found in Isac and Cojocaru [31] (see also [28] and [40]).

Theorem 5.

1. *If* $x \in \mathrm{qi}\ \mathbb{K}$, *then for any* $h \in H$ *it follows that:* $\Pi_{\mathbb{K}}(x,h) = h$;
2. *If* $x \in \mathrm{qbdry}\ \mathbb{K}$, *then for any* $v \in H \setminus T_{\mathbb{K}}(x)$ *there exists* $n^*(x) \in n_{\mathbb{K}}(x)$ *such that*

$$\beta(x) = -\langle v, n^*(x)\rangle > 0,$$
$$\Pi_{\mathbb{K}}(x,v) = v + \beta(x)\, n^*(x).$$

Proof. If $x \in \mathrm{qi}\ \mathbb{K}$, then $T_{\mathbb{K}}(x) = H$, by definition of qi $\mathbb{K}$, and it follows that

$$\Pi_{\mathbb{K}}(x,h) = P_{T_{\mathbb{K}}(x)} h = P_H h = h.$$

If $x \in \mathrm{qbdry}\ \mathbb{K}$, then setting $\hat{v} = \Pi_{\mathbb{K}}(x,v)$, we get:

$$\hat{v} = \Pi_{\mathbb{K}}(x,v) = P_{T_{\mathbb{K}}(x)} v,$$

namely:

$$\langle v - \hat{v}, w - \hat{v}\rangle \le 0, \quad \forall w \in T_{\mathbb{K}}(x).$$

Because $T_{\mathbb{K}}(x)$ is a cone with vertex 0, choosing, in turn, $w = 0$ and $w = 2\hat{v}$, we get:

$$\langle v - \hat{v}, \hat{v}\rangle = 0. \tag{7}$$

Moreover, if we set $w = y + \hat{v}$ with $y \in T_{\mathbb{K}}(x)$, we obtain

$$\langle v - \hat{v}, y + \hat{v} - \hat{v}\rangle = \langle v - \hat{v}, y\rangle \le 0, \quad \forall y \in T_{\mathbb{K}}(x)$$

and, hence,

$$v - \hat{v} \in (T_{\mathbb{K}}(x))^0 . \tag{8}$$

Because $v \neq \hat{v}$, as $v \in H \setminus T_{\mathbb{K}}(x)$ and $\hat{v} \in T_{\mathbb{K}}(x)$ by assumption, then the relation (8) implies the existence of some $n^* \in n(x)$ and $\beta > 0$ such that

$$\hat{v} - v = \beta\, n^*.$$

Moreover, the orthogonality $\langle n^*, \hat{v}\rangle = 0$ implies

$$\beta = -\langle v, n^*\rangle,$$

and the assertion is proved. ■

We also obtain the following characterization (see also [28]).

Corollary 2. *Let* $x \in \mathbb{K}$. *Then for any* $v \in H$*:*

$$\Pi_{\mathbb{K}}(x,v) = P_{v - N_{\mathbb{K}}(x)}(0) = (v - N_{\mathbb{K}}(x))^{\#} .$$

Proof. If $x \in$ qi $\mathbb{K}$, from Theorem 5 we derive

$$\Pi_{\mathbb{K}}(x, v) = v.$$

On the other hand, if $x \in$ qi $\mathbb{K}$, by definition, $T_{\mathbb{K}}(x) = H$ and $N_{\mathbb{K}}(x) = (T_{\mathbb{K}}(x))^{-} = H^{-} = \{0\}$. Let us suppose now that $x \in$ qbdry $\mathbb{K}$. From Theorem 5 we know that

$$v - \hat{v} \in (T_{\mathbb{K}}(x))^{0} = N_{\mathbb{K}}(x),$$

where $\hat{v} = \Pi_{\mathbb{K}}(x, v)$. Then we get

$$\hat{v} \in v - N_{\mathbb{K}}(x).$$

Because $\hat{v} = \Pi_{\mathbb{K}}(x, v) = P_{T_{\mathbb{K}}(x)} v$, then we have $\hat{v} \in T_{\mathbb{K}}(x)$ and, hence, $\langle z, \hat{v}\rangle \leq\ 0$, $\forall z \in (T_{\mathbb{K}}(x))^{0} = N_{\mathbb{K}}(x)$. Taking into account (7), we get

$$\langle \hat{v}, v - \hat{v} - z\rangle \geq 0,\ \ \forall z \in N_{\mathbb{K}}(x)$$

and, thus, $\hat{v} = P_{v - N_{\mathbb{K}}(x)}(0)$. ■

3 Proof of Theorem 3

We shall now present a new proof of Theorem 3, in light of our results in the previous sections. Theorem 3 is crucial in the study of projected dynamics and perturbed equilibria. It also has an interesting history: the first proof of this theorem appears in [25] in Euclidean space. In more general spaces, such as Hilbert spaces (finite- or infinite-dimensional), there already exist several proofs of this result, as one can see in [9], Theorem 2.2 [31], Proposition 6. However, we give here a novel proof, independent of the previous ones (see [28]).

Let x^* be a solution to the variational inequality

$$\langle F(x^*), x - x^*\rangle \geq 0, \quad \forall x \in \mathbb{K}. \tag{9}$$

Using the characterization of the solution by means of the projection, we get

$$x^* = P_{\mathbb{K}}\left(x^* - \lambda F(x^*)\right), \quad \forall \lambda > 0.$$

Hence,

$$\Pi_{\mathbb{K}}(x^*, -F(x^*)) = \lim_{\lambda \to 0^+} \frac{P_{\mathbb{K}}(x^* - \lambda F(x^*))}{\lambda} = \lim_{\lambda \to 0^+} \frac{x^* - x^*}{\lambda} = 0.$$

vice versa, let x^* be a stationary point of the projected dynamical system, namely, x^* is such that

$$0 = \Pi_{\mathbb{K}}(x^*, -F(x^*)) = P_{T_{\mathbb{K}}(x)}(-F(x^*)).$$

First, let us consider the case when $x^* \in \text{qbdry}\,\mathbb{K}$ and $-F(x^*) \notin T_{\mathbb{K}}(x)$. By virtue of Theorem 5, there exist $\beta^* > 0$ and $n^* \in n_{\mathbb{K}}(x^*)$ such that:

$$F(x^*) = \beta^*\, n^*.$$

Because $n^* \in n_{\mathbb{K}}(x^*)$, we have

$$\langle \beta^*\, n^*, x^* - y\rangle \leq 0, \quad \forall y \in \mathbb{K}$$

and, therefore,

$$\langle F(x^*), y - x^*\rangle \geq 0, \quad \forall y \in \mathbb{K}.$$

Let us consider now the case when $x^* \in \text{qbdry}\,\mathbb{K}$ and $-F(x^*) \in T_{\mathbb{K}}(x^*)$. In this case we get

$$0 = \varPi_{\mathbb{K}}(x, -F(x^*)) = P_{T_{\mathbb{K}}(x^*)}(-F(x^*)) = -F(x^*)$$

and, hence, the variational inequality (9) is satisfied.

Finally, if $x^* \in \text{qi}\,\mathbb{K}$, then $T_{\mathbb{K}}(x^*)$ coincides with H and we get

$$0 = P_H(-F(x^*)) = -F(x^*)$$

as above. This completes the proof. □

Remark 1. By virtue of Corollary 2, we derive that

$$\frac{d\,\dot{x}(t)}{d\,t} = \varPi_{\mathbb{K}}(x, -F(x)) = P_{-F(x)-N_{\mathbb{K}}(x)}(0)$$

$$= \left\{ \hat{v} \in -\left(F(x) + N_{\mathbb{K}}(x)\right) : \|\hat{v}\| = \min_{y \in -(F(x)+N_{\mathbb{K}}(x))} \|y\| \right\}.$$

Then, the initial value problem

$$\begin{cases} \dfrac{d\,\dot{x}(t)}{d\,t} = \varPi_{\mathbb{K}}(x(t), -F(x(t))) \\ \\ x(0) = x_0 \in \mathbb{K} \end{cases} \tag{10}$$

consists of finding the "slow" solution (the solution of minimal norm) to the differential variational inequality

$$\dot{x}(t) \in -\left(N_{\mathbb{K}}(x(t)) + F(x(t))\right)$$

under the initial condition

$$x(0) = x_0.$$

Because

$$\varPi_{\mathbb{K}}(x(t), -F(x(t))) = P_{T_{\mathbb{K}}(x(t))}(-F(x(t))),$$

problem (10) is equivalent to finding the "slow" solution to the problem

$$\begin{cases} \dot{x}(t) \in P_{T_{\mathbb{K}(x)}}(-F(x(t))) \\ \\ x(0) = x_0 \end{cases} \tag{11}$$

where the operator F is single–valued.

Then, as already observed in the Introduction, the results of [2] Chapter 6, Section 6, and of [1] Theorem 2, can be applied to our projected dynamical system.

Remark 2. It is worth noting that the variational inequality (4) is equivalent to the problem:

$$\text{find } u \in \mathbb{K}: \ \langle F(u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall v \in \mathbb{K}, \ \text{a.e. in } [0,T]. \tag{12}$$

Moreover, this remark is interesting because it means that we may have the possibility of applying to (12), among others, the direct method (that is, finding the explicit closed form solution) in order to find solutions to the variational inequality (4). We illustrate this in the case of a numerical example in Section 6 (see also [20, 34], and [18]).

4 Computational Procedure

We now consider the time-dependent variational inequality (5) where $\mathbb{K}$ is given by (3). From Remark 2, it is equivalent to (12). Let the operator F be strictly monotone (see, e.g., [32] and [35]), so that the solution u is unique and assume that the regularity assumptions on the data introduced by Barbagallo in [3] and [4] are satisfied in order to have $u(t) \in C^0([0,T], \mathbb{R}^q)$. Hence, it follows that:

$$\langle F(u(t)), v(t) - u(t)\rangle \geq 0, \quad \forall t \in [0,T].$$

Consider now a sequence of partitions π_n of $[0,T]$, such that:

$$\pi_n = \left(t_n^0, \ldots, t_n^{N_n}\right), \quad 0 = t_n^0 < t_n^1 < \cdots < t_n^{N_n} = T$$

and

$$k_n = \max\left\{t_n^j - t_n^{j-1} : \ j = 1, \ldots, N_n\right\}$$

with $k_n \to 0$ when $n \to \infty$. Then, for each value t_n^{j-1}, we consider the variational inequality

$$\left\langle F\left(u\left(t_n^{j-1}\right)\right), v - u\left(t_n^{j-1}\right)\right\rangle \geq 0, \quad \forall v \in \mathbb{K}\left(t_n^{j-1}\right) \tag{13}$$

where

$$\mathbb{K}\left(t_n^{j-1}\right) = \left\{v \in \mathbb{R}^q : \lambda\left(t_n^{j-1}\right) \leq v \leq \mu\left(t_n^{j-1}\right), \ \sum_{i=1}^{q} \xi_{ji}\, v_i = \rho_j\left(t_n^{j-1}\right)\right\}.$$

We can compute now the unique solution to the finite-dimensional variational inequality (13) by means of the critical point of the projected dynamical system

$$\Pi_{\mathbb{K}}\left(u\left(t_n^{j-1},\tau\right),-F\left(u\left(t_n^{j-1},\tau\right)\right)\right)=0$$

and we can construct an interpolation function $u_n(t)$ such that

$$\lim \|u_n(t)-u(t)\|_{L^\infty([0,T],\mathbb{R}^q)}=0.$$

Remark 3. We can overcome the regularization assumption on the solution u, by considering a discretization procedure and by computing the solution to the finite-dimensional variational inequality obtained after the discretization (see [40]), using the corresponding projected dynamical system. We will demonstrate how to accomplish this in Section 5.

5 Proof of the Convergence

The discretization procedure for the calculus to the solution of the evolutionary variational inequality (5) runs as follows.

We consider a sequence $\{\pi_n\}$ of partitions of $[0,T]$, such that:

$$\pi_n=(t_n^0,\ldots,t_n^{N_n}),\ 0=t_n^0<t_n^1<\cdots<t_n^{N_n}=T$$

and

$$k_n:=\max\left\{t_n^j-t_n^{j-1}:\ j=1,\ldots,N_n\right\}$$

with $k_n\to 0$ when $n\to\infty$.

We consider the space of $\mathbb{R}^m$-value piecewise constant functions induced by π_n:

$$\begin{aligned}P_n\left([0,T],\mathbb{R}^m\right):=\Big\{&v\in L^\infty\left([0,T],\mathbb{R}^m\right):\\ &v_{(t_n^{j-1},t_n^j]}=v_j\in\mathbb{R}^m,\ j=1,\ldots,N_n\Big\}\end{aligned}\tag{14}$$

where v_j denotes the constant value of v on $(t_n^{j-1},t_n^j]$.

The mean value operators $\mu_n: L^1\left([0,T],\mathbb{R}^m\right)\to P_n\left([0,T],\mathbb{R}^m\right)$ are then introduced by:

$$\mu_n\, v_{(t_n^{j-1},t_n^j]}:=\frac{1}{t_n^j-t_n^{j-1}}\int_{t_n^{j-1}}^{t_n^j}v(s)\,ds.\tag{15}$$

The following Lemma (see, for instance, [7]) will be useful:

Lemma 1. *Let $1\le r<\infty$. Then, the linear operators*

$$\mu_n: L^r\left([0,T],\mathbb{R}^m\right)\to L^r\left([0,T],\mathbb{R}^m\right)$$

are uniformly bounded with norm 1 and:

$$\mu_n v\to v\ \text{in}\ L^r\left([0,T],\mathbb{R}^m\right)$$

as $n\to\infty$, $\forall v\in L^r\left([0,T],\mathbb{R}^m\right)$.

Consider now the following closed and convex set:

$$\mathbb{K} := \Big\{ F(t) \in L^2\left([0,T], \mathbb{R}^m\right) : \ \lambda \le F(t) \le \nu, \ \text{a.e. in } [0,T], \\ \Phi\, F(t) = \rho(t), \ \lambda, \ \nu \ge 0, \Big\} \tag{16}$$

where, for the time being, the upper and lower bounds and the $\rho(t)$ are constant (i.e., not time-dependent) functions, and and an affine-linear mapping $C : [0,T] \times \mathbb{K} \to L^2\left([0,T], \mathbb{R}^m\right)$:

$$C[t, F(t)] = A(t)\, F(t) + B(t), \quad A(t) \in L^\infty, \ B(t) \in L^2.$$

Thus, we are led to solve the problem of finding $H(t) \in \mathbb{K}$:

$$\int_0^T \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt \ge 0, \ \forall F(t) \in \mathbb{K}. \tag{17}$$

In correspondence with each partition we can write:

$$\int_0^T \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt \\ = \sum_{j=1}^{N_n} \int_{t_n^{j-1}}^{t_n^j} \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt. \tag{18}$$

Thus, in each interval $[t_n^{j-1}, t_n^j]$ we can consider the problem of finding $u_j^n(t) \in \mathbb{K}$:

$$\int_{t_n^{j-1}}^{t_n^j} \langle A(t)\, H_j^n(t) + B(t), F_j^n(t) - H_j^n(t) \rangle \, dt \ge 0, \quad \forall F_j^n(t) \in \mathbb{K}. \tag{19}$$

Instead of (19), consider now the finite-dimensional problem of finding $H_j^n \in \mathbb{K}_m \subset \mathbb{R}^m$:

$$\langle A_j^n\, H_j^n + B_j^n, F_j^n - H_j^n \rangle \ge 0, \quad \forall F_j^n \in \mathbb{K}_m \tag{20}$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t)\, dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t)\, dt \tag{21}$$

and consider H_j^n as constant approximations of the solutions $H_j^n(t)$ of (19). Here $\mathbb{K}_m$ is the convex subset of $\mathbb{R}^m$ with same lower and upper bounds and the same demand of $\mathbb{K}$.

Our aim is to prove that the functions:

$$H_n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1}, \, t_n^j)\, H_j^n \tag{22}$$

are, in a suitable sense, piecewise constant approximations to solutions to the original problem (17). We can then prove the following theorem (see [40]):

Theorem 6. *Let $\mathbb{K}$ be as in (16) and, moreover, let $A(t)$ be positive definite a.e. in $[0,T]$. Then, the set $U = \{H_n\}_{n\in\mathbb{N}}$ is (weakly) compact and its cluster points are feasible. Moreover, if $\bar{H}$ is a weak cluster point for U, then $\bar{H}$ solves (17).*

In Theorem 6, we have considered the constant convex set (16). Now we turn back to the case of a time-dependent convex set:

$$\begin{aligned}\mathbb{K} := \big\{F(t) \in L^2\left([0,T],\mathbb{R}^m\right):\ \lambda(t) \le F(t) \le \nu(t),\ \text{a.e. in } [0,T],\\ \lambda(t),\ \nu(t) \ge 0,\ \varPhi F(t) = \rho(t) \text{ a.e. in } [0,T]\big\}\end{aligned} \tag{23}$$

and consider piecewise constant approximations for it. For the sake of clarity and completeness, let us recall some basic definitions of set convergence.

Definition 8. *Let S be a metric space and $\{\mathbb{K}_n\}$ a sequence of sets of S. We say that $\mathbb{K}_n$ is Kuratowsky-convergent to $\mathbb{K}$ if and only if:*

$$\liminf_n \mathbb{K}_n = \limsup_n \mathbb{K}_n = \mathbb{K},$$

where

$$\limsup_n \mathbb{K}_n := \Big\{y \in S : \exists n_1 < n_2 < \ldots,\ \textit{with } y_{n_i} \in \mathbb{K}_n,\ y = \lim_i y_{n_i}\Big\}$$

$$\liminf_n \mathbb{K}_n := \Big\{y \in S : \exists n_0 \in \mathbb{N} : \forall n > n_0\ \exists y_n \in \mathbb{K}_n,\ \textit{and } \lim_n y_n = y\Big\}.$$

Definition 9. *Let S be a normed space and $\{\mathbb{K}_n\}$ a sequence of closed and convex subsets therein. We say that $\mathbb{K}_n$ is Mosco convergent to $\mathbb{K}$ if and only if:*

$$w - \limsup_n \mathbb{K}_n \subset \mathbb{K} \subset s - \liminf \mathbb{K}_n \tag{24}$$

where w and s mean weak and strong topology, respectively.

We now turn to our set (23) and, in correspondence with each partition π_n of $[0,T]$ consider the sets:

$$\begin{aligned}\mathbb{K}_j^n := \big\{F(t) \in L^2\left([0,T],\mathbb{R}^m\right),\ \text{piecewise constant:}\\ \bar{\lambda}_{j,n} \le F_j(t) \le \bar{\nu}_{j,n},\ \text{a.e. in } (t_{j-1},t_j),\\ \varPhi\, F(t) = \bar{\rho}_{j,n},\ \text{a.e. in } (t_{j-1},t_j)\big\},\end{aligned} \tag{25}$$

where $\bar{\lambda}_{j,n} = \mu_{j,n}\,\lambda(t)$, $\bar{\nu}_{j,n} = \mu_{j,n}\,\nu(t)$ and $\bar{\rho}_{j,n} = \mu_{j,n}\,\rho(t)$ are the mean values of $\lambda(t)$, $\nu(t)$ and $\rho(t)$ on (t_{j-1},t_j). Thus, we can consider the set $\mathbb{K}^n = \cap\mathbb{K}_j^n$, which, $\forall n \in \mathbb{N}$, has piecewise constant lower and upper bounds and demand, which we denote by $\bar{\lambda}_n$, $\bar{\nu}_n$ and $\bar{\rho}_{j,n}$, respectively. Then, the following result holds (see [28]).

Lemma 2. *The set sequence $\mathbb{K}^n$ converges to $\mathbb{K}$ (in Mosco sense).*

We come back now to our problem of finding $H(t) \in \mathbb{K}(t)$:

$$\int_0^T \langle C[t, H(t)], F(t) - H(t) \rangle \, dt \geq 0, \quad \forall F(t) \in \mathbb{K}(t) \tag{26}$$

and, $\forall F(t) \in \mathbb{K}(t)$, consider $F^n(t) \in \mathbb{K}^n$ such that $F^n(t) \to F(t)$ (strongly). Such $F^n(t)$ does exist thanks to the first part of the proof of Lemma 2. Now, $\forall n \in \mathbb{N}$, let us consider a solution $H^n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1}, t_n^j) H_j^n$, where H_j^n is the solution to the finite-dimensional variational inequality:

$$\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \rangle \geq 0, \quad \forall F_j^n \in \mathbb{K}_j^n.$$

We are now able to present the final result (see [28]).

Theorem 7. *Let $A(t)$ be positive definite a.e. in $[0, T]$. Then the sequence $H^n(t)$ defined in (22) admits weak cluster points. Each cluster point is feasible and solves the original variational inequality.*

6 A Numerical Dynamic Traffic Network Example

In this section, we present a numerical example that is taken from transportation science. For additional background, we refer the reader to [10], [21], [22], and the references therein. We consider a transportation network consisting of a single origin/destination pair of nodes and two paths connecting these nodes of a single link each, as depicted in Figure 1.

The feasible set $\mathbb{K}$ is as in (3), where we take $p := 2$. We also have that $q := 2$, $j := 1$, $T := 2$, $\rho(t) := t$, and $\xi_{ji} := 1$ for $i \in \{1, 2\}$:

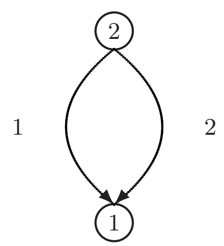

Figure 1. Network structure of the numerical example

$$\mathbb{K} = \left\{ u \in L^2([0,2], \mathbb{R}^2) | \right.$$

$$(0,0) \le (u_1(t), u_2(t)) \le \left(t, \frac{3}{2}t\right) \text{ a.e. in } [0,2];$$

$$\left. \sum_{i=1}^{2} u_i(t) = t \text{ a.e. in } [0,2] \right\}.$$

In this application, $u(t)$ denotes the vector of path flows at t. The cost functions on the paths are defined as: $u_1(t)+1$ for the first path and $u_2(t)+2$ for the second path. We consider a vector field F defined by

$$F : L^2([0,2], \mathbb{R}^2) \to L^2([0,2], \mathbb{R}^2);$$

$$(F_1(u(t), F_2(u(t))) = (u_1(t)+1, u_2(t)+2).$$

The theory of EVI (as described above) states that the system has a unique equilibrium, as F is strictly monotone, for any arbitrarily fixed point $t \in [0,2]$. Indeed, one can easily see that $\langle F(u_1,u_2) - F(v_1,v_2), (u_1 - v_1, u_2 - v_2)\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2 > 0$, for any $u \neq v \in L^2([0,2], \mathbb{R}^2)$. With the help of PDS theory, we can compute an approximate curve of equilibria, by selecting $t_0 \in \left\{ \frac{k}{4} | k \in \{0, \ldots, 8\} \right\}$. Hence, we obtain a sequence of PDS defined by the vector field $-F(u_1(t_0), u_2(t_0)) = (-u_1(t_0)+1, -u_2(t_0)+2)$ on nonempty, closed, convex, 1-dimensional subsets:

$$\mathbb{K}_{t_0} := \left\{ \left\{ [0, t_0] \times \left[0, \frac{3}{2}t_0\right] \right\} \cap \{x + y = t_0\} \right\}.$$

For each, we can compute the unique equilibrium of the system at the point t_0, that is, the point:

$$(u_1(t_0), u_2(t_0)) \in \mathbb{R}^2 \text{ such that } -F(u_1(t_0), u_2(t_0)) \in N_{\mathbb{K}_{t_0}}(u_1(t_0), u_2(t_0)).$$

Proceeding in this manner, we obtain the equilibria consisting of the points:

$$\left\{ (0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), (1,0), \left(\frac{9}{8}, \frac{1}{8}\right), \right.$$

$$\left. \left(\frac{5}{4}, \frac{1}{4}\right), \left(\frac{11}{8}, \frac{3}{8}\right), \left(\frac{3}{2}, \frac{1}{2}\right) \right\}.$$

The interpolation of these points yields the curve of equilibria.

We note that due to the simplicity of the network topology in Figure 1 and the linearity (and separability of the cost functions in this example), we can also obtain explicit formulae for the path flows over time as given below:

$$\begin{cases} u_1(t) = t, \\ & \text{if } 0 \le t \le 1 \\ u_2(t) = 0 \end{cases}$$

and

$$\begin{cases} u_1(t) = \dfrac{t+1}{2}, \\ & \text{if } \ 1 \leq t \leq 2 \\ u_2(t) = \dfrac{t-1}{2}. \end{cases}$$

The above results demonstrate how the two theories of projected dynamical systems and evolutionary variational inequalities that have been developed in parallel can be connected to enhance the modeling, analysis, and computation of solutions to a plethora of time-dependent equilibrium problems that arise in such disciplines as engineering, operations research/management science, economics, and finance.

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Strategic Audit Policies Without Commitment

Kalyan Chatterjee,[1] Sanford Morton,[2] and Arijit Mukherji[*]

[1] Department of Economics, The Pennsylvania State University, University Park, Pennsylvania, USA
kchatterjee@psu.edu

[2] Private Consultant, Seattle, Washington, USA
smorton@pobox.com

Abstract This chapter constructs and analyzes a simple auditing model in order to answer questions concerning three principal issues: (i) the information contained in the report, (ii) commitment to the audit policy, and (iii) audit effort. The approach taken is based on the concept of perfect Bayesian equilibrium. We attempt to examine the nature of such equilibria and arguments as to which equilibrium one would expect to observe.

Key words: audit policies, audit game, commitment, perfect Bayesian equilibrium

1 Introduction

This paper constructs and analyzes a simple model of auditing in which three principal issues are explored, namely: (i) *The information contained in the report.* An audit is a process of verification of a report of private information available to the reporter but not to the auditor. What information is contained in a report? Is it sufficient for the auditor to infer the private information exactly or is it imperfect? How does this affect what the auditor does? (ii) *Commitment to the audit policy: How does its absence affect reporting and investigation decisions?* Can the auditor commit in advance to an audit policy, even when it may not be optimal to carry out the policy at the time of implementation, and, related to this, does auditing have only a purely deterrent role or can it lead to recovery of assets as well? (iii) *Audit effort.* How is audit intensity or effort determined?

Our attempts to answer these questions will involve using the concept of Perfect Bayesian Equilibrium; we will attempt to examine the nature of such equilibria and arguments as to which equilibrium one would expect to observe.

[*] Arijit Mukherji tragically passed away in October 2000. This paper is dedicated to his memory.

The paper therefore also serves the purpose of introducing the ideas of equilibrium refinements (and the effects of assuming a player can commit to a sequence of actions in advance) to the audience for this book and illustrating the usefulness of these refinements through an important application. The partially expository nature of our objectives mean that we have explained proofs and examples in more detail than we would normally have chosen to do.

We present the basic structure of our model in an informal fashion in this introduction and compare it with earlier work. In the next section, the model is specified, emphasizing the nature of the ability to commit or its absence. We formally derive results for the case when auditing results in perfect discovery in Sections 3 and 4. In order to examine audit effort or intensity, Section 5 assumes that an audit is imperfect so that repeated auditing may be necessary to verify the report. We conclude by discussing potential empirical implications of the competing theories of auditing. All proofs are contained in an appendix.

The game that we consider has two players—a manager, who observes the true value of the firm he or she manages and who decides whether to consume some part of this value as perquisites, and an auditor, who does not know the firm's true value but is retained by the firm's shareholders to monitor the manager's report of the value. The difference between the true and the reported value then constitutes the unauthorized consumption of perquisites by the manager. To focus on the interaction between the audit and reporting strategy, we assume the auditor has no moral hazard problem in auditing and acts in the interests of shareholders.

In our model, "nature" moves first and draws a value for the firm from a commonly known probability distribution. The manager observes this value and decides how much to report, retaining the residual as a perquisite. The auditor observes the report and decides whether or not to audit at a cost c. As an important distinguishing feature of our model, we assume that the auditor cannot commit to the audit strategy in advance of the report but must use the information conveyed by the report in the audit decision. Assume for the moment that an audit consists of a single observation that leads to perfect discovery (this is relaxed later). If there is no audit, the manager obtains a payoff corresponding with the difference between true and reported value, and the auditor (shareholders) obtains the reported amount.[1] If there is an audit, the auditor obtains the whole value of the firm less the cost of observation, and the manager must pay a penalty proportional to the amount

[1] We assume that both the manager and the auditor are rational economic agents. In a tax audit context, Erard and Feinstein [7] consider the implications of assuming that some taxpayers are intrinsically honest and will not misreport their true taxable income. In a model of analytical review, Newman et al. [19] consider a model in which the auditee is honest with some probability and fraudulent with the complementary probability. In our model, if the manager were able to consume the residual, undetected, then he would prefer to do so. Graetz et al. [12] were the first to consider intrinsically honest taxpayers.

of underreporting. This penalty could be thought of as being nonpecuniary in nature and hence not accruing to any individual.

We characterize a large number of equilibria in this model, including some that resemble the audit policy obtained if the auditor *can* commit in advance to the audit policy. This multiplicity of equilibria results from the many different interpretations that the auditor can place on a report, corresponding with different reporting strategies, all of which provide the same information to the auditor about the value of an audit. The equilibria of this model all involve some pooling—managers with different true values make the same report. Therefore auditing potentially has an information acquisition role in any pooling equilibrium. The report identifies, in equilibrium, a range of values that the manager may have observed. If the report is audited, the actual value is discovered in these cases, therefore producing information that was unavailable before the audit. Some equilibria also involve partially separating reporting strategies—managers with different values make different reports. In the range of values in which the equilibrium is separating, the true value can be inferred exactly from the report, and the role of the audit is purely to deter. However, the auditor would still want to audit, as recovery of the fraud amount involves verification that stealing has actually occurred.

Among all these equilibria, we show how to choose one as most plausible. The equilibrium that we will argue for involves pooling only at the lower end of the range of values and separation at all values above a cutoff. This maximally separating—and therefore maximally informative—equilibrium is chosen by using the D1 refinement of sequential equilibrium proposed by Banks and Sobel [2]. A pooling reporting strategy will have many out-of-equilibrium reports that should never be sent by the manager, and D1 places restrictions on what interpretation the auditor can make if such reports are received. These interpretations must be credible because it is the auditor's response to those reports that ensure that they are never sent. In our context, what matters is the strong monotonicity property this refinement associates with beliefs. For any "unexpected move" (deviation from equilibrium), D1 requires that the auditor believe that the manager observed that value of the firm that would make such a deviation most desirable. This will rule out all but the maximally informative equilibrium.[2]

The potential empirical implications of our analysis use the maximally separating equilibrium as the basic prediction. We compare the qualitative features of this equilibrium with those when the auditor can commit. In the maximally separating equilibrium, every type of manager understates the value of the firm. Audit probabilities are responsive and strictly decreasing in the report. The prior distribution of firm values affects the equilibrium only toward the two ends of the support. In contrast, the commitment equilibria have audit probabilities that are constant over a lower range of reports. The manager understates the report only when it will never be audited.

[2] Reinganum [20] uses a similar device in a different context of plea bargaining.

The prior distribution plays a crucial role in these other equilibria by changing the intervals of reports that characterize the equilibrium.

We now consider how this model helps us to pose the questions we are interested in exploring. Like Fellingham and Newman [8], an auditor and a manager choose their strategies optimally given the conjectures each has of the other's behavior (in other words, the problem is formulated as an explicit extensive form game and therefore amenable to equilibrium analysis). In Fellingham and Newman's version, the manager has no private information and the auditor and manager move simultaneously, one choosing whether to commit fraud and the other whether to audit. Their framework does not allow auditing to have any informational role, only one of pure deterrence. It is clear that adding a reporting stage to their game without introducing private information will not be enough to induce any qualitatively different conclusions, as both the manager who has committed fraud and the one who has not will find it optimal to deny fraud. In our approach, on the other hand, the potential informational role of auditing, in an environment where strategic misreporting could occur, can be examined along with its deterrent aspect. This leads to a richer and more complete strategic analysis.

The second major issue is that of commitment. In the tax audit literature especially, models have been proposed with features similar to ours except that the order of moves between auditor and manager (or taxpayer) is reversed. In these papers (Morton [16], Sanchez and Sobel [23], Border and Sobel [2], Reinganum and Wilde [21], for example), the auditor announces a policy to which he or she is committed no matter what information is conveyed by the manager's report. The equilibrium in such a model consists of the auditor auditing every report below a certain cutoff with the same constant probability and auditing reports above the cutoff with zero probability. The manager (or taxpayer) reports the value truthfully up to the cutoff. If the value of the firm is above the cutoff, the manager reports the cutoff value. Thus only those who, in equilibrium, do not commit fraud are audited. The auditor expects not to find any underreporting when he or she audits, though she is committed to incur the costs of such an audit. There are several means by which such a commitment can be sustained, such as bonding, reputation effects, or delegation. We will discuss these below, but each appears to be somewhat problematic. Our model offers an exploration of the policies that may result in the absence of such commitment and an elucidation of the distinctions between the two approaches.

The paper most similar to ours is Reinganum and Wilde [22] (especially the appendix), who analyze a similar reporting game in a tax audit context. The differences between their analysis and ours are as follows:

1. They consider only a single, perfectly informative equilibrium, whereas we find all the pure reporting strategy equilibria and show how to refine these to a unique equilibrium.

2. Their model allows for an unbounded amount of fraud, by assuming that there is a negative income tax. Because the manager cannot steal more than the value of the firm, we place a lower bound on the amount of fraud. We show this will rule out all perfectly informative equilibria.
3. We also directly model the audit technology, based on the nature of audit sample information, and consider two distinct ways of modeling audit intensity. (Although the appendix of Reinganum–Wilde deals with the single-audit case we consider in the text, the main body of their paper can be interpreted as an analysis of audit intensity, though this is not linked to the single audit case as is done here. They refer to the single-audit case as the costs being linear with respect to probability.)

We should say, however, that we acknowledge that Reinganum and Wilde [22] was the first paper to raise the commitment issue and to analyze the consequences of no commitment. Theirs is clearly the pioneering paper in this area, though we feel that we too have made a contribution as described above.

Another recent paper, Khalil [13], has a title very similar to ours, though the model he discusses is somewhat different. His paper is in the context of regulation, modeled as a principal–agent problem with monitoring. The principal first proposes a contract, the agent who could be one of two types either accepts or rejects the contract, and if she accepts produces a level of output. Given the output, the principal could choose to audit or not, to determine if in fact the agent has produced the contractual output corresponding with his type. Our model is with a continuum of types and we do not have a contracting or production stage. The paper does, of course, address similar issues of commitment and incentives to audit.

The third major area, imperfect audits and audit effort, is analyzed in Section 5. When one unit of audit cost will discover a misstatement only probabilistically, the audit may be repeated in order to gain higher confidence in the report. This results in an equilibrium intensity of auditing and a model of audit effort in which the auditor does not obtain perfect assurance in the manager's report. Baiman et al. [1] model a three-agent contracting problem between the owner and manager of a firm and an independent auditor. Although contracting issues among these agents are of high interest, their results seem too strong, as they show that the auditor will always be motivated to choose effective auditing to obtain full information whenever he is engaged. This prevents the auditor from using a strategy that is contingent on the manager's report as well as partial auditing to obtain less than full information. Our approach is to make exogenous but plausible assumptions about the contracting relationship in order to focus on the details of the audit and reporting strategy.

In a previous version of this paper, we showed that our framework applies also to reporting value to the financial markets. The risk in this case is that the manager's report will be overstated (e.g., higher income or assets than is permitted by accounting principles) so that the manager can show better performance in order to obtain bonuses or promotion. We argue that the risks and benefits to both the auditor and manager for such misstatements are qualitatively the same as for asset fraud. There is a perfectly informative equilibrium for reporting fraud. The other main difference is that the audit probability schedule is now an increasing function of the manager's report. In the interests of space, this extension does not appear in the paper.

2 The Perfect Audit Game

This section describes the benchmark case of asset fraud and perfect auditing. Imperfect auditing and reporting fraud will be considered later. The game has two players, a manager of a firm and an auditor. As an insider, only the manager knows the value, v, of the firm, so let v be a random variable with a continuous probability density function $f(\cdot)$, on bounded support $[0, V]$. The manager must issue a report, r, on the value of the firm. By underreporting the value, $r < v$, the manager can obtain rents from the firm in which the residual $v - r$ is appropriated to his own use, i.e., asset fraud. The amount the manager can report is restricted to lie in the interval $[0, v]$: the manager will not contribute to the firm from his own pocket and cannot take more than the value of the firm for his own use.

The auditor observes the report and may then choose to conduct an audit at a cost $c > 0$, which will perfectly reveal v and the amount of the fraud. If an audit reveals a misreport, then the manager must return the amount of the fraud and will suffer some penalty that is assumed to be in proportion to the amount of the fraud, $M(v - r)$, with $M > 0$. Acting in the interests of the owner, the auditor wishes to minimize the expected amount of misreporting net of audit cost. Formally, the expected payoffs to the manager and expected costs to the owner, respectively, when the manager observes v, reports r, and the auditor audits with probability p, are

$$U = (1 - p)(v - r) - pM(v - r) = [1 - p(M + 1)](v - r)$$

$$C = pc + (1 - p)(v - r) = (v - r) + p[r - (v - c)].$$

The basic incentives in this game are straightforward to describe. The manager wishes to report as little as possible, except to the extent that the audit deters him. In particular, the manager will be attracted to low reports that are never audited and carry no risk of discovery. Further, if a report is always audited or, in fact, audited with any probability greater than $1/(1 + M)$ (the probability that makes the manager's expected payoffs identically zero), the manager will never choose that report unless he is being truthful. As for the auditor, the manager's report may convey some information about the value of the firm, so the auditor may wish to use this report to calculate the expected value of the firm. A costly audit will be undertaken only when all available information suggests that a sufficient amount of misreporting will be discovered to justify the audit cost.

The equilibrium concept that formally captures this is the sequential equilibrium of Kreps and Wilson [15]. (This paper does not define the equilibrium concept for infinite strategy spaces, so effectively we use the Perfect Bayesian Equilibrium concept of Fudenberg and Tirole – see their textbook [10] for an exposition.) In most of what follows, we confine our attention to equilibria with pure reporting strategies.

Definition 1. *A pure reporting strategy* equilibrium *of this game consists of an audit probability schedule, $p(r)$, reporting strategy, $r(v)$, and posterior updating rule, $f(v|r)$, such that*

1. *for every v, $r(v)$ maximizes the manager's expected payoffs, U, for $r \in [0, v]$ and given $p(r)$,*
2. *for every r, $p = p(r) \in [0, 1]$ minimizes the auditor's expected costs, $E(C|r)$, where E is the expectation over v given the posterior $f(v|r)$, and*
3. *for every equilibrium report, r, $f(v|r)$ is the Bayes posterior for the prior, $f(v)$, given the reporting strategy $r(v)$.*

In general, an equilibrium will require that the expected amount of misstatement in a report just be equal to the audit cost.[3] This is because if there is too much expected fraud in a report, the auditor will wish to audit with probability one, in which case, the manager would not issue that report unless he is being truthful and the report would not be misstated. And if there is too little fraud in a report, the auditor will not audit, and (if it is a low report) the manager will wish to send that report, thereby increasing the expected misstatement. When reports are misstated just by the audit cost, the auditor will be willing to audit probabilistically; an equilibrium audit policy must then motivate the desired reporting behavior from the manager.

There is an alternate way of modeling this problem. Although an audit occurs after the manager issues his report, in some circumstances it may be possible for the auditor to formulate the audit policy prior to the report. Morton [16] and Sanchez and Sobel [23] have analyzed auditing in this case and have found that the following audit policy is optimal:

Definition 2. *A* commitment audit policy *is an audit probability schedule*

$$p(r) = \begin{cases} \frac{1}{M+1} & \text{if } r < r^* \\ 0 & \text{if } r \geq r^* \end{cases}$$

for some cutoff report $r^* \in [0, V]$ that the auditor chooses optimally. When v is less than the cutoff, the manager cannot avoid being audited with a probability

[3] This accounts for our decision to use a continuous, rather than discrete, formulation for the value of the firm. In general, with discrete values and reports, the manager must use a random reporting strategy to ensure the expected amount of misstatement is equal to the audit cost, and there will seldom be a pure reporting strategy equilibrium. Because of this, the continuous formulation is in fact more tractable.

just sufficient to deter fraud and so will be willing to report truthfully; when v is greater than the cutoff, the manager will report the lowest amount, r^*, which carries no risk of discovery. Thus the corresponding commitment reporting strategy *is*

$$r(v) = \begin{cases} v & \text{if } v \leq r^* \\ r^* & \text{if } v \geq r^*. \end{cases}$$

We have called this a commitment policy because it requires the auditor to commit himself to a policy that he will later wish to abandon. To see why, note that the auditor would not be willing to follow this policy after he receives the report, as it calls for an audit of reports that are known not to be misstated, and so is not an equilibrium in our sense. Even if the auditor announced this policy in advance, if the manager knows it can be revised at the time of audit, the manager may not find it credible and would instead predict the auditor will use a policy that satisfies (2) and (3) above. It would be ideal for the auditor to announce a policy, and have it be believed, but then follow a different policy at the time of audit, but this is unlikely to fool a sophisticated and rational reporter who understands the nature of the game.

If there is some mechanism by which the auditor can costlessly commit himself, then the auditor would generally wish to do this, because, according to the results of Morton [16] and Sanchez and Sobel [23], he could have committed himself to an equilibrium policy in our sense but chose not to, evidently to do better. However, the plausibility of such mechanisms need to be considered carefully. Theoretically, one could publicly post a large bond with a reliable third party, guaranteeing that the audit policy would be implemented, on penalty of forfeiting the bond. Alternatively, one might argue that long-run reputation effects might enable a commitment to audit reports that are truthful (see Schelling [24] for a classic discussion of commitment techniques). However, neither bonding nor public proclamation of the audit policy is typically observed in practice, perhaps because of the difficulty of verifying a probabilistic strategy. Another idea, suggested by Fershtman et al. [9], and by Mookherjee and Png [17], is that the audit policymaker could delegate the implementation of the audit to a computer, or to a subordinate with an incentive structure to follow the policy rather than discover fraud. Delegation is very common in practice but, conceptually, it appears to push back the incentive problem one level: what sustains a commitment to the computer program or the incentive structure or how does the policymaker prevent himself from altering the policy at the time of audit? Because of these difficulties with commitment, we believe our equilibrium without commitment is plausible in many circumstances.

In addition, a commitment audit policy is not a sensitive forum for exploring the role of information in reporting and auditing. With commitment, whenever a report is audited, a misreport is never discovered because the audit is done with sufficient intensity to deter fraud from that report. The auditor does not attempt to extract information from a report, and the audit never reveals any new information. Thus the deterrent effect of auditing has overwhelmed any use of information in the audit. In contrast, the sequential equilibrium we use here is designed to explore just these issues. The next section shows that an equilibrium audit policy does require the auditor to use the information contained in the manager's report.

3 Equilibria of Perfect Audits

This section analyzes the equilibria of the perfect audit game. We begin by characterizing an equilibrium in which the manager's reports are very informative. Like the commitment policy, the auditor will be able to infer the value of the firm before auditing, yet unlike commitment, this is not because the report is truthful as the auditor will be unwilling to incur the audit cost to merely verify a truthful report. This audit policy is also qualitatively different from the commitment policy in that lower reports will be audited with strictly higher probability. There are additional equilibria of this audit game, including some that resemble commitment audit policies (although the manager is almost never truthful in his reporting strategy). All of these equilibria have monotone reporting strategies in which the manager's report is nondecreasing in the value of the firm[4]. With a multiplicity of equilibria it is important to select one as most plausible, so this section concludes by showing how to eliminate all but the most informative equilibrium by using a refinement of sequential equilibrium.

In analyzing this game, it is useful to focus on the nature of the manager's reporting strategy, which may be *separating* (the manager makes distinct reports for distinct firm values) or *pooling* (the manager sometimes makes the same report for distinct values). The auditor will use the report to determine the updated value of the firm in deciding whether to audit, so separating reports will give the auditor perfect information about the value of the firm. For a separating report, to induce the auditor to audit, it must be that the amount of fraud in each report is just equal to the audit cost, so this immediately suggests that a separating equilibrium have reporting strategies $r(v) = v - c$. The audit probability schedule must then be chosen to induce this strategy from the manager.

However, for $v < c$, this strategy calls for the manager to make a negative report, which was assumed not to be feasible. Therefore, the perfectly revealing reporting strategy must be modified to allow for pooling at the lowest report, $r = 0$. This report must be audited with positive probability because otherwise the manager would always report 0. With pooling, the auditor will not be able to infer the firm value, but to induce auditing it must still be that the average amount of fraud in this report is just equal to the audit cost. Thus define a lower interval of firm values whose average is equal to c.

[4] In the appendix, we discuss examples of both nonmonotone and probabilistic reporting strategies.

Definition 3. *Let* $I_0 = [0, v_1]$ *be a lower interval such that* $E(v \mid v \in I_0) = c$.

I_0 is the unique lower interval of types that, if all and only types in that interval chose the report $r = 0$, the auditor's expected recovery from auditing $r = 0$ would just equal the audit cost. We will assume that $Ev > c$, as otherwise there will exist only the trivial equilibrium in which it is never worthwhile for the auditor to audit, even when the manager always defrauds the firm of its entire value. Because $f(v)$ is a continuous probability density function, the interval I_0 exists. We can now state

Proposition 1. *There exists an equilibrium (unique as to the audit probabilities of equilibrium reports) in which types* $v > v_1$ *use the separating strategy* $r(v) = v - c$ *and types* $v < v_1$ *report* $r = 0$. *The equilibrium audit probability schedule is given by*

$$1 - (M+1)p(r) = \exp \frac{r - (V - c)}{c}$$

for reports $r \in (v_1 - c, V - c]$ *and by*

$$1 - (M+1)p(0) = c \exp - \frac{V - v_1}{c}$$

for the report $r = 0$.

Because every report that is sent contains an average of c amount of misstatement, the auditor is indifferent to auditing or not and is willing to audit with these probabilities. It is straightforward to verify that this audit schedule will induce the required reporting strategy from the manager. The more difficult part of the proof is uniqueness, which relies on an argument by construction using an envelope technique.

This equilibrium is very different from a commitment audit policy as it is a strictly decreasing audit schedule. From an *ex ante* perspective, the lowest reports are most likely to contain fraud in relation to the prior expected value of the firm, and these reports are audited most intensively. Further, in contrast to the commitment audit policy, which never discovers a misreport, an audit always discovers some amount of misreport except in the zero probability case that $v = 0$.

There are other equilibria of this game, but the maximally informative equilibrium just described will be the unique one to survive refinement. To show this, we will characterize two additional classes of pure reporting strategy equilibria. The most convenient way to categorize equilibria in signaling games is by the nature of the pooling in the reporting strategy. We begin by asking, when can the maximally separating equilibrium be perturbed by adding some additional pooling? The answer is that, subject to mild conditions, almost any interval partition, P, of the set of possible firm values, $[0, V]$, can be an equilibrium, in which values in each interval in the partition pool by making the same report[5].

[5] Partition equilibria have been explored in the accounting literature on financial disclosure by Gigler [11] and, most recently, Morgan and Stocken [18].

Proposition 2. *Suppose there is an equilibrium in which $p(r) < 1/(M+1)$, for all equilibrium reports, r. Then there is an interval partition, P, of $[0, V]$ (in which the intervals may overlap at their end points) such that*

1. $I_0 = [0, v_1] \in P$,
2. *for every other* I, $\inf I > E(v \mid v \in I) - c$;
3. *every* $v \in I$ *reports* $r = E(v \mid v \in I) - c$, *except for the highest interval for which* r *may be less than this.*

Conversely, suppose that P satisfies (1) and (2). Then there is an equilibrium in which $p(r) < 1/(M+1)$ for all equilibrium reports and in which the reporting strategy is $r(v) = E(v \mid v \in I) - c$ for $v \in I$.

These conditions can be motivated as follows: First, the partition, P, must consist of connected intervals. This is because the usual "single-crossing" property of the manager's indifference curves holds here, so that if a given manager type prefers a higher report to a lower report, then so do all higher types. Second, the lowest interval I_0 must be an element of the partition for any of these equilibria as, also by the single-crossing property, these are all and only the types who will report $r = 0$. Therefore, there are never any perfectly separating equilibria here because pooling at I_0 is necessary. Third, the auditor must be indifferent in order to choose $0 < p(r) < 1$; so the report for each pool or interval, I, of types is set so that

$$r(I) = E(v \mid v \in I) - c.$$

But all types $v > 0$ will obtain strictly positive payoff because $p(r) < 1/(M+1)$, which requires the final condition that $v > r(I)$, for all $v \in I$.

Audit policies that look similar to the one used under commitment can also be observed in the no-commitment equilibrium, although the reporting strategy must be different from the commitment game in order to induce the auditor to audit. Instead of reporting truthfully in the audit region, in which case the auditor would not be willing to audit, the manager must misstate the report by the amount c. Because the commitment audit policy leaves the manager indifferent among reports in the audit region, $r < r^*$, it is often possible to construct the required reporting strategies that leave the auditor indifferent.

Proposition 3. *Suppose an equilibrium exists in which*

(i) $p(r) = \begin{cases} \frac{1}{M+1} & \text{if } r < r^* \\ 0 & \text{if } r \geq r^* \end{cases}$

and the manager's reporting strategy is pure and monotone. Then there is an interval partition P of $[0, V]$ such that

(ii) $I_0 \in P$
(iii) the highest interval of the partition contains $[V - c, V]$, *and*
(iv) for every interval $I \in P$, $\inf I \geq E(v \mid I) - c$.

Furthermore, the highest interval is $[r^*, V]$ *and the manager's equilibrium reporting strategy is*

$$\text{(v) } r(v) = \begin{cases} r^* & \text{if } v \geq r^* \\ E(v \mid I) - c & \text{if } v \in I. \end{cases}$$

Conversely, suppose there is an interval partition of $[0, V]$ *that satisfies (ii), (iii), and (iv). Then let* r^* *be the infimum of the highest interval in the partition and there is an equilibrium in which the audit strategy is given by (i) and the reporting strategy by (v).*

This shows that there are many equilibria in which the auditor appears to use a commitment audit policy. As before, when the reporting strategy is monotone, all and only types in the lower interval I_0 will report $r = 0$. At the other end of the range of firm values, the manager will report r^* whenever $v > r^*$. In between, the manager cannot avoid the audit region and is indifferent among all reports. If each interval of types uses the pooling strategy according to (v) (which never requires the manager to overreport if (iv) is satisfied) then the auditor will also be indifferent and be willing to audit according to the commitment policy (i). This is nevertheless quite different from the commitment equilibrium because (iv) and (v) together imply that, except when $v = 0$, the manager always underreports the value of the firm.

In general, the cutoff value that is optimal in the commitment audit policy may not be an equilibrium cutoff value here. However, even if the two audit policies are identical, the auditor will be strictly better off with commitment because, with commitment, the auditor can enforce truthful reporting and ensure no misreporting when $v < r^*$[6]. Without commitment, the manager will almost always misreport when $v < r^*$, and the auditor will recover this fraud only with probability $\frac{1}{1+M} < 1$. Thus, although the audit policies may be identical, the reporting strategies are very different.

This multiplicity of equilibria (the appendix provides examples that show there are also equilibria with nonmonotone and with mixed reporting strategies) can be resolved by showing that the maximally separating equilibrium of Proposition 1 is the most plausible.[7] All equilibria of this game involve some

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[6] Although the manager is indifferent between truthful and misreporting, an infinitesimally higher audit probability will make the manager strictly prefer to report truthfully.

[7] We have received queries about mixed reporting strategies like the ones found in Crawford and Sobel [6]. We should emphasize that our model is not a cheap talk model like Crawford–Sobel. In Crawford–Sobel, messages are 'cheap talk,' *i.e.,* messages are costless. In our model, messages *are* costly: a report of v involves making a payment of v. In addition, if the true type of the manager is v, he prefers not to report more than v, whereas Crawford–Sobel make no such restriction, and overreporting might occur in equilibrium in their model. The following is obviously not an equilibrium (as has been suggested): Take any interval $I_i = [a_i, b_i]$ in the partition equilibrium discussed in the paper. Let the manager randomize over $a_i - c$ to $b_i - c$. Any reports in this range are supposed by the auditor to

pooling in the manager's report (unlike the model of Reinganum and Wilde [22]) and also contain out-of-equilibrium reports that are never sent by the manager no matter what value he may observe. These off-equilibrium reports are often of crucial importance because the auditor's response to these reports are precisely what prevents them from being sent and make the equilibrium reports rational for the manager. Sequential equilibrium places almost no restrictions on how the auditor can interpret out-of-equilibrium moves, and so he can interpret them in fairly silly ways in order to make an otherwise incredible response to prevent the move from being made. Refinements of sequential equilibrium generally focus on how to interpret out-of-equilibrium moves—what sense will the auditor make if he observes a report that, according to the equilibrium, should never have been sent—and place additional restrictions on these out-of-equilibrium beliefs by asking for the most reasonable interpretation to place on reports that should not have occurred.

To illustrate the qualitative differences between the equilibria, we now present some numerical examples. Suppose $\tilde{v}$ is uniformly distributed on the interval $[0, 100]$, suppose that the verification cost $c = 20$ and that the penalty parameter $M = 10$. We will now construct partition equilibria of the type described in Propositions 2 and 3.

Example 1. For these parameters, $v_1 = 40$. An example of a hypothetical three-element partition is given by the following: $I_0 = [0, 40]$, $I_1 = [40, 70]$, and $I_2 = [70, 100]$. All types $v \in I_0$ report $40 - c = 0$. All types $v \in I_1$ report $55 - c = 35$. All types $v \in I_2$, the equilibrium report is $85 - c = 65$. All reports are audited with probability $p(r) \leq \frac{1}{11}$.

Example 2. In fact this is *not* the only three-element partition. Consider $I_0 = [0, 40]$, $I_1 = [40, 76]$, and, $I_2 = [76, 100]$. All types $v \in I_0$ report $40 - c = 0$. For $v \in I_1$ report $58 - c = 38$. For $v \in I_2$, the equilibrium report would be $88 - c = 68$.

Example 3. One can similarly build a four-element partition $I_0 = [0, 40]$, $I_1 = [40, 60]$, $I_2 = [60, 80]$, $I_3 = [80, 100]$ for the same set of parameters.

Example 4. For all these equilibria, the first and last elements of the partition are fixed. If we just restricted ourselves to partitions where the other elements

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come from I_i. This is impossible to sustain as an equilibrium with nondegenerate mixed auditing strategies, as $E[v \mid I_i] - r = c$, for such a mixed strategy to be in equilibrium, and this cannot be true for two distinct values of the report r. (Everything else in the expression above remains the same.) It is also unclear how the manager with different values of v can be indifferent among such reports, because it might involve reporting more than the actual value—a negative payoff.

With a continuum of types, the restriction to pure reporting strategies is a natural one to make, though the pathological examples with nonmonotone reports show there could be mixed-strategy equilibria. The "disturbed" game interpretation of mixed strategies (due to Harsanyi) in fact uses pure strategies with a continuum of types to purify mixed strategies.

were of equal length, one equilibrium induces the partition $I_0 = [0, 40]$, $I_1 = [40, 60]$, $I_2 = [60, 80]$, $I_3 = [80, 100]$ and the corresponding equilibrium reports are $\{\{0\}, \{30\}, \{50\}, \{70\}\}$. Corresponding audit probabilities are any p such that $p < \frac{1}{11}$ for all reports.

There may be different equilibria for the same parameter set. We will now present the "maximally separating equilibrium" described in Proposition 1.

Example 5. This equilibrium induces the partition $I_0 = [0, 40]$, $I_1 = [40, 100]$. Corresponding equilibrium reporting strategies are $\{0\}, \{v - 20\}$. The corresponding audit probabilities are respectively 0.000387148 if 0 is reported, and $\frac{1-exp(0.05r-4)}{11}$ for reports r in the upper tail. We obtain $p(0) = 0.000387148$ by solving $1 - 11p(0) = 20exp(-\frac{100-40}{20})$.

Example 6. In Proposition 3, we described the "commitment-like" partition equilibria: an example follows. This equilibrium induces the partition $I_0 = [0, 40]$, $I_1 = [40, 60]$, $I_2 = [60, 80]$, and $I_3 = [80, 100]$. Corresponding equilibrium reports are $\{\{0\}, \{30\}, \{50\}, \{80\}\}$, with corresponding audit probabilities $\{\{\frac{1}{11}\}, \{\frac{1}{11}\}, \{\frac{1}{11}\}, \{0\}\}$, respectively.

Example 7. We now show that for the same parameter values, one can construct two distinct equilibria of the type described in Proposition 2, one of which has a five-element partition and the other has a four-element partition, with the additional feature that the "finer" partition (the one with five elements) refines the "coarser" four-element partition. If we assume the same parameter values as before, one can consider an equilibrium that induces the four-element partition $I_0 = [0, 40]$, $I_1 = [40, 60]$, $I_2 = [60, 80], I_3 = [80, 100]$, with corresponding equilibrium reports $\{\{0\}, \{30\}, \{50\}, \{80\}\}$, and another equilibrium that induces the five-element partition $I_0^{'} = [0, 40]$, $I_1^{'} = [40, 60]$, $I_2^{'} = [60, 70]$, $I_3^{'} = [70, 80]$, $I_4^{'} = [80, 100]$, with corresponding equilibrium reports $\{\{0\}, \{30\}, \{45\}, \{55\}, \{80\}\}$. As our previous examples illustrate, partition equilibria cannot, in general, not be ranked in terms of "fineness."

The above examples illustrate the magnitude of the issue of multiple equilibria that we face and lead us to the next issue, that of refinements.

Although there are many refinements available in the literature, we shall use "Divinity" by Banks and Sobel [2] and reformulated as "D1" by Cho and Sobel [5], who show that it is generically equivalent in this sort of signaling game to the "strategic stability" of Kohlberg and Mertens [14], the most powerful refinement criterion available. It is difficult to characterize strategic stability in infinite games like ours, so this equivalence may not hold here. Nevertheless, it gives special plausibility to the use of D1. Also, for a general class of signaling games, Cho and Sobel [5] have shown that D1 selects a unique equilibrium, a result we confirm below.

In our context, D1 can be described as follows: the auditor observes an out-of-equilibrium report, r, and is considering whether manager type v may have sent the report. The answer depends on the relative strength of the manager's incentives to send that report and what he speculates may happen if he sends this report rather than his equilibrium report. If there is some other type v' who would be willing to issue r under a wider range of possible responses by the auditor than would v, then D1 requires that the auditor never believe that v could send report r. More formally,

Definition 4. *For any equilibrium, let $A(v)$ be the set of audit probabilities for r for which type v would either weakly or strictly prefer the report r to his equilibrium report. Let $B(v')$ be the set of audit probabilities for r for which some type v' would strictly prefer the report r to his equilibrium report. If $A(v) \subset B(v')$, then type v' has stronger incentives to deviate from his equilibrium report than v does, because v' would strictly prefer a larger set of possible auditor responses than v weakly prefers. In such a case, D1 requires that the auditor cannot place any weight on the conjecture that type v sent the off-equilibrium report. In particular, the equilibrium satisfies D1 if the Bayes posterior for this report r and type v is zero: $f(v \mid r) = 0$.*

Because the manager always prefers a lower audit probability (and because it is continuous and monotonic in his expected payoff), this condition can be simplified somewhat. All types who are deterred by lower probabilities of audit are also deterred by higher probabilities. For every type, calculate the audit probability for an out-of-equilibrium report that would leave the manager indifferent between that report and his equilibrium report. Then, if that report is observed, the auditor must believe it was sent by the type(s) who have the largest such probability. An equilibrium survives D1 if all of the off-equilibrium beliefs satisfy this criterion, and most importantly, the audit probabilities specified for these out-of-equilibrium reports are optimal given these beliefs.

We can now show that only the maximally informative equilibrium of Proposition 1 survives D1. All of the others fail because D1 prevents the auditor from adopting that allow him to audit with sufficient probability to deter some manager types from making out-of-equilibrium reports.

Proposition 4. *The only equilibrium to survive D1 is the maximally informative equilibrium of Proposition 1.*

The idea of the proof is to consider the reports made by any two adjacent intervals in the pooling partition. The reports in between are out-of-equilibrium. For an out-of-equilibrium report, r', sufficiently close to the higher report, r, D1 requires that the auditor believe it was sent by the type $\inf I$ on the lower boundary of the upper adjacent interval, I, as this type has the strongest incentive to send such a report. But if $r - E(v|I) = c$ so that the auditor is willing to audit r, then $r' - \inf I < c$, if r' is sufficiently close to r, and the auditor is not willing to audit r'. But this cannot be an equilibrium because the manager would then prefer the lower r' that is not audited to the higher report r that he is supposed to make.

This argument rules out the equilibria of Proposition 2, where there is much pooling among high types, and also the commitment-like equilibria of Proposition 3, where reports immediately below the cutoff report are out-of-equilibrium. It does not rule out the maximally informative equilibrium. Although all reports $r' \in (0, v_1 - c)$ are never observed in equilibrium, D1 requires that the auditor believe that v_1 sent such reports. But in this separating equilibrium, $v_1 - r(v_1) = c$, so $v_1 - r' > c$ for lower reports and the auditor will wish to audit with probability one, which is what is required to prevent the manager from sending those reports.

4 Comparative Statics

Because we have selected the maximally separating equilibrium as the most reasonable solution to this game, we will concentrate on this equilibrium to discuss the empirical implications of the analysis.

An *increase in the penalty rate*, M, will uniformly decrease all the audit probabilities, because with higher penalties, one need audit less often to obtain the same reporting behavior. However, M has no effect on the reporting strategy. This is because the manager's reporting strategy must be chosen to leave the auditor indifferent between auditing or not, and the auditor is not directly affected by M. This implies that the penalty rate will not affect the initial amount of misreporting, but will affect the average amount discovered after an audit—a higher penalty rate will increase the expected amount of misreporting remaining after an audit.

Recall that the auditor's costs are $C = v - r + p(r - v + c)$. Given the manager's reporting strategy, the last term is expected to be zero, so the auditor's costs are just the initial amount of underreporting, which is unaffected by M. Although we have held the auditor's fee revenue from auditing constant in this game, this suggests that audit fees will also be unaffected by M if they are determined by the total costs of the auditor. The manager's expected payoff, $U = [1 - p(M + 1)](v - r)$, is also unaffected by M. Although the audit probability will change with M, there is no net effect on the term $[1 - p(M + 1)]$. There is also no effect on the reporting strategy. In sum, changes in the penalty rate will affect the audit strategy, but very little else.

A *change in the audit cost*, c, will have more consequential effects. As c increases, there will be uniformly less auditing and more misreporting, so the expected costs of the auditor and the expected payoffs to the manager will both increase. Therefore, audit fees will increase with c. The reason c plays a more fundamental role in the model than does M is that M enters the model only through the term $1 - (M + 1)p$, in the manager's expected payoff function, so p can adjust to accommodate any changes in M without affecting any other aspect of the solution. On the other hand, c determines the amount of misreporting, as well as the cutoff value, v_1, for types that will report $r = 0$.

There are two ways in which a change in the *prior distribution of firm values* may affect the equilibrium: through a change in v_1 or a change in V. Holding V constant for the moment, if it becomes more likely that the firm's value is larger (in the precise sense of a decrease in the conditional expectation of the lower interval $E(v \mid v \in I_0)$, a condition that is not in general equivalent to first-order stochastic dominance), then v_1 must increase. The only effect on the equilibrium is that a larger interval of firm values will report $r = 0$, and $p(0)$ will decrease to attract this larger interval. For values who continue to make strictly positive reports, there is no change in either the reporting strategy or audit probabilities. Because the expected amount of misreporting at $r = 0$ must still be equal to c, there is no change in the total amount of misreporting, or in the expected costs of the auditor. Less fraud will be discovered, however, because $p(0)$ has decreased.

If the upper bound, V, of the support increases, but without altering the conditional expectation around the lower interval, $[0, v_1]$, then the audit probability schedule will increase, so that $p(r)$ will be higher for every r. This will decrease the manager's expected payoff, but the expected amount of misreporting and the auditor's expected costs will be unchanged at c. It often appears that the highest reports are audited most intensively in practice, particularly in a tax audit context. This may be one explanation of this practice because, even though the wealthy taxpayer reports higher income than an indigent, the auditor may have very different priors for the two taxpayers (this is also the explanation in Reinganum and Wilde [22]).

5 Imperfect Audits and Audit Effort

In this section, we examine how this model may be generalized to include an effort choice for the auditor. To this point, the audit was assumed to be perfect and would perfectly reveal the true value of the firm. The only effort choice concerned the probability with which the auditor would undertake an audit. Because audits are seldom perfect in fact, auditing will frequently be sequential, in which further investigation is done after an initial examination of the manager's report. The details of this process are complicated and will surely vary according to the circumstances, so we wish to make our first step in the analysis of sequential audits as simple as possible. One natural assumption is that an audit is imperfect and will discover a misreport only with some probability less than one, which might be described as the reliability of the audit technology. If the auditor audits only once and fails to discover a misreport, he cannot be sure that the report was correctly stated; but it is now reasonable to repeat the audit because the probability of discovering a misreport (if one exists) is strictly increasing in the number of repetitions. Of course, the total audit cost should also be increasing, so this will enable us to examine the trade-offs the auditor faces in choice of audit effort, that is, in the number of repetitions of the audit.

Suppose π is the probability that an audit will discover a misreport if one exists. Therefore, with probability π, a single audit will publicly reveal the value of the firm, and the auditor can cease any further investigation; but with probability $1-\pi$, the audit will not discover any information to disprove or confirm the manager's report. The auditor's decision problem now reflects the imperfect reliability of the audit technology. Consistent with our earlier approach that the auditor cannot commit in advance to an audit policy, we assume that at each repetition the auditor will decide whether to investigate further.[8] This results in an infinite dynamic programming problem in which, at each stage, i, the auditor will choose the probability of auditing, p_i, to minimize his expected current and future costs. Let E_i be an expectation with respect to v based on whatever information the auditor has acquired up to stage i. If he audits report r at stage i, he will incur the audit cost of c. With probability π he will learn the value of the firm and will stop, but with probability $1-\pi$, he will not discover the value of the firm, and he will face the costs of proceeding. Of course, he will also face these future costs if he does not audit.[9]

Letting C_{i+1} be the expected future costs beyond stage i, the auditor will choose p_i to minimize

$$C_i = E_i p_i[c + \pi \cdot 0 + (1-\pi)C_{i+1}] + (1-p_i)C_{i+1}$$

If the auditor ceases to audit at stage N and beyond, his cost becomes

$$C_N = E_N(v-r).$$

Because the result of each audit is independent, the probability of *not* discovering the value of the firm after auditing the report r with probabilities $p_i(r)$ is $\prod_{i=0}^{\infty}(1-p_i(r)\pi)$, so the probability that the value *will* be discovered is

$$p(r) = 1 - \prod_{i=0}^{\infty}(1-p_i(r)\pi).$$

[8] An alternative to this sequential decision problem might be called batch auditing in which the auditor decides in advance of any auditing, but after observing the manager's report, how many repetitions to make, somewhat in the way sample sizes are often calculated. This batch approach to repeated auditing may not constitute a credible audit policy as the auditor may wish to change the batch size after it is partially collected. Reinganum and Wilde's [22] assumption of an audit cost that is convex in the probability of discovery appears to be equivalent to batch auditing. Other possible interpretations of their formulation are also possible, including the size of the audit team assigned to a particular task.

[9] This implicitly assumes a strict liability rather than negligence standard for the auditor, as he will be subject to penalty whenever he fails to discover a misreport, no matter how intensively he audited. It would be interesting to explore a negligence standard in which the auditor is penalized only if he fails to collect sufficient competent evidence. We are not yet sure how to model such standards of evidence. Also, in practice, auditors are liable to be sued whenever they do not find a misreport, so the strict liability regime may be the more plausible.

The payoffs to the manager are unchanged from the perfect audit game, except that the manager cares now only about the probability of discovery and not the probability or intensity of audit per se:

$$U = (1 - p)(v - r) - pM(v - r) = [1 - p(M + 1)](v - r).$$

The manager will choose his report to maximize U given $p(r)$.

We note two features of this equilibrium that simplify the auditor's decision problem. First, an audit either reveals the firm's value or not. If it does, then auditing and the game terminates. If it does not, then the auditor gets *no additional information about* v.[10] Therefore the auditor's information at every stage that the game continues is identical to his information immediately after the manager's report: $E_i(\cdot) = E(\cdot \mid r)$. Second, suppose in any stage that the auditor strictly desires to audit. If a misreport is not discovered in that stage, then, because the auditor has gained no additional information about v, he will also strictly desire to audit in the next stage. This will continue in every stage until the probability of discovery approaches one, which cannot be an equilibrium as the manager would not misreport when he is certain to be discovered. Therefore, in equilibrium, whenever the auditor is willing to audit a report even once, it must be that he is indifferent to auditing in every stage, and being indifferent, he can ignore the future costs. He will then choose p_i to minimize

$$\begin{aligned} C_i &= Ep_i[c + (1 - \pi)(v - r)] + (1 - p_i)(v - r) \\ &= E(v - r) + p_i[c - \pi E(v - r)]. \end{aligned}$$

This shows that the auditor's dynamic programming problem is, in this game, equivalent to a myopic, one-period problem. With this we can now prove the following:

Proposition 5. *Suppose $p(r), r(v)$ are equilibrium auditing and reporting strategies when the audit is perfect and the audit cost is c'. Let $p_i(r)$ be any audit probabilities such that for every r*

$$p(r) = 1 - \prod_{i=0}^{\infty}(1 - p_i(r)\pi).$$

Then $p_i(r), r(v)$ constitute a sequential sampling equilibrium when the audit cost is $c = c' \cdot \pi$.

Because this proposition shows that every imperfect audit can be translated into a perfect audit, the same selection principles can be used to choose the maximally informative equilibrium as the uniquely plausible outcome of the model. It also shows that, in a qualitative sense, the basic model with

[10] This is an important aspect of the specification.

perfect auditing is very robust to the incorporation of audit effort. Aside from observing repeated audits, imperfect auditing is essentially a change in the audit cost. It is also apparent that many different combinations of $p_i(r)$ can result in the same $p(r)$. It is natural to focus on the monotonic audit strategy in which the auditor audits $n(r)$ times with probability one until the last, which is audited with probability $p_n(r)$. These are uniquely determined by the probability of discovery

$$p(r) = 1 - (1 - \pi)^{n(r)}(1 - p_n(r)\pi).$$

With this convention, as the system becomes more reliable, the number of repetitions declines.

6 Conclusion

In this paper, we have examined a simple model in which the strategic interplay between commitment and informational asymmetry can be studied. This is an alternative to the more common contracting approach in accounting, which assumes commitment and public verifiability. The inability of the auditor to commit leads to a wide variety of equilibrium auditing and reporting behavior. It is not enough for the auditor to merely specify an audit policy; this policy must also be consistent with expectations about the manager's reporting strategy and, in particular, about the interpretations formed when unexpected, out-of-equilibrium reports are observed. These additional restrictions imposed on the auditor have surprising implications in permitting more rather than fewer equilibria. Further, in the equilibria where the auditor uses an audit policy similar to the one used in the commitment equilibrium but the manager never uses a truthful reporting strategy, these restrictions on the auditor can change the manager's behavior but not the auditor's.

By examining the plausibility of various out-of-equilibrium expectations, we were led to a unique equilibrium in which the manager always misreports the value of the firm, and this misreport is sometimes discovered. This is the polar opposite of the commitment model where the value of the firm is always truthfully reported when it is below the cutoff, and the audit never discovers a misreport. Casual empiricism suggests that both results are extreme, as audits sometimes, but not always, discover a misreport. These extreme results are generic in simple models, and to obtain the middle ground it may be necessary to depart from strict rationality assumptions.

Another difference between the commitment and no-commitment models is that comparatively little use is made in the latter of the probability distribution of types. The upper bound and lower tail of the distribution have marginal effects on the equilibrium, but otherwise, the reporting and audit strategies are largely independent of the distribution. The distribution is determined by institutional features, and the equilibrium's robustness here is an attractive feature.

In both models, the informational role of the audit is relatively minor. To a large extent, it is the report that conveys information on the value of the firm. In the commitment model, this is always true, and the audit plays an exclusively deterrent role. Many of the equilibria without commitment have fairly uninformative reports, which leaves room for the audit to discover something, but we have shown that the equilibrium in which the reporting strategy is maximally informative is the most plausible in this case as well. It appears that the informational aspect of auditing is subsidiary to its deterrent role.

Comparative statics on the maximally informative equilibrium highlight the importance of the audit cost. A change in that cost has pervasive effects on both the audit policy and reporting strategy, whereas changes in the penalty rate on the manager or the prior distribution of firm values are of less consequence.

We have also examined a generalization to imperfect auditing and audit effort. Formally, the auditor's problem becomes an infinite dynamic program as he may now wish to repeat the audit to obtain greater confidence that the audit was reliable. With considerable relief, we were able to simplify the problem to a static program and show that an imperfect audit was essentially equivalent to an increase in the audit cost of a perfect audit. In addition to showing how audit effort responds in equilibrium to the reliability of the audit, this robustness gives greater credibility to the basic game.

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Appendix

Because there are many equilibria of this game, it is useful to begin by stating some conditions that will be true of all equilibria.

Lemma 1. *For any equilibrium,*

1. $p(r) > 0$ *when* $r = 0$.
2. $p(r) = 0$ *for* $V - c < r \leq V$.
3. $p(r) \leq 1/(1 + M)$ *for any report chosen by the manager.*
4. $p(r)$ *is nonincreasing among the reports that may be chosen by the manager.*

Proof.

(1) If $p(0) = 0$, then the manager will always choose $r = 0$, as this maximizes fraud and also has no risk of discovery. But then the posterior given $r = 0$ will be identical to the prior, and the expected recovery from auditing will be $Ev - c > 0$. Thus, the auditor will wish to audit with probability one, a contradiction.

(2) Because the manager can never report more than the value of the firm, the maximum amount that can be recovered from a report $r \in (V - c, V]$ is $V - r < c$. Therefore, the expected recovery from such reports is less than the audit cost, and the auditor will never audit.

(3) A probability of audit of $1/(1 + M)$ is just sufficient to deter all underreporting, so if the manager chooses a report with $p(r) > 1/(1+M)$, he must be reporting truthfully. Consequently, there will be nothing to recover in such a report, and the auditor will prefer to choose $p(r) = 0$.

(4) A lower report implies that the manager is appropriating more of the value of the firm to his own use. If he can obtain more rents from the firm at a lower risk of discovery, he will never issue the higher report, contradicting the assumption that the manager will sometimes issue the higher report.

■

Proof of Proposition 1

Because every type $v > v_1$ misreports by exactly c, and because all types $v < v_1$ report $r = 0$, which is an average misreport of c, the auditor is willing to audit with these probabilities. In turn, it is straightforward to verify that it is optimal for the manager to make these reports when faced with this audit policy.

To prove that this audit policy is necessary given this maximally separating reporting strategy, we construct the unique audit probability schedule that will induce this reporting behavior using a general technique based on the envelope theorem. Define the maximum value function of the manager's reporting problem when faced with some audit probability schedule, $p(r)$:

$$u(v) = \max_r [1 - p(r)(M + 1)](v - r) \mid 0 \le r \le V.$$

This is the manager's indirect utility as a function of the firm value. By the Maximum Theorem of Berge [3], u is continuous in v, and its total derivative is equal to the partial of U with respect to v, whenever it exists:

$$u'(v) = \frac{\partial}{\partial v}[1 - p(r(v))(M + 1)](v - r) = 1 - p(r(v))(M + 1) = \frac{u(v)}{v - r(v)}.$$

Using the boundary condition that the highest type is never audited, $p(r(V)) = 0$ or $u(V) = V - r(V)$. The solution to this differential equation is

$$u(v) = [V - r(V)] \exp - \int_v^V \frac{1}{t - r(t)}\, dt$$

for $v > 0$, and for continuity let $u(0) = 0$. When $r(v) = v - c$, then

$$u(v) = c \exp \frac{v - V}{c} = [1 - (M+1)p(r(v))]c.$$

Substituting $v = r + c$ yields

$$1 - (M+1)p(r) = \exp \frac{r - (V - c)}{c}.$$

To find $p(0)$, we use the fact that v_1 is indifferent between, $r = 0$ and $r = v_1 - c$

$$[1 - (M+1)p(0)] = u(v_1) = c \exp - \frac{V - v_1}{c}$$

and this concludes the proof.

Proof of Proposition 2

For the first half of the proposition, suppose $p(r)$ is the audit policy in this equilibrium. Because $p(r) < 1/(M+1)$ for all equilibrium reports r, every type $r > 0$ must obtain strictly positive expected utility. This implies that $p(r)$ must be a strictly decreasing function of equilibrium reports, as otherwise the manager could obtain higher expected payoff by making a lower report without incurring any increased probability of audit.

Let $I(r)$ be the set of types who are willing to choose r:

$$I(r) = \{v \mid \text{for all } r', [1 - (M+1)p(r)](v - t) \geq [1 - (M+1)p(r')](v - r')\}.$$

We will show that $I(r)$ must be a connected interval. Let v and $v'' \in I(t)$ and consider $v'' > v' > v$. If r' is not an equilibrium report, then it cannot be preferred by v' to r, so suppose that r' is an equilibrium report and further that $r' < r$. Because v prefers r to r', some algebra shows that $v' > v$ does also:

$$\begin{aligned}
&[1 - (M+1)p(r)](v' - r) - [1 - (M+1)p(r')](v' - r') \\
&\quad = [1 - (M+1)p(r)](v - r) - [1 - (M+1)p(r')](v - r') \\
&\qquad + (M+1)(v' - v)[p(r') - p(r)] \\
&\quad \geq \ 0
\end{aligned}$$

as v prefers r, $v' > v$, and $p(r)$ is decreasing for equilibrium r. An analogous argument shows that r' will not prefer any $r' > r$ because v'' prefers r and $v > v'$. Thus $I(r)$ is a connected interval. These inequalities also show that if v is indifferent between distinct equilibrium reports r and r', then $v' > v$

strictly prefers one or the other. Thus, two distinct intervals can overlap at most at a singleton.

Because, in an equilibrium, every v must have a maximizing report, every v is in some $I(r)$. Therefore, the set of all $I(r)$, for equilibrium r, form a partition of $[0, V]$, except that the end points of an interval of positive length may overlap with its neighbor. By the Maximum Theorem of Berge [3], the maximum value function

$$u(v) = \max_{r} \, [1 - (M+1)p(r)](v - r) \mid \; 0 \leq r \leq r$$

is continuous in r, so that such an end point does in fact overlap with the neighboring interval and that type is indifferent between the reports of the two intervals. Types in the interior of an interval are in no other interval and so strictly prefer a unique report.

Because $p(r)$ is strictly decreasing among equilibrium reports, only the highest interval can have an audit probability of zero. All other intervals must have a strictly interior audit probability, which requires that the auditor be indifferent:

$$c = E(v \mid v \in I(r)) - r,$$

for all equilibrium reports r. This may hold as a weak inequality for the highest interval. This proves (iii), as well as (i) when $r = 0$.

To prove (ii), recall that whenever $v > 0$, the manager must receive a strictly positive expected payoff and therefore must report less than the full value of the firm. Thus, for the lower end point of each interval

$$\inf I(r) > r \geq E(v \mid v \in I(r)) - c,$$

and (ii) is proved. This shows necessity and completes the first half of the proof.

We now show that these same conditions are sufficient for a partition to be the reporting pools of an equilibrium. The proof is by construction and uses an envelope argument, which is shown to be equally applicable to pooling as well as separating equilibria.

For a interval partition P of $[0, V]$, let $I(v) \in P$ be the interval that contains v. Let $r(v)$ be the pure reporting strategy

$$r(v) = E(v' \mid v' \in I(v)) - c.$$

Because $I(v)$ is an interval, $r(v)$ is a nondecreasing step function; by (ii), $\inf I(v) > r(v) > 0$ for $v > 0$, and by (i), $\inf I(0) = 0 = t(0)$, so reports are nonnegative and no type is required to report an amount greater than the firm value. Note also that $r(V) \leq V - c$.

We can now construct a maximum value function, $u(v)$, for the manager using the envelope condition that the total derivative is equal to the partial derivative of the manager's optimal objective function with respect to v. If $r(v)$ is to solve the reporting problem, then the value function must be

$$u(v) \equiv [1-(M+1)p(r(v))][v-r(v)]$$

and the envelope condition is that

$$u'(v) = \frac{\partial}{\partial v}[1-(M+1)p(r(v))][v-r(v)].$$

We specify as boundary condition that the highest type is never audited, $p(r(V)) = 0$ or $u(V) = V - r(V)$. The solution to this differential equation is

$$u(v) = [V - r(V)] \exp - \int_v^V \frac{1}{t - r(t)}\, dt$$

for $v > 0$, and for continuity let $u(0) = 0$. The remainder of the proof consists of constructing an equilibrium that yields this $u(v)$ when the manager reports $r(v)$.

To specify audit probabilities for equilibrium reports, construct $p(r(v))$ so that if v chooses $r(v)$, he attains $u(v)$:

$$[1-(M+1)p(r(v))][v-r(v)] = u(v).$$

These audit probabilities are strictly less than 1/(M+1), as $u(v)/[v - r(v)] > 0$. For reports off the equilibrium path, let

$$p(r) = \begin{cases} 1 & \text{if } r < V - c \\ r\ 0 & \text{otherwise.} \end{cases}$$

The auditor is willing to choose these probabilities because, for equilibrium reports, $r(v)$ was constructed to leave the auditor indifferent. Probability assessments for out-of-equilibrium reports can easily be constructed so that when $r < V - c$, the auditor believes it is some type $v \geq r + c$ and wishes to audit; when $r > V - c$, the auditor believes it is some type $v \leq r + c$ and prefers not to audit.

Given these audit probabilities, we must show that $r(v)$ is a maximizing report for the manager. He will not choose any off-equilibrium report that is audited with probability zero because there is the lower equilibrium report $r(V)$ that is also never audited. He will not choose any other off-equilibrium report because these are audited with probability one. Therefore, it remains only to show that manager v prefers $r(v)$ to any other equilibrium report $r(v')$:

$$u(v) \geq [1-(M+1)p(r(v'))][v-r(v')].$$

Using the definition of $u(v')$, this is equivalent to

$$\frac{u(v)}{u(v')} \geq \frac{v - r(v')}{v' - r(v')}$$

or,

$$\exp - \int_{v'}^{v} \frac{1}{t - r(t)}\, dt \geq \exp - \int_{v'}^{v} \frac{1}{t - r(v')}\, dt.$$

But this last inequality holds because $r(v)$ is a nondecreasing step function. This proves sufficiency and completes the proof.

Proof of Proposition 3

Because the audit probability $1/(M+1)$ gives the manager an expected payoff of zero, if (i) is the audit strategy, then for $v > r^*$, the manager will report r^*, the lowest report that is not audited, and for $v \leq r^*$, will be indifferent among all reports $r \in [0, v]$. Therefore, the manager will report r^* if v is in the highest interval $I^* = [r^*, V]$. From Proposition 1(ii), reports $r \in (V - c, V)$ will never be audited, so it must be that $r^* \leq V - c$ and this proves (ii). Also, according to (i), the auditor cannot strictly prefer to audit r^*, so $c \geq E(v \mid [r^*, V] - r^*)$, and (iii) holds for $I = I^*$.

If the reporting strategy is monotone, the set of firm values for which the manager issues the same report must be a connected interval. Because $v = 0$ is constrained to issue report $r = 0$, the set of values that issue that report must be the interval $I_0 = [0, v_1]$, in which $E(v \mid I_0) = c$, in order to induce the auditor to audit $r = 0$. This proves (i) and (ii) for $I = I_0$.

For other intervals, I, the reports issued when the firm value is in I will be audited probabilistically, so the expected misstatement must equal the audit cost, $E(v \mid I) - r = c$, when r is reported by the values in I. This establishes the reporting strategy for $I \neq I^*$. Finally, the manager is constrained never to report more than the value of the firm, so if r is reported by the values in I, this reporting strategy implies that $r = E(v \mid I) - c \leq \inf I$, and (iii) is proved.

It is straightforward to verify the second half of the proposition so its proof is omitted.

Example of Nonmonotone and Mixed Reporting Strategies

Proposition 3 shows that there are an extremely large number of well-behaved step function equilibria: subject to mild conditions, any pooling of types by reporting strategy into connected intervals can be an equilibrium. However, even these conditions are not sufficient, as by relaxing the requirement that reporting strategies be monotone, we can also generate an equilibrium in which the pools are not connected intervals and can then generate mixed reporting strategy equilibria.

Example 8. Let $V = 5$ and $c = 1$. Consider the nonmonotone reporting strategies for the nonconnected pools:

$$r(v) = \begin{cases} 0 \text{ if } v \in [0, 1] \cup [2, 3] \\ 1 \text{ if } v \in (1, 2) \cup (3, 4) \\ 4 \text{ if } v \in [4, 5] \end{cases}$$

and audit strategy

$$p(r) = \begin{cases} 1/(M+1) & \text{if } r < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose v is uniformly distributed within each of these five intervals $[n-1, n]$ but that the probability mass of each interval, q_n, may differ. Set the expected recovery from auditing $r = 0$ and $r = 1$, respectively, to be equal to the audit cost

$$\frac{(.5q_1 + 2.5q_3)}{(q_1 + q_5)} = 1$$

$$\frac{(1.5-1)q_2 + (3.5-1)q_4)}{q_2 + q_4)} = 1,$$

or

$$q_1 = 3q_3$$

$$q_2 = 3q_4$$

respectively. Clearly, these can be chosen small enough so that the auditor is willing to audit $r = 0$ and 1, and there is some positive residual probability for q_5. The out-of-equilibrium beliefs necessary to support this as an equilibrium can be specified easily enough so that any report not used in equilibrium is audited with probability 1.

However, in a preview of the argument used in Proposition 4, these equilibria do not survive plausible restrictions on beliefs off the equilibrium path. Suppose a report of 3.99 is made and audited with a probability p. Obviously, this can only be from types in [3.99,5] because all other types will get a negative payoff even if $p = 0$. Consider some $v \in (4, 5]$. The equilibrium payoff for these types is $v - 4$, because a report of 4 is not audited. Therefore such a type will be indifferent between deviating and playing equilibrium if $(v - 3.99)(1 - (M+1)p) = v - 4$. The quantity $\frac{v-4}{v-3.99}$ is increasing in v, therefore $v = 4$ will strictly prefer deviating for values of p for which higher types would be indifferent, so that the deviation to 3.99 can only come from [3.99,4]. Given this, 3.99 will not be audited, so this equilibrium does not survive D1 (see the proof of Proposition 4) below.

Example 9. Let everything be as in Example 1 and assume specifically that $q_1 = q_2 = .10, q_3 = q_4 = .30$, and $q_5 = .20$. The proof of Proposition 4 shows that there is also an equilibrium with monotone reporting strategies and connected pooling intervals. In particular, it is

$$r(v) = \begin{cases} 0 \text{ if } v \in [0, 2] \\ 2 \text{ if } v \in [2, 4] \\ 4 \text{ if } v \in [4, 5] \end{cases}$$

with the same audit strategy as in Example 8. This is an equilibrium because v is uniform on each of the above three intervals, so $E(v \mid v \in [0, 2]) = 1$ and $E(v \mid v \in [2, 4]) = 3$.

To construct a mixed reporting strategy equilibrium, suppose the manager randomizes among the reporting strategies of these two equilibria according to the flip of a coin, i.e., independently of v. If the auditor observes a report of 1 or 2, he knows which equilibrium is being played, but not if he observes 0 or 4. In either equilibrium, he is willing to audit reports less than 4, so he is still willing even without being able to infer the equilibrium.

Proof of Proposition 4

We first show that D1 eliminates the pooling equilibria of Proposition 2, in which the reporting strategy is monotonic and the equilibrium audit probabilities are strictly decreasing. Let $r_n < r_{n+1}$ be two adjacent equilibrium reports, with audit probabilities $p_n > p_{n+1}$, sent by types in the intervals $[v_n, v_{n+1}]$ and $[v_{n+1}, v_{n+2}]$, respectively, with $v_n < v_{n+1} < v_{n+2}$. Consider an out-of- equilibrium report $r \in (r_n, r_{n+1})$.

The largest audit probability, p, for r such that a type $v \in [v_n, v_{n+1}]$ will weakly prefer r to his equilibrium report r_n is given by

$$1 - (M+1)p = [1 - (M+1)p_n]\frac{v - r_n}{v - r}.$$

Because $r > r_n$, the right-hand side is decreasing in v, so the $v \in [v_n, v_{n+1}]$ with the largest such p is the upper bound, v_{n+1}. A parallel argument establishes that the $v \in [v_{n+1}, v_{n+2}]$ who has the largest such p is the lower bound, v_{n+1}. This shows that among types in the two adjacent pools who make reports just above and just below the out-of-equilibrium report, D1 requires that the auditor believe it is only the boundary type who could have sent the report. A similar argument also shows that the auditor will believe it is this boundary type among all other pools.

We now ask what these beliefs imply about the auditor's choice of audit probabilitites for out-of-equilibrium reports. Consider first the out-of-equilibrium reports $r \in (0, v_1 - c)$ that are common to every equilibrium of Proposition 2, including the maximally informative equilibrium. The auditor must believe all these are sent only by v_1. But as $v_1 - r > c$, he will audit all these with probability one. This is fine, because the manager will then be deterred from sending these reports, as was desired.

Now consider any other out-of-equilibrium report $r \in (r_n, r_{n+1})$, bounded by two pools. By Proposition 2, to convince the auditor to audit r_{n+1} with the required probability we must have

$$r_{n+1} = E(v \mid v \in [v_{n+1}, v_{n+2}]) - c.$$

In particular, $r_{n+1} > v_{n+1} - c$, so there is an out-of-equilibrium report $r \in (r_n, r_{n+1})$, but sufficiently close to r_{n+1}, such that $r > v_{n+1} - c$ or $c > v_{n+1} - r$. Therefore, the auditor will not audit this r, and because it is not audited, types who should have chosen higher reports will now choose here, thus destroying every equilibrium containing a higher pool.

The maximally informative equilibrium also contains no reporting at the highest reports, $r \in [V-c, V]$. D1 requires that the auditor believe type V sent these reports and so he will not audit, as in the lemma, which is as specified for the maximally informative equilibrium.

Turning finally to the commitment-like equilibria of Proposition 3, note that reports in $(r^* - c, r^*]$ must never be chosen by the manager. If one were, then $v = r^*$ is the largest that the firm value could be, because these reports are being audited with probability $1/(M+1)$, and values $v \in (r^*, V]$ would prefer to choose r^* and not be audited. Consequently, the auditor would not audit such a report, a contradiction, and so reports in $(r^* - c, r^*]$ are out-of-equilibrium. By similar arguments as above, D1 requires that the auditor believe that $v = r^*$ sent such report, r; so $v - r < c$ and the auditor will not audit r, thus eliminating these equilibria as well.

Proof of Proposition 5

The $p_i(r)$ have been defined so that the probability of discovery is $p(r)$. Because the manager cares only about this probability of discovery, he will be willing to report $r(v)$. Conversely, suppose that the manager reports according to $r(v)$. Then $p(r)$ solves

$$\begin{aligned} \max \quad & E\left[v - r + p(c' - (v - r)) \mid r\right] \\ = & E\left[v - r \mid r\right] + p[c - \pi E(v - r \mid r)]/\pi \end{aligned}$$

so, $p[c - \pi E(v - r \mid r)]$ is zero for every r. Therefore, whenever, $p(r) > 0$ in perfect auditing, the manager will be willing to choose $p_i(r) > 0$ at any stage in imperfect auditing, and conversely as well. In particular, the manager will be willing to choose a combination of $p_i(r)$ such that

$$p(r) = 1 - \prod_{i=0}^{\infty}(1 - p_i(r)\pi)$$

as is required.

Optimality and Efficiency in Auctions Design: A Survey

Roger L. Zhan

Department of Industrial and Systems Engineering,
University of Florida, Gainesville, Florida, USA
zhan@ufl.edu

Abstract Efficiency and optimality are the two primary and generally conflicting goals in any auction design: the former focuses on the social welfare of the whole seller–bidder system, whereas the latter emphasizes revenue-maximizing on the seller side. In this chapter, we review the auctions design problem based on these two aspects in various information structures and circumstances. The most recent results are collected and analyzed. This chapter tends to complement the survey, *Auction Theory: A Guide to the Literature*, by Klemperer [58] in 1999. The main objective of this chapter is to provide a thorough survey on the current auctions design literature and to synthesize the developed theories underlying traditional auctions with the new elements and phenomena from the emerging and rapidly growing areas, such as online auctions.

Key words: optimal auction design, efficient auction design, online auction, optimal reserve price, asymptotic property

1 Introduction

An auction is a game with partial information where a player's valuation of an object is hidden from other players. It serves as a popular method in resources (goods) allocation to specify a set of rules to determine the winner(s) and the related payments. A typical setting of the auction is that a seller attempts to sell one or more items to a set of bidders. The involved players (seller and bidders) do not have complete information about the value of the items on sale in the sense that they do not know others' values but know their own values, which may or may not be affected by others. All players are assumed to be selfish and payoff-maximizing. The auction theory studies the behavior of the players in this noncooperative environment.

Auctions can be used to sell (allocate) almost all kinds of goods. Governments use them to sell public resources such as radio spectrum licenses and oil drilling rights; firms and individuals use them to sell houses, flowers, antiques, etc. Auction theory itself is an important part of economic theory, and it helps to understand properties of the markets, such as the price formulation and information structures. They also find applications in fields of computer science, such as allocating bandwidth in the communication networks.

Depending on what environment an auction takes place in, we can categorize the auctions in various ways in terms of characteristics of goods, bidders, timing of process, and payment rules. Single-object auctions and multiobject auctions are one of the most obvious and important classifications. Auctions can be oral (bidders hear each other's bids and make counter-offers) or written (bidders submit closed, sealed-bids in writing). In an oral auction, the number of bidders may be known, but in a sealed-bid auction it may be unknown. In terms of the characteristics of the bidders, the auctions can be symmetric and asymmetric, depending on whether the bidders' private values or signals are drawn from a common distribution; they can be a private value or common value model, depending on whether the bidder's valuation will change after others' information is revealed. It is also obvious that the situation will be different if bidders are risk averse instead of risk neutral. In terms of payment rules, they can be classified as a first-price and second-price auction, or discriminatory and uniform payment in the multiobject environment; there exists all-pay auction: the bidder with the highest valuation wins, but all pay their bids. Depending on whether the price increases or decreases, they can be ascending price auction (or English auction) or descending price auction (also known as Dutch auction). In multiobject auctions, researchers consider whether the items to be sold are identical or not (homogeneous or heterogeneous); whether bids are allowed only for individual items or for combinations of items (individual or combinatorial); whether items are auctioned one at a time or all at once (sequential or simultaneous). A detailed framework for classifying and describing single-object auctions is provided in [30]. More technical contents and material can be found in the survey by Klemperer [58], the two-volume of critical papers collection in [57], and in the book by Krishan [59].

In this chapter, we categorize the auctions based on the the primary goal of the auctions for the seller (or designer, planner): optimality and efficiency. Optimal auctions are designed to maximize the expected revenue of the seller by a set of tools including posing a reserve price or charging an entry fee, whereas the objective of efficient auctions is to maximize the social welfare, the sum of the players' surplus. In other words, the efficient design aims to maximize the system welfare, whereas the optimal design aims to maximize the seller's individual revenue. Because optimality and efficiency usually cannot be achieved simultaneously, the auction designers have to make the choice before starting to address the rules. A financial self-interested agent may prefer the optimal auctions, whereas a public agent like the government may prefer the efficient auctions to gain more social welfare. Nevertheless, all agents need to balance optimality and sufficiency to make the auctions practical.

Most of the work ([25,46,84,106], among others) in the earlier stage (1960s and 1970s) deals with the efficient auctions (mechanisms) design. Another important strand concerning the design of optimal auctions has evolved starting with Myerson's [89] and Riley and Samuelson's [95] work in 1981. Since that, much effort has been devoted to both issues in auction theory. This chapter will trace the tradition but survey more recent findings in this field, which are mainly discovered after the year 2000.

The rest of this chapter is organized as follows. The efficient auctions design is presented in Section 2. We survey the optimal auctions design in Section 3, where the revenue equivalence theorem and the role of the reserve price are discussed with more detail. The trade-off between optimality and efficiency is discussed in Section 4. We conclude in Section 5 with a discussion on future research directions.

2 Efficient Auctions Design

The simplest and most thoroughly investigated auction model is the symmetric independent private values (SIPV) model with risk-neutral bidders, in which (1) a single indivisible object is for sale (single-object auction); (2) each bidder knows his own valuation about the object but no one else dose (private value). The unknown valuations are independent and identically distributed (independence, symmetry); (3) all bidders are *ex ante* identical (symmetry); (4) bidders are risk-neutral. This model serves as a prototype in the research of auction theory. All results in the literature either can be derived directly from this model or come from relaxing some assumptions or with some other features and information in different situations.

The standard methodology used in auctions design, like other fields in game theory, is to derive and characterize the (Bayesian) Nash equilibrium, which can be used to predict the bidders' behavior and is a function of the auction's rules and usually involves order statistics. Reference [16] and Chapter 8 in [114] give an excellent introduction to these methodologies.

The efficiency problem in the single-object auction where buyers have private values is theoretically solved in Vickrey's pioneering work [106]. The winner is the buyer whose valuation of the good is highest and will pay the second highest value. Truth-telling is a weakly dominant strategy for any buyer. The mechanism is known as Vickrey's auction,[1] which also applies in the case of multiple identical objects. This format is significantly extended to a mechanism called Vickrey–Clarke–Groves (VCG) mechanism.[2] The VCG mechanism works for homogeneous goods as well as heterogenous ones in private value environment. We do not intend to repeat the mechanism description here. The designer usually needs to solve an optimal assignment problem:

[1] It is also referred to as the second-price sealed-bid auction.
[2] It is introduced in Clarke [25] and Groves [46].

$$\max \sum_{j\in N}\sum_{S\subseteq M} v^j(S)y(S,j) \tag{1}$$

$$\text{s.t.} \sum_{S\ni i}\sum_{j\in N} y(S,j) \leq 1, \forall i \in M \tag{2}$$

$$\sum_{S\subseteq M} y(S,j) \leq 1, \forall j \in N \tag{3}$$

$$y(S,j) = \{0,1\}, \forall S \subseteq M, j \in N \tag{4}$$

where N is the set of bidders, M is the set of objects, S is the subset in M for any partition. $v^j(S)$ is the value of S from bidder j. $y(S,j)$ are binary variables: It is one if S is assigned to j; otherwise, it is zero. This assignment allows the bidders to simultaneously submit "all-or-nothing bids" for combinations of the items being sold. It leads to the combinatorial auction design problem. Pekeč and Rothkopf [92] provide an excellent assessment of the current state of the art on the practical combinatorial auction design.

The efficiency of the VCG mechanism comes from the payment rule that a winning bidder pays the opportunity cost for the items won, that is the difference between the total maximum social welfare with and without the winning bidder himself. Some excellent work on VCG mechanism can be found in [109] and [15].

However, because the efficiency heavily depends on the optimal solution to the assignment problem, which is NP-hard, the VCG mechanism is impractical to implement in general. The computation issues are discussed in [97,109], and [15], among others.

Besides the standard efficient auction formats, there are various variations of the VCG mechanism or the Vickrey auction especially in multiple objects setting. Ausubel makes a great contribution to the development of this aspect in a series of papers: He studies a generalized Vickrey auction in [6], an ascending-bid auction for multiple identical objects in [5], and a dynamic auction for heterogeneous commodities where bidders are permitted to strategically exercise their market power in [4]. The ascending-bid auction in [5] and the dynamic auction in [4] share the same idea: A bidder wins a unit at the price[3] when some other bidders reduce their demands, and the sum of total demands of all the others becomes lower than the number of units available. Immediately thereafter, the total number of units available and the demand of the winner are reduced by one. If no other bidder clinches at this price, the auction continues until all units are allocated. In the case of single-object auction, Maskin [71] shows that under a broad class of assumptions, an ascending auction is efficient if bidders' private signals are single-dimensional, even with asymmetries among bidders and common-value components to valuations. Bikhchandani [14] analyzes the simultaneous sealed auctions and shows that the pure strategy Nash equilibria are efficient in this setting.

[3] Price vector, in the case of heterogeneous objects.

The Vickrey auction and most of the standard simple auction formats fail to be efficient if the private value assumption is dropped [70]. Recently, researchers have made progress to accommodate the interdependent values settings—where one buyer's valuation depends on the private information of another buyer. Researchers generalize the standard simple auction formats, such as the Vickrey auction and English auction, to capture this new feature by modifying the bid report process and payment methods. When each bidder's signal is one-dimensional, the mechanisms in Dasgupta and Maskin [31] and Perry and Reny [93, 94] attain efficiency. In [31], the Vickrey auction is extended to allow bidders to make contingent bids—bids that are the functions of other bidders' valuations—whereas in Perry and Reny's method, the efficiency is achieved by using a two-stage auction. In the first stage, bidders' true signals are revealed in any way, then in the second stage, all possible pairs of bidders are formed, and each pair plays the Vickrey auction.

By separating the winners and the prices determination processes, Izmalkov proposes an efficient mechanism that consists of a number of sequential single-unit English auctions in [50]. Mezzetti [82] presents an efficient mechanism in an environment where first, the final allocation of the goods is determined, second, the bidders observe their own outcome-decision payoffs, and last, final transfers are made. The mechanism incorporates the information of bidders' types and their pre-monetary transfer payoffs. However, in a general mechanism design framework, Jehiel and Moldovanu [52] examine the difficulty/impossiblity in implementing the efficient decision rules when types are multidimensional and continuous, which is natural when there are multiple nonidentical objects. When agents' types are independently distributed, efficient design is possible only when a certain "congruence condition relating the social and private rates of information substitution is satisfied" ([52], p. 1237). This is mainly because the bidder's single-dimensional transfers are not able to extract bidders' multiple dimensional information. However, relaxing the Jehiel–Moldovanu's assumption that bidder's private information is independently distributed, McLean and Postlewaite [78] and Johnson *et al.* [54] show the existence of efficient auction mechanisms. Eső and Maskin [37] also investigate the efficient design issue when signals are multidimensional.

The inefficiency of standard auctions even exists in the sale of homogeneous objects as illustrated in [5] and [8]. Particularly, in an ascending multiobject auction, a bidder with large demands has an incentive to reduce demand in order to pay less for those units he won. Ausubel and Cramton [8] demonstrate the inefficiency in various auction settings: flat demand and downward-sloping demand, independent private values and correlated values, and uniform pricing and pay-your-bid pricing. The inefficiency results are demonstrated and measured in many experimental studies that are not included in this survey.

Efficient mechanism design also has been studied under some other new settings. Jackson and Kremer [51] compare the auction formats in a competitive environment where large numbers of agents compete for a limited supply of resources. Chen *et al.* [22] consider the multiunit efficient auctions for procurement in the supply chain settings that incorporate the transportation costs into the auctions.

3 Optimal Auctions Design

The problem of designing an auction that maximizes the seller's revenue (optimal auction) is usually more challenging, especially in the case of multiple heterogeneous objects. However, Myerson's revelation principle (see [89]) allows one, without loss of generality, to restrict attention to a direct-revelation mechanism that is used to prove a surprising result called "revenue equivalence theorem." His paper provides the framework that has become the paradigm for the research of optimal mechanism design for selling one or more homogeneous objects.

In this section, we will review the results on the revenue equivalence theorem under different circumstances. In the case that this theorem does not hold, we compare different auction formats based on the revenue-generating ability. Because the reserve price is one of the main tools to maximize the seller's revenue, we present the main results in this aspect. A buyout price evolving from online auctions is also discussed. We will discuss the trade-off between optimality and efficiency in auctions design in next section.

3.1 Revenue Equivalence Theorem

Vickrey ([105, 106]) examines the possibility that different auction formats might give the same expected revenue to a seller of one or more homogeneous objects. The results are significantly generalized by Myerson [89] and Riley and Samuelson [95]. Myerson is considered to provide a more general treatment.

Consider an auction in which a seller is selling one or more homogeneous objects[4] to N bidders. Each buyer i has unit demand and he values the object at v_i, which is private information, but it is common knowledge that valuations are independent identically distributed with *cdf* F and *pdf* f on the support $[0, \bar{v}]$. v_i is used to represent the ith bidder's type. v_{-i} is the types of all bidders but the ith. That is, $v_{-i} = (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_N)$.

The buyer who is assumed to be risk neutral tends to maximize his own expected surplus, defined as his valuation less the expected amount paid to the seller. Although the auction mechanism consists of both allocation (p) and payment rules (x), the seller's expected revenue can be expressed only as a function of the allocation rule and the expected surplus for the buyer with type zero. More specifically, the allocation function (p) for bidder i can be expressed as

[4] Myerson's original work only consider the single-object case. The extending result about multiple homogeneous objects is due to Maskin and Riley [72] and Engelbrecht-Wiggans [33].

$$p_i(v_i, v_{-i}) = \begin{cases} 1, \text{ if buyer } i \text{ is awarded a unit,} \\ 0, \text{ otherwise.} \end{cases}$$

If the functions $p_i(\cdot, v_{-i})$ is increasing in v_i, and let $U(p, x, 0)$ be the expected surplus for the buyer with type zero, then the seller's expected revenue is given by

$$\mathbf{E}_{v_i, v_{-i}} \left[\sum_{i=1}^{N} \left(v_i - \frac{1 - F(v_i)}{f(v_i)} p_i(v_i, v_{-i}) \right) \right] - N \times U(p, x, 0). \tag{5}$$

It follows that all mechanisms, which result in the same allocation p for each realization of v, yield the same expected revenue provided $U(p, x, 0)$ are the same. This condition can be easily satisfied in most auction formats. This result is called *Revenue Equivalence Theorem*.[5] For example, all the "standard" auctions, such as the first-price and second-price auction, for selling one object yield the same expected revenue as they all award the object to the buyer with highest valuation, and the bidder with the lowest possible type receives zero expected revenue.

Because this result is so fundamental and mathematically nice, researchers explore different ways to derive, extend, and understand it. Engelbrecht-Wiggans [33] extends the original result to multiple-unit auctions. Bulow and Roberts [18] reinterpret it in the language of microeconomic theory, specifically, the logic of the marginal revenue versus the marginal cost.

Furthermore, the result also provides a way to design an optimal mechanism. This is achieved by simply choosing the allocation rule $p(v)$ that maximizes $\sum_{i=1}^{N} \left(v_i - \frac{1-F(v_i)}{f(v_i)} p_i(v_i, v_{-i}) \right)$ in (5). That is, the mechanism will award the object according the rank of so-called virtual value $v_i - \frac{1-F(v_i)}{f(v_i)}$, provided this value is above zero. If the virtual value function is increasing in v_i, we can simply define the reserve price as $\left\{ v_0 : v_0 - \frac{1-F(v_0)}{f(v_0)} = 0 \right\}$. Here we also can see the optimal auction is in conflict with the efficient auction. The seller may keep the object even if his value is zero and the allocation rule is based on the rank of virtual values not the actual values from bidders.

By relaxing one or more assumptions underlying hypotheses of the revenue equivalence theorem, researchers either show that the theorem continues to hold or rank the auction format in terms of revenue-generating ability in more realistic settings. Milgrom and Weber [83] rank the classic auction formats, showing that the second-price auction produces no less revenue than the first-price auction if the bidders valuations are dependent and the signals are affiliated. A complete list of papers published before the year 2000 can be found in Klemperer's extensive survey [58] under the sections such as

[5] The result applies to the private-value models and to a more general common-value models provided bidders' types are independent.

"Risk aversion"[6] and "Correlation and affiliation."[7] The impact of the bidding behavior on the revenue equivalence theorem is surveyed in Shen and Su [100].

If the single-object assumption is relaxed, extra efforts are needed to develop the theories. Maskin and Riley [72] study the optimal selling procedure when the buyer has a multiunit demand. Palfrey [90] considers the bundling decisions that depend on the number of buyers. If the objects are heterogeneous, the substitution and complementarity among the objects can further complicate the problem. The typical approach in the literature is to analyze the simplified models thoroughly, usually the one with two objects or bidders, and get the insight from these models. Rochet and Stole [96] summarize some revenue comparison results in the multiobject setting. Levin [63] identifies the optimal auction formats when all objects are complement for bidders. Armstrong [2] considers the revenue equivalence in the heterogeneous case with restriction to a binary distribution.

Fibich *et al.* [41], using the perturbation analysis, show a weak version of the revenue equivalence theorem in the sense that the revenues differences across the auction mechanisms are only of second order in asymmetric auctions, where the distribution function of each bidder undergoes a mild independent change beyond the initially identical distributed function. Eső and Futó [36] show that for every incentive compatible selling mechanism, there exists a mechanism that provides deterministically the same revenue when the seller is risk averse while bidders are risk neutral with independent private values.

The proof of the revenue equivalence theorem largely relies on the Revelation Principle developed in [89], however, Szentes [102] presents a method to transfers an equilibrium strategy profile from the first-price auction to the second-price auction. Using this method, he shows the revenue equivalence when the independence and the private value assumptions of Vickrey's classic model are dropped.

3.2 Optimal Auctions Design and Revenue Comparison

Another stream in literature is to find a particular auction format to maximize the seller's revenue or to compare different formats in some nonstandard but practical environments.

Krishna and Rosenthal [61] attempt to make revenue comparisons in the case that one bidder is local who is interested in receiving a single item; the other is global, who has the valuation for the entire set that are superadditive. An experiment is provided in [27] to compare a particular simultaneous multiround auction with a particular multiround combinatorial auction. Eliaz *et al.* [32] and Goeree *et al.* [44] study how to increase revenue through the

[6] Relaxing the risk neutral assumption.
[7] Relaxing the private information assumption.

right-to-choose auctions[8] to sell multiple heterogenous goods. Mondere and Tenneholtz [88] address the optimal auction format in computational environments such as the Internet. They present an upper bound on the revenue with a fixed number of risk-averse participants and discuss the conditions that make standard auctions to approach the theoretical bound.

Bundling, i.e., subcollections of objects, is quite commonly used to sell multiple objects. Manelli and Vincent [68] identify environments in which bundling is optimal within the class of all incentive compatible and individually rational mechanisms. Avery and Hendershott [11] propose a probabilistic bundling method in their optimal auctions for multiple products. Like in [4], they use a binary distribution to simplify the analysis. Under the restriction of the binary distribution, Eső [35] studies the expected revenue maximization mechanism with the risk-averse bidders and the correlated private values. He finds that the sufficiently strong correlation of the valuations helps the seller to extract all rents even from the risk-averse bidders, which complements the result in [26].

Without the known demand distribution, Segal [103] investigates how to set a price for each buyer on the basis of the demand distribution inferred statistically from other buyers' bids. Relaxing the single-dimensional private information, Fang and Morris [38] show that the revenue equivalence between the first-price and second-price auctions breaks down and there is no definite revenue ranking in the multidimensional private value auction environments, where each bidder observes his own private valuation as well as noisy signals about his opponents' private valuation. With resale opportunity, the Myerson allocation [89] cannot achieve optimality. Haile [48] analyzes the resale game when new information about the good on sale is revealed after a standard auction, whereas Zheng [116] considers the model without the new information appearing and focuses on the tension between a seller's manipulation and the counterbalance from resale. On the other hand, Skreta [101] studies a two-period model where a seller reserves the right to sell the unsold object. In this model, the seller will implement two revenue maximizing mechanisms with possible different buyer-specific cut-offs in two periods.

The optimal auction design in a multiple objects model with multidimensional bidder valuations is challenging, because it is difficult to represent the monotonicity relationship in the incentive-compatibility condition, which is fairly easy in a single-dimensional case, as shown in [89]. The breakthrough is done by McAfee and McMillan [77] where they use a system of partial differential equations to express the incentive-compatibility constraint to study the nonlinear pricing mechanism. The latter, extended in [3], is adapted by Zheng [115] to provide an explicit formula of optimal auctions in the multi-dimensional setting. Manelli and Vincent [69] instead develop the procedure to check the relationship between a mechanism and an extreme point of the

[8] The winner of each stage has the right to choose one of the available objects. It creates competition when the heterogenous objects are put together.

feasible set in a similar multidimensional setting. Malakhov and Vohra [67] study the optimal auction design with single and multidimensional types by interpreting the problem through a linear program that is an instance of a parametric shortest path problem on a lattice.

3.3 Reserve Price and Buyout Price

Setting a reserve price or charging an entry fee is one of the main ways to implement the optimal auctions. Beginning with Myerson's result [89] on the risk-neutral independent-private-values (IPV) auction model, researchers have extensively explored the role of the reserve price in different circumstances.

In Myerson's original model, the reserve price is independent of the number of bidders, which is notable but puzzling because it conflicts with common practice. Levin and Smith [64] show that the seller's optimal reserve price converges to his true value as the number of bidders grows when bidders' valuation are correlated. The similar phenomena exists in a ranked-item environment as discussed in Feng *et al.* [40]. Ausubel and Cramton [10] generalize the Vickrey auction to allow for reserve pricing in a multiunit auction with interdependent values.

Engelbrecht-Wiggans [34] particularly studies the screening effect on the number of bidders from the reserve price. In his two simple examples, he illustrates that a reduced pool of bidders may outweigh any benefits from a reserve price. Equilibrium reserve prices in sequential ascending auctions is studied in [21], where Caillaud and Mezzetti analyze the bidder behavior in two ascending-price auctions with the seller setting the reserve price before the beginning of each auction.

The reserve price is valuable to a seller but she may have to commit not selling the object below the reserve price in any mechanism. If without this commitment, Menezes and Ryan [81] find that the value of the reserve price may be completely undermined if the seller lacks enough power to negotiate with the highest bidder if the reserve price is not met. However, if potential bidders in an auction have to incur a cost to prepare their bids and thus to learn their valuations, it is optimal to impose an announced reserve price in the first-price auction without commitment as suggested in Burguet and Sákovics [19].

When a seller wants to auction off many similar items over a long period of time, McAfee *et al.* [76] propose methods on how to set reserve price by using data from past auctions and information about the subsequent sales of unsold items.

whereas the literature on the reserve price is large, existing research work on buyout prices is recent and relatively limited. Buyout prices allow bidders to instantaneously purchase an item at a specified price, mainly through online auctions. The comprehensive survey by Klemperer [58] makes no mention of buyout prices, whereas Lucking-Reiley [66] observes its use in online auction practices but points out the lack of the theoretical literature in 2000.

Among the first set of the theoretical papers on buyout prices, Kirkegaard and Overgaad [56], Budish and Takeyama [17], Mathews [74], and Hidvegi *et al.* [49] investigate the reason why the buyout price may increase the seller's revenue under various circumstances, such as the information revelation in the sequential auctions [56]. Gallien and Gupta [45] extensively study the role of the time sensitivity of both sellers and buyers in using the temporary and permanent buyout options. A similar model can also be found in Caldentey and Vulcano [28], which studies a "dual auction and list price channel" resembling an auction with buyout prices. The empirical study can be found in Wang *et al.* [112]. Their main finding is that the combined mechanism can increase both customers' utility and sellers' profit under certain conditions. The result is empirically tested using data collected from eBay.

4 Optimality Versus Efficiency: Trade-off and Asymptotic Properties

No mechanism designers can solely focus on either optimality or efficiency in order to put the auctions in practice. They have to find a way to balance these two aspects. However, not much research has been done in this area. Krishna and Perrey [60] prove that in the independent private values model, the VCG mechanism maximizes the revenue for the seller among all auctions that implement the efficient allocation. Williams has shown the similar result in [111].

It is well-known that optimality and efficiency conflict in general. Jehiel and Moldovanu [53] show that there is a conflict between revenue maximization and efficiency in multiobject auctions even with symmetric bidders.

With the perfect resale option, Ausubel and Cramton's finding [7] reveals that the efficiency is regained in an optimal auction: the seller will assign goods to those with the highest valuation. Likhodedov and Sandholm [65] study the trade-off between the optimality and efficiency objectives by designing a deterministic dominant strategy mechanism that maximizes expected social welfare (efficiency) subject to a minimum constraint on the seller's expected payoff (optimality).

However, when the number of bidders becomes very large (approaches infinity), most standard auctions approach optimality and efficiency simultaneously regardless of whether the bidders are symmetric or asymmetric, risk-neutral or risk-averse. See, for example, Swinkels [104] and Bali and Jackson [12] for a more detailed analysis. Fibich *et al.* [42] provide more accurate characteristics of how large should the number of bidders be and the converge rates. Similar work can also be found in Monderer and Tennenholtz [87] in which they prove that Clarke mechanism is an asymptotically optimal multiobject auction.

5 Conclusion and Future Directions

This chapter provides an overview of the main results on auctions design from the year 2000. It focuses on the two themes in auctions design: efficiency and optimality. This survey can provide readers a comprehensive guide to the ongoing research in this field.

Although the theoretical and empirical research bases for traditional auctions are well established, some current results, in particular, those restricting to the assumptions of binary distributions, two bidders, or two objects, are worth further investigation to handle more general cases.

Another important direction is to continue to incorporate some new elements arising from real-life applications, especially interacting with the supply chain management and the online environment.

Recently, van Ryzin *et al.* [98, 107] studied optimal auction design in the setting of inventory management. More research is needed to understand how auctions can be embedded in the supply chain management, not limiting in procurement, and how they can potentially redefine the supply chain with the help of the Internet.

The features of the online auctions are usually investigated separately in the current research. This trend is expected to continue to grow as we still have not fully gained the understanding of elements involved in the online auctions, such as unknown demand, fairness, reputation, and false identity. The ultimate goal is to synthesize the current results and data from the current online auction practice to design future trading mechanisms.

6 Bibliographic Notes

The double auctions, which are completely ignored here, are reviewed in Friedman and Rust [43]. Their recent development can be found in Chu and Shen [23, 24] and references therein.

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Part II

Multiobjective, KKT, Bilevel

Solution Concepts and an Approximation Kuhn–Tucker Approach for Fuzzy Multiobjective Linear Bilevel Programming

Guangquan Zhang, Jie Lu, and Tharam Dillon

Faculty of Information Technology, University of Technology, Sydney, NSW, Australia
{zhangg, jielu, tharam}@it.uts.edu.au

Abstract When modeling an organizational bilevel decision problem, uncertainty often appears in the parameters of either objective functions or constraints of the leader and the follower. Furthermore, the leader and the follower may have multiple objectives to consider simultaneously in their decision making. To deal with the two issues, this study builds a fuzzy multiobjective linear bilevel programming (FMOLBLP) model. It then proposes the definitions of optimal solutions and related theorems for solving a FMOLBLP problem. Based on these theorems, it develops an approximation Kuhn–Tucker approach to solve the FMOLBLP problem where fuzzy parameters can be described by any form of membership functions of fuzzy numbers. An example illustrates the applications of the proposed approach.

Key words: bilevel programming, bilevel decision making, fuzzy optimization, fuzzy numbers, Kuhn–Tucker approach, multiobjective programming

1 Introduction

The bilevel decision making structure appears naturally in many critical resource planning, management, and policymaking areas, including tourism resource planning, water resource management, financial planning, healthcare planning, land-use planning, production planning (coordination of multidivisional firms, network facility location), transportation planning (network design, trip demand estimation), and power market planning [1–3, 13, 23]. The decision maker at the upper level is termed the leader and at the lower level the follower. When the leader at the upper level attempts to optimize his or her objective, the follower at the lower level tries to find an optimized strategy according to each of the possible decisions made by the leader [4, 7].

Bilevel decision making is handled by the bilevel programming (BLP) technique, introduced by Von Stackelberg [21]. BLP has been applied with remarkable success in various domains [14–17]. The majority of the research on BLP has centered on the linear version of the problem [8–10,23]. A set of approaches and algorithms for the linear BLP, such as the well-known Kuhn–Tucker approach [4,5], Kth-best approach [6,8], and Branch-and-bound algorithm [11], have had successful applications.

There are two situations challenges that still exist in the BLP approaches. The first one is the multiobjective issue. In the practice, multiple conflicting objectives often need to be considered simultaneously by the leader, and/or the follower, for critical resource planning problems. For example, a coordinator of a multidivision firm considers three objectives in making an aggregate production plan: maximise net profits, maximise quality of products, and maximise worker satisfaction. The three objectives could be in conflict with each other but must be considered simultaneously. Any improvement in one objective may be achieved only at the expense of others. The normal multiobjective decision-making problem has been well researched by many researchers such as Hwang and Masud [12]. But in a bilevel model, the selection of an alternative solution for the leader is affected by his or her follower's optimal reaction. Therefore, how to find an optimal solution for the leader that has multiple objectives under the consideration of both its constraints and its follower is a new issue, called a multiobjective linear bilevel programming (MOLBLP) problem.

The second one is the parameter uncertainty issue. The parameters of a bilevel model are sometimes hard to fix at some crisp values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters. The possible values of these parameters are often only imprecisely or ambiguously known to the experts who establish this model. With this observation, it would be certainly more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data that can be represented by means of fuzzy sets [24]. A bilevel problem in which the parameters, either in objective functions and/or in constraints of the leader and/or the follower, are described by fuzzy values is called a fuzzy linear bilevel programming (FLBLP) problem. This study will deal with both issues in the linear version, called a fuzzy multiobjective linear bilevel program (FMOLBLP).

Based on our study reported in [19] and [20], we have first developed an extended Kuhn–Tucker approach [27] to solve FBLP problems. This chapter extends our previous research by developing an approximation Kuhn–Tucker approach to solve FMOLBLP problems. The proposed approximation Kuhn–Tucker approach allows the fuzzy parameters of a FMOLBLP model to be any form of membership functions.

After the introduction, Section 2 reviews related definitions, theorems and properties of fuzzy numbers, and the Kuhn–Tucker approach. A general fuzzy numbers based approximation Kuhn–Tucker approach for solving FMOLBLP problems is presented in Section 3. A numerical example is shown in Section 4 for illustrating the proposed approach. Conclusions and proposals for further study are discussed in Section 5.

2 Preliminaries

In this section, we present some basic concepts, definitions, and theorems that are to be used in the subsequent sections. The work presented in this section can also be found from our recent papers in [25–27].

2.1 Fuzzy Numbers

Let R be the set of all real numbers, R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ be any two vectors, where x_i, $y_i \in R$, $i = 1, 2, \ldots, n$, and T denotes the transpose of the vector. Then we denote the inner product of x and y by $\langle x, y\rangle$. For any two vectors x, $y \in R^n$, we write $x \geqq y$ iff $x_i \geqslant y_i$, $\forall i = 1, 2, \ldots, n$; $x \geq y$ iff $x \geqq y$ and $x \neq y$; $x > y$ iff $x_i > y_i$, $\forall i = 1, 2, \ldots, n$.

Definition 1. *A fuzzy number $\tilde{a}$ is defined as a fuzzy set on R, whose membership function $\mu_{\tilde{a}}$ satisfies the following conditions:*

1. *$\mu_{\tilde{a}}$ is a mapping from R to the closed interval $[0, 1]$;*
2. *it is normal, i.e., there exists $x \in R$ such that $\mu_{\tilde{a}}(x) = 1$;*
3. *for any $\lambda \in (0, 1]$, $a_\lambda = \{x; \mu_{\tilde{a}}(x) \geqslant \lambda\}$ is a closed interval, denoted by $[a^L_\lambda, a^R_\lambda]$.*

Let $\mathscr{F}(R)$ be the set of all fuzzy numbers. By the decomposition theorem of fuzzy sets, we have

$$\tilde{a} = \bigcup_{\lambda \in [0,1]} \lambda [a^L_\lambda, a^R_\lambda], \tag{1}$$

for every $\tilde{a} \in \mathscr{F}(R)$.

Let $\mathscr{F}^*(R)$ be the set of all finite fuzzy numbers on R.

Theorem 1. *Let $\tilde{a}$ be a fuzzy set on R, then $\tilde{a} \in \mathscr{F}(R)$ iff $\mu_{\tilde{a}}$ satisfies*

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & x \in [m, n], \\ L(x), & x < m, \\ R(x), & x > n. \end{cases}$$

where $m, n \in R$, $L(x)$ is a right-continuous monotone increasing function, $0 \leqslant L(x) < 1$ and $\lim_{x \to -\infty} L(x) = 0$, $R(x)$ is a left-continuous monotone decreasing function, $0 \leqslant R(x) < 1$ and $\lim_{x \to \infty} R(x) = 0$.

Corollary 1. *For every $\tilde{a} \in \mathscr{F}(R)$ and λ_1, $\lambda_2 \in [0, 1]$, if $\lambda_1 \leqslant \lambda_2$, then $\tilde{a}_{\lambda_2} \subseteq \tilde{a}_{\lambda_1}$.*

Definition 2. *For any $\tilde{a}$, $\tilde{b} \in \mathscr{F}(R)$ and $0 \leqslant \lambda \in R$, the sum of $\tilde{a}$ and $\tilde{b}$ and the scalar product of λ and $\tilde{a}$ are defined by the membership functions*

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup \min_{t=u+v} \{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \tag{2}$$

$$\mu_{\tilde{a}-\tilde{b}}(t) = \sup \min_{t=u-v} \{\mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v)\}, \tag{3}$$

$$\mu_{\lambda\tilde{a}}(t) = \sup_{t=\lambda u} \mu_{\tilde{a}}(u). \tag{4}$$

Theorem 2. *For any $\tilde{a}, \tilde{b} \in \mathscr{F}(R)$ and $0 \leqslant \alpha \in R$,*

$$\begin{aligned}
\tilde{a} + \tilde{b} &= \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L + b_\lambda^L, a_\lambda^R + b_\lambda^R], \\
\tilde{a} - \tilde{b} &= \tilde{a} + \left(-\tilde{b}\right) = \bigcup_{\lambda\in[0,1]} \lambda[a_\lambda^L - b_\lambda^R, a_\lambda^R - b_\lambda^L], \\
\alpha\tilde{a} &= \bigcup_{\lambda\in[0,1]} \lambda[\alpha a_\lambda^L, \alpha a_\lambda^R].
\end{aligned}$$

Definition 3. *Let $\tilde{a}_i \in \mathscr{F}(R)$, $i = 1, 2, \ldots, n$. We define $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$*

$$\mu_{\tilde{a}} : R^n \to [0,1]$$

$$x \mapsto \bigwedge_{i=1}^{n} \mu_{\tilde{a}_i}(x_i),$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, and $\tilde{a}$ is called an n-dimensional fuzzy number on R^n. If $\tilde{a}_i \in \mathscr{F}^*(R)$, $i = 1, 2, \ldots, n, \tilde{a}$ is called an n-dimensional finite fuzzy number on R^n.

Let $\mathscr{F}(R^n)$ and $\mathscr{F}^*(R^n)$ be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on R^n, respectively.

Proposition 1. *For every $\tilde{a} \in \mathscr{F}(R^n)$, $\tilde{a}$ is normal.*

Proposition 2. *For every $\tilde{a} \in \mathscr{F}(R^n)$, the λ-section of $\tilde{a}$ is an n-dimensional closed rectangular region for any $\lambda \in [0,1]$.*

Proposition 3. *For every $\tilde{a} \in \mathscr{F}(R^n)$ and λ_1, $\lambda_2 \in [0,1]$, if $\lambda_1 \leqslant \lambda_2$, then $a_{\lambda_2} \subset a_{\lambda_1}$.*

Definition 4. *For any n-dimensional fuzzy numbers $\tilde{a}, \tilde{b} \in \mathscr{F}(R^n)$, we define*

1. *$\tilde{a} \gtrapprox \tilde{b}$ iff $a_\lambda^L \geqq b_\lambda^L$ and $a_\lambda^R \geqq b_\lambda^R$, $\lambda \in (0,1]$;*
2. *$\tilde{a} \succsim \tilde{b}$ iff $a_\lambda^L \geq b_\lambda^L$ and $a_\lambda^R \geq b_\lambda^R$, $\lambda \in (0,1]$;*
3. *$\tilde{a} \succ \tilde{b}$ iff $a_\lambda^L > b_\lambda^L$ and $a_\lambda^R > b_\lambda^R$, $\lambda \in (0,1]$.*

We call the binary relations $\gtrapprox$, $\succsim$, and $\succ$ a fuzzy max order, a strict fuzzy max order, and a strong fuzzy max order, respectively.

2.2 The Extended Kuhn–Tucker Approach for Linear Bilevel Programming

Let write a linear programming (LP) as follows.

$$\begin{aligned} &\min f(x) = cx \\ &\text{subject to } Ax \geq b \\ &\qquad\qquad\quad x \geq 0, \end{aligned}$$

where c is an n-dimensional row vector, b an m-dimensional column vector, A an $m \times n$ matrix with $m \leqslant n$, and $x \in R^n$.

Let $\lambda \in R^m$ and $\mu \in R^n$ be the dual variables associated with constraints $Ax \geq b$ and $x \geq 0$, respectively. Bard [4] gave the following proposition.

Proposition 4. (See [4]) *A necessary and sufficient condition that* (x^*) *solves the above LP is that there exist (row) vectors* λ^*, μ^* *such that* (x^*, λ^*, μ^*) *solves:*

$$\begin{aligned} &\lambda A - \mu = -c \\ &Ax - b \leq 0 \\ &\lambda(Ax - b) = 0 \\ &\mu x = 0 \\ &x \geq 0, \lambda \geq 0, \mu \geq 0 \end{aligned}$$

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F: X \times Y \to R^1$, and $f: X \times Y \to R^1$, a linear BLP problem is given by Bard [4]:

$$\min_{x \in X} F(x, y) = c_1 x + d_1 y \tag{5a}$$

$$\text{subject to } A_1 x + B_1 y \leq b_1 \tag{5b}$$

$$\min_{y \in Y} f(x, y) = c_2 x + d_2 y \tag{5c}$$

$$\text{subject to } A_2 x + B_2 y \leq b_2, \tag{5d}$$

where $x \geqslant 0$, $y \geqslant 0$, c_1, $c_2 \in R^n$, d_1, $d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p \times n}$, $B_1 \in R^{p \times m}$, $A_2 \in R^{q \times n}$, $B_2 \in R^{q \times m}$.

Let $u \in R^p$, $v \in R^q$ and $w \in R^m$ be the dual variables associated with constraints (5b), (5d), and $y \geq 0$, respectively. We have the following theorem.

Theorem 3. *A necessary and sufficient condition that* (x^*, y^*) *solves the linear BLP problem* (5) *is that there exist (row) vectors* u^*, v^*, *and* w^* *such that* $(x^*, y^*, u^*, v^*, w^*)$ *solves:*

$$\min F(x, y) = c_1 x + d_1 y \tag{6a}$$

$$\text{subject to } A_1 x + B_1 y \leq b_1 \tag{6b}$$

$$A_2 x + B_2 y \leq b_2 \tag{6c}$$

$$u B_1 + v B_2 - w = -d_2 \tag{6d}$$

$$u(b_1 - A_1 x - B_1 y) + v(b_2 - A_2 x - B_2 y) + wy = 0 \tag{6e}$$

$$x \geq 0, y \geq 0, u \geq 0, v \geq 0, w \geq 0. \tag{6f}$$

Theorem 3 means that the most direct approach to solve (5) is to solve the equivalent mathematical program problem given in (6). One advantage that it offers is that it allows a more robust model to be solved without introducing any new computational difficulties.

3 Fuzzy Multiobjective Linear Bilevel Programming

In this section, we propose a formulation and necessary and sufficient condition for solution of the fuzzy multiobjective linear bilevel programming problem.

3.1 The FMOLBLP Model

Definition 5. *A topological space is compact if every open cover of the entire space has a finite subcover. For example,* $[a, b]$ *is compact in* R *(the Heine–Borel theorem, See* [22]*).*

Consider the following FMOLBLP problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to \mathscr{F}^*(R^s)$, and $f : X \times Y \to \mathscr{F}^*(R^t)$,

$$\min_{x \in X} F(x, y) = \left(\tilde{c}_{11}x + \tilde{d}_{11}y, \tilde{c}_{21}x + \tilde{d}_{21}y, \ldots, \tilde{c}_{s1}x + \tilde{d}_{s1}y\right)^T \tag{7a}$$

$$\text{subject to } \tilde{A}_1x + \tilde{B}_1y \precsim \tilde{b}_1 \tag{7b}$$

$$\min_{y \in Y} f(x, y) = \left(\tilde{c}_{12}x + \tilde{d}_{12}y, \tilde{c}_{22}x + \tilde{d}_{22}y, \ldots, \tilde{c}_{t2}x + \tilde{d}_{t2}y\right)^T \tag{7c}$$

$$\text{subject to } \tilde{A}_2x + \tilde{B}_2y \precsim \tilde{b}_2 \tag{7d}$$

where $\tilde{c}_{i1}$, $\tilde{c}_{j2} \in \mathscr{F}^*(R^n)$, $\tilde{d}_{i1}$, $\tilde{d}_{j2} \in \mathscr{F}^*(R^m)$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$, $\tilde{b}_1 \in \mathscr{F}^*(R^p)$, $\tilde{b}_2 \in \mathscr{F}^*(R^q)$, $\tilde{A}_1 = (\tilde{a}_{ij})_{p \times n}$, $\tilde{a}_{ij} \in \mathscr{F}^*(R)$, $\tilde{B}_1 = \left(\tilde{b}_{ij}\right)_{p \times m}$, $\tilde{b}_{ij} \in \mathscr{F}^*(R)$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q \times n}$, $\tilde{e}_{ij} \in \mathscr{F}^*(R)$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q \times m}$, $\tilde{s}_{ij} \in \mathscr{F}^*(R)$.

For the sake of simplicity, we set

$$\tilde{X} \times \tilde{Y} = \left\{(x, y)\,;\, \tilde{A}_1x + \tilde{B}_1 \precsim \tilde{b}_1, \tilde{A}_2x + \tilde{B}_2y \precsim \tilde{b}_2\right\}$$

and assume that $\tilde{X} \times \tilde{Y}$ is compact. In a FMOLBLP problem, for each $(x, y) \in \tilde{X} \times \tilde{Y}$, the value of the objective functions $F(x, y) = (F_1(x, y), F_2(x, y), \ldots, F_s(x, y))$ and $f(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_t(x, y))$ of leader and follower are s-dimensional and t-dimensional fuzzy numbers, respectively. Thus, we introduce the following concepts of optimal solutions to FMOLP problems.

Definition 6. *A point* $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ *is said to be a complete optimal solution to the FMOLBLP problem if it holds that* $F(x^*, y^*) \precsim F(x, y)$ *and* $f(x^*, y^*) \precsim f(x, y)$ *for all* $(x, y) \in \tilde{X} \times \tilde{Y}$.

Definition 7. *A point* $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ *is said to be a Pareto optimal solution to the FMOLBLP problem if there does not exist* $(x, y) \in \tilde{X} \times \tilde{Y}$ *such that* $F(x^*, y^*) \succsim F(x, y)$ *and*$f(x^*, y^*) \succsim f(x, y)$ *holds.*

Definition 8. *A point* $(x^*, y^*) \in \tilde{X} \times \tilde{Y}$ *is said to be a weak Pareto optimal solution to the FMOLBLP problem if there is no* $(x, y) \in \tilde{X} \times \tilde{Y}$ *such that* $F(x^*, y^*) \succ F(x, y)$ *and* $f(x^*, y^*) \succ f(x, y)$ *holds.*

Associated with the FMOLBLP problem, we now consider the following multiobjective leader multiobjective follower linear bilevel programming (MOLBLP) problem:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to \mathscr{F}^*(R^s)$, and $f : X \times Y \to \mathscr{F}^*(R^t)$,

$$\min_{x \in X} (F(x, y))_\lambda^{L(R)} = \left((F_1(x, y))_\lambda^L, (F_1(x, y))_\lambda^R, \cdots (F_s(x, y))_\lambda^L, (F_s(x, y))_\lambda^R \right)^T, \lambda \in [0, 1] \tag{8a}$$

$$\text{subject to } A_1{}_\lambda^L x + B_1{}_\lambda^L y \leqq b_1{}_\lambda^L, A_1{}_\lambda^R x + B_1{}_\lambda^R y \leqq b_1{}_\lambda^R, \quad \lambda \in [0, 1] \tag{8b}$$

$$\min_{y \in Y} (f(x, y))_\lambda^{L(R)} = \left((f_1(x, y))_\lambda^L, (f_1(x, y))_\lambda^R, \cdots (f_t(x, y))_\lambda^L, (f_t(x, y))_\lambda^R \right)^T, \lambda \in [0, 1] \tag{8c}$$

$$\text{subject to } A_2{}_\lambda^L x + B_2{}_\lambda^L y \leqq b_2{}_\lambda^L, A_2{}_\lambda^R x + B_2{}_\lambda^R y \leqq b_2{}_\lambda^R, \quad \lambda \in [0, 1] \tag{8d}$$

where $(F_i(x, y))_\lambda^L = c_{i1}{}_\lambda^L x + d_{i1}{}_\lambda^L y$ and $(f_j(x, y))_\lambda^L = c_{j2}{}_\lambda^L x + d_{j12}{}_\lambda^L y$, $\lambda \in [0, 1]$, $c_{i1}{}_\lambda^L$, $c_{i1}{}_\lambda^R$, $c_{j2}{}_\lambda^L$, $c_{j2}{}_\lambda^R \in R^n$, $d_{i1}{}_\lambda^L$, $d_{i1}{}_\lambda^R$, $d_{j2}{}_\lambda^L$, $d_{j2}{}_\lambda^R \in R^m$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$, $b_1{}_\lambda^L, b_1{}_\lambda^R \in R^p$, $b_2{}_\lambda^L, b_2{}_\lambda^R \in R^q$, $A_1{}_\lambda^L = \left(a_{ij}{}_\lambda^L\right)$, $A_1{}_\lambda^R = \left(a_{ij}{}_\lambda^R\right) \in R^{p \times n}$, $A_2{}_\lambda^L = \left(e_{ij}{}_\lambda^L\right)$, $A_2{}_\lambda^R = \left(e_{ij}{}_\lambda^R\right) \in R^{q \times n}$, $B_1{}_\lambda^L = \left(b_{ij}{}_\lambda^L\right)$, $B_1{}_\lambda^R = \left(b_{ij}{}_\lambda^R\right) \in R^{p \times m}$, $B_2{}_\lambda^L = \left(s_{ij}{}_\lambda^L\right)$, $B_2{}_\lambda^R = \left(s_{ij}{}_\lambda^R\right) \in R^{q \times m}$.

For the sake of simplicity, we set $\underline{X} \times \underline{Y} = \left\{(x, y); A_1{}_\lambda^L x + B_1{}_\lambda^L \leqq b_1{}_\lambda^L,\ A_1{}_\lambda^R x + B_1{}_\lambda^R \leqq b_1{}_\lambda^R,\ A_2{}_\lambda^L x + B_2{}_\lambda^L \leqq b_2{}_\lambda^L,\ A_2{}_\lambda^R x + B_2{}_\lambda^R \leqq b_2{}_\lambda^R\right\}$ and assume that $\underline{X} \times \underline{Y}$ is compact.

$\tilde{X} \times \tilde{Y} = \underline{X} \times \underline{Y}$ from Definition 4.

The above problem can have (i) a complete optimal solution, (ii) a Pareto optimal solution, and (iii) a weak Pareto optimal solution. The definitions of these solutions are given in Definitions 9, 10, and 11.

Definition 9. *A point* $(x^*, y^*) \in \underline{X} \times \underline{Y}$ *is said to be a complete optimal solution to the MOLBLP problem if it holds that*

$$(F_i(x^*, y^*))_\lambda^L \leqq (F_i(x, y))_\lambda^L,\ (F_i(x^*, y^*))_\lambda^R \leqq (F_i(x, y))_\lambda^R,\ i = 1, 2, \ldots, s$$

and

$$(f_i(x^*, y^*))^L_\lambda \leqq (f_i(x, y))^L_\lambda,\ (f_i(x^*, y^*))^R_\lambda \leqq (f_i(x, y))^R_\lambda,\ i = 1, 2, \ldots, t$$

for $\lambda \in [0, 1]$ *and* $(x, y) \in \underline{X} \times \underline{Y}$.

Definition 10. *A point* $(x^*, y^*) \in \underline{X} \times \underline{Y}$ *is said to be a Pareto optimal solution to the MOLBLP problem if there is no* $(x, y) \in \underline{X} \times \underline{Y}$ *such that*

$$(F_i(x^*, y^*))^L_\lambda \geq (F_i(x, y))^L_\lambda,\ (F_i(x^*, y^*))^R_\lambda \geq (F_i(x, y))^R_\lambda,\ i = 1, 2, \ldots, s$$

and

$$(f_i(x^*, y^*))^L_\lambda \geq (f_i(x, y))^L_\lambda,\ (f_i(x^*, y^*))^R_\lambda \geq (f_i(x, y))^R_\lambda,\ i = 1, 2, \ldots, t$$

hold.

Definition 11. *A point* $(x^*, y^*) \in \underline{X} \times \underline{Y}$ *is said to be a weak Pareto optimal solution to the MOLBLP problem if there is no* $(x, y) \in \underline{X} \times \underline{Y}$ *such that*

$$(F_i(x^*, y^*))^L_\lambda > (F_i(x, y))^L_\lambda,\ (F_i(x^*, y^*))^R_\lambda > (F_i(x, y))^R_\lambda,\ i = 1, 2, \ldots, s$$

and

$$(f_i(x^*, y^*))^L_\lambda > (f_i(x, y))^L_\lambda,\ (f_i(x^*, y^*))^R_\lambda > (f_i(x, y))^R_\lambda,\ i = 1, 2, \ldots, t$$

hold.

Theorems 4 and 5 below will provide a complete solution to the MOLBLP problem.

Theorem 4. *Let* (x^*, y^*) *be the solution of the MOLBLP problem given in expressions* (8)*. Then it is also a solution of the FMOLBLP problem defined by expressions* (7)*.*

Proof. The proof is obvious from Definition 4. ■

Lemma 1. [27] *If there is* (x^*, y^*) *such that* $c^L_\alpha x + d^L_\alpha y \geqq c^L_\alpha x^* + d^L_\alpha y^*, c^L_\beta x + d^L_\beta y \geqq c^L_\beta x^* + d^L_\beta y^*$, $c^R_\alpha x + d^R_\alpha y \geqq c^R_\alpha x^* + d^R_\alpha y^*$, *and* $c^R_\beta x + d^R_\beta y \geqq c^R_\beta x^* + d^R_\beta y^*$, *for any* $(x, y)(0 \leqslant \beta < \alpha \leqslant 1)$ *and fuzzy sets* $\tilde{c}$ *and* $\tilde{d}$ *on* R *have the trapezoidal membership function given by*

$$\mu_{\tilde{e}}(x) = \begin{cases} 0 & x < e^L_\beta \\ \frac{\alpha-\beta}{e^L_\alpha - e^L_\beta}\left(x - e^L_\beta\right) + \beta & e^L_\beta \leqq x < e^L_\alpha \\ \alpha & e^L_\alpha \leqq x \leqq e^R_\alpha \\ \frac{\alpha-\beta}{e^R_\alpha - e^R_\beta}\left(x - e^R_\beta\right) + \beta & e^R_\alpha < x \leqq e^R_\beta \\ 0 & e^R_\beta < x \end{cases}$$

then

$$c_\lambda^L x + d_\lambda^L y \geqq c_\lambda^L x^* + d_\lambda^L y^*,$$
$$c_\lambda^R x + d_\lambda^R y \geqq c_\lambda^R x^* + d_\lambda^R y^*,$$

for any $\lambda \in [\beta, \alpha]$.

Theorem 5. *For* $x \in X \subset R^n$, $y \in Y \subset R^m$, *if all the fuzzy parameters* $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_1$, $\tilde{b}_2$, *and* $\tilde{d}_{ij}$ *have trapezoidal membership functions in the FMOLBLP problem* (7),

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_\beta^L \\ \frac{\alpha-\beta}{z_\alpha^L - z_\beta^L}\left(t - z_\beta^L\right) + \beta & z_\beta^L \leqq t < z_\alpha^L \\ \alpha & z_\alpha^L \leqq t < z_\alpha^R \\ \frac{\alpha-\beta}{z_\beta^R - z_\alpha^R}\left(-t + z_\beta^R\right) + \beta & z_\alpha^R \leqq t \leqq z_\beta^R \\ 0 & z_\beta^R < t \end{cases}, \tag{9}$$

where $\tilde{z}$ *denotes* $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$, *and* $\tilde{d}_{ij}$, *respectively, then* (x^*, y^*) *is a complete optimal solution to the problem* (7) *if and only if* (x^*, y^*) *is a complete optimal solution to the MOLBLP problem:*

$$\begin{aligned} &\min_{x \in X} (F_i(x,y))_\alpha^L = {c_{i1}}_\alpha^L x + {d_{i1}}_\alpha^L y, i = 1, 2, \ldots, s \\ &\min_{x \in X} (F_i(x,y))_\alpha^R = {c_{i1}}_\alpha^R x + {d_{i1}}_\alpha^R y, i = 1, 2, \ldots, s \\ &\min_{x \in X} (F_i(x,y))_\beta^L = {c_{i1}}_\beta^L x + {d_{i1}}_\beta^L y, i = 1, 2, \ldots, s \\ &\min_{x \in X} (F_i(x,y))_\beta^R = {c_{i1}}_\beta^R x + {d_{i1}}_\beta^R y, i = 1, 2, \ldots, s \end{aligned} \tag{10a}$$

$$\begin{aligned} \text{subject to } & {A_1}_\alpha^L x + {B_1}_\alpha^L y \leqq {b_1}_\alpha^L, \\ & {A_1}_\alpha^R x + {B_1}_\alpha^R y \leqq {b_1}_\alpha^R, \\ & {A_1}_\beta^L x + {B_1}_\beta^L y \leqq {b_1}_\beta^L, \\ & {A_1}_\beta^R x + {B_1}_\beta^R y \leqq {b_1}_\beta^R, \end{aligned} \tag{10b}$$

$$\begin{aligned} &\min_{y \in Y} (f_i(x,y))_\alpha^L = {c_{i2}}_\alpha^L x + {d_{i2}}_\alpha^L y, i = 1, 2, \ldots, t \\ &\min_{y \in Y} (f_i(x,y))_\alpha^L = {c_{i2}}_\alpha^L x + {d_{i2}}_\alpha^L y, i = 1, 2, \ldots, t \\ &\min_{y \in Y} (f_i(x,y))_\beta^L = {c_{i2}}_\beta^L x + {d_{i2}}_\beta^L y, i = 1, 2, \ldots, t \\ &\min_{y \in Y} (f_i(x,y))_\beta^R = {c_{i2}}_\beta^R x + {d_{i2}}_\beta^R y, i = 1, 2, \ldots, t \end{aligned} \tag{10c}$$

$$\begin{aligned} \text{subject to } & {A_2}_\alpha^L x + {B_2}_\alpha^L y \leqq {b_2}_\alpha^L, \\ & {A_2}_\alpha^L x + {B_2}_\alpha^L y \leqq {b_2}_\alpha^L, \\ & {A_2}_\beta^L x + {B_2}_\beta^L y \leqq {b_2}_\beta^L, \\ & {A_2}_\beta^R x + {B_2}_\beta^R y \leqq {b_2}_\beta^R. \end{aligned} \tag{10d}$$

Proof. If (x^*, y^*) is a complete optimal solution to the FMOLBLP problem, then for any $(x, y) \in \tilde{X} \times \tilde{Y}$, we have $F(x^*, y^*) \precapprox F(x, y)$. Therefore, for any $\lambda \in [\,\beta, \alpha]$,

$$(F_i(x^*, y^*))_\lambda^L \leqq (F_i(x, y))_\lambda^L \text{ and } (F_i(x^*, y^*))_\lambda^R \leqq (F_i(x, y))_\lambda^R, i = 1, 2, \ldots, s$$

and

$$(f_j(x^*, y^*))_\lambda^L \leqq (f_j(x, y))_\lambda^L \text{ and } (f_j(x^*, y^*))_\lambda^R \leqq (f_j(x, y))_\lambda^R, j = 1, 2, \ldots, t.$$

Hence x^* is a complete optimal solution to the MOLBLP problem given by Definition 8.

If (x^*, y^*) is a complete optimal solution to the MOLBLP problem, from Lemma 1, if (x^*, y^*) satisfies (10a) and (10c), then it satisfies (8a) and (8c). Now we need only to prove, if(x^*, y^*) satisfies (10b) and (10d), then it satisfies (8b) and (8d). In fact, for any $\lambda \in [\beta, \alpha]$,

$$\begin{aligned}
{a_{ij}}_\lambda^L &= \frac{\lambda - \beta}{\alpha - \beta}\left({a_{ij}}_\alpha^L - {a_{ij}}_\beta^L\right) + {a_{ij}}_\beta^L, \\
{b_{ij}}_\lambda^L &= \frac{\lambda - \beta}{\alpha - \beta}\left({b_{ij}}_\alpha^L - {b_{ij}}_\beta^L\right) + {b_{ij}}_\beta^L, \\
{b_1}_\lambda^R &= \frac{\lambda - \beta}{\alpha - \beta}\left({b_1}_\alpha^R - {b_1}_\beta^R\right) + {b_1}_\beta^R,
\end{aligned}$$

we have

$$\begin{aligned}
{A_1}_\lambda^L x^* + {B_1}_\lambda^L y^* &= ({a_{ij}}_\lambda^L) x^* + ({b_{ij}}_\lambda^L) y^* \\
&= \left(\frac{\lambda - \beta}{\alpha - \beta}\left({a_{ij}}_\alpha^L - {a_{ij}}_\beta^L\right) + {a_{ij}}_\beta^L\right) x^* \\
&\quad + \left(\frac{\lambda - \beta}{\alpha - \beta}\left({b_{ij}}_\alpha^L - {b_{ij}}_\beta^L\right) + {b_{ij}}_\beta^L\right) y^* \\
&= \frac{\lambda - \beta}{\alpha - \beta}\left({a_{ij}}_\alpha^L\right) x^* + \left(1 - \frac{\lambda - \beta}{\alpha - \beta}\right)\left({a_{ij}}_\beta^L\right) x^* \\
&\quad + \frac{\lambda - \beta}{\alpha - \beta}\left({b_{ij}}_\alpha^L\right) y^* + \left(1 - \frac{\lambda - \beta}{\alpha - \beta}\right)\left({b_{ij}}_\beta^L\right) y^* \\
&= \frac{\lambda - \beta}{\alpha - \beta}\left(\left({a_{ij}}_\alpha^L\right) x^* + \left({b_{ij}}_\alpha^L\right) y^*\right) \\
&\quad + \left(1 - \frac{\lambda - \beta}{\alpha - \beta}\right)\left(\left({a_{ij}}_\beta^L\right) x^* + \left({b_{ij}}_\beta^L\right) y^*\right) \\
&= \frac{\lambda - \beta}{\alpha - \beta}\left({A_1}_\alpha^L x^* + {B_1}_\alpha^L y^*\right) + \left(1 - \frac{\lambda - \beta}{\alpha - \beta}\right)\left({A_1}_\beta^L x^* + {B_1}_\beta^L y^*\right) \\
&\leqq \frac{\lambda - \beta}{\alpha - \beta} {b_1}_\alpha^L + \left(1 - \frac{\lambda - \beta}{\alpha - \beta}\right) {b_1}_\beta^L = {b_1}_\lambda^L,
\end{aligned}$$

from (10b). Similarly, we can prove

$$\begin{aligned} A_{1\lambda}^{R} x^* + B_{1\lambda}^{R} y^* &\leqq b_{1\lambda}^{R}, \\ A_{2\lambda}^{L} x^* + B_{2\lambda}^{L} y^* &\leqq b_{2\lambda}^{L}, \\ A_{2\lambda}^{R} x^* + B_{2\lambda}^{R} y^* &\leqq b_{2\lambda}^{R}, \end{aligned}$$

for any $\lambda \in [\beta, \alpha]$ from (10b) and (10d). The proof is complete. ■

Corollary 2. *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions in the FMOLBLP problem (7), given by*

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_{\alpha_0}^{L} \\ \frac{\alpha_1 - \alpha_0}{z_{\alpha_1}^{L} - z_{\alpha_0}^{L}} \left(t - z_{\alpha_0}^{L}\right) + \alpha_0 & z_{\alpha_0}^{L} \leqq t < z_{\alpha_1}^{L} \\ \frac{\alpha_1 - \alpha_0}{z_{\alpha_2}^{L} - z_{\alpha_1}^{L}} \left(t - z_{\alpha_1}^{L}\right) + \alpha_1 & z_{\alpha_1}^{L} \leqq t < z_{\alpha_2}^{L} \\ \cdots & \cdots \\ \alpha & z_{\alpha_n}^{L} \leqq t < z_{\alpha_n}^{R} \\ \frac{\alpha_n - \alpha_{n-1}}{z_{\alpha_{n-1}}^{R} - z_{\alpha_n}^{R}} \left(-t + z_{\alpha_{n-1}}^{R}\right) + \alpha_{n-1} & z_{\alpha_n}^{R} \leqq t < z_{\alpha_{n-1}}^{R} \\ \cdots & \cdots \\ \frac{\alpha_0 - \alpha_1}{z_{\alpha_1}^{R} - z_{\alpha_0}^{R}} \left(-t + z_{\alpha_0}^{R}\right) + \alpha_0 & z_{\alpha_1}^{R} \leqq t \leqq z_{\alpha_0}^{R} \\ 0 & z_{\alpha_0}^{R} < t \end{cases}, \tag{11}$$

where $\tilde{z}$ denotes $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij}, \tilde{c}_{ij}, \tilde{b}_1, \tilde{b}_2$, and $\tilde{d}_{ij}$, respectively, then, (x^*, y^*) is a complete optimal solution to the problem (7) if and only if (x^*, y^*) is a complete optimal solution to the MOLBLP problem:

$$\begin{aligned} &\min_{x \in X} \left(F_i(x,y)\right)_{\alpha_j}^{L} = c_{i1}{}_{\alpha_j}^{L} x + d_{i1}{}_{\alpha_j}^{L} y, i = 1, 2, \ldots, s, j = 0, 1, \ldots, n \\ &\min_{x \in X} \left(F_i(x,y)\right)_{\alpha_j}^{R} = c_{i1}{}_{\alpha_j}^{R} x + d_{i1}{}_{\alpha_j}^{R} y, i = 1, 2, \ldots, s, j = 0, 1, \ldots, n \end{aligned} \tag{12a}$$

$$\begin{aligned} \text{subject to } &A_{1}{}_{\alpha_j}^{L} x + B_{1}{}_{\alpha_j}^{L} y \leqq b_{1}{}_{\alpha_j}^{L}, j = 0, 1, \ldots, n \\ &A_{1}{}_{\alpha_j}^{R} x + B_{1}{}_{\alpha_j}^{R} y \leqq b_{1}{}_{\alpha_j}^{R}, j = 0, 1, \ldots, n \end{aligned} \tag{12b}$$

$$\begin{aligned} &\min_{y \in Y} \left(f_i(x,y)\right)_{\alpha_j}^{L} = c_{i2}{}_{\alpha_j}^{L} x + d_{i2}{}_{\alpha_j}^{L} y, i = 1, 2, \ldots, t, j = 0, 1, \ldots, n \\ &\min_{y \in Y} \left(f_i(x,y)\right)_{\alpha_j}^{R} = c_{i2}{}_{\alpha_j}^{R} x + d_{i2}{}_{\alpha_j}^{R} y, i = 1, 2, \ldots, t, j = 0, 1, \ldots, n \end{aligned} \tag{12c}$$

$$\begin{aligned} \text{subject to } &A_{2}{}_{\alpha_j}^{L} x + B_{2}{}_{\alpha_j}^{L} y \leqq b_{2}{}_{\alpha_j}^{L}, j = 0, 1, \ldots, n \\ &A_{2}{}_{\alpha_j}^{R} x + B_{2}{}_{\alpha_j}^{R} y \leqq b_{2}{}_{\alpha_j}^{R}, j = 0, 1, \ldots, n. \end{aligned} \tag{12d}$$

Theorem 6 below gives the solution to the Pareto optimal solution of the MOLBLP problem.

Theorem 6. *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions* (11) *in the FMOLBLP problem* (7)*, then (x^*, y^*) is a Pareto optimal solution to the problem* (7) *if and only if (x^*, y^*) is a Pareto optimal solution to the MOLBLP problem* (12).

Proof. Let (x^*, y^*) be a Pareto optimal solution to the FMOLBLP problem. We carry out a proof by contradiction. Let us suppose that there exists a Pareto optimal solution to MOLBLP problem, which is not a solution of the FMOLBLP problem $(\bar{x}, \bar{y}) \in X \times Y$ such that, for $\lambda = \alpha$, β

$$\begin{aligned}&\left((F_1(x^*,y^*))^L_\lambda, (F_1(x^*,y^*))^R_\lambda, \ldots, (F_s(x^*,y^*))^L_\lambda, (F_s(x^*,y^*))^R_\lambda\right)^T \\ &\quad \geq \left((F_1(\bar{x},\bar{y}))^L_\lambda, (F_1(\bar{x},\bar{y}))^R_\lambda, (F_s(\bar{x},\bar{y}))^L_\lambda, (F_s(\bar{x},\bar{y}))^R_\lambda\right)^T.\end{aligned}$$

Therefore

$$\begin{aligned}0 \geq \Big(&(F_1(\bar{x},\bar{y}))^L_\lambda - (F_1(x^*,y^*))^L_\lambda, (F_1(\bar{x},\bar{y}))^R_\lambda (F_1(x^*,y^*))^R_\lambda, \ldots, \\ &(F_s(\bar{x},\bar{y}))^L_\lambda - (F_s(x^*,y^*))^L_\lambda, (F_s(\bar{x},\bar{y}))^R_\lambda - (F_s(x^*,y^*))^R_\lambda\Big)^T.\end{aligned}$$

Hence

$$0 \geq (F_i(\bar{x},\bar{y}))^L_\lambda - (F_i(x^*,y^*))^L_\lambda, 0 \geq (F_i(\bar{x},\bar{y}))^R_\lambda - (F_i(x^*,y^*))^R_\lambda, i = 1, 2, \ldots, s.$$

That is

$$(F_i(\bar{x},\bar{y}))^L_\lambda \leq (F_i(x^*,y^*))^L_\lambda, (F_i(\bar{x},\bar{y}))^R_\lambda \leq (F_i(x^*,y^*))^R_\lambda, i = 1, 2, \ldots, s.$$

By using Lemma 1, for any $\lambda \in [\beta, \alpha]$, we have

$$(F_i(\bar{x},\bar{y}))^L_\lambda \leq (F_i(x^*,y^*))^L_\lambda, (F_i(\bar{x},\bar{y}))^R_\lambda \leq (F_i(x^*,y^*))^R_\lambda, i = 1, 2, \ldots, s.$$

that is $F(x^*, y^*) \succsim F(\bar{x}, \bar{y})$. However, this contradicts the assumption that (x^*, y^*) is a Pareto optimal solution to the FMOLBLP problem.

Let (x^*, y^*) be a Pareto optimal solution to the MOLBLP problem. If (x^*, y^*) is not a Pareto optimal solution to the problem, then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $F(x^*, y^*) \succsim F(\bar{x}, \bar{y})$. Therefore, for any $\lambda \in [\beta, \alpha]$, we have

$$\begin{aligned}&\left((F_1(x^*,y^*))^L_\lambda, (F_1(x^*,y^*))^R_\lambda, \ldots, (F_s(x^*,y^*))^L_\lambda, (F_s(x^*,y^*))^R_\lambda\right)^T \\ &\quad \geq \left((F_1(\bar{x},\bar{y}))^L_\lambda, (F_1(\bar{x},\bar{y}))^R_\lambda, (F_s(\bar{x},\bar{y}))^L_\lambda, (F_s(\bar{x},\bar{y}))^R_\lambda\right)^T.\end{aligned}$$

that is

$$(F_i(\bar{x},\bar{y}))_\lambda^L \leq (F_i(x^*,y^*))_\lambda^L, (F_i(\bar{x},\bar{y}))_\lambda^R \leq (F_i(x^*,y^*))_\lambda^R, i=1,2,\ldots,s.$$

Hence, for $\lambda=\alpha$ and $\lambda=\beta$, we have

$$(F_i(\bar{x},\bar{y}))_\lambda^L \leq (F_i(x^*,y^*))_\lambda^L, (F_i(\bar{x},\bar{y}))_\lambda^R \leq (F_i(x^*,y^*))_\lambda^R, i=1,2,\ldots,s.$$

which contradicts the assumption that (x^*,y^*) is a Pareto optimal solution to the MOLBLP problem.

Hence we have proved Theorem 6. ■

Theorem 7 below gives the weak Pareto optimal solution to the MOLBLP problem.

Theorem 7. *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions* (11) *in the FMOLBLP problem* (7)*, then (x^*,y^*) is a weak Pareto optimal solution to the problem* (7) *if and only if (x^*,y^*) is a weak Pareto optimal solution to the MOLBLP problem* (12)*.*

Proof. The proof is similar to that for Theorem 6. ■

Theorem 8. *For $x \in X \subset R^n$, $y \in Y \subset R^m$, if all the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_{ij}$, $\tilde{e}_{ij}$, $\tilde{s}_{ij}$, $\tilde{c}_{ij}$, $\tilde{b}_1$, $\tilde{b}_2$, and $\tilde{d}_{ij}$ have piecewise trapezoidal membership functions* (11) *in the FMOLBLP problem* (7)*, then a necessary and sufficient condition that (x^*,y^*) solves the FMOL BLP problem* (7) *is that there exist (row) vectors u^*, v^*, and z^* such that (x^*,y^*,u^*,v^*,z^*) solves:*

$$\min_{x\in X}(F(x,y)) = \sum_{j=1}^{s} w_{j1}\left(\sum_{i=0}^{n}\left({c_{j1}}_{\alpha_i}^{L}x + {d_{j1}}_{\alpha_i}^{L}y\right) + \sum_{i=0}^{n}\left({c_{j1}}_{\alpha_i}^{R}x + {d_{j1}}_{\alpha_i}^{R}y\right)\right) \tag{13a}$$

$$\text{subject to } {A_1}_{\alpha_i}^{L}x + {B_1}_{\alpha_i}^{L}y \leqq {b_1}_{\alpha_i}^{L}, i=0,1,\ldots,n$$

$${A_1}_{\alpha_i}^{R}x + {B_1}_{\alpha_i}^{R}y \leqq {b_1}_{\alpha_i}^{R}, i=0,1,\ldots,n \tag{13b}$$

$${A_2}_{\alpha_i}^{L}x + {B_2}_{\alpha_i}^{L}y \leqq {b_2}_{\alpha_i}^{L}, i=0,1,\ldots,n$$

$${A_2}_{\alpha_i}^{R}x + {B_2}_{\alpha_i}^{R}y \leqq {b_2}_{\alpha_i}^{R}, i=0,1,\ldots,n \tag{13c}$$

$$\begin{aligned} u\left(\sum_{i=0}^{n}{B_1}_{\alpha_i}^{L} + \sum_{i=0}^{n}{B_1}_{\alpha_i}^{R}\right) + v\left(\sum_{i=0}^{n}{B_2}_{\alpha_i}^{L} + \sum_{i=0}^{n}{B_2}_{\alpha_i}^{R}\right) - z \\ = -\sum_{j=1}^{t} w_{j2}\left(\sum_{i=0}^{n}{d_{j2}}_{\alpha_i}^{L} + \sum_{i=0}^{n}{d_{j2}}_{\alpha_i}^{R}\right) \end{aligned} \tag{13d}$$

$$u\left(\left(\sum_{i=0}^{n} b_{1\alpha_i}^{L}+\sum_{i=0}^{n} b_{1\alpha_i}^{R}\right)-\left(\sum_{i=0}^{n} A_{1\alpha_i}^{L}+\sum_{i=0}^{n} A_{1\alpha_i}^{R}\right)x\right.$$
$$\left.-\left(\sum_{i=0}^{n} B_{1\alpha_i}^{L}+\sum_{i=0}^{n} B_{1\alpha_i}^{R}\right)y\right)+v\left(\left(\sum_{i=0}^{n} b_{2\alpha_i}^{L}+\sum_{i=0}^{n} b_{2\alpha_i}^{R}\right)\right. \tag{13e}$$
$$\left.-\left(\sum_{i=0}^{n} A_{2\alpha_i}^{L}+\sum_{i=0}^{n} A_{2\alpha_i}^{R}\right)x-\left(\sum_{i=0}^{n} B_{2\alpha_i}^{L}+\sum_{i=0}^{n} B_{2\alpha_i}^{R}\right)y\right)$$
$$+zy=0$$

$$x\geq 0, y\geq 0, u\geq 0, v\geq 0, z\geq 0, \sum_{j=1}^{s} w_{j1}=1 \text{ and } \sum_{j=1}^{t} w_{j2}=1. \tag{13f}$$

Proof. We can prove this result by combining Theorem 5 and Theorem 3, and use the method of weighting [18]. ■

Based on Theorem 8, we present an approximation Kuhn–Tucker approach for solving the FMOLBLP problem (7).

3.2 The Approximation Kuhn–Tucker Approach

The systematic method to the solution of the FMOLBLP problem that uses an approximation Kuhn–Tucker approach is as follows.

Step 1 Give the weights w_{j1} and w_{j2} for multiple fuzzy objectives of the leader and the follower, respectively, thus $\sum_{j=1}^{s} w_{j1}=1$ and $\sum_{j=1}^{t} w_{j2}=1$.

Step 2 Transform the problem (7) to the problem (8) by using Theorem 5.

Step 3 Decompose the interval [0, 1] into 2^{l-1} mean subintervals with $(2^{l-1}+1)$ nodes $\lambda_i\left(i=0,\ldots,2^{l-1}\right)$, which are arranged in the order $0=\lambda_0<\lambda_1<\cdots<\lambda_{2^{l-1}}=1$, and a range of errors $\varepsilon>0$.

Step 4 Set $l=1$, then solve $(\text{MOLBLP})_2^l$, i.e., (10) by using the Kuhn–Tucker approach when $\beta=0$ and $\alpha=1$, we obtain an optimization solution $(x,y)_{2^l}$.

Step 5 Solve $(\text{MOLBLP})\ _2^{l+1}$ by Theorem 8 and the Kuhn–Tucker approach. We obtain an optimal solution $(x,y)_{2^{l+1}}$.

Step 6 Check if $\|(x,y)_{2^{l+1}}-(x,y)_{2^l}\|<\varepsilon$, then the solution (x^*,y^*) of the FMOLBLP problem is $(x,y)_{2^{l+1}}$, otherwise, update l to $2l$ and go back to Step 4.

Step 7 Show the solution.

An example will be given in Section 4 to illustrate the proposed approach.

4 A Numerical Example

Here we consider an illustrative example to demonstrate the theory developed in the earlier sections. In particular, we illustrate the use of the systematic methodology.

Consider the following FMOLBLP problem with $x \in R^1$, $y \in R^1$, and $X = \{x \geqslant 0\}$, $Y = \{y \geqslant 0\}$,

$$
\begin{aligned}
&\min_{x\in X} F_1(x,y) = \tilde{1}x - \tilde{2}y \\
&\min_{x\in X} F_2(x,y) = \tilde{1}x + \tilde{3}y \\
&\text{subject to } -\tilde{1}x + \tilde{3}y \preccurlyeq \tilde{4} \\
&\qquad\qquad \min_{y\in Y} f_1(x,y) = \tilde{1}x + \tilde{1}y \\
&\qquad\qquad \min_{y\in Y} f_2(x,y) = \tilde{1}x - \tilde{2}y \\
&\qquad\qquad \text{subject to } \tilde{1}x - \tilde{1}y \preccurlyeq \tilde{0} \\
&\qquad\qquad\qquad\qquad -\tilde{1}x - \tilde{1}y \preccurlyeq \tilde{0}
\end{aligned}
$$

where

$$
\mu_{\tilde{1}}(t) = \begin{cases} 0 & t<0 \\ t^2 & 0 \leqslant t < 1 \\ 2-t & 1 \leqslant t < 2 \\ 0 & 2 \leqslant t \end{cases}, \quad \mu_{\tilde{2}}(t) = \begin{cases} 0 & t<1 \\ t-1 & 1 \leqslant t < 2 \\ 3-t & 2 \leqslant t < 3 \\ 0 & 3 \leqslant t \end{cases},
$$

$$
\mu_{\tilde{3}}(t) = \begin{cases} 0 & t<2 \\ t-2 & 2 \leqslant t < 3 \\ 4-t & 3 \leqslant t < 4 \\ 0 & 4 \leqslant t \end{cases}, \quad \mu_{\tilde{4}}(t) = \begin{cases} 0 & t<3 \\ t-3 & 3 \leqslant t < 4 \\ 5-t & 4 \leqslant t < 5 \\ 0 & 5 \leqslant t \end{cases},
$$

$$
\mu_{\tilde{0}}(t) = \begin{cases} 0 & t<-1 \\ t+1 & -1 \leqslant t < 0 \\ 1-t^2 & 0 \leqslant t < 1 \\ 0 & 1 \leqslant t \end{cases}.
$$

We now solve this problem by using the proposed approximation Kuhn–Tucker approach.

Step 1. The weights for the two fuzzy objectives of the leader are (0.5, 0.5) and of the follower (0.5, 0.5).

Step 2. The FMOLBLP problem is first transformed to the following MOLBLP problem by using Theorem 5

$$
\begin{aligned}
&\min_{x\in X} (F_1(x,y))^L_\lambda = \tilde{1}^L_\lambda x + (-\tilde{2})^L_\lambda y, \quad \lambda \in [0,1] \\
&\min_{x\in X} (F_1(x,y))^R_\lambda = \tilde{1}^R_\lambda x + (-\tilde{2})^R_\lambda y, \quad \lambda \in [0,1]
\end{aligned}
$$

$$\min_{x\in X}(F_2(x,y))^L_\lambda = \tilde{1}^L_\lambda x + \tilde{3}^L_\lambda y, \quad \lambda\in[0,1]$$
$$\min_{x\in X}(F_2(x,y))^R_\lambda = \tilde{1}^R_\lambda x + \tilde{3}^R_\lambda y, \quad \lambda\in[0,1]$$
$$\text{subject to } (-\tilde{1})^L_\lambda x + \tilde{3}^L_\lambda y \leqslant \tilde{4}^L_\lambda, (-\tilde{1})^R_\lambda x + \tilde{3}^R_\lambda y \leqslant \tilde{4}^R_\lambda, \quad \lambda\in[0,1]$$
$$\min_{y\in Y}(f_1(x,y))^L_\lambda = \tilde{1}^L_\lambda x + \tilde{1}^L_\lambda y, \quad \lambda\in[0,1]$$
$$\min_{y\in Y}(f_1(x,y))^R_\lambda = \tilde{1}^R_\lambda x + \tilde{1}^R_\lambda y, \quad \lambda\in[0,1]$$
$$\min_{y\in Y}(f_2(x,y))^L_\lambda = \tilde{1}^L_\lambda x + (-\tilde{2})^L_\lambda y, \quad \lambda\in[0,1]$$
$$\min_{y\in Y}(f_2(x,y))^R_\lambda = \tilde{1}^R_\lambda x + (-\tilde{2})^R_\lambda y, \quad \lambda\in[0,1]$$
$$\text{subject to } \tilde{1}^L_\lambda x + (-\tilde{1})^L_\lambda y \leqslant \tilde{0}^L_\lambda, \tilde{1}^R_\lambda x + (-\tilde{1})^R_\lambda y \leqslant \tilde{0}^R_\lambda,\ \lambda\in[0,1]$$
$$(-\tilde{1})^L_\lambda x + (-\tilde{1})^L_\lambda y \leqslant \tilde{0}^L_\lambda, (-\tilde{1})^R_\lambda x + (-\tilde{1})^R_\lambda y \leqslant \tilde{0}^R_\lambda,\ \lambda\in[0,1]$$

Step 3. Let the interval [0, 1] be decomposed into 2^{l-1} mean subintervals with $(2^{l-1}+1)$ nodes λ_i $(i = 0,\dots,2^{l-1})$, which is arranged in the order $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{2^{l-1}} = 1$, and a range of errors $\varepsilon = 10^{-6} > 0$.

Step 4. When $l = 1$, we solve the following MOLBLP problem

$$\min_{x\in X}(F_1(x,y))^{L(R)}_1 = 1x - 2y$$
$$\min_{x\in X}(F_1(x,y))^L_0 = 0x - 3y$$
$$\min_{x\in X}(F_1(x,y))^R_0 = 2x - 1y$$
$$\min_{x\in X}(F_2(x,y))^{L(R)}_1 = 1x + 3y$$
$$\min_{x\in X}(F_2(x,y))^L_0 = 0x + 2y$$
$$\min_{x\in X}(F_2(x,y))^R_0 = 2x + 4y$$
$$\text{subject to } -1x + 3y \leqslant 4$$
$$-2x + 2y \leqslant 3$$
$$0x + 4y \leqslant 5$$
$$\min_{y\in Y}(f_1(x,y))^{L(R)}_1 = 1x + 1y$$
$$\min_{y\in Y}(f_1(x,y))^R_0 = 2x + 2y$$
$$\min_{y\in Y}(f_2(x,y))^{L(R)}_1 = 1x - 2y$$
$$\min_{y\in Y}(f_2(x,y))^L_0 = 0x - 3y$$
$$\min_{y\in Y}(f_2(x,y))^R_0 = 2x - 1y$$

$$\begin{aligned} \text{subject to } & 1x - 1y \leqslant 0 \\ & 0x - 2y \leqslant -1 \\ & 2x - 0y \leqslant 1 \\ & -1x - 1y \leqslant 0 \\ & -2x - 2y \leqslant -1. \end{aligned}$$

Step 5. We solve this MOLBLP problem by using the extended Kuhn–Tucker approach [27] and the method of weighting.

$$\begin{aligned} & \min_{x \in X} F(x,y) = 0.5\,(6x+3y) = 3x + 1.5y \\ & \text{subject to } -1x + 3y \leqslant 4 \\ & \qquad -2x + 2y \leqslant 3 \\ & \qquad 0x + 4y \leqslant 5 \\ & \qquad \min_{y \in Y} f(x,y) = 0.5\,(6x-3y) = 3x - 1.5y \\ & \qquad \text{subject to } 1x - 1y \leqslant 0 \\ & \qquad\qquad 0x - 2y \leqslant -1 \\ & \qquad\qquad 2x - 0y \leqslant 1 \\ & \qquad\qquad -1x - 1y \leqslant 0 \\ & \qquad\qquad -2x - 2y \leqslant -1. \end{aligned}$$

By Theorem 8, we solve the following problem:

$$\begin{aligned} & \min_{x \in X} F(x,y) = 3x + 1.5y \\ & \text{subject to } -1x + 3y \leqslant 4 \\ & \qquad -2x + 2y \leqslant 3 \\ & \qquad 0x + 4y \leqslant 5 \\ & \qquad 1x - 1y \leqslant 0 \\ & \qquad 0x - 2y \leqslant -1 \\ & \qquad 2x - 0y \leqslant 1 \\ & \qquad -1x - 1y \leqslant 0 \\ & \qquad -2x - 2y \leqslant -1 \\ & \qquad 3u_1 + 2u_2 + 4u_3 - u_4 - 2u_5 - 0u_6 - u_7 - 2u_8 - u_9 = 1.5 \\ & \qquad x \geq 0, y \geq 0, u_1 \geq 0, \ldots, u_9 \geq 0. \end{aligned}$$

The result is

$$\begin{aligned} & \min_{x \in X} (F_1(x,y))_1^{L(R)} = -2.5 \\ & \min_{x \in X} (F_1(x,y))_0^{L} = -3.75 \\ & \min_{x \in X} (F_1(x,y))_0^{R} = -1.25 \end{aligned}$$

$$\min_{x\in X}(F_2(x,y))_1^{L(R)} = 3.75$$
$$\min_{x\in X}(F_2(x,y))_0^{L} = 2.5$$
$$\min_{x\in X}(F_2(x,y))_0^{R} = 5$$
$$\min_{y\in Y}(f_1(x,y))_1^{L(R)} = 1.25$$
$$\min_{y\in Y}(f_1(x,y))_0^{R} = 2.5$$
$$\min_{y\in Y}(f_2(x,y))_1^{L(R)} = -2.5$$
$$\min_{y\in Y}(f_1(x,y))_0^{L} = -3.75$$
$$\min_{y\in Y}(f_1(x,y))_0^{R} = -1.25$$
$$x = 0, y = 1.25$$

Step 6. The condition is not met, go to Step 4.
Step 4. When $l = 2$, we solve the following MOLBLP problem

$$\min_{x\in X}(F_1(x,y))_1^{L(R)} = 1x - 2y$$
$$\min_{x\in X}(F_1(x,y))_{\frac{1}{2}}^{L} = \frac{\sqrt{2}}{2}x - \frac{3}{2}y$$
$$\min_{x\in X}(F_1(x,y))_0^{L} = 0x - 3y$$
$$\min_{x\in X}(F_1(x,y))_{\frac{1}{2}}^{R} = \frac{3}{2}x - \frac{5}{2}y$$
$$\min_{x\in X}(F_1(x,y))_0^{R} = 2x - 1y$$
$$\min_{x\in X}(F_2(x,y))_1^{L(R)} = 1x + 3y$$
$$\min_{x\in X}(F_2(x,y))_{\frac{1}{2}}^{L} = \frac{\sqrt{2}}{2}x + \frac{5}{2}y$$
$$\min_{x\in X}(F_2(x,y))_0^{L} = 0x + 2y$$
$$\min_{x\in X}(F_2(x,y))_{\frac{1}{2}}^{L} = \frac{3}{2}x + \frac{7}{2}y$$
$$\min_{x\in X}(F_2(x,y))_0^{R} = 2x + 4y$$
$$\begin{aligned}\text{subject to } & -1x + 3y \leqslant 4 \\ & -\frac{3}{2}x + \frac{5}{2}y \leqslant \frac{7}{2} \\ & -2x + 2y \leqslant 3\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sqrt{2}}{2}x+\frac{7}{2}y\leqslant\frac{9}{2}\\
&0x+4y\leqslant 5\\
&\min_{y\in Y}\left(f_1(x,y)\right)_1^{L(R)}=1x+1y\\
&\min_{y\in Y}\left(f_1(x,y)\right)_{\frac{1}{2}}^{L}=\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\\
&\min_{y\in Y}\left(f_1(x,y)\right)_{\frac{1}{2}}^{R}=\frac{3}{2}x+\frac{3}{2}y\\
&\min_{y\in Y}\left(f_1(x,y)\right)_0^{R}=2x+2y\\
&\min_{y\in Y}\left(f_2(x,y)\right)_1^{L(R)}=1x-2y\\
&\min_{y\in Y}\left(f_2(x,y)\right)_{\frac{1}{2}}^{L}=\frac{\sqrt{2}}{2}x-\frac{3}{2}y\\
&\min_{y\in Y}\left(f_2(x,y)\right)_0^{L}=0x-3y\\
&\min_{y\in Y}\left(f_2(x,y)\right)_{\frac{1}{2}}^{R}=\frac{3}{2}x-\frac{5}{2}y\\
&\min_{y\in Y}\left(f_2(x,y)\right)_0^{R}=2x-1y\\
&\text{subject to } 1x-1y\leqslant 0\\
&\qquad\qquad\quad \frac{\sqrt{2}}{2}x-\frac{3}{2}y\leqslant-\frac{1}{2}\\
&\qquad\qquad\quad 0x-2y\leqslant -1\\
&\qquad\qquad\quad \frac{3}{2}x-\frac{\sqrt{2}}{2}y\leqslant\frac{\sqrt{2}}{2}\\
&\qquad\qquad\quad 2x-0y\leqslant 1\\
&\qquad\qquad\quad -\frac{3}{2}x-\frac{3}{2}y\leqslant-\frac{1}{2}\\
&\qquad\qquad\quad -1x-1y\leqslant 0\\
&\qquad\qquad\quad -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y\leqslant\frac{\sqrt{2}}{2}\\
&\qquad\qquad\quad -2x-2y\leqslant -1.
\end{aligned}
$$

Step 5. We solve this MOLBLP problem by using the Kuhn–Tucker approach [4, 5] and the method of weighting.

$$
\begin{aligned}
&\min_{x\in X}F(x,y)=\left(\sqrt{2}+9\right)x+5y\\
&\text{subject to } -1x+3y\leqslant 4
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}x+\frac{5}{2}y \leqslant \frac{7}{2}\\
&-2x+2y \leqslant 3\\
&-\frac{\sqrt{2}}{2}x+\frac{7}{2}y \leqslant \frac{9}{2}\\
&0x+4y \leqslant 5\\
&\min_{y\in Y} f(x,y)=\left(\sqrt{2}+9\right)x+\left(\sqrt{2}-11\right)y\\
&\text{subject to } 1x-1y \leqslant 0\\
&\qquad \frac{\sqrt{2}}{2}x-\frac{3}{2}y \leqslant -\frac{1}{2}\\
&\qquad 0x-2y \leqslant -1\\
&\qquad \frac{3}{2}x-\frac{\sqrt{2}}{2}y \leqslant \frac{\sqrt{2}}{2}\\
&\qquad 2x-0y \leqslant 1\\
&\qquad -\frac{3}{2}x-\frac{3}{2}y \leqslant -\frac{1}{2}\\
&\qquad -1x-1y \leqslant 0\\
&\qquad -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y \leqslant \frac{\sqrt{2}}{2}\\
&\qquad -2x-2y \leqslant -1
\end{aligned}
$$

The result is

$$
\begin{aligned}
&\min_{x\in X}\left(F_1(x,y)\right)_1^{L(R)}=-2.5\\
&\min_{x\in X}\left(F_1(x,y)\right)_{\frac{1}{2}}^{L}=-1.875\\
&\min_{x\in X}\left(F_1(x,y)\right)_0^{L}=-3.75\\
&\min_{x\in X}\left(F_1(x,y)\right)_{\frac{1}{2}}^{R}=-3.125\\
&\min_{x\in X}\left(F_1(x,y)\right)_0^{R}=-1.25\\
&\min_{x\in X}\left(F_2(x,y)\right)_1^{L(R)}=3.75\\
&\min_{x\in X}\left(F_2(x,y)\right)_{\frac{1}{2}}^{L}=3.125\\
&\min_{x\in X}\left(F_2(x,y)\right)_0^{L}=2.5\\
&\min_{x\in X}\left(F_2(x,y)\right)_{\frac{1}{2}}^{L}=4.375\\
&\min_{x\in X}\left(F_2(x,y)\right)_0^{R}=5
\end{aligned}
$$

$$\min_{y\in Y}(f_1(x,y))_1^{L(R)} = 1.25$$
$$\min_{y\in Y}(f_1(x,y))_{\frac{1}{2}}^{L} = \frac{5\sqrt{2}}{8}$$
$$\min_{y\in Y}(f_1(x,y))_{\frac{1}{2}}^{R} = 1.875$$
$$\min_{y\in Y}(f_1(x,y))_0^{R} = 2.5$$
$$\min_{y\in Y}(f_2(x,y))_1^{L(R)} = -2.5$$
$$\min_{y\in Y}(f_2(x,y))_{\frac{1}{2}}^{L} = -1.875$$
$$\min_{y\in Y}(f_2(x,y))_0^{L} = -3.75$$
$$\min_{y\in Y}(f_2(x,y))_{\frac{1}{2}}^{R} = -3.125$$
$$\min_{y\in Y}(f_2(x,y))_0^{R} = -1.25$$
$$x = 0, y = 1.25$$

Step 6. $x = 0, y = 0.5$ is the optimal solution as the condition $\|(x,y)_{2^2} - (x,y)_{2^1}\| = 0 < \varepsilon$ is met.

Step 7. The solution of the problem is $x = 0, y = 1.25$ such that

$$\min_{x\in X} F_1(x,y) = -\widetilde{2.5}$$
$$\min_{x\in X} F_2(x,y) = \widetilde{3.75}$$
$$\min_{y\in Y} f_1(x,y) = \widetilde{1.25}$$
$$\min_{y\in Y} f_2(x,y) = -\widetilde{2.5}$$

where

$$\mu_{-\widetilde{2.5}}(t) = \begin{cases} 0, & t < -3.75, \\ \frac{t+3.75}{1.25}, & -3.75 \leqslant t < -2.5, \\ \frac{-1.25-t}{1.25}, & -2.5 \leqslant t < -1.25, \\ 0, & -1.25 \leqslant t, \end{cases}$$

$$\mu_{\widetilde{3.75}}(t) = \begin{cases} 0, & t < 2.5, \\ \frac{t-2.5}{1.25}, & 2.5 \leqq t < 3.75, \\ \frac{5-t}{1.25}, & 3.75 \leqq t < 5, \\ 0, & 5 \leqq t, \end{cases}$$

$$\mu_{\widetilde{1.25}}(t) = \begin{cases} 0, & t < 0, \\ \left(\frac{t}{1.25}\right)^2, & 0 \leqslant t < 1.25, \\ \frac{2.5-t}{1.25}, & 1.25 \leqslant t < 2.5, \\ 0, & 1 \leqslant t, \end{cases}$$

This example shows how the approximation Kuhn–Tucker approach is used to solve a FMOLBLP problem.

5 Conclusion and Further Study

Uncertainty often occurs in bilevel decision making. Therefore, fuzzy parameter based bilevel decision models can be more suitable to describe a real-world bilevel decision situation. The leader and the follower often have multiple objectives to consider simultaneously. In this chapter, we first developed a model for FMOLBLP and then provided the complete solution, the Pareto solution, and the weak Pareto solution. Based on these, we presented a general fuzzy number based approximation Kuhn–Tucker approach to solve such fuzzy multi-objective bilevel decision problems.

Further study on this topic includes the development of a model and related approaches for fuzzy multiobjective multifollower bilevel decision problems. In such problems, multiple followers are involved in bilevel decision making. The leader's decision will be affected not only by those followers' individual reactions but also by the relationships among these followers. As uncertain data could occur in the objectives and constraints of both the leader and the followers, it will be a challenge to get an optimal solution for the leader in the complex environment.

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Pareto Optimality

Dinh The Luc

Department of Mathematics, University of Avignon, Avignon, France
dtluc@univ.avignon.fr

Abstract This chapter discusses some selected topics of the theory of Pareto optimality. It includes existence criteria, optimality in product spaces, scalarization via support functions, nonconvex duality, and solution methods.

Key words: efficient solution, nonconvex duality, normal cone method, partial order, product space, vector variational principle

1 Introduction

An example of typical situations in which Pareto optimality is involved is the house purchase problem. Suppose that a real-estate agency suggests to us three houses A, B, and C of the same price that fits our budget. We set three main criteria to evaluate the offers—appearance, comfort, and environment—and range the score from 0 to 5. Here is the table of our evaluation:

	A	B	C
Appearance	3	3	5
Comfort	4	4	4
Environment	5	4	3

By looking at this table, we can eliminate the offer B from our choice because it is worse than A from all points of view. As to the remaining offers A and C, we observe that A is better than C with regard to the appearance criterion but worse than C with regard to the environment criterion. At this stage, it is impossible to say which one is the best with regard to three criteria. Actually they are both optimal according to what we are going to develop in this chapter.

The concept of Pareto optimality originated in the economic equilibrium and welfare theories at the beginning of the past century. The main idea of this concept is that a society is enjoying maximum ophelimity when no one can be made better off without making someone else worse off. Named after Vilfredo Pareto (1848–1923), an Italian economist, this theory is now central in economics with broad range of applications in the social sciences, engineering, management, and informations.

The first rigorous mathematical treatment of Pareto optimality was given in the publication of Kuhn–Tucker [31] in 1951 on the necessary and sufficient conditions for efficiency. At the same time, Koopmans [30] initiated the use of Pareto optimality in operations research. Further contributions are due to Zadeh [56], Klinger [28], DaCunha-Polak [14], Geoffrion [16], and some others in the 1950s and 1960s. However, only in the 1970s and 1980s had an impetuous development of Pareto optimality really begun. Today we count thousands of papers and books on the subject and the research is intensive both in the theory and applications.

The aim of the current chapter is to give an overview on some relevant topics of Pareto optimality. The chapter is organized as follows. In Section 2, we recall the definition of partial orders in a topological vector space and give some instances of orders that are frequently used. In Section 3, the main concepts of Pareto optimality are introduced: ideal minimal point, minimal point, properly minimal point, and related notions. Section 4 is devoted to existence of minimal points; sufficient and necessary conditions for existence are presented in a general setting. In Section 5, particular orders are introduced in a product space. They help us to convert a constrained problem into a problem without constraints and derive a general multiplier rule. An enlarged order in the product space yields also a variational principle of Ekeland's type for set-valued mappings. Section 6 deals with some useful scalarizations. Attention is paid to the scalarization via support functions that allow one to generate all efficient solutions of convex problems. In Section 7, a nonstandard duality approach is presented to solve multiobjective problems. In the last section, two solution methods are provided to generate efficient solutions of linear problems and convex problems in finite dimension. A majority of the results are given without proofs, but with references in case they are not easy to obtain. A few proofs are given because either they have certain interest in understanding the subject or they have not been mentioned in the literature.

2 Partial Orders

Given a set of alternatives, the problem of choosing the best alternative basically depends on the way we classify or evaluate them. The most popular evaluation method is to associate to each alternative a real value, and the best alternative is defined as the one with the largest or the smallest value. This, however, is not always possible as we have already seen in the real-estate purchase problem in which every offer is associated with a triple of real values. In that problem, the classification of offers is carried out in the 3-dimensional space, in which the notion of the smallest and the largest values is not available. In this section, we shall recall the concept of partial order in a multidimensional space and point out some orders of particular interest. Throughout the section, E denotes a locally convex space. For a subset A of E, the notations $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{co}(A)$, and A^c stand for the closure, the interior, the convex hull, and the complement of A, respectively.

Definition 1. *Let R be a binary relation on E that is R is a subset of $E \times E$. It is said to be a partial order on E if it is*

(i) reflexive: $(x, x) \in R$ for every $x \in E$;
(ii) transitive: $(x, y), (y, z) \in R$ imply $(x, z) \in R$.

Being a partial order, R is called linear if it is compatible with the linear structure of the space: $(x, y) \in R$ implies $(x + z, y + z), (tx, ty) \in R$ for all $z \in E$ and $t > 0$.

Here is a simple geometric structure of linear partial orders.

Proposition 1. *If $R \subseteq E \times E$ is a linear partial order, then the set $C := \{x \in E : (x, 0) \in R\}$ is a convex cone in E. Conversely, if C is a convex cone in E, then the relation R defined by $(x, y) \in R$ if and only if $x - y \in C$ is a linear partial order in E.*

Proof. Let R be a linear partial order in E. Let $(x, 0) \in R$. By the linearity, one has $(tx, 0) \in R$ for all $t > 0$. Hence $tx \in C := \{y \in E : (y, 0) \in R\}$ for $t > 0$. When $t = 0$, by the reflectivity one has $(0, 0) \in R$, hence $0 \in C$ and C is a cone. This cone is convex because for $x, y \in C$ we have $(x, 0)$ and $(y, 0)$ belong to R, hence by transitivity $(x + y, 0) \in R$, and therefore $x + y \in C$.

Conversely, assume that C is a convex cone in E. Because $0 \in C$ and $x - x = 0$ for all $x \in E$, we have $(x, x) \in R$. This shows that R is reflexive. Moreover, if $x - y \in C$ and $y - z \in C$, then by the convexity of C we obtain $x - z = x - y + y - z \in C$ or equivalently, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, y) \in R$. In this way R is a partial order in E. It is linear because $x - y \in C$ implies $t(x - y) \in C$ for $t > 0$ and $(x + z) - (y + z) \in C$ for all $z \in E$, which means $(x, y) \in R$ implies $(tx, ty) \in R$ and $(x + z, y + z) \in R$ for all $t > 0$ and $z \in E$. The proof is complete. ■

The order determined by a convex cone C is often written as $x \geq_C y$ if and only if $x - y \in C$. The strict inequality $x >_C y$ is also in use to indicate $x \geq_C y$ and $x \neq y$. Let A be a subset of E. The cone generated by A is denoted by $\mathrm{cone}(A)$ and is defined by

$$\mathrm{cone}(A) := \{tx : t \geq 0, \quad x \in A\}.$$

Given a convex cone $C \subseteq E$, the positive polar cone and the strictly positive polar cone of C are defined respectively by

$$C' := \{\xi \in E' : \langle \xi, x \rangle \geq 0 \quad \text{for all} \quad x \in C\}$$
$$C^+ := \{\xi \in E' : \langle \xi, x \rangle > 0 \quad \text{for all} \quad x \in C \setminus \{0\}\},$$

where E' is the topological dual space of E. The linear part of the cone C is denoted by $\ell(C)$, that is $\ell(C) = C \cap (-C)$. When the linear part of C is trivial, i.e., $\ell(C) = \{0\}$, we say that C is pointed. The following particular cones will be of use.

1. **The Pareto cone:** Let $\mathbb{R}^n_+$ be the positive octant of the n-dimensional Euclidean space $\mathbb{R}^n$. Then, for two vectors $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, one has $x \geq_{\mathbb{R}^n_+} y$ if and only if $x_i \geq y_i$, $i = 1, \ldots, n$. The cone $\mathbb{R}^n_+$ is called the Pareto cone because the original Pareto optimality is defined by the order generated by this cone. When $n = 1$, the usual order of real numbers is exactly the order $(\geq_{\mathbb{R}_+})$. This order is total in the sense that any two numbers x and y are comparable: either $x \geq y$ or $y \geq x$. When $n > 1$, the order $(\geq_{\mathbb{R}^n_+})$ is not total.
2. **Correct cones:** We say that a cone C in E is correct if $\mathrm{cl}C + C \setminus \ell(C) \subseteq C$ or equivalently $\mathrm{cl}C + C \setminus \ell(C) \subseteq C \setminus \ell(C)$. This kind of cone is very useful in establishing the existence of optimal solutions for multiobjective problems. Let us mention some typical cases of correct cones.
   (i) Every closed and convex cone is correct;
   (ii) If $C \setminus \ell(C)$ is open, then C is correct;
   (iii) If C consists of the origin and an intersection of half-spaces that are either open or closed, then C is correct.
3. **Cones with a convex bounded base:** A subset B of a cone $C \subseteq E$ is said to be a base of C if it does not contain the origin in its closure and $C = \{tb : b \in B, t \geq 0\}$. When B is convex and/or bounded, we say that the cone C has a convex and/or bounded base. We list some conditions for a cone to have such a property:
   (i) In a finite dimensional space, every convex cone whose closure is a pointed cone has a convex and bounded base.
   (ii) In a locally convex space, a cone C has a convex base if and only if there is an open half-space containing $C \setminus \{0\}$. In particular, a cone with a convex base is pointed; and a convex cone C has a convex base if and only if $C^+ \neq \emptyset$.
4. **Cones with the Daniell property:** A net $\{x_i\}_{i \in I}$ in the space E is said to be decreasing with respect to the convex cone C if $x_i >_C x_j$ for $i < j$ in I. It is said to be minorized if there is some element $a \in E$ such that $x_i \geq_C a$ for all $i \in I$. The cone C is said to have the Daniell property if every minorized decreasing net has an infimum and converges to this infimum. Here are some sufficient conditions for a convex cone to have the Daniell property.
   (i) Every pointed, closed, and convex cone in a finite dimensional space has the Daniell property;
   (ii) If E is a Banach space and C_E has a closed, convex, and bounded base B, then for $\varepsilon > 0$ sufficiently small, the cone $\mathrm{cl\,cone}(B + B(0, \varepsilon))$, where $B(0, \varepsilon)$ is the ball of radius ε centered at the origin, has the Daniell property.
   (iii) If C has the Daniell property, then any closed subcone of C has the same property.

3 Pareto Efficiency

The totalness property of the usual order on $\mathbb{R}$ makes it sense when speaking about the maximum and the minimum of a set of real numbers. The situation is quite different in a space of higher dimension that is equipped with a nontotal partial order. For instance, in the 2-dimensional Euclidean space $\mathbb{R}^2$ ordered by the Pareto cone $\mathbb{R}^2_+$, given a set of two elements $(0,1)$ and $(1,0)$, neither relation $(0,1) \geq_{\mathbb{R}^2_+} (1,0)$, nor $(1,0) \geq_{\mathbb{R}^2_+} (0,1)$ is valid. So, this set, regardless of being finite, has neither maximum, nor minimum. The nontotalness of partial orders leads us to several concepts of optimality and this makes the topic appealing. Generally speaking, Pareto efficiency can be defined in a space with any order, not necessarily transitive. We restrict, however, our presentation to the case of partial orders only. This is because most of practical models involve partial orders and most interesting theoretical results as well as numerical methods are developed within this framework. Thus, we assume that E is a locally convex space and C_E is a convex, pointed cone that defines a partial order $(\geq_{C_E})$ in E.

Definition 2. *Let $A \subseteq E$ be a nonempty set. We say that*

(i) a point $a \in A$ is an ideal (or utopia) minimal point of A if $x \geq_{C_E} a$ for every $x \in A$. The set of all ideal minimal points of A is denoted by IMin(A) or IMin$(A|C_E)$;
(ii) a point $a \in A$ is a minimal (or Pareto minimal/efficient/nondominated) point of A if whenever $a \geq_{C_E} x$ for some $x \in A$ one has $x \geq_{C_E} a$. The set of all minimal points of A is denoted by Min(A) or Min$(A|C_E)$.

Sometimes one is interested also in the set of minimal points with respect to the order generated by the cone $\{0\} \cup \operatorname{int} C_E$ if $\operatorname{int} C_E \neq \emptyset$. This is the set of weakly minimal points and denoted by $\mathrm{WMin}(A)$ or $\mathrm{WMin}(A|C_E)$. It is clear that a minimal point is weakly minimal, but the converse is not true. The reason for studying weakly minimal points is that they are easier to compute and have nicer properties than minimal points. The limit of a sequence of minimal points of a compact set may be not minimal, but weakly minimal. The concept of maximal points is defined similarly. The set of all maximal points of A is denoted by $\mathrm{Max}(A)$ or $\mathrm{Max}(A|C_E)$. It is clear that a point belongs to $\mathrm{Max}(A|C_E)$ if and only if it belongs to $\mathrm{Min}(A|-C_E)$. A similar conclusion remains true for ideal maximal points and weakly maximal points. Below is an equivalent definition of minimal points.

Proposition 2. *Let $A \subseteq Y$. Then*

(i) $a \in$ IMin(A) if and only if $a \in A$ and $A \subseteq a + C_E$;
(ii) $a \in$ Min(A) if and only if $a \in A$ and $A \cap (a - C_E) = \{a\}$. In other words $a \in$ Min(A) if and only if $a \in A$ and there is no $y \in A$ with $a \geq_{C_E} y$ and $a \neq y$;
(iii) $a \in$ WMin(A) if and only if $a \in A$ and $A \cap (a - \operatorname{int} C_E) = \emptyset$.

In typical situations, ideal minimal and ideal maximal points do not exist. When Y is the space $\mathbb{R}^n$ equipped with the Pareto cone, ideal minimality means minimum of each component. For this reason, ideal points are not a main concern of our study. Furthermore, among minimal points there are some that are stable with respect to perturbations of the order. For instance, the unit disk of the space $\mathbb{R}^2$ ordered by the Pareto cone has minimal points on the negative quarter of the boundary. The points $x = (-1, 0)$ and $y = (0, -1)$ are minimal points of the disk, which are no longer minimal when we use the perturbed partial order determined by the cone

$$C := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + \varepsilon x_2 \geq 0, \varepsilon x_1 + x_2 \geq 0\}$$

for $\varepsilon > 0$. All other minimal points of the disk remain minimal with respect to C if ε is small enough. This suggests the following concept of Pareto minimality.

Definition 3. *Assume that C_E has a convex and bounded base B. If there is a convex neighborhood V of the origin in Y such that the cone generated by $B + V$ is not identical to Y and a point $a \in A$ is minimal with respect to the order ($\geq_{cone(B+V)}$), then we say that a is a properly minimal point of A. The set of all properly minimal points of A is denoted by PMin(A) or PMin($A|C_E$).*

It can be seen that the definition of proper minimality does not depend on the choice of a convex, bounded base of C_E. In other words, if B and B' are two convex, bounded bases of C_E, then the sets of properly minimal points defined by using B and B' coincide.

To illustrate the above definitions, let us consider the following example. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \quad \text{or} \quad x \geq 0,\ |y| \leq 1\} \subseteq \mathbb{R}^2$ and let $C_E = \mathbb{R}^2_+$. Then

$$\begin{aligned}
\text{IMin}\ A &= \emptyset \\
\text{PrMin}\ A &= \{(x, y) \in \mathbb{R}^2 : \quad x^2 + y^2 = 1,\ x < 0,\ y < 0\} \\
\text{Min}\ A &= \{(x, y) \in \mathbb{R}^2 : \quad x^2 + y^2 = 1,\ x \leq 0,\ y \leq 0\} \\
\text{WMin}\ A &= \text{Min}\ A \cup \{(x, -1) : x \geq 0\}.
\end{aligned}$$

The relationship between the different concepts of minimality is seen in the next proposition the proof of which follows from the definitions.

Proposition 3. *For every nonempty set $A \subseteq E$ one has*

(i) *PrMin(A) $\subseteq$ Min(A) $\subseteq$ WMin(A);*
(ii) *If IMin(A) $\neq \emptyset$, then IMin(A) $=$ Min(A) and this set is a singleton.*

The next theorem shows that properly minimal points are good representatives of the set of minimal points. We recall that the weak recession cone of a set $A \subseteq E$ is the cone $A_{w\infty}$ consisting of the weak limits of nets $\{t_\alpha a_\alpha\}_{\alpha \in I}$ where $a_\alpha \in A$ and $t_\alpha > 0$ tends to 0. We refer the interested reader to [35] and [38] for more on recession cones in infinite dimensional spaces.

Theorem 1. *Assume that either of the following conditions holds:*

(i) A is weakly compact and B is a closed, convex, and bounded base of C_E;
(ii) The space E is Banach, A is weakly closed with $A_{w\infty} \cap (-B) = \emptyset$, and B is a weakly compact, convex base of C_E.

Then PMin$(A|C_E)$ is dense in Min$(A|C_E)$.

This theorem has recently been proved in [19]. See also [7] and [12] for some particular cases. The notion of proper minimality encompasses some known concepts of efficiency. First we note that proper minimality can be considered as a direct extension of Henig's proper efficiency [22] to infinite dimension. Recall that in the n-dimensional space $\mathbb{R}^n$ partially ordered by a closed and convex cone C, a point $x_0 \in A \subseteq \mathbb{R}^n$ is called a Henig-efficient point of A with respect to C if there is a convex cone $K \subseteq \mathbb{R}^n$, $K \neq \mathbb{R}^n$ such that $C \setminus \{0\} \subseteq \text{int } K$ and $x_0 \in \text{Min}(A|K)$. As noticed before, in this situation C has a convex and bounded base and there is a convex neighborhood V of 0 such that $\text{cone}(B+V) \subseteq K$. Hence $x_0 \in A$ is properly minimal if and only if it is Henig-efficient. When C is not closed or when the space is infinite dimensional, the existence of a convex cone $K \neq X$ with $C \setminus \{0\} \subseteq \text{ int } K$ and $x_0 \in \text{Min}(A|K)$ does not imply that x_0 is properly minimal.

Now we recall some more concepts of efficiency from [7] and [12] that are important in the study of the geometry of minimal points. Let $A \subseteq X$ be nonempty. A point $x_0 \in A$ is said to be

(a) a superefficient point of A if for every neighborhood V of 0 there is a neighborhood U of 0 such that

$$\text{cl cone}(A - x_0) \cap (-C + U) \subseteq V;$$

(b) a strictly efficient point of A if there is a neighborhood U of 0 such that

$$\text{cl cone}(A - x_0) \cap (-B + U) = \emptyset;$$

(c) a strongly efficient point of A if for every $\xi \in X'$ there are neighborhoods U and V of 0 such that $\langle \xi, \cdot \rangle$ is bounded on the set

$$\text{cone}(A - x_0) \cap [U - \text{cone}(B + V)].$$

Proposition 4. *Let A be a nonempty set and let $x_0 \in A$. The following assertions are equivalent:*

(i) x_0 is a properly minimal point of A;
(ii) x_0 is a strictly efficient point of A;
(iii) x_0 is a strongly efficient point of A;
(iv) x_0 is a superefficient point of A.

The proof of these equivalences is given in [19]. The interest of properly minimal points resides also in the fact that under some convexity hypotheses, properly minimal points are obtained by minimizing certain positive functionals (see Proposition 9 of Section 6). We conclude this section by pointing out that proper minimality is defined under the condition that the ordering cone has a convex and bounded base. This concept can be extended to a more general ordering cone in which case some of the equivalences of Proposition 4 may fail.

4 Existence

As in the previous section, E is a locally convex space partially ordered by a convex and pointed cone C_E. In order to present a general condition on the existence of minimal points, let us define a section of a set $A \subseteq E$ at $x \in E$ by $A_x := A \cap (x - C_E)$. This is the set of all elements of A that are smaller (with respect to the order $\geq_{C_E}$) than x.

Definition 4. *A set $A \subseteq E$ is said to be C_E-complete (resp. strongly C_E-complete) if it has no covering of the form*

$$\{(x_\alpha - \operatorname{cl} C_E)^c : \alpha \in \Gamma\} \quad \big(\text{resp. } \{(x_\alpha - C_E)^c : \alpha \in \Gamma\}\big)$$

where $\{x_\alpha\}_{\alpha \in \Gamma}$ is a decreasing net in A.

We note that every strongly C_E-complete set is C_E-complete. The converse is not always true. When C_E is a closed cone, these two concepts coincide.

Some related notions are also present in the literature. Let us recall two of them that are frequently cited:

(i) A subset A of E is said to be C_E-compact if any cover of A of the form $\{U_i + C_E : i \in I, U_i \text{ are open}\}$ admits a finite subcover (see [20, 34]);
(ii) A subset $A \subseteq E$ is said to be C_E-semicompact if any cover of A of the form $\{(x_i - \operatorname{cl} C_E)^c : i \in I, x_i \in A\}$ admits a finite subcover (see [13]).

It is clear that every compact set is C_E-compact, and every C_E-compact set is C_E-semicompact. The converse is not true in general. Here are some sufficient conditions of C_E-complete sets:

(i) Every C_E-semicompact set is C_E-complete. In particular every weakly compact set in a locally convex space is C_E-complete.
(ii) Every compact set is strongly C-complete if C has Sterna–Karwat's property: for every linear subspace $L \subseteq E$, the set $C \cap L$ is a linear subspace if and only if $\ell(C \cap L)$ is a linear subspace. In particular, when E is finite dimensional, every compact set is strongly C-complete whatever C is.

The first condition and the second part of the second one can be found in [34], the second condition can be derived from [50]. The following result is already known (see [36]).

Theorem 2. *Let A be a nonempty set in E. The following assertions hold:*

(i) $Min(A) \neq \emptyset$ if and only if there is $x \in E$ such that A_x is nonempty and strongly C_E-complete;
(ii) When C is correct, $Min(A) \neq \emptyset$ if and only if there is $x \in E$ such that A_x is nonempty and C_E-complete.

The next particular case is useful in practice.

Corollary 1. *If A is a nonempty compact set in a finite dimensional space, then $Min(A) \neq \emptyset$ whatever the cone C_E is. If A is a nonempty compact set in an infinite dimensional space and the cone C_E is closed, then $Min(A) \neq \emptyset$.*

Note that in an infinite dimensional space, a compact set may have no efficient points if the cone C_E is not correct. In fact, let E be the space of sequences whose terms are all zero except for a finite number. Elements of E are written as infinite dimensional vectors $(x^1, x^2, \ldots)$. The space is equipped with the max-norm: $\|(x^1, x^2, \ldots.)\| = \max\{|x^i| : i = 1, 2, \ldots\}$. The ordering cone C_E consists of sequences whose last nonzero term is positive. This cone is convex and pointed but not correct because its closure is the whole space E. Let $x_0 = (1, 0, 0, \ldots), x_n = (1, -\frac{1}{2^n}, \ldots, -\frac{1}{2^n}, 0, \ldots 0)$, and $A = \{x_i : i = 0, 1, 2, \ldots\}$. It is evident that $\lim_{n\to\infty} x_n = x_0$. Hence A is a compact set. Despite this, $\mathrm{Min}(A) = \emptyset$ because $x_0 >_{C_E} x_1 >_{C_E} x_2 \cdots$

When the cone C_E is closed, the following result is a useful criterion for a point to be minimal. We shall say that a net $\{x_i : i \in I\}$ is loosely decreasing if $x_i \geq_{C_E} x_j$ for $i < j$. When $E = \mathbb{R}$ and $C_E = \mathbb{R}_+$, loosely decreasing nets are exactly nonincreasing nets.

Theorem 3. *Assume that C_E is closed. The following assertions are equivalent*

(i) A point $a \in A$ is minimal;
(ii) The set $A \setminus \{a\}$ is covered by the family $\{(x_\lambda - C_E)^c : \lambda \in \Lambda\}$ for every loosely decreasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ in A converging to a;
(iii) The set $A \setminus \{a\}$ is covered by the family $\{(x_\lambda - C_E)^c : \lambda \in \Lambda\}$ for some loosely decreasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ in A converging to a.

Proof. We first establish the implication $(i) \Rightarrow (ii)$. Suppose to the contrary that (ii) does not hold. There exist a loosely decreasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ converging to a and an element $b \in A \setminus \{a\}$ such that $b \notin (x_\lambda - C_E)^c$, or equivalently $b \in x_\lambda - C_E$ for all $\lambda \in \Lambda$. By passing to the limit and due to the closedness of C_E, we derive $b \in a - C_E$. This contradicts (i).

The implication $(ii) \Rightarrow (iii)$ is clear. We finally prove the implication $(iii) \Rightarrow (i)$. Suppose to the contrary that a is not minimal. Then one can find some element $b \in A \setminus \{a\}$ such that $b \in a - C_E$. In view of (iii), there is some index $\lambda_0 \in \Lambda$ such that $b \in (x_{\lambda_0} - C_E)^c$. This implies that $a \in (x_{\lambda_0} - C_E)^c$, and hence

$$a \notin x_{\lambda_0} - C_E. \tag{1}$$

On the other hand, the net $\{x_\lambda\}_{\lambda \in \Lambda}$ being loosely decreasing, we have $x_\lambda \in x_{\lambda_0} - C_E$ for all $\lambda \geq \lambda_0$. By passing to the limit in the above inclusion, we obtain $a \in x_{\lambda_0} - C_E$, which contradicts (1). The proof is complete. ■

As a consequence of the preceding theorem, we derive the following result of [23], which was used to obtain a vector version of Ekeland's variational principle.

Corollary 2. *Let E be a sequentially complete locally convex space, let $\{p_\lambda : \lambda \in \Lambda\}$ be a base of seminorms determining the locally convex topology of E, and let C_E be pointed, closed, and convex cone in E. Assume that there is a sequence $\{x_n\}_{n=0}^\infty \subseteq A$ such that*

(i) $x_{n+1} \in x_n - C_E$, $n = 0, 1, 2, \ldots$
(ii) $\lim_{n\to\infty} \sup_{x,y \in A_{x_n}} p_\lambda(x - y) = 0$ *for all* $\lambda \in \Lambda$.

Then $\{x_n\}_{n=0}^\infty$ is convergent and its limit is a minimal point of A.

Proof. The condition (i) shows that the sequence $\{x_n\}_{n=0}^\infty$ is loosely decreasing. The condition (ii) shows that it is fundamental, hence it converges to some element $a \in A$, and the family $\{(x_n - C_E)^c : n = 0, 1, \ldots\}$ is a covering of the set $A \setminus \{a\}$. In view of Theorem 3, the limit a is minimal. ■

We now derive an existence condition for properly minimal points in a Banach space.

Corollary 3. *Let E be a Banach space and let C_E be a cone with a closed, bounded, and convex base B. Then every closed, bounded set has a properly minimal point.*

Proof. Let V_ε be a ball centered at the origin with radius $\varepsilon > 0$. When ε is small enough, the cone $K :=\text{cl cone}(B + V_\varepsilon)$ has the Daniell property. Let A be a closed and bounded set in E. As K has interior, the set A is minorized in the sense that it is contained in $a + K$ for some element $a \in E$. Thus, A is K-complete and as K is closed, it is correct. Applying Theorem 2, we conclude that A has a properly minimal point. ■

Some more results on existence of minimal points can be found in [17, 20, 23, 43, 44] and the references given in these.

5 Optimality in Product Spaces

5.1 Product Order and Set-Valued Optimization

Let X be a nonempty set, let Y and Z be two locally convex spaces, and let $C_Y \subset Y$ and $C_Z \subset Z$ be convex cones. In this section, we study the following vector (or multiobjective) minimization problem with set-valued data

$$(V) \qquad \begin{array}{l} \text{Min } F(x) \\ \text{s.t. } x \in X,\ G(x) \cap C_Z \neq \emptyset \end{array}$$

where F and G are set-valued maps from X to Y and Z, respectively.

The feasible set of problem (V) is given by

$$X_0 := \{x \in X : G(x) \cap C_Z \neq \emptyset\}.$$

A couple $(x_0, y_0) \in X_0 \times Y$ with $y_0 \in F(x_0)$ is called a minimizer of (V) if $y_0 \in \mathrm{Min}(F(X_0)|C_Y)$. The point x_0 is often called an efficient solution of (V) and y_0 is called an efficient value of (V). Properly efficient solutions and weakly efficient solutions are defined in a similar manner. To avoid misunderstanding, sometimes one writes Min_{C_Y} instead of Min in the formulation of the problem (V) to underline with respect to which order one minimizes the map. When no confusion likely occurs, one omits the subscript C_Y. The vector maximization problem associated to F and G is written as

$$\begin{array}{ll} \mathrm{Max}_{C_Y}\ F(x) & \\ \text{s.t.} & x \in X,\ G(x) \cap C_Z \neq \emptyset \end{array}$$

which is nothing, but the vector minimization problem with respect to the ordering cone $-C_Y$:

$$\begin{array}{ll} \mathrm{Min}_{-C_Y}\ F(x) & \\ \text{s.t.} & x \in X,\ G(x) \cap C_Z \neq \emptyset. \end{array}$$

The constraint $G(x) \cap C_Z \neq \emptyset$ is a generalized form of the usual equality and inequality constraints in mathematical programming. Indeed, if the space Z is the product space $\mathbb{R}^p \times \mathbb{R}^q$ and the cone C_Z is the product cone $\mathbb{R}^p_+ \times \{0\}$, and the map G is single-valued, say $G(x) = (g_1(x), \ldots, g_{p+q}(x))$, then the constraint becomes $G(x) \in C_Z$, which is expressed by the system of p inequalities and q equalities:

$$\begin{array}{l} g_i(x) \geq 0, i = 1, \ldots, p \\ g_j(x) = 0, j = p+1, \ldots, p+q. \end{array}$$

In mathematical programming, there are techniques to link the objective function and the constraints of a given problem. For instance, given a mathematical programming problem

$$(P) \qquad \begin{array}{ll} \min f(x) & \\ \text{s.t.} & g_i(x) \geq 0, i = 1, \ldots, p \\ & g_j(x) = 0, j = p+1, \ldots, p+q. \end{array}$$

We can transform the objective function f into a constraint by introducing an additional variable $t \in \mathbb{R}$ and a constraint $f(x) \leq t$. Thus, (P) is equivalent to

$$(P') \qquad \begin{array}{l} \min t \\ \text{s.t.} \quad t - f(x) \geq 0 \\ g_i(x) \geq 0, i = 1, \dots, p \\ g_j(x) = 0, j = p+1, \dots, p+q. \end{array}$$

A converse technique is to incorporate constraints into the objective function so that a resulting problem is an optimization problem without constraints. This is a penalty method that consists of adding a term $\phi(g_1(x), \dots, g_{p+q}(x))$ to the objective function, where ϕ is a function that takes the null value if all the constraints are satisfied and very big values (even the value $+\infty$) when the constraints are violated.

In this subsection, we shall see that by studying the problem in a product space it is also possible to convert a constrained problem into an unconstrained problem without using penalty functions. Let us first define some new orders in the product space. We assume that C_Y and C_Z have bounded and convex bases B_Y and B_Z, respectively. The cone $C_Y \times C_Z$ is convex in $Y \times Z$ and has a bounded base $B = \{(tb_1, (1-t)b_2) : t \in [0,1], b_1 \in B_Y, b_2 \in B_Z\}$; the space $Y \times Z$ being equipped with the product topology. We shall, however, be interested in a smaller cone

$$K := \operatorname{cone}(B_Y \times C_Z).$$

In general this cone is convex but not closed even when C_Y and C_Z are closed. The following inclusions are immediate:

$$\begin{array}{l} K \subseteq C_Y \times C_Z \\ \mathrm{cl}K = \mathrm{cl}(C_Y \times C_Z). \end{array}$$

A bounded base B_K of K can be given by

$$B_K = \{(tb_1, (1-t)b_2) : t \in (0,1], b_1 \in B_Y, b_2 \in B_Z\}.$$

We give here an example to clearly see the distinction between these ordering cones. Let the space Y be $\mathbb{R}^2$ with an ordering cone $C_Y = \mathbb{R}^2_+$ and a base $B_Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$. Let Z be $\mathbb{R}$ with $C_Z = \mathbb{R}_+$ and a base $B_Z = \{1\}$. Then the product cone $C_Y \times C_Z$ is the Pareto cone $\mathbb{R}^3_+$, and the cone K consists of the origin and the vectors $(y_1, y_2, y_3) \in \mathbb{R}^3$ satisfying $y_1 \geq 0, y_2 \geq 0$ and $y_3 > 0$. The base B_K is given by $B_K = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 \geq 0, y_2 \geq 0, y_3 > 0, y_1 + y_2 + y_3 = 1\}$.

Given a nonempty set Q in the product space $Y \times Z$, we are interested in a link between the minimal points of Q with respect to the ordering cone $C_Y \times C_Z$ and the ones with respect to the cone K. The following inclusions are clear:

(i) $\mathrm{Min}(Q|\{0\} \times C_Z) \cap \mathrm{Min}(Q|K) \subseteq \mathrm{Min}(Q|C_Y \times C_Z) \subseteq \mathrm{Min}(Q|K)$
(ii) $\mathrm{Min}(Q|K) \subseteq \mathrm{Min}(Q|C_Y \times C_Z) - \{0\} \times C_Z$ provided that Q has the domination property with respect to $C_Y \times C_Z$, that is $\mathrm{Min}([(x,y) + C_Y \times C_Z] \cap Q | C_Y \times C_Z) \neq \emptyset$ for every $(x,y) \in Q$.

Sometimes the structure of the set Q is complicated, but the structure of its enlargement by means of the cones $C_Y \times C_Z$ is simple; for instance when the set Q is not convex, but the set $Q - C_Y \times C_Z$ is convex or polyhedral. In such situations, the next characterization is useful in establishing optimality conditions of minimal points of Q (see [19] for the proof).

Proposition 5. *Let $Q \subseteq Y \times Z$ be a nonempty set and $(x_0, y_0) \in Q$. Then the following assertions are equivalent*

(i) $(x_0, y_0) \in Min(Q|K)$
(ii) $(0,0) \in Min(Q - (x_0, y_0) - C_Y \times C_Z|K)$
(iii) $(0,0) \in Min(cone(Q - (x_0, y_0) - C_Y \times C_Z)|K)$.

When the cones C_Y and C_Z have interior, we have also the following equivalent assertions:

(i) $(x_0, y_0) \in Min(Q|\text{cone}(\text{int}K))$;
(ii) $(0,0) \in Min(\text{cone}(Q - (x_0, y_0) - C_Y \times C_Z)|\text{cone}(\text{int}K))$;
(iii) $(0,0) \in Min(\ \text{clcone}(Q - (x_0, y_0) - C_Y \times C_Z)|\text{cone}(\text{int}K))$.

Let us now return to the vector problem (V). The following observation (Corollary 5.9 of [34], page 60) shows why it is important to study minimal points in the product space $Y \times Z$ ordered by the special cone K. We write $(a, b) \in P_Y Min(Q|K)$ when there is a small convex neighborhood V of the origin in Y such that $B_Y + V$ does not contain the origin in its closure, and (a, b) is a minimal point of Q with respect to the ordering cone cone$[(B_Y + V) \times C_Z]$.

Lemma 1. *A couple (x_0, y_0) is a minimizer (resp., proper minimizer) of (V) if and only if $(y_0, 0) \in Min(Q|K)$ (resp., $(y_0, 0) \in P_Y\ Min(Q|K)$) where $Q = \bigcup_{x \in X} \{(F(x), G(x)) + C_Y \times C_Z\}$.*

The next result gives a general multiplier rule for set-valued vector problems (see also [54]). We shall denote by $B^\sharp$ the set of linear functionals $\xi \in Y'$ such that $\inf_{y \in B} \langle \xi, y \rangle > 0$.

Theorem 4. *Assume that (x_0, y_0) is a minimizer of (V). If cone $(Q - (y_0, 0))$ has a nonempty convex interior, then there exists $(\xi, \gamma) \in [C_Y \times C_Z]' \setminus \{(0,0)\}$ such that*

$$\langle \xi, y \rangle + \langle \gamma, z \rangle \geq \langle \xi, y_0 \rangle$$

for every $y \in F(x)$, $z \in G(x)$ and $x \in X$. Moreover, if int $C_Z \neq \emptyset$ and there is some $x \in X$ such that $G(x) \cap$ int $C_Z \neq \emptyset$, then $\xi \neq 0$. If, in addition (x_0, y_0) is a proper minimizer of (V), then $\xi \in B_Y^\sharp$.

Proof. Apply Lemma 1 and the classic separation theorem of convex sets. ■

A sufficient condition can also be given when a multiplier rule holds.

Proposition 6. *Assume that x_0 is feasible and $y_0 \in F(x_0)$. If there is $(\xi, \gamma) \in C_Y^+ \times C'_Z$ (resp., $(\xi, \gamma) \in B_Y^\sharp \times C'_Z$) such that*

$$\langle \xi, y \rangle + \langle \gamma, z \rangle \geq \langle \xi, y_0 \rangle$$

for every $y \in F(x)$, $z \in G(x)$, and $x \in X$, then (x_0, y_0) is a minimizer (resp., proper minimizer) of (V).

Proof. Invoke Lemma 1 and Proposition 9 in the next section. ■

5.2 Enlarged Order and Variational Principle for Set-Valued Maps

Another case of minimality in a product space was recently studied by Isac and Tammer [23] (see also [17]). Let X and Y be Banach spaces whose norms are denoted by the same symbol $\|.\|$. The product space $X \times Y$ is equipped with the sum norm. Let C_Y be a closed, pointed, and convex cone in Y. We fix an element $e \in C_Y$ and consider the following cone in the product space $X \times Y$ for $\varepsilon \in (0, 1)$:

$$C(\varepsilon) := \{(x, y) \in X \times Y : y + \epsilon e\|(x, y)\| \in -C_Y\}.$$

It can be proven (see [23]) that this cone has a closed, convex, and bounded base. The next corollary is an improvement of the maximal point theorem of [23].

Corollary 4. *Assume that $A \subseteq X \times Y$ is a nonempty, closed set such that its projection on Y is minorized. Then the section of A at every point in A has a properly minimal point with respect to the ordering cone $-C(\varepsilon)$.*

Proof. Let (a, b) be an element of A. Let (x, y) be in the section $A_{(a,b)} := A \cap ((a, b) + C(\varepsilon))$ (with respect to the ordering cone $-C(\varepsilon)$). By the definition of the cone $C(\varepsilon)$, one has

$$\begin{aligned} y &\in b - \varepsilon e\|(x - a, y - b)\| - C_Y \\ &\subseteq b - C_Y. \end{aligned} \tag{2}$$

Hence, the projection of the section $A_{(a,b)}$ on Y is majorized by b. It follows from this and the hypothesis that the projection of $A_{(a,b)}$ on Y is bounded. The relation (2) shows that the projection of $A_{(a,b)}$ on X is bounded, too, and so the section $A_{(a,b)}$ is bounded itself. It is clear that it is closed because the cone $C(\varepsilon)$ is closed. It remains to apply Corollary 3 to complete the proof. ■

Let F be a set-valued map from X to Y. The graph of F is the set

$$graph(F) := \{(x, y) \in X \times Y : y \in F(x), x \in Y\}.$$

Let $\varepsilon > 0$ and $e \in C_Y \setminus \{0\}$. We say that $(x_0, y_0) \in graph(F)$ is an (ε, e)-minimizer of F if

$$F(X) \cap (y_0 - \varepsilon e - C_Y \setminus \{0\}) = \emptyset.$$

The following result is a set-valued version of the variational principle given in [23].

Theorem 5. *Assume that the graph of F is closed and the image $F(X)$ is minorized in Y. Then for every real number $\varepsilon \in (0,1)$ and every (ε^2, e)-minimizer (x_0, y_0) of F, there exists $(x_\varepsilon, y_\varepsilon)$ in the graph of F such that*

(i) $y_\epsilon \in y_0 - \varepsilon e\|x_\varepsilon - x_0\| - C_Y$*;*
(ii) $\|x_\varepsilon - x_0\| \leq \varepsilon$*;*
(iii) $(x_\varepsilon, y_\varepsilon)$ *is a minimizer of the set-valued map* $x \mapsto \hat{F}(x) := \{y + \varepsilon e\|x - x_\varepsilon\| + \|y - y_\varepsilon\| : y \in F(x))\}$*.*

Proof. Set $A = graph(F)$ and apply Corollary 3 to obtain a minimal point $(x_\varepsilon, y_\varepsilon) \in A_{(x_0,y_0)}$ of the set A with respect to the cone $-C(\varepsilon)$ in the product space $X \times Y$. We have

$$(x_\varepsilon - x_0, y_\varepsilon - y_0) \in C(\varepsilon) \tag{3}$$
$$(x - x_\varepsilon, y - y_\varepsilon) \notin C(\varepsilon) \setminus \{0\} \quad \text{for}(x, y) \in A \setminus \{(x_\varepsilon, y_\varepsilon)\} \tag{4}$$

The relation (3) implies

$$\begin{aligned} y_\varepsilon &\in y_0 - \varepsilon e\|(x_\varepsilon - x_0, y_\varepsilon - y_0)\| - C_Y \\ &\subseteq y_0 - \varepsilon e\|x_\varepsilon - x_0\| - C_Y \end{aligned}$$

which yields (i). The relation (ii) is obtained from the first one and from the fact that (x_0, y_0) is an (ε^2, e)-minimizer of F.
The relation (4) gives

$$y + \varepsilon e(\|x - x_\varepsilon\| + \|y - y_\varepsilon\|) \notin y_\varepsilon - C_Y\{0\}$$

for every $x \in X$ and $y \in F(x)$. This is (iii). ∎

For other variants of vector variational principle, see [10, 21], and the references given in [23].

6 Scalarization

Consider the following multiobjective optimization problem:

$$(VP) \qquad \begin{array}{ll} \text{Max } f(x) & \\ \text{s.t.} \quad x \in X_0 & \end{array}$$

where X_0 is a nonempty set in the space X and f is a vector-valued function from X_0 to Y. The set of all efficient (resp. properly efficient, weakly efficient) solutions of (VP) is denoted by $S(f, X_0)$ (resp. $PrS(f, X_0)$, $WS(f, X_0)$).

A frequently used method in the study of the problem (VP) is to convert it into a scalar optimization problem of the form

$$(P) \qquad \begin{array}{ll} \max & g \circ f(x) \\ \text{s.t.} & x \in X_0 \end{array}$$

where g is a real-valued function on $f(X_0)$, called a scalarizing function. In this section, we shall develop a class of scalarizing functions with such a property that optimal solutions of (P) furnish efficient solutions of (VP).

6.1 General Case

Let us recall that a function $g : Y_0 \to R \cup \{\pm\infty\}$ is said to be increasing (with respect to the cone C_Y) on the set $Y_0 \subseteq Y$ if $a >_{C_Y} b$ for $a, b \in Y_0$ implies $g(a) > g(b)$. An increasing function with respect to the cone $(\text{int}C_Y) \cup \{0\}$ is called weakly increasing.

The following standard result expresses a relationship between the solution set $S(g \circ f, X)$ of problem (P) and that of (VP) (see [34] for the proof).

Proposition 7. *The following assertions are true:*

(i) If g is increasing (respectively, weakly increasing), then every optimal solution of (P) is an efficient solution (respectively, weakly efficient solution) of (VP);
(ii) Conversely, for every weakly efficient solution x of (VP), there exists a continuous weakly increasing function g such that x is an optimal solution of (P).
(iii) There exists a continuous weakly increasing function g such that the weakly efficient solution set of (VP) coincides with the solution set of (P).
(iv) If Y is a normed space, C_Y has a compact base, and if both $f(A)$ and $Max(f(A)|C_Y)$ are compact, then there is a continuous, increasing function g such that the efficient solution set of (VP) coincides with the solution set of (P).

Below are some useful scalarizations.

(1) **Linear scalarization:** Let ξ be a continuous linear function on Y. Then ξ is increasing (respectively weakly increasing) if and only if ξ belongs to the strictly polar cone C_Y^+ (resp. $C' \setminus \{0\}$).

When Y is the space $\mathbb{R}^n$ and the ordering cone is $\mathbb{R}^n_+$, the scalarized problem that uses a linear increasing function is written as

$$(P_\xi) \qquad \begin{array}{ll} \max & \sum_{i=1}^n \xi_i f_i(x) \\ \text{s.t.} & x \in X_0 \end{array}$$

for some positive vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_+$. The positive numbers $\xi_1, \ldots, \xi_n$ are called weights. Each number ξ_i expresses the importance of the criterion (component) f_i with respect to the others. For instance, by choosing the first component of ξ equal to one and the other components equal to zero, we mean to take only the first criterion f_1 into the consideration and neglect the others. The scalar problem (P_ξ) is called the weighted problem of (VP) associated with the weight vector ξ. Certain generalizations of (P_ξ) are also in use. They are of the form:

$$(P'_\xi) \qquad \begin{array}{ll} \max & \sum_{i=1}^{n} \xi_i [f_i(x)]^\rho \\ \text{s.t.} & x \in X_0 \end{array}$$

and

$$(P''_\xi) \qquad \begin{array}{ll} \max & \sum_{i=1}^{n} [\xi_i f_i(x)]^\rho \\ \text{s.t.} & x \in X_0 \end{array}$$

where ρ is a positive number *a priori* chosen and it is assumed that $f_i(x) \geq 0$ for all $i = 1, \ldots, n$ and $x \in X_0$.

Linear scalarization is particularly helpful when the vector problem is linear or convex.

Proposition 8. *Assume that (VP) is a linear problem with values in a finite dimensional space. Then there exists a finite number of weight vectors* $\xi^i : i = 1, \ldots, k$ *such that the solution set of* (VP) *is exactly the union of the solution sets of the scalarized problems* $(P_{\xi^i}), i = 1, \ldots, k$.

Several methods exist (see [3, 26, 55, 57]) that determine the weight vectors stated in Proposition 8. For a convex problem, that is problem (VP) with X_0 being a convex set and f a concave function, we have the following standard result that can be proved by a separation of convex sets (see [9, 34]).

Proposition 9. *Assume that (VP) is a convex problem. Then*

(i) *a point* $x \in X_0$ *is a properly efficient solution of (VP) if and only if it is an optimal solution of (P) with* $g \in C_Y^+$;
(ii) *a point* $x \in X_0$ *is a weakly efficient solution of (VP) if only if it is an optimal solution of (P) with* $g \in C'_Y \setminus \{0\}$.

It should be noticed that an efficient solution of a convex problem, which is not proper, can be a solution of no scalar problem (P) with $g \in C_Y^+$. Thus we have the following inclusions for a convex problem

$$\begin{aligned} PrS(f, X_0) &= \cup_{g \in C_Y^+} S(g \circ f, A) \subseteq S(f, X_0) \\ &\subseteq WS(f, X_0) = \cup_{g \in C'_Y \setminus \{0\}} S(g \circ f, X_0). \end{aligned}$$

Under some additional hypotheses on C_Y, X_0, and f (for instance when C_Y is closed and has a bounded convex base, X_0 is compact, and f is continuous), one may have

$$S(f, X_0) \subseteq cl[PrS(f, X_0)].$$

The equality

$$S(f, X_0) = WS(f, X_0)$$

can be realized if f is strictly convex. Another important feature of linear scalarization is that for $g \in C_Y^{'}$, problem (P) is convex whenever (VP) is convex. This allows one to apply convex optimization techniques to solve the vector problem.

(2) **Scalarization by the smallest weakly increasing functions:** Let $a \in Y$ and $e \in \operatorname{int} C_Y$. Set

$$h_{e,a}(y) = \sup\{t \in \mathbb{R} : y \in a + te + C_Y\} \text{ for } y \in Y.$$

This function is weakly increasing and has the property that if g is any other weakly increasing function on Y, then the upper level set of g at $g(a)$ must contain the level set of $h_{e,a}$ at 0 for some $a \in Y$.

In the finite dimensional case $Y = \mathbb{R}^n$ and $C_Y = \mathbb{R}^n_+$, the function $h_{e,a}$ is expressed by

$$h_{e,a}(y) = \min\{\frac{y_i - a_i}{e_i} : i = 1, \dots, n\}.$$

The associated scalar problem is of the form

$$(P_{e,a}) \qquad \begin{array}{ll} \min & \min\{\frac{f_i(x) - a_i}{e_i} : i = 1, \dots, n\} \\ \text{s.t.} & x \in X_0 \end{array}$$

The usefulness of this scalarization is seen in the next result, which concretizes the second assertion of Proposition 7.

Recall that f is quasiconvex on a convex set X_0 if for every point x, y in X_0 and for every $a \in \mathbb{R}^n$ with $a \geq_{C_Y} f(y)$ and $a \geq_{C_Y} f(x)$, one has $a \geq_{C_Y} f(\lambda x + (1 - \lambda) y)$ for all $\lambda \in [0, 1]$. This amounts to say that the composite function $\xi \circ f$ is quasiconvex for all extreme directions of the polar cone C_Y'. We refer to [2] and [36] for the proof.

Proposition 10. *For each vector $e \in int(C_Y)$, every weakly efficient solution x of (VP) is an optimal solution of the scalarized problem $(P_{h_{f(x),e}})$. Moreover, this scalar problem is quasiconvex whenever f is quasiconvex.*

6.2 Scalarization via Support Functions

The assertion (iii) of Proposition 7 is quite general. It states that by solving one scalarized problem, one is able to obtain all solutions of the vector problem. The construction of a scalarizing function with such a property is not easy. We shall develop a constructive method for such a kind of scalarizing functions.

Given a nonempty set $A \subseteq Y$, its support function s_A is defined on Y' by

$$s_A(\xi) = \sup_{y \in A} \langle \xi, y \rangle \quad \text{for} \quad \xi \in Y';$$

and the polar set of A is a subset $A^\circ \subseteq Y'$ defined by

$$A^\circ = \{\xi \in Y' : s_A(\xi) \leq 1\}.$$

Let $\mathcal{A}$ be the family of subsets A of Y that satisfy $A \cap C \neq \emptyset$. We shall adopt the following convention

$$\frac{r}{0} = \begin{cases} +\infty \text{ if } & 0 < r \quad \text{or} \quad r = +\infty \\ -\infty \text{ if } & 0 > r \quad \text{or} \quad r = -\infty \\ 0 \quad\ \text{ if } & r = 0. \end{cases}$$

Let Λ be a closed base of C^+, that is $C^+ = \{t\xi : \xi \in \Lambda,\ t \geq 0\}$ and $0 \notin \Lambda$. For every $A \in \mathcal{A}$, we define a function $g_A : Y \longrightarrow [0, \infty]$ by

$$g_A(y) = \sup_{\xi \in \Lambda} \frac{\langle \xi, y \rangle^+}{s_A(\xi)},$$

where $\langle \xi, y \rangle^+ = \max\{\langle \xi, y \rangle, 0\}$. This function will play a crucial role in solving problem (VP). Below we list some of its properties which can be verified without difficulty.

Proposition 11. *Let $A \in \mathcal{A}$. Then the following assertions hold:*

(i) g_A *is nondecreasing, lower semicontinuous, and sublinear on* Y *and coincides with the support function of the set* $A^o \cap C^+$.
(ii) g_A *is increasing on* C *provided that* Λ *is weakly compact and* $+\infty > \sup_{\xi \in \Lambda} s_A(\xi) > 0$.

To simplify the presentation, we shall make the following assumption:

(H) $f(X)$ is a bounded set with $f(X) \cap \mathrm{int}C \neq \emptyset$.

This assumption implies the existence of some $\xi_* \in \Lambda$ such that $s_{f(X)}(\xi_*) > 0$. Denote by $\mathcal{A}_0 \subseteq \mathcal{A}$ the family of nonempty bounded subsets $A \subseteq Y$ satisfying

$$f(X) \subseteq A \subseteq \{\xi_*/s_{f(X)}(\xi_*)\}^\circ.$$

For each $A \in \mathcal{A}_0$ by using the scalarizing function g_A, we consider the following scalarized problem of (VP):

$$(P_A) \qquad \begin{array}{ll} \max & \displaystyle\sup_{\xi \in \Lambda} \frac{\langle \xi, f(x) \rangle}{s_A(\xi)} \\ \text{s.t.} & x \in X. \end{array}$$

The following result shows the importance of this particular scalarization (see [40] for the proof).

Theorem 6. *Under hypothesis (H) one has*

(i) the optimal value of problem (P_A) is equal to 1;
(ii) every optimal solution of (P_A) is a weakly efficient solution of (VP) provided that Λ is weakly compact and that $0 < \sup_{\xi \in \Lambda} s_A(\xi) < +\infty$;
(iii) every weakly efficient solution of (VP) is an optimal solution of (P_A) provided $A = f(X)$ and $f(X) - C_Y$ is a convex set.

We notice that the convexity required in the above theorem is satisfied when X is a convex set and f is concave in the sense that $f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_1) \in C_Y$ for any $t \in [0,1]$ and for all $x_1, x_2 \in X$. Later on we shall exploit the third conclusion of Theorem 6 to solve problem (VP). The idea as we shall see in Section 8 is to approximate the set $f(X)$ by a sequence of polytopes A_k for which the scalarizing functions g_{A_k} are easy to construct. The function g_A will be the limit of g_{A_k}, and the solution set of (P_A) will be obtained by limits of the approximate solution sets of (P_{A_k}).

7 Nonconvex Duality and Stability

There exists a well-developed duality theory for vector optimization problems. Duality for linear problems is found in [24, 29], convex duality via geometric approach is given in [47], axiomatic duality is treated in [37], convex duality with D.C constraints is presented in [8], and other approaches are presented in [34, 49], and some others. In this section, we consider the multiobjective problem (VP) as described in the previous section. We do not construct dual problems directly for (VP), but for the special scalarized problem (P_A). The approach allows us to develop a method to generate all solutions of the problem (VP). It is to note that even when the set X is convex and the function f is concave, the scalarized problem (P_A) is no longer a concave maximization problem. Therefore, the usual Fenchel–Moreau–Rockafellar duality approach of convex analysis is not suitable to our case. The construction below of [39] is much inspired by the approach of Toland's dualization (see [41, 48], and [53] related to abstract duality and duality in optimization of the difference of convex functions).

We assume the hypothesis (H) of the previous section. The problem (P_A) can be written as

$$(P'_A) \qquad \begin{array}{l} \max\, g_A \circ f(x) \\ \text{s.t.} \quad x \in X \end{array}$$

which is equivalent to

$$\begin{array}{l} \max\, s_{A^\circ \cap C^+} \circ f(x) \\ \text{s.t.} \qquad x \in X. \end{array}$$

We exchange the suprema in this latter problem to obtain the dual of (P_A):

$$(Q_A) \qquad \begin{array}{ll} \max s_{f(X)}(\xi) \\ \text{s.t.} \qquad \xi \in A^\circ \cap C^+. \end{array}$$

This duality construction is very simple, yet it has useful properties. The first one is that there is no gap between the optimal values of (P_A) and (Q_A), which is a common feature of Toland type duality. Thus, in view of Theorem 6, the optimal value $v(Q_A)$ of the dual problem is equal to 1, too. The question that remains is how the optimal solutions of (P_A) are linked with the optimal solutions of (Q_A). Before tackling this question, let us give an example to illustrate the construction of a scalarized problem and its dual. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and let

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq \frac{1}{4},\ x_1 \geq 0,\ x_2 \geq 0\},$$
$$A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq \frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2}\}.$$

Let $f : X \to \mathbb{R}^2$ be the identity function. The dual cone C^+ coincides with $\mathbb{R}^2_+$ and the standard simplex $\Lambda = \{(\xi_1, \xi_2) \in \mathbb{R}^2_+ : \xi_1 + \xi_2 = 1\}$ can be used as a base of C^+. The support function s_A satisfies

$$s_A(\xi) = \frac{1}{2} \quad \text{for every } \ \xi \in \Lambda.$$

The problem (P_A) is written as

$$\begin{array}{ll} \max & \sup_{(\xi_1,\xi_2)\in\Lambda}(\xi_1 x_1 + \xi_2 x_2) \\ \text{s.t.} & (x_1, x_2) \in X \end{array}$$

which is simplified as

$$\begin{array}{ll} \max & \max\{x_1, x_2\} \\ \text{s.t.} & (x_1, x_2) \in X \end{array}$$

because $\xi_1 x_1 + \xi_2 x_2$ is a linear function of $\xi \in \Lambda$ for every fixed (x_1, x_2). In order to construct the problem (Q_A), let us compute the polar of A:

$$\begin{aligned} A^\circ &= \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 x_1 + \xi_2 x_2 \leq 1,\ \forall\, (x_1, x_2) \in A\} \\ &= \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 + \xi_2 \leq 2,\ \xi_1 \leq 2,\ \xi_2 \leq 2\}. \end{aligned}$$

The support function $s_{f(X)}$ is given by

$$s_{f(X)} = \sup_{x\in X}[\xi_1 x_1 + \xi_2 x_2] = \frac{\sqrt{\xi_1^2 + \xi_2^2}}{2}.$$

The dual problem (Q_A) is then written as

$$\begin{array}{ll} \max & \frac{\sqrt{\xi_1^2+\xi_2^2}}{2} \\ \text{s.t.} & \xi_1 + \xi_2 \leq 2,\ 0 \leq \xi_1 \leq 2 \text{ and } 0 \leq \xi_2 \leq 2. \end{array}$$

Observe that in this example, (P_A) is a maximization problem over a convex set X whereas (Q_A) is a maximization problem over a polytope of the dual space Y'.

As promised, let us now derive a relationship between the solutions of problem (P_A) and those of (Q_A). Consider the following auxiliary problems

$$(P_x) \qquad \begin{array}{ll} \max & \langle \xi, f(x) \rangle \\ \text{s.t.} & \xi \in A^\circ \cap C^+ \end{array}$$

for a fixed $x \in X$, and

$$(Q_\xi) \qquad \begin{array}{ll} \max & \langle \xi, f(x) \rangle \\ \text{s.t.} & x \in X, \end{array}$$

for a fixed $\xi \in A^\circ \cap C^+$.

Recall that the normal cone to a convex set $D \subseteq Y$ at $a \in D$ is defined by

$$N(D,a) = \{\xi \in Y' : \langle \xi, y - a \rangle \leq 0 \quad \text{for every } y \in D\}.$$

Below are some optimality conditions for problem (P_x) and (Q_ξ) that can be obtained from the construction of the auxiliary problems and from the subdifferential theory of convex analysis. The notation ∂h stands for the convex subdifferential of h.

Proposition 12. *The following assertions hold:*

(i) $v(P_A) = \sup_{x \in X} v(P_x)$ *and* $v(Q_A) = \sup_{\xi \in A^\circ \cap C^+} v(Q_\xi)$.

(ii) $\xi \in A^\circ \cap C^+$ *is an optimal solution of* (P_x) *if and only if there is some* $y \in -C$ *such that* $\langle \xi, y \rangle = 0$ and $f(x) - y \in N(A^\circ, \xi)$.

(iii) Assume that X *is convex and* f *is concave and continuous. Then* $x \in X$ *is an optimal solution of* (Q_ξ) *if and only if*

$$-N(X,x) \cap \partial(-\xi \circ f)(x) \neq \emptyset.$$

A duality relation between the optimal solutions of (P_A) and those of (Q_A) is given next (see [40] for the proof).

Theorem 7. *The following assertions hold*

(i) *If* $x \in S(P_A)$ *and* $\xi \in S(P_x)$, *then* $\xi \in S(Q_A)$.

(ii) *If* $\xi \in S(Q_A)$ *and* $x \in S(Q_\xi)$, *then* $x \in S(P_A)$.

In both cases,

$$v(P_A) = v(P_x) = \langle f(x), \xi \rangle = v(Q_x) = v(Q_A).$$

The expressions of the solution sets $S(P_A)$ and $S(VP)$ given in the next corollary of Theorem 7 are helpful in development of numerical methods for solving problem (VP).

Corollary 5. *Assume that either of the following conditions holds:*

(i) A *and* C *are polyhedral;*
(ii) $0 \in \mathrm{int}\overline{\mathrm{co}}(A \cup (-C))$.

Then one has

$$S(P_A) = \bigcup_{\xi \in S(Q_A)} S(Q_\xi) \;=\; \bigcup_{\xi \in S(Q_A)} \{x \in X : \langle f(x), \xi \rangle = 1\}.$$

Moreover, if Λ *is weakly compact and if* $f(X) \subseteq C$ *is bounded, closed with* $f(X) - C$ *being convex, then*

$$S(VP) = \bigcup_{\xi \in S(Q_{f(X)})} \{x \in X : \langle \xi, f(x) \rangle = 1\}.$$

In the remaining part of this section, we study the stability of problem (P_A) and its dual (Q_A). To this purpose, let us recall the convergence in the sense of Kuratowski and Painlevé and the convergence with respect to the Hausdorff distance. Let us fix a base Λ of the cone C'_Y, which is assumed bounded, and denote $\delta := \sup_{\xi \in \Lambda} ||\xi||$. Given two nonempty closed subsets $A_1, A_2 \subseteq Y$, the Hausdorff distance between them is defined by

$$h(A_1, A_2) = \inf\{t > 0 : A_1 \subseteq A_2 + tB,\ A_2 \subseteq A_1 + tB\},$$

where B denotes the closed unit ball in Y. Let $\{A_n\}_{n=1}^{\infty} \subseteq Y$ be a sequence of nonempty closed sets. Its upper limit and lower limit in the sense of Kuratowski and Painlevé are defined as

$$\limsup_{n\to\infty} A_n := \{\lim_{i\to\infty} a_{n_i} : a_{n_i} \in A_{n_i}, i = 1, 2, \ldots\},$$

$$\liminf_{n\to\infty} A_n := \{\lim_{n\to\infty} a_n : a_n \in A_n, n = 1, 2, \ldots\}.$$

We say that this sequence H-converges to a closed set A if

$$\lim_{n\to\infty} h(A_n, A) = 0,$$

and KP-converges to A if

$$\limsup_{n\to\infty} A_n \subseteq A \subseteq \liminf_{n\to\infty} A_n.$$

For a nonempty set $A \subseteq Y$, together with the polar set A° we shall consider two other polar sets:

$$A^{-1} := \{\xi \in Y' : \langle \xi, a\rangle < -1 \quad \text{for all} \quad a \in A\}$$

$$A^{-} := \{\xi \in Y' : \langle \xi, a\rangle \leq 0 \quad \text{for all} \quad a \in A\}.$$

It is evident that $A^{-1} \subseteq A^{-} \subseteq A^{\circ}$, and when A^{-1} is nonempty, the cone A^{-} is nontrivial and the set A° is unbounded. Direct verification gives the following formula to compute the polar set of the sum $A+\varepsilon B$. Let $A \subseteq Y$ be nonempty and let $\varepsilon > 0$, then the polar set $(A+\varepsilon B)^{\circ}$ is the set

$$\left\{\frac{\xi}{1+\varepsilon\|\xi\|} : \xi \in A^{\circ}\right\} \cup \left\{\frac{\xi}{\varepsilon\|\xi\|-1} : \xi \in A^{-1}, \|\xi\| > \frac{1}{\varepsilon}\right\} \cup \left\{\xi : \xi \in A^{-}, \|\xi\| = \frac{1}{\varepsilon}\right\}.$$

With this formula, one can prove the following convergence of the function g_A defined in Subsection 6.2.

Lemma 2. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of closed sets H-converging to a closed set A with $0 \in \operatorname{int}(A-C)$. Then $\{g_{A_n}\}_{n=1}^{\infty}$ pointwise converges to g_A.*

We obtain a convergence property for solutions of scalarized problems and dual problems.

Theorem 8. *Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of closed sets H-converging to a closed set A with $0 \in \operatorname{int}(A-C)$. Then*

$$\limsup_{n\to\infty} S(P_{A_n}) \subset S(P_A),$$

$$\limsup_{n\to\infty} S(Q_{A_n}) \subset S(Q_A).$$

Proof. Apply Lemma 2 and Theorem 6. ∎

The inclusions of this theorem does not allow one to obtain all solutions of the problem (P_A) when A_n are approaching A. The concept of approximate solutions will be helpful. Given $\varepsilon > 0$, we say that $x_0 \in X$ is an ε-solution of problem (P'_A) if

$$g_A \circ f(x_0) + \varepsilon \geq g_A \circ f(x) \ \text{ for every } x \in X.$$

The set of all ε-solutions of (P'_A) (hence of (P_A) as well) is denoted by $S_\varepsilon(P'_A)$.

Theorem 9. *Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of nonempty, closed sets, which H-converges to a closed set A with $0 \in \operatorname{int}(A-C)$ and let X be compact. Then*

$$S(P_A) \subseteq \cap_{\varepsilon>0} \liminf_{n\to\infty} S_\varepsilon(P_{A_n}).$$

If in addition the sequence $\{A_n\}_{n=1}^{\infty}$ is monotone (either increasing or decreasing by inclusions), then for every $\varepsilon > 0$ there is some $n_0 > 0$ such that

$$S(P_A) \subseteq S_\varepsilon(P_{A_n}) \textit{ for all } n \geq n_0$$

and in particular

$$\lim_{n\to\infty,\varepsilon\downarrow 0} h(S_\varepsilon(P_{A_n}), S(P_A)) = 0.$$

We see from this theorem (given in [39]) that by choosing $\varepsilon > 0$ and n appropriately, one can obtain all solutions of problem (P_A) with help of problem (P_{A_n}).

8 Generating the Solution Set

To date, there exists a large variety of methods for solving multiobjective problems, starting with the classic ones (the weighting method, the ϵ-constraint method, the goal programming method, etc.) that can be found in classic textbooks ([9, 55, 57]) and more recent ones, such as the outer approximation method, the normal-boundary intersection method, and the normal cone method for continuous case and evolutionary algorithm, metaheuristic algorithm for combinatoric case (see [3–5, 15, 25, 26, 32, 42, 45–47, 51, 55, 58] and the references therein). Most of these methods allow us to find one or a few solutions, the others are aimed at generating the whole solution set. The latter are numerically difficult because except for the linear case, existing algorithms of mathematical programming allow one to obtain a few solutions only. In this section, we present a method based on normal cones to solve linear problems and a method based on duality and scalarization to solve convex problems. We refer the readers to [15, 18, 27] for nonconvex problems.

8.1 Linear Problems

Linear multiobjective problems are well studied. Known solution methods such as the simplex method ([55, 57]), the outcome space based method of [3], etc., are quite suitable to solve them. In this subsection, we present the normal cone method of [26] to generate all efficient solutions of linear problems. Let us consider the following problem denoted by (LP):

$$(VP) \qquad \begin{array}{ll} \operatorname{Min} Cx & \\ \text{s.t.} & Ax \geq b \end{array}$$

where C is an $m \times n-$matrix with m rows $C^1, \ldots, C^m$ and A is an $p \times n-$matrix with p rows $a^1, \ldots, a^p$, and $b \in \mathbb{R}^p$. The order in $\mathbb{R}^m$ is given by the Pareto cone $\mathbb{R}^m_+$.

Denote by $M := \{x : Ax \geq b\}$. We recall that the normal cone to M at $x_0 \in M$ is the set $N(M, x_0)$ and given by

$$N(M, x_0) := \{v \in R^n : \langle v, x - x_0 \rangle \leq 0,\ x \in M\}.$$

The normal cone can be explicitly calculated by the following rule.

Lemma 3. *Let $I(x_0)$ be the active index set at $x_0 \in M$, i.e.,*

$$I(x_0) = \{i \in \{1, \dots, p\} : \langle a^i, x_0 \rangle = b_i\}$$

and $\langle a^j, x_0 \rangle > b_j$ if $j \notin I(x_0)$. Then $N(M, x_0) = \operatorname{cone}\{-a^i : i \in I(x_0)\}$.

Proof. By a direct verification. ■

Definition 5. *Let $I \subseteq \{1, \dots, p\}$. We say that I is normal if there is $x_0 \in M$ such that $N(M, x_0) = \operatorname{cone}\{-a^i : i \in I\}$, and I is positive if cone $\{-a^i : i \in I\}$ contains a vector of the form $-\sum_{i=1}^{m} \lambda_i C^i$ with $\lambda_1 > 0, \dots, \lambda_m > 0$.*

Let F be a face of the polyhedral convex set M. We say that F is an efficient solution face if every point of F is an efficient solution of (LP).

Theorem 10. *Assume that there are no redundant constraints among $\langle a^i, x \rangle \geq b_i,\ i = 1, \dots, p$. Let F be a face of M determined by the system*

$$\langle a^i, x \rangle \ = b_i,\ i \in I_F \subseteq \{1, \dots, p\}$$

$$\langle a^j, x \rangle \ \geq b_j,\ j \in \{1, \dots, p\} \setminus I_F.$$

Then F is an efficient solution face if and only if I_F is positive and normal.

The next three procedures allow to completely solve the problem (LP).

Procedure 1 (Finding an initial efficient solution vertex.)

Step 1. Solve the system

$$\sum_{i=1}^{p} \mu_i a^i = \sum_{j=1}^{m} \lambda_j \cdot C^j,\ \mu_i \geq 0,\ \lambda_j \geq 1.$$

If it has no solutions, STOP ((VP) has no efficient solutions). Otherwise go to Step 2.

Step 2. Let λ be a solution of the above system. Put $v = C^T \lambda$. If $v = 0$, STOP (every feasible solution of (VP) is efficient). Otherwise solve the scalar linear problem

$$\max_{x \in M} \langle v, x \rangle.$$

It is sure that this problem has optimal solutions. An optimal solution vertex of this problem is an efficient solution vertex of (VP).

Procedure 2 (Determining all efficient edges emanating from an initial efficient vertex x_0.)

Step 1. Determine the active index set

$$I(x_0) := \{i \in \{1, \dots, p\} : \langle a^i, x_0 \rangle = b_i\},$$

and pick $I \subseteq I(x_0)$ with $|I| = n - 1$ not previously considered.
If rank $\{a^i : i \in I\} = n - 1$, go to Step 2.
Otherwise pick another $I \subseteq I(x_0)$.

Step 2. Verify whether I is positive by solving the system

$$\sum_{I \subset I} \mu_i a^i = \sum_{j=1}^{m} \lambda_j C^j,\ \mu_i \geq 0,\ i \in I,\ \lambda_j \geq 1,\ j = 1, \dots, m.$$

If it has a solution, then go to Step 3 (I is positive).
Otherwise return to Step 1.

Step 3. Verify whether I is normal, which implies that the edge determined by I is efficient.
Find $v \neq 0$ by solving

$$\langle a^i, v \rangle = 0,\ i \in I.$$

Solve the system

$$\langle a^i, x_0 + tv \rangle \geq b_i,\ i = 1, \dots, p.$$

Let the solution set be $[t_0, 0]$ or $[0, t_0]$ (t_0 may be ∞ or $-\infty$).
If $t_0 = 0$, then Return to Step 1 (I is not normal).
If $t_0 \neq 0$, then $[x_0, x_0 + t_0 v]$ is an efficient edge. Store it and return to Step 1 until no subset $I \subseteq I(x_0)$ with power $(n - 1)$ left.

Procedure 3 (Finding an ℓ-dimensional efficient solution face adjacent to x_0.)
Let $\{[x_0, x_0 + t_i v_i];\ i = 1, \dots, k\}$ be the family of all efficient edges emanating from x_0 that have been obtained by Procedure 2 (assume $t_i > 0$).

Step 1. Pick $J \subseteq \{1, \dots, k\}$ with $|J| = \ell$, not previously considered and set

$$x_J = \frac{x_0}{\ell + 1} + \sum_{j \in J} \lambda_j \frac{x_j}{\ell + 1}$$

where $x_j = x_0 + t_j v_j$ and $\lambda_j = t_j$ if t_j is finite, $\lambda_j = 1$ if $t_j = \infty$.

Step 2. Determine the active index set $I(x_J)$.
If $I(x_J) = \emptyset$, then Return to Step 1.
Otherwise go to Step 3.

Step 3. (Verify whether $I(x_J)$ is positive.)

Solve the system of Step 2 (*Procedure 2*) with $I = I(x_J)$.
If it has a solution, go to Step 4 ($I(x_J)$ is positive).
Otherwise return to Step 1.

Step 4. (Find an ℓ-dimensional efficient face containing $[x_0, x_0 + t_j v_j]$: $j \in J$.)
Determine $J_0 := \{j \in \{1, \ldots, k\} : I_J \supseteq I(x_J)\}$.
Then the convex hull of $\{[x_0, x_0 + t_j v_j] : j \in J_0\}$ is an ℓ-dimensional efficient face adjacent to x_0.
Store it and pick J not containing J_0 with $|J| = \ell$ and continue Step 1.

Note that the set of efficient solutions of (VP) is pathwise connected, the above procedures allow one to generate all efficient solutions of (VP) in a finite number of iterations. *Procedure 3* also gives a method generating all maximal efficient faces adjacent to a given efficient vertex.

8.2 Convex Problem

The method we are going to describe is based on the scalarization and duality we gave in the two preceding sections. This method is interesting from the theoretical point of view and numerically implementable. Let us consider the problem:

$$(VP) \qquad \begin{array}{l} \text{Max } f(x) \\ \text{s.t.} \quad x \in X, \end{array}$$

where X is a nonempty subset of $\mathbb{R}^n$, $f = (f_1, \ldots, f_m)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\mathbb{R}^m$ is equipped with the ordering cone $\mathrm{int}(\mathbb{R}^m_+) \cup \{0\}$. Thus, we are interested in finding all efficient solutions of the problem, namely the set

$$S(VP) = \{x \in X : (f(X) - f(x)) \cap \ \mathrm{int}\, \mathbb{R}^m_+ = \emptyset\}$$

which are weakly efficient with respect to the Pareto cone $\mathbb{R}^m_+$. It is to note that the weakly efficient set is bigger than the efficient set and it is more stable than the efficient set. A similar method can be developed for efficient solutions, but the convergence analysis is more complicated.

Before presenting the solution method, let us introduce approximate solutions for multiobjective problems and derive some scalarization and duality properties from the analysis of Section 7.

Given $\varepsilon > 0$, we say that $x_0 \in X$ is an ε-solution of (VP) if

$$f(X) \cap ([f(x_0) + (\varepsilon, \ldots, \varepsilon)] + \ \mathrm{int}\, \mathbb{R}^m_+) = \emptyset.$$

The set of all ε-solutions of (VP) is denoted by $S_\varepsilon(VP)$. The standard simplex

$$\Lambda = \{\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+ \ : \ \sum_{i=1}^m \xi_i = 1\}$$

will serve as a base of the nonnegative polar cone $\mathbb{R}^m_+$. We shall make use of the notation $A^\diamondsuit = (A - \mathbb{R}^m_+) \cap \mathbb{R}^m_+$ for $A \subseteq \mathbb{R}^m_+$ and assume throughout the following hypothesis:

(H′) $f(X) \subset \mathrm{int}\mathbb{R}^m_+$ and the set $[f(X)]^\diamondsuit$ is nonempty, compact and convex.

This hypothesis is fulfilled, for instance, when X is nonempty compact and convex, f is continuous and concave, and with help of a translation. By setting $A = [f(X)]^\diamondsuit$, we have $f(X) \subseteq A$ and $A^\diamondsuit = A$.

Lemma 4. *Let $A = [f(X)]^\diamondsuit$, one has $S(P_A) = S(VP)$. Moreover, for $\varepsilon \geq 0$, $x_0 \in X$ is an ε-solution of (VP) if and only if there is some vector $\xi \in \Lambda$ such that x_0 is an ε-solution of (Q_ξ).*

Proof. That the sets $S(P_A)$ and $S(VP)$ coincide is immediate from Theorem 6. Now, let x_0 be an ε-solution of (VP). Then the set $f(X) - \mathbb{R}^m_+$, which is convex according to the hypothesis (H′), does not meet the convex set $f(x_0) + (\varepsilon, \ldots, \varepsilon) + \mathrm{int}\mathbb{R}^m_+$. Separating them, we find some $\xi \in \Lambda$ such that

$$\langle \xi, f(x) \rangle \leq \langle \xi, f(x_0) \rangle + \varepsilon$$

for every $x \in X$. This shows that x_0 is an ε-solution of (Q_ξ). Conversely, if x_0 is not an ε-solution of (VP), then there are some $x \in X$ and $c \in \mathrm{int}\mathbb{R}^m_+$ such that $f(x) = f(x_0) + (\varepsilon, \ldots, \varepsilon) + c$. We derive for each $\xi \in \Lambda$ that

$$\langle \xi, f(x) \rangle \geq \langle \xi, f(x_0) \rangle + \varepsilon + \langle \xi, c \rangle > \langle \xi, f(x_0) \rangle + \varepsilon.$$

Hence x_0 cannot be an ε-solution of (Q_ξ). The proof is complete. ■

The next result is obtained from Theorem 6, Lemma 4, and Corollary 5.

Proposition 13. *Under the hypothesis (H′), one has*

$$S(VP) = \bigcup_{\xi \in bd_+(A^\circ)} S(Q_\xi) = \bigcup_{\xi \in bd_+(A^\circ)} \{x \in X : \langle \xi, f(x) \rangle = 1\},$$

where $bd_+(A^\circ)$ denotes the intersection of the boundary of the set A° with the octant $\mathbb{R}^m_+$. In particular, if A is a polytope and Γ is a set of those vertices of A° that lie in $\mathbb{R}^m_+$, then

$$S(VP) = \bigcup_{\xi \in \Gamma} \{x \in X : \langle \xi, f(x) \rangle = 1\}$$

In view of the above proposition, for generating the solution set of (VP), it suffices to determine A° or more precisely $bd_+(A^\circ)$ and then solve the scalar problems (Q_ξ). In what follows, we present an algorithm to solve problem (VP) by approximating the set A° from outside. The idea is to start up with a polyhedron $S_1 \supset A^\circ$ and to build up a sequence of polyhedra

$$S_1 \supset S_2 \supset \ldots \supset S_k \supset \ldots \supset A^\circ.$$

This can be done by the dual relation between a polyhedron and its polar, which states that if two full-dimensional polyhedra P and S containing 0 are polar to each other, then there exists a 1-1 correspondence between the set of facets of P not containing 0 and the set of nonzero vertices of S.

Denote by q the vector $(q_1, \ldots, q_m)$, where $q_1, \ldots, q_m$ are the optimal values of the following problems

$$\begin{array}{ll} \max & f_i(x) \\ \text{s. t.} & x \in X. \end{array}$$

It is easy to see that the vectors $y^1 = (q_1, \ldots, 0), \ldots, y^m = (0, \ldots, q_m)$ belong to $[f(X)]^\diamondsuit$. Set

$$\begin{aligned} B_1 &= \mathrm{co}\{0, y^1, \ldots, y^m\} \\ \hat{B}_1 &= B_1 - \mathbb{R}^m_+ \\ S_1 &= \{\xi \in \mathbb{R}^m : \langle \xi, y^i \rangle \leq 1,\ i = 1, \ldots, m\} \cap \mathbb{R}^m_+ \\ \hat{S}_1 &= S_1 - \mathbb{R}^m_+. \end{aligned}$$

Denote by V_1 the vertex set of S_1, which actually consists of the vectors

$$v = (v_1, \ldots, v_m) \quad \text{where} \quad v_i \in \big\{0, \frac{1}{q_i}\big\},\ i = 1, \ldots, m.$$

We construct S_{k+1} by induction. Assume that S_k is known together with its vertex set V_{k+1}. Define

$$V_k^* = \{v \in V_k : s_A(v) > 1\}.$$

If $V_k^* = \emptyset$, we set $S_{k+1} = S_k$. Otherwise define

$$S_{k+1} = S_k \cap \{\xi \in \mathbb{R}^m : \langle \xi, y_v \rangle \leq 1,\ v \in V_k^*\}$$

where y_v is a maximum of the linear function $\langle v, . \rangle$ on A and

$$\begin{aligned} \hat{S}_{k+1} &= S_{k+1} - \mathbb{R}^m_+ \\ B_{k+1} &= \mathrm{co}(B_k \cup \{y_v : v \in V_k^*\}) \\ \hat{B}_{k+1} &= B_{k+1} - \mathbb{R}^m_+. \end{aligned}$$

Here are some important properties of the sets $S_k, \hat{S}_k, B_k$, and $\hat{B}_k$ (see [40] for the proof).

(i) For every $k \geq 1$, the sets B_k and $\hat{B}_k$ are polar to $\hat{S}_k$ and S_k, respectively.
(ii) $S_{k+1} \subseteq S_k$.
(iii) $A^\circ \cap \mathbb{R}^m_+ \subseteq S_k$ and $A^\circ = (A^\circ \cap \mathbb{R}^m_+) - \mathbb{R}^m_+ \subseteq \hat{S}_k$.
(iv) If $V_k^* = \emptyset$ for some $k \geq 1$, then $\hat{S}_k = A^\circ$.

(v) The sequence $\{\hat{S}_k\}_{k=1}^{\infty}$ H-converges to A°.

Now we are in the position to present the algorithm for solving problem (VP).

Step 1 (Initialization). Choose a small $\varepsilon > 0$. Find a_i and $b_i \in \mathbb{R}$ such that $a_i \geq f_i(x) > b_i$ for all $x \in X$ and $i = 1, \dots, m$. Set $k = 1$ and for $i = 1, \dots, m$:

$$f_i(x) = f_i(x) - b_i, \quad q_i = a_i - b_i, \quad y^i = q_i e_i$$

where $e_1 = (1, \dots, 0), \dots, e_m = (0, \dots, 1)$. Define

$$S_k = \{\xi \in \mathbb{R}^m_+ : \langle \xi, y^i \rangle \leq 1, i = 1, \dots, m\}$$

and set $V_k = \{v = (v_1, \dots, v_m) : v_i \in \left\{0, \frac{1}{q_i}\right\}, i = 1, \dots, m\}$.

Step 2. For each $v \in V_k$, solve problem (P_v):

$$\begin{array}{l} \max \ \langle v, y \rangle \\ \text{s. t. } \ y \in [f(X)]^{\diamondsuit}. \end{array}$$

to obtain $s_A(v)$ and an optimal solution y_v.

Step 3. Set $V_k^* = \{v \in V_k : s_A(v) > 1 + \varepsilon\}$.

If $V_k^* = \emptyset$, then stop. Set

$$E_{\varepsilon} = \bigcup_{v \in V_k} \{x \in X : \langle f(x), v \rangle \geq 1\}$$

Otherwise, go to the next step.

Step 4. Set

$$S_{k+1} = S_k \cap \{z \in \mathbb{R}^m : \langle z, y_v \rangle \leq 1, \ v \in V_k^*\}$$

and find the vertex set V_{k+1} of S_{k+1}. Set $k = k + 1$ and return to Step 2.

The convergence of this algorithm is seen in the next theorem (see [40] for the proof).

Theorem 11. *Assume that $f(X) \subseteq \mathrm{int}\mathbb{R}^m_+$ and $[f(X)]^{\diamondsuit}$ is a nonempty, compact, and convex set. Then*

(i) for a given $\varepsilon > 0$, the algorithm terminates after a finite number of iterations and

$$S(VP) \subseteq E_{\varepsilon} \subseteq S_{\varepsilon\delta_k}(VP)$$

where $\delta_k = 1/(\min_{v \in V_k} \sum_{i=1}^{m} v_i)$;

(ii) if problem (VP) is linear, then the algorithm terminates after a finite number of iterations with zero tolerance $\varepsilon = 0$ and

$$S(VP) = \bigcup_{v \in V_k} \{x \in X : \langle f(x), v \rangle = 1\}.$$

We would like to point out that when the algorithm terminates, in view of the convergence theorem, all elements of the set E_ε are $\varepsilon\delta_k$-solutions of (VP). Moreover, because A contains the origin of the space in its interior, there is a positive γ such that all coordinates of elements of V_k are greater then γ. Consequently, δ_k is majorized by $1/(m\gamma)$, and $\varepsilon\delta_k$ converges to 0 as soon as ε tends to 0. In practice, when implementing the algorithm, one normally does not obtain the whole set E_ε as no existing solvers are able to do it. So instead, one finds some $x_v \in X$ such that $y_v = f(x_v)$ solves Problem (P_v) and stores the set $\hat{E}_\varepsilon = \{x_v : v \in V_k\}$. This set is the best representative portion of the weakly efficient solution set of (VP) in the sense that each element of this set is a weakly efficient solution, and for every weakly efficient solution x there exists some element x_v of the set $\hat{E}_\varepsilon$ such that $|\langle v, f(x_v)\rangle - \langle v, f(x)\rangle| \leq \varepsilon$ where v is a vertex of the polyhedron $\hat{S}_k$ that approximates the polar of $[f(X)]^\diamond$ from outside.

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Multiobjective Optimization: A Brief Overview

Massimo Pappalardo

Department of Mathematics, University of Pisa, Pisa, Italy
pappalardo@dma.unipi.it

Abstract To define optimality when we are in presence of several conflicting objectives, we need "good" definition of order in $\mathbb{R}^p$. After this, we can study many types of problems: existence of solutions, optimality conditions, and solution methods.

Key words: multiobjective optimization, orders and cones, Pareto optimality, scalarization

1 Introduction

Multiobjective optimization problems typically have conflicting objectives, and a gain in one objective very often is an expense of another. Therefore the definition of optimality is not obvious as in the scalar case. However in many contexts, mathematical models involving more than one objective seem much more adherent to the real problems. This happens, for example, in many engineering design problems (aircraft gas turbine engine, unmanned vehicle configuration, pumpling assemblies, networks management, and so on), where it is necessary to increase productivity, strength, and efficiency, but at the same time they want to decrease noise, vibration, production and maintenance costs, and in many economics or finance problems like optimal portfolio where we are in presence of many variance criteria.

The chapter is organized as follows. In Section 2, we introduce our problem and we outline the relationships between multiobjective optimization, partial orders, and cones. In Section 3, we deal with results concerning the existence of solutions of multiobjective optimization. In Section 4, optimality conditions are illustrated both in the differentiable and in the nondifferentiable case. Section 5 is devoted to scalarization techniques. Section 6 deals with a brief survey of solution methods. A wide bibliography concludes the chapter.

2 Multiobjective Optimization

The definition of optimal solution for a scalar (i.e., single objective) optimization problem is based on the usual order relation of the set $\mathbb{R}$. In an analogous way, in order to define optimal solution for a multiobjective optimization problem, we need to introduce an order relation on $\mathbb{R}^p$.

A binary relation on $\mathbb{R}^p$ is a subset A of $\mathbb{R}^p \times \mathbb{R}^p$. An element $x \in \mathbb{R}^p$ is said to be in relation with $y \in \mathbb{R}^p$ if and only if $(x, y) \in A$.

Definition 1. *Let A be a binary relation on $\mathbb{R}^p$. We say that it is*

1. *reflexive if $(x, x) \in A$ for every $x \in \mathbb{R}^p$;*
2. *antisymmetric if $(x, y) \in A$ and $(y, x) \in A$ imply $x = y$;*
3. *transitive if $(x, y) \in A$, $(y, z) \in A$ imply $(x, z) \in A$;*
4. *complete if $(x, y) \in A$ or $(y, x) \in A$ for each $x, y \in \mathbb{R}^p$, $x \neq y$.*

A binary relation is said to be a partial order on $\mathbb{R}^p$ if it is reflexive, transitive, and antisymmetric. If a partial order is complete then it is called total order. A partial order is called linear if it satisfies the following two conditions:

1. $(x, y) \in A$ implies $(x + z, y + z) \in A$ for every $x, y, z \in \mathbb{R}^p$;
2. $(x, y) \in A$ implies $(tx, ty) \in A$ for every $x, y \in \mathbb{R}^p$ and for every $t > 0$.

We can define a partial order on $\mathbb{R}^p$ by means of pointed convex cones.

Definition 2. *A subset X of $\mathbb{R}^p$, is said to be a cone if $\lambda x \in X$ for every $x \in X$ and for every $\lambda > 0$.*

Let us consider a convex cone $C \subset \mathbb{R}^p$; it gives the binary relation

$$A_C = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^p : \; x - y \in C\},$$

and in what follows we will write

$$x \geq_C y \;\; \text{instead of} \quad x - y \in C.$$

This binary relation is reflexive if $0 \in C$. If we require that C be pointed, that is $C \cap (-C) = \{0\}$, then there are not $x, y \in \mathbb{R}^p$, with $x \neq y$, such that

$$x \geq_C y \;\; \text{and} \;\; y \geq_C x,$$

therefore relation $\geq_C$ is antisymmetric.

When $C = \mathbb{R}^p_+$, then $\geq_C$ is obviously reflexive, antisymmetric, transitive, but not complete; moreover for $x = (x_1, \ldots, x_p),\; y = (y_1, \ldots, y_p)$ we have:

$$x \geq_C y \iff x_i \geq y_i \qquad \forall\, i = 1, \ldots, p.$$

The partial order generated by this cone is called Pareto-order.

Now we can define the minimum points of a subset of $\mathbb{R}^p$ with respect to the partial order generated by a pointed convex cone C.

Definition 3. *Let X be a nonempty subset of $\mathbb{R}^p$ and C a pointed convex cone. We say that*

(i) $x \in X$ is an efficient point of X with respect to C if $x \geq_C y$, for some $y \in X$, then $y \geq_C x$; we denote the set of efficient points by X_{eff};
(ii) supposing that $\operatorname{int} C$ (i.e., the interior of C) is nonempty, $x \in X$ is a weakly efficient point of X with respect to C if x is an efficient point of X with respect to $(\operatorname{int} C \cup \{0\})$; we denote the set of weakly efficient points by X_{weff}.

A classic result establishes a one-to-one correspondence between linear partial orders and pointed convex cones:

Theorem 1. *Let A be a binary relation on $\mathbb{R}^p$. Then A is a linear partial order if and only if there exists a pointed convex cone C such that*

$$(x, y) \in A \ \Leftrightarrow (y - x) \in C.$$

We are now able to introduce a multiobjective minimization problem. From now on, suppose that a convex and pointed cone $C \subset \mathbb{R}^p$ is given. Let us consider the vector-valued functions $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^k$. We will consider the following multiobjective minimization problem:

$$\begin{cases} \min_C \ f(x), \\ x \in R := \{x \in \mathbb{R}^p : \ g(x) \leq 0, \ h(x) = 0\}, \end{cases} \tag{1}$$

where $\min_C$ marks vector minimum with respect to the cone C; $y \in R$ is a (global) vector minimum point (for short v.m.p.) of (1) if and only if $f(y)$ is an efficient point of the image of the feasible region through the objective function f, that is

$$Z = f(R) = \{z \in \mathbb{R}^p : \exists \ x \in R : f(x) = z\},$$

with respect to C. At $C = \mathbb{R}^p_+$, (1) becomes the classic Pareto multiobjective problem.

Definition 4. *A decision vector x^* is Pareto optimal if there does not exist another decision vector $x \in R$ such that $f_i(x) \leq f_i(x^*)$, $\forall i = 1, \ldots, p$, and $f_j(x) < f_j(x^*)$, for at least one index j.*

Analogous definition can be given for local Pareto optimality. As in scalar optimization, global optima are obviously also local and the opposite holds if, for example, the feasible region is convex and the objective functions f_i are quasiconvex. When $(\operatorname{int} C \neq \emptyset)$, a multiobjective minimization problem that is often associated with (1) is the following one, called weak problem:

$$\begin{array}{c} w - \min_C \ f(x), \\ x \in R \end{array} \tag{2}$$

where $w - \min_C$ marks vector minimum with respect to the cone $\operatorname{int} C \cup 0$; $y \in R$ is a weak v.m.p. of (2) if and only if $f(y)$ is a weakly efficient point of the image $f(R)$ with respect to C. We remark that the solutions of (1) are solutions also of (2), but not necessarily vice versa.

These definitions of optimality (via cones) arise because it is not possible to adopt the following and more natural definition. Given

$$z_i^{id} = \min_R \; f_i(x),$$

we could define optimal solution of (1) the vector $z^{id} = (z_1^{id}, \ldots, z_p^{id})$. Unfortunately, it very often happens that

$$z^{id} \notin Z.$$

The point z^{id} is called ideal point and, if $z^{id} \in Z$, then it is surely the optimal solution of our multiobjective problem. In any case, to compute it is very useful because it is always a lower bound of the Pareto optimal set in the sense that for every efficient point y, we have

$$z_i^{id} \leq y_i, \quad \forall i.$$

Analogously, we can define the nadir point

$$z_i^{nd} = \max_R \; f_i(x),$$

and we obtain an upper bound in the sense that for every efficient point y, we have

$$z_i^{nd} \geq y_i, \quad \forall i.$$

As noted, when we are in presence of a Pareto solution, we could improve the value of an objective function but we may worsen another objective function. It is easy to propose examples where such a worsening can go to infinity. In any case, when this does not happen, we are in presence of proper Pareto optimal solution:

Definition 5. *A Pareto solution x^* is called proper if there exists $M > 0$ such that, for every index i and for every $x \in R$ such that $f_i(x) < f_i(x^*)$, there exists an index j such that $f_j(x^*) < f_j(x)$ and moreover*

$$[(f_i(x^*) - f_i(x))/(f_j(x) - f_j(x^*))] \leq M.$$

Proper minima are, in a certain sense, those Pareto-minima whose optimality is preserved under small perturbations of the ordering cone. Thus, proper minima require some kind of stability.

For the sake of completeness, we must recall that there are also other concepts of optimality, some of them even referred to nonconical dominance structure, but they will be not considered here.

3 Existence of Solutions

The main condition that ensures the existence of efficient solutions for a multiobjective problem is some kind of compactness in the value space of the objective functions.

Definition 6. *We say that a set $Y \subseteq \mathbb{R}^p$ is called $\mathbb{R}^p_+$-compact if $\forall y \in Y$ the set $(y - \mathbb{R}^p_+) \cap Y$ is compact.*

It is possible to prove the following results:

Theorem 2. *If $Y \subseteq \mathbb{R}^p$ is nonempty and $\mathbb{R}^p_+$-compact, then $Y_{eff} \neq \emptyset$.*

Theorem 3. *If $Y \subseteq \mathbb{R}^p$ is nonempty, closed, convex, and $\mathbb{R}^p_+$-compact, then Y_{eff} is connected.*

Indeed in Theorems 2 and 3, it would be sufficient to assume the semicompactness as we show.

Definition 7. *A set Y is said to be semicompact if every open cover of Y of the form $\{y^a - \mathbb{R}^p_+ : y^a \in Y, a \in A\}$ has a finite subcover.*

Thus in order to obtain an existence result for problem (1), we need some assumptions ensuring the compactness or the semicompactness of the set $f(R)$. To do this we give the following definition:

Definition 8. *A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be $\mathbb{R}^p_+$-semicontinuous if $f^{-1}(y - \mathbb{R}^p_+)$ is closed for all $y \in \mathbb{R}^p$.*

Then we have:

Theorem 4. *If R is compact and f is $\mathbb{R}^p_+$-semicontinuous, then $f(R)$ is $\mathbb{R}^p_+$-semicompact.*

It is also possible to prove that f is $\mathbb{R}^p_+$-semicontinuous if and only if every f_i is lower semicontinuous. In this way, we have obtained a generalization to multiobjective problems of the classic Weierstrass theorem.

In the applications, a relevant case is the linear multiobjective problem that is the case where f, g, and h are affine functions. In this case, there are many results concerning the structure of solutions-set. One of the most important is the following: two efficient extreme points are edge-connected in the sense that they are connected by means of a path of efficient edges of the polyhedron. This result is very useful for the construction of a solution method of simplex-type.

4 Optimality Conditions

In this section, we present optimality conditions for multiobjective optimization both in the differentiable and in the nondifferentiable case. The literature about this topic is very wide.

4.1 Differentiable Case

The starting point is the classic generalization of the Fermat theorem, and it concerns a sort of steepest descent result:

Theorem 5. *If f is differentiable, $R = \mathbb{R}^p$, and x^* is a weak Pareto optimal solution, then*

$$\max_{k=1,..,p} \langle \nabla f_k(x^*), w\rangle \geq 0, \quad \forall w \in \mathbb{R}^p.$$

Suppose, now, that we are in presence of constraints. Taking into account the classic definition of Bouligand tangent cone $T(R; x^*)$ of the set R at the point x^*, we have:

Theorem 6. *If f is differentiable and x^* is a weak Pareto optimal solution, then*

$$\max_{k=1,..,p} \langle \nabla f_k(x^*), w\rangle \geq 0, \quad \forall w \in T(R; x^*).$$

The analysis of the structure of the set $T(R, x^*)$ in terms of the functions g and h leads us to the Karush–Kuhn–Tucker theory for multiobjective optimization.

Theorem 7. *Let us suppose that f, g, h are continuously differentiable and that x^* is a weak Pareto optimal solution. Then there exist multipliers $\lambda \in \mathbb{R}^p_+$, $\mu \in \mathbb{R}^m_+$, $\nu \in \mathbb{R}^k$ such that*

$$\begin{cases} \lambda \nabla f(x^*) + \mu \nabla g(x^*) + \nu \nabla h(x^*) = 0 \\ \langle \mu, g(x^*)\rangle = 0. \end{cases} \tag{3}$$

As in the scalar case, a fundamental question arises in establishing when the multiplier λ is regular, i.e., it is not equal to 0. But, in contrast with the scalar case, we have two different types of regular multiplier: $\lambda \neq 0$ and $\lambda_i > 0$.

Let us consider the following system in the unknown w:

$$\begin{cases} \langle \nabla f_i(x^*), w\rangle < 0 & \forall i = 1, \ldots, p \\ \langle \nabla g_i(x^*), w\rangle \leq 0 & \forall i \in I(x^*) \\ \langle \nabla h_i(x^*), w\rangle = 0 & \forall i = 1, \ldots, k. \end{cases} \tag{4}$$

where $I(x^*)$ is the active constraint set.

When system (4) has no solution, we say that Abadie constraint qualification holds; it ensures the existence of a multiplier $\lambda \neq 0$ in the KKT theorem.

For obtaining the second case we need, obviously, more stronger constraint qualifications. Given $s \in \{1, \ldots, p\}$, let us consider the following system in the unknown w

$$\begin{cases} \langle \nabla f_s(x^*), w\rangle < 0 & \\ \langle \nabla f_i(x^*), w\rangle \leq 0 & \forall i \neq s \\ \langle \nabla g_i(x^*), w\rangle \leq 0 & \forall i \in I(x^*) \\ \langle \nabla h_i(x^*), w\rangle = 0 & \forall i = 1, \ldots, k. \end{cases} \tag{5}$$

When system (5) has no solution, we say that Guignard constraint qualification holds; it ensures the existence of a multiplier λ in the KKT theorem such that $\lambda_i > 0, \quad \forall i = 1, \ldots, p$.

Other important topics developed and analyzed in literature are those concerning boundedness and uniqueness of KKT multipliers. This depends on the fact that in scalar optimization, results of such types are fundamental in showing convergence properties of solution methods. Other results have been established in literature for first-order sufficient conditions and second-order necessary conditions.

4.2 Nondifferentiable Case

Several necessary optimality conditions for nondifferentiable multiobjective problems have been proposed in literature in these past two decades. Generalized derivatives are necessary. A way to rely on nonsmooth tools is some kind of componentwise approach; that is, considering generalized derivatives of the components of the given functions. Because components are taken into account, this approach has one drawback: practically it works only in the Pareto-case.

In this approach, we can consider classic Dini–Hadamard derivatives of a real function $f : \mathbb{R}^n \mapsto \mathbb{R}$.

Definition 9. *Let $\bar{x}$ and $v \in \mathbb{R}^n$. We call*

$$\mathrm{D}^+ f(\bar{x})(v) := \limsup_{v' \to v, t \to 0^+} \frac{f(\bar{x} + tv') - f(\bar{x})}{t},$$

the upper Dini–Hadamard derivative. By changing limsup with liminf, we have the definition of lower Dini–Hadamard derivative.

The following result holds:

Theorem 8. *If $\bar{x}$ is a weak Pareto optimal solution, then*

$$\max_{k=1,..,p} \mathrm{D}^+ f_k(\bar{x}; w) \geq 0, \quad \forall w \in T(R; \bar{x}).$$

Relying on the concept of Kuratowski limit, a very different approach has been presented in recent papers [20]. It can be considered somehow a global one because set-valued directional derivative of vector functions are introduced without relying on the component.

Definition 10. *The global Dini–Hadamard derivatives, $\mathrm{D}^+ f(\bar{x})(v)$ and $\mathrm{D}^- f(\bar{x})(v)$ are, respectively, the set of Pareto maximum and Pareto minimum points (when they are finite) of*

$$\bigcap_{k=1}^{\infty} \mathrm{cl} \left\{ \frac{f(\bar{x} + tv') - f(\bar{x})}{t} \;;\; (t, v') \in \tilde{B}_{\frac{1}{k}}(v) \right\},$$

where $\tilde{B}_{\frac{1}{k}}(v) = (0, \frac{1}{k}) \times B_{\frac{1}{k}}(v)$.

The following result holds:

Theorem 9. *If $\bar{x}$ is a weak Pareto optimal solution, then*

$$\bigcap_{k=1}^{\infty} \mathrm{cl}\left\{\frac{f(\bar{x}+tv')-f(\bar{x})}{t}\ ;\ (t,v')\in \tilde{B}_{\frac{1}{k}}(v)\right\}\cap(-\mathrm{int}\mathbb{R}^p_+)=\emptyset, \quad \forall v$$

Another frequently used concept is the Clarke's generalized Jacobian of a Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}^m$:

$$\partial f(x)=conv\{A:\exists x_i\longrightarrow x,\exists\nabla f(x_i)\quad and\ \ \nabla f(x_i)\longrightarrow A\}$$

where *conv* denotes the convex hull. In this case, we have the following:

Theorem 10. *Assume that $f,\ g,$ and h are Lipschitz functions and $\bar{x}$ is a Pareto weak optimal solution. Then there exist multipliers $\lambda\in\mathbb{R}^p_+,\mu\in\mathbb{R}^m_+,\nu\in\mathbb{R}^k$ such that:*

$$\begin{cases} 0\in\partial(\lambda f+\mu g+\nu h)(\bar{x})\\ \langle\mu,g(\bar{x})\rangle=0.\end{cases} \tag{6}$$

The proof of this theorem needs more sophisticated instruments like Ekeland's variational principle, generalized mean value theorem, and topological properties of Clarke's gradient.

In any case, we observe that necessary optimality conditions using the global approach of the Dini–Hadamard derivatives are sharper than those using Clarke's Jacobian because we have the following result:

Theorem 11.

$$\mathrm{D}^+f(\bar{x})(v)\subseteq\partial f(x)v.$$

5 Scalarization Approach

Generally, there are many (possibly infinite) efficient points of a multiobjective problem. One of the most analyzed topics in multiobjective optimization is the scalarization of (1), namely how to build a scalar minimization problem, which leads one to find all the solutions of (1).

The classic scalarization, which is called weight-method, consists in considering the following scalar minimum problem

$$\begin{cases}\min\ \langle q,f(x)\rangle\\ x\in R\end{cases} \tag{7}$$

where $q\in\mathbb{R}^p_+$ and $\sum q_i=1$.

Every solution of (7) is a weak Pareto solution of (1). Moreover, if, for a fixed weight-vector $q\geq 0$, (7) admits a unique solution, then it is a Pareto solution of (1). If $q_i>0,\ \ \forall i,$ then every solution of (7) is a Pareto solution of (1).

The definition of convexlike function will be useful in the next theorem.

Definition 11. *Let C be a convex cone in $\mathbb{R}^p$ and consider a function $F : \mathbb{R}^n \to \mathbb{R}^p$. We say that F is C-convexlike if for every $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$, there is $z \in \mathbb{R}^n$ such that*

$$\alpha F(x) + (1-\alpha) F(y) - F(z) \in C.$$

Theorem 12. *Let f be $\mathbb{R}^p_+$-convexlike and R convex. If y is a v.m.p. of (1), then there exists $\bar{q} \in \mathbb{R}^p_+$, $\bar{q} \neq 0$, such that y is a solution of (7), at $q = \bar{q}$.*

Unfortunately, we remark that if we solve (7) varying the parameter $p \in \operatorname{int} \mathbb{R}^p_+$, we do not find necessarily all the solution of (1).

This approach can be generalized to the case of ordering cones different from the Pareto one. In such cases, the weight must be chosen in the polar cone:

Definition 12. *Let $C \subset \mathbb{R}^p$ be a convex cone. The polar cone of C is given by*

$$C^* := \{z \in \mathbb{R}^p : \ \langle z, c \rangle \geq 0, \qquad \forall\, c \in C\}.$$

There are other scalarization techniques. Among them, one of the most popular is the so-called ϵ-constraints method. We must select an objective function and we transform the other objectives in constraints by imposing on them some upper bounds (the ϵ_i) and obtaining the following scalar problem:

$$\begin{cases} \min \ f_r(x), \\ f_i(x) \leq \epsilon_i \quad \forall i = 1, \ldots, p, \quad i \neq r \\ x \in R. \end{cases} \tag{8}$$

We have the following result:

Theorem 13.

(i) *Every solution of (8) is a weak Pareto optimal solution of (1).*
(ii) *A decision vector $x^* \in R$ is a Pareto optimal solution of (1) if and only if it is solution of (8) for every $r = 1, \ldots, p$ with $\epsilon_i = f_i(x^*)$, $i \neq r$.*
(iii) *If $x^* \in R$ is the unique optimal solution of (8) for some r and with $\epsilon_i = f_i(x^*)$, $i \neq r$, then it is a Pareto optimal solution of (1).*

6 Solution Methods

To find all Pareto optimal solutions is the main aim of multiobjective optimization problem. Nevertheless, when and if we have all the Pareto optimal solutions, we have the problem of "choosing" the best according to some criterion. This second phase needs a so-called decision maker.

Generally the solution methods are divided in four classes:

(i) No-preference methods.
We do not have a decision maker and the method terminates when it produces a Pareto solution.
(ii) *A posteriori* methods.
They produce Pareto optimal solutions (possibly all) and a decision maker chooses the best according his personal criterion.
(iii) *A priori* methods.
The decision maker specifies his preference criterion and the method produces a Pareto optimal solution that is the best according the decision-maker preference.
(iv) Interactive methods.
The decision maker specifies his preference while the algorithm is producing Pareto optimal solutions and the algorithm continues taking into account this preference.

Analyze briefly these methods.

No-preference methods: One of the most popular no-preference methods is the so-called goal method. Such a method consists in finding, in the objectives-space, a Pareto solution that minimizes the distance with a "landmark," which, very often, is the ideal point. Therefore the method consists in solving the following scalar problem:

$$\begin{cases} \min \ ||f(x) - z^{id}||_s, \\ x \in R, \end{cases} \tag{9}$$

where $1 \leq s \leq +\infty$. This means that we are choosing the point x^*, which is the nearest to z^{id} according to the s-distance. We have the following result:

Theorem 14. *Every solution of (9) is a Pareto solution of (1).*

When (1) is linear, it is useful to choose $p = 1$ or $p = +\infty$, In fact the problem (9), in this case, becomes a linear programming problem by adding some auxiliary variables.

***A posteriori* methods:** In order to obtain Pareto solutions (possibly all), we may use weight-method or ϵ-constraint method, which we have described in Section 5. Another interesting method of such a class is the so-called goal-attainment method. To the objectives vector we associate a vector of "goals," say $(F_1, \dots, F_p)$, and we define the following scalar problem:

$$\begin{cases} \min \ \gamma \\ f_i(x) - w_i\gamma \leq F_i \\ x \in R, \end{cases} \tag{10}$$

where γ is a real variable. The coefficients w_i control the attaining (by defect or by surplus) of the goals and they permit us to express the trade-off among the objectives. In fact if $w_i = 0$, for some i, this means that the objective i must be respected; if $w_i = F_i$, we shall obtain for every objective the same percentage of variance.

***A priori* methods:** Two of the most used *a priori* method are the value-function method and the lexicographic method. In the first, the decision maker establishes his value function $V : \mathbb{R}^p \to \mathbb{R}$, and we solve the scalar problem:

$$\begin{array}{c} \min \; V(f(x)) \\ x \in R. \end{array} \tag{11}$$

If the function V is linear, we obtain the weight-method. In the lexicographic method, the decision maker arranges the objectives in a decreasing order, and we solve the scalar problem:

$$\begin{array}{c} \min \; f_1(x) \\ x \in R. \end{array} \tag{12}$$

If this problem has a unique solution, then this is solution of (1) and the algorithm terminates; otherwise, take a solution x^* of (12) and solve the problem:

$$\begin{cases} \min \; f_2(x) \\ f_1(x) \leq f_1(x^*) \\ x \in R. \end{cases} \tag{13}$$

The procedure continues iteratively.

Interactive methods: These methods combine the preceding methods in some interactive procedure.

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Parametric Multiobjective Optimization

Rentsen Enkhbat,[1] Jurgen Guddat,[2] and Altannar Chinchuluun[3]

[1] Department of Mathematics, School of Economics Studies, National University of Mongolia, Ulaanbaatar, Mongolia
renkhbat46@ses.edu.mn
[2] Institute of Mathematics, Humboldt University Berlin, Berlin, Germany
guddat@mathematik.hu-berlin.de
[3] Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida, USA
altannar@ufl.edu

Abstract The weighted sum approach for finding Pareto optimal solutions in multiobjective optimization has been presented depending on a parameter value. We show that the one-parametric optimization techniques can be applied to parametric multiobjective optimization.

Key words: multiobjective optimization, parametric optimization, Pareto optimality

1 Introduction

Consider the multiobjective optimization problem:

$$\min_{x\in D} f(x)$$

with the vector function $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ and a given set $D \in \mathbb{R}^n$.

Multiobjective optimization problems are widely used not only in mathematics but also in engineering and economics. History of multiobjective optimization goes back to F.Y. Edgeworth (1881) and V. Pareto (1896), who has already given the definition of standard optimality concept in multiobjective optimization.

There are many works [3, 4, 7, 13–15, 17, 20–22] devoted to theoretical and numerical aspects of multiobjective optimization. One of the main approaches of finding Pareto optimal solutions is to solve the scalarized optimization problems with given weights:

$$\min_{x\in D} \sum_{i=1}^{m} \alpha_i f_i(x), \tag{1}$$

where $f_i : D \to \mathbb{R}, i = 1, 2, \dots, m$, are functions and $D \subset \mathbb{R}^n$, $\alpha_i \geq 0, i = 1, 2, \dots, m$, are given.

In general, Problem (1) is a nonlinear optimization problem where the global minimizer has to be found. Depending on structure of the functions and the feasible set, Problem (1) can be classified into a specific class of global optimization problems. For instance,

- If $f_i, i = 1, 2, \dots, m$ are convex and D is convex, then Problem (1) is convex programming,
- If $f_i, i = 1, 2, \dots, m$ are concave and D is convex, then Problem (1) is concave programming,
- If $f_i, i = 1, 2, \dots, s$ are convex and $f_i, i = s + 1, \dots, m$ are concave, then Problem (1) is DC programming.

If $f_i, i = 1, 2, \dots, m$ are nonconvex, then Problem (1) belongs to the class of global optimization problem. Finally, if $\alpha_i = \alpha_i(t)$, $t \in [t_A, t_B]$, Problem (1) reduces to parametric multiobjective optimization.

The main purpose of this paper is to consider parametric multiobjective optimization problems and propose appropriate algorithms for solving them. The paper is organized as follows.

In Section 2, we recall basic concepts of multiobjective optimization. Section 3 is devoted to multiobjective optimization problems with parametric weights. Parametric multiobjective linear programming is presented in Section 4.

2 Basics of Multiobjective Optimization

Let us consider multiobjective optimization problems in finite dimensional spaces of the general form

$$\min_{x \in D} f(x), \tag{2}$$

where D is a nonempty subset of $\mathbb{R}^n$ and f is a given vector function with $f(x) = (f_1(x), \dots, f_m(x))$.

Definition 1. *$\overline{x} \in D$ is called a Pareto optimal point (or an efficient solution) if there is no $\overline{x} \in D$ with*

$$f_i(x) \leq f_i(\overline{x}) \quad \textit{for all} \quad i \in \{1, 2, \dots, m\}$$

and $f(x) \neq f(\overline{x})$.

The Pareto optimal concept is the main optimality notion used in multiobjective optimization. The main approach for determination of Pareto optimal points is the weighted sum approach. If we introduce appropriate weights $\alpha_1, \alpha_2, \dots, \alpha_m$, we obtain the scalarized optimization problem:

$$\min_{x \in D} \sum_{i=1}^{m} \alpha_i f_i(x) \tag{3}$$

Definition 2. *$\overline{x} \in D$ is called a properly Pareto optimal point of problem (2), if $\overline{x}$ is a Pareto optimal point and there is a real number $\mu > 0$ such that, for every $i \in \{1, 2, \dots, m\}$ and every $x \in D$ with $f_i(x) < f_i(\overline{x})$, there exists at least one $j \in \{1, 2, \dots, m\}$ with $f_j(x) > f_j(\overline{x})$ and*

$$\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} \leq \mu.$$

Theorem 1. *[13]. Let $\alpha_1, \alpha_2, \dots, \alpha_m > 0$ be given real numbers. If $\overline{x} \in D$ is a solution of the scalar optimization problem (3), then $\overline{x}$ is a (properly) Pareto optimal point of the multiobjective optimization problem (2).*

Definition 3. *$\overline{x}$ is called a strongly Pareto optimal point of Problem (2) (or a strongly efficient solution) if*

$$f_i(\overline{x}) \leq f_i(x) \quad \textit{for all} \quad x \in D \quad \textit{and} \quad i \in \{1, 2, \dots, m\}.$$

Lemma 1. *Every strongly Pareto optimal point of Problem (2) is a Pareto optimal point.*

Proof. Let $\overline{x} \in D$ be a strongly Pareto optimal point, i.e.,

$$f_i(\overline{x}) \leq f_i(x) \quad \text{for all} \quad x \in D \quad \text{and} \quad i \in \{1, 2, \dots, m\}.$$

Then there is no $x \in D$ with $f(x) \neq f(\overline{x})$ and $f_i(x) \leq f_i(\overline{x})$ for all $i \in \{1, 2, \dots, m\}$. Hence, $\overline{x}$ is a Pareto optimal point. ∎

3 Multiobjective Optimization Problems

Consider the multiobjective optimization problems of the following type:

$$\min_{x \in D} f(x), \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$, and $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, m$, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \dots, s$, are twice continuously differentiable convex functions. Let

$$D = \{x \in R^n | g_i(x) \leq 0, j \in J\}, J = \{1, 2, \dots, s\}.$$

Consider the problem of finding a Pareto optimal solution for problem (4) where all weights depend on a parameter. Then the scalarized optimization problem is

$$\min_{x \in D} f(x, t), t \in [t_A, t_B] \tag{5}$$

where $f(x, t) = \sum_{i=1}^{m} \alpha_i(t) f_i(x)$; $\alpha_i : [t_A, t_B] \to \mathbb{R}_+$, $i = 1, 2, \dots, m$, are positive defined continuous functions and t_A, t_B are given. According to Theorem 1, it is clear that every solution to (5) for $t \in [t_A, t_B]$ is a Pareto optimal solution to (4) for $t \in [t_A, t_B]$. Now Problem (5) can be considered as a one-parametric convex minimization problem. Assume that:

(H1:) There exists a continuous function $x : [t_A, t_B] \to \mathbb{R}^n$ such that $x(t)$ is a global minimizer for (5).
(H2:) $x(t_A)$ is known.

The KKT conditions for Problem (5) state that

$$\begin{cases} D_x f(x,t) + \sum_{j\in J} \mu_j D_x g_j(x,t) = 0 \\ g_j(x,t) \le 0, \mu_j \ge 0, j \in J \\ \mu_j g_j(x,t) = 0, j \in J \\ t \in [t_A, t_B], \end{cases} \tag{6}$$

where $D_x f(x,t) = (\frac{\partial f(x,t)}{\partial x_1}, \dots, \frac{\partial f(x,t)}{\partial x_n})$.

Consider the auxiliary parametric optimization problem

$$\min f(x,t),\ t \in [t_A, t_B] \tag{7}$$

$$s.t.\ g_j(x) = 0,\ j \in \widetilde{J} \subset J \tag{8}$$

Let $\upsilon^0 = (x^0(t), \mu^0(t))$ satisfy the KKT conditions for Problem (7)–(8) with $\widetilde{J} = J_0$.

This system can be written in the following compact notation:

$$F(\upsilon, t) = 0, \quad t \in [t_A, t_B], \tag{9}$$

where $\upsilon = (x^0(t), \mu^0(t))$.

In order to apply Newton's method to System (9), we have to solve a linear system $D_\upsilon F(\upsilon(t), t)$ as matrix. The same matrix is used to compute $\dot{\upsilon}(t)$:

$$D_\upsilon F(\upsilon(t), t)\dot{\upsilon}(t) = -D_t F(\upsilon(t), t) \tag{10}$$

Therefore, using Newton's method as corrector, we have [9]:

$$D_\upsilon F(\upsilon_i^{k_i-1}, t_i)(\upsilon_i^{k_i} - \upsilon_i^{k_i-1}) = -F(\upsilon_i^{k_i-1}, t_i)$$

and

$$D_\upsilon F(\upsilon_i^{k_i-1}, t_i)(\dot{\upsilon}_i^{k_i-1}) = -D_t F(\upsilon_i^{k_i-1}, t_i).$$

Now we can adapt algorithm PATH1 ([9]) for solving Problem (5) as follows.

Algorithm PATH1

Step 1. Given $x^0, J_0, \mu^0, \varepsilon_t, \varepsilon_\upsilon, \varepsilon_{\dot{\upsilon}}, \Delta t_{min}, \Delta t_{max}, t_0 := t_A, k := 1$.
Step 2. Determine a step size $\Delta t_k \in [\Delta t_{min}, \Delta t_{max}]$.
Step 3. Find an approximate KKT point $\upsilon^k = (x^k, \mu^k)$ solving Problem (7)–(8) for $t = t_k$ with $\|\upsilon^k - \upsilon(t_k)\| < \varepsilon_\upsilon, \upsilon(t) = (x(t_k), \mu(t_k))$.
Step 4. If

$$\begin{cases} g_j(x^k(t_k)) < -\|D_x g_j(x^k)\|\varepsilon_\upsilon, \quad j \notin J_{k-1} \\ (\mu^k)_j > \varepsilon_\upsilon, \quad j \in J_{k-1} \end{cases}$$

then $J_k := J_{k-1}$ and go to Step 6.

Step 5. Find $\bar{t}$ solving the system:

$$\begin{cases} g_j(x^k(t)) \le 0, & j \in J \backslash J_{k-1} \\ \mu_j^k(t) \ge 0, & j \in J_{k-1} \end{cases}$$

Step 6. Solve system (9) approximately, i.e.,

$$\begin{aligned} |\widetilde{t} - \bar{t}| &< \varepsilon_t, \\ \|\widetilde{\upsilon}^k - \upsilon^k(\widetilde{t})\| &\le \varepsilon_\upsilon, \\ \|\dot{\widetilde{\upsilon}}^k - \dot{\upsilon}^k(\widetilde{t})\| &\le \varepsilon_{\dot{\upsilon}}, \\ \widetilde{\upsilon} &= (\widetilde{x}^k, \widetilde{\mu}^k). \end{aligned}$$

Step 7. From index sets:

$$\begin{aligned} \widetilde{J} = J_{k-1} \bigcup \{ & j \notin J_{k-1} : g_j(\widetilde{x}^k, \widetilde{t}) + |D_x g_j(\widetilde{x}^k, \widetilde{t}) \dot{\widetilde{x}}^k + D_t g_j(\widetilde{x}^k, \widetilde{t})| \varepsilon_t \\ & \ge -\varepsilon_t - [\|D_x g_j(\widetilde{x}^k, \widetilde{t})\| + 1] \varepsilon_\upsilon \}, \\ \widetilde{J}^+ :&= \{ j \in J_{k-1} : \widetilde{\mu}_j^k \ge |\dot{\widetilde{\mu}}_j^k| \varepsilon_t + \varepsilon_\upsilon + \varepsilon_t \}, \\ \widetilde{J}^0 :&= \widetilde{J} \backslash \widetilde{J}^+. \end{aligned}$$

Step 8. If $|\widetilde{J}^0| = 1$, then construct the index set J^k as:

$$J_k = \{ j : g_j(\widetilde{x}^k) = 0 \}.$$

Otherwise, go to the next step.
Step 9. Solve Problem (7)–(8) for $t = t_k$ and $J_k := \{ j : g_j(\widetilde{x}) = 0 \}$.
Step 10. Set $k := k + 1$ and go to Step 2.

We note that Algorithm PATH1 generates a sequence of approximate global minimizers in Problem (5). The convergence of this algorithm is given by the following theorem.

Theorem 2. *[9] Assume that the assumptions (H1)–(H2) hold. Then for $\varepsilon, \varepsilon_t, \varepsilon_\upsilon, \varepsilon_{\dot{\upsilon}}, \Delta t_{max}$ sufficiently small, algorithm PATH1 generates a discretization*

$$t_A = t_0 < t_1 < \cdots < t_i < t_{i+1} < t_N = t_B$$

with corresponding points $(\widetilde{x}^i, \widetilde{\mu}^i)$ such that $\|(\widetilde{x}^i, \widetilde{\mu}^i) - (x(t_i), \mu(t_i))\| < \varepsilon,\ i = 1, 2, \ldots, N$.

Remark 1. A search for $\bar{t}$ in Step 5 is carried out by bisection strategy in [9].

Remark 2. Choice of parameters $\varepsilon_t, \varepsilon_\upsilon, \varepsilon_{\dot{\upsilon}}$ and $\Delta t_{min}, \Delta t_{max}$ is done according to [9].

Consider the following example.

Example 1.

$$\begin{aligned}
&\min \begin{pmatrix} (x_1-4)^2 \\ x_2^2 \end{pmatrix} \\
&s.t.\ -2x_1 + x_2 \le 2 \\
&\quad\ \ 4x_1 - 3x_2 \le 12 \\
&\quad\ -4x_1 - x_2 \le 4.
\end{aligned}$$

Formulate Problem (5) for $f_1(x_1,x_2) = (x_1-4)^2$, $f_2(x) = x_2^2$, $\alpha_1(t) = t$, $\alpha_2(t) = 1-t$, $t \in (0,1)$ as follows:

$$\begin{aligned}
&\min\ (1-t)(x_1-4)^2 + tx_2^2, \quad t \in (0,1) \\
&s.t.\ -2x_1 + x_2 \le 2 \\
&\quad\ \ 4x_1 - 3x_2 \le 12 \\
&\quad\ -4x_1 - x_2 \le 4.
\end{aligned}$$

We can check that the KKT conditions give the following solution for this problem:

$$\begin{cases} x_1^* = \dfrac{84-36t}{9+7t}, \\ x_2^* = \dfrac{76(1-t)}{9+7t} \end{cases} \quad t \in (0,1).$$

4 Parametric Multiobjective Optimization in Linear Programming

Consider multiobjective linear programming problems as a special case of Problem (4)

$$\min_{x \in D} f(x), \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^s$ with $f(x) = (f_1(x), \ldots, f_p(x))$, $f_i(x) = < c^i, x >$, $c^i = (c_1^i, c_2^i, \ldots, c_n^i) \in \mathbb{R}^n$, $i = 1, 2, \ldots, p$, and $D = \{x \in \mathbb{R}^n \mid Ax = b, x \ge 0\}$.

A is a real $(m \times n)$ matrix, and $b \in \mathbb{R}^m$ is a given vector, $< \cdot >$ denotes the scalar product in $\mathbb{R}^n$.

Introduce the scalarized function $F(x,t)$ with linear parametric weights.

$$F(x,t) = \sum_{i=1}^{p} (\alpha_i t + \beta_i) < c^i, x >, \ t \in [t_A, t_B].$$

Then the corresponding scalarized minimization problem is

$$\min_{x \in D} F(x,t), \ t \in [t_A, t_B], \tag{12}$$

where the coefficients α_i and β_i $(i = 1, \ldots, p)$ are given and $\alpha_i t + \beta_i > 0$ for all $t \in [t_A, t_B]$ and $i = 1, \ldots, p$.

It is clear that all assumptions of Theorem 1 are satisfied and a solution to linear programming Problem (12) for any $t \in [t_A, t_B]$ is a Pareto optimal solution. We can see that Problem (12) is a parametric linear programming and there exist special path following methods [18]. The following statement allows us to solve Problem (12) numerically.

Theorem 3. *[5]. Assume that Problem (12) has a nondegenerate basic solution for each $t \in [t_A, t_B]$. Then Problem (12) can be solved in a finite number of discretization $t_A < t_1 < \cdots < t_B$. Now we can apply algorithm LPT ([5]) to Problem (12) as follows.*

Algorithm LPT

Input: Nondegenerate problem (4.2). Output $t_A < t_1 < \cdots < t_B$.

Step 1. Set $k := 1$ and solve (4.2) for $t = t_k := t_A$. Let $x^k = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ be a basic solution and $B_k = (a^{i_1}, \ldots, a^{i_m})$ is its corresponding basic.

Step 2. Compute the optimality criteria $\Delta^k_{j_1}$ and $\Delta^k_{j_2}$ at the basis B_k :

$$\Delta^k_{j_1} = \sum_{s=1}^{m} c^{'}_{i_s} x_{i_s j} - c^{'}_j,$$

$$\Delta^k_{j_2} = \sum_{s=1}^{m} c^{''}_{i_s} x_{i_s j} - c^{''}_j, \quad j = 1, \ldots, n,$$

where

$$x_{i_s j} = (B_k^{-1} a^j)_s, \quad s = 1, \ldots, m, \quad j = 1, \ldots, n;$$

$$c^{'}_j = \sum_{i=1}^{p} \beta_i c^i_j, \quad c^{''}_j = \sum_{i=1}^{p} \alpha_i c^i_j, \ j = 1, \ldots, n.$$

Step 3. Find $\widetilde{t}$ and l from the condition:

$$\widetilde{t} = \min_{\Delta_{j_2} > 0} \left(-\frac{\Delta_{j_1}}{\Delta_{j_2}}\right) = -\frac{\Delta_{l_1}}{\Delta_{l_2}}.$$

If $\widetilde{t}$ does not exist, then $\widetilde{t} = +\infty$ and terminate.

Step 4. If $\widetilde{t} \geq t_B$ then stop. Discretization process is

$$t_A < t_1 < \cdots < t_B.$$

Step 5. Compute $x_{i_s l}$ as

$$x_{i_s l} = (B_k^{-1} a^l)_s, \quad s = 1, \ldots, m,$$

Step 6. Find a number r as follows:

$$\frac{x_{i_r}}{x_{i_r l}} = \min_{x_{i_s l}>0} \frac{x_{i_s}}{x_{i_s l}}$$

Step 7. Construct a new basis B_{k+1} by replacing a^{i_r} in B_k by a^l. x^{k+1} is a corresponding basic solution.

Step 8. Set $k = k+1$ and $t_k = \widetilde{t}$, then go to Step 2.

Let us illustrate algorithm LPT on the following examples.

Example 2.

$$\begin{aligned} \min\ & F(x) = (-2-3t)x_1 + (1-2t)x_2 - 3tx_3 - 4x_4, \quad t \in [-20, 20] \\ s.t.\ & x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 7 \\ & -3x_1 + 4x_2 + 3x_3 - x_4 + x_6 = 15 \\ & 2x_1 - 5x_2 + 2x_3 + 2x_4 + x_7 = 2 \\ & x_j \geq 0,\ j = 1, \ldots, 7. \end{aligned}$$

The algorithm LPT provides the following solution:

$$x^* = \begin{cases} (0,0,0,1,4,16,0) \ \ if \ -20 \leq t < -\frac{9}{2} \\ (0, \frac{8}{19}, 0, \frac{39}{19}, 0, \frac{292}{19}, 0) \ if \ -\frac{9}{2} \leq t < \frac{2}{65} \\ (\frac{13}{3}, \frac{4}{3}, 0, 0, 0, \frac{68}{3}, 0) \ \ if \ \frac{2}{65} \leq t \leq 20 \end{cases}$$

Example 3. [14] Determine all Pareto optimal solutions of the following problem:

$$\begin{aligned} \min\ & \begin{cases} -4x_1 - 2x_2 \\ 8x_1 - 10x_2 \end{cases} \\ s.t.\ & x_1 + x_2 \leq 70 \\ & x_1 + 2x_2 \leq 100 \\ & x_1 \leq 60 \\ & x_2 \leq 40 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Let the vector t of the weights be given as $(t, 1-t)$ with $t \in (0,1)$. Then the scalarized parametric optimization problem is

$$\begin{aligned} \min\ & (-8+4t)x_1 + (8t-10)x_2, \quad t \in (0,1) \\ s.t.\ & x_1 + x_2 \leq 70 \\ & x_1 + 2x_2 \leq 100 \\ & x_1 \leq 60 \\ & x_2 \leq 40 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

All solutions of this problem are

$$x^* = \begin{cases} (40,30), \quad if\ 0 < t < \frac{1}{2} \\ (40\lambda + 60(1-\lambda), 30\lambda + 10(1-\lambda)),\ \lambda \in [0,1]\ \ if\ t = \frac{1}{2} \\ (60,10)\ \ if\ \frac{1}{2} < t < 1 \end{cases}$$

Example 4. [17] Find all Pareto optimal solutions of the following problem:

$$\min \begin{pmatrix} -2x_1 - x_2 + 25 \\ \\ x_1 - 2x_2 + 18 \end{pmatrix}$$

$$\begin{aligned} s.t.\ & -x_1 + 3x_2 \leq 21 \\ & x_1 + 3x_2 \leq 27 \\ & 4x_1 + 3x_2 \leq 45 \\ & 3x_1 + x_2 \leq 30 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

The scalarized optimization problem with parametric weight $(t, 1-t) \in (0,1)$ is

$$\begin{aligned} \min F(x,t) &= t(-2x_1 - x_2 + 25) + (1-t)(x_1 - 2x_2 + 18) \\ &= (1-3t)x_1 + (-2+t)x_2 + 43t + 18,\ t \in (0,1) \\ s.t. \quad & -x_1 + 3x_2 \leq 21 \\ & x_1 + 3x_2 \leq 27 \\ & 4x_1 + 3x_2 \leq 45 \\ & 3x_1 + x_2 \leq 30 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

All solutions of this problem found by algorithm LPT are given as follows:

$$x^* = \begin{cases} (0,7)\ \ if\ 0 < t \leq \frac{1}{8} \\ (3,8)\ \ if\ \frac{1}{8} < t \leq \frac{1}{2} \\ (6,7)\ \ if\ \frac{1}{2} < t \leq \frac{11}{13} \\ (9,3)\ \ if\ \frac{11}{13} < t < 1 \end{cases}$$

5 Conclusion

We have proposed methods and algorithms for solving parametric multiobjective optimization problems in a finite number of discretization intervals. Some numerical examples are provided.

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Part III

Applications

The Extended Linear Complementarity Problem and Its Applications in Analysis and Control of Discrete-Event Systems

Bart De Schutter

Delft Center for Systems and Control, Delft University of Technology,
Delft, The Netherlands
b.deschutter@dcsc.tudelft.nl

Abstract In this chapter, we give an overview of complementarity problems with a special focus on the extended linear complementarity problem (ELCP) and its applications in analysis and control of discrete-event systems such as traffic signal controlled intersections, manufacturing systems, railway networks, etc. We start by giving an introduction to the (regular) linear complementarity problem (LCP). Next, we discuss some extensions, with a particular emphasis on the ELCP, which can be considered to be the most general linear extension of the LCP. We then discuss some algorithms to compute one or all solutions of an ELCP. Next, we present a link between the ELCP and max-plus equations. This is then the basis for some applications of the ELCP in analysis and model-based predictive control of a special class of discrete-event systems. We also show that — although the general ELCP is NP-hard — the ELCP-based control problem can be transformed into a linear programming problem, which can be solved in polynomial time.

Key words: linear complementarity problem, extended linear complementarity problem, algorithms, control applications, discrete-event systems, max-plus-linear systems

Introduction

The linear complementarity problem (LCP) is one of the fundamental problems in optimization and mathematical programming [10, 54]. Several authors have introduced (both linear and nonlinear) extensions of the LCP, and some of these linear extensions will be discussed in more detail below. The importance of the LCP and its generalizations is evidenced by a broad range of applications in the fields of engineering and economics such as quadratic programming, determination of Nash equilibriums, nonlinear obstacle problems, and problems involving market equilibriums, invariant capital stock, optimal stopping, contact and structural mechanics, elastohydrodynamic lubrication, traffic equilibriums, operation planning in deregulated electricity markets, manufacturing systems, etc. (see the other chapters of this book, the books and overview papers [10, 29–31, 38], and the references therein).

Apart from the LCP, the focus of this chapter will be on yet another extension of the LCP, which we have called the extended linear complementarity problem (ELCP) [17], and which can in some way be considered as the most general linear extension of the LCP. This problem arose from our research on discrete-event systems (max-plus-linear systems, max-plus-algebraic applications, and min-max-plus systems [18, 24]) and hybrid systems (traffic signal control, and first-order hybrid systems with saturation [15, 21]). Furthermore, the ELCP can also be used in the analysis of several classes of hybrid systems such as piecewise-affine systems [35, 61], max-min-plus-scaling systems [26], and linear complementarity systems [20, 36].

This chapter is organized as follows: In Section 1, we present the LCP and the ELCP, and we discuss how they are related. In Section 2, we present some other (linear) generalizations of the LCP, and we show that they can be considered as special cases of the ELCP. Next, we discuss some algorithms to compute one or all solutions of an ELCP in Section 3. In Section 4, we then explain the relation between systems of max-plus equations and the ELCP, which is the basis for several applications of the ELCP in analysis and control of discrete-event systems, some of which are then discussed in more detail in Section 5. We conclude this chapter with a summary.

As this chapter is mainly intended to be an overview, the proofs will be reduced to a minimum (with appropriate references to the papers where the full proofs can be found) and only be given in case they are functional.

1 Linear Complementarity Problem

1.1 Notation

All vectors used in this paper are assumed to be column vectors. The transpose of a vector a is denoted by a^{T}. Furthermore, inequalities for vectors have to be interpreted entrywise. We use I_n to denote the n by n identity matrix and $0_{m\times n}$ to denote the m by n zero matrix. If the dimensions of the identity matrix or the zero matrix are omitted, they should be clear from the context.

1.2 Linear Complementarity Problem

One of the possible formulations of the LCP is the following [10]:

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, find vectors w, $z \in \mathbb{R}^n$ such that

$$w = Mz + q \tag{1}$$

$$w, z \geqslant 0 \tag{2}$$

$$w^{\mathrm{T}} z = 0. \tag{3}$$

Note that if w and z are solutions of the LCP, then it follows from (2) and (3) that

$$z_i w_i = 0 \qquad \text{for } i = 1, \ldots, n\,,$$

i.e., for each i we have the following conditions: if $w_i > 0$, then we should have $z_i = 0$, and if $z_i > 0$, then $w_i = 0$. So the zero patterns of w and z are complementary. Therefore, condition (3) is called the *complementarity condition* of the LCP.

For an extensive state-of-the-art overview of the LCP (and related problems), we refer the interested reader to [10, 29–31, 38].

1.3 Extended Linear Complementarity Problem

The ELCP is defined as follows [17]:

Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$, and m index sets $\phi_1, \ldots, \phi_m \subseteq \{1, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that

$$Ax \geqslant c \tag{4}$$

$$Bx = d \tag{5}$$

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i \;=\; 0. \tag{6}$$

The feasible set of the ELCP (4)–(6) is defined by

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid Ax \geqslant c, Bx = d\}.$$

The surplus variable $\operatorname{surp}(i, x)$ of the ith inequality of $Ax \geqslant c$ is defined as $\operatorname{surp}(i, x) = (Ax - c)_i$.

Condition (6) represents the *complementarity condition* of the ELCP. One possible interpretation of this condition is the following: Because $Ax \geqslant c$, all the terms in (6) are nonnegative. Therefore, (6) is equivalent to

$$\prod_{i \in \phi_j} (Ax - c)_i = 0 \qquad \text{for } j = 1, \ldots, m\,.$$

So we could say that each set ϕ_j corresponds with a group of inequalities in $Ax \geqslant c$, and that in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

The solution set of an ELCP can be characterized as follows [17]:

Theorem 1. *In general the solution set $\mathcal{S}$ of an ELCP consists of the union of faces of a polyhedron.*

This solution set can be represented using four sets:

- a set $\mathcal{X}^{\text{fin}}$ of finite vertices of $\mathcal{S}$,
- a set $\mathcal{X}^{\text{ext}}$ of generators for the extreme rays of $\mathcal{S}$,
- a basis $\mathcal{X}^{\text{cen}}$ for the linear subspace associated with the maximal affine subspace of $\mathcal{S}$,
- and a set Λ of pairs of so-called maximal cross-complementary subsets of $\mathcal{X}^{\text{ext}}$ and $\mathcal{X}^{\text{fin}}$ (where each pair corresponds with a face of $\mathcal{S}$).

In Section 3.1, we will present an algorithm to compute these sets. Then x is a solution of the ELCP if and only if there exists an ordered pair $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}}) \in \Lambda$ such that

$$x = \sum_{x_k^{\text{cen}} \in \mathcal{X}^{\text{cen}}} \lambda_k x_k^{\text{cen}} + \sum_{x_k^{\text{ext}} \in \mathcal{X}_s^{\text{ext}}} \kappa_k x_k^{\text{ext}} + \sum_{x_k^{\text{fin}} \in \mathcal{X}_s^{\text{fin}}} \mu_k x_k^{\text{fin}} \tag{7}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ for all k and $\sum_k \mu_k = 1$.

We can also reverse Theorem 1 [17]:

Theorem 2. *The union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP.*

Remark 1. The complementarity conditions of both the LCP and the ELCP consist of a sum of products. However, in contrast with the ELCP where the products may contain one, two, or more factors, the products in complementarity condition of the LCP always contain exactly two factors. Moreover, any variable in the LCP is contained in precisely one index set ϕ_j, whereas in the ELCP formulation it may be contained in any number of index sets.

We also have the following complexity result:

Theorem 3. *In general the ELCP with rational data is an NP-hard problem.*

The proof of this result is based on the fact that in general, the LCP with rational data is also NP-hard [7].

1.4 The Link between the LCP and ELCP

It is easy to verify that the following lemma holds:

Lemma 1. *The LCP is a special case of the ELCP.*

Moreover, we also have a reverse statement [23]:

Theorem 4. *If the surplus variables of the inequalities of an ELCP are bounded (from above*[1]*) over the feasible set of the ELCP, then the ELCP can be rewritten as an LCP.*

[1] We only need boundedness from above because the surplus variables are always nonnegative due to the condition $Ax \geqslant c$.

Proof. Consider the ELCP (4)–(6). If there is an equality condition $Bx = d$ present, then we remove it using the following procedure: we can replace $Bx = d$ by $Bx \geqslant d$, and impose equality conditions on these inequalities by adding the index sets $\phi_{m+1} = \{p+1\}, \ldots, \phi_{m+q} = \{p+q\}$. So from now on we consider the following formulation of the ELCP[2]:

$$Ax \geqslant c \tag{8}$$

$$\sum_{i=1}^{m} \prod_{j \in \phi_i} (Ax - c)_j = 0. \tag{9}$$

The proof of the theorem consists of two main steps:

1. First, we transform the ELCP into a mixed integer problem to get rid of the ELCP complementarity condition at the cost of introducing some additional binary variables.
2. Next, we transform all variables (both binary and real-valued ones) into nonnegative real ones, which will lead to an LCP.

Step 1: Transformation into a mixed integer problem

Define a diagonal matrix $D^{\text{upp}} \in \mathbb{R}^{p \times p}$ with $(D^{\text{upp}})_{ii} = d_{ii}^{\text{upp}}$ an upper bound for $\text{surp}(i, x) = (Ax - c)_i$ over the feasible set $\mathcal{F}$ of the ELCP. So for each $i \in \{1, \ldots, p\}$ we have $d_{ii}^{\text{upp}} \geqslant (Ax - c)_i$ for all $x \in \mathcal{F}$. Now consider the following system of equations:

$$\delta \in \{0,1\}^p, \;\; x \in \mathbb{R}^n \tag{10}$$

$$0 \leqslant (Ax - c)_i \leqslant d_{ii}^{\text{upp}} \delta_i \qquad \text{for } i = 1, \ldots, p \tag{11}$$

$$\sum_{i \in \phi_j} \delta_i \leqslant \#\phi_j - 1 \qquad \text{for } j = 1, \ldots, m\,, \tag{12}$$

where $\#\phi_j$ denotes the number of elements of the set ϕ_j. Problem (10)–(12) will be called the equivalent *mixed integer linear feasibility problem* (MILFP).

Now we show that the MILFP is equivalent to the ELCP (8)–(9) in the sense that x is a solution of the ELCP (8)–(9) if and only if there exists a δ such that (x,δ) is a solution of (10)–(12). Equation (8) is implied by (11). Note that (10) and (12) imply that for each j, at least one of the δ_i's with $i \in \phi_j$ is equal to 0. If $\delta_{i'} = 0$, then it follows from (11) that $(Ax - c)_{i'} = 0$. This implies that in each index set ϕ_j, there is at least one index for which the corresponding surplus variable equals 0. Hence, the complementarity condition (9) is also

[2] Note, however, that if we want to solve an ELCP using, e.g., the algorithm of Section 3.1, then the formulation (5)–(6) leads to a more efficient solution than does the reformulation (8)–(9).

implied by (10)–(12). So (10)–(12) imply (8)–(9), and it is easy to verify that the reverse statement also holds. As a consequence, the MILFP is equivalent to the ELCP.

Define $S \in \mathbb{R}^{m \times p}$ with $s_{ji} = 1$ if $i \in \phi_j$ and $s_{ji} = 0$ otherwise, and $t \in \mathbb{R}^m$ with $t_j = \#\phi_j - 1$. The MILFP can now be rewritten compactly as

Find $x \in \mathbb{R}^n$ and $\delta \in \{0,1\}^p$ such that

$$0 \leqslant Ax - c \leqslant D^{\mathrm{upp}}\delta \tag{13}$$

$$S\delta \leqslant t. \tag{14}$$

Step 2: Now we transform the MILFP into an LCP.

This will be done in three steps.

(a) First we transform condition $\delta \in \{0,1\}^p$ into the LCP framework. All the variables of an LCP should be real-valued, but the vector δ in the MILFP is a binary vector. However, the condition $\delta_i \in \{0,1\}$ is equivalent to the set of conditions

$$\delta_i \in \mathbb{R}, \quad \delta_i \geqslant 0, \quad 1 - \delta_i \geqslant 0, \quad \delta_i(1-\delta_i) = 0.$$

So if we introduce a vector $v_\delta \in \mathbb{R}^p$ of auxiliary variables, then the condition $\delta \in \{0,1\}^p$ is equivalent to

$$\delta, v_\delta \in \mathbb{R}^p, \quad \delta, v_\delta \geqslant 0, \quad v_\delta = \mathbf{1}_p - \delta, \quad \delta^{\mathrm{T}} v_\delta = 0,$$

where $\mathbf{1}_p$ is a p-component column vector consisting of all 1's.

(b) The inequality $0 \leqslant Ax - c$ can be adapted to the LCP framework by introducing an auxiliary vector $v_A \in \mathbb{R}^p$ with $v_A = Ax - c \geqslant 0$. To obtain a complementarity condition for v_A, we introduce $w_A \in \mathbb{R}^p$ such that $v_a, w_A \geqslant 0$ and $v_A^{\mathrm{T}} w_A = 0$ (note that we can always take $w_A = 0$ to get these conditions satisfied). Hence, $0 \leqslant Ax - c$ can be rewritten as

$$v_a, w_A \geqslant 0, \quad v_A = Ax - c, \quad v_A^{\mathrm{T}} w_A = 0,$$

with $v_A, w_A \in \mathbb{R}^p$. The inequalities $Ax - c \leqslant D^{\mathrm{upp}}\delta$ and $S\delta \leqslant t$ can be dealt with in a similar way.

(c) All variables in an LCP are nonnegative whereas this condition is not present in the MILFP. Therefore, we split x in its positive part $x^+ = \max(x,0)$ and its negative part $x^- = \max(-x,0)$. So $x = x^+ - x^-$ with $x^+, x^- \geqslant 0$ and $(x^+)^{\mathrm{T}} x^- = 0$. To obtain independent LCP-like complementarity conditions for x^+ and x^-, we introduce additional auxiliary vectors $v^+, v^- \in \mathbb{R}^n$ with $v^+ = x^+$ and $v^- = x^-$ such that $(v^-)^{\mathrm{T}} x^+ = 0$ and $(v^+)^{\mathrm{T}} x^- = 0$ with $x^+, x^-, v^+, v^- \geqslant 0$.

Combining these three steps results in the following equivalent LCP:

$$\underbrace{\begin{bmatrix} v_\delta \\ v^- \\ v^+ \\ v_A \\ v_{D^{\mathrm{upp}}} \\ v_S \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} -I_p & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & I_n & 0\ 0\ 0 \\ 0 & I_n & 0 & 0\ 0\ 0 \\ 0 & A & -A & 0\ 0\ 0 \\ D^{\mathrm{upp}} & -A & A & 0\ 0\ 0 \\ -S & 0 & 0 & 0\ 0\ 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \delta \\ x^+ \\ x^- \\ w_A \\ w_{D^{\mathrm{upp}}} \\ w_S \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} \mathbf{1}_p \\ 0 \\ 0 \\ -c \\ c \\ t \end{bmatrix}}_{q} \tag{15}$$

$$w, z \geqslant 0 \tag{16}$$

$$w^{\mathrm{T}} z = 0, \tag{17}$$

with $w, z \in \mathbb{R}^{3p+2n+m}$. The solution of the original ELCP can be extracted from the solution of the LCP (15)–(17) by setting $x = x^+ - x^-$. ■

The introduction of the MILFP in this proof was inspired by the paper [5], in which a class of hybrid systems is discussed consisting of mixed logical dynamic systems, which can be shown to be equivalent to systems with an ELCP-based model description [35].

2 Other Extensions of the LCP

Several authors have introduced linear and nonlinear extensions and generalizations of the LCP. Some examples of "linear" extensions of the LCP are

- the Horizontal LCP [10]:
  Given M, $N \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, find w, $z \in \mathbb{R}^n$ such that

$$\begin{aligned} & w, z \geqslant 0 \\ & Mz + Nw = q \\ & z^{\mathrm{T}} w = 0. \end{aligned}$$

- the Vertical LCP [10] (also known as the Generalized LCP of Cottle and Dantzig [9]):
  Let $M \in \mathbb{R}^{m\times n}$ with $m \geqslant n$ and let $q \in \mathbb{R}^m$. Suppose that M and q are partitioned as follows:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix},$$

  with $M_i \in \mathbb{R}^{m_i\times n}$ and $q_i \in \mathbb{R}^{m_i}$ for $i = 1, \ldots, n$ and with $\sum_{i=1}^{n} m_i = m$.
  Now find $z \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& z \geqslant 0 \\
& q + Mz \geqslant 0 \\
& z_i \prod_{j=1}^{m_i} (q_i + M_i z)_j \; = \; 0 \quad \text{for } i = 1, \ldots, n.
\end{aligned}
$$

- the Extended LCP of Mangasarian and Pang [33, 48]:
  Given M, $N \in \mathbb{R}^{m \times n}$ and a polyhedral set $\mathcal{P} \subseteq \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& x, y \geqslant 0 \\
& Mx - Ny \in \mathcal{P} \\
& x^{\mathrm{T}} y = 0.
\end{aligned}
$$

- the Extended Horizontal LCP of Sznajder and Gowda [62]:
  Given $k+1$ matrices $C_0, C_1, \ldots, C_k \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n$ and $k-1$ vectors $d_1, d_2, \ldots, d_{k-1} \in \mathbb{R}^n$ with positive components, find $x_0, x_1, \ldots, x_k \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& x_0, x_1, \ldots, x_k \geqslant 0 \\
& d_j - x_j \geqslant 0 && \text{for } j = 1, \ldots, k-1 \\
& C_0 x_0 = q + \sum_{j=1}^{k} C_j x_j \\
& x_0^{\mathrm{T}} x_1 = 0 \\
& (d_j - x_j)^{\mathrm{T}} x_{j+1} = 0 && \text{for } j = 1, \ldots, k-1.
\end{aligned}
$$

- the Generalized LCP of Eaves [27]:
  Given n positive integers $m_1, m_2, \ldots, m_n$, n matrices $A_1, A_2, \ldots, A_n \in \mathbb{R}^{p \times m_i}$, and a vector $b \in \mathbb{R}^p$, find $x_1, x_2, \ldots, x_n \in \mathbb{R}^{m_i}$ such that

$$
\begin{aligned}
& x_1, x_2, \ldots, x_n \geqslant 0 \\
& \sum_{i=1}^{n} A_i x_i \leqslant b \\
& \sum_{i=1}^{n} \prod_{j=1}^{m_i} (x_i)_j = 0.
\end{aligned}
$$

- the Generalized LCP of Ye [66]:
  Given A, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times k}$ and $q \in \mathbb{R}^m$, find x, $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$ such that

$$
\begin{aligned}
& x, y, z \geqslant 0 \\
& Ax + By + Cz = q \\
& x^{\mathrm{T}} y = 0.
\end{aligned}
$$

- the Generalized LCP of De Moor and Vandenberghe [13]:
  Given $Z \in \mathbb{R}^{p \times n}$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, find $u \in \mathbb{R}^n$ (with $u \neq 0$) such that

$$\begin{aligned}
&u \geqslant 0 \\
&Zu = 0 \\
&\sum_{j=1}^{m} \prod_{i \in \phi_j} u_i \;=\; 0.
\end{aligned}$$

- the (Extended) Generalized Order LCP of Gowda and Sznajder [34]:
  Given $B_0, B_1, \ldots, B_k \in \mathbb{R}^{n \times n}$, and $b_0, b_1, \ldots, b_k \in \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that

$$(B_0 x + b_0) \wedge (B_1 x + b_1) \wedge \ldots \wedge (B_k x + b_k) \;=\; 0$$

  where $\wedge$ is the entrywise minimum: if $x,\ y \in \mathbb{R}^n$, then $(x \wedge y)_i = \min(x_i, y_i)$ for $i = 1, \ldots, n$.
  This problem is the Extended Generalized Order LCP. If we take $B_0 = I_n$ and $b_0 = 0_{n \times 1}$ we get the (regular) Generalized Order LCP.
- the mixed LCP [10]:
  Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, find $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{aligned}
&v \geqslant 0 \\
&a + Au + Cv = 0 \\
&b + Du + Bv \geqslant 0 \\
&v^{\mathrm{T}}(b + Du + Bv) = 0.
\end{aligned}$$

It is quite easy[3] to show [17] that all these generalizations are special cases of the ELCP. Furthermore, in [20] we have shown that the following extension of the LCP is also a special case of the ELCP:

- the Linear Dynamic Complementarity Problem [59], which is defined as follows:
  Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$ and $D \in \mathbb{R}^{k \times k}$, find for a given $x_0 \in \mathbb{R}^n$ sequences $\{y_l\}_{l=0}^{n-1}$, $\{u_l\}_{l=0}^{n-1}$ with $y_l, u_l \in \mathbb{R}^k$ for all l such that

$$\begin{aligned}
y_0 &= Cx_0 + Du_0 \\
y_1 &= CAx_0 + CBu_0 + Du_1 \\
&\;\vdots \\
y_{n-1} &= CA^{n-1}x_0 + CA^{n-2}Bu_0 + \cdots + CBu_{n-2} + Du_{n-1},
\end{aligned}$$

[3] Regarding the Extended LCP of Mangasarian and Pang, note that we may assume without loss of generality that $\mathcal{P}$ can be represented as $\mathcal{P} = \{u \in \mathbb{R}^m \mid Su \geqslant t\}$ for some matrix $S \in \mathbb{R}^{l \times m}$ and vector $t \in \mathbb{R}^l$.

and such that for each index $i \in \{1, 2, \dots, k\}$ at least one of the following statements is true:

$$\left[(y_0)_i \ \dots \ (y_{n-1})_i\right]^{\mathrm{T}} = 0 \quad \text{and} \quad \left[(u_0)_i \ \dots \ (u_{n-1})_i\right]^{\mathrm{T}} \succeq 0$$
$$\left[(y_0)_i \ \dots \ (y_{n-1})_i\right]^{\mathrm{T}} \succeq 0 \quad \text{and} \quad \left[(u_0)_i \ \dots \ (u_{n-1})_i\right]^{\mathrm{T}} = 0,$$

where $z \succeq 0$ for a vector $z \in \mathbb{R}^n$ indicates that z is lexicographically nonnegative, i.e., either $z_i = 0$ for all i or the first nonzero component of z is positive.

Hence, we have

Theorem 5. *The ELCP can be considered as a unifying framework for the LCP and its various generalizations.*

The underlying geometrical explanation for the fact that all the generalizations of the LCP mentioned above are particular cases of the ELCP is that they all have a solution set that consists of the union of faces of a polyhedron, and that the union of any arbitrary set of faces of an arbitrary polyhedron can be described by an ELCP (see Theorem 2). More generally, if we define a "linear" generalization of the LCP as a problem consisting of an explicit or implicit system of linear (in)equalities in combination with a "general" complementarity condition, i.e., an ELCP-like complementarity condition that constrains the solutions of the problem to lie on the (relative) boundary of the feasible set, then the solution set of this "linear" generalization will consist of the union of faces of a polyhedron, which implies that such a "linear" generalization of the LCP is a special case of the ELCP.

For more information on the generalizations discussed above and for applications and methods to solve these problems, the interested reader may consult the references cited above and [1, 28, 37, 39, 47, 50, 52, 54, 64, 68] and the references therein.

3 Algorithms for the ELCP

In this section, we present some algorithms to compute all or just one solution of an ELCP. For algorithms to solve a (regular) LCP, we refer to [4, 6, 10, 40–43, 51, 54, 56, 58, 60, 65, 67] and the references therein.

3.1 An Algorithm to Compute All Solutions

In order to compute the entire solution set of the ELCP (4)–(6), we first homogenize the ELCP by introducing a scalar $\alpha \geqslant 0$ and defining

$$u = \begin{bmatrix} x \\ \alpha \end{bmatrix}, \quad P = \begin{bmatrix} A & -c \\ 0_{1\times n} & 1 \end{bmatrix} \text{ and } Q = [\, B \ \ -d \,].$$

Then we get a homogeneous ELCP of the following form:

Given $P \in \mathbb{R}^{p \times n}$, $Q \in \mathbb{R}^{q \times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find $u \in \mathbb{R}^n$ (with $u \neq 0$) such that

$$Pu \geqslant 0 \tag{18}$$

$$Qu = 0 \tag{19}$$

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Pu)_i = 0. \tag{20}$$

So now we have a system of homogeneous linear equalities and inequalities subject to a complementarity condition. Recall that the complementarity condition (20) can also be written as

$$\prod_{i \in \phi_j} (Pu)_i = 0 \qquad \text{for } j = 1, \ldots, m\,. \tag{21}$$

The solution set of the system of homogeneous linear inequalities and equalities (18)–(19) is a polyhedral cone $\mathcal{P}$ and can be described using two sets of generators: a set of central generators $\mathcal{C}$ and a set of extreme generators $\mathcal{E}$. The set $\mathcal{C}$ can be considered as a basis for the linear subspace of $\mathcal{P}$. The generators in $\mathcal{E}$ generate the extreme rays of $\mathcal{P}$. Now u is a solution of (18)–(19) if and only if it can be written as

$$u = \sum_{c_k \in \mathcal{C}} \alpha_k \, c_k \; + \sum_{e_k \in \mathcal{E}} \beta_k \, e_k , \tag{22}$$

with $\alpha_k \in \mathbb{R}$ and $\beta_k \geqslant 0$.

To calculate the sets $\mathcal{C}$ and $\mathcal{E}$, we use an iterative algorithm that is an adaptation of the double description method of Motzkin [53]. During the iteration, we already remove generators that do not satisfy the (partial) complementarity condition because such rays cannot yield solutions of the ELCP. In the kth step of the algorithm, the partial complementarity condition is defined as follows:

$$\prod_{i \in \phi_j} (Pu)_i = 0 \qquad \text{for all } j \text{ such that } \phi_j \subset \{1, 2, \ldots, k\}. \tag{23}$$

So we only consider those groups of inequalities that have already been processed entirely. For $k \geqslant p$, the partial complementarity condition (23) coincides with the full complementarity condition (21) or (20). This leads to the following algorithm:

Algorithm 1 : Calculation of the Central and Extreme Generators

Initialization:

- $\mathcal{C}_0 := \{c_i \mid c_i \text{ is the } i\text{th column of } I_n \text{ for } i = 1, \ldots, n\}$
- $\mathcal{E}_0 := \emptyset$

Iteration:

for $k := 1, 2, \ldots, p + q\,,$

- Calculate the intersection of the current polyhedral cone (described by $\mathcal{C}_{k-1}$ and $\mathcal{E}_{k-1}$) with the half-space or hyperplane determined by the kth inequality or equality of (18)–(19). This yields a new polyhedral cone described by $\mathcal{C}_k$ and $\mathcal{E}_k$.
- Remove the generators that do not satisfy the partial complementarity condition.

Result: $\mathcal{C} := \mathcal{C}_{p+q}$ and $\mathcal{E} := \mathcal{E}_{p+q}$

Not every combination of the form (22) satisfies the complementarity condition. Although every linear combination of the central generators satisfies the complementarity condition, not every positive combination of the extreme generators satisfies the complementarity condition. Therefore, we introduce the concept of cross-complementarity:

Definition 1. (Cross-complementarity) *Let $\mathcal{E}$ be the set of extreme generators of an homogeneous ELCP. A subset $\mathcal{E}_s$ of $\mathcal{E}$ is cross-complementary if every combination of the form*

$$u = \sum_{e_k \in \mathcal{E}_s} \beta_k \, e_k ,$$

with $\beta_k \geqslant 0$, satisfies the complementarity condition.

In [17], we have proved that in order to check whether a set $\mathcal{E}_s$ is cross-complementary, it suffices to test only one strictly positive combination of the generators in $\mathcal{E}_s$, e.g., the combination with $\beta_k = 1$ for all k. Now we can determine Γ, the set of maximal cross-complementary sets of extreme generators: $\Gamma = \{ \mathcal{E}_s \mid \mathcal{E}_s \text{ is maximal and cross-complementary} \}$.

Algorithm 2: Determination of the Cross-Complementary Sets of Extreme Generators

Initialization:
- $\Gamma := \emptyset$
- Construct the cross-complementarity graph $\mathcal{G}$ with a node e_i for each generator $e_i \in \mathcal{E}$ and an edge between nodes e_k and e_l if the set $\{e_k, e_l\}$ is cross-complementary.
- $\mathcal{S} := \{e_1\}$

Depth-first search in $\mathcal{G}$:
- Select a new node e^{new} that is connected by an edge to all nodes of the set $\mathcal{S}$ and add the corresponding generator to the test set: $\mathcal{S}^{\text{new}} := \mathcal{S} \cup \{e^{\text{new}}\}$.
- **if** $\mathcal{S}^{\text{new}}$ is cross-complementary
  then Select a new node and add it to the test set.
  else Add $\mathcal{S}$ to Γ: $\Gamma := \Gamma \cup \{\mathcal{S}\}$, and go back to the last point where a choice was made.
  Continue until all possible choices have been considered.

Result: Γ

Now u is a solution of the homogeneous ELCP if and only if there exists a set $\mathcal{E}_s \in \Gamma$ such that u can be written as

$$u = \sum_{c_k \in \mathcal{C}} \alpha_k c_k + \sum_{e_k \in \mathcal{E}_s} \beta_k e_k, \tag{24}$$

with $\alpha_k \in \mathbb{R}$ and $\beta_k \geqslant 0$.

Finally, we have to extract the solution set $\mathcal{S}$ of the original ELCP (cf. equation (7)), i.e., we have to retain solutions of the form (24) that have an α component equal to 1 ($u_\alpha = 1$). So we transform the sets $\mathcal{C}$, $\mathcal{E}$, and Γ as follows:

- If $c \in \mathcal{C}$, then $c_\alpha = 0$. We drop the α component and put the result in $\mathcal{X}^{\text{cen}}$ (i.e., the basis the linear subspace associated with the maximal affine subspace of $\mathcal{S}$).
- If $e \in \mathcal{E}$, then there are two possibilities:
  - If $e_\alpha = 0$, then we drop the α component and put the result in $\mathcal{X}^{\text{ext}}$ (i.e., the set of generators for the extreme rays of $\mathcal{S}$).
  - If $e_\alpha > 0$, then we normalize e such that $e_\alpha = 1$. Next, we drop the α component and put the result in $\mathcal{X}^{\text{fin}}$ (i.e., the set of finite vertices of $\mathcal{S}$).
- For each set $\mathcal{E}_s \in \Gamma$, we construct the set of corresponding extreme generators $\mathcal{X}_s^{\text{ext}}$ and the set of corresponding finite vertices $\mathcal{X}_s^{\text{fin}}$. If $\mathcal{X}_s^{\text{fin}} \neq \emptyset$, then we add the pair $(\mathcal{X}_s^{\text{ext}}, \mathcal{X}_s^{\text{fin}})$ to Λ, the set of pairs of maximal cross-complementary sets of finite vertices and extreme generators (where each pair corresponds with a face of $\mathcal{S}$).

For a more detailed and precise description of these algorithms and a worked example, the interested reader is referred to [17]. Also note that the running time and memory requirements of the algorithms presented above increase exponentially with the size of the ELCP (see [14] for more details). This implies that the above ELCP algorithm, which determines the entire solution set of the ELCP, is not well suited for large ELCPs with a large number of variables and (in)equalities, or a complex solution set. Therefore, we will now present some method to compute only one solution of an ELCP.

3.2 Algorithms to Compute One Solution

Some of the methods that could be used to compute one solution of an ELCP are

- via global minimization [49]:
  We could minimize the left-hand side of the complementarity condition (6) subject to the linear equality and inequality constraints (4)–(5). This results in a nonlinear nonconvex optimization problem with linear constraints, that could, e.g., be solved using multistart local optimization (SQP), simulated annealing, tabu search, etc. [56].

- as a system of multivariate polynomial equations:
  if we introduce a dummy variable s_i, then the ith inequality of the system $Ax \geqslant c$ can be transformed into an equality: $A_{i,.}x - s_i^2 = c_i$. Note that $s_i = 0$ if and only if $A_{i,.}x = c_i$. If we repeat this reasoning for each inequality, then we find that the complementarity condition (6) results in $\sum_{j=1}^{m} \prod_{i \in \phi_j} s_i = 0$. The resulting system of multivariate polynomial equations could then be solved using, e.g., a homotopy method [44].
- using a combinatorial approach:
  We could select one index i_j out of each set ϕ_j for $j = 1, \ldots, m$. Each index i_j then corresponds with an inequality of $Ax \geqslant c$ that should hold with equality. So in that case, we just get a system of linear equalities and inequalities. If this system has a solution, we have obtained a solution of the ELCP; if not, we have to select another combination of indices, and repeat the process.
- using a mixed-integer linear programming approach:
  This approach is based on Theorem 4 and applies if the surplus variables of the inequalities of the ELCP are bounded over the feasible set. Note that a sufficient condition for this is that the feasible set of the ELCP is bounded. For engineering problems, such bounds are often available, e.g., as a consequence of physical or other constraints, operating ranges, etc. If we add a dummy linear objective function to the MILFP (10)–(12), we obtain a mixed-integer linear programming problem. This problem can then be solved using, e.g., a branch-and-bound method [32, 63] or a branch-and-cut method [8]. Moreover, there exist good commercial and free solvers for mixed-integer linear programming problems (such as, e.g., CPLEX, Xpress-MP, GLPK, lp_solve, etc.; see [2, 45] for an overview).

Note that all these approaches are essentially of combinatorial nature. However, based on our own experiences, the bests results are usually obtained using the mixed-integer linear programming approach.

4 Link with Max-Plus Equations

In this section, we consider max-plus equations as they arise in various applications in the max-plus algebra and in the analysis and control of max-plus-linear systems. But first we give a short introduction to the basic concepts of the max-plus algebra.

4.1 Max-Plus Algebra

The basic operations of the max-plus algebra [3, 11] are maximization and addition, which are represented by $\oplus$ and $\otimes$, respectively,

$$x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y$$

for $x, y \in \mathbb{R}_\varepsilon \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$. The structure $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra. The operations $\oplus$ and $\otimes$ are called the max-plus-algebraic addition and max-plus-algebraic multiplication, respectively, as many properties and concepts from linear algebra can be translated to the max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. Note that 0 is the identity element for $\otimes$ and that $-\infty$ is absorbing for $\otimes$.

The matrix E_n is the $n \times n$ max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = -\infty$ for all i, j with $i \neq j$. The basic max-plus-algebraic operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$, $C \in \mathbb{R}_\varepsilon^{n \times p}$ then

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_k (a_{ik} + c_{kj})$$

for all i, j. Note the analogy with the definitions of matrix sum and product in conventional linear algebra.

The max-plus-algebraic matrix power of $A \in \mathbb{R}_\varepsilon^{n \times n}$ is defined as follows: $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ for $k = 1, 2, \ldots$ For scalar numbers $x, r \in \mathbb{R}$ we have $x^{\otimes^r} = r \cdot x$.

4.2 Systems of Max-Plus-Polynomial Equations

In the next section, we shall see that many max-plus-algebraic problems can be written in the following form:

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, \ldots, p_1 \tag{25}$$

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \qquad \text{for } k = p_1 + 1, \ldots, p_1 + p_2\,, \tag{26}$$

i.e., the max-plus-algebraic equivalent of a system of polynomial equations. Therefore, we call (25)–(26) a system of multivariate polynomial equalities and inequalities in the max-plus algebra, or a system of max-plus-polynomial equations for short. Note that the exponents can be negative or real. Using the notations introduced in Section 4.1, it is easy to verify that in conventional algebra, this problem can be rewritten as follows:

> Given a set of integers $\{m_k\}$ and three sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p_1, p_1 + 1, \ldots, p_1 + p_2\}$, find $x \in \mathbb{R}^n$ such that

$$\max_{i=1,\ldots,m_k} \Bigl(a_{ki} + \sum_{j=1}^{n} c_{kij} x_j \Bigr) = b_k \qquad \text{for } k = 1, \ldots, p_1 \tag{27}$$

$$\max_{i=1,\ldots,m_k} \Bigl(a_{ki} + \sum_{j=1}^{n} c_{kij} x_j \Bigr) \leqslant b_k \qquad \text{for } k = p_1 + 1, \ldots, p_1 + p_2 \,. \tag{28}$$

Let us now we show that (27)–(28) can be recast as an ELCP.

4.3 Translation into an ELCP

Clearly, the kth equation of (27) is equivalent to the system of linear inequalities

$$a_{ki} + c_{ki1} x_1 + c_{ki2} x_2 + \cdots + c_{kin} x_n \leqslant b_k \qquad \text{for } i = 1, \ldots, m_k \,,$$

where at least one inequality should hold with equality. So equation (27) will lead to p_1 groups of linear inequalities, where in each group at least one inequality should hold with equality.

Using the same reasoning, equations of the form (28) can also be transformed into a system of linear inequalities, but without an extra condition.

If we define $p_1 + p_2$ matrices C_k and $p_1 + p_2$ column vectors d_k such that $(C_k)_{ij} = c_{kij}$ and $(d_k)_i = b_k - a_{ki}$, then our original problem is equivalent to $p_1 + p_2$ groups of linear inequalities $C_k x \leqslant d_k$, where there has to be at least one inequality that holds with equality in each group $C_k x \leqslant d_k$ for $k = 1, \ldots, p_1$.

Now we define

$$\tilde{A} = \begin{bmatrix} -C_1 \\ -C_2 \\ \vdots \\ -C_{p_1+p_2} \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} -d_1 \\ -d_2 \\ \vdots \\ -d_{p_1+p_2} \end{bmatrix},$$

and p_1 sets ϕ_j such that $\phi_j = \{s_j + 1, \ldots, s_j + m_j\}$ for $j = 1, \ldots, p_1$, where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, \ldots, p_1 - 1$. Our original problem (27)–(28) is then equivalent to the following ELCP:

Find $x \in \mathbb{R}^n$ such that

$$\tilde{A} x \geqslant \tilde{c}$$

$$\sum_{j=1}^{p_1} \prod_{i \in \phi_j} (\tilde{A} x - \tilde{c})_i = 0.$$

Conversely, we can also show that any ELCP can be written as a system of max-plus equations of the form (27)–(28), which yields the following theorem [19]:

Theorem 6. *A system of multivariate polynomial equalities and inequalities in the max-plus algebra is equivalent to an ELCP.*

Proof. We have already shown that (27)–(28) can be recast as an ELCP. To show that the ELCP (4)–(6) can also be recast as a system of the form (27)–(28), we consider the equivalent ELCP of the form (8)–(9), and we rewrite the ELCP inequalities into the form $c - Ax \leqslant 0$, and we note that if in a group of several homogeneous inequalities of this form at least one inequality should hold with equality, then the maximum of the left-hand sides of the inequalities in this group should be equal to 0. Hence, we get one equation of the form (27) for the ELCP inequalities that belong to some subset ϕ_j and an equation of the form (28) for the other ELCP inequalities. ∎

5 Applications: Analysis and Control of Max-Plus-Linear Systems

5.1 Max-Plus-Linear Discrete Event Systems

Typical examples of discrete-event systems are flexible manufacturing systems, telecommunication networks, parallel processing systems, traffic control systems, and logistic systems. The class of discrete-event systems essentially consists of man-made systems that contain a finite number of resources (e.g., machines, communications channels, or processors) that are shared by several users (e.g., product types, information packets, or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation) [3].

In general, models that describe the behavior of a discrete-event system are nonlinear in conventional algebra. However, there is a class of discrete-event systems – the max-plus-linear discrete-event systems – that can be described by a model that is "linear" in the max-plus algebra [3]. The max-plus-linear discrete-event systems can be characterized as the class of discrete-event systems in which only synchronization and no concurrency or choice occurs. More specifically, these systems can be described by a model of the form

$$\begin{aligned} x_i(k) = \max\Big(&\max_{j=1,\dots,n}(a_{ij} + x_j(k-1)), \\ &\max_{j=1,\dots,m}(b_{ij} + u_j(k))\Big) && \text{for } i = 1,\dots,n \qquad (29) \\ y_i(k) = &\max_{j=1,\dots,n}(c_{ij} + x_j(k)) && \text{for } i = 1,\dots,l, \qquad (30) \end{aligned}$$

where $x(k)$ represents the time instants at which the internal processes of the system start for the kth time (i.e., the state of the system), $u(k)$ represents the time instants at which the system is fed with new data or products for the kth (i.e., the input of the system), and $y(k)$ represents the time instants at which the kth batch of final data or finished products leave the system (i.e., the output of the system). The additions with a_{ij}, b_{ij}, and c_{ij} in (29)–(30) correspond with the time delays like processing times, production times, traveling times, etc. The maximizations correspond with synchronization: a new activity can only start as soon as all predecessor activities are finished.

In a manufacturing context, $x(k)$ contains the time instants at which the processing units start working for the kth time, $u(k)$ the time instants at which the kth batch of raw material is fed to the system, and $y(k)$ the time instants at which the kth batch of finished product leaves the system.

Using the notations from max-plus algebra introduced in Section 4.1, the model (29)–(30) can be written as

$$x_i(k) = \bigoplus_{j=1}^{n} a_{ij} \otimes x_j(k-1) \;\oplus\; \bigoplus_{j=1}^{m} b_{ij} \otimes u_j(k) \qquad \text{for } i = 1, \ldots, n$$

$$y_i(k) = \bigoplus_{j=1}^{n} c_{ij} \otimes x_j(k) \qquad \text{for } i = 1, \ldots, l,$$

or in a more compact matrix-vector format as

$$x(k) = A \otimes x(k-1) \;\oplus\; B \otimes u(k) \tag{31}$$

$$y(k) = C \otimes x(k). \tag{32}$$

This latter form also illustrates where the name “max-plus-linear” systems comes from: for these systems, the state and the output are a linear combination (in the max-plus sense) of the previous state and the input.

Using the model (31)–(32), we can compute the output sequence $y(1), \ldots,$ $y(N)$ of the system for a given input sequence $u(1), \ldots, u(N)$ and initial state $x(0)$ as follows:

$$\begin{aligned} y(k) = C \otimes A^{\otimes^k} \otimes x(0) \;\oplus\; C \otimes A^{\otimes^{k-1}} \otimes B \otimes u(1) \oplus \\ C \otimes A^{\otimes^{k-2}} \otimes B \otimes u(2) \;\oplus\; \cdots \;\oplus\; C \otimes B \otimes u(k) \end{aligned} \tag{33}$$

for $k = 1, \ldots, N$.

To illustrate the definition presented above, we now consider a simple (max-plus-linear) manufacturing system, determine its evolution equations, and write them in the forms (29)–(30) and (31)–(32).

Example 1. Consider the production system of Figure 1.

This manufacturing system consists of three processing units (P_1, P_2, and P_3) and works in batches (one batch for each finished product). Raw material is fed to P_1 and P_2, processed and sent to P_3 where assembly takes place. The processing times for P_1, P_2, and P_3 are respectively $d_1 = 11$, $d_2 = 12$, and $d_3 = 7$ time units. It takes $t_1 = 2$ time units for the raw material to get from the input source to P_1, and $t_3 = 1$ time unit for a finished product of P_1 to

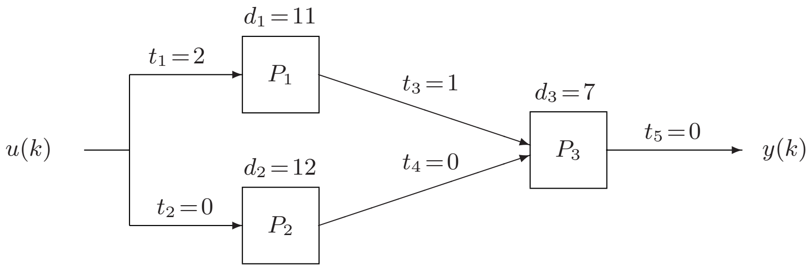

Figure 1. A simple manufacturing system

get to P_3. The other transportation times and the set-up times are assumed to be negligible. A processing unit can only start working on a new product if it has finished processing the previous product. Each processing unit starts working as soon as all parts are available.

Let us now determine the time instant at which processing unit P_1 starts working for the kth time. If we feed raw material to the system for the kth time, then this raw material is available at the input of processing unit P_1 at time $t = u(k) + 2$. However, P_1 can only start working on the new batch of raw material as soon as it has finished processing the previous, i.e., the $(k-1)$th batch. Because the processing time on P_1 is $d_1 = 11$ time units, the $(k-1)$th intermediate product will leave P_1 at time $t = x_1(k-1)+11$. Because P_1 starts working on a batch of raw material as soon as the raw material is available and the current batch has left the processing unit, this implies that we have

$$x_1(k) = \max(x_1(k-1) + 11,\, u(k) + 2\,). \tag{34}$$

Using a similar reasoning, we find the following expressions for the time instants at which P_2 and P_3 start working for the kth time and for the time instant at which the kth finished product leaves the system:

$$x_2(k) = \max(x_2(k-1) + 12,\, u(k) + 0\,) \tag{35}$$

$$x_3(k) = \max(x_1(k) + 11 + 1, x_2(k) + 12 + 0,\, x_3(k-1) + 7\,) \tag{36}$$

$$= \max(x_1(k-1) + 23,\, x_2(k-1) + 24,\, x_3(k-1) + 7,\, u(k) + 14\,) \tag{37}$$

$$y(k) = x_3(k) + 7 + 0. \tag{38}$$

Let us now rewrite the evolution equations of the production system using the symbols $\oplus$ and $\otimes$. It is easy to verify that (34) can be rewritten as

$$x_1(k) = 11 \otimes x_1(k-1) \;\oplus\; 2 \otimes u(k).$$

Equations (35)–(38) result in

$$
\begin{aligned}
x_2(k) &= 12 \otimes x_2(k-1) \ \oplus \ u(k) \\
x_3(k) &= 23 \otimes x_1(k-1) \ \oplus \ 24 \otimes x_2(k-1) \ \oplus \ 7 \otimes x_3(k-1) \ \oplus \ 14 \otimes u(k) \\
y(k) &= 7 \otimes x_3(k).
\end{aligned}
$$

If we rewrite these evolution equations in max-algebraic matrix notation, we obtain the description

$$
\begin{aligned}
x(k) &= \begin{bmatrix} 11 & -\infty & -\infty \\ -\infty & 12 & -\infty \\ 23 & 24 & 7 \end{bmatrix} \otimes x(k-1) \ \oplus \ \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix} \otimes u(k) \\
y(k) &= \begin{bmatrix} -\infty & -\infty & 7 \end{bmatrix} \otimes x(k).
\end{aligned}
$$

5.2 Max-Plus-Algebraic Problems and Analysis of Max-Plus Systems

It is easy to verify that the following max-plus-algebraic problems can be recast as a system of max-plus-polynomial equations and inequalities and thus also as an ELCP [14, 19]:

- solving two-sided max-plus-linear equations:
  Given $A, B \in \mathbb{R}_\varepsilon^{m \times n}$, and $c, d \in \mathbb{R}_\varepsilon^m$, find $x \in \mathbb{R}_\varepsilon^n$ such that
  $$A \otimes x \ \oplus \ c = B \otimes x \ \oplus \ d.$$
- max-plus-algebraic matrix decomposition:
  Given a matrix $A \in \mathbb{R}_\varepsilon^{m \times n}$ and an integer $p > 0$, find $B \in \mathbb{R}_\varepsilon^{m \times p}$ and $C \in \mathbb{R}_\varepsilon^{p \times n}$ such that
  $$A = B \otimes C.$$
- determining state space realizations of max-plus-linear systems:
  Given a partial impulse response $\{G_k\}_{k=1}^N$ of a max-plus-linear system with unknown system matrices A, B, and C, and a system order n, determine the system matrices of the system.
  For a single-input system, the impulse response is the output of the system for the input sequence given by $u(1) = 0$ and $u(k) = -\infty$ for all $k > 0$ (i.e., an impulse signal), and for the initial state $x(0) = \begin{bmatrix} -\infty & -\infty & \cdots & -\infty \end{bmatrix}^{\mathrm{T}}$. In general, for a multiinput system, the sequence of the ith columns of the G_k's corresponds with the output sequence obtained when an impulse signal is applied to the ith input and the other inputs are kept at $-\infty$. Using (33), it is then easy to verify that the impulse response satisfies
  $$G_k = C \otimes A^{\otimes^{k-1}} \otimes B \quad \text{for all } k.$$
  If $\{G_k\}_{k=1}^N$ is known, this results in a system of max-plus-polynomial equations in A, B, and C.

- transformation of state space models:

  Given system matrices A, B, C, find L, $\hat{A}$, and $\hat{C}$ such that

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \otimes L.$$

  If we can find such a decomposition, and if we define

$$\tilde{A} = L \otimes \hat{A}, \quad \tilde{B} = L \otimes B, \quad \tilde{C} = \hat{C},$$

  then it is easy to verify that the state space models corresponding with the triplets (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ of systems matrices have the same impulse response, i.e.,

$$C \otimes A^{\otimes^k} \otimes B = \tilde{C} \otimes \tilde{A}^{\otimes^k} \otimes \tilde{B} \quad \text{for all } k.$$

  In that case, we say that (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are equivalent realizations of the same max-plus-linear system.

  An alternative transformation is the following:

  Given system matrices A, B, C, find M, $\hat{A}$, and $\hat{B}$ such that

$$\begin{bmatrix} A \; B \end{bmatrix} = M \otimes \begin{bmatrix} \hat{A} \; \hat{B} \end{bmatrix}.$$

  In this case, we should consider

$$\tilde{A} = \hat{A} \otimes M, \quad \tilde{B} = \hat{B}, \quad \tilde{C} = C \otimes M.$$

Other applications related to the max-plus algebra that result in an ELCP include computing singular value decompositions, QR decompositions, and other matrix factorizations in the extended max-plus-algebra, and systems of max-min-plus equations [22].

5.3 Model-Based Predictive Control of Max-Plus-Linear Systems

Framework

As a final application, we consider model predictive control (MPC) of max-plus-linear systems. MPC [46] was pioneered simultaneously by Richalet *et al.* [57] and Cutler and Ramaker [12]. Since then, MPC has probably become the most applied advanced control technique in the process industry. A key advantage of MPC is that it can accommodate constraints on the inputs and outputs. Usually MPC uses linear or nonlinear discrete-time models. However, we now consider the extension of MPC to max-plus-linear discrete-event systems [25].

In MPC, we determine at each event step k the optimal input sequence $u(k)$, $u(k+1), \ldots, u(k+N_{\mathrm{p}}-1)$ over a given prediction horizon N_{p}. We assume that at event step k, the previous value $x(k-1)$ of the state can be measured or estimated using previous measurements. We can then use (33) to estimate the evolution of the output of the system for the input sequence $u(k), \ldots, u(k+N_\mathrm{p}-1)$:

$$\hat{y}(k+j|k) = C \otimes A^{\otimes^j} \otimes x(k-1) \oplus \bigoplus_{i=0}^{j} C \otimes A^{\otimes^{j-i}} \otimes B \otimes u(k+i), \qquad (39)$$

where $\hat{y}(k+j|k)$ is the estimate of the output at event step $k+j$ based on the information available at event step k. If the due dates r for the finished products are known and if we have to pay a penalty for every delay, a well-suited output cost criterion is the tardiness:

$$J_\mathrm{out}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{l} \max(\hat{y}_i(k+j|k) - r_i(k+j), 0). \qquad (40)$$

On the other hand, we also want to keep the throughput time and the internal buffer levels as low as possible. Therefore, we will *maximize* the input time instants. For a manufacturing system, this would correspond with a scheme in which raw material is fed to the system as late as possible. This results in the following input cost criterion

$$J_\mathrm{in}(k) = \sum_{j=0}^{N_\mathrm{p}-1} \sum_{i=1}^{m} u(k+j). \qquad (41)$$

The input and output cost criteria are combined as follows in the overall performance function J:

$$J(k) = J_\mathrm{out}(k) + \lambda J_\mathrm{in}(k),$$

with $\lambda > 0$.

Because for discrete-event systems the inputs $u(k)$ correspond with consecutive feeding times, this sequence should be nondecreasing, resulting in the constraint

$$u(k+j) \geqslant u(k+j-1) \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1\,.$$

Furthermore, we sometimes also have constraints such as minimum or maximum separation between input and output events:

$$\begin{aligned} a_1(k+j) &\leqslant u(k+j) - u(k+j-1) \leqslant b_1(k+j) && \text{for } j = 0, \ldots, N_\mathrm{p}-1 \\ a_2(k+j) &\leqslant \hat{y}(k+j|k) - \hat{y}(k+j-1|k) \leqslant b_2(k+j) && \text{for } j = 0, \ldots, N_\mathrm{p}-1\,, \end{aligned}$$

maximum due dates for the output events:

$$\hat{y}(k+j|k) \leqslant r(k+j) \qquad \text{for } j = 0, \ldots, N_\mathrm{p}-1\,,$$

or maximum deviations from the due dates:

$$\begin{aligned} r(k+j) - \delta^{-}(k+j) &\leqslant \hat{y}(k+j|k) \\ &\leqslant r(k+j) + \delta^{+}(k+j) \quad \text{for } j = 0, \ldots, N_{\mathrm{p}} - 1\,, \end{aligned}$$

If we define

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N_{\mathrm{p}}-1) \end{bmatrix}, \quad \tilde{y}(k) = \begin{bmatrix} \hat{y}(k|k) \\ \hat{y}(k+1|k) \\ \vdots \\ \hat{y}(k+N_{\mathrm{p}}-1|k) \end{bmatrix},$$

we can collect all the above constraints into one system of linear equations of the form

$$A_{\mathrm{c}}(k)\tilde{u}(k) + B_{\mathrm{c}}(k)\tilde{y}(k) \leqslant c_{\mathrm{c}}(k). \tag{42}$$

The MPL-MPC Problem and its Link with the ELCP

If we combine the material of the previous subsection, we finally obtain the following problem:

> At event step k, find the input sequence vector $\tilde{u}(k)$ that minimizes $J(k) = J_{\mathrm{out}}(k) + \lambda J_{\mathrm{in}}(k)$ subject to the evolution equations (39) and the constraints (42).

This problem will be called the max-plus-linear MPC (MPL-MPC) problem for event step k. MPL-MPC also uses a receding horizon principle, which means that at event step k, the future control sequence $u(k), \ldots, u(k+N_{\mathrm{p}}-1)$ is determined such that the cost criterion is minimized subject to the constraints. At event step k, the first element of the optimal sequence (i.e., $u(k)$) is then applied to the system. At the next event step, the horizon is shifted, the model is updated with new information of the measurements, and a new optimization at event step $k+1$ is performed, and so on.

Let us now have a closer look at the MPL-MPC problem. We could consider both $\tilde{u}(k)$ and $\tilde{y}(k)$ as optimization variables. Clearly, as the constraints of the MPL-MPC problem are a combination of max-plus-polynomial constraints and linear constraints, they can be recast as an ELCP. This implies that the optimal sequence $\tilde{u}(k)$ can be determined by optimizing $J(k)$ over the solution set of this ELCP.

Algorithms for the MPL-MPC Problem

Now we discuss some methods to solve the MPL-MPC problem. The material in this section is inspired by [25], but due to the fact that we focus on one particular performance function (i.e., (40)–(41)), we can make some significant simplifications in our explanation and in our approach with respect to [25].

As indicated above, we can solve the MPL-MPC problem by first determining the entire solution set of the ELCP that corresponds with the constraints of the MPL-MPC problem in a parameterized way using the algorithms of Section 3.1, and then optimizing $J(k)$ over this solution set. However, as the MPL-MPC problem has to be solved at each event step, this approach is not feasible in practice.

Alternatively, we could consider the MPL-MPC problem as a nonlinear nonconvex optimization problem and use standard multistart nonlinear nonconvex local optimization methods to compute the optimal control policy. However, in practice this approach is also often not feasible.

We could also apply the mixed-integer programming approach as follows: note that because A, B, and C are known, the evolution equations (39) can be rewritten as

$$\tilde{y}_i(k) = \max_{j=1,\ldots,mN_{\mathrm{p}}} (h_{ij} + \tilde{u}_j(k), g_j(k)) \qquad \text{for } i = 1, \ldots, lN_{\mathrm{p}}, \tag{43}$$

for some matrix H and a vector $g(k)$ that depends on $x(k-1)$ (see [25] for the exact expressions). We can now eliminate $\tilde{y}(k)$ from the objective function $J(k)$, resulting in an expression of the form

$$\begin{aligned} J(k) &= \sum_{i=1}^{lN_{\mathrm{p}}} \big(\max_{j=1,\ldots,mN_{\mathrm{p}}} (h_{ij} + \tilde{u}_j(k), g_j(k)) - \tilde{r}_i(k) \big) + \lambda \sum_{j=1}^{mN_{\mathrm{p}}} \tilde{u}_j(k) \\ &= \max_{i=1,\ldots,K} \max_{j=1,\ldots,mN_{\mathrm{p}}} (p_{ij}\tilde{u}_j(k) + q_j(k)) \\ &= \max_{i=1,\ldots,K} (P\tilde{u}(k) + q(k))_i \end{aligned}$$

for an appropriately defined matrix P, vector q, and constant K where $\tilde{r}(k)$ is defined in a similar way as $\tilde{y}(k)$ and where we have made recursive use of the following basic property: for $\alpha, \beta, \gamma \in \mathbb{R}$ we have $\max(\alpha, \beta) + \gamma = \max(\alpha + \gamma, \beta + \gamma)$. If we now introduce a scalar dummy variable t such that

$$t = \max_{i=1,\ldots,K} (P\tilde{u}(k) + q(k))_i, \tag{44}$$

then the MPL-MPC problem reduces to minimizing a linear objective function ($J(k) = t$) subject to the constraints (42), (43), and (44). Note that these constraints are a combination of max-plus and linear constraints, i.e., they correspond with an ELCP. As shown in the proof of Theorem 4, these constraints thus can be rewritten as a system of mixed-integer linear equations (in fact, the detour via the ELCP is not necessary, and the equations can directly be transformed into mixed-integer linear constraints). Hence, the MPL-MPC problem can be recast as a mixed-integer linear programming problem.

If in addition the matrix $B_{\mathrm{c}}(k)$ in (42) only has nonnegative entries, we can make a further simplification, which will ultimately result in a linear programming problem. In fact, if all entries of $B_c(k)$ are nonnegative (this occurs, e.g., when there are no constraints on $\tilde{y}(k)$, or if there are only upper bound constraints on $\tilde{y}(k)$), then we can also easily eliminate $\tilde{y}(k)$ from the linear constraints (42), resulting in

$$(A_c(k)\tilde{u}(k))_\ell + \sum_{i=1}^{lN_p} (B_c)_{\ell i} \max_{j=1,\ldots,mN_p} (h_{ij} + \tilde{u}_j(k), g_j(k)) \leqslant (c_c(k))_\ell \quad \text{for all } \ell,$$

or equivalently an expression of the form

$$\max_{i=1,\ldots,L} (S_{(\ell)}(k)\tilde{u}(k) + s_{(\ell)}(k))_\ell \leqslant (c_c(k))_\ell \qquad \text{for all } \ell,$$

for an appropriately defined matrix $S_{(\ell)}(k)$ and vector $s_{(\ell)}(k)$, or even more simply

$$S_\ell(k)\tilde{u}(k) + s(k) \leqslant c_{(\ell)}(k) \qquad \text{for all } \ell, \tag{45}$$

for an appropriately defined vector $c_{(\ell)}(k)$. As now we have eliminated $\tilde{y}(k)$ completely, we have to minimize

$$J(k) = \max_{i=1,\ldots,K} (P\tilde{u}(k) + q(k))_i$$

over the linear constraint (45). If we again introduce a dummy variable t and solve the following linear optimization problem

$$\begin{aligned} &\min_{t,\tilde{u}(k)} t \\ &\text{subject to (45) and } \ t \geqslant (P\tilde{u}(k) + q(k))_i \ \text{ for } i = 1, \ldots, K\,, \end{aligned}$$

then it is easy to verify that in the optimal solution, at least one of the bounds on t is tight, i.e., (44) holds. So in this case, we can find the optimal solution of the MPL-MPC problem via linear programming, for which efficient algorithms exist such as (variants of) the simplex method or interior point methods [55, 56].

For a worked example and a comparison of several of these alternative MPL-MPC algorithms, we refer the interested reader to [25].

In [24, 26], we have extended the above results to max-min-plus-scaling systems, a class of discrete-event systems that can be modeled using the operations maximization, minimization, addition and scalar multiplication. Related work involving the determination of optimal switching times for traffic signals and for first-order linear hybrid systems with saturation is described in [15, 16].

6 Conclusion

In this chapter, we have presented the extended linear complementarity problem (ELCP) and its relation to the regular linear complementarity problem (LCP) and to various linear generalizations of the LCP. We have shown that the ELCP can in a way be considered to be the most general linear extension of the LCP. We have also discussed some properties of the solution set of an ELCP and presented some algorithms to solve an ELCP: we have considered an algorithm for determining the complete solution set of an ELCP and also several algorithms to determine only one solution. Next, we have shown that a system of max-plus-polynomial equations is equivalent to an ELCP, which allows us to solve several problems that arise in the max-plus algebra and in the analysis and control of max-plus-linear systems. In particular, for the model-based predictive control of max-plus-linear systems, the original ELCP-based problem can be reduced to a linear programming problem, which can be solved very efficiently.

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Traffic Assignment: Equilibrium Models

Michael Florian[1] and Donald W. Hearn[2]

[1] Center for Research on Transportation, University of Montreal, Montreal, Canada
mike@crt.umontreal.ca
[2] Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida, USA
hearn@ise.ufl.edu

Abstract The formulation of static assignment models, with variable and fixed demand, based on Wardrop's first principle, are presented for deterministic and stochastic models. The main algorithms used to obtain solutions for these network equilibrium models are given for each class of assignment problems. Calibration and validation issues are considered.

Key words: traffic assignment, equilibrium models, variational inequalities, stochastic traffic assignment, linear approximation method, partial linear approximation method, PARTAN, restricted simplicial decomposition, cyclic decomposition, path equilibration algorithm, relaxation algorithm

1 Introduction

Traffic equilibrium models are commonly in use for the prediction of traffic patterns on transportation networks that are subject to congestion phenomena. Even though their application in various transportation planning contexts has increased dramatically over the past 25 years, due to the development of efficient solution algorithms and the increasing power of various computing platforms, they are based on concepts that were stated more than 70 years ago. The idea of traffic equilibrium originated as early as 1924, when Knight [20] gave a simple and an intuitive description of a postulate of traffic behaviour under congested conditions, as follows:

> Suppose that between two points here are two highways, one of which is broad enough to accommodate without crowding all the traffic which may care to use it, but is poorly graded and surfaced, while the other is a much better road, but narrow and quite limited in capacity. If a large number of trucks operate between the two termini and are free to choose either of the two routes, they will tend to distribute themselves between

> the roads in such proportions that the cost per unit of transportation, or effective returns per unit of investment, will be the same for every truck on both routes. As more trucks use the narrower and better road, congestion develops, until a certain point it becomes equally profitable to use the broader but poorer highway.

Some 28 years later, Wardrop ([30]) stated two principles that formalize this notion of equilibrium and introduced the alternative behaviour postulate of the minimization of the total travel costs. His first principle states that "the journey times on all routes actually used are equal and less than those which would be experienced by a simple vehicle of any unused route." Under certain assumptions, another interpretation of this principle is that the routes actually used are the shortest in time under prevailing traffic conditions and their perception by the travellers. Wardrop's first principle of equilibrium of route choice, which is identical with the notion postulated by Knight, has become accepted over the past 40 years as a sound behavioural principle to describe the spreading of trips over alternative routes. The traffic flows that satisfy this principle are usually referred to as "user optimal" flows, as each user chooses the route that is perceived to be the best. On the other hand, the "system optimal" flows are characterized by Wardrop's second principle, which states the "the average journey time is minimum."

The first mathematical model of network equilibrium was formulated by [3]. This seminal contribution was the starting point for other research and then application of such route choice models. The purpose of this paper is to present the elements of the network equilibrium assignment models used in transportation planning, to review their mathematical properties and most commonly used solution methods, and to outline past and current applications. For more complete presentations of the topic of this paper, see references [13] and [25].

2 Model Formulation: Deterministic Models

The network models that are most commonly used are steady-state models, in spite of the fact that all traffic phenomena are temporal. A given period of time for which the demand for travel is quantified is considered and then the flow pattern that results from the action of the demand and the performance of the transport infrastructure available needs to be determined.

A deterministic network equilibrium assignment model of route choice may be formulated by using the following notation. The transportation network consists of nodes $n \in N$, which represent origins and destinations of traffic and intersections and arcs $a \in A$, which represent the road network. The number of vehicles on link a is v_a $(a \in A)$ and the cost of travelling on a link is given by a user cost function $s_a(v)$ $(a \in A)$, where v is the vector of link flows over the entire network. These cost functions may model the time delay for travel on that arc, in which case it is referred to as a volume-delay function; however, it may model other costs such as tolls or fuel consumption. The vector valued user cost function $s(v)$ is assumed to be monotonic (strictly monotonic), i.e.,

$$[s(v^1) - s(v^2)](v^1 - v^2) \geq (>) 0, \quad a \in A,\ v^1 \neq v^2 \tag{1}$$

continuous, and differentiable. The origin to destination demands g_i, $i \in I$, where I is the set of origin–destination (O-D) pairs, are distributed over directed paths $k \in K_i$, where K_i is the set of paths for O-D pair i and it is assumed that $k_i \neq \emptyset$. Also $K = \bigcup_{i \in I} K_i$. The flows on paths k, h_k satisfy conservation of flow and non-negativity constraints:

$$\sum_{k \in K_i} h_k = g_i, \quad i \in I,\ h_k \geq 0,\ k \in K. \tag{2}$$

The corresponding link flows v_a are given by

$$v_a = \sum_{k \in K} \delta_{ak} h_k, \quad a \in A, \tag{3}$$

where

$$\delta_{ak} = \begin{cases} 1 & \text{if link } a \text{ belong to path } k, \\ 0 & \text{otherwise.} \end{cases}$$

The link costs are additive in the sense that the cost of a path $s_k(v)$ is the sum of the user costs on the links of the path, that is,

$$s_k(v) = \sum_{a \in A} \delta_{ak} s_a(v), \quad k \in K. \tag{4}$$

If u_i $(= u_i(v))$, $i \in I$, are the costs of shortest paths for O-D pairs i, so that

$$u_i = \min_{k \in K_i} [s_k(v)], \quad i \in I, \tag{5}$$

the demands for travel g_i, $i \in I$, are given by functions $G_i(u)$, where u is the vector of least cost travel times for all the O-D pairs of the network:

$$g_i = G_i(u) \geq 0, \quad i \in I. \tag{6}$$

The vector of demand functions, $G(u)$, is assumed to be strictly monotonic decreasing, i.e.,

$$[G(u^1) - G(u^2)](u^1 - u^2) < 0, \quad i \in I, \tag{7}$$

continuous, and bounded from above.

A network equilibrium model that satisfies Wardrop's user optimal principle is formulated by stating that

$$s_k(v^*) - u_i^* \begin{cases} = 0 \text{ if } h_k^* > 0 \\ \geq 0 \text{ if } h_k^* = 0 \end{cases}, \quad k \in K_i, \ i \in I \tag{8}$$

over the feasible set (2) and (3). It is relatively straightforward to show that equation (8) may be restated in the "complementarity" form

$$u_i^* \leq s_k(v^*) \text{ and } [s_k(v^*) - u_i^*]h_k^* = 0, \quad k \in K_i, \ i \in I \tag{9}$$

and the equations (8) and (5) are equivalent to

$$s_{k_1}^* \leq s_{k_2}^*, \quad \text{if } h_{k_1}^* > 0, \quad k_1, k_2 \in K_i, \ i \in I. \tag{10}$$

Another very useful restatement of Wardrop's first principle serves to convert the model to a variational inequality as done by Smith [28] and Dafermos [7]. This is accomplished by noting that equation (8) is equivalent to

$$[s_k(v^*) - u_i^*](h_k - h_k^*) \geq 0, \quad k \in K_i. \tag{11}$$

Above h_k, $k \in K_i$, is any feasible path flow. If $h_k^* > 0$, then $s_k(v^*) = u_i^*$, as h_k may be smaller than h_k^*. If $h_k^* = 0$, then equation (11) is satisfied when $s_k(v^*) - u_i^* \geq 0$. By summing equation (11) over $k \in K_i$; $i \in I$, it is found that

$$\sum_{i \in I} \sum_{k \in K_i} s_k(v^*)(h_k - h_k^*) \geq \sum_{i \in I} u_i^*(g_i - g_i^*). \tag{12}$$

By using equations (3) and (4), a change of summation on the left-hand side yields

$$\sum_{a \in A} s_a(v^*)(v_a - v_a^*) \geq \sum_{i \in I} u_i^*(g_i - g_i^*). \tag{13}$$

Because the vector demand function $G(u)$ is strictly monotonic decreasing, it is invertible. Let $w_i(g)$ denote the inverse of the demand function. Substituting for u_i gives

$$\sum_{a \in A} s_a(v^*)(v_a - v_a^*) - \sum_{i \in I} w_i(g^*)(g_i - g_i^*) \geq 0 \tag{14}$$

over the feasible set (2) and (3), which may be rewritten in vector notation as

$$s(v^*)(v - v^*) - w(g^*)(g - g^*) \geq 0. \tag{15}$$

It can be verified that equation (14) implies equation (8) by constructing a flow pattern that differs from the equilibrium flow on only one path $k_1 \in K_i$, for which $h_{k_1} = h_{k_1}^* + \delta$, $0 \leq |\delta| \leq h_{k_1}^*$.

The existence of a solution of the network equilibrium model is ensured by the continuity of the cost and demand functions and the fact that the feasible set is compact if cycle flows do not occur and the demand functions are bounded from above [1, 7].

The following example illustrates that a solution may not exist when the link cost functions are not continuous. The network consists of one O-D pair

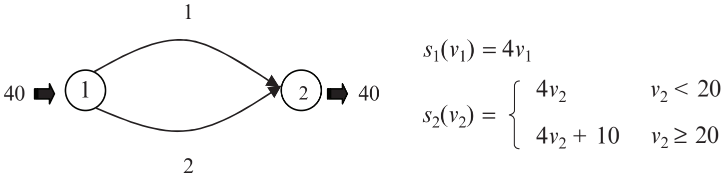

Figure 1. Network with discontinuous cost functions

and two links as shown in Figure 1. The demand from 1 to 2 is 40. When $v_1 = v_2 = 20$, the cost of link 2 is higher than the cost of link 1, but for $v_1 = 20 + \epsilon$ and $v_2 = 20 - \epsilon > 0$, the cost of link 1 is higher than the cost of link 2. In practical applications, the continuity requirement is usually satisfied.

It is important that a network model, which is used to predict traffic flows for different network and demand scenarios, yields unique link flows and origin to destination costs. If this were not so, the comparison of the different future situations would be difficult to carry out as differences between scenarios would depend on the nonuniqueness of the flows. Fortunately, for many applications, the network equilibrium models have unique flows and origin to destination demands. This is ensured when the link cost functions are strictly monotonic and the demand functions (and their inverses) are strictly monotonic decreasing, as assumed above.

To demonstrate this ([1,28]), suppose that there are two distinct solutions (v^1, g^1) and (v^2, g^2). Writing equation (15) once with $v = v^1$, $g = g^1$ and $v^* = v^2$, $g^* = g^2$ and once with $v^* = v^1$, $g^* = g^1$ and $v = v^2$, $g = g^2$ and adding the two inequalities gives

$$[s(v^2) - s(v^1)](v^1 - v^2) - [w(g^2) - w(g^1)](g^1 - g^2) \geq 0. \tag{16}$$

By imposing equations (1) and (7), it follows that equation (16) is satisfied if and only if each term is equal to zero. Hence

$$[s(v^1) - s(v^2)](v^1 - v^2) = 0, \quad a \in A, \tag{17}$$

$$[w(g^1) - w(g^2)](g^1 - g^2) = 0, \quad i \in I \tag{18}$$

with the conclusion that $g^1 = g^2$ and $s_a(v^1) = s_a(v^2)$ if the link cost functions are monotonic and $v^1 = v^2$ if the link cost functions are strictly monotonic. Because u_i^*, $i \in I$, are the lengths of shortest paths for each O-D pair i based on the link costs $s_a(v^*)$, it follows that they are unique as well.

It is worthwhile to note that the path flows h_k^*, $k \in K$, are not unique, in general. Given the link flows v_a^*, $a \in A$, the corresponding path flows are given as the solution of the simultaneous linear equations

$$v_a^* = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} h_k^*, \quad a \in A. \tag{19}$$

Because in most applications $|K| \geq |A|$, the number of variables h_k^* far exceeds the number of constraints and the decomposition of link flows into path flows is not unique. The consequence is that the analysis of the path flows h_k^*, $k \in K$, requires some care, but they contain nevertheless valuable information.

Another important property of the network equilibrium model is that it is stable. Roughly speaking, the equilibrium flows depend continuously upon the travel demands and link cost functions. Small changes in the travel demands result in small changes in the traffic flows. This was demonstrated by Dafermos and Nagurney [8] for the model of this section and by Hall [17] for the fixed-demand variant of the network equilibrium model. This property is very desirable in applications, provided that the model is a suitable representation of the observed link flows.

Most of the applications of network equilibrium assignment in practice have been achieved for simpler versions of the model (15) subject to equations (2) and (3). They were facilitated by the fact that, if the Jacobians $\nabla s(v)$ and $\nabla w(g)$ of the cost functions and inverse demand functions $w(u)$ are symmetric,

$$\frac{\partial s_a(v)}{\partial v_{\tilde{a}}} = \frac{\partial s_{\tilde{a}}(v)}{\partial v_a}, \quad \text{for all } a, \tilde{a} \in A$$

and

$$\frac{\partial w_i(g)}{\partial g_{\tilde{i}}} = \frac{\partial w_{\tilde{i}}(g)}{\partial g_i}, \quad \text{for all } i, \tilde{i} \in I,$$

then equation (17) is equivalent to a convex cost optimization problem as, by Green's lemma, the vectors $s(v)$ and $w(g)$ can be viewed as gradients of the line integrals $\int_0^u s(x)dx$ and $\int_0^g w(y)dy$, respectively. The assumptions made on $s(v)$ and $g(u)$ imply that

$$Z(v, g) = \int_0^v s(x)dx - \int_0^g w(y)dy \tag{20}$$

is a convex function in (v, g), and the minimization of $Z(v, g)$ is equivalent to solving equation (17). If, furthermore, the link cost functions are separable, i.e., $s_a(v) = s_a(v_a)$, $a \in A$, and so are the inverse demand functions, $w_i(u) = w_i(u_i)$, $i \in I$, the strict monotonicity assumptions on $s(v)$ and $w(u)$ imply that $s_a(v_a)$ are strictly increasing and $w_i(u_i)$ are strictly decreasing and their Jacobians are diagonal matrices. The equivalent convex optimization problem becomes simply

$$\min[Z(v, g)] = \sum_{a \in A} \int_0^{v_a} s_a(x)dx - \sum_{i \in I} \int_0^{g_i} w_i(y)dy \tag{21}$$

subject to equations (2) and (3).

In the case of fixed demand, the problem takes the classic form

$$\min[S(v)] = \sum_{a \in A} \int_0^{v_a} s_a(x)dx \tag{22}$$

subject to equations (2) and (3) with $g_i = \bar{g}_i$, $i \in I$, i.e., constant demand.

3 Model Formulation: Stochastic Models of Traffic Assignment

Stochastic network equilibrium models are based on the hypothesis that travellers make systematic errors in their perception of the travel costs, in contrast with the deterministic models where it is assumed that the travellers have perfect knowledge of the costs. The choice of the probability density functions, which are postulated to represent the systematic perception errors, result in different models. Stochastic network equilibrium models are preferred for applications when the network is not subject to a high level of congestion and the choice of paths is not determined solely by the travel times or costs, but also by preference variations.

In order to formulate stochastic network equilibrium models, it is necessary to introduce pr_k, the probability that an individual chosen from the population g_i will choose path $k \in K_i$. This is defined as

$$pr_k = pr_k(Z_i), \quad k \in K_i, \; i \in I, \tag{23}$$

where Z_i is the vector of *perceived* travel times of all paths k for an O-D pair i. The perceived travel times on link a are assumed to be given by a probability density function

$$z_a \sim D(s_a, \theta s_a),$$

where s_a is the actual travel cost, θs_a is its variance, and θ is a constant. Thus, the probability of choosing path k is given by

$$pr_k = Pr\left[z_k = \min_{k' \in K_i}(z_{k'})\right], \quad k \in K_i, \; i \in I, \tag{24}$$

the probability that the path is perceived to be the shortest.

If $D(s_a, \theta s_a)$ is assumed to be the normal distribution, then the vector of perceived travel times, z_i, is multivariate normally distributed. The weak law of large numbers implies that, on the average, the path flows h_k satisfy

$$pr_k = \frac{h_k}{g_i}, \quad k \in K_i, \; i \in I. \tag{25}$$

It is possible to show [11] that the stochastic network equilibrium model is equivalent to solving the optimization problem

$$\min_{v}\left(\sum_{i\in I} g_i E\left\{\min_{k\in K_i}[z_k|s^k(v)]\right\}\right)+\sum_{a\in A} v_a s_a(v_a)-\sum_{a\in A}\int_0^{v_a} s_a(x)dx \qquad (26)$$

subject only to non-negativity constraints. The objective (26) is not convex in general, but it can be shown that there is only one stationary point and, in the neighborhood of this point, the objective function is strictly convex in the flow variables. Hence the resulting link flows are unique.

An interesting special case of stochastic network equilibrium models occurs when the path probabilities are given by a logit function:

$$pr_k=\frac{\exp[-\theta s_k(v)]}{\sum_{k'\in K_i}\exp[-\theta s_{k'}(v)]},\quad k\in K_i,\ i\in I. \qquad (27)$$

In this particular case, it is easy to show that the equivalent optimization problem is

$$\min_{h}\left(\sum_{i\in I}\sum_{k\in K_i} h_k \ln h_k\right)+\theta\sum_{a\in A}\int_0^{v_a} s_a(x)dx \qquad (28)$$

subject to the usual constraints (2) and (3).

When the link travel costs are constant, i.e., $s_a(v_a)=\bar{s}_a$, then the solutions of this model are dependent on the way that the network is represented as all paths are perceived to be independent, even if they share links.

4 Solution Algorithms for Network Equilibrium Assignment Models

4.1 Deterministic Symmetric Models

As shown in Section 2, if the link cost functions and the demand functions are separable, a network equilibrium model that has a unique solution may be reformulated as an equivalent convex cost differentiable optimization problem. Because the feasible flows satisfy equations (2) and (3), the conservation of flow and non-negativity constraints and the only interactions between the link flows for different origins or different O-D pairs occur in the objective function. This makes it possible to construct a wide range of algorithms for solving the problem, each based on a particular decomposition of the flows.

It is possible to classify the algorithms for the symmetric network equilibrium problem according to the way that the problem is decomposed, which may be by O-D, by origin, or by using simplicial decomposition of the problem based on the extreme points of the feasible region (2) and (3). The most commonly used algorithms are based on the linear approximation algorithm [16] and operate in the space of link flows. The adaptation of this algorithm and some of its variants are described first. Then algorithmic approaches that were devised for the solutions in the space of path flows, which are referred to as path equilibration algorithms, are briefly discussed. The convergence properties of the algorithms are noted but not proved in detail as they may be found in nonlinear programming texts such as [23].

One of the simplest convergent algorithms for minimizing a convex function subject to linear constraints is the linear approximation method. Bruynooghe et al. [4] were the first to propose the method; however, the later work of LeBlanc et al. [22] and Nguyen [24] made this method popular in practice. Computer codes are widely available for solving both the fixed demand and the variable demand version of the network equilibrium models. The fixed-demand problem is

$$\begin{aligned}
\min\ [S(v)] &= \sum_{a\in A}\int_0^{v_a} s_a(x)dx \\
\text{s.t. } &\sum_{k\in K_i} h_k = \bar{g}_i, \quad i \in I, \\
&h_k \geq 0,\ k \in K, \\
&v_a = \sum_{k\in K} \delta_{ak} h_k, \quad a \in A.
\end{aligned}$$

Given an initial feasible solution, a feasible direction of descent is generated by solving a subproblem that is obtained by a first-order approximation of the objective function. The linearized approximation for an intermediate iteration l at a solution v^l is

$$S(v^l) + \nabla S(v^l)(y - v^l). \tag{29}$$

By eliminating the constant terms $S(v^l)$ and $\nabla S(v^l)v^l$, the linearized subproblem simplifies to

$$\min \left[\sum_{i\in I}\sum_{k\in K_i}\sum_{a\in A} s_a(v_a^l)\delta_{ak} y_k\right] \tag{30}$$

$$\text{s.t. } \sum_{k\in K_i} y_k = \bar{g}_i, \quad i \in I, \tag{31}$$

$$y_k \geq 0, \quad k \in K. \tag{32}$$

By changing the order of summation in equation (30) and by using equation (4), the objective becomes

$$\min \left[\sum_{i\in I}\sum_{k\in K_i} s_k(v^l) y_k\right] \tag{33}$$

subject to equations (31) and (32).

The solution of this problem is obtained by computing shortest paths for each O-D pair i and allocating the demand $\bar{g}_i$ to that path ('all-or-nothing' assignment). This yields the link flow

$$z_a^l = \sum_{k \in K} \delta_{ak} y_k^l, \quad a \in A \tag{34}$$

and the direction of descent is

$$d_a^l = (z_a^l - v_a^l), \quad a \in A. \tag{35}$$

An iteration of the linear approximation algorithm is completed by finding the solution of

$$\min_{0 \le \lambda \le 1} [S(v^l + \lambda d^l)] \tag{36}$$

or, equivalently, by finding λ, $0 < \lambda < 1$, for which

$$\sum_{a \in A} s_a(v_a^l + \lambda d_a^l) d_a^l = 0 \tag{37}$$

unless the minimum of equation (36) is attained for $\lambda = 0$ or $\lambda = 1$.

The following algorithm results.

Linear approximation method
Step 0. Find v^1; $s^1 = s(v^l)$, $l = 1$.
Step 1. Perform an 'all-or-nothing' assignment based on the current arc costs $s(v^l)$ and obtain y^l. Let $d^l = y^l - v^l$.
Step 2. Verify whether a predetermined stopping criterion is satisfied. If it is, stop; otherwise continue to step 3.
Step 3. Find optimal step size λ^l by solving equation (37).
Step 4. Update arc flows $v^{l+1} = v^l + \lambda^l d^l$ and arc costs $s^{l+1} = s(v^{l+1})$, set $l = l + 1$ and return to step 1.

The algorithm generates paths at each iteration, but these are not kept. Hence the storage requirements are modest and do not increase with the number of iterations. Also, it is easy to obtain a lower bound on the value of the optimal objective function. Because $S(v)$ is a convex function and $\nabla S(v) = s(v)$,

$$S(v^*) \ge S(v^{l'}) + s(v^{l'})(y^{l'} - v^{l'}), \quad l' = 1, 2, \ldots, l. \tag{38}$$

The right-hand side of equation (38) provides a lower bound on $S(v^*)$ at each iteration. The best lower bound (BLB) at a current iteration l is

$$BLB = \max_{l'=1,2,\ldots,l} [S(v^{l'}) + s(v^{l'})(y^{l'} - v^{l'})]. \tag{39}$$

As a consequence, a natural stopping criterion, denoted the relative gap (RGAP) is

$$RGAP = \frac{S(v^l) - BLB}{S(v^l)} \times 100. \tag{40}$$

Because $S(v^l) - BLB$ is an estimate of the difference between an optimal solution and the current solution, the computations are terminated when $RGAP \le \epsilon_1$, where $\epsilon_1 > 0$ is a predetermined parameter. Other stopping criteria that are used may be a maximum number of iterations, $l^{\max}$, or the quantity $[s(v^l)v^l - s(v^l)y^l]$, which tends to zero as the optimum solution is approached.

This last stopping criterion is

$$\frac{s(v^l)v^l - s(v^l)y^l}{\sum_i \bar{g}_i} \leq \epsilon_2, \tag{41}$$

where $\epsilon_2 > 0$ is another predetermined parameter. The left-hand side of equation (41) has the physical interpretation of the difference between average trip costs on currently used paths and the average trip costs on current shortest paths. This quantity does not decrease monotonically with increasing number of iterations.

An intuitive interpretation of this algorithm is that the travellers adjust their route choice from congested routes to less congested routes until all routes are of about equal length. This explains its resemblance to many heuristic algorithms that have been suggested and used to solve this problem. On the other hand, the linear approximation algorithm exhibits slow convergence in the vicinity of the optimal solution because its asymptotic rate of convergence is sublinear. This has motivated the development of variants of this algorithm that attempt to improve the rate of convergence. One of these variants is presented later in this section.

The variable-demand network equilibrium model ([18]) may be solved by a *partial* linear approximation method, first suggested by Evans [11], which employs the linearization of only some of the variables of the objective function. In equation (21), only the link cost functions are linearized. The resulting subproblem at iteration l is

$$\min \Big[\sum_{i\in I}\sum_{k\in K_i}\sum_{a\in A} s_a(v_a^l)\delta_{ak}y_k\Big] - \sum_{i\in I} w_i(g_i^l)x_i \tag{42}$$

$$\text{s.t.} \sum_{k\in K_i} y_k - x_i = 0, \quad i \in I, \tag{43}$$

$$y_k \geq 0,\ k \in K,\ y_i \geq 0,\ i \in I. \tag{44}$$

This subproblem is solved by determining u_i^l, $i \in I$, to be the costs of the shortest paths based on the current link costs $s(v^l)$ and then simplifying (42) by using equations (43) and (44) to solve

$$\min \sum_{i\in I}[u_i^l - w_i(g_i^l)]x_i \tag{45}$$

$$\text{s.t. } x_i \geq 0, i \in I. \tag{46}$$

By applying the Kuhn–Tucker conditions (see [23]), x_i^l are determined analytically as follows:

$$x_i^l = \begin{cases} G_i(u_i^l) & \text{if } G_i(u_i^l) \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

The demands x_i^l are then assigned to the shortest paths in order to obtain y_a^l, $a \in A$, and the direction of descent is $d^l = [(y^l - v^l); (x^l - g^l)]$.

Even though the solutions obtained with the linear approximation method are usually acceptable for the solution of the fixed or variable network equilibrium models for large-scale problems, the slow convergence of the method in the neighborhood of the solution has motivated the development of several variants that improve its asymptotic rate of convergence. These include the adaptation of the PARTAN (parallel tangents) method [2, 12, 21] and the restricted simplicial decomposition method, which are described in more detail below.

The motivation for exploring the PARTAN variant of the linear approximation method is that, for unconstrained minimization problems, the PARTAN algorithm is equivalent to the conjugate gradient algorithm [23]. This algorithm alternates a regular iteration of the linear approximation algorithm with a direction generated by using every other solution, v^{l-1} and v^{l+1}. The solution at this alternate iteration is obtained by finding α^l, $\alpha^l \leq \alpha^l_{\max}$, which minimizes the objective function for the solution $v^{l-1} + \alpha(v^{l+1} - v^{l-1})$ where $\alpha^l_{\max}$ is the largest step size that maintains the non-negativity of the path flow. It can be shown that, at a current iteration l, the largest step size that may be taken is given by the formula

$$\alpha^l_{\max} = \frac{1}{1 - \bar{\lambda}^l \bar{\lambda}^{l-1} \bar{\alpha}^{l-1}},$$

where $\bar{\lambda}^l = 1 - \lambda^l$ and $\bar{\alpha}^{l-1} = 1 - \alpha^{l-1}$. The algorithm may be stated as follows.

Linear approximation with PARTAN

Step 0. Find a feasible solution v^l; $s^1 = s(v^1)$; $l = 1$.

Step 1. Find the linear approximation direction, $d^l = y^l - v^l$.

Step 2. If a predetermined stopping criterion is satisfied, stop; otherwise continue to step 3.

Step 3. Find optimal step size λ^l.

Step 4. Update arc flows $\tilde{v}^l = v^l + \lambda^l d^l$.

Step 5. If $l = 1$, then $v^{l+1} = \tilde{v}^l$; $s^{l+1} = s(v^{l+1})$; $l = l + 1$ and return to step 1; otherwise, the PARTAN direction is $d_p^l = \tilde{v}^l - v^{l-1}$.

Step 6. Find the optimal PARTAN step size α^l as the solution of

$$\min S(v^{l-1} + \alpha d_p^l)$$
$$\text{s.t. } 0 \leq \alpha \leq \alpha^l_{\max}.$$

Step 7. Update arc flows $v^{l+1} = v^{l-1} + \alpha^l d_p^l$ and arc costs $s^{l+1} = s(v^{l+1})$; $l = l + 1$, and return to step 1.

The restricted simplicial decomposition algorithm ([18,19]) is an extension of the simplicial decomposition methods proposed by von Hohenbalken [29] for solving nonlinear programs with pseudo-convex, continuously differentiable objective functions and linear constraints. Its application for solving the fixed-demand network equilibrium problem (22) subject to equations (2) and (3) is as follows. Because the feasible region Θ is bounded, there are a finite number of extreme points, and every flow in Θ can be written as a convex combination of these extreme points. If Θ_z denotes a set of retained extreme points of Θ and Θ_v a set that is empty or contains the flows at a current iteration, and $q \geq 1$ denotes an integer parameter that controls the number of extreme points at an iteration, the algorithm is as follows.

Restricted simplicial decomposition
Step 0. v^0 is a feasible solution; set $\Theta_z^0 = \emptyset$, $\Theta_v^0 = v^0$ and $l = 0$.
Step 1. Solve the subproblem

$$\begin{aligned} \min\ & \Big[\sum_{a\in A} s_a(v_a^l)y_a\Big] \\ \text{s.t.}\ & \sum_{k\in K_i} y_k = \bar{g}_i, \quad i \in I,\ y_k \geq 0,\ k \in K, \\ & z_a = \sum_{k\in K} \delta_{ak} y_k. \end{aligned}$$

which is the same as step 1 of the linear approximation method that is solved by an 'all-or-nothing' allocation of the demands $\bar{g}_i$ to shortest paths for each O-D pair i. Let the solution be z^l.
Step 2. If $s(v^l)(z^l - v^l) \geq 0$, stop; v^l is the optimal solution. Otherwise, if $|\Theta_z^l| < q$, then $\Theta_z^{l+1} = \Theta_z \cup z^l$ and $\Theta_v^{l+1} = \Theta_v^l$. If $|\Theta_v^l| = q$, replace the extreme point of Θ_z^l that has the minimal weight in the expression for v^l in the convex combination of elements of Θ^l with z^l to obtain Θ_z^{l+1} and let $\Theta_v^{l+1} = v^l$. Set $\Theta^{l+1} = \Theta_z^{l+1} \cup \Theta_v^{l+1}$. Go to step 3.
Step 3. Let $v^{l+1} = \text{argmin}\{S(v) |\ v \in \text{convex hull of } \Theta^{l+1}\}$. $v^{l+1} = \sum_{j=1}^{m} \lambda_j z_j$, where $m = |\Theta^{l+1}|$ and $z_j \in \Theta^{l+1}$. Remove all elements z_j with $\lambda_j = 0$ from Θ_z^{l+1} and Θ_v^{l+1}. Set $l = l + 1$ and return to step 1.

The efficiency of the algorithm depends to a large extent on the solution of the (master) problem in step 3. In order to achieve convergence, the master problem need not be solved exactly. It is only necessary to ensure that a sufficient decrease in the objective is achieved in successive iterations. This may be achieved by approximating the objective of the master problem with a quadratic function. Then, under appropriate assumptions, there is an almost closed form solution for the (approximate) master problem.

Path equilibration algorithms for the symmetric fixed-demand network equilibrium are considered next. In this approach, the problem is decomposed and a sequence of problems, for each O-D pair i, is solved in the space of path flows. This general approach, which is equivalent to a Gauss–Seidel decomposition (or relaxation), is also known as 'cyclic decomposition,' because in a step of the algorithm, a single O-D problem is solved by keeping the flows of all other O-D pairs fixed. The algorithm terminates when there is no improvement in the solution for all O-D pairs i, which constitute a 'cycle.'

The subproblem solved for each O-D pair i is the fixed-demand network equilibrium problem

$$\min \Big[\sum_{a\in A} \int_0^{v_a^i+\bar{v}_a} s_a(x)dx\Big] \tag{48}$$

$$\text{s.t.} \sum_{k\in K_i} h_k = \bar{g}_i, \quad i \in I, \tag{49}$$

$$h_k \geq 0, \quad k \in K_i, \tag{50}$$

where

$$\bar{v}_a = \sum_{i'\neq i} \sum_{k\in K_i} \delta_{ak} h_k \tag{51}$$

and

$$v_a^i = \sum_{k\in K_i} \delta_{ak} h_k. \tag{52}$$

The Gauss–Seidel solution strategy may be stated as follows.

Cyclic decomposition by O-D pair
Step 0. Given an initial solution, set $i = 0$, $i' = 0$.
Step 1. If $i' = |I|$, stop; otherwise set $i = i \mod |I| + 1$ and continue.
Step 2. If the current solution is optimal for the ith subproblem (48) to (52), set $i' = i' + 1$ and return to step 1; otherwise solve the ith subproblem, update flows, set $i' = 0$ and return to step 1.

The convergence of the Gauss–Seidel strategy is ensured as the objective function is convex and any local minimum is a global minimum as well.

Path equilibration algorithms used to solve equations (48) to (52) operate in the space of path flow and obtain a solution where all used paths are of equal cost. Because the number of paths grows exponentially with the network size $(|N|, |A|)$, path equilibration algorithms are usually implemented by using a restriction strategy, where the paths that carry flow are generated as required. Let $K_i^+ = \{k \in K_i | h_k > 0\}$ be the set of paths with positive flows. The simplest such algorithm, due to Dafermos [5], finds the shortest path and longest path and transfers flow between these paths in order to equalize their cost. The algorithm may be stated as follows.

Path equilibration algorithm
Step 0. Find an initial solution v_a^i, $s_a(v_a^i + \bar{v}_a)$ and determine the initial K_i^+.
Step 1. Find k_1 such that $s_{k_1} = \min_{k\in K_i} s_k$ and k_2 such that $s_{k_2} = \max_{k\in K_i^+} s_k$.
If $s_{k_2} - s_{k_1} \leq \epsilon$, stop.

Step 2. Find λ by solving the one variable problem

$$\min_\lambda \Big[\sum_{a\in A}\int\limits_0^{y_a} s_a(x)dx\Big] \tag{53}$$

$$\text{s.t.}\quad 0 \le \lambda \le h_{k_2}, \tag{54}$$

$$y_a = v_a^i + (\delta_{ak_1} - \delta_{ak_2})\lambda + \bar{v}_a. \tag{55}$$

Step 3. Using the λ obtained, update $h_{k_1} = h_{k_1} + \lambda$, $h_{k_2} = h_{k_2} - \lambda$ and recalculate the v_a, s_a, and K_i^+. Go to 1.

This algorithm is just one of many path equilibration schemes possible. To generate a direction of descent for the subproblem (48) to (52), the reduced gradient or the projected gradient algorithm may be used, as they are well-known convergent nonlinear programming methods.

4.2 Deterministic Asymmetric Models

For simplicity, only algorithms for the fixed-demand network equilibrium problem are presented, i.e., for finding v^* feasible that satisfies

$$s(v^*)(v - v^*) \ge 0, \quad \text{for all feasible } v. \tag{56}$$

A large class of algorithms for this problem, which are referenced as relaxation methods, result when the cost function is modified at each iteration by fixing the interaction between blocks of variables and thus removing, at each iteration, the asymmetry of the cost functions. These algorithms include the nonlinear Jacobi method and the nonlinear Gauss–Seidel method. They are sometimes referred to as diagonalization methods, because the resulting Jacobians of the relaxed vector of cost functions are diagonal.

In order to describe a relaxation algorithm, it is convenient to introduce a smooth function

$$\hat{s}(v, \tilde{v}) : \Theta \times \Theta \to \mathbb{R}^n$$

with the property that $\hat{s}(v, v) = s(v)$ and $\nabla_v \hat{s}(v, \tilde{v})$ is positive definite and symmetric. Hence, if $v^{l+1} = v^l$, then v^{l+1} is a solution of the asymmetric network equilibrium model and is the unique solution of the variational inequality problem

$$\hat{s}(v^{l+1}, v^l)(v - v^{l+1}) \ge 0, \quad v, v^{l+1} \text{ feasible} \tag{57}$$

which is obtained by solving the strictly convex differentiable optimization problem

$$v^{l+1} = arg\min_v \left[\sum_{a\in A}\int_0^{v_a} \hat{s}_a(x, v^l)dx\right]. \tag{58}$$

Different algorithms result from the choices made for the function $\hat{s}(v, \tilde{v})$: the nonlinear Jacobi method obtained for

$$\hat{s}_a(v, v^l) = s_A(v_1^l, \ldots, v_a, \ldots, v_{|A|}^l) \tag{59}$$

and the nonlinear Gauss–Seidel method results for

$$\hat{s}_a(v, v^l) = s_A(v_1^{l+1}, \ldots, v_a^{l+1}, v_a, \ldots, v_{a+1}^l, \ldots, v_{|A|}^l). \tag{60}$$

An algorithm defined by equation (58) is globally convergent [6] if

$$\|\nabla_v s^{-1/2}(v^1, \tilde{v}^1)\nabla_{\tilde{v}} s(v^2, \tilde{v}^2)\nabla_v s^{-1/2}(v^3, \tilde{v}^3)\|_2 < 1$$
$$\text{for } v^i, \tilde{v}^i \text{ feasible}, \ i = 1, 2, 3. \tag{61}$$

This sufficient condition for the convergence of the relaxation methods is difficult to verify and rather restrictive. The intuitive interpretation of this condition is that the Jacobian of the vector of cost functions is weakly asymmetric. In summary, one way to state this class of relaxation algorithms is the following.

Relaxation algorithm
Step 0. Find a feasible solution v^l, $l = 1$.
Step 1. Determine v^{l+1} as the solution of equation (58).
Step 2. If $\|v^{l+1} - v^l\| \leq \epsilon$, stop; otherwise, $l = l + 1$ and go to step 1.

Among the algorithms that have been proposed for solving the asymmetric network equilibrium models are the simplicial decomposition method, gap descent methods, projection algorithms, and a dual cutting plane method. These methods are not presented here, but the interested reader should consult references [13] and [25].

4.3 Stochastic Symmetric Models

The solution algorithms for this problem employ a simulation in order to obtain a direction of descent for the objective function (26). In order to evaluate the objective function exactly, an exhaustive path enumeration for all O-D pairs of the network would be necessary. This is clearly prohibitive from a computational perspective. An algorithm that implements the basic step

$$v^{l+1} = v^l + \alpha_l d^l, \tag{62}$$

where the step size α_l satisfy

$$\sum_{l=1}^{\infty} \alpha_l = \infty, \quad \sum_{i=1}^{\infty} \alpha_l^2 < \infty \tag{63}$$

and d^l, the direction of descent, is determined by a Monte Carlo simulation ([26]). This simulation is performed by sampling all links for a travel time realization, computing shortest paths and performing an 'all-or-nothing' assignment on these paths. This procedure is repeated several times and then a flow vector $\hat{y}_a$ is obtained by averaging the link flows. The number of times that the procedure is repeated determines the variance of $\hat{y}_a$. The direction of descent, d_a^l, is $\hat{y}_a - v_a$.

It can be proved that, when the step sizes satisfy equation (63), this algorithm converges to the unique solution of the stochastic network equilibrium model. Because $\alpha_l = 1/l$ satisfies equation (63), this choice is often made, and the method is referred to as the 'method of successive averages.' This method lacks a natural stopping criterion, and the descent in the values of the objective function is not monotonic.

Attention is now directed to the logit-based stochastic network equilibrium model

$$\min_h \left(\sum_{i \in I} \sum_{k \in K_i} h_k \ln h_k \right) + \theta \sum_{a \in A} \int_0^{v_a} s_a(x) dx \tag{64}$$

subject to equations (2) and (3). This problem may also be solved by the method of successive averages, without requiring simulation in order to obtain a direction of descent. At each iteration l, the direction of descent is obtained by computing shortest paths based on current link costs and by performing an 'all-or-nothing' assignment on these paths.

It can be shown theoretically and empirically that the flow pattern that results from stochastic network equilibrium models tends toward the flow pattern obtained with the deterministic model as the network becomes congested. This may be recognized intuitively by inspecting the terms of the corresponding objective functions (26) and (64).

5 Combined Mode Models

The network equilibrium models presented so far do not distinguish between different classes or different modes of traffic. In many applications, the network equilibrium models are more complex and lead to more elaborate models that identify explicitly different modes, such as private car and public transit, or different classes of traffic that may correspond with different vehicle types or different socioeconomic classes. Some examples of such models are presented next.

Suppose that the vehicles traveling on the network are subdivided into $|M|$ different types. The link cost function on each link is different for each vehicle type and depends on the different types of vehicle that use the link, $s_a^m(v)$, $a \in A$, where v is the vector of flows (v_a^m), $a \in A$, $m \in M$. The demand for each vehicle type $\bar{g}_i^m$ is known. The corresponding deterministic network equilibrium model is given by the variational inequality

$$\sum_{m \in N} \sum_{a \in A} s_a^m(v^*)(v_a^m - v_a^{m^*}) \geq 0 \tag{65}$$

$$\text{s.t.} \sum_{k \in K_i^m} h_k = \bar{g}_i^m, \quad i \in I,\ m \in M, \tag{66}$$

$$h_k \geq 0, \ k \in K_i^m, \ i \in I, \ m \in M, \tag{67}$$

$$v_a^m = \sum_{i \in I} \sum_{k \in K_i^m} \delta_{ak} h_k. \tag{68}$$

Unless some simplifying assumptions are made regarding the link cost functions, the solution of this model requires an efficient algorithm for a very large scale variational inequality model.

One way to simplify this multiclass model is to induce symmetry and separability in the link cost functions. For instance, it is postulated that the user cost functions simplify to

$$s_a^m(v) = s_a\left(\sum_{m \in M} v_a^m\right) + t_a^m, \quad a \in A, \ m \in M \tag{69}$$

which implies that the travel time depends on the total number of vehicles on the link and that only a constant term, t_a^m, differentiates between the various classes of traffic. Then, with the appropriate manipulation, an equivalent convex cost minimization problem is obtained:

$$\min \left[\sum_{a \in A} \int_0^{v_a} s_a(x)dx\right] + \sum_{a \in A} \sum_{m \in M} t_a^m v_a^m \tag{70}$$

$$\text{s.t.} \sum_{k \in K_i^m} h_k = g_i^m, \quad i \in I, \ m \in M, \tag{71}$$

$$h_k \geq 0, \quad k \in K, \tag{72}$$

$$v_a^m = \sum_{k \in K_i^m} \delta_{ak} h_k, \quad a \in A, \ m \in M, \tag{73}$$

$$v_a = \sum_{m \in M} v_a^m. \tag{74}$$

This model may be solved efficiently by an adaptation of the linear approximation method and has been used extensively in applications. Other variants of equations (71) to (74) are possible as well.

Another example of a combined model is a two-mode model of traffic [15] where one mode is the private car and the other mode is public transit. A mode choice function

$$G_i(u_i) = \frac{1}{1 + \exp[\alpha + \beta(u_i^1 - u_i^2)]}, \quad i \in I \tag{75}$$

gives the probability (or proportion) of trips that will use mode 1, which has a travel cost of u_i^1, and the competing mode 2 has a travel cost of u_i^2. A two-mode network equilibrium model may be formulated by assuming that a 'user optimal' route choice is made on both mode 1 and mode 2 (which may correspond with a transit mode):

$$s_k(v^*) - u_i^{m^*} \begin{cases} = 0 \text{ if } h_k^* > 0 \\ \geq 0 \text{ if } h_k^* = 0 \end{cases}, \quad k \in K_i^m,\ i \in I,\ m = 1, 2 \tag{76}$$

subject to conservation of flow and non-negativity constraints

$$\sum_{k \in K_i^m} h_k = g_i^m, \quad i \in I,\ m \in 1, 2, \tag{77}$$

$$h_k \geq 0, \quad k \in K_i^m,\ i \in I,\ m \in 1, 2 \tag{78}$$

and

$$u_i^m(v) = \min_{k \in K_i^m} [s_k(v)], \quad i \in I,\ m = 1, 2. \tag{79}$$

The link cost functions $s_a^m(v)$, $m = 1, 2$, $a \in A$, are asymmetric and not separable, in general.

This model may be cast in the form of a variational inequality by carrying out the usual derivation by using equation (11). The resulting variational inequality is

$$\sum_{a \in A} s_a^1(v^*)(v_a^1 - v_a^{1^*}) + \sum_{a \in A} s_a^2(v_a^*)(v_a^2 - v_a^{2^*}) - \sum_{i \in I} w_i(g_i^{1^*})(g_i^1 - g_i^{1^*}) \geq 0, \tag{80}$$

where $w_i(g_i^{1^*})$ is the inverse of equation (75). The Jacobi method may be used to obtain a solution to this model.

The two multiclass multimode models presented above are just two examples of the multitude of combined models that are formulated for particular transportation planning applications. Some of these models are so complex that their solution is obtained by *ad hoc* equilibration procedures that are inspired by the method of successive averages.

6 Application and Validation of Network Equilibrium Assignment Models

The validation of network equilibrium models has been reported in relatively few published empirical studies in spite of the fact that literally thousands of applications have been successfully carried out. The early studies of Florian and Nguyen [14] on the urban network of the City of Winnipeg, Canada, and of Dow and Van Vliet [10] on the urban network of the City of Leeds, UK, are examples of successful validation exercises. A practical problem that arises when applying the fixed-demand network equilibrium models is the determination of the O-D matrix. Synthetic demand models or O-D surveys are used to determine the demand for travel with various degrees of success. Even when survey data are available, they do not include information for all the trips taken during the peak hour, and various adjustment methods are used to reconcile the differences between the flow predictions and the observed link

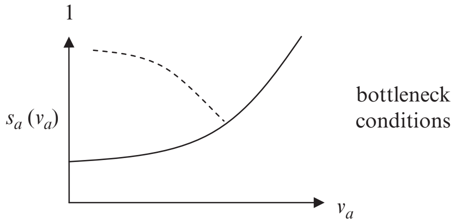

Figure 2. Volume-delay function with 'spill back' effect

counts. The process of calibrating a network equilibrium model involves the network representation, the calibration and allocation of user cost functions (known also as volume-delay functions) to the links of the network, as well as eventual adjustments of the O-D matrix. Often significant approximations are made in order to build the necessary database for the application of the model. Yet, it may still be the best predictive tool available for evaluating the impact of network changes in the short and medium term. The link flows obtained in the validation studies mentioned above simulate the average hourly flows during the peak period quite well, and the origin to destination travel times are satisfactorily reproduced as well.

The static nature of the network equilibrium model, which renders its solution to be efficient, is also one of its main drawbacks in the simulation of traffic flows. The application of network equilibrium models is based implicitly on the assumption that the traffic is not subject to severe bottlenecks that may cause the traffic to back up and 'spill back' to upstream links. When this assumption is not satisfied, monotonically increasing user cost functions do not model properly the phenomenon of increased travel time and reduced flow on links that contain traffic bottlenecks as indicated in Figure 2.

The use of the network equilibrium model is not particularly demanding in computer expenditure. The planning network of the Southern California Association of Governments has (in 2004) 3339 centroids (zones), 30,678 nodes and 109,770 directional links. There are 6 classes of traffic (3 private and 3 commercial vehicle types). An iteration on an IBM Thinkpad notebook (Intel Centrino, 1.6 MHz) takes 4 minutes, and about 50 iterations are required for a reasonable convergence of less than 1% relative gap. The ever-increasing power of processors that are used to build personal computers and workstations will render the computation of equilibrium flows on even larger networks possible in elapsed times of the order minutes.

7 Conclusion

The study of network equilibrium assignment models and related solution algorithms may be considered to have reached a mature stage. A variety of models may be formulated and solved efficiently on contemporary computing platforms. Applications of network equilibrium models are abundant and relatively common in the practice of transportation planning. However, some of the basic premises of the formulation of these models such as the additivity of link costs to form the cost of a path and the static analysis of 'average flows' during a selected time period open the way to the study of more complex models.

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Investment Paradoxes in Electricity Networks

Mette Bjørndal and Kurt Jørnsten

Department of Finance and Management Science, Norwegian School of Economics and Business Administration, Bergen, Norway
{mette.bjorndal,kurt.jornsten}@nhh.no

Abstract The famous Braess' paradox, which sometimes occurs in user-optimal traffic equilibrium models, demonstrates that adding a new link to the network may lead to a degradation in network performance. User-equilibrium is characterized by each user competing noncooperatively for the network resources by choosing the path that is best for himself, without paying attention to the effect this has on the other users (eventually including himself). Similar effects occur in congested electricity networks, where flows follow Kirchhoff's junction rule and loop rule. Thus, due to the special nature of electricity networks, we show that grid investments, which at first sight seem an improvement of the grid, may prove to be detrimental to social surplus, even without considering investment costs. Moreover, some agents will have incentives to advocate these changes. It is also demonstrated that a thermal limit, which is internal to a market, may result in market integration being disadvantageous. The possibility of such paradoxical effects, and the incentives that they provide to different agents, should clearly be taken into consideration both in the process of grid development and market development.

Key words: congested electrical networks, Kirchhoff's junction and loop rules, user-equilibrium flows, Braess' paradox

1 Introduction

In this article, we are addressing grid investments. In general, there are two aspects of this question, the first is that of detecting beneficial investments, and the second is how to induce them under the chosen market regime. We will show that network "improvements," i.e., strengthening a line or building a new line, may in fact be detrimental to social surplus, and that some agents will have incentives to advocate these changes. The reason that electricity markets possess such characteristics is that in a deregulated electricity market model, the economic equilibrium model must include physical equilibrium constraints in the form of Kirchhoff's junction rule and loop rule. This means that electrons behave "noncooperatively," and hence, given injections and withdrawals, power cannot be routed. The main reason for the occurrence of the paradoxical behavior, which is addressed in this article, is due to the fact that, in general, Nash equilibria are Pareto inefficient.

2 Braess' Paradox and Generalizations

In user-optimizing traffic assignment problems where each individual user chooses the path with the lowest travel cost, it is well-known that the equilibrium flow in a network is generally different from the system optimal flow, i.e., the flow minimizing total travel cost. In his original example, Braess [3] showed that adding a new link to a congested network may in fact increase travel cost for all, and this phenomenon is referred to as the Braess' paradox. Braess' paradox and variations of it have been the subject of several papers, like Murchland [15], Stewart [20], Frank [11], Dafermos and Nagurney [9], Steinberg and Zangwill [19], and Steinberg and Stone [18], among others.

Hallefjord et al. [12] discuss paradoxes in traffic networks in the case of elastic demand. When travel demand is elastic, it is not evident what a paradoxical situation is, and in this case there is a need for characterizations of different paradoxes. An example is given, where total flow decreases while travel time increases due to adding a new link to the network. This is a rather extreme type of paradox. A different paradox is when the network "improvement" leads to a reduction in social surplus.

The reason for the traffic equilibrium paradoxes is the behavioral assumption that a traveler chooses the path that is best for himself, without paying attention to the effect this has on the other users (eventually including himself). In user-equilibrium, a user cannot decrease travel time by unilaterally changing his travel route, leading us to seeing the equilibrium as a Nash equilibrium of an underlying game. Korilis et al. [14] investigate the noncooperative structure of certain networks, where the term noncooperative emphasizes that the networks are "operated according to a decentralized control paradigm, where control decisions are made by each user independently, according to its own individual performance objectives." Nash equilibria are generally Pareto inefficient as demonstrated by Dubey [10], and Korilis et al. [14] use the Internet as an example while referring more generally to queuing networks.

Cohen and Horowitz [8] give examples of Braess' paradox for other noncooperative networks like mechanical systems (strings) and hydraulic and electrical networks, and point to the need for specifications of conditions under which general networks behave paradoxically. This is partly provided by Calvert and Keady [7], and Korilis et al. [14] propose methods for avoiding degradation of performance when adding resources to noncooperative networks.

In the following sections, we will give examples of paradoxical situations that can occur in deregulated electricity markets due to the fact that electrons behave "noncooperatively." This behavior is reflected by the power flow equations. When computing the economic equilibria, we assume competitive markets, i.e., the given supply and demand functions reflect truthfully marginal cost and willingness to pay. When assessing different grid configurations, we compare social surpluses, i.e., consumers' willingness to pay less production cost.

3 Grid Investments in Electricity Networks

In Wu et al. [21], a 3-node example is given, showing that strengthening a line by increasing its admittance may lead to larger minimum cost. The network and initial optimal dispatch is displayed in Figure 1 (assuming a linear lossless "DC" approximation of the power flow equations). In optimal dispatch, the nodal prices will be related by $p_1 < p_2 < p_3$ because line 1-3 is congested in direction from node 1 to node 3 (for an argument, see Wu et al. [21]). When the admittance of line 2-3 is increased, the power flow equations change, and flow will increase on path 1-3-2 if injections are maintained. This will result in line 1-3 becoming overloaded, and injection in node 1 must be reduced. If consumption is to be maintained, injection in node 3 must increase, leading to larger minimum cost.

In a similar 3-node example exhibited in Figure 2, Bushnell and Stoft [4] show that a new line hurts the network but still collects congestion rent, defined as the merchandizing surplus, i.e., the price difference of the end, points of the line times the flow over the line.

In the example there is high cost production in node 1 and relatively lower cost production in node 2. Consumption takes place in node 3 where there is

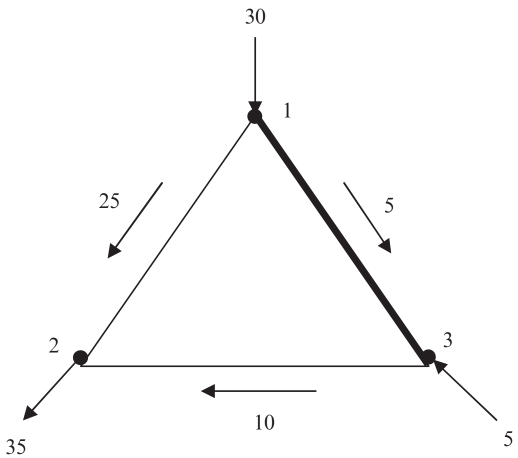


Figure 1. Increasing admittance increases cost

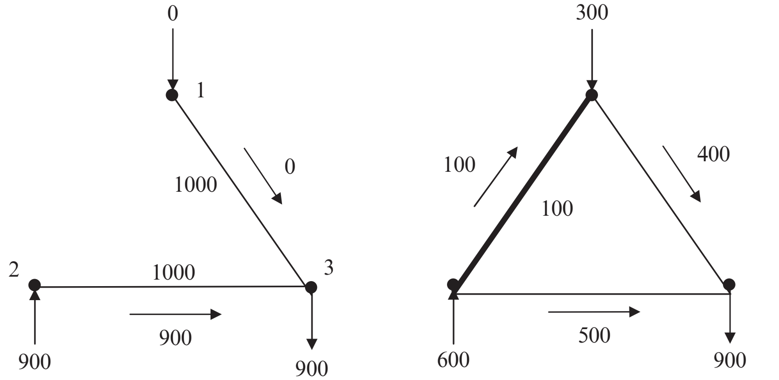


Figure 2. New line increases cost

a fixed demand equal to 900 MW. Initially, there are only two links, 1-3 and 2-3, each with a capacity of 1000 MW, and demand is supplied entirely by the low cost producers in node 2.

In part B of Figure 2, a new line has been built between nodes 1 and 2. This is a weak line with a capacity of only 100 MW, and it introduces loop flow having as a consequence that the transfer capacity between nodes 2 and 3 is greatly reduced. Assuming reactances equal to 1 on every link and no production in node 1 to generate counter flow on line 1-2, it is reduced from 1000 to 300 MW. By inducing injections in node 1, the minimum cost of supplying 900 MW to node 3 is obtained by injecting 600 MW in node 2 and 300 MW in node 1, which is obviously a more costly dispatch. The new line is congested in direction 2 to 1, and as $p_1 > p_2$, the new line receives congestion rent $(p_1 - p_2)q_{21} > 0$.

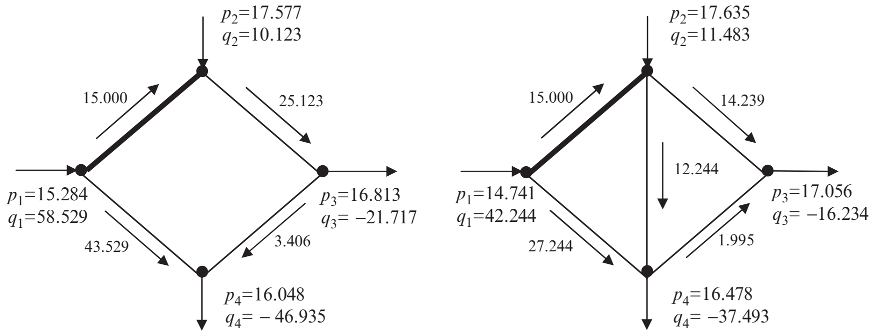
In the following, we will give examples of paradoxes in a 4-node network with the classic Braess' configuration, sometimes referred to as the Wheatstone bridge topology, and with elastic demand and production in every node. We assume linear cost and demand functions, represented by $p_i = c_i q_i^s$ and $p_i = a_i - b_i q_i^d$ where p_i is the price in node i, q_i^s is the quantity produced in node i, q_i^d is the quantity consumed in node i, and a_i, b_i, and c_i are positive constants. Net injection in node i is given by $q_i = q_i^s - q_i^d$. With input data given in Table 1 and a thermal capacity of 15 units on line 1-2, optimal dispatch and optimal prices are given in Figure 3. Part A shows the situation without line 2-4, whereas part B includes this line. We use a linear and lossless "DC" approximation with reactances equal to 1 one every line.

By introducing the new line, total production and consumption has been reduced together with social surplus. On the other hand, grid revenue defined as the merchandizing surplus, MS, where

$$MS = -\sum_i p_i q_i = \frac{1}{2}\sum_i \sum_j (p_j - p_i) q_{ij},$$

Table 1. Cost and Demand Parameters

Node	Consumption		Production
	a_i	b_i	c_i
1	20	0.05	0.1
2	20	0.05	0.3
3	20	0.05	0.4
4	20	0.05	0.5



Part A: No Line between Nodes 2 and 4
Social Surplus: 2878.526
Grid Revenue: 45.848

Part B: New Line between Nodes 2 and 4
Social Surplus: 2852.660
Grid Revenue: 69.444

Figure 3. Optimal dispatch before and after line 2-4

Table 2. Allocation Effects of New Line

	Node 1		Node 2		Node 3		Node 4	
	Before	After	Before	After	Before	After	Before	After
Production	152.843	147.415	58.589	58.783	42.031	42.641	32.097	32.955
Consumption	94.314	105.171	48.466	47.300	63.749	58.874	79.031	70.448
Net Exports	58.529	42.244	10.123	11.483	-21.717	-16.234	-46.935	-37.493
Producer Surplus	1168.048	1086.554	514.901	518.321	353.328	363.646	257.552	271.511
Consumer Surplus	222.379	276.522	58.724	55.933	101.597	86.655	156.149	124.075
Surplus of Region	1390.427	1363.076	573.624	574.254	454.925	450.301	413.701	395.585

increases. The effect on individual agents varies, i.e., some agents lose whereas others are better off, as displayed in Table 2. If surplus changes for an agent, it means that the nodal price that he faces has been altered. More specifically, if the price of node i increases as a consequence of the new line, producer i gains while consumer i loses. If the price falls, the opposite is valid.

Considering the surplus of each region (i.e., the combined producer and consumer surpluses of each node), it is evident that in general, some regions are better off due to the new line, whereas others lose. However, it is not

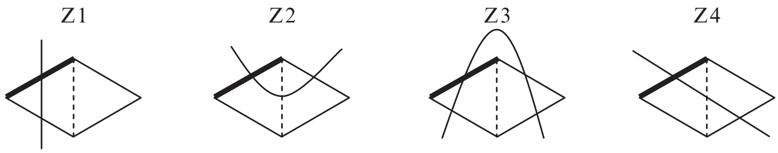


Figure 4. Allocations to two zones

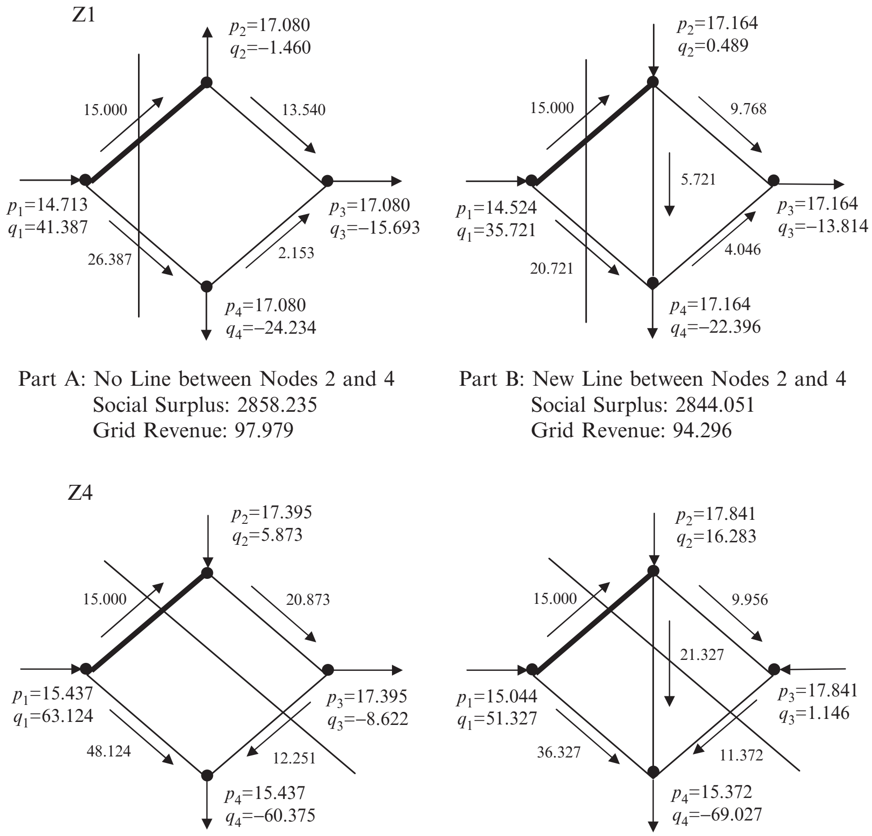
difficult to construct examples in which every region loses because of the new line. For instance, changing the example above by letting $c_2 = 0.37$ makes every region worse off, whereas the grid revenue increases when line 2-4 is introduced.

In the discussion so far, we have considered optimal nodal pricing as the means of managing congestion. Zonal pricing, which is an approximation of optimal nodal prices, where we require prices to be uniform within specified zones, constitutes an alternative, and is used in for instance the Nordic power market (see Bjørndal and Jørnsten [2]). In the given example, assuming only two zones, there are four zone allocations that separate nodes 1 and 2. These are displayed in Figure 4. Different zone allocations affect social surplus, and for the parameters of our example, $Z4$ is best without the new line, whereas $Z1$ is best when the new line is included. This illustrates that modifications to the grid should lead to a reconsideration of zone allocations.

Prices, net injections and power flows for $Z1$ and $Z4$ are displayed in Figure 5, together with total social surplus and grid revenue. As is evident from the numbers, also under zonal pricing total social surplus is reduced when the new line is built. This is so for fixed zone allocations (i.e., the partition of nodes into zones remains the same after the new line is in place), but it is also valid even if the best zone allocation is chosen at every point. For fixed zone allocations, grid revenue is reduced when building the new line. However, if the new line changes the partition of nodes from $Z4$ to $Z1$, grid revenue increases considerably, thus it may provide a strong incentive on the part of the grid owners to build the line.

In Table 3, we show the surpluses for each region. In general, the change of surplus for individual agents can be positive or negative. In $Z1$, every region surplus as well as the grid revenue decreases due to the new line. If parameters are changed so that $c_2 = 0.35$ and the thermal capacity of line 1-2 is 5 units, the effect of the new line on every region would be negative when choosing the social beneficial zone allocations (i.e., switching from $Z4$ to $Z1$ when building line 2-4). Grid revenue on the other hand would increase.

In the examples cited so far, the reductions in social surplus are relatively minor. In the original example in Table 1, the reduction in total social surplus is equal to 25.866, or 0.9%. This is partly due to the assumption of identical demand functions in every node. By allowing more unequal distributions of consumption, the reductions can be of considerable size. For instance, increasing b_i, $i = 1, 2$ to 0.25, i.e., the size of the markets in nodes 1 and 2 are



Part A: No Line between Nodes 2 and 4
Social Surplus: 2858.235
Grid Revenue: 97.979

Part B: New Line between Nodes 2 and 4
Social Surplus: 2844.051
Grid Revenue: 94.296

Part C: No Line between Nodes 2 and 4
Social Surplus: 2869.871
Grid Revenue: 5.380

Part D: New Line between Nodes 2 and 4
Social Surplus: 2821.270
Grid Revenue: −49.495

Figure 5. Zonal solutions Z1 and Z4 before and after the new line

Table 3. Region Surpluses

	$Z1$		$Z4$	
	Before	After	Before	After
Node 1	1361.881	1354.600	1399.745	1377.242
Node 2	571.474	571.434	572.168	577.110
Node 3	449.917	448.685	446.096	444.489
Node 4	376.983	375.036	446.482	471.924

assumed to be only 20% of the markets in nodes 3 and 4, social surplus in optimal dispatch is reduced from 2541.968 to 2394.397, i.e., by 5.8%, when the new line is built. This is more than 2.5 times the cost of the thermal limit itself, as social surplus in unconstrained dispatch is equal to 2600.506. If there is no consumption in nodes 1 and 2, social surplus is reduced from 2395.869 to 2129.125, i.e., by 11.1%. Also when increasing demand by shifting the demand curves positively (for instance by raising the a_i's), the paradox becomes more severe.

The persistence of the paradox depends on cost parameters as well. Consider for instance varying c_2. When $c_2 \in [0, 0.080)$, the new line improves social surplus. When $c_2 \in [0.080, 0.102)$, the new line has no effect on social surplus because the thermal limit is not binding in optimal dispatch, neither with nor without line 2-4. Finally, when $c_2 \geq 0.102$, the new line reduces social surplus, implying that the paradox also occurs when production in node 2 is so costly that it is not being used. The reduction reaches a maximal value at $c_2 = 0.350$. Varying c_4 in the same manner, the thermal capacity is binding for all values of c_4. When $c_4 < 0.179$, line 2-4 improves social surplus, whereas the paradox arises for $c_4 > 0.179$.

4 Admittance Versus Thermal Capacity

From the derivation of the "DC"-approximation in Wu et al. [21], assuming voltage magnitudes equal to 1 for every node, we have that

$$q_{ij} = \frac{1}{x_{ij}} \sin(\delta_i - \delta_j) = Y_{ij} \sin(\delta_i - \delta_j)$$

where δ_i is the phase angle at node i, q_{ij} is the power flow over line ij, x_{ij} is the reactance of line ij, and the admittance Y_{ij} of line ij is equal to the reciprocal of the reactance of the line. Because the sine function has a maximal value of 1, we must have that $q_{ij} \leq Y_{ij}$. Considering also the thermal limit C_{ij} of line ij, q_{ij} is bounded by $\min\{C_{ij}, Y_{ij}\}$. This means that "strengthening" a line has two interpretations: increasing the admittance or increasing the thermal limit.

From the optimal dispatch problem (refer for instance Wu et al. [21]), we know that the shadow price of the thermal limit $q_{ij} \leq C_{ij}$ is non-negative, i.e., $\mu_{ij} \geq 0$, which means that social surplus cannot be reduced by improving the thermal limit of any line. What we have shown by the previous examples is that whenever there is at least one binding thermal limit, say on line ij,

$$\frac{\partial \texttt{Social Surplus}}{\partial Y_{kl}}$$

may be negative for some link kl. That is, by either increasing the admittance of an existing line or by building a new line,[1] we may reduce social surplus.

Consider now varying the thermal capacity of line 1-2. In Figure 6, social surpluses are shown as functions of C_{12}. The functions are concave and increasing, and the difference between the curves is the greatest for $C_{12} = \epsilon$ and

[1] That is, increasing the admittance from the 0 level

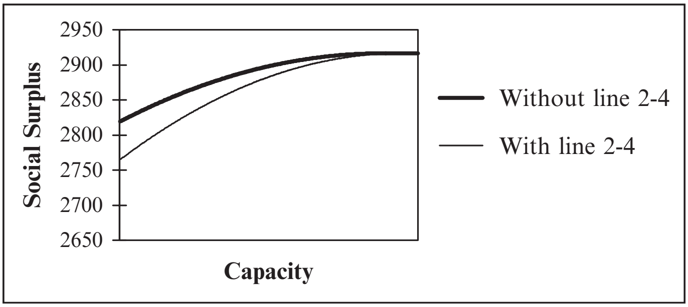


Figure 6. Social surplus and thermal capacity of line 1-2

vanishes when C_{12} is so large that the thermal limit is no longer binding in any of the network configurations considered. This occurs at $C_{12} = 42.587$, which is the flow over line 1-2 in unconstrained dispatch, assuming line 2-4 is included in the network. From this point, social surplus is constant and equal to 2916.525, and increasing the thermal capacity is not beneficial in either network configuration.

As is shown by Wu et al. [21], in optimal dispatch, the merchandizing surplus MS is equal to the *congestion rent* defined by

$$CR = \sum_i \sum_j \mu_{ij} C_{ij}.$$

Because line 1-2 is the only congested line in our example,[2] grid revenue is equal to $\mu_{12}C_{12}$, i.e., for a given thermal capacity C_{12}, the size of the grid revenue is determined by the value of

$$\mu_{12} = \frac{\partial \texttt{Social Surplus}}{\partial C_{12}}.$$

As is indicated by the curves of Figure 6, building line 2-4 will increase grid revenue because at every $C_{12} < 42.587$, the social surplus function *with* line 2-4 is steeper than the function depicting social surplus *without* line 2-4.

Note however that whether the grid revenue increases due to the new line is not indicative of whether the paradox occurs. Grid revenue may increase also when the new line is beneficial. For instance, letting $c_4 = 0.15$, total social surplus increases from 3448.992 to 3457.022 when the new line is built. Grid revenue increases from 58.969 to 64.530, i.e., total social surplus increases more than the grid revenue, leaving a net increase for the market participants as well, due to the new line.

In Figure 7, social surplus is shown as a function of the admittance of line 2-4. For reference, social surplus without line 2-4 is also exhibited. We note that the difference between social surplus with and without line 2-4 increases

[2] Assuming $\mu_{12} > 0$, while $\mu_{ij} = 0$ for $ij \neq 12$

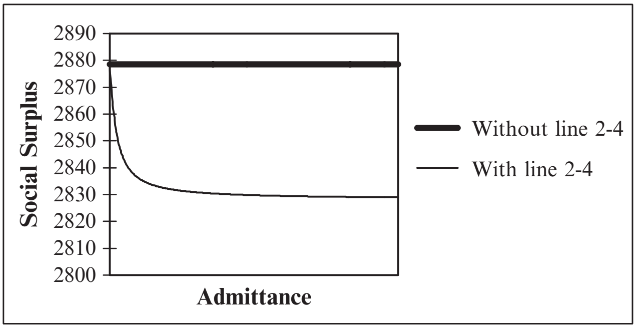


Figure 7. Social surplus and admittance of line 2-4

with the admittance Y_{24}. When $Y_{24} \to \infty$, social surplus approaches the value 2828.161 asymptotically, signifying that the paradox becomes more severe the stronger is the new line, but there is a maximal degradation of social surplus equal to $2878.526 - 2828.161 = 50.365$.

5 Interpretation in Terms of Load Factors

The load factors of line 1-2 for different trades can be expressed as functions of Y_{24}. When the new line is introduced with an admittance of Y_{24}, the power flow equations become the following:

Kirchhoff's junction rules:

$$\begin{aligned} q_1 &= q_{12} + q_{14} \\ q_2 &= -q_{12} + q_{23} + q_{24} \\ q_3 &= -q_{23} + q_{34} \end{aligned}$$

Kirchhoff's loop rules:

$$\begin{aligned} q_{24} &= -Y_{24} q_{12} + Y_{24} q_{14} \\ q_{24} &= Y_{24} q_{23} + Y_{24} q_{34} \end{aligned}$$

Conservation of energy:

$$q_1 + q_2 + q_3 + q_4 = 0.$$

By solving the power flow equations for different trades, we find the load factor matrix

$$\mathbf{B}_{12}^{Y_{24}} = \begin{pmatrix} 0 & \frac{3+2Y_{24}}{4(1+Y_{24})} & \frac{1}{2} & \frac{1+2Y_{24}}{4(1+Y_{24})} \\ -\frac{3+2Y_{24}}{4(1+Y_{24})} & 0 & -\frac{1}{4(1+Y_{24})} & -\frac{1}{2(1+Y_{24})} \\ -\frac{1}{2} & \frac{1}{4(1+Y_{24})} & 0 & -\frac{1}{4(1+Y_{24})} \\ -\frac{1+2Y_{24}}{4(1+Y_{24})} & \frac{1}{2(1+Y_{24})} & \frac{1}{4(1+Y_{24})} & 0 \end{pmatrix},$$

where the entry of row k and column l is β_{12}^{kl}, which is the load factor of a trade from node k to node l on line 1-2 (in direction from 1 to 2). In the linear "DC" approximation, load factors are constants for given admittances, and $\beta_{ij}^{kl} = -\beta_{ij}^{lk}$. The negative numbers indicate that the corresponding trades generate counter flows on line 1-2.

When $Y_{24} \to \infty$, trades between nodes 2, 3, and 4 have no influence on line 1-2, which can be seen from

$$\mathbf{B}_{12}^{\infty} = \lim_{Y_{24}\to\infty} \mathbf{B}_{12}^{Y_{24}} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$

Nodes 2, 3, and 4 thus become one market with identical nodal prices. Net injection in node 1, on the other hand, distributes equally on lines 1-2 and 1-4 (load factors are equal to $\frac{1}{2}$), implying that the maximal export from region 1 is equal to $30 = 2C_{12}$. An interpretation of this situation is that nodes 2 and 4 are electrically "the same," which is similar to a cost of zero on line 2-4 in a traffic equilibrium network. In the case of our electrical network, this makes the paradox maximal.

The paradox of the example of Table 1 and Figure 3 can be interpreted in terms of the load factors. The load factor matrix without line 2-4 is equal to

$$\mathbf{B}_{12}^{0} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix},$$

whereas the load factor matrix *with* line 2-4 (with admittance equal to 1) is equal to

$$\mathbf{B}_{12}^{1} = \begin{pmatrix} 0 & \frac{5}{8} & \frac{1}{2} & \frac{3}{8} \\ -\frac{5}{8} & 0 & -\frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{8} & 0 & -\frac{1}{8} \\ -\frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0 \end{pmatrix}.$$

Considering optimal dispatch without line 2-4, $q_1 = 58.529$, $q_2 = 10.123$, $q_3 = -21.717$, and $q_4 = -46.935$. As is evident from matrices $\mathbf{B}_{12}^{0}$ and $\mathbf{B}_{12}^{1}$, the load factors of trades between net injection and net consumption nodes have developed unfavorably when introducing line 2-4. The positive load factors β_{12}^{13} and β_{12}^{14} stays the same or increases, meaning that the corresponding trades use as much or more of the capacity of line 1-2 under the new network configuration. The negative load factors β_{12}^{23} and β_{12}^{24} have decreased in absolute value, indicating that the trades that they represent produce smaller counter flows on line 1-2, thus relieving the capacity constraint to a lesser extent. Under the new network configuration, the injection vector $(58.529, 10.123, -21.717, -46.935)$ is no longer feasible. According to the characterization used by Bushnell and Stoft [6], the old dispatch belongs to the "newly infeasible region," and the "newly feasible" region that follows from the new line provides no better dispatch, thus the paradox.

6 Example: Market Integration

A consequence of the paradoxical characteristics of certain electricity networks is that in the presence of congestion constraints, social surplus can be reduced when markets are integrated. In Figure 8, market 1 consists of nodes 1, 2, and 3 and market 2 consists of nodes 4 and 5. We assume linear cost and demand functions, with parameters given in Table 4. We want to consider integrating the markets by building lines 2-4 and 3-5. Disregarding any thermal constraints, we find that social surplus would increase from 3126.177 to 3157.895. The system price settles on 16.842, which is higher than the price of market 1 and lower than the price of market 2.

Assume now there is a capacity limit of 10 units on line 1-2. In Figure 9, we show optimal dispatch without the connecting lines. Social surplus is equal

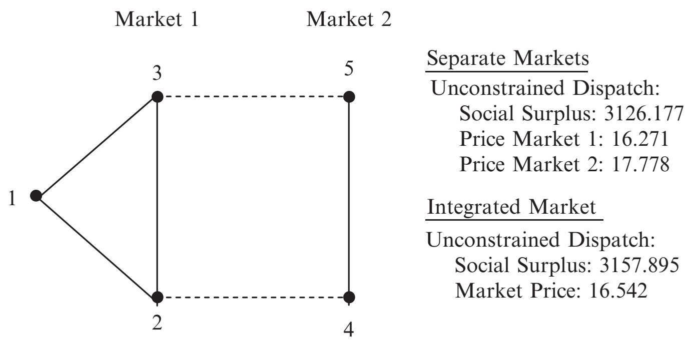


Figure 8. Market integration: unconstrained dispatch

Table 4. Market Integration: Unconstrained Dispatch

Node	Consumption		Production
	a_i	b_i	c_i
1	20	0.05	0.1
2	20	0.05	0.8
3	20	0.05	0.4
4	20	0.05	0.6
5	20	0.05	0.3

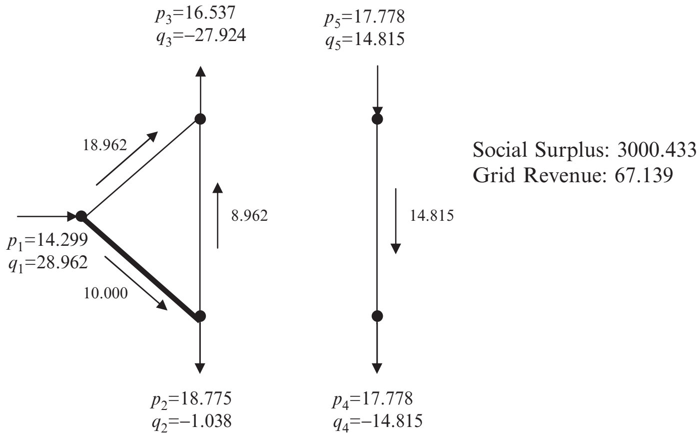


Figure 9. Optimal dispatch: before integration

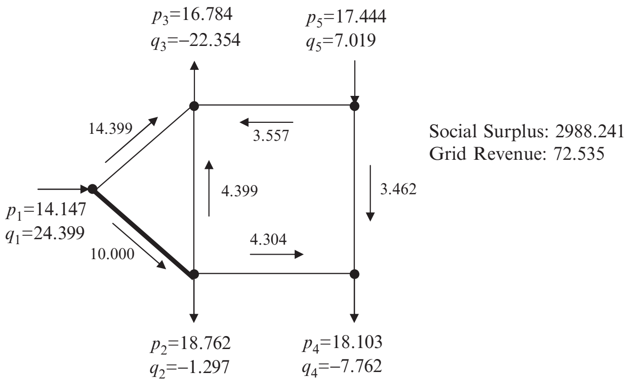


Figure 10. Optimal dispatch: after integration

to 3000.433. In Figure 10, the new lines have been built, and social surplus is reduced to 2988.241, implying that the thermal limit on line 1-2, which is *internal* to market 1, prevents the realization of potential benefits from market integration.

7 Suggested Cures

Given that an investment has already been carried out, in traffic equilibrium networks marginal cost pricing can lead to improved overall system performance from the grid modification, even when Braess' paradox occurs in user-equilibrium (Pas and Principio [16]). Penchina [17] discusses different cures, including various tolls and reversible one-way signs, showing that the "best" remedy depends on traffic, and although system optimum is achieved under marginal cost pricing, in some cases there is a trade-off between the optimality and complexity of the suggested cure. In electricity networks there is no equivalent methodology, as electrons do not respond to marginal cost pricing. To alter flows for a given set of injections, we would have to alter line impedances.

Considering the investment decision itself, the obvious way to avoid the paradox in our Wheatstone bridge example is to build line 1-3 instead of line 2-4. This would resolve the capacity problem of line 1-2 but may be unacceptable for other reasons, for instance investment cost. Generally, the issue of how to encourage beneficial investments and discourage detrimental investments has been treated in the literature, for instance by Baldick and Kahn [1], Bushnell and Stoft [4–6], and Hogan [13]. As is also demonstrated by Bushnell and Stoft [4,5], transmission congestion contracts (TCCs), where new contracts are allocated according to a *feasibility rule*, which helps internalizing the external effects of detrimental grid investments, can provide at least a partial solution. However, the results depend on transmission rights matching actual dispatch, and hourly TCCs are so far not available for instance in the Norwegian power market, thus the problem posed is not completely solved, neither in theory nor practice.

As is confirmed by some of the examples in this chapter, and also pointed to by Bushnell and Stoft [6], the performance of a network depends on expected dispatch, which is influenced by future supply and demand conditions, which are constantly changing and subject to uncertainty. Thus, as market conditions change, so can the performance of the different network configurations considered. This is further complicated by typically long asset lifetimes and the lumpiness of the investment decisions, which sometimes makes it desirable to expand the network in a manner that is not immediately beneficial but will be in the long run. Ideally, we should compare different expansion *paths* rather than various fixed networks, as the investment problem is dynamic in nature.

8 Conclusion

Depending on the parameters of the problem considered (cost, demand, thermal capacity, and admittance), a new line may be detrimental to social surplus. In general, some agents are better off whereas others lose in such a situation. In this article, we provide examples where, in optimal dispatch, every region loses while the grid revenue increases. For fixed zone allocations, there is the possibility that every region-surplus *and* grid revenue is reduced as a consequence of a new line. We have also demonstrated that a thermal limit, which is internal to a market, may result in market integration being disadvantageous. The possibility of such paradoxical effects and the incentives that they provide to different agents must clearly be taken into consideration both in the process of grid development and market development.

There are many interesting research questions that can be raised based on the possible occurrence of paradoxical effects. To mention a few: Can a regulatory regime be designed to reduce the negative effects of the paradoxical phenomena? Given a certain number of grid investments, that when added to the existing network yield a desired effect, does there exist a network expansion path, consisting of expanding the network with one link at a time, that yields positive effects in each step? When investigating the regulatory regime, it is important to consider not only whether the regulation is based on price-caps or revenue-caps, or whether we have a rate-of-return-regulation. The incentive-effects of a specific regulation may also be influenced by the mechanisms for relieving congestion — the opposite also being true.

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Algorithms for Network Interdiction and Fortification Games

J. Cole Smith[1] and Churlzu Lim[2]

[1] Department of Industrial and Systems Engineering, The University of Florida, Gainesville, Florida, USA
cole@ise.ufl.edu

[2] Systems Engineering and Engineering Management, University of North Carolina at Charlotte, 9201 University City Blvd., Charlotte, NC 28223, USA
clim2@uncc.edu

Abstract This chapter explores models and algorithms applied to a class of Stackelberg games on networks. In these *network interdiction* games, a network exists over which an operator wishes to execute some function, such as finding a shortest path, shipping a maximum flow, or transmitting a minimum cost multicommodity flow. The role of the interdictor is to compromise certain network elements before the operator acts, by (for instance) increasing the cost of flow or reducing capacity on an arc. We begin by reviewing the field of network interdiction and its related theoretical and mathematical foundations. We then discuss recent applications of stochastic models, valid inequalities, continuous bilinear programming techniques, and asymmetric analysis to network interdiction problems. Next, note that interdiction problems can be extended to a three-stage problem in which the operator fortifies the network (by increasing capacities, reducing flow costs, or defending network elements from the interdictor) before the interdictor takes action. We devote one section to ongoing research in this area and conclude by discussing areas for future research.

Key words: network interdiction, network fortification, bilinear programming, Stackelberg games, mixed-integer programming, duality

1 Introduction

In this chapter, we discuss developments in the fields of network interdiction and fortification problems, with a particular focus on advances in mathematical and algorithmic contributions made in the past decade. We describe these problems in the context of a two-player game. One player, called the "operator," wishes to optimally utilize some network, for instance, by conducting a maximum flow across the network or by sending messages via a least-cost path.

The "interdictor" will attempt to compromise the network in some manner in order to worsen the operator's optimal solution. Some typical interdiction actions include reducing the capacity of certain arcs (perhaps to zero, which disables the use of the arc altogether), increasing transportation costs on arcs, and monitoring traffic arriving on certain links and nodes. The interdictor is usually constrained by some budget that restricts the amount of damage or alteration that can be done to the network.

Hence, one classic interdiction game is a two-stage, two-player game in which the interdictor acts first to strategically compromise certain elements of the network, and the operator follows with an optimal decision based on the resulting network. That is, these problems take on the form of Stackelberg games [96], which involve a leader (interdictor) and follower (operator). Even if there does not exist an actual human interdictor, natural disasters or accidental failures can serve as the interdictor in worst-case analyses. In this case, "Murphy's Law," the pessimistic philosophy that the worst possible event will transpire, determines the set of network elements compromised by nature.

This scenario can be expanded for the case in which the operator has the option of expending limited resources to fortify some aspects of the network, thus providing a measure of protection for the network against the interdictor. These fortification measures can prohibit the interdictor from taking interdiction actions on certain aspects of the network or can install additional capacity (and perhaps new links) in anticipation of the interdictor's next move. In this case, we now consider a three-stage game with two players, in which the operator acts first to fortify the network, followed by the interdictor's action, which is in turn followed by the operator's optimal recourse solution on the resulting network.

This line of research differs from the field of survivable network design, which is geared more toward the design of networks that can withstand accidental disruptions or attacks on limited portions of the network infrastructure. A common assumption on such networks is that link failures are rare, and hence the network must be able to withstand any single failure. References [30, 52, 94] regard network design problems for scenarios in which any one link can fail. Grötschel and Monma [44] and Grötschel, Monma, and Stoer [45] study mathematical programming techniques for multiconnectivity problems by identifying a set of strong valid inequalities for the problem, along with separation routines for their identification within a branch-and-cut algorithm. Some systems, such as SONET ring networks [107, 108], are inherently survivable due to their topologies. (See [13, 41, 42, 62, 90, 92, 97] for a series of network design optimization studies on SONET rings.) Each of these network design problems fall under the fortification rubric presented here, where one "fortifies" a network that initially has no links and protects against an enemy that could destroy any single link. By contrast, we will not confine our study to any particular topology, as done in SONET ring network design studies, and will allow for more general disruptions than single-link (or double-link, etc.) failures.

It is of interest to note that some survivable network design studies (see [35], for example) enumerate sets of possible network failures, called "failure scenarios," and then employ Benders decomposition to decompose the problem into a master network design problem and a recourse network flow problem for each possible failure scenario. However, the set of possible interdiction actions that can be taken by the interdictor is potentially too large to enumerate. For instance, if the interdictor can compromise any five out of twenty links, there exist 15,504 possible scenarios that must be enumerated. Thus, we require fundamentally new algorithms for interdiction/fortification problems.

Before proceeding to discuss classic and emerging techniques for these problems, we illustrate how three different applications fall into this framework.

Example 1 (Nuclear smuggling interdiction). Our first example arises in the context of defense against thieves who intend to move from one point in a network to another. The genesis of this line of work study arose in the deployment of nuclear sensors in the Former Soviet Republics. These nuclear sensors are designed to interdict thieves, who intend to steal nuclear material and transport it across national boundaries. This work has been pioneered by Morton, Pan, and Charlton in various studies over the past few years [72, 76, 77].

A joint American/Russian program called the Second Line of Defense (SLD) was established in 1998 to help mitigate the success of these thieves in smuggling nuclear material. One of the responsibilities of the SLD was to study the deployment of nuclear sensors on links or vertices of the smuggler's transportation network. The deterministic version of this problem is modeled by considering a directed network having a single source and destination, where each link (i, j) is associated with two probabilities: p_{ij} is the probability of avoiding detection with no sensor monitoring link (i, j), and q_{ij} is the probability of avoiding detection with a sensor monitoring link (i, j). This model assumes $0 \leq q_{ij} < p_{ij} \leq 1$, and assumes that all of these link probabilities are independent of one another. Because the location of these sensors is public knowledge (which is unfortunately all too common in such applications, as demonstrated in [18]), the smuggler has full knowledge of the location of each sensor and will be assumed to follow a path that maximizes the probability of traversing from the origin to the destination undetected. The interdiction problem is thus to deploy a (limited) set of sensors such that the smugger's maximum-reliability path is minimized. Hence, in this context, the network operator is the smuggler, and the interdictor is the SLD team that deploys nuclear monitors.

For instance, consider the situation depicted in Fig. 1. In this example, a smuggler wishes to transit from Los Angeles to Washington, using only the arcs depicted in the network. We permit the deployment of two sensors on any of the six intermediate nodes (all nodes but Los Angeles and Washington). If a

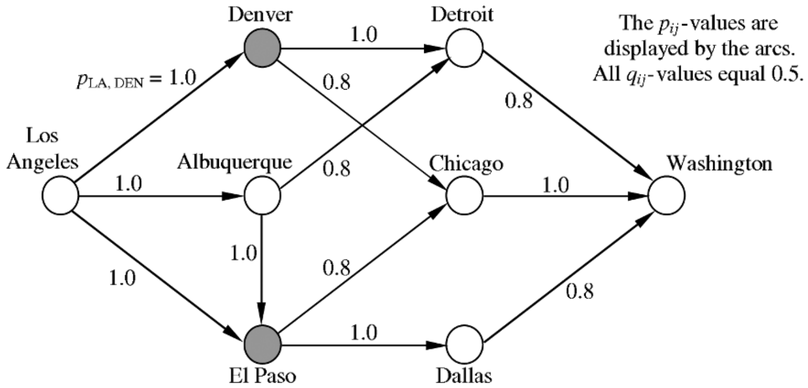


Figure 1. Example nuclear interdiction instance with two deployed sensors

sensor is installed at a node, then the outgoing arcs are all considered to be monitored. Because of the network topology and the limited number of sensors that can be deployed, it is impossible to force the smuggler to pass through a monitored node. However, it is possible to force the smuggler to avoid sensors by traversing two links that have a p-value of 0.8. The only deployment of sensors that force the smuggler to utilize two such links places these sensors in Denver and El Paso. The smuggler will then maximize his evasion probability by traversing the path Los Angeles – Albuquerque – Detroit – Washington, whose reliability (with no monitored links) is given by 0.64.

Example 2 (Distributed packet filtering in computer networks). The dramatic increase in the use of computer communications has carried with it a troublesome array of attacks on critical networks. One of the pressing problems facing major computer networks appears in the form of Distributed Denial of Service (DDoS) attacks, wherein a set of compromised hosts concurrently sends large amounts of traffic targeted at a server, gateway, or network [25,32]. The aim of the attack is to disrupt normal operation of the targeted network system by depleting its resources. Many DDoS attacks disguise their true origin by inscribing bogus information in the source address field of the IP (Internet Protocol) packet header, referred to as IP source address spoofing. This causes recovery to take of the order hours and days, at which point damage has already been done.

A proactive approach to DDoS defense called "route-based distributed packet filtering" [82] is aimed at preventing spoofed DDoS packets from reaching their targets. We deploy filters on nodes in the network, which determine whether a packet, given its purported source and destination address and the set of routes used in the network, must necessarily be misrepresenting its true origin. We also assume that attacks do not originate from nodes on which a filter is located. See [71] for a detailed study of this problem.

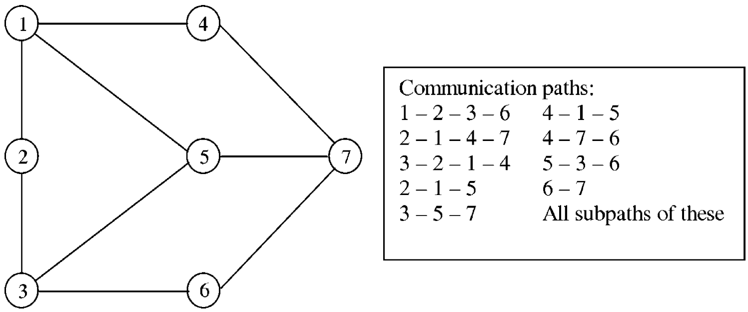


Figure 2. Computer network topology and routes

Note that these filters are only capable of testing necessary conditions for the true packet origin address to match its purported origin address: The filters could fail to prevent a spoofed packet from reaching its destination if the packet encounters filters that cannot prove that the purported origin has indeed been spoofed. To illustrate this scenario, consider the network and routing plan depicted in Fig. 2. Intuitively, routing is done along a path having the fewest number of links. Ties are broken on a path from node i to node j, $i < j$, by using the path that visits the lowest-possible indexed node first. Suppose that a filter is placed on node 2. Then if an attack is launched from node 1 by sending a spoofed packet with purported origin 5 to destination 3, the filter will stop the attack: Node 5 does not use node 2 as an intermediate node in its routes. However, if the forged address is node 4 instead of node 5, the filter will not stop the attack, because node 4 does indeed use node 2 as an intermediate routing node.

In this scenario, the operator plays the role of the DDoS attacker, trying to launch the most effective attack possible, while the interdictor deploys filters to limit the worst attack possible from the operator. Armbruster et al. [4] analyze the case in which the interdictor seeks to place the fewest number of filters possible in the network such that all attacks are halted. This is an NP-hard problem in general, and many filters are usually necessary to obtain such "perfect security." In the example of Fig. 2, filters must be placed at all nodes except node 6 to achieve perfect security. A more interesting metric is the "traceback number," as described in [68–70]. In this problem, we seek to deploy the fewest number of filters such that for any pair of nodes u and v in the network, the number of nodes that can attack node v using the purported origin address of u is not more than some threshold value τ. (This guarantees that a node under attack need not investigate more than τ suspects.)

Example 3 (Emergency service protection). Finally, consider the situation in which a set of emergency services is responsible for providing service to several clients. For instance, these emergency services might be fire stations whose clients are residences and businesses. A typical model assumes that when a node requires emergency services, a service facility located closest to the node

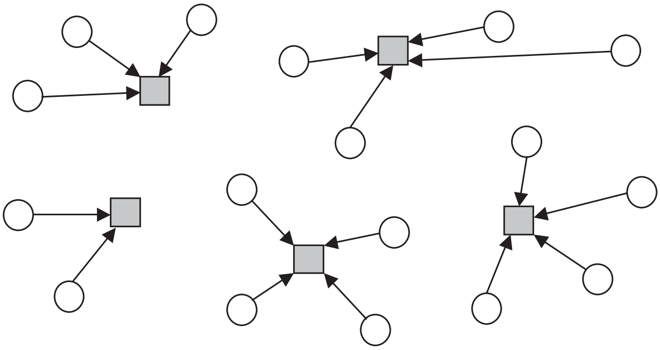


Figure 3. Node to facility assignment

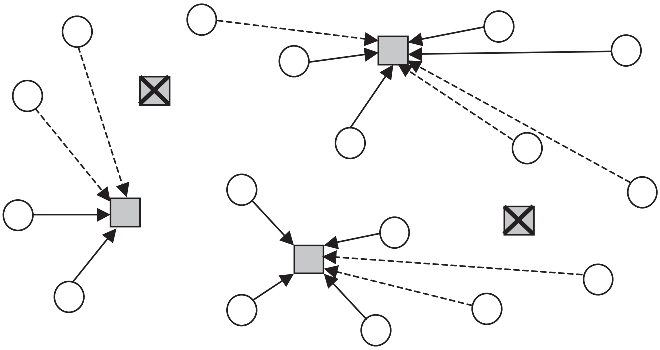


Figure 4. Revised assignment after interdiction of two facilities

will respond. One classic facility location problem seeks to establish a set of p emergency facilities such that the sum of shortest distances from each node to its closest facility is minimized. Figure 3 depicts an example in which the facilities are located at the gray squares, followed by the assignment of nodes to their closest facility.

Two recent works [22, 23] analyze the situation in which an interdictor can attack up to k out of these p facilities, removing them from serving client nodes. This situation is shown in Fig. 4 after the interdictor has removed two of the five nodes. Hence, in this scenario, the interdictor removes facilities from the network, while the operator assigns each node to its closest facility. However, research performed in [22, 23] expands this game to include a fortification stage, where the operator acts first to protect a subset of facilities from the interdictor. The interdictor must then confine all attacks to those unprotected facilities.

The remainder of this chapter is organized as follows. In Section 2, we provide a literature review of interdiction research and the theoretical/methodological fundamentals behind solving interdiction and fortification problems. We discuss advances in interdiction algorithms in Section 3, and then provide an analysis of fortification optimization techniques in Section 4. Finally, we conclude the chapter in Section 5 with a discussion of future research directions.

2 Literature Review

This section describes classic literature on network interdiction and fortification models and algorithms. Section 2.1 recaps models for interdiction problems where the network operator is faced with solving a shortest path, maximum flow, or (more generally) a minimum cost network flow problem. Section 2.2 provides theory and algorithms that have arisen from various domains including survivable network design, bilinear programming, and bilevel programming. These tools form a partial foundation for addressing contemporary interdiction and fortification problems.

2.1 Development of Fundamental Interdiction Models

Since Wollmer [104] proposed an algorithm that discretely interdicts a prescribed number of arcs in a network in order to minimize the follower's maximal flow, the network interdiction problem has been studied for more than four decades with broad applicability to military and homeland security operations (see [3, 5, 18, 26, 28, 34, 36, 40, 53, 61, 67, 103, 105, 106]). A generic form of the interdiction problem in which the leader minimizes the follower's maximum objective can be formulated as follows.

$$\underset{\mathbf{x}\in\mathbf{X}}{\text{Minimize}} \quad \text{maximize} \quad \mathbf{p}(\mathbf{x})^\top \mathbf{y} \tag{1}$$

$$\text{subject to} \quad \mathbf{D}\mathbf{y} \leq \mathbf{r}(\mathbf{x}) \tag{2}$$

$$\mathbf{y} \geq \mathbf{0}\,, \tag{3}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the leader and the follower variables, respectively, and where $\mathbf{p}(\mathbf{x})$ and $\mathbf{r}(\mathbf{x})$ are the vector of unit flow profits and the vector of available resources, respectively. Typically, $\mathbf{y}$ represents flows, and the set of constraints $\mathbf{D}\mathbf{y} \leq \mathbf{r}(\mathbf{x})$ includes flow conservation, arc capacity constraints, and side constraints as necessary. Note that $\mathbf{p}(\mathbf{x})$ and $\mathbf{r}(\mathbf{x})$ depend on the leader's decisions $\mathbf{x}$. Hence, subject to the leader's own feasible solution set $\mathbf{X}$, the leader controls the unit flow profits and/or resources in order to minimize the follower's maximum profit. It is common that $\mathbf{X}$ takes the form of $\mathbf{X} = \{\mathbf{x} : \mathbf{b}^\top \mathbf{x} \leq B,\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}\}$ where $\mathbf{e}$ is a vector of ones, that is, a single budget constraint with variables having values between 0 and 1. When $x_j = 1$, the resource j is interdicted, whereas $x_j = 0$ refers to survival of the resource. If $\mathbf{x}$ can take on any continuous value between 0 and 1, then a resource can be partially interdicted. Else, if a resource must be entirely destroyed under the leader's attack (or not damaged at all), then $\mathbf{x}$ is further restricted to take on binary values. Hence, the budget constraint corresponds with $\mathbf{X} = \{\mathbf{x} : \mathbf{b}^\top \mathbf{x} \leq K,\ \mathbf{x} \text{ binary}\}$. A special case in which the leader can interdict at most K resources is when $\mathbf{b} = \mathbf{e}$.

Among various network problems, the interdiction literature is mainly focused on the three types of follower's network problems: shortest path, maximum flow, and minimum cost (or maximum profit) flow problems. From now on, we consider a directed graph $G(N, A)$ with a node set N and an arc set A. While a single index is assigned for each node, we will use either a single index or a pair of node indices to represent each arc, as necessary for ease of presentation.

Shortest Path Interdiction

In the shortest path interdiction problem, the follower traverses from a source node s to a sink node t using the shortest path. By means of interdiction, the leader can increase the length of arc $j \in A$ from c_j to $c_j + d_j$. Note that if d_j is sufficiently large, the arc j is practically unavailable when completely interdicted.

This problem description is similar to the situation encountered in Example 2 by taking logarithms of the p_{ij}- and q_{ij}-values. However, placing a sensor on a node in Example 2 reduced the evasion probability of all arcs exiting that node, while this problem only permits altering the cost of one arc at a time.

Define $RS(i)$ and $FS(i)$, $\forall i \in N$, to be the reverse star and forward star of node i, respectively, i.e., $RS(i) = \{j \in A : \text{arc } j \text{ enters node } i\}$ and $FS(i) = \{j \in A : \text{arc } j \text{ leaves node } i\}$. Define $N_0 = N \setminus \{s, t\}$ to be the set of intermediate nodes. Then, this problem can be formulated as follows.

$$\underset{\mathbf{x} \in \mathbf{X}}{\text{Maximize}} \quad \text{minimize} \quad \sum_{j \in A} (c_j + d_j x_j) y_j \tag{4}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{5}$$

$$y_j \geq 0 \ \forall j \in A\,, \tag{6}$$

where $\mathbf{X} = \{\mathbf{x} : \mathbf{b}^\top \mathbf{x} \leq B,\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}\}$. In the objective function in (4), the length of arc $j \in A$ is increased by d_j when interdicted, that is, when $x_j = 1$. The set of constraints in (5) represents the follower's flow conservation restrictions. Note that the leader's variables $\mathbf{x}$ appear only in the objective function, while the right-hand-side of the follower's problem is free of the leader's variables. Letting $\boldsymbol{\pi}$ denote the dual variables associated with the flow conservation constraints, the problem can be written as a single maximization linear program as follows.

$$\text{Maximize} \quad \pi_t - \pi_s \tag{7}$$

$$\text{subject to} \quad \pi_i - \pi_j - d_k x_k \leq c_k \quad \forall k = (i, j) \in A \tag{8}$$

$$\mathbf{x} \in \mathbf{X}\,. \tag{9}$$

Note that the resulting problem is a linear program. Fulkerson and Harding [34] solve the dual of (7)–(9) via the parametric solution of the minimum cost flow problem. Instead of maximizing the shortest path, Golden [40] solves a minimum interdiction cost problem while ensuring the shortest path is increased to a certain threshold value τ. That is, the objective function is given by $\mathbf{b}^\top \mathbf{x}$ while a threshold constraint $\pi_t - \pi_s \geq \tau$ is added to the problem in (7)–(9) together with $\mathbf{X}$ now having only nonnegativity constraints. Assuming that interdiction actions must be binary, Israeli and Wood [53] propose a Benders decomposition algorithm enhanced by so-called *supervalid inequalities* in order to solve the mixed-integer program.

Closely related to the shortest path interdiction problem is the problem of identifying the most vital arc, which when removed, induces the greatest increase in the length of the shortest path. Corley and Sha [29] present an algorithm that repeatedly investigates $s-t$ paths in the order from the shortest to the longest until finding an arc that is commonly used in those paths investigated so far but not in the next shortest path. For the same problem, noting that such an arc is in the shortest path, Malik et al. [63] propose a more efficient algorithm that examines arcs only in the shortest path. Recently, the shortest path interdiction problem under uncertainty has been studied by Hemmecke et al. [49], Held et al. [47], and Held and Woodruff [48].

Maximum Flow Interdiction

The maximum flow interdiction problem concerns minimizing the maximum flow between designated source and sink nodes, s and t, on a capacitated network. Letting u_j denote the capacity of arc j, $\forall j \in A$, and adding a dummy arc (t, s) having $u_{ts} = \infty$ into the arc set A, the problem can be stated as follows.

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbf{X}}{\text{minimize}} \quad \text{maximize} \quad & y_{ts} && (10)\\
\text{subject to} \quad & \sum_{j\in FS(i)} y_j - \sum_{j\in RS(i)} y_j = 0 \ \forall i \in N && (11)\\
& y_j \leq u_j(1 - x_j) \ \forall j \in A \setminus \{(t, s)\} && (12)\\
& y_j \geq 0 \ \forall j \in A\ . && (13)
\end{aligned}
$$

The inner maximization problem is a capacitated maximum flow problem, which includes the flow conservation constraints in (11) and arc capacity constraints in (12). Note that the leader's variables appear in the right-hand side of the arc capacity constraints. Hence, interdicting an arc has an effect of decreasing the corresponding arc capacity. Assuming a nonempty feasible region of the inner maximization problem, we can use the same reformulation as done for the shortest path interdiction problem. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ denote dual variables associated with the flow conservation and the arc capacity constraints, respectively. Then, we have that

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j \in A} u_j(1 - x_j)\beta_j && (14)\\
\text{subject to} \quad & \alpha_i - \alpha_j + \beta_k \geq 0 \quad \forall k = (i,j) \in A \setminus \{(t,s)\} && (15)\\
& \alpha_t - \alpha_s \geq 1 && (16)\\
& \boldsymbol{\beta} \geq \mathbf{0} && (17)\\
& \mathbf{x} \in \mathbf{X}\,. && (18)
\end{aligned}$$

Unlike the shortest path interdiction problem in (7)–(9), this problem is not linear due to the bilinear terms $x_j\beta_j$ in the objective function. However, observe that we can further restrict the dual variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the follower problem to be binary, because changing the right-hand side of the structural constraints in the primal problem induces the change of at most one unit of the maximum flow (see Wood [106]). In fact, at optimality, the variables β_{ij} will equal to 1 if arc $(i,j) \in A$ is a member of the minimum cut corresponding with the optimal maximum flow solution, and zero otherwise. We will also obtain $\alpha_s = 0$, $\alpha_t = 1$, and $\alpha_i = 0$ if i belongs to the same side of the optimal cut-set as s, and $\alpha_i = 1$ otherwise. Therefore, a standard linearization technique can be used in order to reformulate the problem as a linear mixed-integer program by substituting $x_j\beta_j$ with a single variable w_j and adding linear constraints $w_j \leq x_j$, $w_j \leq \beta_j$, $w_j + (1 - \beta_j) \geq x_j$, and $w_j \geq 0$. In fact, the latter two constraints are unnecessary because the bilinear terms appear only in the objective function with negative signs. Wood [106] also extends this discussion to the multicommodity flow case.

Some earlier studies include those by Wollmer [104] and McMasters and Mustin [67], in which a prescribed number of arcs are removed in order to maximize the reduction in the maximum flow via the topological dual problem under the assumption that the network is planar. Ghare et al. [36] propose a branch-and-bound method for the problem in which the planarity assumption is dropped but the leader's variables are still binary. (See [28, 84] for other approaches to find vital arcs.) Burch et al. [19] present an approximation algorithm for the maximum flow interdiction problem, and Cormican et al. [26] study a stochastic version of the maximum flow interdiction problem.

Minimum Cost Flow Interdiction

The minimum cost flow interdiction problem is the most general case among interdiction problems in this section. In the minimum cost flow interdiction problem, the follower minimizes a linear cost flow function, and the leader takes interdiction actions to maximize the follower's minimum value. Define d_i to be the supply (if positive) or demand (if negative) present at node i, $\forall i \in N$ where $\sum_{i \in N} d_i = 0$. The minimum cost flow interdiction problem can be formulated as follows.

$$\underset{\mathbf{x} \in \mathbf{X}}{\text{Maximize}} \quad \text{minimize} \quad \sum_{j \in A} c_j y_j \tag{19}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = d_i \ \forall i \in N \tag{20}$$

$$y_j \leq u_j(1 - x_j) \ \forall j \in A \tag{21}$$

$$y_j \geq 0 \ \forall j \in A \, . \tag{22}$$

Note that the above problem is similar to the maximum flow interdiction problem except for the objective function and right-hand side of (20). Using the same reformulation as in the maximum flow interdiction problem, we have that

$$\text{Maximize} \quad \sum_{i \in N} d_i \alpha_i - \sum_{j \in A} u_j (1 - x_j) \beta_j \tag{23}$$

$$\text{subject to} \quad \alpha_i - \alpha_j - \beta_k \leq c_k \quad \forall k = (i, j) \in A \tag{24}$$

$$\boldsymbol{\beta} \geq \mathbf{0} \tag{25}$$

$$\mathbf{x} \in \mathbf{X} \, . \tag{26}$$

Recall that we were able to further restrict dual variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be binary in the maximum flow interdiction problem. However, this cannot be applied to the current problem because the change in the objective function value per unit flow change is dependent on the cost vector $\mathbf{c}$. Due to its requirement of sophisticated global optimization techniques, very few studies have been conducted for this type of problem. Wollmer [105] presents an approximation algorithm that repeatedly employs a single arc search algorithm. Recently, Lim and Smith [61] propose exact algorithms for solving discrete as well as continuous interdiction problems via linear mixed-integer programs. This technique is developed in more detail in Section 3.

2.2 Related Areas of Research

In this section, we provide a review of topics that are related to the development of network interdiction and fortification problems, including survivable network design, bilinear programming, and bilevel programming theory and methodology.

Survivable Network Design

Most modern network design algorithms have begun to take into account the survivability aspect of a network, which measures the susceptibility of a network to failures of small subsets of its arcs. It is important to design networks that are robust with respect to accidental failures like transportation breakdowns, road closures, and telephone line breaks, or to failures made maliciously by enemy entities. The Survivable Network Design (SND) problem seeks a minimum-cost robust network configuration that provides a number of alternative paths between nodes of the network. Many SND studies have focused on telecommunication applications (see [2, 24, 30, 74, 75, 83, 85, 87] for example). Given point-to-point traffic demands, one general SND problem assigns capacities to arcs (perhaps from among a finite set of alternatives) in order to minimize construction/expansion costs, while satisfying certain survivability constraints under any single node or arc failure. A critical survivability requirement is the minimum degree of flow feasibility after a network component failure. That is, the network is survivable if specified percentages of demands can be satisfied when any single node or arc fails. Some alternative survivability conditions impose upper bounds on path lengths [2] and/or require the existence of multiple paths between each point-to-point demand [30].

We remark that the aforementioned studies only consider the random failure of network components. Furthermore, most of these research efforts analyze single component failures (some exceptions can be found in [75], who consider the simultaneous failure of pairs of arcs, and [35], who consider the case of multiple arc failures in a Benders decomposition scheme). However, when the network is maliciously attacked, this random failure assumption might not be reasonable because the enemy will attempt to make the maximum impact on the network (see [79] for example). Also, as mentioned before, for the common scenario in which the enemy can simultaneously attack multiple arcs, enumerating failure scenarios consisting of "any k arcs" for some integer k becomes computationally intractable.

Bilinear Programming

As discussed in [61], the disjointly constrained bilinear programming problem formulation plays an important role in solving interdiction problems. These problems minimize a bilinear objective function over two polytopes, which constrain disjoint sets of variables that comprise bilinear terms in the objective function. Bilinear programs arise in a wide range of applications such as bimatrix games [64], quadratic assignment problems [37, 59], and global logistics systems [43] (see [58] for more applications of bilinear programming). Nonetheless, because the optimization of bilinear programs is strongly NP-hard [80], conventional local optimization methods do not guarantee global optimality. However, due to the well-known property that there exists a globally optimal solution among combinations of extreme points taken from each polytope (see [58, 89] for example), bilinear programming has received a great deal of attention in the global optimization area.

Konno [58] proposes a cutting plane algorithm for optimizing bilinear programs that employs *concavity cuts*, also known as *Tuy cuts*, in one of polytopes to yield an ε-optimum, at which the optimality gap is no more than a prescribed positive value $\varepsilon > 0$. A similar cutting plane method of Vaish and Shetty [101] is convergent to a globally optimal solution, but a finite convergence is not guaranteed. Using concavity cuts in conjunction with negative edge extensions (see [38]) and disjunctive cuts, Sherali and Shetty [89] design a finitely convergent cutting plane algorithm. Audet et al. [6] propose a finitely convergent branch-and-bound algorithm, in which the problem is reformulated to equivalent min-max and max-min problems and branching is performed based on the complementarity slackness conditions of these problems. This method is further strengthened by a cutting plane generation phase as a preprocessing step [1]. In addition to the above solution methods, bilinear programming problems can be solved via general concave minimization algorithms such as extreme point ranking [21,57,65,66,73,78,86,88,95,98], cutting plane [39, 54, 81, 99, 100, 109, 110], and branch-and-bound [11, 31, 50, 51, 93, 99] methods. We refer the reader to [8,27,102] for excellent comprehensive reviews of general bilinear programming methods.

Bilevel Programming

When the leader's and the follower's objectives are different, interdiction problems are closely related to *bilevel programming* (also known as *two-level programming* or *hierarchical optimization*) [7, 17, 33, 60], which is a mathematical programming version of a Stackelberg game [96]. In this game, two players sequentially make decisions in a noncooperative manner. The bilevel program consists of two sets of decision variables. One set, the *lower-level* variables, is constrained to be the solution of an optimization problem given the values of the other set of decision variables, called *upper-level* variables. Even for a linear bilevel program in which all objectives and constraints are linear, this problem is known to be strongly NP-hard [12, 46, 55]. Two notable solution methods for solving this type of problem are implicit enumeration of extreme points [14, 20] and reformulation using Karush–Kuhn–Tucker (KKT) optimality conditions [9, 33].

Similar to bilinear programs, it is known that an optimal solution to the linear bilevel problem can be found among the extreme points of the intersection of lower- and upper-level polytopes [8]. Candler and Townsley [20] propose an implicit enumeration of all bases for the lower-level problem by sequentially solving two linear programs: one is the lower-level problem given the values of upper-level variables, and the other is the problem having upper-level variables and lower-level variables associated with an optimal basis for the previous lower-level problem. Bialas and Karwan [15] design a more explicit enumeration method, in which the next best adjacent extreme point of the relaxed linear program is repeatedly searched until the values of lower-level variables are optimal to the lower-level problem.

3 Advances in Interdiction Models and Algorithms

In this section, we discuss four emerging aspects of network interdiction studies. We begin by analyzing extensions needed for stochastic network interdiction in Section 3.1, in the context of maximum flow interdiction. A review of cutting plane advances in solving stochastic shortest path (or maximum-reliability, as described in Example 2) is discussed in Section 3.2. Then, the shortest path interdiction in which the follower has a different perception about the arc lengths is presented in Section 3.3. Finally, we focus on the problem of handling continuous multicommodity network flow interdiction problems in Section 3.4, employing classic bilinear and bilevel programming concepts.

3.1 Stochastic Maximum Flow Interdiction

Cormican et al. [26] analyze stochastic optimization problems in discrete interdiction of maximum flow. In their work, the effectiveness of interdicting arc $j \in A$ is not deterministic: Interdiction either succeeds perfectly or fails completely. Letting $\tilde{I}_j$ be a random variable that equals to 1 if an attempted interdiction of $j \in A$ is successful and zero otherwise, the interdiction problem (14)–(18) now becomes:

$$\text{Minimize } \left\{E\left[h(\mathbf{x}, \tilde{\mathbf{I}})\right] | \mathbf{x} \in \mathbf{X}\right\}, \text{ where} \tag{27}$$

$$\begin{aligned}
h(\mathbf{x}, \tilde{\mathbf{I}}) = \text{Minimize} \quad & \sum_{j \in A} u_j(1 - \tilde{I}_j x_j)\beta_j && (28)\\
\text{subject to} \quad & \alpha_i - \alpha_j + \beta_k \geq 0 \quad \forall k = (i,j) \in A \setminus \{(t,s)\} && (29)\\
& \alpha_t - \alpha_s \geq 1 && (30)\\
& \boldsymbol{\beta} \geq \mathbf{0}\ . && (31)
\end{aligned}$$

Call this problem **StochOBJ** (as interdiction actions are reflected in the objective). An alternative and equivalent reformulation of this problem, called **StochCON**, is given by

$$\text{Minimize } \left\{E\left[g(\mathbf{x}, \tilde{\mathbf{I}})\right] | \mathbf{x} \in \mathbf{X}\right\}, \text{ where} \tag{32}$$

$$\begin{aligned}
g(\mathbf{x}, \tilde{\mathbf{I}}) = \text{Minimize} \quad & \sum_{j \in A} u_j \beta_j && (33)\\
\text{subject to} \quad & \alpha_i - \alpha_j + \beta_k \geq -\tilde{I}_k x_k \quad \forall k = (i,j) \in A \setminus \{(t,s)\} && (34)\\
& \alpha_t - \alpha_s \geq 1 && (35)\\
& \boldsymbol{\beta} \geq \mathbf{0}\ , && (36)
\end{aligned}$$

which is derived by relaxing the primal restriction $x_j \leq u_j(1-x_j)$ to $x_j \leq u_j$, but penalizing primal flow that takes place on interdicted arcs. This penalty term is given by $-\sum_{j \in A} y_j x_j \tilde{I}_j$. Because each unit of flow from s to t yields an objective reward of only one, and because the penalty for using a successfully interdicted arc also equals to one, there will exist an optimal solution in which no flow is transmitted over an interdicted arc.

One method for solving this problem lies in scenario decomposition. In this approach, one enumerates a set of Q scenarios $\mathbf{I}^1, \ldots, \mathbf{I}^Q$, that describe whether or not interdiction is successful on each arc. That is, $\mathbf{I}^q$ is an $|A|$-dimensional binary vector whose components I_j^q equal to 1 if interdiction would be successful on arc $j \in A$ and 0 otherwise, for each $q = 1, \ldots, Q$. The probability of scenario q occurring is given by $0 < r^q \leq 1$, for all $q = 1, \ldots, Q$, where $\sum_{q=1}^{Q} r^q = 1$. One might then use Benders decomposition to solve a master problem of the form:

$$\text{Minimize} \quad \sum_{q=1}^{Q} r^q \theta^q \tag{37}$$

$$\text{subject to} \quad \theta^q \geq \sum_{j \in A} \beta_j^\star u_j (1 - I_j^q) x_j \quad \forall q = 1, \ldots, Q, \ \boldsymbol{\beta}^\star \in \boldsymbol{\Psi} \tag{38}$$

$$\mathbf{x} \in \mathbf{X}\,, \tag{39}$$

where $\boldsymbol{\Psi}$ is the set of extreme points to (29)–(31) enumerated thus far in the decomposition process. The Benders subproblem would then take on the form (28)–(31) given a value of $\mathbf{x}$ obtained from the master problem solution. Also, one could derive a similar algorithm using (33)–(36), in which the dual feasible regions change depending on the scenario.

Naturally, this process suffers computationally from the vast number of required scenarios, and worse, the effectiveness of the obtained solution depends on the sampling of scenarios. A more sophisticated process derives lower and upper bounds on the optimal objective function value to StochOBJ. Define the deterministic problem, **DetCON**, as the modification to StochCON in which binary random variables $\tilde{I}_j$ in the right-hand side of (34) is replaced by their expected values $\overline{I}_k$, $\forall k \in A$. The optimal objective function value to DetCON is a lower bound on the true optimal objective of StochOBJ (this result is attributable to Jensen's inequality [16, 56]). Furthermore, letting $\hat{\mathbf{x}}$ be the solution to DetCON, $h(\mathbf{x}, \overline{\mathbf{I}})$ is an upper bound on the true optimal objective of StochOBJ.

Cormican et al. [26] analyze the case in which the probability of successfully interdicting arc $j \in A$ is given by $0 < p_j \leq 1$. (These interdiction probabilities are assumed to be independent of one another.) Using the foregoing bounding scheme, they observe that bounds provided by DetCON can be tightened by incorporating scenarios that partition the set of possible outcomes, and whose realization probabilities are determined according to the independent interdiction probabilities p_j, $\forall j \in A$. Hence, the foregoing Benders decomposition scheme is applied to generate lower and upper bounds on the optimal objective for StochCON, and these bounds are subsequently refined by creating finer partitions of the set of possible outcomes. This process is also extendable to the cases in which capacities are uncertain, and in which multiple attempts at interdicting an arc are permitted to take place.

3.2 Step Inequalities for Shortest Path Interdiction

In this subsection, we consider a recent strategy intended to improve the efficacy of solving integer programming models for shortest path interdiction problems. Because tighter linear programming (LP) relaxation is a crucial factor in improving the solution efficiency of integer programming problems, we discuss a class of valid inequalities for shortest path interdiction problems.

Morton et al. [72] and Pan and Morton [77] examine nuclear smuggling interdiction models and algorithms (see Example 2 in Section 1), in which the origin-destination pair is uncertain. Define a set Ω of scenarios (as done in Section 3.1) that enumerate all possible origin-destination pairs. For each scenario $\omega \in \Omega$, the origin is given by s^ω and the destination is given by t^ω, and the probability of realizing this origin-destination pair is given by ρ^ω. In these studies, individual links $(i,j) \in A$ can be interdicted; recall that interdiction decreases the evasion probability on link (i,j) from p_{ij} to q_{ij}, where $0 \leq q_{ij} < p_{ij} \leq 1$.

The Benders decomposition strategy used to solve these problems is still valid in this case. A set of master problem variables θ^ω are defined for each $\omega \in \Omega$, which represent the maximum evasion probability in scenario ω given the interdiction decisions, and an objective of

$$\text{minimize} \sum_{\omega \in \Omega} \rho^\omega \theta^\omega$$

is defined to minimize the expected maximum evasion probability. However, the resulting Benders master problem is an integer program whose linear relaxation can be quite loose. To combat this problem, a set of valid inequalities has been proposed to improve the integer programming model.

Morton et al. [72] consider a transformation of this problem to a bipartite network $G(N, A)$ with bipartition $N = \{N_1 \cup N_2\}$, $N_1 \cap N_2 = \emptyset$. This transformation is useful for the case in which evader sources and destinations are disjoint locations, and where interdiction can only be performed at nodes along some border that separates the possible origins and destinations. Hence, an evader will pass through exactly one border point on the way to the destination. For each $\omega \in \Omega$, the transformation to a bipartite network would thus compute the most reliable path from s^ω to a boundary node $k \in N$ (using only intermediate nodes on the "source" side of the border), and the most reliable path from k to the destination node d^ω (using only intermediate nodes on the "destination" side of the border). These paths can be computed by Dijkstra's algorithm after taking the logarithm of each arc reliability. Let γ_k^ω represent the product of these path reliabilities. If a sensor is placed on node k, then the evasion probability is given by $\gamma_k^\omega q_k$; else, it is given by $\gamma_k^\omega p_k$. After this transformation, one can construct a bipartite network in which $N_1 = \{\text{all possible origin nodes}\}$ and in which $N_2 = \{\text{all border checkpoints}\}$, and where the evasion probabilities (which now depend on the scenario ω) are given as above.

Given a set of sensor locations $\hat{\mathbf{x}}$, let $N_2(0) = \{k \in N_2$: no sensor is located at $k\}$, and $N_2(1) = \{k \in N_2$: a sensor is located at $k\}$. The evader then chooses a route corresponding with $\max\{\max_{k\in N_2(0)}\{\gamma_k^\omega p_k\}, \max_{k\in N_2(1)}\{\gamma_k^\omega q_k\}\}$. This strategy yields a rule that can be exploited in the form of valid inequalities. First, define

$$k^\star \in \operatorname{argmin}_{k\in N_2} \{\gamma_k^\omega q_k\} . \tag{40}$$

Note that the evader will discard any link $(s^\omega, \hat{k})$ such that $\gamma_{\hat{k}}^\omega p_{\hat{k}} \le \gamma_{k^\star}^\omega q_{k^\star}$, as the maximum evasion probability is always bounded below by $\gamma_{k^\star}^\omega q_{k^\star}$. Suppose there remain H arcs incident from s^ω after this preprocessing, and that these arcs connect s^ω to nodes $1, \ldots, H$ in N_2 such that $\gamma_1^\omega p_1 \ge \gamma_2^\omega p_2 \ge \cdots \ge \gamma_H^\omega p_H$, and for convenience, define $\gamma_{H+1}^\omega p_{H+1} = \gamma_{k^\star}^\omega q_{k^\star}$. The evader will check arc $(s^\omega, 1)$ to see if it is interdicted, and will traverse this link if not. Else, the evader will check arc $(s^\omega, 2)$, and so on, until a link is found that is not interdicted. If all arcs from s^ω to nodes $1, \ldots, H$ are interdicted, the evader will select the interdicted arc $(s^\omega, k^\star)$.

A *step inequality* is used to capture this decision-making process. The foregoing greedy algorithm can be captured by the logical inequality:

$$\theta^\omega \ge \gamma_1^\omega p_1 - (\gamma_1^\omega p_1 - \gamma_2^\omega p_2)x_{s^\omega 1} - \cdots - (\gamma_H^\omega p_H - \gamma_{H+1}^\omega p_{H+1})x_{s^\omega H} \ , \tag{41}$$

where $x_{s^\omega i}$ is a binary variable that equals to 1 if arc (s^ω, i) has been interdicted and 0 otherwise. Observe that if $x_{s^\omega 1} = x_{s^\omega 2} = \cdots = x_{s^\omega h} = 1$ for $0 \le h \le H$, then (41) reduces to

$$\begin{aligned}\theta^\omega \ge \gamma_{h+1}^\omega p_{h+1} - (\gamma_{h+1}^\omega p_{h+1} - \gamma_{h+2}^\omega p_{h+2})x_{s^\omega, h+1} \\ - \cdots - (\gamma_H^\omega p_H - \gamma_{H+1}^\omega p_{H+1})x_{s^\omega H} \ ,\end{aligned}$$

which yields a valid inequality as it implies that $\theta^\omega \ge \gamma_{h+1}^\omega p_{h+1}$. Furthermore, Morton et al. [72] observe that this inequality is valid for any subset $\{\kappa_1, \ldots, \kappa_m\} \subseteq \{1, \ldots, H+1\}$, as long as this subset includes the element $H+1$. Assuming that $\kappa_1 < \cdots < \kappa_m$, and defining $r_i^\omega = \gamma_{\kappa_i}^\omega p_{\kappa_i}$ for $i = 1, \ldots, m$, we then get the valid inequality

$$\theta^\omega \ge r_1^\omega - (r_1^\omega - r_2^\omega)x_{s^\omega \kappa_1} - \cdots - (r_{m-1}^\omega - r_m^\omega)x_{s^\omega \kappa_m} \ . \tag{42}$$

Given linear relaxation values of $\hat{\mathbf{x}}$, it is possible to determine a set $\{\kappa_1, \ldots, \kappa_m\} \subseteq \{1, \ldots, H+1\}$ that corresponds with a most-violated inequality (in terms of the left-hand side of (42) given $\hat{\mathbf{x}}$, minus the current master problem estimate of θ^ω) in polynomial time. Pan and Morton [77] then extend this logic to handle the case of nonbipartite graphs, showing also that their generalized inequality can also be separated in polynomial time.

3.3 Shortest Path Interdiction with Asymmetric Information

In this subsection, we consider the deterministic shortest path interdiction problem when the follower has a different perception than the leader about arc length data. To formulate the problem, let $\bar{c}_j$ and $\bar{c}_j + \bar{d}_j \; \forall j \in A$ denote the follower's perception of arc lengths before and after interdiction, respectively. One important consideration in solving these problems regards the action of the follower when alternative optimal solutions are present. Despite the fact that two solutions may be equally preferable to the follower, those solutions might have very different objectives for the leader. Bayrak and Bailey [10] model this problem under these two considerations when there exist alternative choices to the follower. First, assuming that the follower is *cooperative* when alternative solutions exist (i.e., the follower breaks ties among alternative optimal solutions in favor of the leader), the problem can be formulated as the following bilevel program:

$$\text{Maximize} \quad \sum_{j \in A} (c_j + d_j x_j) y_j \tag{43}$$

$$\mathbf{x} \in \mathbf{X}, \tag{44}$$

where

$$y = \text{argmin} \quad \sum_{j \in A} (\bar{c}_j + \bar{d}_j x_j) y_j \tag{45}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N \setminus \{s, t\} \end{cases} \tag{46}$$

$$y_j \geq 0 \; \forall j \in A \, . \tag{47}$$

Note that the follower's decision $\mathbf{y}$ is a solution of the shortest path problem that has a different objective function in (45).

Bayrak and Bailey [10] reformulate the leader's optimization problem as follows, using concepts from KKT conditions and strong duality.

$$\text{Maximize} \; \sum_{j \in A} (c_j + d_j x_j) y_j \tag{48}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{49}$$

$$\alpha_i - \alpha_j \leq \bar{c}_k + \bar{d}_k x_k \; \forall k = (i, j) \in A \tag{50}$$

$$\alpha_t - \alpha_s = \sum_{j \in A} (\bar{c}_j + \bar{d}_j x_j) y_j \tag{51}$$

$$y_j \geq 0 \; \forall j \in A \tag{52}$$

$$\mathbf{x} \in \mathbf{X} \, . \tag{53}$$

Note that (49) and (50) represent primal and dual feasibility conditions of the shortest path problem in (45)–(47). Constraint (51) enforces strong duality. The problem is now a nonlinear mixed-integer program due to the bilinear terms x_jy_j, $\forall j \in A$. When the interdiction actions are discrete, the problem can be linearized by substituting $w_j = x_jy_j$ and adding linearization constraints $w_j \le x_j$, $w_j \le y_j$, $w_j + (1 - y_j) \ge x_j$, and $w_j \ge 0$. We refer to [10] for an alternative linear formulation.

This formulation is optimistic for the leader, assuming cooperation on the part of the follower. In the presence of alternative shortest paths to the follower, the solution to (48)–(53) would select a KKT solution that yields the maximum objective value according to (48). In reality, using such a solution can be risky because the follower may not act cooperatively. Hence, one can assume the worst-case scenario when the follower has alternative choices. To find this pessimistic solution, consider the following problem given the leader's variables $\mathbf{x}$.

$$\text{Minimize} \quad \sum_{j \in A} \left[M(\bar{c}_j + \bar{d}_j x_j) + (c_j + d_j x_j) \right] y_j \tag{54}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{55}$$

$$y_j \ge 0 \ \forall j \in A\ . \tag{56}$$

When M is sufficiently large, the problem would yield an optimal solution based on $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$, but among all alternative optimal solutions with respect to $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$, would choose a solution that is optimal with respect to $\mathbf{c}$ and $\mathbf{d}$. Using the model (54)–(56) [10], formulate the pessimistic problem as follows.

$$\underset{\mathbf{x} \in \mathbf{X}}{\text{Maximize}}\, f(\mathbf{x}) - g(\mathbf{x})\ , \tag{57}$$

where

$$f(\mathbf{x}) = \text{ minimize} \quad \sum_{j \in A} \left[M(\bar{c}_j + \bar{d}_j x_j) + c_j + d_j x_j \right] \hat{y}_j \tag{58}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} \hat{y}_j - \sum_{j \in RS(i)} \hat{y}_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{59}$$

$$\hat{y}_j \ge 0 \ \forall j \in A \tag{60}$$

and

$$g(\mathbf{x}) = \text{ minimize} \quad \sum_{j \in A} M(\bar{c}_j + \bar{d}_j x_j) y_j \tag{61}$$

$$\text{subject to} \quad \sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{62}$$

$$y_j \ge 0 \ \forall j \in A\ . \tag{63}$$

Using the dual of the problem in $f(\mathbf{x})$ and rewriting $-g(x)$ as a maximization problem, the problem in (57) can be written as follows.

$$\text{Maximize} \quad \alpha_s - \alpha_t - \sum_{j \in A} M(\bar{c}_j + \bar{d}_j x_j) y_j \tag{64}$$

$$\text{subject to} \quad \alpha_i - \alpha_j \leq M(\bar{c}_k + \bar{d}_k x_k) + c_k + d_k x_k \ \forall k = (i,j) \in A \tag{65}$$

$$\sum_{j \in FS(i)} y_j - \sum_{j \in RS(i)} y_j = \begin{cases} 1 & \text{for } i = s \\ -1 & \text{for } i = t \\ 0 & \text{for } i \in N_0 \end{cases} \tag{66}$$

$$y_j \geq 0 \ \forall j \in A \tag{67}$$

$$\mathbf{x} \in \mathbf{X} \, . \tag{68}$$

This problem can be solved using the same standard linearization technique when $\mathbf{x}$ is binary.

3.4 Continuous Interdiction on Multicommodity Flow Networks

Finally, we discuss new developments in solving continuous interdiction problems. As demonstrated several times in this chapter, one often converts minimax models to minimization models by taking the dual of the inner maximization problem and combining the minimization of the interdictor's problem and the follower's dual problem. If interdiction terms appear on the right-hand side of the follower's problem, the resulting single minimization problem is a bilinear programming problem, whose bilinear terms can be linearized if at least one variable in each bilinear term is restricted to be binary-valued. However, if both variables in a bilinear term can take on fractional values, linearization constraints are no longer directly applicable.

In this subsection, we describe techniques to solve a continuous multicommodity network flow interdiction problem, which we term as **MFNIP**. Define a set of commodities K, where for each $k \in K$, S^k and D^k represent the set of supply nodes and demand nodes for commodity k, respectively, and define $N_0^k = N \setminus \{S^k \cup D^k\}$ to be the set of transshipment nodes for commodity k. Let s_l^k be the amount of supply for commodity $k \in K$ present at $l \in S^k$, and let d_l^k be the amount of demand for commodity $k \in K$ present at $l \in D^k$.

A flow profit of r_j^k is obtained for each unit of commodity $k \in K$ transmitted across arc $j \in A$. The sum of commodities that can be shipped on arc $j \in A$ is limited by u_j. However, the interdictor can reduce this amount to $u_j(1 - x_j)$ for any value of $x_j \in [0, 1]$. The interdictor is given a budget of B, and interdicting all of arc $j \in A$ incurs a cost of b_j. Assuming that partial interdictions are proportional to the cost of interdicting an entire arc, we define the interdiction space $\mathbf{X}$ as

$$\mathbf{X} \equiv \left\{ \mathbf{x} \in R^{|A|} : \sum_{j \in A} b_j x_j = B, \ 0 \leq x_j \leq 1 \ \forall j \in A \right\} . \tag{69}$$

The equality in (69) is due to the assumption that $\mathbf{b}^\top \mathbf{e} \geq B$ and the observation that an optimal solution to MFNIP exists in which the interdictor exhausts all of the interdiction budget.

Model MFNIP minimizes $\{\Lambda(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\}$, where

$$\Lambda(\hat{\mathbf{x}}) = \max \quad \sum_{j\in A}\sum_{k\in K} r_j^k y_j^k \tag{70}$$

$$\text{subject to} \quad \sum_{i\in FS(l)} y_i^k - \sum_{j\in RS(l)} y_j^k = 0 \qquad \forall k \in K,\ \forall l \in N_0^k \tag{71}$$

$$\sum_{i\in FS(l)} y_i^k - \sum_{j\in RS(l)} y_j^k = s_l^k \qquad \forall k \in K,\ \forall l \in S^k \tag{72}$$

$$\sum_{i\in FS(l)} y_i^k - \sum_{j\in RS(l)} y_j^k = -d_l^k \qquad \forall k \in K,\ \forall l \in D^k \tag{73}$$

$$\sum_{k\in K} y_j^k \le u_j(1-\hat{x}_j) \qquad \forall j \in A \tag{74}$$

$$y_j^k \ge 0 \quad \forall j \in A,\ \forall k \in K. \tag{75}$$

Taking the dual of the inner problem (70)–(75) and combining its dual minimization with the minimization of $\Lambda(\mathbf{x})$, we get the following nonlinear programming problem called **BLPI**, which is equivalent to MNFIP:

$$\text{Minimize} \quad \sum_{k\in K}\sum_{l\in S^k} s_l^k \alpha_l^k - \sum_{k\in K}\sum_{l\in D^k} d_l^k \alpha_l^k + \sum_{j\in A} u_j\beta_j - \sum_{j\in A} u_j x_j \beta_j \tag{76}$$

$$\text{subject to} \quad \alpha_i^k - \alpha_j^k + \beta_h \ge r_h^k,\ \forall k \in K,\ \forall h = (i,j) \in A \tag{77}$$

$$\alpha_l^k \ \text{ unrestricted } \forall k \in K,\ \forall l \in N, \quad \beta_j \ge 0\ \forall j \in A \tag{78}$$

$$\mathbf{x} \in \mathbf{X}. \tag{79}$$

Let $g(\mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the objective function given by (76), and let $\boldsymbol{\Theta}$ denote the dual feasible region constrained by (77) and (78). Note that fixing $\mathbf{x}$ yields a linear program in terms of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and vice versa. Hence, the problem BLPI has an optimal solution $(\mathbf{x}^\star, \boldsymbol{\alpha}^\star, \boldsymbol{\beta}^\star)$ such that $\mathbf{x}^\star$ and $(\boldsymbol{\alpha}^\star, \boldsymbol{\beta}^\star)$ are extreme points of $\mathbf{X}$ and $\boldsymbol{\Theta}$, respectively (see [89] for example).

Lim and Smith [61] prescribe an algorithm that partitions MFNIP into $|A|$ subproblems. The partitioning concept stems from the fact that for each extreme point $\hat{\mathbf{x}}$ of $\mathbf{X}$, there exists a single basic variable $\hat{x}_r$ such that $\hat{x}_r \in [0, 1]$, whereas all other variables are nonbasic at their lower bounds of zero or upper bounds of one. (This fact is due to the presence of a single constraint, aside from the simple bounding constraints on the x-variables.)

Suppose that we designate x_r as a basic variable (that is, the one variable that can take on any value between 0 and 1). Then the remaining nonbasic variables are confined to take on binary values if $\mathbf{x}$ is to be an extreme point of $\mathbf{X}$. Consider the following subproblem:

$$\begin{aligned}
\textbf{SP:}\ \text{Minimize} \quad & g(\mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) && (80)\\
\text{subject to} \quad & \mathbf{b}^\top \mathbf{x} = B && (81)\\
& 0 \le x_r \le 1 && (82)\\
& x_j \in \{0,1\} \quad \forall j \in A \setminus \{r\} && (83)\\
& (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Theta}. && (84)
\end{aligned}$$

From the equality in (81), we can substitute x_r in terms of the binary x-variables using the following transformation:

$$x_r = \frac{B - \sum_{j \in A\setminus\{r\}} b_j x_j}{b_r}. \tag{85}$$

Employing this substitution and noting that $B - b_r \le \sum_{j \in A\setminus\{r\}} b_j x_j \le B$, we can now formulate the following mixed-integer bilinear program, in which all quadratic terms involve at least one binary variable:

$$\begin{aligned}
\textbf{SP}(\boldsymbol{r})\textbf{:}\ \text{Minimize} \quad & \sum_{k\in K}\sum_{l\in S^k} s_l^k \alpha_l^k - \sum_{k\in K}\sum_{l\in D^k} d_l^k \alpha_l^k \\
& + \sum_{j\in A} u_j \beta_j - \sum_{j\in A\setminus\{r\}}^{n} u_j x_j \beta_j \\
& + \left(\frac{u_r \beta_r}{b_r}\right) \sum_{j\in A\setminus\{r\}} b_j x_j - \frac{B u_r \beta_r}{b_r} && (86)\\
\text{subject to} \quad & \sum_{j\in A\setminus\{r\}} b_j x_j \ge B - b_r && (87)\\
& \sum_{j\in A\setminus\{r\}} b_j x_j \le B && (88)\\
& x_j \in \{0,1\} \quad \forall j \in A \setminus \{r\} && (89)\\
& (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Theta}. && (90)
\end{aligned}$$

Using linearization to eliminate the nonlinear terms $x_j\beta_j$ with w_j $\forall j \in A\setminus\{r\}$, and $x_j\beta_r$ with v_j $\forall j \in A \setminus \{r\}$, we have the following linear mixed-integer program.

$$\begin{aligned}
\textbf{MILP}(\boldsymbol{r})\textbf{:}\ \text{Minimize} \quad & \sum_{k\in K}\sum_{l\in S^k} s_l^k \alpha_l^k - \sum_{k\in K}\sum_{l\in D^k} d_l^k \alpha_l^k \\
& + \sum_{j\in A} u_j \beta_j - \sum_{j\in A\setminus\{r\}} u_j w_j \\
& + \left(\frac{u_r}{b_r}\right) \sum_{j\in A\setminus\{r\}} b_j v_j - \frac{B u_r \beta_r}{b_r} && (91)\\
\text{subject to} \quad & \text{Constraints (87)} - \text{(90)} && (92)
\end{aligned}$$

$$w_j - \beta_j \leq 0 \quad \forall j \in A \setminus \{r\} \tag{93}$$
$$w_j - \overline{\beta}_j x_j \leq 0 \quad \forall j \in A \setminus \{r\} \tag{94}$$
$$\beta_r + \overline{\beta}_r x_j - v_j \leq \overline{\beta}_r \quad \forall j \in A \setminus \{r\} \tag{95}$$
$$v_j \geq 0 \quad \forall j \in A \setminus \{r\}, \tag{96}$$

where $\overline{\beta}_j$ is an upper bound on β_j, $\forall j \in A$.

Observe that in the linearization of $v_j = \beta_r x_j$, $\forall j \in A \setminus \{r\}$, we need not state the upper bounding constraints on v_j, as these variables appear only in the objective function multiplied by a positive number, and in the bounding constraints. Hence, the equations $v_j \leq \overline{\beta}_r x_j$ and $v_j \leq \beta_r$ that would normally be included in a linearization are removed in this formulation. The typical lower bounding constraints on w_j, $\forall j \in A \setminus \{r\}$, are also not present in the formulation, as they too will not be binding at optimality.

Given a solution to $\mathrm{SP}(r)$, we recover the value of x_r according to (85). Let $[\mathbf{x}(r), \boldsymbol{\alpha}(r), \boldsymbol{\beta}(r)]$ and $z(r)$ denote an optimal solution and optimal objective value of $\mathrm{SP}(r)$, respectively. Then, $r^\star \in \operatorname{argmin}_{r \in A}\{z(r)\}$ yields an overall optimal solution $\mathbf{x}(r^\star)$ and objective value $z(r^\star)$. Lim and Smith [61] pursue computational details regarding the implementation of this algorithm, including the derivation of tight $\overline{\beta}$-values, and the description of heuristics tailored for this problem and its variant in which the x-variables are discrete.

4 Fortification in Network Design

In this section, we consider the problem of building or fortifying a multi-commodity flows network to defend against network interdiction in various scenarios. As mentioned in Section 1, this problem takes the form of a three-stage, two-player game, in which the operator acts first to construct a network and transmit an initial set of flows through the network. The interdictor acts next to destroy a set of constructed arcs, and the operator acts last to transmit a final set of flows in the network.

Smith et al. [91] prescribe optimal network design algorithms for three different profiles of interdiction: an interdictor acting optimally to minimize the operator's maximum profits obtained from transmitting flows, or destroying arcs based on capacities and based on initial flows. The first profile assumes that the interdictor knows all the information that the operator has, whereas only partial information is exposed to the interdictor in the two latter scenarios. (Recall that this information asymmetry is presented in Section 3.3 in the context of shortest path network interdiction.) We discuss their developments in this chapter, with a focus on the case in which both players have full information regarding the network data and act optimally.

4.1 Problem Description

We first define notation for the current problem and describe the three interdiction scenarios. Each arc $i \in A$ is associated with a nonnegative construction cost c_i, and the operator is limited by a budget of C for constructing arcs in the network. Furthermore, define K to be the set of commodities, and let d_k^j denote the supply/demand of commodity $k \in K$ at node $j \in N$. Without loss of generality, we assume that $\sum_{j \in N} d_k^j = 0,\ \forall k \in K$. We are given a per-unit flow profit p_k^i for transmitting a unit of commodity k over arc i for each $i \in A$ and $k \in K$. This value includes the (nonpositive) per-unit flow cost on arc i, plus (positive) revenue for successfully shipping a unit of commodity k if it enters a destination node of k. This reward is subtracted from the flow profit if arc i exits a destination node for k.

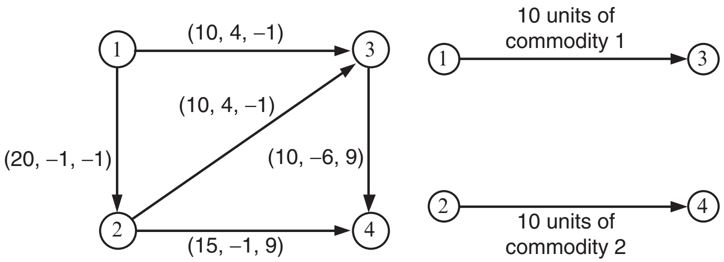
Define the following set of decision variables. Let x_i, $\forall i \in A$, be a binary decision variable that equals to 1 if arc i is constructed and 0 otherwise. For the flow decision variables, we let f_i^k and g_i^k, $\forall i \in A$, $\forall k \in K$, represent the flow of commodity k on arc i before and after interdiction, respectively. Also, let $w_i \in [0, 1]$, $\forall i \in A$, represent the remaining percentage of arc i's capacity after interdiction. Although w_i is determined by the interdictor, we view it as a decision variable induced by the operator's choice of x-variables.

It is possible that there may not exist a feasible multicommodity flow in the network (especially after interdiction), or that it might not be cost-effective to route all of the requested demands for some commodity. Thus, we create dummy arcs in the network to allow some origin nodes to send less than their supplies and destination nodes to receive less than their demands. These dummy arcs have zero construction and flow profits (unless some disposal or shortage costs are appropriate) and large enough interdiction costs and capacities to ensure that they cannot be destroyed by the interdictor.

The objective function maximizes some convex combination of the profit obtained from transmitting flows across the network before and after interdiction, where profit is measured by revenues gained from successful shipment of goods minus arc construction costs. Suppose that $100\rho\%$ of flows occur before interdiction for some $\rho \in [0, 1]$. Then our total profit is ρ times our pre-interdiction flow profits plus $(1 - \rho)$ times our post-interdiction flow profits, minus the arc construction costs.

We consider three interdiction scenarios. In case 1, the interdictor acts to (optimally) minimize the operator's maximum post-interdiction flow profit. In case 2, the interdictor destroys arcs having the largest capacity until his budget is exhausted or until no arcs remain. Finally, in case 3, the interdictor destroys arcs on which the largest pre-interdiction flows exist.

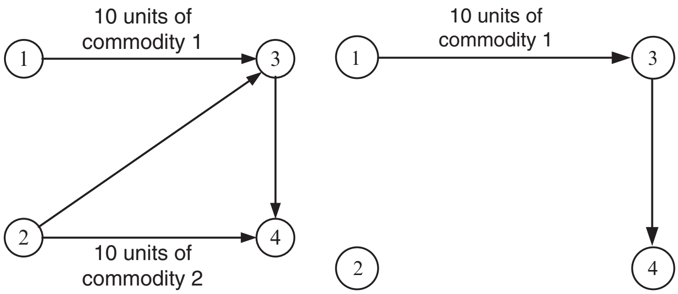
To illustrate these interdiction schemes, suppose that the network operator can build arcs on the network depicted in Fig. 5. There exist two commodities in the problem whose origin-destination pairs are (1, 3) and (2, 4). Each origin (destination) node can supply (receive) at most 10 units. Recall that we permit shortages in supplies and demands by drawing dummy arcs from node 1 to node 3, and from node 2 to node 4. These arcs are omitted in Fig. 5(a) for ease of readability.



(a) Example network design instance.

(b) Optimal network design and flows without interdiction.

Figure 5. Network topology and optimal solution without interdiction



(a) Optimal design and pre-interdiction flows.

(b) Optimal post-interdiction flows.

Figure 6. Network design and flows under case 1 interdiction

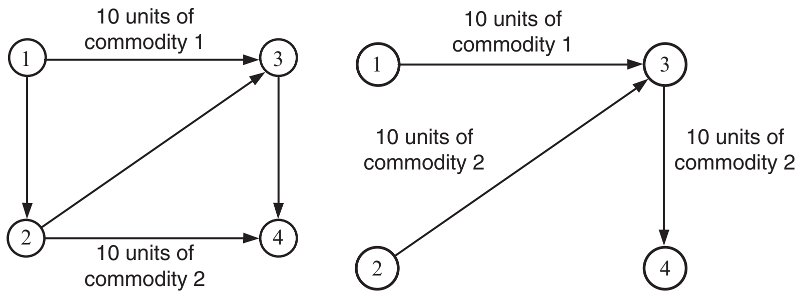
The arcs are labeled by their capacity, unit flow profit of commodity 1, and unit flow profit of commodity 2, i.e., $(u_i, p_i^1, f_p^2)\ \forall i = 1, \ldots, 5$. For this instance, suppose that the operator achieves a revenue of 5 for each unit of commodity 1 delivered, and a revenue of 10 for each unit of commodity 2 delivered, while the unit flow cost is 1 for any commodity on any arc. Assume that each arc can be constructed by the operator and destroyed by the interdictor with a common cost of 10, i.e., $c_i = b_i = 10\ \forall i = 1, \ldots, 5$. Let the operator's budget be $C = 50$ and the interdictor's budget be $B = 20$. Finally, let the profit weight be $\rho = 0.5$. If no interdiction action is taken, the optimal design and flows would be given by those in Fig. 5(b), yielding a profit of 110 $(= 130 - 20)$ in this scenario.

Figure 6 depicts the optimal network design for the first case, where the interdictor optimally destroys arcs to minimize the operator's flow profit after interdiction. While the network design is same as the one in case 2, the interdictor optimally disrupts arcs (2, 3) and (2, 4) by observing that commodity 2 yields a greater profit to the network operator than commodity 1 does. Then, the operator can send post-interdiction flows only on arc (1, 3) as in Fig. 6(b).

The profit is 45 (= (0.5)(130) + (0.5)(40) − 40). (The interdictor also has an alternative optimal solution in which arcs (2, 4) and (3, 4) are destroyed.) Despite the fact that arc (3,4) does not carry any pre- or post-interdiction flow, the operator must build this arc to prevent the interdictor from simply destroying arcs (1,3) and (2,4), which, without the presence of (3,4), would leave the operator with no post-interdiction flows. (The operator also has an alternative optimal solution in which only arcs (1,3) and (2,4) are constructed and in which no post-interdiction flows exist.)

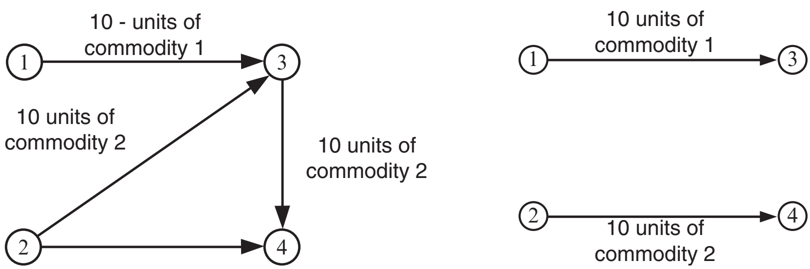
For the second case in which interdiction is made based on the arc capacities, Fig. 7(a) depicts the optimal network design and pre-interdiction flows, and Fig. 7(b) depicts the optimal post-interdiction flows. From this network design, the interdictor would destroy arcs (1,2) and (2,4), as they have the largest capacities. Note that the operator needs to alter the initial flows of commodity 2 using arcs (2,3) and (3,4) after interdiction. The overall profit is now reduced to 75 (= (0.5)(130) + (0.5)(120) − 50).

Finally, in the third case, the interdictor will destroy arcs having the largest initial flows. Figure 8 illustrates the optimal network design and flow patterns. Note that the initial flow on arc (1, 3) is slightly smaller than those on arcs



(a) Optimal design and pre-interdiction flows. (b) Optimal post-interdiction flows.

Figure 7. Network design and flows under case 2 interdiction



(a) Optimal design and pre-interdiction flows. (b) Optimal post-interdiction flows.

Figure 8. Network design and flows under case 3 interdiction

(2, 3) and (3, 4) by some $\varepsilon > 0$ units so that the interdictor can be induced to destroy the latter two arcs as desired by the network operator. The network operator's profit is $85 - 2\varepsilon$ $(= (0.5)(120 - 4\varepsilon) + (0.5)(130) - 40)$.

4.2 Case 1: Optimal Interdiction

First, suppose that the interdictor optimally disrupts arcs so as to minimize the operator's profit. The interdictor has complete information of the network design, including arc capacities, flow profits, and demands. Given a network design x, therefore, the interdictor solves the continuous multicommodity flow network interdiction problem. Using the formulation as a bilinear program (BLP) presented in Section 3.4, the interdiction problem is given by

$$\text{Minimize} \quad \sum_{k \in K} \sum_{j \in N} d_j^k \alpha_j^k + \sum_{i \in A} (u_i x_i) w_i \beta_i \tag{97}$$

$$\text{subject to} \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \boldsymbol{\Theta} \tag{98}$$

$$\mathbf{w} \in \mathbf{W}\,, \tag{99}$$

where $\boldsymbol{\Theta}$ is the dual feasible region of the operator's multicommodity flows problem, and where $\mathbf{W} = \{\mathbf{w} \in R^{|A|} : \sum_{i \in A} b_i(1 - w_i) = B,\ 0 \le w_i \le 1, \forall i \in A\}$. Note that the x-variables appear only in the objective function. As discussed in the previous section, a global optimum of this problem can be found among pairs of extreme points from respective feasible regions. Let $\boldsymbol{\Theta}_E$ and $\mathbf{W}_E$ denote sets of extreme points of $\boldsymbol{\Theta}$ and $\mathbf{W}$, respectively. Accordingly, let $\boldsymbol{\Pi} \subseteq \boldsymbol{\Theta}_E \times \mathbf{W}_E$ be the set of pairs of such extreme points. Furthermore, let $\phi_{\boldsymbol{\pi}}(\mathbf{x})$ denote the objective function value of (97) at $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ given $\mathbf{x}$. Then, the network design problem can be formulated as the following bilevel program.

$$\begin{aligned} \text{Maximize} \quad & \rho \sum_{k \in K} \sum_{i \in A} p_i^k f_i^k + (1 - \rho)\min\{\phi_\pi(x) : \pi \in \mathit{\Pi}\} \\ & \qquad - \sum_{i \in A} c_i x_i \end{aligned} \tag{100}$$

$$\text{subject to} \quad \sum_{i \in A} c_i x_i \le C \tag{101}$$

$$\sum_{i \in FS(j)} f_i^k - \sum_{i \in RS(j)} f_i^k = d_j^k \quad \forall k \in K \quad \forall j \in N \tag{102}$$

$$\sum_{k \in K} f_i^k \le u_i x_i \quad \forall i \in A \tag{103}$$

$$f_i^k \ge 0 \quad \forall i \in A \quad \forall k \in K. \tag{104}$$

$$x_i \in \{0, 1\} \quad \forall i \in A. \tag{105}$$

The objective (100) minimizes the pre-interdiction flow costs weighted by ρ, plus the post-interdiction flow costs weighted by $(1 - \rho)$, minus the arc construction costs. The arc construction budget constraint is given in (101), and the flow constraints before interdiction are given in (102)–(104).

Observing the linearity of $\phi_{\boldsymbol{\pi}}(\mathbf{x})$ with respect to $\mathbf{x}$, we have that $\min\{\phi_{\boldsymbol{\pi}}(\mathbf{x}) : \boldsymbol{\pi} \in \mathbf{\Pi}\}$ is concave with respect to $\mathbf{x}$. Therefore, we can prescribe a cutting-plane algorithm (or outer-linearization method), which we call **BCPA**, that generates Benders cuts in an iterative fashion. At iteration j of BCPA, we have the following master problem.

$$\begin{aligned} \text{Maximize} \quad & \rho \sum_{k \in K} \sum_{i \in A} p_i^k f_i^k + (1-\rho) z - \sum_{i \in A} c_i x_i && (106) \\ \text{subject to} \quad & (100)\text{–}(105) && (107) \\ & z \leq \phi_{\boldsymbol{\pi}}(\mathbf{x}) \quad \forall \boldsymbol{\pi} \in \mathbf{\Pi}_j \ , && (108) \end{aligned}$$

where $\mathbf{\Pi}_j \subseteq \mathbf{\Pi}$ is the set of $\boldsymbol{\pi}$-vectors obtained in prior iterations by solving the interdiction problem. The BCPA algorithm can be summarized as follows.

Algorithm BCPA

Step 0. Set $\mathbf{\Pi}_1 = \emptyset$ and $j = 1$.
Step 1. Solve the problem (97)–(99) to obtain a solution $\mathbf{x}^j$ and $\mathbf{z}^j$.
Step 2. Given $\mathbf{x}^j$, solve the problem (97)–(99) to obtain a solution $\boldsymbol{\pi}$ and its objective value $\phi_{\boldsymbol{\pi}}(\mathbf{x}^j)$.
Step 3. If $z^j \leq \phi_{\boldsymbol{\pi}}(\mathbf{x}^j)$, then $\mathbf{x}^j$ is optimal and stop. Else, put $\mathbf{\Pi}_{j+1} = \mathbf{\Pi}_j \cup \{\boldsymbol{\pi}\}$, increment $j \leftarrow j+1$, and return to Step 1.

4.3 Case 2: Interdiction Based on Capacity

Suppose that the interdictor repeatedly destroys arcs having the largest capacity until the budget B is exhausted. For our initial discussion, we assume that all arc capacities are unique (we discuss the implication of this assumption further at the end of this subsection). Hence, we can order the arc indices $i = 1, \ldots, |A|$ so that $u_i < u_{i+1}$ $\forall i = 1, \ldots, |A|-1$, and so the interdictor will prefer to destroy arc i before destroying $i+1$. Note that in general, this overall ordering can be based on any input data criteria.

Smith et al. [91] present a model that makes use of the fact that there will exist only one w-variable that can be fractional, as the interdictor is essentially solving a linear knapsack problem. They define binary decision variables δ_i, $\forall i \in A$, equal to one if and only if all constructed arcs with an index smaller than i are completely destroyed, and all constructed arcs with an index greater than arc i are not affected by the interdictor. Arc i itself may be completely or partially interdicted or unaffected by the interdictor.

We assume that the interdictor exhausts the entire interdiction budget. This assumption forces us to build at least enough capacity so that the interdictor can destroy B units; we handle this assumption by adding a dummy arc between two dummy nodes disconnected from the rest of the network.

This arc will have a zero arc construction cost, zero flow profit, any arbitrary capacity, and an interdiction cost of B. Then, the constraints that govern the interdictor's decision can be written as follows.

$$\sum_{i \in A} \delta_i = 1 \tag{109}$$

$$w_i \leq \sum_{h=1}^{i} \delta_h \quad \forall i \in A \tag{110}$$

$$w_i \geq x_i - \sum_{h=i}^{|A|} \delta_h \quad \forall i \in A \tag{111}$$

$$\sum_{i \in A} b_i(x_i - w_i) = B \tag{112}$$

$$\delta_i \in \{0,1\} \quad \forall i \in A. \tag{113}$$

Constraint (109) requires that exactly one variable serves as the dividing point, such that all arcs having a higher capacity than u_i are destroyed (enforced by (110)), and all arcs having a smaller capacity than u_i are not interdicted at all (enforced by (111)).

4.4 Case 3: Interdiction Based on Flow

Finally, suppose that the interdictor destroys the arcs having the largest pre-interdiction flows. Despite the similarity with the previous case in the interdictor's greedy strategy, the interdiction decision does not depend on a simple set of binary decision variables $\mathbf{x}$, and thus the optimization model in the previous section must be modified for this case.

Once again, we determine an index r such that arc r may be partially destroyed, implying that every arc having a greater flow than the flow on arc r must be completely destroyed, and all arcs with a smaller flow than the flow on arc r cannot be interdicted. Unlike the previous case, we do not define a decision variable to determine the identity of r, but instead we must solve one integer program for each possible value that r can take. Note that arcs having a larger flow than the flow on arc r will be completely destroyed whereas those with smaller flows will survive. Define $\varepsilon > 0$ as an arbitrarily small constant. Then, we replace (109)–(113) with the following constraints:

$$\sum_{k \in K} f_j^k - \sum_{k \in K} f_r^k \leq M_{jr}(x_j - w_j) - \varepsilon x_j \; \forall j \in A, \; j \neq r \tag{114}$$

$$\sum_{k \in K} f_j^k - \sum_{k \in K} f_r^k \geq -M_{rj}(1 - (x_j - w_j)) + \varepsilon x_j \; \forall j \in A, \; j \neq r \tag{115}$$

$$\sum_{i \in A} b_i(x_i - w_i) = B \tag{116}$$

$$0 \le w_r \le 1 \quad \forall r \in A \tag{117}$$

$$w_i \in \{0,1\} \quad \forall i \in A - \{r\} \tag{118}$$

$$x_r = 1, \tag{119}$$

where $M_{ij} = u_i + \varepsilon$. The addition of the constraints in (114)–(119) captures the flow-based greedy interdiction model, assuming that the interdictor completely destroys all arcs having more flow than the flow on r and does not interdict arcs having less flow than arc r. By the disjunction of (114) and (115), the flow of arc j must satisfy either $\sum_{k\in K} f_j^k \le \sum_{k\in K} f_r^k - \varepsilon x_j$ or $\sum_{k\in K} f_j^k \ge \sum_{k\in K} f_r^k + \varepsilon x_j$. This guarantees the uniqueness of the interdictor's solution (the implications of which are discussed in more detail below). Note that if $x_j = 0$, then $w_j = 0$ as well. The right-hand side of (114) would then be zero, and because $\sum_{k\in K} f_j^k$ must be zero, this inequality is valid. The right-hand side of (115) is $-M_{rj}$, so this inequality remains valid as well even if $\sum_{k\in K} f_r^k = u_r$. Now assume that $x_j = 1$. If there exists more initial flow on arc j than on r (i.e., $\sum_{k\in K} f_j^k \ge \sum_{k\in K} f_r^k + \varepsilon x_j$), then as $x_j = 1$, we have by (114) that arc j must be interdicted (by setting $w_j = 0$). Similarly, (115) forces $w_j = x_j$ if there is less flow on j than r (i.e., $\sum_{k\in K} f_j^k \le \sum_{k\in K} f_r^k - \varepsilon x_j$).

In order to find an optimal solution, we optimize a network fortification model in which the interdictor's decisions are governed by (114)–(119) for each $r = 1, 2, \ldots, |A|$. Out of these $|A|$ solutions, we select the best solution as the global optimum. More details about this process can be found in [91].

5 Future Research

This chapter offers an introduction to network interdiction modeling and algorithms and provides a discussion of new lines of research that have been initiated in this field over the past decade. However, we believe that this collection of work is only the start of mathematical investigation in this field.

Many interdiction studies have traditionally been limited to those problems for which linearization constraints can be applied to eliminate troublesome nonlinear terms. However, there seem to exist numerous applications in which interdiction actions are continuous and in which a simple application of linearization constraints is not sufficient to solve the problem. We discussed one study in Section 3.4 (regarding the work of Lim and Smith [61]) in which linearization is not applicable, but even their study captures only knapsack-constrained interdiction decisions. Future research in the field must address the case in which interdiction decisions are more general than the cases described in this chapter. We anticipate that lessons gleaned from bilinear programming and global optimization theory will be required to obtain the most effective algorithms for this class of problems.

We have also assumed in this chapter that information in the network game is symmetric. Naturally, this is not always the case. Although asymmetric situations do arise in some interdiction studies (see [10, 72], in addition to our discussion in Section 3.3), most of these studies seem to regard an incidental difference in the perception of data. We believe that in many cases, it is possible to alter our opponent's perception of data, in order to induce an ineffective action. For instance, in the case of shortest path interdiction, the network operator could potentially have the option of fortifying network arcs at a substantial cost, or perhaps simply creating the *perception* that the network has been fortified at certain areas. Substantially different optimization challenges arise in our preliminary analyses in this case, which will also necessitate the investigation of a new line of optimization studies and will probably require the development of sophisticated and effective heuristic techniques.

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Game Theoretical Approaches in Wireless Networks

Manki Min

Department of Electrical Engineering and Computer Science, South Dakota State University, Brookings, South Dakota, USA
manki.min@sdstate.edu

Abstract In this chapter, we review game theoretical approaches in wireless networks, mainly the issues of power control, cooperation between mobile terminals, security, and radio channel access control.

Key words: game theory, wireless networks, power control, cooperation, security, radio channel access control

1 Introduction

The promising wireless networking environment has been studied extensively in various aspects. It has intrinsic features such as limited resource (computational resource, network resource, battery power, etc.) and mobility of terminals in the network. These limitations have been setbacks to direct applications of traditional (wired-network) approaches to wireless networks. Many problems are NP-hard, and as a result, heuristics and approximation schemes have been developed to get good (but may not be optimal) solutions to the problems. Some of the problems are tackled with optimization techniques such as integer programming or global optimization. However, as opposed to the situation in optimization where no interactions of players are assumed, in reality the interactions of players may influence the choices for an equilibrium or a stable operating point of systems. We focus on game theoretical approaches to the problems of power control, cooperation between hosts, and channel access control. Several variations are also discussed briefly.

The rest of this chapter is as follows. In Section 2, problems in wireless networks are identified, and in Section 3, game theoretical approaches to the problems are discussed. Section 4 concludes this chapter.

2 Problems in Wireless Networks

In this section, we introduce problems in game theoretic wireless network fields and review some related works not using game theoretic approaches.

2.1 Power Control

In wireless networks, most devices work on their own batteries. The limitation of battery power gets worse when devices communicate because much more power is consumed to transmit than to compute. As a result, optimal power consuming networks have been studied for various scenarios such as broadcasting, multicasting, routing, and target coverage (for wireless sensor networks). The power control problem for wireless data is to assign each device's transmission power to promote the quality and efficiency of the wireless networking system.

Ariyavisitakul [6] studied signal-to-interference ratio (SIR) based power control in a Code-Division Multiple Access (CDMA) system (see Figure 1). The author pointed out that uplink power control models based on absolute signal strength at every base station are impractical. Instead, the author proposed a SIR-based periodic power control model and showed its capability of responding to any change of interference by each user in the system. The numerical results show that a CDMA system using power control based on SIR has an advantage of higher performance than a system with power control based on signal strength. The main advantage is its capability to make use of interference from other coverage areas to stabilize individual control processes. Research in [53, 54] confirms the improved performance of SIR-based power control models. Reference [54] provides the simulation results showing stepwise removal algorithm using SIR outperforms the fixed transmitting power. Reference [53] proves that iterative power control algorithm converges both synchronously and asynchronously when outdated or incorrect interference

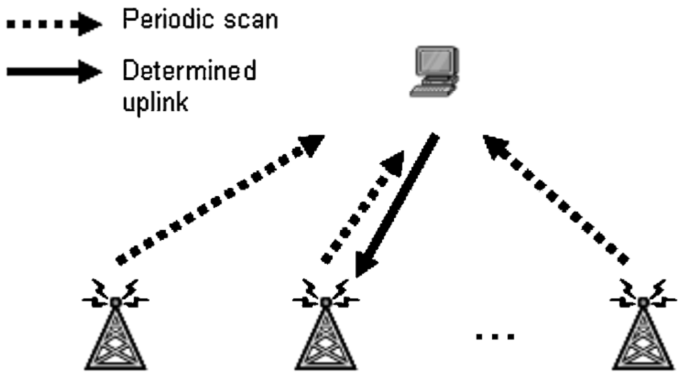

Figure 1. Feedback power control

measurements are used. Moreover, SIR-based power control methods converge to a fixed unique point at which total transmitting power is minimized.

Game theoretical approaches can be found in many papers in the literature [1–3, 5, 18–21, 40–43].

2.2 Cooperation and Security

A wireless ad hoc network consists of wireless mobile hosts without any infrastructure or any central administration. Such a networking environment requires each mobile host in the network to cooperate by forwarding received packets. Many ad hoc routing protocols have been proposed, assuming absence of selfish nodes that do not cooperate for the network operation [13, 16, 23, 35–38, 46]. However due to the high consumption of battery power for transmissions, some hosts may act selfishly, in other words instead of cooperating they may try to preserve their own battery as much as possible. The cooperation problem is to design a cooperation enforcing mechanism in the presence of selfish nodes.

References [29, 32] study, through simulations, the impact of selfish behaviors on the network performance in terms of network throughput, average delay of correctly delivered packets, and routing overhead. The simulation results show even in the presence of a small percentage of selfish nodes, the network performance suffers from a severe degradation. The observation provides the necessity of a cooperative security scheme through the collaboration between nodes in order to detect selfish behaviors.

Buchegger and Boudec [9,10] applied the *grudgers* concept in Dawkins [15] to cooperative routings. Dawkins divided birds into three categories: *sucker*, *cheats*, and *grudgers*. Suckers always help (groom parasites off from other birds' heads), and cheats make other birds help them but do not help others. Grudgers are originally suckers but they bear a grudge against cheats and eventually do not help cheats. The CONFIDANT protocol in [9] uses a reputation system to locally manage the feedbacks on misbehaviors, and once noncooperative behavior exceeding threshold value has been detected, it is informed to other nodes and eventually the misbehaving nodes are excluded from the routing (see Figure 2). The system consists of reputation system, trust manager, monitor, and path manager. Reference [10] provides performance analysis of the protocol proposed in [9] by extending Dynamic Source Routing (DSR) protocol [23]. The simulation results show the network with up to 60% of misbehaving nodes performs almost as good as the network without any misbehaving node using the proposed protocol. The performance metrics of throughput, overhead, and utility clarifies the significance of cooperative mechanism on routing. Yang *et al* [52] proposed a token-based cooperative solution applied to Ad-Hoc on Demand Distance Vector (AODV) protocol [38]. Every network operation requires a periodically renewed token, and the renewal of token is done by the collaborative monitoring at local neighbors.

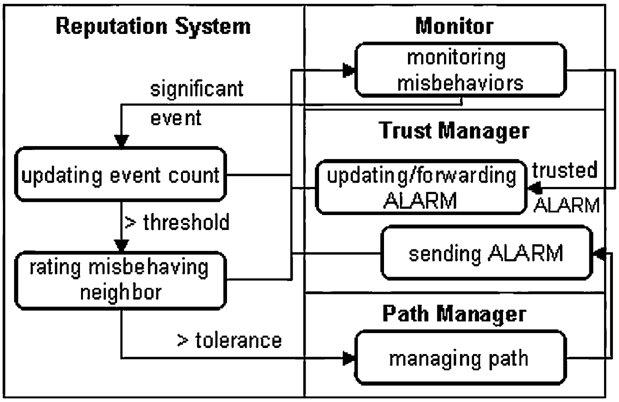

Figure 2. CONFIDANT protocol

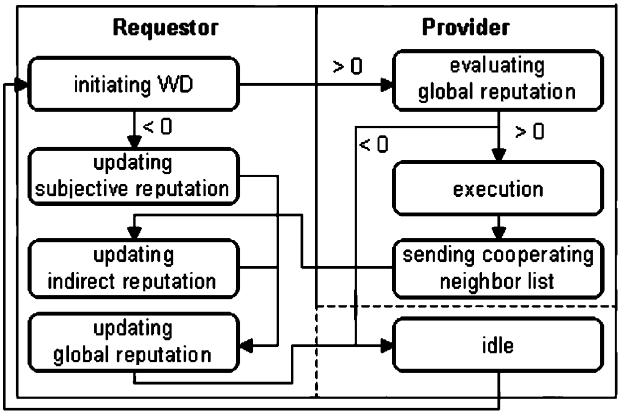

Figure 3. CORE protocol

Michiardi and Molva [31] proposed a generic cooperation enforcing mechanism based on reputation system. The authors introduced three types of reputation concepts: subject reputation, indirect reputation, and functional reputation. The indirect reputation will be updated only when indirect information about cooperating neighbors has been received in order to prevent denial-of-service type attacks. In other words, the indirect reputation is only used to increase the reputation of a neighbor. The functional reputation enables the calculation of a global value of reputation that takes into account different aspects/observations. The request of a node with the negative reputation rating factor (which is a combination of the reputation concepts) will be denied of the execution (see Figure 3). Buttyán and Hubaux [11] introduced the concept of *nuglet* protected by tamper-resistant security module. The nuglet counter counts the number of forwarding packets minus the number of requests to send a packet. In order for a node to send a packet, the nuglet counter must be positive, in other words the node must have contributed enough to its network community. The proposed scheme does not require explicit collaboration among nodes, but it ensures its proper operation by integrating public-key infrastructure into the packet forwarding mechanism.

Crowcroft *et al* [14] proposed an incentive model for cooperation enforcement in mobile ad hoc networks. The model provides a mechanism to allow users to make choice of the flows on potential routes in a decentralized fashion. The decisions are based on the congestion prices (consisting of power and bandwidth factors) of the relevant nodes and the willingness-to-pay parameter. In order to encourage the cooperation, whenever a node acts as a transit node for other nodes, the node gains credit balance, which can be used for payment later. The routes for series of traffic are assumed to be determined by ad hoc routing such as AODV or DSR. The total flow rate generated by user s is given by the following equation:

$$x_s(t) = \sum_{r \in R^S(s)} y_r(t) = \frac{w_s(t)}{\min_{r \in R^S(s)} \sum_{j \in r} \mu_{jr}(t)}. \tag{1}$$

The credit balance of the user s is used to adjust the willingness-to-pay parameter of the user s as follows:

$$w_s(t) = \alpha_s b_s(t), \text{for some parameter } \alpha_s > 0. \tag{2}$$

And the credit balance itself is discounted over time as follows:

$$\frac{db_s}{dt} = -\beta(b_s(t) - 1) - w_s(t) + \sum_{r:s \in r} y_r \mu_{sr}(t). \tag{3}$$

The variables in the above equations are summarized in Table 1. The paper includes simulation results for 10-node network under three circumstances: static, dynamic (where nodes join or leave), and mobile (where some nodes move).

Game theoretical approach has been used in the literature [30, 33, 34, 44, 45, 47].

Table 1. Variables Used in [14]

$x_s(t)$:	the total flow rate of user s at time t
$R^S(s)$:	the subset of routes that originates at source s
$y_r(t)$:	the flow along the route r at time t
$w_s(t)$:	the willingness-to-pay parameter of user s at time t
$\mu_{jr}(t)$:	the congestion price that user j charges for forwarding the unit flow along the route r at time t; the price consists of power and bandwidth prices and is defined differently depending on j's role (source, transit, or destination) on r
$b_s(t)$:	the credit balance of user s at time t
y_r:	the traffic flow along the route r

2.3 Channel Access Control

Wireless communication shares the communication medium, and this feature puts importance on the radio channel access control problem to guarantee the quality of communication. Various performance metrics such as throughput, channel access delay, channel utilization, and fairness can be considered.

Reference [25] analyzes the performance of the Slotted Aloha (S-Aloha) scheme for broadband wireless networks. The authors showed that due to the lack of available resources, a complex mechanism such as CDMA would not provide the better performance, but may increase the network complexity. Also, the activity of users will not tend to be simultaneous, and hence a simple and efficient mechanism such as S-Aloha will provide good performance with low network complexity.

In [48], Genetic Algorithm (GA) is used to implement a Fixed Channel Allocation (FCA) scheme in BFWA (Broadband Fixed Wireless Access) networks (see Figure 4). The solution (channel assignment) is modeled as a string of nodes assigned at each channel, and the aim is to achieve uniform channel utilization, which is defined as the average number of APs (Access Points) using a particular channel. To be fair to all APs, the APs are divided equally among the C channels in the system. The fitness function at the i-th iteration is defined as:

$$F(t) = \sum_{i=1}^{C} F_i(t), \tag{4}$$

where C is the total number of the channels in the system, and $F_i(t)$ is defined as:

$$F_i(t) = \sum_{j=1}^{c_i} \sum_{k=1, k\neq j}^{c_i} [P_{A_j,A_k}(t) + P_{A_j,S_k}(t) + P_{S_j,A_k}(t) + P_{S_j,S_k}(t)], \tag{5}$$

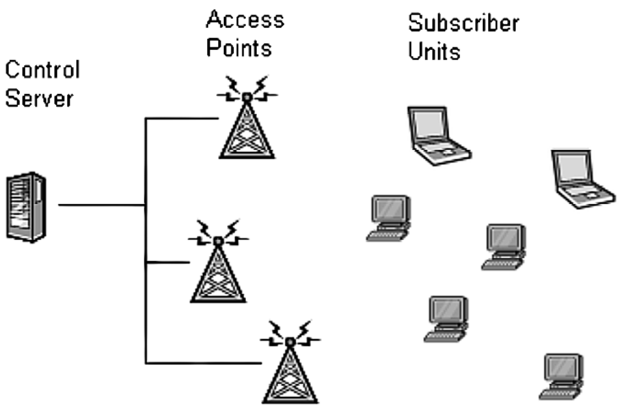

Figure 4. BFWA network

where c_i is the total number of APs using channel i, and $P_{X,Y}(t)$ is the received interference power at X from Y at iteration t. Here X, Y can be either APs (A_j, A_k) or SUs (S_j, S_k). The implemented scheme FCA-GA is used as a benchmark to measure the suboptimality of the proposed schemes such as Dynamic Channel Allocation using GA (DCA-GA), Least Interfered Method (LI), etc. In the LI method, each AP scans the interference power of each channel and selects the channel with the lowest interference power. The simulation shows a quite interesting result that FCA-GA has zero channel fluctuation whereas other schemes have relatively high fluctuation. In contrast with FCA-GA, which requires complete knowledge of the entire network, DCA-GA requires partial and local interference information and performs the best among the rest of the schemes. However, LI also requires partial information only.

Reference [26] considers the burst error properties of wireless communication channels and proposes a Time-Division Multiple Access (TDMA)-based uplink access control scheme that makes use of Channel State Information (CSI) of each mobile device (see Figure 5). One interesting result of the simulation is the performance of the proposed scheme (CHARISMA) under different mobile device speeds. The results are described to be almost static for different speed situations due to the adaptive CSI refresh mechanism.

A scalable modeling framework is introduced in [12] for the analytical study of medium access control protocols. The proposed model allows individual modeling of each node with many layer-specific parameters. Regarding the node placement, no spatial probability distribution or any particular arrangement is assumed. It also allows the computation of individual performance metrics. The proposed framework can model three types of impacts: physical layer, MAC layer, and topology. Signals used at physical layer are modeled by the successful frame reception probability as follows:

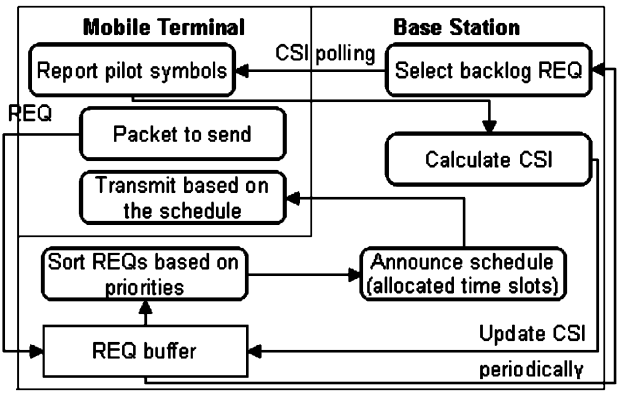

Figure 5. CHARISMA protocol

Table 2. Variables Used in [12]

n_r:	number of users in the network
$\{c_{ik}^r\}_{k=1}^{2^{n_r-1}}$:	the k-th combination of active transmitting nodes except i
$\overline{c_{ik}^r}$:	the complement set of t_{ik}^r
$\mathcal{C}_i^r$:	a random variable about the occurrence of a specific combination c_{ik}^r of transmitters
q_i^r:	the probability that a frame transmitted by i is successfully received at r
$f(c_{ik}^r)$:	$P\{\text{successful frame reception} \mid \mathcal{C}_i^r = c_{ik}^r\}$
τ_i:	the probability that node i transmits a frame at any time, according to the MAC protocol in place
q_i:	the probability of successful transmission of node i
$h_i(\cdot)$:	a time-invariant function relating q_i and τ_i
π:	$[\pi_1\pi_2...\pi_n]^T$; $\pi_i = f(c_{i0}^r)f(c_{r0}^i)$
$\mathbf{\Phi}$:	the interference matrix $[\phi_{ij}]$; $\phi_{ij} = h_i'(0)\pi_i$ if $j \neq i$, 0 o.w.

$$q_i^r = \sum_k f(c_{ik}^r) \prod_{m \in \overline{c_{ik}^r}} (1 - \tau_m) \prod_{n \in c_{ik}^r} \tau_n. \tag{6}$$

The schedulings established at MAC layer are modeled by the transmission probability as follows:

$$q_i = \sum_k \sum_l f(c_{ik}^r) f(c_{rl}^i) P(\mathcal{C}_i^r = c_{ik}^r) P(\mathcal{C}_r^i = c_{rl}^i) \tag{7}$$

and

$$\tau_i = h_i(q_i), \quad i \in V, \tag{8}$$

where $P(\mathcal{C}_i^r = c_{ik}^r) = \prod_{m \in \overline{c_{ik}^r}}(1 - \tau_m) \prod_{n \in c_{ik}^r} \tau_n$. The topology of the network is modeled by a linear model as follows:

$$(\mathbf{I} + \mathbf{\Phi})\mathbf{q} = \pi \tag{9}$$

and

$$\tau = h_i'(0)(\mathbf{I} + \mathbf{\Phi})^{-1}\pi. \tag{10}$$

For the validation of the model, they modeled IEEE 802.11 DCF (distributed coordination function) and compared the analytical results with simulation results using Qualnet [39]. The results show pretty accurate prediction by the model in terms of throughput.

Game theoretical approach has been used in the literature [4,17,22,28,49,51]

3 Game Theoretical Approaches

In this section, we discuss game theoretical approaches to the problems discussed in Section 2.

3.1 Power Control

Ji and Huang [21] studied the uplink power control problem as a noncooperative N-person game in many different cases. In the uplink power control problem, each user wants to choose the transmitting power that maximizes its own utility (defined as a function of power and SIR). The authors proposed two algorithms: one for searching individual equilibrium and the other for individual optimal power control. The second algorithm is based on the simplified assumption: each base station will control the transmitting power of its carried users without consideration of the interferences The algorithm has only two steps: (I) the base station informs each user of its total receiver power $r_k(P)$ and (II) each user selects its power according to

$$p_j = \frac{r_k(P)}{h_{kj}} - \lambda_j, \tag{11}$$

where h_{kj} represents the path gain from the j-th user to the k-th base station, and λ_j denotes the user j's relative preference of good Quality of Service (QoS) over saving of power. Special cases of unconstrained or constrained power control for linear utility functions are studied, and the application to indoor wireless communication is proposed. As a more general case, an exponential utility function is presented.

Shah *et al* [43] proposed a power control model using utility and pricing based on noncooperative game. The utility and pricing concepts are borrowed from economic concepts. As the utility function, SIR is used to represent the level of satisfaction. However as opposed to the discrete definition of the utility function for voice communications, the utility function for data communications is modeled as continuous. The rationale behind this definition is that the throughput is generally proportional to SIR. The authors pointed out that Nash equilibrium achieved by noncooperative game using only utility (using SIR) is Pareto inefficient. This observation motivates the introduction of pricing function. The power control game based on utility and (linear) pricing can be formulated as follows.

$$\max_{p_i} u_i - F_i, \qquad \forall\ i = 1, \ldots, N, \tag{12}$$

where u_i is the utility function for a node i and $F_i = \beta p_i$, in other words the price of a node i is linear to the node i's power p_i. Saraydar *et al* [40] investigated the Pareto efficiency of a pricing policy and characterized the Nash equilibria achieved using pricing based on supermodularity. Characterized by *strategic complementarities*, in supermodular games, each player's increase of its strategy tends to increase other player's strategies. These games have Nash equilibria, and there are an upper and a lower bound on Nash strategies of each user. The price function c for node i is defined as follows.

$$c_i(p_i, \mathbf{p}_{-i}) = c\alpha_i p_i, \tag{13}$$

where $\mathbf{p}_{-i} = [p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N]$ is the vector consisting of elements other than the i-th element. The experiment results show significant utility benefit of using pricing policy against power control game without pricing. Goodman and Mandayam [18, 19] defined *net utility* function, which is the difference of utility and price. More simulation results for similar approach are provided in the paper.

Heikkinen [20] studied power control game against nature under incomplete information. The power control game is formulated as follows.

$$\alpha^* = \max_x \min_p \frac{p'Gx}{p'Fx}, \tag{14}$$

where G is the link gain matrix and F is the interference matrix. The game is defined by two players: power vector selector and nature. Power vector selector chooses the transmit power vector x and nature chooses a distribution p. The author verified that a mixed strategy equilibrium capacity (SIR) can be interpreted as the optimal outcome from an incomplete information game against nature by constructing a belief system such that the given mixed strategy Nash-equilibrium with complete information coincides with a Bayesian Nash-equilibrium of a game against nature under incomplete information. The *belief system* is defined to consist of:

- a set S_i of types, for each player i
- a probability distribution on the set S^{-i} of the types of the other players, for each type s_i
- an action a_i, for each type s_i
- a payoff function $u_i : A \to R$, for each type s_i

The author also provided a game-theoretic rationale for no power control under lack of information, in other words it can be optimal for power selector to play any mixed strategy, preferably one with minimal sum of powers.

Saraydar *et al* [41] studied base station assignment problem based on maximum received signal strength (MRSS) and base station assignment problem based on maximum SIR (MSIR). The authors formulated four power control games: MCPG-MRSS using MRSS without pricing, MCPGP-MRSS using MRSS and pricing, MCPG-MSIR using MSIR without pricing, and MCPGP-MSIR using MSIR and pricing. Each problem is defined as follows:

$$\text{(MCPG-MRSS)} \quad \max_{p_j \in P_j} u_{a_j,j}(\mathbf{p}), \quad \forall j \in N \tag{15}$$

$$\text{(MCPGP-MRSS)} \quad \max_{p_j \in P_j} u_{a_j,j}(\mathbf{p}) - c_{a_j} p_j, \quad \forall j \in N \tag{16}$$

$$\text{(MCPG-MSIR)} \quad \max_{p_j \in P_j} \max_{i \in \mathcal{K}} u_{ij}(\mathbf{p}), \quad \forall j \in N \tag{17}$$

$$\text{(MCPGP-MSIR)} \quad \max_{p_j \in P_j} \max_{i \in \mathcal{K}} u_{ij}(\mathbf{p}) - c_{a_j} p_j, \quad \forall j \in N, \tag{18}$$

where $u_{a_j,j}(\mathbf{p})$ is the utility of terminal j at its assigned base station a_j, u_j is the utility of user j, and c_i is a constant *pricing factor* announced by base station i. The authors proposed two pricing schemes: global pricing and local pricing. In local pricing, a base station broadcasts the pricing factor locally. The authors proved the inefficiency of NPG (noncooperative power control game without pricing) in [42]. Based on the supermodularity theory, introduced in [40], the authors proved that when NPGP (noncooperative power control game with pricing) has Nash equilibria, the one making highest net utility is the Nash equilibrium with the minimum total transmit powers. An algorithm producing the Pareto-dominant equilibrium (minimum power vector yielding Nash equilibrium with highest net utility) is introduced in the paper.

Alpcan *et al* [1] proposed two uplink power update algorithms: Parallel Update Algorithm (PUA) and Random Update Algorithm (RUA). In PUA, the users optimize their power levels periodically using the response function, which represents the optimal response of the user to the parameters in the model. RUA is a stochastic modification of PUA, in which the users periodically optimize their power levels with a predefined probability. The authors presented the sufficient conditions of each algorithm for global stability and convergence to the unique equilibrium solution from any feasible starting point. For the pricing, two pricing schemes are presented: centralized pricing scheme and decentralized, market-based pricing scheme. In centralized pricing scheme, the users are divided into classes and the base station sets the prices for the classes. In decentralized, market-based pricing scheme, the base station sets a single price and the users choose whether to pay or not. The simulation results show RUA performs better than PUA in a delay-free system, and PUA performs better than RUA in a system with delay. Whereas [1] deals with a single-cell system, [2] models a multicell wireless data network as a switched hybrid system where handoffs of mobiles between the individual base stations are discrete switching events between different subsystems. For the dynamic case where mobiles connect to base stations dynamically, the authors use the concept of *dwell-time*, which is the minimum amount of time between two switches, and *average dwell-time*. The paper also studies the stability under feedback delays to conclude that the feedback delay does not affect the stability, but it may affect convergence rates, i.e., larger delays may slow convergence rates and hence they may decrease the robustness of the system. Reference [3] proposes two iterative algorithms (synchronous and asynchronous update schemes) based on the approach in [2].

Reference [5] studies S-modular (meaning either submodular or supermodular) games and its application to power control. The authors defined a General Updating Algorithm (GUA) and proved its convergence to equilibrium. GUA is defined as follows: There are N infinite increasing sequences $\{T_k^i\}$, $i = 1, \ldots, N, k = 1, 2, 3, \ldots$. At time T_k^i, player i uses the best response policy to the policies used by all other players just before T_k^i. This scheme includes parallel updates when $\{T_k^i\}$ does not depend on i. The convergence of GUA to equilibrium explains the properties of the S-modular games, namely that a Nash equilibrium exists and can be obtained by greedy best-response type algorithms, and best-response policies are monotonic in other players' policies. The result is applied to power control and leads to sufficient conditions for convergence of GUA.

3.2 Cooperation and Security

Michardi and Molva [30] applied an m-dimensional version of Albert Tucker's *Prisoner's Dilemma* (PD) game to the cooperation problem in mobile ad hoc networks. The PD game can be illustrated with an example: two men who are charged with a conspiracy are held separately by the police. Each has two options (see Table 3), to confess or not, and (1) if one confesses and the other does not, the former will be set free and the other will get a long sentence. (2) If both confess, both will get a medium-length sentence. (3) If neither confesses, both will get a short sentence. Obviously the mutually beneficial strategy is that both do not confess, and as a result, both get a short sentence. However from each prisoner's point of view, confess strategy will give the prisoner shorter sentences regardless of the other prisoner's strategy. Hence the unique Nash equilibrium is the confess strategy for both. In the m-dimensional PD game, each node can cooperate, 'c', or defect, 'd'. The payoffs for 'c' and 'd' are defined based on the total number of cooperating nodes, and the Nash equilibrium is analyzed. The authors presented a sufficient condition for at least the half of the nodes in the network to cooperate. Reference [33] presents the utility function to model the selfishness problem based on the energy that a node spends for its own communications and the energy that the node used for network operations.

In [34], the same authors studied the formal assessment of the features of cooperation enforcement mechanisms including their mechanism CORE [31,33] using both cooperative and noncooperative game theory. Using cooperative game theory, cooperation enforcement mechanisms guarantee a coalition of cooperating nodes of size at least half of the size of the network. In addition, the necessary condition for a node to join in a coalition is presented. Meanwhile using noncooperative game theory, strategies are introduced and compared using simulation. The game is modeled as an infinitely repeated PD game, and three strategies (*spiteful*, *TIT-FOR-TAT*, and *CORE*) are introduced. In all the three strategies, a node cooperates on first move. In spiteful, a node cooperates with the other node only if both have always cooperated. In TIT-FOR-TAT, a node copies the opponent's last move. In CORE, the decision is made based on the reputation. One interesting point is that the authors considered *imperfect private monitoring* assumption where noise exists in the

Table 3. A's Outcome, Prisoner's Dilemma

A's action \ B's action	c	d
c	Medium sentence	Set free
d	Long sentence	Short sentence

network environment. The simulation results show significantly different evolution graph between perfect private monitoring assumption (which assumes a noise-free environment) and imperfect private monitoring assumption.

Srinivasan *et al* [44] proposed a distributed and scalable acceptance algorithm (to decide whether to accept or to reject a relay request) based on the Generous TIT-FOR-TAT (GTFT) strategy [7]. In the GTFT strategy, each player cooperates based on reciprocal cooperation, but each is a little generous and on occasion one cooperates even when the others did not cooperate in the previous game. The GTFT algorithm adopts this idea, and the condition for each node to accept a relay request is as follows:

$$\psi_h^j(k) \leq \tau_j \text{ and } \phi_h^j(k) \geq L_{ij}\psi_h^j(k) - \epsilon, \tag{19}$$

meaning a relay request for the type j is accepted only when the node h has relayed less traffic for type j sessions than what it should and the node h has relayed less traffic than others have done for h. By using a small positive value ϵ, the small generosity for each node is generated. The GTFT algorithm has the following advantages: each action is based on locally gathered information and it is scalable as the number of variables it requires is independent of the number of the nodes. Reference [45] discusses some implementation issues such as NAR (Normalized Acceptance Rate) calculation, security issues with the presence of malicious users, implementation of m-GTFT as an extension of AODV, and acknowledgment messages.

Urpi *et al* [47] proposed a general model describing selfishness. The proposed model assumes that time is discretely divided into frames $t_1, \ldots, t_n$ and defines the following variables for each frame t_k as in Table 4. Nodes are divided into n energy classes, $e_1, \ldots, e_n$, and to represent the importance of the energy resource, α_{e_k} is defined to have a value between 0 and 1. The authors discussed strategies in the literature [9–11, 31, 44, 45] based on the proposed model.

Table 4. Variables Used in [47]

$N_i(t_k)$:	the set of node i's neighbors
$B_i(t_k)$:	the remaining energy of i
$T_i^j(t_k)$:	the number of packets that i generated as source and that it has to send to neighbor j
$F_j^i(t_{k-1})$:	the number of packets that j forwarded for i during the previous frame
$R_i^j(t_{k-1})$:	the number of packets that i received as final destination from j during the previous frame
$\tilde{R}_i^j(t_{k-1})$:	the number of packets that i received from a source j during the previous frame

3.3 Channel Access Control

MacKenzie and Wicker [28] used game theory to analyze S-Aloha scheme from the perspective of a selfish user. The authors modeled S-Aloha protocol as a stochastic game where the current state of the game is represented by the number of users (n) who currently have packets to send. Each player has two possible actions: transmit (T) or wait (W). The constant transmission cost c is used, and payoff for a transmission is either $1 - c$ or c depending on whether a single or multiple users transmitted at a given slot. The objective is defined to maximize each user's discounted expected payoff over the slots from now until she transmits successfully. The discount factor δ is set very close to 1 due to nearly insignificant delay for a single slot. The equilibria of the selfish Aloha game were proved using the existence of Markov perfect equilibrium (MPE) in general stochastic games with a countable number of states and actions. The authors compared the selfish Aloha system with the centrally controlled Aloha system. The steady-state probability's drop rate by the selfish Aloha system converges in contrast with the rapid drop rate by the centrally controlled Aloha system. This result supports that the selfish Aloha system is more robust than a centrally controlled system where the selfish Aloha shows performance comparable with that of a centrally controlled system.

Jin and Kesidis [22] also studied the equilibria of an S-Aloha system. The authors considered a noncooperative users group sharing a channel via S-Aloha. Each user selects a desired throughput based on her QoS requirement and willingness to pay. At m-th iteration of the game, each user n then adjusts her transmission-probability q_n^m to attain her desired throughput y_n. The next transmission-probability is determined as

$$q_n^{m+1} = \min\{y_n/x_n^m, 1\}, \tag{20}$$

where $x_n^m = \prod_{i \neq n}(1 - q_i^m)$. By eliminating undesirable cases, the equilibrium point q_n^* satisfies

$$q_n^* = \min\{y_n/x_n^*, Q\}, \tag{21}$$

for some large $Q < 1$. The authors discussed local convergence to an equilibrium point using a modified game, where the next transmission-probability is computed by a convex function of current probability and the right-hand side of (20).

Wong and Wassell [49] proposed a game theoretical approach for Dynamic Channel Allocation (DCA) problem using a payoff function. Similar to LI (Least Interfered Method) introduced in [48], in DCA using GT (Game Theory), AP (Access Point) will scan all available channels at the start of a MAC frame and select the channel with the lowest interference power. The authors defined the payoff function $\pi_{j,k}(t)$ for a pair of APs j and k at time t as follows:

$$\pi_{j,k}(t) = G_j(t)((1 - P_I(t))O_{j,k}(t) + S_{j,k}(t)), \tag{22}$$

where $G_j(t)$ is the packet throughput for AP j defined as the percentage of time a packet is transmitted or received. $P_I(t)$ is the probability that AP j and AP k use the same channel, $O_{j,k}(t)$ is the average fraction of $T_j(t)$ (the period between two scans for AP j) that would coincide with $T_k(t)$, and $S_{j,k}(t)$ is the average fraction of $T_j(t)$ that coincides with the SCAN portion of AP k. A mixed strategy is used and an AP plays strategy s_1 with probability p and s_2 with $1-p$. Using the function $U_{j,k}(x,y)$ that is defined as the payoff for AP j when AP j plays strategy x and AP k plays strategy y, the authors provided the probability p that maximizes the mixed payoff:

$$\pi_{MIX} = p^2U(s_1,s_1) + p(1-p)U(s_1,s_2) + (1-p)pU(s_2,s_1) + (1-p)^2U(s_2,s_2) \tag{23}$$

and hence

$$p = \frac{2U(s_2,s_2) - U(s_1,s_2) - U(s_2,s_1)}{2(U(s_1,s_1) + U(s_2,s_2) - U(s_1,s_2) - U(s_2,s_1))}. \tag{24}$$

The IEEE 802.11 distributed coordination function (DCF), which has been the de facto access standard, suffers from the fairness problem and is studied by game theoretical approaches in [17, 51]. Fang and Bensaou [17] modeled the fair bandwidth allocation problem as a constrained maximization problem. Using Lagrange relaxation and duality theory, they provided both noncooperative and cooperative game formulations. The problem is based on the flow contention graph where a vertex represents an active flow/link, and an edge between two vertices represents wireless proximity between each other (see Figure 6). In a flow contention graph, flows in the same maximal clique (considered as a *channel resource*) cannot transmit at the same time, and this property is used in cooperative game approach. The authors proved that

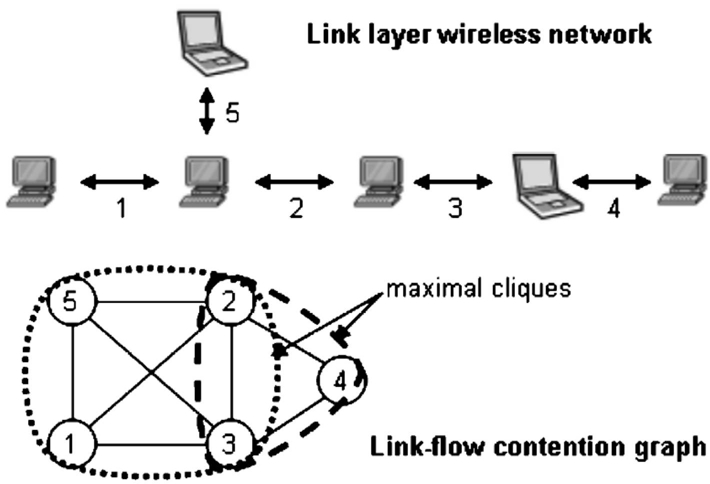

Figure 6. Link layer wireless network and its flow contention graph

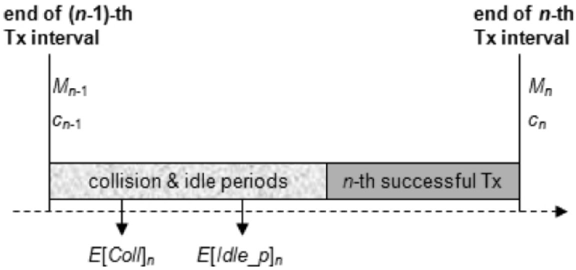

Figure 7. Price-based Congestion Control policy

the noncooperative game has a unique Nash equilibrium. The simulation results show better fairness by the noncooperative game approach and better throughput by the cooperative game approach. Xiao *et al* [51] presented a new backoff algorithm based on Nash equilibrium strategy aiming for fairness and compared its performance with that of the backoff algorithm in DCF.

Batiti *et al* [8] proposed a Price-based Congestion Control (PCC) policy to achieve a better channel utilization (see Figure 7). The PCC policy operates in two steps. In the first step, the AP determines the percentage increase/decrease in the number of active stations to drive the system to the optimal operating point. In the second step, the AP computes the cost levels that stimulate/discourage new users to achieve the percentage determined in the first step. The authors evaluated the performance under saturated/not-saturated scenarios using long/mixed packets. In saturated mode, each station always has a nonempty queue, and in not-saturated mode, each station generates new packets to transmit according to Poisson distribution. The results support the feasibility of a price-based congestion control policy in a Wi-Fi hot spot.

4 Conclusions

In this chapter, we reviewed game theoretical approaches in wireless networks. The restrictive natures of wireless communication, for example limited battery, scarce and shared communication medium, together with the resulting economical issues, make game theory a reasonable fit of approach to the problems. Game theoretical approaches will play an important role in driving wireless networking in bloom.

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Multiobjective Control of Time-Discrete Systems and Dynamic Games on Networks

Dmitrii Lozovanu*

Moldova State University, Faculty of Mathematics and Informatics;
Institute of Mathematics and Computer Science, Moldovan Academy of Sciences, Chisinau, Moldova
lozovanu@math.md

Abstract We consider time-discrete systems with a finite set of states. The starting and the final states of the dynamical system are fixed. We assume that the dynamics of the system is controlled by p actors (players), and each of them intends to optimize his own integral-time cost of the system's passages by a certain trajectory. Applying Nash and Pareto optimality principles for such a model, we obtain multiobjective control problems, solutions of which correspond with solutions of noncooperative and cooperative dynamic games, respectively. Necessary and sufficient conditions for the existence of Nash equilibrium and Pareto optimum in considered game control models are derived. Such conditions for stationary and nonstationary cases of the dynamic games are formulated. In the following, we extend dynamic programming technique for determining Nash equilibrium and Pareto optimum for dynamic games in positional form, especially for dynamic games on networks. Efficient polynomial-time algorithms are elaborated for finding optimal strategies of players in dynamic games on networks. These algorithms are applied for studying and solving cyclic games. In addition, computational complexity of the proposed algorithms for the considered class of dynamic problems is discussed. Some extensions and generalizations of obtained results are suggested.

Key words: time-discrete systems, multiobjective control, dynamic games, Nash equilibria, Pareto optima, dynamic programming, polynomial-time algorithms

1 Problems Formulation

We formulate the multiobjective control problems applying the game-theoretical concept to the following classic discrete control problem [2, 4, 44, 65].

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1.1 Single-Objective Discrete Control Problem

Let us consider a discrete dynamical system L with a finite set of states $X \subset R^n$. At every time-step $t = 0, 1, 2, \dots$, the state of the system L is $x(t) \in X$. Two states x_0 and x_f are given in X, where $x_0 = x(0)$ represents the starting point of L and x_f is the state in which the system L must be brought, i.e., x_f is the final state of L. We assume that the system L should reach the final state x_f at the time-moment $T(x_f)$ such that

$$T_1 \le T(x_f) \le T_2,$$

where T_1 and T_2 are given. The dynamics of the system L is described as follows

$$x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \dots \tag{1}$$

where

$$x(0) = x_0 \tag{2}$$

and

$$u(t) = (u_1(t), u_2(t), \dots, u_m(t)) \in R^m$$

represents the vector of control parameters (see [2, 4, 33, 65]). For any time step t, the feasible set $U_t(x(t))$ for the vector $u(t)$ of control parameters is given, i.e.,

$$u(t) \in U_t(x(t)), \quad t = 0, 1, 2, \dots. \tag{3}$$

We assume that in (1) the vector functions

$$g_t(x(t), u(t)) = (g_t^1(x(t), u(t)), g_t^2(x(t), u(t)), \dots, g_t^n(x(t), u(t)))$$

are determined uniquely by $x(t)$ and $u(t)$ at every time-step $t = 0, 1, 2, \dots$. So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$. In addition, we assume that at each moment of time t, the cost

$$c_t(x(t), x(t+1)) = c_t(x(t), g_t(x(t), u(t)))$$

of the system's passage from the state $x(t)$ to the state $x(t+1)$ is known.

Let

$$x_0 = x(0), x(1), x(2), \dots, x(t), \dots$$

be a trajectory generated by given vectors of control parameters

$$u(0), u(1), \dots, u(t-1), \dots.$$

Then either this trajectory passes through the state x_f at the time-moment $T(x_f)$ or it does not pass through x_f.

We denote by

$$F_{x_0 x_f}(u(t)) = \sum_{t=0}^{T(x_f)-1} c_t(x(t), g_t(x(t), u(t))) \tag{4}$$

the integral-time cost of system's passage from x_0 to x_f if $T_1 \le T(x_f) \le T_2$; otherwise we put

$$F_{x_0 x_f}(u(t)) = \infty.$$

Problem 1. To find vectors of control parameters

$$u(0), u(1), u(2), \ldots, u(t), \ldots,$$

which satisfy condition (3) and minimize functional (4).

If $T_1 = T_2$, we obtain the discrete control problem with fixed number of stages, i.e., the problem from [4]; if $T_1 = 0$, $T_2 = \infty$, we have the discrete control problem with free number of stages [2, 33, 65].

1.2 Multiobjective Control Based on Concept of Noncooperative Games: Nash Equilibria

Consider the dynamic system L with the finite set of states X, where at every time-step t the state of L is $x(t) \in X$. The dynamics of the system L is controlled by p players, and it is described as follows

$$x(t+1) = g_t(x(t), u^1(t), u^2(t), \ldots, u^p(t)),\ t = 0, 1, 2, \ldots. \tag{5}$$

where

$$x(0) = x_0$$

is a starting point of the system, L and $u^i(t) \in R^{m_i}$ represents the vector of control parameters of player $i \in \{1, 2, \ldots, p\}$. The state $x(t+1)$ of the system L at the time-step $t+1$ is obtained uniquely if the state $x(t)$ at the time-step t is known, and players $1, 2, \ldots, p$ independently fix their vectors of control parameters $u^1(t), u^2(t), \ldots, u^p(t)$, respectively. For each player $i \in \{1, 2, \ldots, p\}$, the admissible sets $U_t^i(x(t))$ for the vectors of control parameters $u^i(t)$ are given, i.e.,

$$u^i(t) \in U_t^i(x(t)),\ t = 0, 1, 2, \ldots;\ i = \overline{1, p}. \tag{6}$$

We assume that $U_t^i(x(t)),\ t = 0, 1, 2, \ldots;\ i = \overline{1, p}$, are nonempty finite sets and

$$U_t^i(x(t)) \cap U_t^j(x(t)) = \emptyset,\ i \ne j,\ t = 0, 1, 2, \ldots.$$

Let us consider that players $1, 2, \ldots, p$ fix their vectors of control parameters

$$u^1(t), u^2(t), \ldots, u^p(t);\ t = 0, 1, 2, \ldots,$$

respectively, and the starting state $x(0) = x_0$ and the final state x_f are known. Then for the fixed vectors of control parameters $u^1(t), u^2(t), \ldots, u^p(t)$, either a unique trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from x_0 to x_f exists and $T(x_f)$ represents the time-moment when the state x_f is reached, or such a trajectory from x_0 to x_f does not exist. We denote by

$$F^i_{x_0x_f}(u^1(t), u^2(t), \dots, u^p(t)) = \sum_{t=0}^{T(x_f)-1} c^i_t(x(t), g_t(x(t), u^1(t), u^2(t), \dots, u^p(t)))$$

the integral-time cost of the system's passage from x_0 to x_f for the player $i \in \{1, 2, \dots, p\}$ if the vectors $u^1(t), u^2(t), \dots, u^p(t)$ satisfy condition (6) and generate a trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from x_0 to x_f such that

$$T_1 \le T(x_f) \le T_2;$$

otherwise we put

$$F^i_{x_0x_f}(u^1(t), u^2(t), \dots, u^p(t)) = \infty.$$

Note that $c^i_t(x(t), g_t(x(t), u^1(t), u^2(t), \dots, u^p(t))) = c^i_t(x(t), x(t+1))$ represents the cost of the system's passage from the state $x(t)$ to the state $x(t+1)$ at the stage $[t, t+1]$ for the player i.

Problem 2. To find vectors of control parameters

$$u^{1^*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t),\ u^{i^*}(t), u^{i+1^*}(t), \dots, u^{p^*}(t),$$

which satisfy the condition

$$\begin{gathered} F^i_{x_0x_f}(u^{1^*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t),\ u^{i^*}(t), u^{i+1^*}(t), \dots, u^{p^*}(t)) \\ \le F^i_{x_0x_f}(u^{1^*}(t), u^{2^*}(t), \dots, u^{i-1^*}(t), u^i(t), u^{i+1^*}(t), \dots, u^{p^*}(t)) \\ \forall\, u^i(t) \in R^{m_i},\ t = 0, 1, 2, \dots;\ i = \overline{1, p}. \end{gathered}$$

So, we consider the problem of finding the solution in the sense of Nash [50, 53].

An important particular case of Problem 2 represents the zero-sum control problem of two players with given costs

$$c_t(x(t), x(t+1)) = c^2_t(x(t), x(t+1)) = -c^1_t(x(t), x(t+1))$$

of the system's passage from the state $x(t)$ to the state $x(t+1)$, which determine the payoff function

$$F_{x_0x_f}(u^1(t), u^2(t)) = F^2_{x_0x_f}(u^1(t), u^2(t)) = -F^1_{x_0x_f}(u^1(t), u^2(t)).$$

In this case, we seek for a saddle point $(u^{1*}(t),\ u^{2*}(t))$ of the function $F_{x_0x_f}(u^1(t), u^2(t))$, i.e., we consider the following problem.

Problem 3. To find vectors of control parameters $u^{1*}(t)$, $u^{2*}(t)$ such that

$$F_{x_0x_f}(u^{1*}(t), u^{2*}(t)) = \max_{u^1(t)} \min_{u^2(t)} F_{x_0x_f}(u^1(t), u^2(t))$$
$$= \min_{u^2(t)} \max_{u^1(t)} F_{x_0x_f}(u^1(t), u^2(t)).$$

We will describe the classification of necessary and sufficient conditions for the existence of Nash equilibria in such dynamic games, which has been obtained in [37–46]. Furthermore we will describe the classification of dynamical games for which Nash equilibria exist, and algorithms for solving such kind of problems will be proposed.

1.3 Multiobjective Control Based on Concept of Cooperative Games: Pareto Optima

We consider the dynamical system L, which is controlled by p players $1, 2, \dots, p$. Assume that players coordinate their actions in the control processes by using common vector of control parameters $u(t) = (u^1(t), u^2(t), \dots, u^p(t)) \in R^m$ (see [4, 10, 33]). So, the dynamics of the system is described according to (1)–(3).

Let

$$u(0), u(1), u(2), \dots, u(t-1), \dots$$

be a players' coordinated control, which generates a trajectory

$$x(0), x(1), x(2), \dots, x(t), \dots .$$

Then either this trajectory passes through the state x_f at the finite moment $T(x_f)$ or it does not pass through x_f. We denote by

$$F^i_{x_0x_f}(u(t)) = \sum_{t=0}^{T(x_f)-1} c^i_t(x(t), g_t(x(t), u(t))),\ i = \overline{1,p}$$

the integral-time cost of the system's passage from x_0 to x_f if

$$T_1 \le T(x_f) \le T_2;$$

otherwise we put

$$F^i_{x_0x_f}(u(t)) = \infty.$$

Here $c^i_t(x(t), g_t(x(t), u(t))) = c^i_t(x(t), x(t+1))$ represents the cost of the system's passage from the state $x(t)$ to the state $x(t+1)$ at the stage $[t, t+1]$ for the player i, $i \in \{1, 2, \dots, p\}$.

Problem 4. To find the vectors of control parameters $u^*(t)$ such that there is no other control vector $u(t) \neq u^*(t)$, for which

$$(F^1_{x_0x_f}(u(t)), F^2_{x_0x_f}(u(t))\dots, F^p_{x_0x_f}(u(t)))$$

$$\le (F^1_{x_0x_f}(u^*(t)), F^2_{x_0x_f}(u^*(t))\dots, F^p_{x_0x_f}(u^*(t)))$$

and for any $i_0 \in \{1, 2, \dots, p\}$

$$F^{i_0}_{x_0x_f}(u(t)) < F^{i_0}_{x_0x_f}(u^*(t)).$$

So, we consider the problem of finding Pareto solution [50, 52, 58].

Unlike Nash equilibria, Pareto optima for multiobjective discrete control always exists if there is an admissible solution $u(t)$, $t = 0, 1, 2, \dots, T(x_f)$, which generates a trajectory $x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$ from x_0 to x_f.

2 Alternate Players Control and Nash Equilibria for Dynamic Games in Positional Form

In order to formulate the theorem of the existence of Nash equilibria for the considered multiobjective control problem from Section 1.2, we will use the following condition.

We assume that an arbitrary state $x(t) \in X$ of the dynamic system L at the time-moment t represents a position $(x, t) \in X \times \{0, 1, 2, \dots\}$ for one of the players $i \in \{1, 2, \dots, p\}$. This means that in the control process, the next state $x(t+1) \in X$ is determined (chosen) by the player i if the dynamic system L at the time-moment t has the state $x(t)$, which corresponds with the position (x, t) of the player i. This situation corresponds with the case when the expression

$$g_t(x(t), u^1(t), u^2(t), \dots, u^{i-1}(t), u^i(t), u^{i+1}(t), \dots, u^p(t))$$

in (5) for a given position (x, t) of player i depends only on the control vector $u^i(t)$, i.e.,

$$g_t(x(t), u^1(t), u^2(t), \dots, u^{i-1}(t), u^i(t), u^{i+1}(t), \dots, u^p(t)) = \overline{g}_t(x(t), u^i(t)).$$

So, further the notations (x, t) and $x(t)$ have the same sense.

Definition 1. *We say that the alternate players control condition is satisfied for the multiobjective control problems if for any fixed $(x, t) \in X \times \{0, 1, 2, \dots\}$ the equations in (5) depend only on one of the vectors of control parameters. The multiobjective control problems with such an additional condition are called game control models in positional form.*

The following lemma presents a necessary and sufficient condition for the alternate players control condition to hold.

Lemma 1. *The alternate players control condition for the multiobjective control problems holds if and only if at every time-step $t = 0, 1, 2, \dots$ for the set of states X there exists a partition*

$$X = X_1(t) \cup X_2(t) \cup \dots \cup X_p(t); \ (X_i(t) \cap X_j(t) = \emptyset, \ i \neq j) \tag{7}$$

such that the equations in (5) can be represented as follows

$$x(t+1) = \overline{g}_t(x(t), u^i(t)) \quad \text{if} \ \ x(t) \in X_i(t); \ t = 0, 1, 2, \dots; \ i = \overline{1, p}, \tag{8}$$

i.e.,

$$\begin{aligned} &g_t(x(t), u^1(t), u^2(t), \dots, u^i(t), u^{i+1}(t), \dots, u^p(t)) \\ &= \overline{g}_t(x(t), u^i(t)) \quad \text{if} \ \ x(t) \in X_i(t); \ t = 0, 1, 2, \dots; \ i = \overline{1, p}. \end{aligned}$$

Here, $X_i(t)$ corresponds with the set of positions of the player i at the time-step t (note that some of $X_i(t)$ in (7) can be empty sets).

Proof. $\Rightarrow$ Let us assume that the alternate players control condition for the multiobjective control problem holds. Then for a fixed time-step t, the equations in (5) depend only on one of the vectors of control parameters $u^i(t), \ i \in \{1, 2, \dots, p\}$. Therefore, if we denote by $X_i(t)$ the set of states of the dynamical system that corresponds with the positions of player i at time-step t, the equation in (5) can be regarded as a solution that satisfies (8).

$\Leftarrow$ Let us assume that the partition (7) is given for any $t = 0, 1, 2, \dots,$ and the expression in (5) is represented in form (8). This means that the equation at every time-step t depends only on one of the vectors of the control parameters. ■

On the basis of these results, we can prove the important fact that the set of positions can be characterized in the following way:

Corollary 1. *If the alternate player control condition for the multiobjective control problem holds, then the set of positions $Z_i \subseteq X \times \{0, 1, 2, \dots\}$ of player i can be represented as follows*

$$Z_i = \bigcup_t (X_i(t), t), \quad i = \overline{1, p}.$$

Let us assume that the alternate players control for the problem from Section 1.2 holds. Then the set of possible system's transactions of the dynamical system L can be described by a directed graph $\overline{G} = (Z, \overline{E})$ with the set of vertices $Z = \bigcup_{i=1}^{p} Z_i$, where Z_i, $i = \overline{1, p}$, represents the set of positions of player i. An arbitrary vertex $z \in Z$ in $\overline{G}$ corresponds with a position (x, t) for one of the players $i \in \{1, 2, \dots, p\}$, and a directed edge $e = (z', z'')$ reflects the possibility of the system's transaction from the state $z' = (x, t)$ to the state $z'' = (y, t+1)$ determined by $x(t)$ and the control vector $u^i(t) \in U_t^i(x(t))$ such that

$$y = x(t+1) = \overline{g}_t(x(t), u^i(t)) \text{ if } x(t) \in Z_i.$$

If we associate to edges $e = ((x,t),(y,t+1))$ of this graph the costs $c^i_{((x,t),(y,t+1))} = c^i((x,t),(y,t+1))$, we obtain the control problem on network.

Here graph G has a structure of a T_2-partited directed graph with the given starting position $(x_0, 0)$ and the final position $(x_f, T(x_f))$, where $T_1 \le T(x_f) \le T_2$.

Taking into account this representation of the dynamics of the system L, the following theorem is proved in [43–45].

Theorem 1. *Let us assume that for the multiobjective control problem there exists a trajectory*

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from the starting state x_0 to the final state x_f generated by the vectors of control parameters

$$u^1(t), u^2(t), \dots, u^p(t), \quad t = \overline{0, T(x_f) - 1},$$

where $u^i(t) \in U^i_t(x(t))$, $i = \overline{1,p}$, $t = \overline{0, T(x_f) - 1}$ and $T_1 \le T(x_f) \le T_2$. Moreover, we assume that the alternate players control condition is satisfied. Then for this problem there exists the optimal solution in the sense of Nash $u^{1^*}(t), u^{2^*}(t), \dots, u^{p^*}(t)$.

Further, in Section 4 we prove this theorem in the more general case, when the dynamics of the system L is determined by a directed graph of transactions that may contain cycles.

As an important result from Theorem 1, we obtain the following corollary.

Corollary 2. *Assume that for any $u^1(t) \in U^1_t(x(t))$, $t = 0, 1, 2, \dots$ in the max-min control problem there exists a control $u^2(t) \in U^2_t(x(t))$, $t = \overline{0, T(x_f) - 1}$ such that $u^1(t)$ and $u^2(t)$ generate a trajectory*

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from the starting state x_0 to the final state x_f, where $T_1 \le T(x_f) \le T_2$. Moreover we assume that the alternate player control condition is satisfied. Then for the payoff function $F_{x_0 x_f}(u^1(t), u^2(t))$ in the max-min control problem there exists a saddle point $(u^{1*}(t),\ u^{2*}(t))$, i.e.,

$$\begin{aligned} F_{x_0 x_f}(u^{1*}(t), u^{2*}(t)) &= \max_{u^1(t)} \min_{u^2(t)} F_{x_0 x_f}(u^1(t), u^2(t)) \\ &= \min_{u^2(t)} \max_{u^1(t)} F_{x_0 x_f}(u^1(t), u^2(t)). \end{aligned}$$

All results related to existence theorems and algorithms for solving the problems on networks can be transferred for the problems from Sections 1, 2. Therefore in the following we will study the control problems on networks.

3 Algorithms for Solving Single-Objective Control Problems on Networks

We describe two algorithms for solving single-objective control problem, which further will be developed for multiobjective control models.

3.1 Dynamic Programming Algorithm for Solving Optimal Control Problem on Networks

We consider optimal control problem for which the dynamics of the system L is described by a directed graph $G = (X, E)$, where the vertices $x \in X$ correspond with the states of L, and an arbitrary edge $e = (x, y) \in E$ signifies the possibility of the system's passage from the state $x = x(t)$ to the state $y = x(t + 1)$ at every moment of time $t = 0, 1, 2, \ldots$. So, the set $E(x) = \{e = (x, y) | (x, y) \in E\}$ of edges originated in x corresponds with an admissible set of control parameters, which determines the next possible state $y = x(t+1)$ of L, if the stage $x = x(t)$ at the moment of time t is given. Therefore we consider $E(x) \neq \emptyset$, $\forall x \in X$. In addition, we assume that to each edge $e = (x, y) \in E$ there is an associated cost function $c_e(t)$, which depends on time and which expresses the cost of the system L to pass from the state $x = x(t)$ to the state $y = x(t + 1)$ at the stage $[t, t + 1]$ (like a transition). So, this graph of states' transitions contains edges, which represent the time-depending cost functions. In addition, in G two vertices x_0 and x_f, which correspond with the starting and the final states of the system L, are given. We call such a special graph a dynamic network [43].

For a given dynamic network, we regard the following problem:

Problem 5. To find a sequence of system's transitions $(x(0), x(1))$, $(x(1), x(2)), \ldots, (x(T - 1), x(T))$, which transfer the system L from the starting state $x_0 = x(0)$ to the final state $x_f = x(T)$ such that T satisfies the condition

$$T_1 \leq T \leq T_2$$

and the integral-time cost

$$F_{x_0 x_f}(T) = \sum_{t=0}^{T-1} c_{(x(t), x(t+1))}(t)$$

of system's transitions by a trajectory

$$x_0 = x(0), x(1), x(2), \ldots, x(T) = x_f$$

is minimal.

This problem generalizes the well-known shortest path problem in a weighted directed graph [12] and arose as an auxiliary one when solving the minimum-cost flow problem on dynamic networks [17–21, 48].

We describe the dynamic programming algorithm for solving this problem and the problem from Section 1.1.

First we describe the algorithm for solving the problem in the case $T_1 = T_2 = T$. Denote by

$$F_{x_0 x_f}(T) = \min_{x_0=x(0),x(1),\ldots,x(T)=x_f} \sum_{t=0}^{T-1} c_{(x(t),x(t+1))}(t)$$

the minimal integral-time cost of system's transaction from x_0 to x_f with T stages. If x_f cannot be reached by using T stages, then we put $F_{x_0 x_f}(T) = \infty$. For $F_{x_0 x(t)}(t)$, the following recursive formula can be gained

$$F_{x_0 x(t)}(t) = \min_{x(t-1) \in X_G^-(x(t))} \Big\{ F_{x_0 x(t-1)}(t-1) + c_{(x(t-1),x(t))}(t-1) \Big\},$$

where $X_G^-(y) = \{x \in X \mid e = (x,y) \in E\}$

If we put

$$F_{x_0 x(0)}(0) = 0,$$

then it is easy to observe that using dynamical programming method, we can tabulate the values $F_{x_0 x(t)}(t)$, $t = 1, 2, \ldots, T$. So, if $T_1 = T_2 = T$, then the problem can be solved in time $O(|X|^2 T)$ (here we do not take into account the number of operations for calculations of values of functions $c_e(t)$ for the given t).

In the case when $T(x_f) \in [T_1, T_2]$, with $T_1 \neq T_2$, the problem can be reduced to $T_2 - T_1 + 1$ problems with $T = T_1, T = T_1 + 1, T = T_1 + 2, \ldots, T = T_2$, respectively; by comparing the minimal integral-costs of these problems, we find the best one and $T(x_f)$.

An important case of the considered problem is the one with $T_1 = 0$, $T_2 = \infty$. This case has sense only for positive and nondecreasing cost functions $c_e(t)$ on edges $e \in E$. It is obvious that for this case $0 \le T(x_f) \le |X|$, and the problem can be solved in time $O(|X|^3)$ (the case with free number of stages).

3.2 An Extension of Dijkstra's Algorithm for Optimal Control Problem with a Free Number of Stages

Let us assume that in the dynamic network, all cost functions $c_e(t)$, $e \in E$, are positive and $T_1 = 0$, $T_2 = \infty$, i.e., we have the problem with free number of stages.

For this case, we describe an algorithm, which extends Dijkstra's algorithm for finding the tree of optimal paths in a weighted directed graph [12,14]. Such an algorithm will find the optimal paths in dynamic network for our problem if the following additional condition is satisfied.

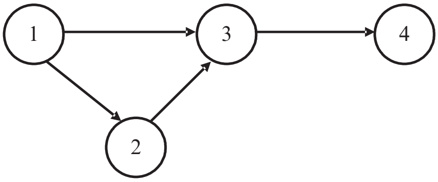

Figure 1. The Network for which Optimization Principle is not satisfied

Let us assume that the cost functions $c_e(t),\ e \in E$, in the dynamic network have the following property. If $P^*(x_0,x)$ is an arbitrary optimal path from x_0 to x that can be represented as $P^*(x_0,x) = P_1^*(x_0,y) \cup P_2^*(y,x)$, where $P_1^*(x_0,y)$ and $P_2^*(y,x)$ have no common edges, then the leading part $P_1^*(x_0,y)$ of the path $P^*(x_0,x)$ is also an optimal path of the problem in G with the given starting state x_0 and final state y. If such a property holds, then we say that for the dynamic network, the optimization principle is satisfied. As example, the graph on Fig. 1 with cost functions on edges $c_{(1,2)}(t) \equiv c_{(1,3)}(t) \equiv c_{(2,3)}(t) = 1, c_{(3,4)}(t) = 3t$ determines a network for which the optimization principle is satisfied. In the case $c_{(1,2)}(t) \equiv c_{(2,3)}(\mathrm{t}){=}1$; $c_{(1,3)}(t) = 3$; $c_{(3,4)}(t) = 3t$, the network does not satisfy the optimization principle because the leading part $P_1^*(1,3) = \{(1,3)\}$ of the optimal path $P^*(1,4) = \{(1,3),(3,4)\}$ is not optimal. In the case, when on the network the cost functions $c_e(t), e \in E$, are positive and the optimization principle is satisfied, the following algorithm determines all optimal paths $P^*(x_0,x)$ from x_0 to each $x \in X$, which correspond with the optimal strategies in the problem for $p = 1$.

Algorithm 1. Determining the Tree of Optimal Paths

Preliminary step (Step 0): Set $Y = \{x_0\},\ E^* = \emptyset$. Assign to every vertex $x \in X$ two labels $t(x)$ and $F(x)$ as follows:

$$\begin{aligned} t(x_0) = 0;\quad t(x) = \infty,\quad \forall\, x \in X \setminus \{x_0\};\\ F(x_0) = 0;\quad F(x) = \infty,\quad \forall\, x \in X \setminus \{x_0\}. \end{aligned}$$

General step (Step k, $k \geq 1$): Find the set

$$E' = \{(x',y') \in E(Y) \,|\, F(x') + c_{(x',y')}(t(x')) = \min_{x \in Y} \min_{y \in \overline{X}(x)} \{F(x) + c_{(x,y)}(t(x))\},$$

where

$$E(Y) = \{(x,y) \in E \,|\, x \in Y,\ y \in X \backslash Y\}, \quad \overline{X}(x) = \{y \in X \backslash Y \,|\, (x,y) \in E(Y)\}.$$

Find the set of vertices $X' = \{y' \in X \setminus Y |\, (x',y') \in E'\}$. For every $y' \in X'$ select one edge $(x',y') \in E'$ and build the union $\overline{E}'$ of such edges. After that, change the labels $t(y')$ and $F(y')$ for every vertex $y' \in X'$ as follows

$$t(y') = t(x') + 1,\quad F(y') = F(x') + c_{(x',y')}(t(x')),\quad \forall (x',y') \in \overline{E}'.$$

Replace the set Y by $Y \cup X'$ and E^* by $E^* \cup \overline{E}'$. Note $X^k = Y,\ E^k = E^*$. If $X^k \neq X$ then fix the tree $GT^k = (X^k, E^k)$ and go to next step $k+1$, otherwise fix the tree $GT = (X, E^*)$ and STOP.

Note, that the tree $GT = (X, E^*)$ contains optimal paths from x_0 to each $x \in X$. After k steps of the algorithm, the tree $GT^k = (X^k, E^k)$ represents a part of GT. If it is necessary to find the optimal path from x_0 to x_f, then the algorithm can be interrupted after k steps as soon as the condition $x_f \in X^k$ is satisfied, i.e., in this case the condition $X^k \neq X$ in the algorithm must be replaced by $x_f \in X^k$. The labels $F(x)$, $x \in X$, indicate the costs of optimal paths from x_0 to $x \in X$, and $t(x)$ represents the number of edges in these paths.

The correctness of the algorithm is based on the following theorem:

Theorem 2. *Let $(G, c(t), x_0, x_f)$ be a dynamic network, where the vector-function $c(t) = (c_{e_1}(t), c_{e_2}(t), \ldots, c_{e_{|E|}}(t))$ has positive and bounded components for $t \in [0, |X|-1]$. Moreover, let us assume that the optimization principle on the dynamic network is satisfied. Then the tree $GT^k = (X^k, E^k)$ obtained after k steps of the algorithm gives the optimal paths from x_0 to every $x \in X^k$, which correspond with optimal strategies in the problem for $p = 1$.*

Proof. We prove the theorem by using the induction principle on the number of steps k of the algorithm. In the case when $k = 0$, the assertion is evident.

Let us assume that the theorem holds for any $k \leq r$ and let us show that it is true for $k = r+1$. If $GT^r = (X^r, E^r)$ is the tree obtained after r steps and $GT^{r+1} = (X^{r+1}, E^{r+1})$ is the tree obtained after $r+1$ steps of the algorithm, then $X^\circ = X^{r+1} \setminus X^r$ and $E^\circ = E^{r+1} \setminus E^r$ represent the vertex set and edge set obtained by the algorithm at step $r+1$. Let us show that if y' is an arbitrary vertex of X°, then in GT^{r+1} the unique directed path $P^*(x_0, y')$ from x_0 to y' is optimal. Indeed, if this is not the case, then there exists an optimal path $Q(x_0, y')$ from x_0 to y', which does not contain the edge $e = (z', y') \in E^\circ$. The path $Q(x_0, y')$ can be represented as $Q(x_0, y') = Q^1(x_0, x') \cup \{(x', y)\} \cup Q^2(y, y')$, where x' is the last vertex of the path $Q(x_0, y')$ belonging to X^r when we pass from x_0 to y'. It is easy to observe that if the conditions of the theorem hold, then

$$\text{cost}\,(Q(x_0, y')) \geq \text{cost}\,(P^*(x_0, y')),$$

where

$$\text{cost}\,(Q(x_0, y')) = \sum_{t=0}^{m_Q} c_{e_t}(t),$$

$e_0, e_1, \ldots, e_{m_Q}$ are the corresponding edges of the directed path $Q(x_0, y')$ when we pass from x_0 to y' and

$$\text{cost}\,(P^*(x_0, y')) = \sum_{t=0}^{m_p} c_{e'_t}(t),$$

were $e'_0, e'_1, \ldots, e'_{m_p}$ are the corresponding edges of the directed path $P^*(x_0, y')$ when we pass from x_0 to y'.

According to the algorithm, we can state that

$$F(x') + c_{(x',y')}(t(x')) > F(z') + c_{(z',y')}(t(z')) = F(y'),$$

where $e' = (z', y')$ is the last edge of the path $P^*(x_0, y')$. Then

$$\text{cost}\,(Q^1(x_0, x') \cup \{(x', y)\}) > \text{cost}\,(P^*(x_0, y')),$$

because

$$F(x') + c_{(x',y)}(t(x')) = \text{cost}\,(Q^1(x_0, x') \cup \{(x', y)\})$$

and $F(y') = \text{cost}\,(P^*(x_0, y'))$.

The cost functions $c_e(t)$, $\forall\, e \in E$, are positive, therefore,

$$\text{cost}\,(Q(x_0, y')) = \text{cost}\,(Q^1(x_0, x') \cup \{(x', y)\} \cup Q^2(y, y'))$$

$$> \text{cost}\,(Q^1(x_0, x') \cup \{(x', y)\}) > \text{cost}\,(P^*(x_0, y')),$$

i.e., $Q(x_0, y')$ is not an optimal path from x_0 to y'. This means that the tree $GT^{r+1} = (X^{r+1}, E^{r+1})$ contains an optimal path from x_0 to every $y' \in X^{r+1}$. ■

All results described in this section are given in [37, 38].

4 Multiobjective Control and Noncooperative Games on Dynamic Networks

In this section, we use the concept of noncooperative games for our problem and formulate the following two multiobjective control models concerning stationary and nonstationary strategies [43–45].

4.1 The Problem of Determining the Optimal Stationary Strategies in Dynamic *c*-Game

Let $G = (X, E)$, be the graph introduced in Section 2 with the given starting and final states $x_0, x_f \in X$. Assume that the vertex set X is divided into p disjoint subsets $X_1, X_2, \ldots, X_p$ $\left(X = \bigcup_{i=1}^{p} X_i,\ X_i \cap X_j = \emptyset,\ i \neq j\right)$ and regard vertices $x \in X_i$ as states of player i, $i = \overline{1, p}$. Moreover we assume that to each edge $e = (x, y)$ of the graph, p functions $c_e^1(t), c_e^2(t), \ldots, c_e^p(t)$ are assigned, where $c_e^i(t)$ expresses the cost of system's passage from the state $x = x(t)$ to the state $y = x(t+1)$ at the stage $[t, t+1]$ for player i.

We define the stationary strategies of players $1, 2, \dots, p$ as maps:

$$\begin{array}{l} s_1\colon x \to y \in X(x) \ \text{ for } \ x \in X_1 \setminus \{x_f\}, \\ s_2\colon x \to y \in X(x) \ \text{ for } \ x \in X_2 \setminus \{x_f\}, \\ \quad - - - - - - - - - - - - \\ s_p\colon x \to y \in X(x) \ \text{ for } \ x \in X_p \setminus \{x_f\}, \end{array}$$

where $X(x) = \{y \in X \mid e = (x, y) \in E\}$.

Taking into account that $G = (X, E)$ is a finite graph, we obtain that the set of strategies of player i

$$S_i = \{s_i : x \to y \in X(x) \text{ for } x \in X_i \setminus \{x_f\}\},\ i = \overline{1, p}$$

is a finite set.

Let $s_1, s_2, \dots, s_p$ be an arbitrary set of strategies of players. We denote by $G_s = (X, E_s)$ the subgraph generated by edges $e = (x, s_i(x))$ for $x \in X_i \setminus \{x_f\}$ and $i = \overline{1, p}$. Obviously, for fixed $s_1, s_2, \dots, s_p$, either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists in G_s or such a path does not exist in G_s. The set of edges of path $P_s(x_0, x_f)$ is denoted by $E(P_s(x_0, x_f))$.

For fixed strategies $s_1, s_2, \dots, s_p$ and fixed states x_0 and x_f, we define the quantities

$$H^1_{x_0 x_f}(s_1, s_2, \dots, s_p), H^2_{x_0 x_f}(s_1, s_2, \dots, s_p), \dots, H^p_{x_0 x_f}(s_1, s_2, \dots, s_p)$$

in the following way.

Let us assume that the path $P_s(x_0, x_f)$ exists in G_s. Then it is unique and we can assign to its edges numbers $0, 1, 2, 3, \dots, k_s$, starting with the edge that begins in x_0. These numbers characterize the time steps $t_e(s_1, s_2, \dots, s_p)$ when the system passes from one state to another, if the strategies $s_1, s_2, \dots, s_p$ are applied. We put

$$H^i_{x_0 x_f}(s_1, s_2, \dots, s_p) = \sum_{e \in E(P_s(x_0, x_f))} c^i_e(t_e(s_1, s_2, \dots, s_p)),$$

if

$$T_1 \le |E(P_s(x_0, x_f))| \le T_2; \tag{9}$$

otherwise we put $H^i_{x_0 x_f}(s_1, s_2, \dots, s_p) = \infty$.

We regard the problem of finding maps $s^*_1, s^*_2, \dots, s^*_p$ for which the following conditions are satisfied

$$\begin{aligned} & H^i_{x_0 x_f}(s^*_1, s^*_2, \dots, s^*_{i-1}, s^*_i, s^*_{i+1}, \dots, s^*_p) \\ & \le H^i_{x_0 x_f}(s^*_1, s^*_2, \dots, s^*_{i-1}, s_i, s^*_{i+1}, \dots, s^*_p),\ \forall\, s_i \in S_i,\ i = \overline{1, p}. \end{aligned}$$

So, we consider the problem of finding the optimal solutions in the sense of Nash.

This problem can be regarded as dynamic game on network $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f, T_1, T_2)$ determined by the graph G, the partition $X_1, X_2, \ldots, X_p$, the vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_{|E|}}(t))$, $i = \overline{1,p}$, the starting and final states x_0, x_f, and the time-span $[T_1, T_2]$. Note that in the considered problem. T_1 and T_2 satisfy the conditions: $0 \leq T_1 \leq |X| - 1$, $T_1 \leq T_2$; for $T_1 \geq |X|$, the problem has no sense. If $T_1 = 0$, $T_2 = \infty$, then we shall use the notation $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$. The last version of the problem has been studied in [3]. In [45] this problem is named dynamic c-game.

Note that this problem is NP-hard even in the case $p = 1$. Indeed, if $T_1 = T_2 = |X| - 1$ and $c^1_e(t) = 1$, $\forall e \in E$, then we obtain the problem of finding the Hamiltonian path from x_0 to x_f in the graph $G = (X, E)$. If $T_1 = 0$ and $T_2 \geq |X| - 1$, then a polynomial-time algorithm for determining optimal stationary strategies of players in dynamic c-game with constant costs $c^i_e(t)$ on edges $e \in E$ can be derived.

4.2 The Problem of Determining the Optimal Nonstationary Strategies in Dynamic *c*-Game

We define the nonstationary strategies of players as maps:

$$
\begin{array}{l}
u_1\colon (x,t) \to (y, t+1) \in X(x) \times \{t+1\} \ \text{ for } \ X_1 \setminus \{x_f\},\ t = 0,1,2,\ldots;\\
u_2\colon (x,t) \to (y, t+1) \in X(x) \times \{t+1\} \ \text{ for } \ X_2 \setminus \{x_f\},\ t = 0,1,2,\ldots;\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
u_p\colon (x,t) \to (y, t+1) \in X(x) \times \{t+1\} \ \text{ for } \ X_p \setminus \{x_f\},\ t = 0,1,2,\ldots.
\end{array}
$$

Here (x, t) has the same meaning as the notation $x(t)$, i.e., $(x, t) = x(t)$.

For any set of nonstationary strategies $u_1, u_2, \ldots, u_p$, we define the quantities

$$
F^1_{x_0 x_f}(u_1, u_2, \ldots, u_p),\ \ F^2_{x_0 x_f}(u_1, u_2, \ldots, u_p), \ldots, F^p_{x_0 x_f}(u_1, u_2, \ldots, u_p)
$$

in the following way.

Let $u_1, u_2, \ldots, u_p$ be an arbitrary set of strategies. Then either $u_1, u_2, \ldots, u_p$ generate in G a finite trajectory

$$
x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f
$$

from x_0 to x_f and $T(x_f)$ represents the time moment when x_f is reached, or $u_1, u_2, \ldots, u_p$ generate in G an infinite trajectory

$$
x_0 = x(0), x(1), x(2), \ldots, x(t), x(t+1), \ldots
$$

which does not pass through x_f, i.e., $T(x_f) = \infty$. In such trajectories, the next state $x(t+1)$ is determined uniquely by $x(t)$ and a map u_k, $k \in \{1, 2, \ldots, p\}$ as follows

$$x(t+1) = u_k(x(t),t),\ x(t) \in X_k.$$

If the state x_f is reached at finite moment of time $T(x_f)$ and

$$T_1 \le T(x_f) \le T_2,$$

then we set

$$F^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \sum_{t=0}^{T(x_f)-1} c^i_{(x(t),x(t+1))}(t),\ i = \overline{1,p};$$

otherwise we put

$$F^i_{x_0x_f}(u_1, u_2, \ldots, u_p) = \infty,\ i = \overline{1,p}.$$

Thus we regard the problem of finding the nonstationary strategies u_1^*, $u_2^*, \ldots, u_p^*$ for which the following condition is satisfied

$$F^i_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_p^*)$$

$$\le F^*_{x_0x_f}(u_1^*, u_2^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_p^*),\ \forall\, u_i,\ i = \overline{1,p}.$$

So, we consider the problem of finding the optimal solution in the sense of Nash [44–47].

In the following, we show that a polynomial-time algorithm for solving this problem can be elaborated.

5 The Main Results for Dynamic *c*-Game with Constant Costs of Edges and Determining Optimal Stationary Strategies of Players

In this section, we study the dynamic c-game with constant costs $c^i_e(t) = c^i_e$, $i = \overline{1,p}$, on edges $e \in E$ for the network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x_f, T_1, T_2)$. First we stress our attention to the case of the problem without restriction on the number of stages for dynamical system, i.e., $T_1 = 0$, $T_2 = \infty$. Namely, this case is important for elaboration of polynomial-time algorithms for determining Nash equilibria in the multiobjective control problem in positional form. On the basis of results for this particular case, we will extend algorithms for the general case of the problem.

So, let us consider the dynamic c-game with constant costs of edges $c^i_e(t) = c^i_e$, $i = \overline{1,p}$, $e \in E$, and without restriction on the number of stages by a trajectory from x_0 to x_f. In this case, the dynamic c-game is determined by the network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x_f)$, where $G = (X, E)$ is a directed graph with sink vertex $x_f \in X$. Note that if G contains a vertex $x \in X$, for which there is no directed path from x to x_f, then it can be deleted without changing the sense of the problem.

The Nash equilibria condition and the algorithm for determining optimal stationary strategies of players have been obtained in [3].

First of all, we note that the definition of payoff functions $H^i_{x_0x_f}(s_1, s_2, \dots, s_p)$, $i = \overline{1,p}$, differs here a little from the definition from [3, 5]. In [3, 5] $H^i_{x_0x_f}(s_1, s_2, \dots, s_p)$ for every $s_1, s_2, \dots, s_p$ is defined in the following way. If $s_1, s_2, \dots, s_p$ generate in G a subgraph G_s, which contains a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f, then

$$H^i_{x_0x_f}(s_1, s_2, \dots, s_p) = \sum_{e \in E(P_s(x_0,x_f))} c^i_e. \tag{10}$$

If in G_s there is no directed path from x_0 to x_f, then a unique directed cycle C_s with the set of edges $E(C_s)$ can be obtained when we pass through directed edges from x_0. Therefore there exists a unique directed cycle C_s, which we can get from x_0, and a unique directed path $P'_s(x_0, x')$, which connects x_0 and C_s (the vertex x' is a unique common vertex of $P'_s(x_0, x')$ and C_s). In this case, $H^i_{x_0x_f}(s_1, s_2, \dots, s_p)$ is defined as follows

$$H^i_{x_0x_f}(s_1, s_2, \dots, s_p) = \begin{cases} +\infty, & \text{if } \sum_{e \in E(C_s)} c^i_e > 0; \\ \sum_{e \in E(P'_s(x_0,x'))} c^i_e, & \text{if } \sum_{e \in E(C_s)} c^i_e = 0; \\ -\infty, & \text{if } \sum_{e \in E(C_s)} c^i_e < 0. \end{cases} \tag{11}$$

For positive costs c^i_e on edges $e \in E$ of the network, the problems from [3, 5] and Section 2 coincide. Therefore the results we formulate below are related to all problems with positive and constant costs on edges.

Further, we need the following definitions.

Definition 2. *Let s^0_k and s^1_k de two different strategies of player $k \in \{1, 2, \dots, p\}$ in the dynamic c-game. We say that the strategy s^0_k dominates the strategy s^1_k if for every $x \in X$ the following condition holds*

$$\begin{aligned} &H^i_{xx_f}(s_1, s_2, \dots, s_{k-1}, s^0_k, s_{k+1}, \dots, s_p) \\ &\le H^i_{xx_f}(s_1, s_2, \dots, s_{k-1}, s^1_k, s_{k+1}, \dots, s_p) \\ &\forall (s_1, s_2, \dots, s_{k-1}, s_{k+1}, \dots, s_p) \in S_1 \times S_2 \\ &\times \dots \times S_{k-1} \times S_{k+1} \times \dots \times S_p;\ i = \overline{1,p} \end{aligned} \tag{12}$$

and there exist strategies $s_1, s_2, \dots, s_{k-1}, s_{k+1}, \dots, s_p$ such that

$$\begin{aligned} &H^{i_0}_{xx_f}(s_1, s_2, \dots, s_{k-1}, s^0_k, s_{k+1}, \dots, s_p) \\ &< H^{i_0}_{xx_f}(s_1, s_2, \dots, s_{k-1}, s^1_k, s_{k+1}, \dots, s_p) \end{aligned} \tag{13}$$

for one of the players $i_0 \in \{1, 2, \dots, p\}$ and at least for a vertex $x \in X$.

Definition 3. *The strategy s^1_k is called not essential strategy for the player $k \in \{1, 2, \dots, p\}$ in the dynamic c-game if there exists a strategy $s^0_k \in S_k$, which dominates s^1_k; otherwise the strategy s^1_k is called an essential one.*

The following theorem represents one of the most important results we shall use for determining Nash equilibria in the considered multiobjective control problems on networks.

Theorem 3. *Let $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x_f)$ be a dynamic network for which the vertex x_f in G is attainable from every $x \in X$. Assume that the vectors $c^i = (c^i_{e_1}, c^i_{e_2}, \dots, c^i_{e_{|E|}})$, $i \in \{1, 2, \dots, p\}$ have positive and constant components. Then in the dynamic c-game on network $(G, X_1, X_2, \dots, X_p,\, c^1, c^2, \dots, c^p, x_0, x_f)$ for the players $1, 2, \dots, p$, there exists an optimal solution in the sense of Nash $s_1^*, s_2^*, \dots, s_p^*$, which satisfies the following properties:*

- the graph $G_{s^*} = (X, E_{s^*})$ generated by $s_1^*, s_2^*, \dots, s_p^*$ has a structure of the directed tree with the sink vertex x_f;

- $s_1^*, s_2^*, \dots, s_p^*$ represent the solution of the dynamic c-game on network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x, x_f)$ with an arbitrary starting position $x \in X$ and the given final position x_f.

This theorem has been formulated in [3]. Moreover, in [3] the sketch of its proof is given. Here we give the proof of this theorem in more detailed form.

In order to prove this theorem, we need the following auxiliary result.

Lemma 2. *Let s'_k be a strategy of player k, $k \in \{1, 2, \dots, p\}$, in the dynamic c-game with the network satisfying conditions of Theorem 3. In addition let $G^{s'_k} = (X, E^{s'_k})$ be a graph obtained from G by deleting all edges $e = (x, y) \in E$, originating in $x \in X_k$, except edges $(x, s'_k(x))$. If in $G^{s'_k}$ the vertex x_f is not attainable from at least one of the vertices $x \in X$, then the strategy s'_k is not essential.*

Proof. Assume that for a given strategy s'_k of player k in the corresponding graph $G^{s'_k}$, the vertex x_f is not attainable from vertices $x \in X'$ and it is attainable from the rest of the vertices $x \in X \setminus X'$, where $X' \neq \emptyset$. Fix a strategy $s_k^0 \in S_k$, for which the graph $G^{s_k^0} = (X, E^{s_k^0})$ has the property that x_f is attainable from every $x \in X$, and let us show that s_k^0 dominates s'_k, i.e., the strategy s'_k is not essential.

It is easy to observe that in $G^{s'_k}$, there are no edges $e = (x, y)$ directed from $x \in X'$ to $y \in X \setminus X'$. This means that $H^{i_0}_{x x_f}(s_1, s_2, \dots, s_{k-1}, s'_k, s_{k+1}, \dots, s_p) = \infty$ for $(s_1, s_2, \dots, s_{k-1}, s_{k+1}, \dots, s_p) \in S_1 \times S_2 \times \dots \times S_{k-1} \times S_{k+1} \times \dots \times S_p$, which involves the validity of condition (12) for $x \in X'$.

Moreover, if here we take into account that in the graph $G^{s_k^0}$ the vertex x_f is attainable from every $x \in X'$, then we obtain that at least for a set of strategies $s_1, s_2, \dots, s_{k-1}, s_k^0, s_{k+1}, \dots, s_p$, the values $H^{i_0}_{x x_f}(s_1, s_2, \dots, s_{k-1}, s_k^0, s_{k+1}, \dots, s_p)$ are finite and therefore condition (13) holds.

In the following, let us show that condition (12) also holds for $x \in X \setminus X'$. Indeed, let $s_1, s_2, \dots, s_{k-1}, s_{k+1}, \dots, s_p$ be an arbitrary set of strategies of players $1, 2, \ldots, k-1, k+1, \ldots, p$. If $s_1, s_2, \ldots, s_{k-1}, s'_k, s_{k+1}, \ldots, s_p$ generate in $G^{s'_k}$ a directed path $P_s(x, x_f)$ from $x \in X \setminus X'$ to x_f, then this path does not pass through vertices $x \in X'$. This means that in $G^{s^0_k}$, the set of strategies $s_1, s_2, \ldots, s_{k-1}, s_{k+1}, \ldots, s_p$ generates the same path $P_s(x, x_f)$ from $x \in X \setminus X'$ to x_f. So, condition (12) for $x \in X \setminus X'$ is also satisfied. This proves that the strategy s^0_k dominates s'_k, i.e., s'_k is not essential. ∎

Corollary 3. *Let $G = (X, E)$ be a directed graph in which the vertex x_f is attainable from every $x \in X$. Then for an arbitrary essential strategy s'_k of player $k \in \{1, 2, \ldots, p\}$, the corresponding graph $G^{s'_k} = (X, E^{s'_k})$ has the property that x_f is attainable from every $x \in X$.*

Corollary 4. *Let a dynamic c-game with the network satisfying conditions of Theorem 3 be given. Assume that in this dynamic c-game Nash equilibria exist. Then for the considered game, there exist such a Nash equilibrium $s^*_1, s^*_2, \ldots, s^*_{k-1}, s^*_k, s^*_{k+1}, \ldots, s^*_p$ that the corresponding graph $G_{s^*} = (X, E_{s^*})$ has a structure of the directed tree with the sink vertex x_f.*

Proof of Theorem 3. We prove this theorem by using the induction principle on the number of players in the dynamic c-game. It is easy to observe that for $p = 1$, our problem becomes a well-known optimal paths problem in a weighted directed graph G with sink vertex x_f. For this problem, there exists a tree of optimal paths $G_{s^*} = (X, E_{s^*})$ with sink vertex x_f, which determines a strategy $s^* : x \to y \in X$, where $s^*(x) = y$, $(x, y) \in E_{s^*}$. So, for $p = 1$ the theorem holds.

Let us assume that the assertion holds for any $p \leq k$, $k \geq 1$ and let us show that it is true for $p = k + 1$. We regard the dynamic c-game on the network with $p = k + 1$ players. Without loss of generality, we may assume that $x_0 \in X_1$.

We consider the two following cases:

Case 1. The set X_1 contains only one position, $X_1 = \{x_0\}$ ($|X_1| = 1$), and for the starting position x_0 there are no entering edges $(x, x_0) \in E$.

Case 2. The set X_1 may contain more than one position ($|X_1| \geq 1$), and for the starting position x_0 there may exist entering edges $(x, x_0) \in E$.

At first, let us prove the theorem in case 1. We denote possible admissible strategies of the first player by $s^1_1, s^2_1, \ldots, s^q_1$. Each strategy $s^k_1 : x_0 \to y \in X(x_0)$, $k = \overline{1, q}$, corresponds with an edge $e_{s^k_1} = (x_0, s^k_1(x_0)) \in E(x_0)$. We call a strategy s'_1 of player 1 an admissible strategy, if for the rest of players $2, 3, \ldots, p$ there exist strategies $s'_2, s'_3, \ldots, s'_p$ such that the corresponding graph $G_{s'} = (X, E_{s'})$, generated by strategies $s'_1, s'_2, \ldots, s'_p$, contains a directed path $P(x_0, x_f)$ from x_0 to x_f. It is easy to observe that for each admissible strategy s'_k in the graph $G^{s'_k} = (X, E^{s'_k})$, the vertex x_f is attainable from every $x \in X$. So, an arbitrary admissible strategy, s'_1 is an essential one.

Let us state that the first player fixes his first possible strategy $s_1 = s^1_1$ and we consider the problem of finding the optimal solutions in the sense of Nash with respect to the rest of the players $2, 3, \ldots, p$. Then in the positional form, the obtained game can be regarded as the dynamic c-game with $p-1$ players, because the position x_0 of the first player can be considered as a position of any other player (we consider it as a position of the second player). So, for $s_1 = s_1^1$ we obtain a new dynamic c-game with $p-1$ players on the network $(G^{s_1^1}, X_2^1, X_3, \ldots, X_p, c_1^2, c_1^3, \ldots, c_1^p, x_0, x_f)$, where $X_2^1 = X_1 \cup X_2$ and $G^{s_1^1} = (X, E^{s_1^1})$ is the graph, obtained from G by deleting edges $e = (x_0, y) \in E$, for which $y \neq s_1^1(x_0)$; $c_1^i : E^{s_1^1} \to R^1$ are the functions obtained, respectively, from the function c^i as a result of the contraction of the set E to the set $E^{s_1^1}$, i.e., $c_{1e}^i = c_e^i$, $\forall e \in E^{s_1^1}$, $i = \overline{2,p}$. If we consider the game in the normal form, then it is a game with $p-1$ players, determined by $p-1$ payoff functions $H_{x_0x_f}^2(s_1^1, s_2, s_3, \ldots, s_p)$, $H_{x_0x_f}^3(s_1^1, s_2, s_3, \ldots, s_p), \ldots, H_{x_0x_f}^p(s_1^1, s_2, s_3, \ldots, s_p)$, where $s_2 \in S_2$, $s_3 \in S_3, \ldots, s_p \in S_p$. According to the induction principle for this game with $p-1 = k$ players, there exist optimal by Nash strategies $s_2^{1*}, s_3^{1*}, \ldots, s_p^{1*}$, and the graph $G_{s^*}^1 = (X, E_{s^*}^1)$, which corresponds with the strategies $s_1^1, s_2^{1*}, \ldots, s_p^{1*}$, has a structure of a directed tree with the sink x_f.

In analogous way, we consider the case when the first player fixes his second admissible strategy s_1^2. Then, according to the induction principle, we find the optimal by Nash strategies $s_2^{2*}, s_3^{2*}, \ldots, s_p^{2*}$ of players $2, 3, \ldots, p$ in the dynamic c-game, which in the normal form is determined by the payoff functions $H_{x_0x_f}^i(s_1^2, s_2, s_3, \ldots, s_p)$, $i = \overline{2,p}$. The strategies $s_1^2, s_2^{2*}, s_3^{2*}, \ldots, s_p^{2*}$ generate the graph $G_{s^*}^2 = (X, E_{s^*}^2)$, which has a structure of the tree with the sink x_f.

Further, we consider the case when the first player fixes his third admissible strategy s_1^3 and we find the optimal by Nash strategies $s_1^3, s_2^{3*}, \ldots, s_p^{3*}$.

Continuing this process, we find the following set of strategies of players $1, 2, \ldots, p$

$$s_1^1, s_2^{1*}, s_3^{1*}, \ldots, s_p^{1*};$$

$$s_1^2, s_2^{2*}, s_3^{2*}, \ldots, s_p^{2*};$$

$$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$$

$$s_1^q, s_2^{q*}, s_3^{q*}, \ldots, s_p^{q*}$$

and the corresponding directed trees $G_{s^*}^1, G_{s^*}^2, \ldots, G_{s^*}^q$ with sink vertex.

Among these sets of players' strategies in the dynamic c-game, we choose the set $s_1^{j*}, s_2^{j*}, s_3^{j*}, \ldots, s_p^{j*}$, for which

$$H_{x_0x_f}^1(s_1^{j*}, s_2^{j*}, \ldots, s_p^{j*}) = \min_{1 \le i \le q} H_{x_0x_f}^i(s_1^i, s_2^{i*}, \ldots, s_p^{i*}) \tag{14}$$

Let us show that $s_1^{j*}, s_2^{j*}, \ldots, s_p^{j*}$ are optimal by Nash strategies for the players $1, 2, \ldots, p$ in the dynamic c-game.

Indeed,

$$H_{x_0x_f}^i(s_1^{j*}, s_2^{j*}, \ldots, s_{i-1}^{j*}, s_i^{j*}, s_{i+1}^{j*}, \ldots, s_p^{j*})$$

$$\le H^i_{x_0x_f}(s_1^{j*}, s_2^{j*}, \dots, s_{i-1}^{j*}, s_i, s_{i+1}^{j*}, \dots, s_p^{j*}),\ \forall s_i \in S_i,\ i = \overline{1,p},$$

as $s_2^{j*}, s_3^{j*}, \dots, s_p^{j*}$ are the optimal by Nash strategies in the dynamic c-game for $s_1 = s_1^{j*}$. Taking into account that the graph $G^{j*}_{s^*} = (X, E^{j*}_{s*})$, generated by the strategies $s_1^{j*}, s_2^{j*}, s_3^{j*}, \dots, s_p^{j*}$, has a structure of a directed tree with sink vertex and $j*$ is chosen according to (14), we have

$$H^1_{x_0x_f}(s_1^{j*}, s_2^{j*}, \dots, s_p^{j*}) \le H^1_{x_0x_f}(s_1, s_2^{j*}, \dots, s_p^{j*}),\ \forall s_1 \in S_1.$$

So, in case 1 the theorem holds.

Note that the given proof of case 1 takes place also if the vertex x_0 contains entering edges. For the proof of the general statement of the theorem, we shall use the case when x_0 does not contain entering edges.

Now let us prove the theorem in case 2. We assume that the set X_1 may contain more than one position ($|X_1| \ge 1$), and for the starting position x_0 there may exist entering edges (x, x_0).

Let us show that this case can be reduced to case 1.

On the basis of Lemma 2, if in the dynamic c-game Nash equilibria exists, then an optimal strategy of the first player s_1^* will correspond with the case when the graph $G^{s_1^*} = (X, E^{s_1^*})$ has the property that x_f is attainable from every $x \in X$. Therefore we select all possible strategies $s_1^1, s_1^2, \dots, s_1^q$, for which the corresponding graphs $G^{s_1^1} = (X, E^{s_1^1})$, $G^{s_1^2} = (X, E^{s_1^2}), \dots, G^{s_1^q} = (X, E^{s_1^q})$ have the property that x_f is attainable from every $x \in X$. After that, we construct an auxiliary graph $\overline{G} = (\overline{X}, \overline{E})$, which is obtained from the graphs $G^{s_1^1}$, $G^{s_1^2}, \dots, G^{s_1^q}$ by using a special construction.

In order to describe how to obtain $\overline{G}$ from $G^{s_1^1}, G^{s_1^2}, \dots, G^{s_1^q}$, we will distinguish the vertex sets from different graphs $G^{s_1^i}$, $G^{s_1^j}$ by using the notations X^i and X^j, which mean that X^i is a vertex set of $G^{s_1^i}$ and X^j is a vertex set of $G^{s_1^j}$; for vertices of the corresponding graphs, we also use the notation $x^i \in X^i$ and $x^j \in X^j$.

The graph $\overline{G}$ is obtained from $G^{s_1^1}$, $G^{s_1^2}, \dots, G^{s_1^q}$ in the following way: the sink vertices $x_f^1, x_f^2, \dots, x_f^q$ of the corresponding graphs we identify in $\overline{G}$ by a common sink vertex x'_f (see Fig. 2).

After that, we add a new vertex x'_0, which is connected by directed edges $e'_{s_1^i} = (x'_0, x_0^i)$, $i = \overline{1,q}$, with corresponding vertices $x_0^i \in X^i$. We associate costs $c'_{e_1^i} = \varepsilon$, where $\varepsilon > 0$ is a small value, to these edges $e'_{s_1^i}$, $i = \overline{1,q}$; all costs on edges from $G^{s_1^i}$ are preserved as in the initial graphs.

On $\overline{G}$, we consider the dynamic c-game with the starting position x'_0 and the final position x_f. According to case 1 there exists Nash equilibrium $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$, for which the corresponding graph $\overline{G}_{\overline{s}^*} = (\overline{X}, \overline{E}_{\overline{s}^*})$ has a structure of directed tree with sink vertex. If we fix in $\overline{G}_{\overline{s}^*}$ the edge $e'_{s_1^{i*}} = (x'_0, x_0^{i*})$ for which $x_0^{i*} = s^*(x'_0)$, then we find the subtree $\overline{G}^{i*}_{\overline{s}^*} = (X^{i*}, E^{i*}_{s^*})$ of $\overline{G}_{s*}$, generated by the set X^{i*}. This tree corresponds with the tree $G_{s*} = (X, E_{s*})$ of optimal by Nash strategies $s_1^*, s_2^*, \dots, s_q^*$.

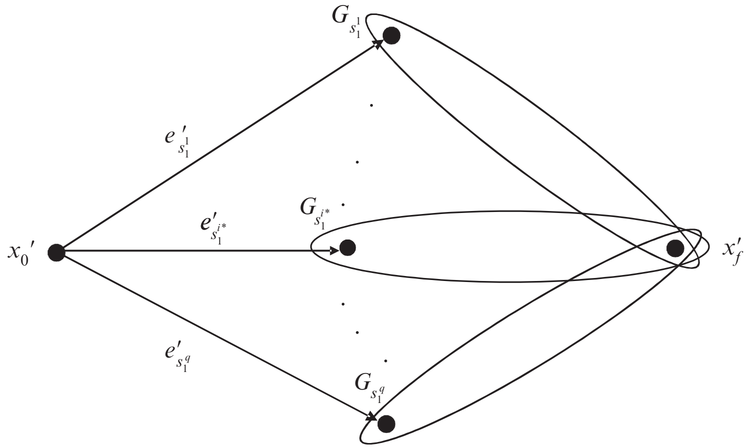

Figure 2. The auxiliary network with starting position x_0'

Remark 1. For the dynamic c-game with payoff functions $H^i_{x_0x_f}(s_1, s_2, \ldots, s_p)$, $i = \overline{1,p}$, defined according to (10), (11), Theorem 3 holds for nonnegative costs c^i_e, $e \in E$, $i = \overline{1,p}$, if $\sum_{e\in E(C_s)} c^i_e \neq 0$ for every directed cycle C_s in G. For the dynamic c-game from Section 4.1, Theorem 3 holds for arbitrary nonnegative costs c^i_e, $e \in E$, $i = \overline{1,p}$.

Theorem 4. *Let $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x_f)$ be a network for which the vertex x_f in G is attainable from every $x \in X$. Assume that the vectors $c^i = (c^i_{e_1}, c^i_{e_2}, \ldots, c^i_{e_{|E|}})$, $i \in \{1, 2, \ldots, p\}$ have positive and constant components. Then on the vertex set X of the network game, there exist p real functions*

$$\varepsilon^1 : X \to R^1,\ \varepsilon^2 : X \to R^1, \ldots, \varepsilon^p : X \to R^1,$$

which satisfy the conditions:

(a) $\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} \geq 0,\ \forall (x,y) \in E_i,\ i = \overline{1,p},$
where $E_i = \{e = (x,y) \in E | x \in X_i,\ y \in X\}$*;*

(b) $\min\limits_{y \in X_G(x)} \{\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)}\} = 0,\ \forall x \in X_i,\ i = \overline{1,p}$*;*

(c) the subgraph $G^0 = (X, E^0)$ generated by the edge set $E^0 = E^0_1 \cup E^0_2 \cup \cup \cdots \cup E^0_p$, $E^0_i = \{e = (x,y) \in E_i | \varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} = 0\}$, $i = \overline{1,p}$, has the property that the vertex x_f is attainable from any vertex $x \in X$, and G^0 contains a subgraph $\overline{G}^0 = (X, \overline{E}^0)$, $\overline{E}^0 \subset E$, which possesses the same property, and besides that

$$\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} = 0,\ \forall (x,y) \in \overline{E}^0,\ i = \overline{1,p}.$$

If $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^p$ are arbitrary real functions, which satisfy conditions (a)–(c), then the optimal by Nash strategies in the dynamic c-game with the network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x_f)$ can be found as follows:

choose in $\overline{G}^0$ *an arbitrary directed tree* $GT = (X, E^*)$ *with the sink vertex* x_f *and fix in* GT *the following maps:*

$$s_1^* : x \to y \in X_{GT}(x) \text{ for } x \in X_1;$$

$$s_2^* : x \to y \in X_{GT}(x) \text{ for } x \in X_2;$$

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$$s_p^* : x \to y \in X_{GT}(x) \text{ for } x \in X_p,$$

where $X_{GT}(x) = \{y \in X | (x, y) \in E^*\}$.

Proof. According to Theorem 3 in the dynamic c-game with network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x_0, x_f)$, there exist optimal by Nash strategies s_1^*, $s_2^*, \ldots, s_p^*$ of players $1, 2, \ldots, p$, and these strategies generate in G a directed tree $GT_{s^*} = (X, E_{s^*})$ with sink vertex x_f. In this tree, we find the functions

$$\varepsilon^1 : X \to R^1,\ \varepsilon^2 : X \to R^1, \ldots, \varepsilon^p : X \to R^1,$$

where $\varepsilon^i(x) = H^i_{xx_f}(s_1^*, s_2^*, \ldots, s_p^*)$, $\forall x \in X$, $i = \overline{1,p}$. It is easy to verify that $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^p$ satisfy conditions (a) and (b). In addition we can see that in G^0 there exists the graph $\overline{G}^0 = (X, \overline{E}^0)$, which satisfies condition (c), because $GT \subseteq \overline{G}^0$. Moreover, if in $\overline{G}^0$ a directed tree $GT_{s'} = (X, E_{s'})$, different from GT_{s^*}, with sink vertex is chosen, then $GT_{s'}$ generates another optimal by Nash solution s'_1, $s'_2, \ldots, s'_p$.

Now let us show that if

$$\varepsilon^1 : X \to R^1,\ \varepsilon^2 : X \to R^1, \ldots, \varepsilon^p : X \to R^1,$$

are arbitrary functions, which verify conditions (a)–(c), then an arbitrary directed tree $GT = (X, E_{s^*})$ of $\overline{G}^0$ generates the maps

$$s_1^* : x \to y \in X_{GT}(x) \text{ for } x \in X_1;$$

$$s_2^* : x \to y \in X_{GT}(x) \text{ for } x \in X_2;$$

$$- - - - - - - - - - - -$$

$$s_p^* : x \to y \in X_{GT}(x) \text{ for } x \in X_p,$$

which correspond with an optimal by Nash solution.

We use induction on the number p of players in the dynamic c-game. In the case $p = 1$, the statement is true, because $X_1 = X$ and conditions (a)–(c) for positive c_e^1 provide existence of the tree $GT = (X, E_{s^*})$ of optimal paths, which correspond with the solution s_1^* for the problem of finding the shortest paths from $x \in X$ to x_f in G.

Assume that the statement holds for $p \leq k$, $k \geq 1$, and let us prove it for $p = k + 1$. We consider that the first player fixes his strategy $s_1 = s_1^*$ and consider the problem of finding optimal by Nash strategies in the network game with respect to other players. The obtained game in the positional form can be interpreted as a c-game with $p-1$ players, as the positions of the first player can be considered as the positions of any other player. Further, we consider them as the positions of the second player.

Thus, if $s_1 = s_1^*$, we obtain a new game with $p-1$ players in the network game $(G^1, X_2^1, X_3, \dots, X_p, c_1^2, c_1^3, \dots, c_1^p, x_0, x_f)$, where X_2^1, G^1 and the functions c_1^i, $i = 2, \dots, p$, are defined as in the proof of Theorem 3. In the normal form, this game is determined by the functions

$$H^2_{x_0 x_f}(s_1^*, s_2, \dots, s_p),\ H^3_{x_0 x_f}(s_1^*, s_2, \dots, s_p), \dots, H^p_{x_0 x_f}(s_1^*, s_2, \dots, s_p),$$

$s_2 \in S_2,\ s_3 \in S_3, \dots, s_p \in S_p$, where $S_2, S_3, \dots, S_p$ are the respective sets of admissible strategies of players $2, 3, \dots, p$.

In the new network game $(G^1, X_2^1, X_3, \dots, X_p, c_1^2, c_1^3, \dots, c_1^p, x_0, x_f)$, consider $p-1$ functions

$$\varepsilon^2 : X \to R^1,\ \varepsilon^3 : X \to R^1, \dots, \varepsilon^p : X \to R^1,$$

which satisfy the conditions

(a) $\varepsilon^i(x) - \varepsilon^i(y) + c^i_{1(x,y)} \geq 0,\ \forall (x,y) \in E_i^1,\ i = \overline{2,p}$, where
$E_2^1 = \{e = (x,y) \in E^1 | x \in X_2^1,\ y \in X\}$,
$E_i^1 = \{e = (x,y) \in E^1 | x \in X_i,\ y \in X\},\ i = \overline{3,p}$;

(b) $\min\limits_{y \in X_{G^1}(x)} \{\varepsilon^2(x) - \varepsilon^2(y) + c^2_{1(x,y)}\} = 0,\ \forall x \in X_2^1$,
$\min\limits_{y \in X_{G^1}(x)} \{\varepsilon^2(x) - \varepsilon^2(y) + c^i_{1(x,y)}\} = 0,\ \forall x \in X_i,\ i = \overline{3,p}$;

(c) the subgraph $G^{1^0} = (X, E^{1^0})$ generated by the edge set $E^{1^0} = E_2^{1^0} \cup E_3^0 \cup \dots \cup E_p^0$, $E_2^{1^0} = \{e = (x,y) \in E_2^1 | \varepsilon^2(x) - \varepsilon^2(y) + c^2_{1(x,y)} = 0\}$, $E_i^0 = \{e = (x,y) \in E_i | \varepsilon^i(x) - \varepsilon^i(y) + c^i_{1(x,y)} = 0\}$, $i = \overline{3,p}$, has the property that the vertex x_f is attainable from any vertex $x \in X$, and G^{1^0} contains a subgraph $\overline{G}^{1^0} = (X, \overline{E}^{1^0})$, which possesses the same property, and besides that

$$\varepsilon^i(x) - \varepsilon^i(y) + c^i_{1(x,y)} = 0,\ \forall (x,y) \in \overline{E}^{1^0},\ i = \overline{2,p}.$$

According to the induction assumption, in the network game $(G^1, X_2^1, X_3, \dots, X_p, c_1^2, c_1^3, \dots, c_1^p, x_0, x_f)$, the solution $\overline{s}_2^*, s_3^*, \dots, s_p^*$ generated by the directed tree $GT = (X, E_{s^*})$,

$$\overline{s}_2^* : x \to y \in X_{GT}(x) \text{ for } x \in X_2^1;$$

$$s_3^* : x \to y \in X_{GT}(x) \text{ for } x \in X_3;$$

$$- - - - - - - - - - - -$$

$$s_p^* : x \to y \in X_{GT}(x) \text{ for } x \in X_p,$$

where $\overline{s}_2^*(x) = s_1^*(x)$ for $x \in X_1$ and $\overline{s}_2^*(x) = s_2^*(x)$ for $x \in X_2$, is optimal by Nash.

Thus

$$H^i_{xx_f}(s_1^*, s_2^*, s_3^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_p^*)$$
$$\leq H^i_{xx_f}(s_1^*, s_2^*, s_3^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_p^*),$$
$$\forall s_i \in S_i, 2 \leq i \leq p.$$

Also, it is easy to verify that

$$H^1_{xx_f}(s_1^*, s_2^*, \dots, s_p^*) \leq H^1_{xx_f}(s_1, s_2^*, \dots, s_p^*),\ \forall s_1 \in S_1,$$

because for fixed $s_2^*, s_3^*, \dots, s_p^*$ in G, the problem of finding

$$\min_{s_1 \in S_1} H^1_{xx_f}(s_1, s_2^*, \dots, s_p^*) \ \text{ for } x \in X$$

becomes the problem of finding the shortest paths from x to x_f in the graph $G' = (X, E')$, generated by set E_1 and edges $(x, s_i^*(x))$, $x \in X_i$, $i = \overline{2, p}$, with the costs c_e^1 on edges $e \in E'$. On this graph, the following condition is satisfied

$$\varepsilon^1(x) - \varepsilon^1(y) + c^1_{(x,y)} \geq 0;\ \forall (x, y) \in E',$$

which involves

$$H^1_{xx_f}(s_1^*, s_2^*, \dots, s_p^*) \leq H^1_{xx_f}(s_1, s_2^*, \dots, s_p^*),\ \forall s_1 \in S_1,$$

because $H^1_{xx_f}(s_1^*, s_2^*, \dots, s_p^*) = \varepsilon^1(x)$, $\forall x \in X$.

Hence $s_1^*, s_2^*, \dots, s_p^*$ is an optimal solution in the sense of Nash in the dynamic c-game. ■

Remark 2. Let

$$\varepsilon^1 : X \to R^1,\ \varepsilon^2 : X \to R^1, \dots, \varepsilon^p : X \to R^1,$$

be arbitrary real functions on X in G, and $\overline{c}^1, \overline{c}^2, \dots, \overline{c}^p$ are p new cost functions on edges $e \in E$ obtained from $c^1, c^2, \dots, c^p$ as follows:

$$\overline{c}^i_{(x,y)} = \varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)},\ \forall (x, y) \in E,\ i = \overline{1, p}. \tag{15}$$

Then the dynamic c-games determined on networks $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x_f)$ and $(G, X_1, X_2, \dots, X_p, \overline{c}^1, \overline{c}^2, \dots, \overline{c}^p, x_0, x_f)$, respectively, are equivalent, because the payoff functions $H^i_{xx_f}(s_1, s_2, \dots, s_p)$ and $\overline{H}^i_{xx_f}(s_1, s_2, \dots, s_p)$ in such games differ only by a constant, i.e.,

$$H^i_{xx_f}(s_1, s_2, \dots, s_p) = \overline{H}^i_{xx_f}(s_1, s_2, \dots, s_p) + \varepsilon^i(x) - \varepsilon^i(x_f).$$

In [3,5], transformation (15) is named the potentional transformation of edges' costs of players in G.

Remark 3. The conditions of Theorem 4 guarantee the existence of optimal stationary strategies $s_1^*, s_2^*, \ldots, s_p^*$ of players $1, 2, \ldots, p$ for every starting position $x \in X$ in the dynamic c-game on network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x, x_f)$ with positive and constant cost functions $c^1, c^2, \ldots, c^p$. If $c^1, c^2, \ldots, c^p$ are arbitrary constant functions, then the conditions of Theorem 4 represent necessary and sufficient conditions for the existence of optimal stationary strategies $s_1^*, s_2^*, \ldots, s_p^*$ in the dynamic c-game on network $(G, X_1, X_2, \ldots, X_p, c^1, c^2, \ldots, c^p, x, x_f)$ for every starting position $x \in X$.

On the basis of the obtained results, we can propose the following algorithm for determining Nash equilibria in the considered dynamic game with constant costs of edges on networks.

Algorithm 2. Determining Nash Equilibria for the Dynamic c-Game on Acyclic Network

Let us consider a dynamic c-game for which the graph $G = (X, E)$ has a structure of acyclic directed graph with sink vertex x_f.

Preliminary step (Step 0): Fix $X^0 = \{x_f\}$ and put $\varepsilon^i(x_f) = 0$, $\forall i = \overline{1,p}$;

General step (Step k, $k \geq 1$): If $X \setminus X^{k-1} = \emptyset$ then STOP; otherwise find a vertex $x^k \in X \setminus X^{k-1}$ for which $X_G(x^k) \subseteq X^{k-1}$, where $X_G(x^k) = \{y \in X | (x^k, y) \in E\}$. If $x^k \in X_{i_k}$, $i_k \in \{1, 2, \ldots, p\}$, then find an edge (x^k, y^k) for which

$$\varepsilon^{i_k}(y^k) + c^{i_k}_{(x^k, y^k)} = \min_{y \in X_G(x^k)} \{\varepsilon^{i_k}(y) + c^{i_k}_{(x^k, y)}\}.$$

After that put

$$\varepsilon^i(x^k) = \varepsilon^i(y^k) + c^i_{(x^k, y^k)}, \ i = \overline{1,p}$$

and

$$X^k = X^{k-1} \cup \{x^k\}.$$

Then go to the next step.

If the functions ε^i, $i = \overline{1,p}$, are known, then the optimal strategies of players $s_1^*, s_2^*, \ldots, s_p^*$ can be found as follows. Find a tree $GT_{s^*} = (X, E_{s^*})$ in the graph $\overline{G}^0 = (X, \overline{E}^0)$ and fix the strategies

$$s^i(x) : x \to y \in X_i, \ (x, y) \in E_{s^*}, \ i = \overline{1,p}.$$

Algorithm 3. Determining Nash Equilibria in Dynamic c-Game on Arbitrary Network, Based on Reduction to the Case with Acyclic Network

Let us have a dynamic c-game with p players and let the directed graph G have an arbitrary structure, i.e., G may contain directed cycles. Moreover, we consider that for x_f there are no leaving edges $(x_f, x) \in E$. We show that the problem in this case can be reduced to the problem of finding the optimal strategies in an auxiliary game in a network without directed cycles.

We construct an auxiliary directed graph $\overline{G} = (Z, \overline{E})$ without directed cycles, where Z and $\overline{E}$ are defined as follows:

$$Z = Z^0 \cup Z^1 \cup Z^2 \cup \cdots \cup Z^{|X|-1},$$

where

$$Z^j = \{z_0^j, z_1^j, z_2^j, \ldots, z_{|X|-1}^j\},\ j = \overline{0, |X|-1},$$

so, $Z^0, Z^1, \ldots, Z^{|X|-1}$ represent the copies of the set X;

$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \cdots \cup E^{|X|-2} \cup E^f,$$

where

$$E^j = \{(z_k^j, z_l^{j+1}) | (x_k, x_l) \in E\},\ j = \overline{0, |X|-2};$$

$$E^f = \{(z_k^j, z_f^{|X|-1}) | (x_k, x_f) \in E,\ j = \overline{0, |X|-3}\}.$$

It is easy to observe that the vertex $z_f^{|X|-1}$ is attainable in this graph from any $z_k^0 \in Z^0$. If we delete in $\overline{G}$ all vertices z_k^i, for which there is no directed path from z_k^i to z_f^i, then we obtain an acyclic directed graph $\overline{G}' = (Z', \overline{E}')$ with sink vertex $z_f^{|X|-1}$. In the following, we divide vertex set Z' into p subsets $Z'_1, Z'_2, \ldots, Z'_p$ corresponding with the position sets of players $1, 2, \ldots, p$, respectively:

$$Z'_1 = \{z_k^j \in Z' | x_k \in X_1,\ j = \overline{0, |X|-1}\}$$

$$Z'_2 = \{z_k^j \in Z' | x_k \in X_2,\ j = \overline{0, |X|-1}\}$$

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$$Z'_p = \{z_k^j \in Z' | x_k \in X_p,\ j = \overline{0, |X|-1}\}.$$

We define on the edge set $\overline{E}'$ the cost functions as follows:

$$\overline{c}^i_{(z_k^j, z_l^{j+1})} = c^i_{(x_k, x_l)},\ \forall (z_k^j, z_l^{j+1}) \in E^j,\ j = \overline{0, |X|-2},\ i = \overline{1, p};$$

$$\overline{c}^i_{(z_k^j, z_f^{|X|-1})} = c^i_{(x_k, x_f)},\ \forall (z_k^j, z_f^{|X|-1}) \in E^f,\ j = \overline{1, |X|-3};$$

After that, we consider the dynamic c-game with the network $(\overline{G}' Z'_1, Z'_2, \ldots, Z'_p, \overline{c}^1, \overline{c}^2, \ldots, \overline{c}^p, z_0^0, z_f^{|X|-1})$, where $\overline{G}'$ is an acyclic directed graph with sink vertex $z_f^{|X|-1}$. If we use Algorithm 3, then we find the values $\varepsilon^i(z_k^j)$, $\forall z_k^j \in Z'$, $i = \overline{1, p}$. It is easy to observe that if we put $\varepsilon^i(x_f) = 0$, $i = \overline{1, p}$, and $\varepsilon^i(x_k) = \varepsilon^i(z_k^{|X|-1})$, $\forall x_k \in X \setminus \{x_f\}$, $i = \overline{1, p}$, then we obtain functions $\varepsilon^i : X \to R$, which satisfy conditions (a)–(c) from Theorem 4. Thus, we find the tree $GT = (X, E_s)$, which corresponds with optimal strategies $s_1^*, s_2^*, \ldots, s_p^*$ of players in our dynamic c-game.

Algorithm 3 is inconvenient because of the great number of vertices in the auxiliary network.

Further, we present a simpler algorithm for finding the optimal strategies of players.

Algorithm 4. Determining Nash Equilibria for the Dynamic c-Game with an Arbitrary Network

Preliminary step: Assign to every vertex $x \in X$ a set of labels $\varepsilon^1(x)$, $\varepsilon^2(x), \dots, \varepsilon^p(x)$ as follows:

$$\varepsilon^i(x_f) = 0, \ \forall i = 1, \dots, p,$$
$$\varepsilon^i(x) = \infty, \ \forall x \in X \setminus \{x_f\}, i = 1, \dots, p.$$

General step (step k ($k \geq 1$)): For every vertex $x \in X \setminus \{x_f\}$, change labels $\varepsilon^i(x)$, $i = 1, \dots, p$, in the following way. If $x \in X_k$, then find the vertex $\overline{x}$ for which

$$\varepsilon^k(\overline{x}) + c^k_{(x,\overline{x})} = \min_{y \in X(x)} \{\varepsilon^k(y) + c^k_{(x,y)}\}.$$

If $\varepsilon^k(x) > \varepsilon^k(\overline{x}) + c^k_{(x,\overline{x})}$, then replace $\varepsilon^i(x)$ by $\varepsilon^i(\overline{x}) + c^i_{(x,\overline{x})}$, $i = 1, \dots, p$. If $\varepsilon^k(x) \leq \varepsilon^k(\overline{x}) + c^k_{(x,\overline{x})}$, then do not change the labels.

Repeat the general step n times. Then labels $\varepsilon^i(x)$, $i = 1, \dots, p$, $x \in X$, become constant.

Let us note that these labels satisfy the conditions of Theorem 4. Hence, using labels $\varepsilon^i(x)$, $i = 1, \dots, p$, $x \in X$, and Theorem 4, we construct optimal by Nash strategies of players $1, 2, \dots, p$. Algorithm 4 has the computational complexity $O(p|X|^2|E|)$.

For the general case of the dynamic c-game, the following theorem holds.

Theorem 5. *Let $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x_f, T_1, T_2)$ be a dynamic network for which in G there exists a directed path $P_s(x_0, x_f)$ from x_0 to x_f such that condition (9) holds ($0 \leq T_1 \leq |X| - 1$, $T_1 \leq T_2$). In addition, assume that in the network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x_f, T_1, T_2)$ vectors $c^i = (c^i_{e_1}, c^i_{e_2}, \dots, c^i_{e_{|E|}})$, $i \in \{1, 2, \dots, p\}$ have positive and constant components. Then in the dynamic c-game on network $(G, X_1, X_2, \dots, X_p$, $c^1, c^2, \dots, c^p, x_0, x_f, T_1, T_2)$, there exists an optimal solution in the sense of Nash $s_1^*, s_2^*, \dots, s_p^*$.*

This theorem can be proved by using the constructive scheme of the proof of Theorem 3 with some modifications.

6 Computational Complexity of Problem of Determining Optimal Stationary Strategies in Dynamic c-Game

The results from Section 5 allow us to describe a class of dynamic c-games for which polynomial-time algorithms for determining the optimal stationary strategies of players can be elaborated. This class is related to dynamic

c-games with constant cost functions on edges of the network and without restrictions on the number of stages.

In general, if additional condition (9) on the number of stages for the considered problem is given, then it is NP-hard. This problem remains NP-hard even when $p = 1$, $T_1 = T_2 = |X| - 1$ and the costs of edges are constant, because in this case it becomes a Hamiltonian path problem in G with the given starting vertex x_0 and the final vertex x_f.

In the following, we can see that if G has a structure of acyclic directed graph, then the problem of determining the optimal stationary strategies in the dynamic c-game with the given restriction on the number of stages and constant costs of edges can be reduced to a similar problem on an auxiliary time-expanded network (see Section 8.1).

7 On Determining the Optimal Stationary Strategies for Dynamic c-Game with Nonconstant Cost Functions on Edges

Again we consider the problem of determining Nash equilibria for dynamic c-game without restrictions on the number of stages by a trajectory from x_0 to x_f. The dynamic c-game is determined by the network $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ where the components $c^i_{e_k}(t)$ of the vector functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_{|E|}}(t))$, $i = \overline{1,p}$, may be nonconstant functions.

First of all, we note that Nash equilibria in such games may fail to hold even in the case of positive and nondecreasing cost functions $c^i_e(t)$, $i = \overline{1,p}$, $e \in E$, defined on edges of the dynamic network.

An example, which confirms this affirmation, is the following: we consider the dynamic network, represented by Fig. 3, which consists of the directed graph $G = (X, E)$, for which the partition of vertex set $X = X_1 \cup X_2$, $X_1 = \{1, 2\}$, $X_2 = \{3, 4, 5\}$, where $x_0 = 1$, $x_f = 5$, is given.

All cost functions on edges are constantly equal to 1 except the following:

$$c^1_{(1,3)}(t) \equiv c^2_{(1,3)}(t) \equiv 3;$$

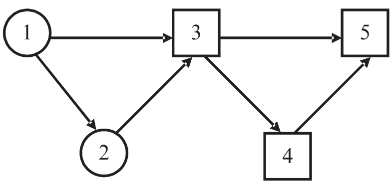

Figure 3. The network for which Nash equilibria may not exist

$$c^2_{(3,5)}(t) = \begin{cases} 1 & \text{if } t \le 1, \\ M & \text{if } t > 1; \end{cases}$$

$$c^1_{(4,5)}(t) = \begin{cases} 1 & \text{if } t \le 2, \\ M & \text{if } t > 2, \end{cases}$$

where M is a great number. It is easy to check that the optimal stationary strategies in the sense of Nash for the dynamic c-game on this network do not exist. Nevertheless, we describe a class of dynamic c-game with nonconstant costs $c^i_e(t)$, $i = \overline{1,p}$, on edges $e \in E$ of the network for which Nash equilibria exist.

At first, we extend the optimization principle for the stationary case of the problem on dynamic networks with p players. We define the optimization principle with respect to player i, $i \in \{1, 2, \ldots, p\}$, on dynamic networks $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$.

Let E^i be a subset of edges from E starting in vertices $x \in X_i$, i.e., $E^i = \{(x, y) \in E \,|\, x \in X_i\}$, $i = \overline{1,p}$. Hereby, the set E^i represents the admissible set of the system's passages from the state $x \in X_i$ to the state $y \in X$ for the player i. Furthermore, the set E^i indicates the set of edges of player i. By E_{s_i} we denote the subset of E^i generated by a fixed strategy s_i of player $i, i \in \{1, 2, \ldots, p\}$, i.e., $E_{s_i} = \{(x, y) \in E^i \,|\, x \in X_i, y = s_i(x)\}$.

Let $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$ be a set of strategies of players $1, 2, \ldots, i-1, i+1, \ldots, p$ and let $G_{S \setminus s_i} = (X, E_{S \setminus s_i})$ be the subgraph of G, where

$$E_{S \setminus s_i} = E_{s_1} \cup E_{s_2} \cup \cdots \cup E_{s_i - 1} \cup E^i \cup E_{s_i + 1} \cup \cdots \cup E_{s_p}.$$

The graph $G_{S \setminus s_i}$ represents the subgraph of G generated by the set of edges of player i and edges of E when the players $1, 2, \ldots, i-1, i+1, \ldots, p$ fix their strategies $s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$, respectively. On $G_{S \setminus s_i}$, we consider the single objective control problem with respect to cost functions $c^i_e(t)$ of player i, starting vertex x_0 and final vertex x_f.

Definition 4. *Let us assume that for any given set of strategies*

$$s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p$$

the cost functions $c^i_e(t)$, $e \in E_{S \setminus s_i}$ in $G_{S \setminus s_i}$ have the property that if an arbitrary optimal path $P^*(x_0, x)$ can be represented as $P^*(x_0, x) = P^*_1(x_0, z) \cup P^*_2(z, x)$ ($P^*_1(x_0, z)$ and $P^*_2(z, x)$ have no common edges), then the leading part $P^*_1(x_0, z)$ of $P^*(x_0, x)$ is an optimal one. We call this property the optimization principle for dynamic networks with respect to player i.

Theorem 6. *Let $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ be a dynamic network with p players for which the vertex x_f in G is attainable from any vertex $x \in X$. Assume that the vector-functions $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_{|E|}}(t))$, $i = \overline{1,p}$ have non-negative and nondecreasing components. Moreover, let us assume that the optimization principle on the dynamic network is* *satisfied with respect to each player. Then, in the dynamic c-game on network* $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f)$ *for players* $1, 2, \ldots, p$*, there exists an optimal solution in the sense of Nash* $s_1^*, s_2^*, \ldots, s_p^*$.

This theorem can be proved in the same way as Theorem 3. The proof of this theorem is given in [38].

In general, if for the dynamic c-game with positive and nondecreasing cost functions $c_e^i(t)$, $e \in E$, $i = \overline{1,p}$, Nash equilibria $s_1^*, s_2^*, \ldots, s_p^*$ exists, then the optimal trajectory $x_0, x_1, \ldots, x_f$, generated by these strategies, corresponds with the optimal trajectory for the nonstationary dynamic c-game. In the next section, we show that for the nonstationary dynamic c-game, Nash equilibria exists if at least one directed path from x_0 to x_f exists. A polynomial time algorithm for determining optimal trajectory from x_0 to x_f is proposed in Section 8.

Here it is important to note that the optimal strategies of players s_1^*, $s_2^*, \ldots, s_P^*$ in the dynamic c-game depend on starting position x_0. In addition, in [37] the following result is proved:

Theorem 7. *Let* $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots c^p(t), x_0, x_f)$ *be a dynamic network with p players for which in G any vertex* $x \in X$ *is attainable from* x_0 *and vector-functions* $c^i(t) = (c^i_{e_1}(t), c^i_{e_2}(t), \ldots, c^i_{e_{|E|}}(t))$, $i = \overline{1,p}$*, have non-negative and nondecreasing components. Moreover, let us consider that the optimization principle for the dynamic network is satisfied with respect to each player. Then, in G there exists a tree* $GT^* = (X, E^*)$ *for which any vertex* $x \in X$ *is attainable from* x_0*, and a unique directed path* $P_{GT^*}(x_0, x)$ *from* x_0 *to x in* GT^* *corresponds with optimal strategies* $s_1^*, s_2^*, \ldots, s_p^*$ *of players in the dynamic c-game on network* $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x)$ *with starting position* x_0 *and final position x. For different vertices x and y, the optimal paths* $P'_{GT^*}(x_0, x)$ *and* $P''_{GT^*}(x_0, y)$ *correspond with different strategies of players* $\overline{s}_1^*, \overline{s}_2^*, \ldots, \overline{s}_p^*$ *and* $\overline{\overline{s}}_1^*, \overline{\overline{s}}_2^*, \ldots, \overline{\overline{s}}_p^*$ *in different games with starting vertex* x_0 *and final positions* x, y*, respectively.*

If the optimization principle in the dynamic c-game is satisfied with respect to each player, then the following algorithm finds the tree of optimal paths $GT^* = (X, E^*)$ in G, when G has no directed cycles, i.e., G is an acyclic graph. We assume that the positions of the network are numbered with $0, 1, 2, \ldots, |X|-1$ according to partial order determined by the structure of the acyclic graph G. This means that if $y > x$, then there is no directed path $P(y, x)$ from y to x. The algorithm consists of $|X|$ steps and constructs a sequence of trees $GT^k = (X^k, E^k)$, $k = \overline{0, |X|-1}$, such that at the final step $k = |X| - 1$, we obtain $GT^{|X|-1} = GT^*$.

Algorithm 5. Determining the Tree of Optimal Paths in an Acyclic Network

Preliminary step (step 0): Set $GT^\circ = (X^\circ, E^\circ)$, where $X^\circ = \{x_0\}$, $E^\circ = \emptyset$. Assign to every vertex $x \in X$ a set of labels $H^1(x), H^2(x), \ldots, H^p(x), t(x)$ as follows:

$$
\begin{array}{ll}
H^i(x_0) = 0, & i = \overline{1,p}, \\
H^i(x) = \infty, & \forall x \in X \setminus \{x_0\},\ i = \overline{1,p}, \\
t(x_0) = 0, & \\
t(x) = \infty, & \forall x \in X \setminus \{x_0\}.
\end{array}
$$

General step (step k, $k \geq 1$): Find in $X \setminus X^{k-1}$ the least vertex x^k and the set of incoming edges $E^-(x^k) = \{(x^r, x^k) \in E \,|\, x^r \in X^{k-1}\}$ for x^k. If $|E^-(x^k)| = 1$ then go to (a); otherwise go to (b):

(a) Find a unique vertex y such that $e' = (y, x^k) \in E^-(x^k)$ and calculate

$$
\begin{array}{l}
H^i(x^k) = H^i(y) + c^i_{(y,x^k)}(t(y)), \quad i = \overline{1,p}; \\
t(x^k) = t(y) + 1.
\end{array}
$$

After that, form the sets $X^k = X^k \cup \{x^k\}$, $E^k = E^{k-1} \cup \{(y, x^k)\}$ and put $GT^k = (X^k, E^k)$. If $k < |X| - 1$, then go to next step $k+1$; otherwise fix $E^* = E^{|X|-1}$, $GT^* = (X, E^*)$ and STOP.

(b) Select the greatest vertex $z \in X^{k-1}$ such that in graph $GT^k = \left(X^{k-1} \cup \{x^k\},\ E^{k-1} \cup E^-(x^k)\right)$ there exist at least two parallel directed paths $P'(z, x^k)$, $P''(z, x^k)$ from z to x^k without common edges, i.e., $E(P'(z, x^k)) \cap E(P''(z, x^k)) = \emptyset$. Let $e' = (x^r, x^k)$ and $e'' = (x^s, x^k)$ be respective edges of these paths with common end vertex in x^k. So, $e', e'' \in E^-(x^k)$. For vertex z determine i_z such that $z \in X_{i_z}$.

If

$$
H^{i_z}(x^r) + c^{i_z}_{(x^r,x^k)}(t(x^r)) \leq H^{i_z}(x^s) + c^{i_z}_{(x^s,x^k)}(t(x^s))
$$

then delete the edge $e'' = (x^s, x^k)$ from $E^-(x^k)$ and from G; otherwise delete edge $e' = (x^r, x^k)$ from $E^-(x^k)$ and from G. After that, check again the condition $|E^-(x^k)| = 1$? If $|E^-(x^k)| = 1$, then go to (a) otherwise go to (b).

Remark 4. The values $H^i(x)$, $i = \overline{1,p}$ for $x \in X$ in Algorithm 5 express the respective costs of the players in the dynamic c-game with the starting position x_0 and the final position x.

8 Determining Nash Equilibria for Nonstationary Dynamic c-Games

Now we study the problem of finding Nash equilibria for nonstationary dynamic c-games. For this case of the problem, we will use a time-expanded network utilized in [43–48].

8.1 Time-Expanded Networks for Nonstationary Dynamic c-Games

Let $(G, X_1, X_2, \ldots, X_p, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f, T_1, T_2)$ be a network that determines our dynamic c-game. We assume that in $G = (X, E)$, the vertex

$x_f \in X$ is attainable from every $x \in X$. Additionally, we construct an auxiliary network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$ where the graph $\overline{G} = (Y, \overline{E})$ is obtained as follows:

$$Y = Y^0 \cup Y^1 \cup Y^2 \cup \dots \cup Y^{T_1} \cup Y^{T_1+1} \cup \dots \cup Y^{T_2} \quad (Y^t \cap Y^k = \emptyset,\ t \neq k);$$

$Y^t = (X, t)$ corresponds with the set of states at the time-step t, $t = \overline{0, T_2}$;

$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \dots \cup E^{T_1} \cup E^{T_1+1} \cup \dots \cup E^{T_2-1} \cup E^f;$$

where

$$E^t = \{((x,t),(y,t+1)) | (x,t) \in Y^t, (y,t+1) \in Y^{t+1}, (x,y) \in E\}, t = \overline{0, T_2-1};$$

$$E^f = \{((x,t),(y,T_2)) | (x,t) \in Y^t, (y,T_2) \in Y^{T_2}, (x,y) \in E, t = \overline{T_1-1, T_2-2}\}.$$

In the case $T_1 = T_2$, we obtain a T_2-partied network. So, the sets $Y^t = (X, t)$, $t = \overline{0, T_2}$, represent $T_2 + 1$ copies of the set X where level sets (*layers*) Y^t and Y^{t+1} are linked by the edges of the form $\Big((x,t),(y,t+1)\Big)$ if $(x,y) \in E$. Additionally, in $\overline{G}$ there exist edges $\Big((x,t),(y,T_2)\Big)$ that connect the set (X,t) and (X,T_2), $t = \overline{T_1, T_2-2}$. The cost functions on the edges $\Big((x,t),(y,t+1)\Big)$, $\Big((x,t),(y,T_2)\Big)$ in $\overline{G}$ can be interpreted as an in the initial network, i.e.,

$$\overline{c}_{((x,t),(y,t+1))}(t) = c_{(x,y)}(t), \quad t = \overline{0, T_2-1};$$

$$\overline{c}_{((x,t),(y,T_2))}(t) = c_{(x,y)}(t), \quad t = \overline{T_1-1, T_2-2}.$$

The sets Z_i of the players' position in that auxiliary network are $Z_i = \bigcup_t (X_i, t),\ i = \overline{1,p}$.

Lemma 3. *Let $\overline{P}(y_0, y_f)$ be an arbitrary directed path from y_0 to y_f in the graph $\overline{G}$. Then the number of edges $|\overline{E}(\overline{P}(y_0, y_f))|$ of the directed path satisfies the condition*

$$T_1 \leq |\overline{E}(\overline{P}(y_0, y_f))| \leq T_2.$$

Moreover in $\overline{G}$ there exists a directed path $\overline{P}(y_0, y_f)$ from y_0 to y_f if and only if in G there exists a directed path $P(x_0, x_f)$ from x_0 to x_f ($P(x_0, x_f)$ may contain directed cycles), which contains the same number of edges

$$|E(P(x_0, x_f)| = |\overline{E}(\overline{P}(y_0, y_f))|.$$

Proof. Let $\overline{P}(y_0, y_f)$ be an arbitrary path from y_0 to y_f in $\overline{G}$ and let us show that

$$T_1 \leq |\overline{E}(\overline{P}(y_0, y_f))| \leq T_2,$$

where $\overline{E}(\overline{P}(y_0, y_f))$ is the set of edges of the path $\overline{P}(y_0, y_f)$. Indeed the path $\overline{P}(y_0, y_f)$ contains at least T_1 edges because it passes through all the layers $Y^0, Y^1, Y^2, \dots, Y^{T_1-1}$ and then goes to one of positions $y \in \bigcup_{i=T_1}^{T_2} Y^i$. On the other hand, the number of the edges of the path $\overline{P}(y_0, y_f)$ cannot exceed T_2 because each vertex (x,t) of the level sets $Y^{T_1}, Y^{T_1+1}, \dots, Y^{T_2}$ is connected with (x_f, T_2) in $\overline{G}$ (if in G there exists an edge (x, x_f)). ∎

Corollary 5.

- *Let G be a graph without directed cycles. Then $\overline{G}$ is an acyclic graph, and in $\overline{G}$ there exists a directed path $\overline{P}(y_0, y_f)$ from y_0 to y_f with the property*

$$T_1 \le |\overline{E}(\overline{P}(y_0, y_f))| \le T_2$$

if and only if in G there exists a path $P(x_0, x_f)$ from x_0 to x_f with the property

$$T_1 \le |E(P(x_0, x_f))| \le T_2.$$

- *Let G be a graph that may contain directed cycles. If $\overline{P}(y_0, y_f)$ is an arbitrary path from y_0 to y_f in $\overline{G}$ with the vertex set*

$$\overline{X}(\overline{P}(y_0, y_f)) = \{y_0, y_1, y_2, \dots, y_{T(x_f)} = y_f\},$$

where $y_t = (x_t, t)$, $t = \overline{0, T(x_f)}$, then $\{x_0, x_1, x_2, \dots, x_{T(x_f)} = x_f\}$ generates in G a directed path $P(x_0, x_f)$ from x_0 to x_f ($\overline{P}(x_0, x_f)$ may contain directed cycles).

So one may conclude that the auxiliary time-expanded network gives all admissible directed paths from x_0 to x_f for the considered problems (for non-cooperative and cooperative case). On the basis of this result, an algorithmic solution is presented.

8.2 Determining Nash Equilibria

Now we show that the problem of finding Nash equilibria for nonstationary dynamic c-games can be reduced to the stationary case of the game on an auxiliary time-expanded network with constant cost functions on the edges.

Theorem 8. *Let $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$ be a network with positive and nondecreasing cost functions $c_e^i(t)$, $i = \overline{1, p}$, on edges $e \in E$. Moreover, let us assume that in $G = (X, E)$ there exists a directed path $P_G(x_0, x_f)$ from x_0 to x_f such that*

$$T_1 \le |E(P_G(x_0, x_f))| \le T_2,$$

i.e., $P_G(x_0, x_f) = \{x_0, e_0, x_1, e_1, x_2, \dots, x_{T(x_f)-1}, e_{T(x_f)-1}, x_{T(x_f)}\}$, where $T_1 \le T(x_f) \le T_2$ (here $P_G(x_0, x_f)$ may contain directed cycles). Then for the nonstationary dynamic c-game on the network, there exist nonstationary strategies in the sense of Nash $u_1^*, u_2^*, \dots, u_p^*$.

Proof. Let us consider arbitrary stationary strategies $\overline{s}_1, \overline{s}_2, \dots, \overline{s}_p$ of the players in the dynamic c-game on the auxiliary time-expanded network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$. It is obvious that in the initial dynamic c-game on $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$, we uniquely can determine the nonstationary strategies $u_1, u_2, \dots, u_p$ of the players as follows:

$$\overline{s}_i(x,t) = u_i(x,t) \ \text{ for } \ (x,t) \in X_i \times \{1,2,\dots,T\},\ i = \overline{1,p}.$$

So, between the set of stationary strategies of the players on the network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$ and the set of nonstationary strategies of the players on $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$, there exists a bijective mapping, which preserves integral-time costs on certain trajectories: If $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ is an equilibrium solution in the sense of Nash for the stationary case of the problem on the network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$, then we observe that

$$u^*(x,t) = \overline{s}^*(x,t) \ \text{ for } \ (x,t) \in X_i \times \{1,2,\dots,T_2\}, \quad i = \overline{1,p},$$

is an equilibrium solution in the sense of Nash for the nonstationary case of the game on the network $(G, X_1, X_2, \dots, X_p, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$. Because the time t on the time expanded network for every position is determined by the level set, the cost functions $\overline{c}^i_{(x,t)}(t)$, $i = \overline{1,p}$, on the auxiliary network can be considered as constant. Therefore, if in $\overline{G}$ there exists a directed path $P_{\overline{G}}(y_0, y_f)$ from y_0 to y_f, then for the dynamic c-game on the auxiliary network, Nash equilibria exist.

According to Lemma 3, such a path $P_{\overline{G}}(y_0, y_f)$ exists in $\overline{G}$ because in G there exists a directed path

$$P_G(x_0, x_f) = \{x_0, e_0, x_1, e_1, x_2, \dots, x_{T(x_f)-1}, e_{T(x_f)-1}, x_{T(x_f)}\}$$

where $T_1 \le T(x_f) \le T_2$ $\ (P_G(x_0, x_f))$ may contain directed cycles). ∎

On the basis of this theorem, now we can propose the following algorithm for determining the equilibrium nonstationary strategies of the players in such dynamic c-games.

Algorithm 6. Determining the Optimal Nonstationary Strategies in Dynamic c-Game

1. We construct the auxiliary time-expanded network

$$(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f).$$

2. Define the equilibrium stationary strategies $\overline{s}_1^*, \overline{s}_2^*, \dots, \overline{s}_p^*$ in the dynamic c-game on

$$(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f).$$

3. Put

$$u_i^*(x,t) = \overline{s}_i^*(x,t) \text{ for } \ (x,t) \in X_i \times \{1,2,\dots,T_2\}, \quad i = \overline{1,p}.$$

In the next section, we extend our approach for multiobjective control problems on networks with Pareto optimality principles.

9 Multiobjective Control and Cooperative Games on Dynamic Networks

Now we shall use the concept of cooperative games and will formulate multiobjective control problems on networks applying the Pareto optimality principle. In an analogous way as in the previous section, we distinguish two versions of the problem concerning stationary and nonstationary strategies.

9.1 Stationary Strategies on Networks and Pareto Solutions

On G, we consider now the following cooperative game:

The stationary strategies of players $1, 2, \ldots, p$ are defined as a map

$$s\colon x \to y \in X(x) \ \text{ for } \ x \in X \setminus \{x_f\}.$$

For an arbitrary stationary strategy $s \in S = \{s | s : x \to y \in X(x) \text{ for } x \in X \setminus \{x_f\}\}$, we denote by $G_s = (X, E_s)$ the subgraph of G generated by edges $e = (x, s(x))$ for $x \in X \setminus \{x_f\}$. Then for every $s \in S$ in G, either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists or such a path does not exists in G. For a given s and fixed x_0 and x_f, we define the quantities $\overline{H}^1_{x_0x_f}(s), \overline{H}^2_{x_0x_f}(s), \ldots, \overline{H}^p_{x_0x_f}(s)$ in the following way.

Let us assume that the path $P_s(x_0, x_f)$ exists in G. Then it is unique and we can assign to its edges, starting with the edge that begins in x_0, numbers $0, 1, 2, \ldots, k_s$. These numbers determine the time steps $t_e(s)$ when the system passes from one state to another if the stationary strategy s is applied. We put

$$\overline{H}^i_{x_0x_f}(s) = \sum_{e \in E(P_s(x_0,x_f))} c^i_e(t_e(s)), \ \text{if} \ \ T_1 \le |E(P_s(x_0, x_f))| \le T_2;$$

otherwise we put $\overline{H}^i_{x_0x_f}(s) = \infty$.

We consider the problem of finding the set S^*_P of Pareto solutions (or a Pareto solution) in the set of stationary strategies S. Note that s^*, where $s^* \in S^*_P$, is called a Pareto solution if in $S \setminus S^*_p$ there is no strategy s' such that

$$\overline{H}^i_{x_0x_f}(s') \le \overline{H}^i_{x_0x_f}(s^*), \quad i = \overline{1, p},$$

and $\overline{H}^{i_0}_{x_0x_f}(s') < \overline{H}^{i_0}_{x_0x_f}(s^*)$ for an index $i_0 \in \{1, 2, \ldots, p\}$.

9.2 Pareto Solution for the Problem with Nonstationary Strategies on Networks

The nonstationary strategy for our cooperative dynamic game is defined as a map

$$u\colon (x, t) \to (y, t+1) \in X(x) \times \{t+1\} \ \text{ for } \ x \in X \setminus \{x_f\}, \ t = 0, 1, 2, \ldots$$

The payoff functions

$$\overline{F}^1_{x_0x_f}(u), \overline{F}^2_{x_0x_f}(u), \ldots, \overline{F}^p_{x_0x_f}(u)$$

of the game are defined in the following way:

Let u be an arbitrary strategy. Then u either generates in G a finite trajectory

$$x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$$

from x_0 to x_f and $T(x_f)$ represents the time moment when x_f is reached, or u generates in G an infinite trajectory

$$x_0 = x(0), x(1), x(2), \ldots, x(t), x(t+1), \ldots$$

which does not pass through x_f, i.e., $T(x_f) = \infty$. In both cases, the next state $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$ as follows:

$$x(t+1) = u(x(t), t), \quad t = 0, 1, 2, \ldots.$$

If the state x_f is reached at a finite moment of time $T(x_f)$ (i.e., the attainability $T_1 \leq T(x_f) \leq T_2$ is guaranteed), then we set

$$\overline{F}^i_{x_0x_f}(u) = \sum_{t=0}^{T(x_0)-1} c^i_{(x(t),x(t+1))}(t), \quad i = \overline{1,p};$$

otherwise we put

$$\overline{F}^i_{x_0x_f}(u) = \infty, \quad i = \overline{1,p}.$$

10 Determining Pareto Solutions for Multiobjective Control Problems on Networks

Note that in the considered multiobjective control problems, Pareto solution always exists if in G there exists at least one directed path $P(x_0, x_f)$ from x_0 to x_f. We propose algorithms for solving the stationary and nonstationary cases of the problems.

10.1 Determining Pareto Stationary Strategies

First of all, an algorithm for determining Pareto stationary solutions for multiobjective control problems on networks without restrictions on the number of stages when the costs on the edges are constant and positive functions is proposed:

Algorithm 7. Determining Pareto Solutions for the Problem with Constant Costs on Edges

Preliminary step (step 0): Set $X^0 = \{x_f\}$; $E^0 = \emptyset$; $\overline{H}^i_{x_f x_f} = 0, \quad i = \overline{1,p}$.

General step (step k, $k \geq 1$): If $X^{k-1} \neq X$, then find the set of edges

$$E(X^{k-1}) = \Big\{ e = (y,x) \in E \,\Big|\, x \in X^{k-1},\, y \in X \setminus X^{k-1} \Big\}.$$

Then find an edge $e' = (y', x')$ in $E(X^{k-1})$ such that the following conditions are satisfied:

(a) $\overline{H}^{i_r}_{y' x_f} = \overline{H}^{i_r}_{x' x_f} + c^{i_r}_{(y',x')} = \min\limits_{(y,x) \in E(X^{k-1})} \left\{ \overline{H}^{i_r}_{x x_f} + c^{i_r}_{(y,x)} \right\}$ for an index $i_r \in \{1, 2, \dots, p\}$;

(b) there is no edge $(\overline{y}, \overline{x}) \in E(X^{k-1})$ such that

$$\overline{H}^i_{\overline{x} x_f} + c^i_{(\overline{y},\overline{x})} \leq \overline{H}^i_{x' x_f} + c^i_{(y',x')}, \quad i = \overline{1,p} \quad \text{and}$$
$$\overline{H}^{i_0}_{\overline{x} x_f} + c^{i_0}_{(\overline{y},\overline{x})} < \overline{H}^{i_0}_{x' x_f} + c^{i_0}_{(y',x')} \text{ for an index } i_0 \in \{1, 2, \dots, p\}.$$

For given y', fix

$$\overline{H}^i_{y' x_f} = \overline{H}^i_{x' x_f} + c^i_{(y',x')},\ i = \overline{1,p}.$$

After that, we put $X^k = X^{k-1} \cup \{y'\}$, $E^k = E^{k-1} \cup \{(y', x')\}$ and go to the next step. If $X^{k-1} = X$ $(k = n)$, then find the tree $GT^{n-1} = (X, E^{n-1})$, which determines the optimal Pareto strategy s^* of players as follows:

$$s^*(y) = x \quad \text{for} \quad y \in X \setminus \{x_f\} \text{ if } (y,x) \in E^{n-1}.$$

Algorithm 7 is an extension of Dijkstra's algorithm [12, 14] for a multi-objective version of the optimal paths problem in a weighted directed graph. The algorithm determines the Pareto stationary strategy s^* of players for the multiobjective control problem on the network $(G, X, c^1, c^2, \dots, c^p, x, x_f, T_1, T_2)$ with an arbitrary starting position $x \in X$ and a given final position $x_f \in X$, i.e., the tree $GT^{n-1} = (X, E^{n-1})$ gives all Pareto optimal paths from every $x \in X$ to x_f.

Theorem 9. *Algorithm 7 finds Pareto stationary strategies of the players in the multiobjective control problem on the network $(G, X, c^1, c^2, \dots, c^p, x, x_f)$ for every given starting position x and final position x_f. The running-time of the algorithm is $O(|X|^3 p)$.*

Proof. We prove this theorem by using induction principle on number of players p. In the case $p = 1$, Algorithm 7 becomes Dijkstra's algorithm for determining the tree of shortest paths in a weighted directed graph, therefore the theorem holds.

Let us assume that the theorem holds for any $p \leq q$, $q \geq 1$, and let us show that it is true for $p = q + 1$.

We consider an auxiliary graph $G'_{q+1} = (X, E_{q+1} \bigcup (E \setminus E(X_{p+1})))$, where E_{q+1} represents the set of edges $e' = (y', x')$ found at the iterations of Algorithm 7 with $i_r = q + 1$ and

$$X_{q+1} = \{y' \in X \mid e' = (y', x') \in E_{q+1}\},$$

$$E(X_{p+1}) = \{e \in E \mid e = (y', x) \in E\}.$$

Based on conditions (a) and (b) of Algorithm 7, we may conclude that if we find Pareto solution of the multiobjective problem on G' with respect to players $1, 2, \ldots, q, q+1$, then we obtain the same solution of the problem as the one on G. Taking into account that in G every vertex $y' \in X_{q+1}$ has only one leaving edge, we may regard our problem on G' as a multiobjective one with respect to players $1, 2, \ldots, q$. According to induction principle, Algorithm 7 finds Pareto solution for the multiobjective problem with respect to players $1, 2, \ldots, q$. In such a way, we obtain the Pareto solution s^* of the problem on the auxiliary graph G', which at the same time is the Pareto solution of the problem on G with respect to players $1, 2, \ldots, q, q+1$.

It is also easy to observe that the number of elementary operations at the general step of the algorithm is $O(|X|^2 p)$. Therefore, the running-time of the algorithm is $O(|X|^3 p)$. ■

Remark 5. Other approach for solving multiobjective control problem with the Pareto optimality principle is based on its reducing to the single objective control problem with a certain convolution criterion

$$\overline{H}_{x_0 x_f}(s) = \sum_{i=1}^{p} \alpha_i \overline{H}^{i}_{x_0 x_f}(s),$$

where

$$\sum_{i=1}^{p} \alpha_i = 1; \;\; \alpha_i \geq 0, \;\; i = \overline{1, p}.$$

Note that in such a way, we can find a Pareto solution for some classes of the problem with positive nondecreasing costs $c_e^i(t)$, $i = \overline{1, p}$, on the edges of the networks. But in the general case, via such an approach, not all Pareto solutions can be determined.

10.2 Pareto Solution for the Nonstationary Case of the Problem

In order to solve the nonstationary case of the problem, we shall use the time-expanded networks. The time-expanded network for the cooperative case of the problem is defined in the same way as for the noncooperative case. There is only one single exception: we do not take into account the partition $Y = Z_1 \cup Z_2 \cup \cdots \cup Z_p$.

The problem of determining nonstationary Pareto strategies for the multiobjective control problem on the network $(G, X, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f,$

$T_1, T_2)$ can be reduced to the problem from Section 10.1 on the auxiliary time-expanded network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$. This reduction is based on the following theorem:

Theorem 10. *Let $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$ be an auxiliary time-expanded network for a given network $(G, X, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$. If s^* is a Pareto stationary strategy of the players for the multiobjective control problem on the network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$, then*

$$u^*(x,t) = s^*(x,t) \quad \textit{for} \quad (x,t) \in X \times \{1, 2, \dots, T_2\}$$

is a nonstationary Pareto strategy for the multiobjective control problem on the network $(G, X, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$.

This theorem can be proved in an analogous way as Theorem 8.

For finding nonstationary Pareto strategies of the players for the multiobjective control problem, the following algorithm can be used:

Algorithm 8. Determining Pareto Solutions for the Nonstationary Case of the Problem

1. For a given network $(G, X, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$, construct the auxiliary time-expanded network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$.
2. Determine Pareto stationary strategies for the multiobjective control problem on the network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$ using Algorithm 7.
3. Put $u^*(x,t) = s^*(x,t)$ for $(x,t) \in X \times \{1, 2, \dots, T_2\}$.

10.3 Computational Complexity of the Stationary Case of the Problem and an Algorithm for Its Solving on Acyclic Networks

Note that our stationary multiobjective control problem on the general network is NP-complete [22], even in the case $p = 1$, $T_1 = T_2 = |X| - 1$, because it becomes the Hamiltonian path problem in a directed graph where all cost functions on the network are constantly equal to 1. Therefore, in the general case this problem is NP-hard. But if G has the structure of an acyclic graph, then the stationary Pareto solution s^* on the network $(G, X, c^1(t), c^2(t), \dots, c^p(t), x_0, x_f, T_1, T_2)$ can be found by using Pareto stationary strategy $\overline{s}^*$ for the problem on the auxiliary network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \dots, \overline{c}^p(t), y_0, y_f)$ in the following way.

Algorithm 9. Determining Stationary Pareto Solution on Acyclic Networks

Preliminary step (step 0): Fix an arbitrary Pareto solution

$$\overline{s}^* : y \to z \in Y(y) \text{ for } y \in Y$$

for the problem on network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \ldots, \overline{c}^p(t), y_0, y_f)$ by using Algorithm 7. Then put $W^0 = \{(x_0, 0), (x_1, 1), \ldots, (x_{T(x_f)}, T(x_f))\}$, $X^0 = \{x_0, x_1, \ldots, x_{T(x_f)}\}$ and fix $s^*(x_t) = x(t+1)$ for $x_t \in X^0$, $t = \overline{0, T(x_f) - 1}$, where $(x_0, 0)$, $(x_1, 1), \ldots, (x_{T(x_f)}, T(x_f))$ is a trajectory generated by Pareto stationary strategy $\overline{s}^*$ in the auxiliary time-expanded network $(\overline{G}, Y, \overline{c}^1(t), \overline{c}^2(t), \ldots, \overline{c}^p(t), y_0, y_f, T_1, T_2)$. If in the auxiliary time-expanded network there is no directed path from y_0 to y_f, then for the considered problem on the network $(G, X, c^1(t), c^2(t), \ldots, c^p(t), x_0, x_f, T_1, T_2)$, Pareto stationary strategy does not exist.

General step (step k, $k \geq 1$): If $X^{k-1} = X$, then STOP, otherwise determine the set

$$W_{\overline{s}^*}(X^{k-1}) = \Big\{(x,t) \in (X \setminus X^{k-1}) \times \{1, 2, \ldots, T_2\} \,\Big|\, \overline{s}^*(x,t) \in W^{k-1}\Big\}.$$

If $W_{\overline{s}^*}(X^{k-1}) \neq \emptyset$, then find a vertex $(x', t') \in W_{\overline{s}^*}(X^{k-1})$ with a minimal t' for a given x' and fix $\overline{s}^*(x') = z$ if $\overline{s}^*(x', t') = (z, t+1)$. After that, construct sets $X^k = X^{k-1} \cup \{x'\}$, $W^k = W^{k-1} \cup \{(x', t')\}$ and go to the next step.

Some similar multiobjective problems on dynamic networks have been studied in [31].

11 Application of the Dynamic c-Game for Studying and Solving Multiobjective Control Problems

Let us show that the results from Section 8 can be used for studying and solving the multiobjective control problems from Section 1.

At first, we consider the problem from Section 1.2 (Problem 2), for which the alternate players control condition is satisfied. We regard this problem as a dynamic c-game determined by an acyclic network $(\overline{G}, Z_1, Z_2, \ldots, Z_p, \overline{c}^1, \overline{c}^2, \ldots, \overline{c}^p, y_0, y_f)$, where the graph $\overline{G} = (Y, \overline{E})$ with partition $Y = \bigcup_{i=1}^p Z_i$ and constant functions $\overline{c}^i : \overline{E} \to R$, $i = \overline{1, p}$, are defined in the following way.

The set of vertices Y consists of $T_2 + 1$ copies of the set of states corresponding with moments of time $t = 0, 1, 2, \ldots, T_2$, i.e., $Y = \bigcup_{t=0}^{T_2}(X, t)$ with the partition $Y = \bigcup_{i=1}^p Z_i$, determined by the alternate players condition $Z_i = \bigcup_{t=0}^{T_2}(X^i(t), t)$.

The set of edges $\overline{E}$ is also represented as

$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \cdots \cup E^{T_1} \cup E^{T_1+1} \cup \cdots \cup E^{T_2-1} \cup E^f;$$

where

$$Y^t = (X, t),\ t = \overline{0, T_2};$$

$$E^t = \{((x,t),(y,t+1)) | (x,t) \in Y^t, (y,t+1) \in Y^{t+1}, (x,y) \in E\}, t = \overline{0, T_2 - 1};$$

$E^f = \{((x,t),(y,T_2)) | (x,t) \in Y^t, (y,T_2) \in Y^{T_2}, (x,y) \in E, t = \overline{T_1 - 1, T_2 - 2}\}.$

Note that $(x,t) \in (X,t)$, and here the notation (x,t) has the same meaning as $x(t)$, i.e., $(x,t) = x(t)$.

In our network, we fix $y_0 = (x_0, 0) \in (X, 0)$ and $y_f = (x_f, T_2) \in (X, T_2)$.

We define the cost functions $\overline{c}^i$, $i = \overline{1,p}$, as follows:

$$\overline{c}^i_{((x,t),(y,t+1))} = c^i_{(x(t),y(t+1))}, \text{ if } y(t+1) = \overline{g}_t(x(t), u^i(t))$$

$$\text{for given } u^i(t) \in U^i_t(x(t)),\ x(t) \in Z_i,\ i = \overline{1,p},\ t = \overline{0, T_2 - 1};$$

$$\overline{c}^i_{((x,t),(y,T_2))} = c^i_{(x(t),y(t+1))}, \text{ if } y(t+1) = \overline{g}_t(x(t), u^i(t))$$

$$\text{for given } u^i(t) \in U^i_t(x(t)),\ x(t) \in Z_i,\ i = \overline{1,p},\ t = \overline{T_1 - 1, T_2 - 2}.$$

It is easy to see that in this network, every directed path $P_{\overline{G}}(z_0, z_f)$ from z_0 to z_f contains $|\overline{E}(P_{\overline{G}}(z_0, z_f))|$ edges such that $T_1 \le |\overline{E}(P_{\overline{G}}(z_0, z_f))| \le T_2$. So, if we define the admissible solution $u^1(t), u^2(t), \dots, u^p(t)$ as a set of vectors of control parameters, which satisfy conditions (5), (6) and $T_1 \le T(x_f) \le T_2$, then we may conclude that there exists a bijective mapping between the set of admissible solutions of the control problem in positional form and the set of admissible strategies in the dynamic c-game. This means that Theorem 1 holds and the following algorithm can be used.

Algorithm 10. Determining Nash Equilibria for Multiobjective Control Problem in Positional Form

1. Construct the auxiliary network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1, \overline{c}^2, \dots, \overline{c}^p, y_0, y_f)$ according to the rules described above.

2. Find the optimal stationary strategy in the dynamic c-game determined by the network $(\overline{G}, Z_1, Z_2, \dots, Z_p, \overline{c}^1, \overline{c}^2, \dots, \overline{c}^p, y_0, y_f)$ and the directed path

$$P^*_{s^*} = \{z_0 = (x^*_0, 0), (x^*_1, 1), \dots, (x^*_t, t), (x^*_{t+1}, t+1)\}, \dots, z_f = (x^*_f, T_2)\}.$$

3. Starting from the final position (x^*_f, T_2), find recursively

$$u^{i*}(t),\ t = T_2 - 1, T_2 - 2, \dots, 1, 0,$$

such that

$$x^*(t+1) = g_t(x^*(t), u^{1*}(t), u^{2*}(t), \dots, u^{p*}(t)).$$

Then $u^{1*}(t), u^{2*}(t), \dots, u^{p*}(t)$ is a solution of the problem.

In an analogous way, the problem of determining Pareto solution for the multiobjective control problem from Section 1.3 can be reduced to the control problem on network $(\overline{G}, \overline{c}^1, \overline{c}^2, \dots, \overline{c}^p, z_0, z_f)$. Here we should not take into account the partition $Z = \bigcup_{i=1}^p Z_i$.

12 Zero-Sum Games on Networks and Polynomial Time Algorithm for Max-Min Paths Problem

In the previous section, we have studied dynamic c-games with positive cost functions on edges. Therefore, we cannot use those results for zero-sum games. In the following, we study zero-sum games of two players with arbitrary cost functions on edges and propose polynomial-time algorithms for their solution. The main results related to this problem have been obtained in [34–36, 39, 40, 49].

12.1 Problem Formulation

In this section, we study the antagonistic dynamic c-game of two players on network with arbitrary constant cost functions on edges. This case of the problem corresponds with the max-min paths problem on networks, which generalizes classic combinatorial problems of the shortest and the longest paths in weighted directed graphs. This max-min paths problem arose as an auxiliary one when searching optimal stationary strategies of players in cyclic games. In addition, we shall use the considered dynamic c-game for studying and solving zero-sum control problem from Section 1.2. The main results are concerned with the existence of polynomial-time algorithms for determining max-min paths in networks as well as the elaboration of such algorithms.

Let $G = (X, E)$ be a directed graph with vertex set X and edge set E. Assume that G contains a vertex $x_f \in X$ such that it is attainable from each vertex $x \in X$, i.e., x_f is a sink in G. On edge set E, the function $c : E \to R$ is given, which assigns a cost c_e to each edge $e \in E$. In addition, the vertex set is divided into two disjoint subsets X_A and X_B ($X = X_A \cup X_B$, $X_A \cap X_B = \emptyset$), which we regard as position sets of two players.

On G, we consider a game of two players. The game starts at the position $x_0 \in X$. If $x_0 \in X_A$, then the move is done by the first player, otherwise it is done by the second one. The move means the passage from the position x_0 to the neighbor position x_1 through the edge $e_1 = (x_0, x_1) \in E$. After that, if $x_1 \in X_A$, then the move is done by the first player, otherwise it is done by the second one and so on. As soon as the final position is reached, the game is over. The game can be finite or infinite. If the final position x_f is reached in finite time, then the game is finite. In the case when the final position x_f is not reached, the game is infinite. The first player in this game has the aim to maximize $\sum_i c_{e_i}$ whereas the second one has the aim to minimize $\sum_i c_{e_i}$.

Strictly the considered game in normal form can be defined as follows. We identify the strategies s_A and s_B of players with the maps

$$s_A : x \to y \in X(x) \text{ for } x \in X_A;$$
$$s_B : x \to y \in X(x) \text{ for } x \in X_B,$$

where $X(x)$ represents the set of extremities of edges $e = (x, y) \in E$, i.e., $X(x) = \{y \in X | e = (x, y) \in E\}$. Because G is a finite graph, then the sets of strategies of players

$$S_A = \{s_A : x \to y \in X(x) \text{ for } x \in X_A\};$$
$$S_B = \{s_B : x \to y \in X(x) \text{ for } x \in X_B\}$$

are finite sets. The payoff function $H_{x_0}(s_A, s_B)$ on $S_A \times S_B$ is defined in the following way.

Let be in G a subgraph $G_s = (X, E_s)$ generated by edges of form $(x, s_A(x))$ for $x \in X_A$ and $(x, s_B(x))$ for $x \in X_B$. Then either a unique directed path $P_s(x_0, x_f)$ from x_0 to x_f exists in G_s or such a path does not exist in G_s. In the second case in G_s, there exists a unique directed cycle C_s, which can be reached from x_0.

For given s_A and s_B, we set

$$H_{x_0}(s_A, s_B) = \sum_{e \in E(P_s(x_0, x_f))} c_e,$$

if in G_s there exists a directed path $P_s(x_0, x_f)$ from x_0 to x_f, where $E(P_s(x_0, x_f))$ is a set of edges of the directed path $P_s(x_0, x_f)$. If in G there is no directed path from x_0 to x_f, then we define $H_{x_0}(s_A, s_B)$ as follows. Let $P'_s(x_0, y_0)$ be a directed path, which connects the vertex x_0 with the cycle C_s, and $P'_s(x_0, y_0)$ has no other common vertices with C_s except y_0. Then we put

$$H_{x_0}(s_A, s_B) = \begin{cases} +\infty, & \text{if } \sum\limits_{e \in E(C_s)} c_e > 0; \\ \sum\limits_{e \in E(P'_s(x_0, y_0))} c_e, & \text{if } \sum\limits_{e \in E(C_s)} c_e = 0; \\ -\infty, & \text{if } \sum\limits_{e \in E(C_s)} c_e < 0. \end{cases}$$

This game is related to zero-sum positional games of two players, and it is determined by the graph G with the sink vertex x_f, the partition $X = X_A \cup X_B$, the cost function $c : E \to R$, and the starting position x_0. We denote the network, which determines this game, by $(G, X_A, X_B, c, x_0, x_f)$.

In [39, 40], it is shown that if G does not contain directed cycles, then for every $x \in X$ the following equality holds

$$v(x) = \max_{s_A \in S_A} \min_{s_B \in S_B} H_x(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} H_x(s_A, s_B), \tag{16}$$

which means the existence of optimal strategies of players in the considered game. Moreover, in [39, 40] it is shown that in G there exists a tree $GT = (X, E^*)$ with the sink vertex x_f, which gives the optimal strategies of players in the game for an arbitrary starting position $x_0 \in X$. The strategies of players are obtained by fixing

$$s^*_A(x) = y, \text{ if } (x, y) \in E^* \text{ and } x \in X_A \setminus \{x_f\};$$
$$s^*_B(x) = y, \text{ if } (x, y) \in E^* \text{ and } x \in X_B \setminus \{x_f\}.$$

In the general case for an arbitrary graph G, equality (16) may fail to hold. Therefore, we formulate necessary and sufficient conditions for the existence of optimal strategies of players in this game and propose a polynomial-time algorithm for determining the tree of max-min paths from every $x \in X$ to x_f. Furthermore, we show that our max-min paths problem on the network can be regarded as a zero value ergodic cyclic game. So, the proposed algorithm can be used for solving such games.

In [34,35], the formulated game on network $(G, X_A, X_B, c, x_0, x_f)$ is named the dynamic c-game. Some preliminary results related to this problem have been obtained in [39,40]. More general models of positional games on networks with p players have been studied in [43].

12.2 Algorithm for Solving the Problem on Acyclic Networks

The formulated problem for acyclic networks has been studied in [35, 39, 40].

Let $G = (X, E)$ be a finite directed graph without directed cycles and a given sink vertex x_f. The partition $X = X_A \cup X_B$ $(X_A \cap X_B = \emptyset)$ of the vertex set of G is given, and the cost function $c : E \to R$ on edges is defined. We consider the dynamic c-game on G with a given starting position $x \in X$.

It is easy to observe that for fixed strategies of players $s_A \in S_A$ and $s_B \in S_B$, the subgraph $G_s = (X, E_s)$ has a structure of directed tree with sink vertex $x_f \in X$. This means that the value $H_x(s_A, s_B)$ is determined uniquely by the sum of edge costs of the unique directed path $P_s(x, x_f)$ from x to x_f. In [39, 40], it is proved that for acyclic c-game on network $(G, X_A, X_B, c, x_0, x_f)$, there exist the strategies of players s_A^*, s_B^* such that

$$v(x) = H_x(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} H_x(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} H_x(s_A, s_B) \tag{17}$$

and s_A^*, s_B^* do not depend on starting position $x \in X$, i.e., (17) holds for every $x \in X$.

The equality (17) is evident in the case when $\mathrm{ext}(c, x) = 0$, $\forall x \in X \setminus \{x_f\}$, where

$$\mathrm{ext}(c, x) = \begin{cases} \max\limits_{y \in X(x)} c_{(x,y)}, & x \in X_A; \\ \min\limits_{y \in X(x)} c_{(x,y)}, & x \in X_B. \end{cases}$$

In this case, $v(x) = 0$, $\forall x \in X$, and the optimal strategies of players can be obtained by fixing the maps $s_A^* : X_A \setminus \{x_f\} \to X$ and $s_B^* : X_B \setminus \{x_f\} \to X$ such that $s_A^* \in \mathrm{VEXT}(c, x)$ for $x \in X_A \setminus \{x_f\}$ and $s_B^* \in \mathrm{VEXT}(c, x)$ for $x \in X_B \setminus \{x_f\}$, where

$$\mathrm{VEXT}(c, x) = \{y \in X(x) | c_{(x,y)} = \mathrm{ext}(c, x)\}.$$

If the network $(G, X_A, X_B, c, x_0, x_f)$ has the property that $\mathrm{ext}(c, x) = 0$, $\forall x \in X \setminus \{x_f\}$, then it is named the network in canonic form. So, for the acyclic c-game on network in canonic form, equality (17) holds and $v(x) = 0$, $\forall x \in X$.

In the general case, equality (17) can be proved by using properties of the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $e = (x, y)$ of the network, where $\varepsilon : X \to R$ is an arbitrary real function on X (the potentional transformation for positional games has been introduced in [5, 35]). The fact is that such transformation of the costs on edges of the acyclic network in c-game does not change the optimal strategies of players, although values $v(x)$ of positions $x \in X$ are changed by $v(x) + \varepsilon(x_f) - \varepsilon(x)$. It means that for an arbitrary function $\varepsilon : X \to R$, the optimal strategies of the players in acyclic c-games on the networks $(G, X_A, X_B, c, x_0, x_f)$ and $(G, X_A, X_B, c', x_0, x_f)$ are the same.

Taking into account that the vertices $x \in X$ of the acyclic graph G can be numbered with $1, 2, \ldots, |X|$, such that if $x > y$, then in G there is no directed path from y to x. Therefore, we can use the following recursive formula

$$\varepsilon(x_f) = 0;$$

$$\varepsilon(x) = \begin{cases} \max\limits_{y \in X(x)} \{c_{(x,y)} + \varepsilon(y)\} \text{ for } x \in X_A \setminus \{x_f\}; \\ \min\limits_{y \in X(x)} \{c_{(x,y)} + \varepsilon(y)\} \text{ for } x \in X_B \setminus \{x_f\} \end{cases} \tag{18}$$

to tabulate the values $\varepsilon(x)$, $\forall x \in X$. It is evident that the transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ satisfies condition $\operatorname{ext}(c', x) = 0$, $\forall x \in X$. This means that the following theorem holds.

Theorem 11. *For an arbitrary acyclic network $(G, X_A, X_B, c, x_0, x_f)$ with a sink vertex x_f, there exists a function $\varepsilon : X \to R$, which determines the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $e = (x, y)$ such that the network $(G, X_A, X_B, c, x_0, x_f)$ has the canonic form. The values $\varepsilon(x)$, $x \in X$, which determine function $\varepsilon : X \to R$, can be found by using recursive formula (18).*

On the basis of this theorem, the following algorithm for determining optimal strategies of players in the c-game is proposed in [35].

Algorithm 11. Determining Optimal Strategies of Players on an Acyclic Network

1. Find the values $\varepsilon(x)$, $x \in X$, according to recursive formula (18) and the corresponding potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x, y) \in E$.
2. Fix arbitrary maps $s_A^*(x) \in \mathrm{VEXT}(c', x)$ for $x \in X_A \setminus \{x_f\}$ and $s_B^*(x) \in \mathrm{VEXT}(c', x)$ for $x \in X_B \setminus \{x_f\}$.

Remark 6. The values $\varepsilon(x)$, $x \in X$, represent the values of the acyclic c-game on $(G, X_A, X_B, c, x_0, x_f)$ with starting position x, i.e., $\varepsilon(x) = v(x)$, $\forall x \in X$.

Algorithm 11 needs $O(|X|^2)$ elementary operations because the tabulation of the values $\varepsilon(x)$, $x \in X$, using formula (18) for acyclic networks needs this number of operations.

12.3 The Main Results for the Problem on an Arbitrary Network

First of all, we give an example showing that equality (16) may fail to hold. On Fig. 4 is given the network with the starting position $x_0 = 1$ and the final position $x_f = 4$, where positions of the first player are represented by circles and positions of the second player are represented by squares; values of cost functions on edges are given alongside them.

It is easy to observe that

$$\max_{s_A \in S_A} \min_{s_B \in S_B} H_1(s_A, s_B) = 2, \quad \min_{s_B \in S_B} \max_{s_A \in S_A} H_1(s_A, s_B) = 3.$$

The following theorem gives conditions for the existence of settle point with finite $v(x)$ for each $x \in X$ in the c-game.

Theorem 12. *Let $(G, X_A, X_B, c, x_0, x_f)$ be an arbitrary network with the sink vertex $x_f \in X$. Moreover, let us consider that $\sum_{e \in E(C_s)} c_e \neq 0$ for every directed cycle C_s from G. Then for the c-game on $(G, X_A, X_B, c, x_0, x_f)$, condition (16) holds for every $x \in X$ if and only if there exists a function $\varepsilon : X \to R$, which determines the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x, y) \in E$ such that $\mathrm{ext}(c', x) = 0$, $\forall x \in X$. If $\sum_{e \in E(C_s)} c_e \neq 0$ for every directed cycle and in G there exists the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x, y) \in E$, then $\varepsilon(x) = v(x)$, $\forall x \in X$.*

Proof. $\Longrightarrow$ Let us consider that $\sum_{e \in E(C_s)} c_e \neq 0$ for every directed cycle C_s in G and condition (16) holds for every $x \in X$. Moreover, we consider that $v(x)$ is a finite value for every $x \in X$. Taking into account that the potentional transformation does not change the cost of cycles, we obtain that this transformation does not change optimal strategies of players although values $v(x)$ of positions $x \in X$ are changed by $v(x) - \varepsilon(x) + \varepsilon(x_f)$. It is easy to observe that if we put $\varepsilon(x) = v(x)$ for $x \in X$, then the function $\varepsilon : X \to R$ determines the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x, y) \in E$ such that $\mathrm{ext}(c', x) = 0$, $\forall x \in X$.

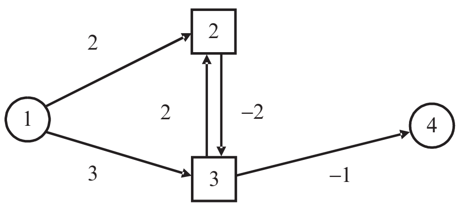

Figure 4. The network for which saddle point may not exist

$\Longleftarrow$ Let us consider that there exists the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x,y) \in E$ such that $\text{ext}(c', x) = 0$, $\forall x \in X$. The value $v(x)$ of the game after the potentional transformation is zero for every $x \in X$, and optimal strategies of players can be found by fixing s_A^* and s_B^* such that $s_A^*(x) \in \text{VEXT}(c', x)$ for $x \in X_A \setminus \{x_f\}$ and $s_B^*(x) \in \text{VEXT}(c', x)$ for $x \in X_B \setminus \{x_f\}$. Because the potentional transformation does not change optimal strategies of players, we put $v(x) = \varepsilon(x) - \varepsilon(x_f)$ and obtain (16). ∎

Corollary 6. *The values $v(x)$, $x \in X$, can be found as follows: $v(x) = \varepsilon(x) - \varepsilon(x_f)$, i.e., the difference $\varepsilon(x) - \varepsilon(x_f)$ is equal to the cost of the max-min path from x to x_f. If $\varepsilon(x_f) = 0$, then $v(x) = \varepsilon(x)$, $\forall x \in X$.*

Corollary 7. *If for every directed cycle C_s in G the condition $\sum_e c_e \neq 0$ holds, then the existence of the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x,y) \in E$ such that*

$$\text{ext}(c', x) = 0, \forall x \in X \tag{19}$$

represents necessary and sufficient conditions for validity of equality (16) for every $x \in X$. In the case when in G there exists cycle C_s with $\sum_{e \in E(C_s)} c_e = 0$, condition (19) becomes only necessary one for validity of equality (16) for every $x \in X$.

Corollary 8. *If in the c-game there exist the strategies s_A^* and s_B^*, for which (16) holds for every $x \in X$, and these strategies generate in G a tree $T_{s^*} = (X, E_{s^*})$ with the sink vertex x_f, then there exists the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x,y) \in E$ such that the graph $G^0 = (X, E^0)$, generated by the set of edges $E^0 = \{(x,y) \in E \mid c'_{(x,y)} = 0\}$, contains the tree T_{s^*} as a subgraph.*

Taking into account the above-mentioned results, we propose the following algorithm for determining the optimal strategies of players in the c-game based on the constructing of the tree of min-max paths if such a tree exists in G.

Algorithm 12. Determining Optimal Strategies of Players on an Arbitrary Network

Preliminary step (step 0): Set $X^* = \{x_f\}$, $\varepsilon(x_f) = 0$.

General step (step k, $k \geq 1$): Find the set of vertices

$$X' = \{x \in X \setminus X^* \mid (x,y) \in E, y \in X^*\}.$$

For each $x \in X'$ calculate

$$\varepsilon(x) = \begin{cases} \max\limits_{y \in X_{X^*}(x)} \{\varepsilon(y) + c_{(x,y)}\}, & x \in X_A \cap X'; \\ \min\limits_{y \in X_{X^*}(x)} \{\varepsilon(y) + c_{(x,y)}\}, & x \in X_B \cap X', \end{cases} \tag{20}$$

where $X_{X^*}(x) = \{y \in X^* \mid (x,y) \in E\}$. Then in $X^* \cup X'$ find the subset

$$U^k = \left\{ x \in X^* \cup X' \;\middle|\; \operatorname*{ext}_{y \in X_{X^* \cup X'}(x)} \{\varepsilon(y) - \varepsilon(x) + c_{(x,y)}\} = 0 \right\}$$

and change X^* by U^k, i.e., $X^* = U^k$. After that we check $X^* = X$? If $X^* \neq X$, then go to the next step. If $X^* = X$, then define the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x,y) \in E$ and find the graph $G^0 = (X, E^0)$, generated by the set of edges $E^0 = \{(x,y) \in E | c'_{(x,y)} = 0\}$. In G^0, fix an arbitrary tree $GT = (X, E^*)$, which determines the optimal strategies of players as follows:

$$\begin{array}{l} s_A^*(x) = y, \text{ if } (x,y) \in E^* \text{ and } x \in X_A \setminus \{x_f\}; \\ s_B^*(x) = y, \text{ if } (x,y) \in E^* \text{ and } x \in X_B \setminus \{x_f\}. \end{array}$$

Now let us show that this algorithm finds the tree of max-min paths $GT = (X, E^*)$ if such a tree exists in G.

Denote by X^i the subset of X, where $x \in X^i$ if in GT there exists the directed path $P_T(x, x_0)$ from x to x_0 that contains i edges, i.e., $X^i = \{x \in X \;\big|\; |E(P_{GT}(x, x_0))| = i\}$. So, $X = X^0 \cup X^1 \cup X^2 \cup \dots \cup X^r$ $(X^i \cap X^j = \emptyset)$, where $X^0 = \{x_f\}$ and X^i, $i \in \{1, 2, \dots, r\}$, represents level i of the vertex set of GT. If in G there exist several max-min trees $GT_1 = (X, E_1^*)$, $GT_2 = (X, E_2^*), \dots, GT_q = (X, E_q^*)$, then we select the one that has $r = \min\limits_{1 \le i \le q} r_i$ number of levels.

Theorem 13. *If in G there exists a tree of max-min path $GT = (X, E^*)$ with the sink vertex x_f, then Algorithm 12 finds it using $k = r$ iterations. The running time of the algorithm is $O(|X|^3)$.*

Proof. We prove the theorem by using the induction principle on number r of levels of max-min tree. If $r = 1$, the theorem is evident. Assume that the theorem is true for any $r \le q$ and let us show that it is true for $r = q + 1$.

Denote by $X^0, X^1, \dots, X^r$ the level sets of the tree $GT = (X, E^*)$, $X = X^0 \cup X^1 \cup X^2 \cup \dots \cup X^r$ $(X^i \cap X^j = \emptyset)$. It is easy to observe that if we delete from GT the vertex set X^r and corresponding pendant edges $e = (x,y)$ for every $x \in X^r$, then we obtain a tree $\overline{GT}^* = (\overline{X}, \overline{E}^*)$, $\overline{X} = X \setminus X^r$. This tree $\overline{GT}^*$ represents the tree of max-min paths for the subgraph $\overline{G} = (\overline{X}, \overline{E})$ of G generated by vertex set $\overline{X}$.

If we apply Algorithm 12 with respect to $\overline{G}$, then according to the induction principle we find the tree of max-min paths $\overline{GT}^*$, which determines $\varepsilon : \overline{X} \to R$ and the potentional transformation $\overline{c}_{(x,y)} = c_{(x,y)} - \varepsilon(x) + \varepsilon(y)$ on edges $(x,y) \in \overline{E}$ such that $\operatorname{ext}(c', y) = 0$, $\forall y \in \overline{X}$. So, Algorithm 12 on $\overline{G}$ determines uniquely the values $\varepsilon(x)$ according to (20).

It is easy to observe that in G, an arbitrary vertex $x \in X^r$ determined on the basis of (20) satisfies the condition:

$$\varepsilon(x) = \begin{cases} \max\limits_{y \in X(x)} \{\varepsilon(y) + c_{(x,y)}\}, \ x \in X^r \cap X_A; \\ \min\limits_{y \in X(x)} \{\varepsilon(y) + c_{(x,y)}\}, \ x \in X^r \cap X_B. \end{cases}$$

This means that if we apply Algorithm 12 on G, then after $r-1$ iterations the vertex set U^{r-1} coincides with $X \setminus X^r$. So, Algorithm 12 determines uniquely the values $\varepsilon(x)$, $x \in X$. Nevertheless, here we have to note that in the process of the algorithm $X^k \subset U^k$ and X^k may differ from U^k for some $k = 1, 2, \ldots, r$.

Taking into account that one iteration of the general step of the algorithm needs $O(|X|^2)$ elementary operations and $k \leq r$ ($r \leq |X|$), we obtain that the running time of the algorithm is $O(|X|^3)$. ■

Remark 7. The considered max-min paths problem can be used for the zero-sum control problem with alternate players control (see Corollary 2). For $p = 2$ on the basis of construction from Section 11, we obtain the network $(\overline{G}, Z_1, Z_2, \overline{c}, z_0, z_f)$, where $\overline{G} = (Z, \overline{E})$, $Z = Z_1 \cup Z_2$ $(Z_1 \cap Z_2 = \emptyset)$, and $\overline{c} = c_e^1 = -c_e^2$, $\forall e \in E$. This network determines the max-min paths problem, solution of which corresponds with the solution of the zero-sum control problem.

13 Polynomial Time Algorithm for Solving Acyclic *l*-Game on Networks

Acyclic l-game on networks has been introduced in [35, 36] as an auxiliary problem for studying and solving cyclic games, which we will consider in the next section.

13.1 Problem Formulation

Let (G, X_A, X_B, c) be a network, where $G = (X, E)$ represents a directed acyclic graph with the sink vertex $x_f \in X$. On E is defined a function c: $E \rightarrow R$ and on X is given a partition $X = X_A \cup X_B$ $(X_A \cap X_B = \emptyset)$ where X_A and X_B correspond with positions sets of two players A and B, respectively.

We consider the following acyclic game from [35]. Again, we define the strategies of players as maps

$$s_A: x \rightarrow y \in X(x) \ \text{ for } \ x \in X_A \setminus \{x_f\};$$
$$s_B: x \rightarrow y \in X(x) \ \text{ for } \ x \in X_B \setminus \{x_f\}.$$

We define the payoff function $\overline{H}_{x_0}$: $S_A \times S_B \rightarrow R$ in this game as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be fixed strategies of players. Then the graph $G_s = (X, E_s)$, generated by edges $(x, s_A(x))$, $x \in X \setminus \{x_f\}$, and $(x, s_B(x))$, $x \in X \setminus \{x_f\}$, has a structure of directed tree with the sink vertex x_f. Therefore it contains a unique directed path $P_s(x_0, x_f)$ with $n(P_s(x_0, x_f))$ edges. We put

$$\overline{H}_{x_0}(s_A, s_B) = \frac{1}{n(P_s(x_0, x_f))} \sum_{e \in E(P_s(x_0, x_f))} c_e.$$

The payoff function $\overline{H}_{x_0}(s_A, s_B)$ on $S_A \times S_B$ defines a game in normal form, which is determined by the network $(G, X_A, X_B, c, x_0, x_f)$.

We consider the problem of finding the strategies s_A^* and s_B^*, for which

$$\overline{v}(x_0) = \overline{H}_{x_0}(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \overline{H}_{x_0}(s_A, s_B).$$

13.2 The Main Properties of Optimal Strategies in Acyclic l-Games

First of all, let us show that for the considered max-min problem there exists a saddle point.

Denote

$$\overline{\overline{v}}(x_0) = \overline{H}_{x_0}(s_A^0, s_B^0) = \min_{s_B \in S_B} \max_{s_A \in S_A} \overline{H}_{x_0}(s_A, s_B)$$

and let us show that $\overline{v}(x_0) = \overline{\overline{v}}(x_0)$.

Theorem 14. *For an arbitrary acyclic l-game, the following equality holds:*

$$\overline{v}(x_0) = \overline{H}_{x_0}(s_A^*, s_B^*) = \max_{s_A \in S_A} \min_{s_B \in S_B} \overline{H}_{x_0}(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \overline{H}_{x_0}(s_A, s_B)$$

Proof. First of all, let us note the following property of acyclic l-game, determined by $(G, X_A, X_B, c, x_0, x_f)$: If the cost function c is changed by $c' = c + h$ (h is an arbitrary real number), then we obtain an equivalent acyclic l-game determined by $(G, X_A, X_B, c', x_0, x_f)$ for which $\overline{v}'(x_0) = \overline{v}(x_0) + h$ and $\overline{\overline{v}}''(x_0) = \overline{\overline{v}}(x_0) + h$. It is easy to observe that if $h = -\overline{v}(x_0)$, then for the acyclic l-game with network $(G, X_A, X_B, c', x_0, x_f)$ we obtain $\overline{v}'(x_0) = 0$. This means that acyclic l-game becomes acyclic c-game for which the following property holds:

$$0 = \overline{v}'(x_0) = \max_{s_A \in S_A} \min_{s_B \in S_B} \overline{H}'_{x_0}(s_A, s_B) = \min_{s_B \in S_B} \max_{s_A \in S_A} \overline{H}'_{x_0}(s_A, s_B) = 0.$$

Taking into account that

$$\overline{H}'_{x_0}(s_A, s_B) = \overline{H}_{x_0}(s_A, s_B) - \overline{v}(x_0)$$

we obtain that

$$\min_{s_B \in S_B} \max_{s_A \in S_A} \left(\overline{H}_{x_0}(s_A, s_B) - \overline{v}(x_0) \right) = \max_{s_A \in S_A} \min_{s_B \in S_B} \left(\overline{H}_{x_0}(s_A, s_B) - \overline{v}(x_0) \right)$$

$$= \overline{\overline{v}}(x_0) - \overline{v}(x_0),$$

i.e. $\overline{\overline{v}}(x_0) - \overline{v}(x_0) = 0$. So, $\overline{\overline{v}}(x_0) = \overline{v}(x_0)$. ∎

Theorem 15. *Let be given an acyclic l-game determined by the network $(G, X_A, X_B, c, x_0, x_f)$ with the starting position x_0. Then there exists the value $\overline{v}(x_0)$ and the function ε: $X \to R$, which determines the potential transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(x) - \varepsilon(y)$ of costs on edges $e = (x, y) \in E$ such that the following conditions hold*

(a) $\overline{v}(x_0) = \text{ext}(c', x), \ \forall x \in X \setminus \{x_f\}$;
(b) $\varepsilon(x_0) = \varepsilon(x_f)$.

The optimal strategies of players in acyclic l-game can be found as follows: fix the arbitrary maps s^*_A: $X_A \setminus \{x_f\} \to X$ and s^*_B: $X_B \setminus \{x_f\} \to X$ such that $s^*_A(x) \in \text{VEXT}(c', x)$ for $x \in X_A \setminus \{x_f\}$ and $s^*_B(x) \in \text{VEXT}(c', x)$ for $x \in X_B \setminus \{x_f\}$.

Proof. The proof of the theorem follows from Theorem 11 if we regard the acyclic l-game as acyclic c-game on network $(G, X_A, X_B, c', x_0, x_f)$ with the cost function $c' = c - \overline{v}(x_0)$. ■

Corollary 9. *The difference $\varepsilon(x) - \varepsilon(x_0)$, $x \in X$, represents the costs of max-min path from x to x_f in the acyclic c-game on network $(G, X_A, X_B, c', x_0, x_f)$ with $c'_{(x,y)} = c_{(x,y)} - \overline{v}(x_0), \ \forall (x, y) \in E$.*

13.3 Polynomial Time Algorithm for Finding the Value and the Optimal Strategies in the Acyclic *l*-Game

The algorithm, which we describe below, is based on results from Section 13.2. In this algorithm, we shall use the following properties:

1. The value $\overline{v}(x_0)$ of acyclic l-game on network $(G, X_A, X_B, c, x_0, x_f)$ is nonnegative if and only if the value $v(x_0)$ of acyclic l-game on network $(G, X_A, X_B, c, x_0, x_f)$ is nonnegative; moreover $\overline{v}(x_0) = 0$ if and only if $v(x_0) = 0$.
2. If $M^1 = \min_{e \in E} c_e$ and $M^2 = \max_{e \in E} c_e$, then $M^1 \leq \overline{v}(x_0) \leq M^2$.
3. If in the network $(G, X_A, X_B, c, x_0, x_f)$ the cost function $c : E \to R$ is changed by the function $c^h : E \to R$, where
$$c^h_e = c_e - h, \ \forall e \in E \tag{21}$$
(h is an arbitrary constant), then the acyclic l-games on $(G, X_A, X_B, c, x_0, x_f)$ and $(G, X_A, X_B, c^h, x_0, x_f)$, respectively, have the same optimal strategies s^*_A, s^*_B. In addition, the values $\overline{v}(x_0)$ and $\overline{v}_h(x_0)$ of these games differ by a constant h: $\overline{v}_h(x_0) = \overline{v}(x_0) - h$. So, the acyclic l-games on $(G, X_A, X_B, c, x_0, x_f)$ and $(G, X_A, X_B, c^h, x_0, x_f)$ are equivalent.
According to the above-mentioned properties, if $\overline{v}(x_0)$ is known, then the acyclic l-game can be reduced to the acyclic c-game by using shift transformation (18) with $h = \overline{v}(x_0)$. After that, we can find the optimal strategies in the game with network $(G, X_A, X_B, c^h, x_0, x_f)$ by

using Algorithm 11. The most important moment for us in the proposed algorithm represents the problem of finding value h, for which $\overline{v}_h(x_0) = 0$. Taking into account properties 1 and 2, we will seek for this value by using dichotomy method on segment $[M^1, M^2]$, such that at each step of this method we will solve a dynamic c-game with network $(G, X_A, X_B, c^k, x_0, x_f)$, where $c^k = c - h_k$. The main idea of the general step of the algorithm is the following. We make shift transformation (21) with $h = h_k$, where h_k is a midpoint of the segment $[M_k^1, M_k^2]$ at step k. After that, we apply Algorithm 11 for the dynamic c-game on network $(G, X_A, X_B, c^{h_k}, x_0, x_f)$ and find $v_{h_k}(x_0)$. If $v_{h_k}(x_0) > 0$, then we fix segment $[M_{k+1}^1, M_{k+1}^2]$, where $M_{k+1}^1 = M_k^1$ and $M_{k+1}^2 = \frac{M_k^1+M_k^2}{2}$; otherwise we put $M_{k+1}^1 = \frac{M_k^1+M_k^2}{2}$ and $M_{k+1}^2 = M_k^2$. If $v_{h_k}(x_0) = 0$ then STOP.

Algorithm 13. Determining the Value and Optimal Strategies in an Acyclic *l*-Game

Preliminary step (step 0): Find the value $v(x_0)$ and optimal strategies s_A^* and s_B^* of the dynamic c-game on $(G, X_A, X_B, c, x_0, x_f)$ by using Algorithm 11. If $v(x_0) = 0$, then fix s_A^* and s_B^* as the solution of l-game, put $\overline{v}(x_0) = 0$, and STOP; otherwise fix $M_1^1 = \min_{e\in E} c_e$, $M_1^2 = \max_{e\in E} c_e$, $L = \max_{e\in E} |c_e|$.

General step (step k, $k \geq 1$): Find $h_k = \frac{M_k^1+M_k^2}{2}$ and make the shift transformation of edges costs

$$c_e^k = c_e - h_k \text{ for } e \in E.$$

Solve the dynamic c-game on network $(G, X_A, X_B, c^k, x_0, x_f)$ and find the value $v_k(x_0)$ and the optimal strategies s_A^*, s_B^*. If $v_k(x_0) = 0$, then fix the optimal strategies s_A^* and s_B^* and put $\overline{v}(x_0) = h_k$. If $|v_k(x_0)| \leq \frac{1}{4|X|^2 L}$, then fix s_A^* and s_B^*; find $\overline{v}(x_0) = \frac{\overline{H}_{x_0}(s_A^*, s_B^*)}{n(P_{s^*}(x_0, x_f))}$ and STOP. If $v_k(x_0) > \frac{1}{4|X|^2 L}$, then fix $M_{k+1}^1 = M_k^1$, $M_{k+1}^2 = h_k$ and go to step $k+1$. If $v_k(x_0) < -\frac{1}{4|X|^2 L}$, then fix $M_{k+1}^1 = h_k$, $M_{k+1}^2 = M_k^2$ and go to step $k+1$.

Theorem 16. *Let $(G, X_A, X_B, c, x_0, x_f)$ be a network with integer cost function $c : E \to R$, and $L = \max_{e\in E} |c_e|$. Then Algorithm 13 finds correctly the value $\overline{v}(x_0)$ and optimal strategies s_A^*, s_B^* in the acyclic l-game. The running time of the algorithm is $O(|X|^2 \log L + 2|X|^2 \log |X|)$.*

Proof. Let $(G, X_A, X_B, c^k, x_0, x_f)$ be a network after final step k of Algorithm 13. Then

$$|v_k(x_0)| \leq \frac{1}{4|X|^2 L}$$

and the number $\varepsilon_k(x)$, $x \in X$, determined according to Algorithm 11 (when we solve acyclic c-game), represents the approximation solution of the system

$$\begin{cases} \varepsilon(y) - \varepsilon(x) + c_{(x,y)}^k \leq 0 \text{ for } x \in X_A, \ (x,y) \in E; \\ \varepsilon(y) - \varepsilon(x) + c_{(x,y)}^k \geq 0 \text{ for } x \in X_B, \ (x,y) \in E; \\ \varepsilon(x_0) = \varepsilon(x_f). \end{cases}$$

This means that $\varepsilon_k(x)$, $x \in X$, and h_k represent the approximative solution of the system

$$\begin{cases} \varepsilon(y) - \varepsilon(x) + c_{(x,y)} \leq h \text{ for } x \in X_A,\ (x,y) \in E; \\ \varepsilon(y) - \varepsilon(x) + c_{(x,y)} \geq h \text{ for } x \in X_B,\ (x,y) \in E; \\ \varepsilon(x_0) = \varepsilon(x_f). \end{cases}$$

According to [29, 30], the exact solution $h = \overline{v}(x)$, $\varepsilon(x)$, $x \in X$, of this system can be obtained from h_k, $\varepsilon_k(x)$, $x \in X$, by using the special round-off procedure in time $O(\log(L+1))$. Therefore, the strategies s_A^*, s_B^* after the final step k of the algorithm correspond with the optimal solution of acyclic l-game.

Taking into account that the tabulation of values $\varepsilon(x)$, $x \in X$, in G needs $O(|X|^2)$ operations and the number of iterations of the algorithm is $O(\log(L+1) + 2\log|X|)$, we obtain that the running time of the algorithm is $O(|X|^2 \log(L+1) + 2|X|^2 \log|X|)$. ■

14 Cyclic Games: Algorithms for Finding the Value and the Optimal Strategies of Players

Cyclic games have been introduced in [15,24,51] as extension of control models for discrete systems with infinite time horizon and mean integral-time cost by a trajectory. Here we show that the problem of finding optimal strategies of players in such games is tightly connected with the problem of finding optimal strategies of players in the dynamic c-game and the acyclic l-game. On the basis of these results, we propose algorithms for determining the value and the optimal strategies in cyclic games.

14.1 Problem Formulation and the Main Properties

Let $G = (X, E)$ be a finite directed graph in which every vertex $x \in X$ has at least one leaving edge $e = (x, y) \in E$. On edge set E is given a function c: $E \rightarrow\ R$, which assigns a cost c_e to each edge $e \in E$. In addition, the vertex set X is divided into two disjoint subsets X_A and X_B ($X = X_A \cup X_B,\ X_A \cap X_B = \emptyset$), which we will regard as positions sets of two players.

On G, we consider the following two-person game from [15, 24, 64, 66]. The game starts at position $x_0 \in X$. If $x_0 \in X_A$, then the move is done by the first player, otherwise it is done by the second one. The move means the passage from the position x_0 to the neighbor position x_1 through the edge $e_1 = (x_0, x_1) \in E$. After that, if $x_1 \in X_A$, then the move is done by the first player, otherwise it is done by the second one and so on indefinitely. The first player has the aim to maximize $\lim_{t\rightarrow\infty} \inf \frac{1}{t}\sum_{i=1}^{t} c_{e_i}$ whereas the second player has the aim to minimize $\lim_{t\rightarrow\infty} \sup \frac{1}{t}\sum_{i=1}^{t} c_{e_i}$.

In [15], it is proved that for this game there exists a value $\overline{v}(x_0)$ such that the first player has a strategy of moves that ensures $\lim_{t\to\infty} \inf \frac{1}{t}\sum_{i=1}^{t} c_{e_i} \geq \overline{v}(x_0)$ and the second player has a strategy of moves that ensures $\lim_{t\to\infty} \sup \frac{1}{t}\sum_{i=1}^{t} c_{e_i} \leq \overline{v}(x_0)$. Furthermore, in [15] it is shown that the players can achieve the value $\overline{v}(x_0)$ applying the strategies of moves that do not depend on t. This means that the considered game can be formulated in the terms of stationary strategies. Such statement of the game in [24] is named the cyclic game.

The strategies of players in the cyclic game are defined as maps

$$s_A\colon x \to y \in X(x) \text{ for } \; x \in X_A,$$
$$s_B\colon x \to y \in X(x) \;\text{ for } \; x \in X_B,$$

where $X(x) = \{y \in X \mid e = (x, y) \in E\}$. Because G is a finite graph, then the sets of strategies of players

$$S_A = \{s_A\colon x \to y \in X(x) \text{ for } x \in X_A\};$$
$$S_B = \{s_B\colon x \to y \in X(x) \text{ for } x \in X_B\}$$

are finite sets. The payoff function $\overline{H}_{x_0}\colon S_A \times S_B \to R$ in the cyclic game is defined as follows.

Let $s_A \in S_A$ and $s_B \in S_B$ be fixed strategies of players. Denote by $G_s = (X, E_s)$ the subgraph of G generated by edges of form $(x, s_A(x))$ for $x \in X_A$ and $(x, s_B(x))$ for $x \in X_B$. Then G_s contains a unique directed cycle C_s, which can be reached from x_0 through the edges $e \in E_s$. We consider that the value $\overline{H}_{x_0}(s_A, s_B)$ is equal to mean edges cost of cycle C_s, i.e.,

$$\overline{H}_{x_0}(s_A, s_B) = \frac{1}{n(C_s)} \sum_{e \in E(C_s)} c_e,$$

where $E(C_s)$ represents the set of edges of cycle C_s and $n(C_s)$ is a number of the edges of C_s. So, the cyclic game is determined uniquely by the network (G, X_A, X_B, c) and the starting position x_0. In [15, 24], it is proved that there exist the strategies $s_A^* \in S_A$ and $s_B^* \in S_B$ such that

$$\begin{aligned} \overline{v}(x) = \overline{H}_x(s_A^*, s_B^*) &= \max_{s_A \in S_A} \min_{s_B \in S_B} \overline{H}_x(s_A, s_B) \\ &= \min_{s_B \in S_B} \max_{s_A \in S_A} \overline{H}_x(s_A, s_B), \forall x \in X. \end{aligned}$$

So, the optimal strategies s_A^*, s_B^* of players in cyclic games do not depend on a starting position x_0, although for different positions $x, y \in X$ the values $\overline{v}(x)$ and $\overline{v}(y)$ may be different. It means that the positions set X can be divided into several classes $X = X^1 \cup X^2 \cup \cdots \cup X^k$ according to values of positions $\overline{v}^1, \overline{v}^2, \ldots, \overline{v}^k$, i.e., $x, y \in X^i$ if and only if $\overline{v}^i = \overline{v}(x) = \overline{v}(y)$. In the case $k = 1$

the network (G, X_A, X_B, c) is named the ergodic network [24]. In [34,39], it is shown that every cyclic game with an arbitrary network (G, X_A, X_B, c) and a given starting position x_0 can be reduced to a cyclic game on an auxiliary ergodic network (G', X'_A, X'_B, c').

It is well-known [28, 66] that the decision problem associated with cyclic game is in $NP \bigcap$ co-NP. Some exponential and pseudo-polynomial algorithms for finding the value and the optimal strategies of players in cyclic game are proposed in [66]. Our aim is to propose polynomial time algorithms for determining optimal strategies of players in cyclic games. We argue such algorithms on the basis of results that have been announced in [39, 40].

14.2 Determining the Best Response of the First Player for the Fixed Strategy of the Second One

In order to find the best response of the first player for the fixed strategy of the second one, we shall use the model from Section 14.1 in the case $X_B = \emptyset$, i.e., $X = X_A$. This case of the model corresponds with the problem of finding in G the maximal mean cost cycle, which can be reached from x_0. An efficient polynomial time algorithm for finding maximal mean cost cycle in a weighted directed graph is proposed in [11, 27]. In [34, 61, 62], it is shown that for a strongly connected graph, this problem can be represented as the following linear programming problem:
to maximize the object function

$$\overline{H} = \sum_{e \in E} c_e \alpha_e$$

on subject

$$\begin{cases} \sum\limits_{e \in E^-(x)} \alpha_e - \sum\limits_{e \in E^+(x)} \alpha_e = 0, \ \forall x \in X; \\ \sum\limits_{e \in E} \alpha_e = 1; \\ \qquad \alpha_e \geq 0, \ e \in E, \end{cases}$$

where $E^-(x)$ is a set of edges $e = (y, x) \in E$, which have their extremities in x, and $E^+(x)$ is a set of edges $e = (x, y) \in E$, originated in x. The variable α_e is associated with each edge $e \in E$.

An arbitrary admissible solution α of the considered linear programming problem determines in G a flow circulation with the constant (equal to 1) sum of flow values by edges of the directed weighted graph G. It is easy to show that any admissible solution of the linear programming problem can be represented in the form of convex combination of flow values of elementary directed cycles with the constant (equal to 1) sum of flow values by edges of these cycles. Thus, associating to each solution α of polyhedral admissible set Z_α of the problem the directed subgraph $G_\alpha = (X_\alpha, E_\alpha)$ generated by the edges $e \in E$ with $\alpha_e > 0$, we obtain that any of the extreme points α' of the polyhedral set Z_α will correspond with the subgraph $G_{\alpha'}$ of G, which has the structure of an elementary directed cycle. So, the following lemma holds.

Lemma 4. *If α' is a solution of the problem, which corresponds with an extreme point of Z_α, then the graph $G_{\alpha'}$ represents an elementary cycle in G.*

On the basis of this lemma in [34, 61], the following theorem is proved.

Theorem 17. *If α^* is an optimal basic solution of the considered linear programming problem, then the cycle E_{α^*} is the maximal mean cost cycle in G.*

So, the problem of finding the maximal mean cost cycle in G can be found by using the polynomial algorithm. Moreover, on the basis of the duality theory, we can find the condition for determining the value v of maximal mean cycle and the solution. Indeed, if for our linear programming problem we define the dual problem:
to minimize

$$z = v$$

on subject

$$\varepsilon(x) - \varepsilon(y) + v \geq c_{(x,y)}, \ \forall (x,y) \in E,$$

then we obtain the following result, which is similar to the one from [5].

Theorem 18. *For a given strongly connected directed graph $G = (X, E)$, there exist the value v and the function $\varepsilon : X \to R$ such that*

$$c'_{(x,y)} = \varepsilon(y) - \varepsilon(x) + c_{(x,y)} - v \leq 0, \ \forall (x,y) \in E$$

and

$$\max_{y \in X(x)} c'_{(x,y)} = 0, \ \forall x \in X.$$

Moreover, if we fix in G an arbitrary map $s^* : x \to y \in X(x)$ such that $c'_{(x,s(x))} = 0$, $\forall x \in X$, then an arbitrary directed cycle C in $G_{\alpha^*} = (X, E_{\alpha^*})$ is a solution of the problem.

Let us show that if $\overline{s}_B$ is an arbitrary fixed strategy of the second player, then the best response $\overline{s}^*_A$ of the first player can be found by using the approach described above. Indeed, if the second player fixes his strategy $\overline{s}_B$, then this means that in G the set of edges $E_{\overline{s}_B} = \{(x, \overline{s}_B(x)) | x \in X_B\}$ is fixed. Therefore we obtain the subgraph $\overline{G} = (X, \overline{E})$, where $\overline{E} = E_A \cup E_{s_B}$, where $E_A = \{(x,y) \in E | x \in X_A\}$, and in order to obtain the best response of the first player, we have to find in this graph the maximal mean cost cycle, which corresponds with solution

$$\overline{s}^*_A : \overline{H}_x(\overline{s}^*_A, \overline{s}_B) = \max_{s_A} \overline{H}_x(s_A, \overline{s}_B) \text{ for } \forall x \in X.$$

The approach based on the alternate best response of players in cyclic games of course can be used for solving some classes of cyclic games. But such approach cannot be estimated from the computational point of view. Therefore in the following, we will propose another approach for determining the optimal strategies in cyclic games.

14.3 Some Preliminary Results

First of all, we need to recall some preliminary results from [24,34,35,39,40].

Let (G, X_A, X_B, c) be a network with the properties described in Section 14.1. In the analogous way as for dynamic c-games, here we denote

$$\operatorname{ext}(c,x) = \begin{cases} \max\limits_{y\in X(x)} c_{(x,y)} & \text{for} \quad x \in X_A, \\ \min\limits_{y\in X(x)} c_{(x,y)} & \text{for} \quad x \in X_B, \end{cases}$$

$$\operatorname{VEXT}(c,x) = \{y \in X(x) \mid c_{(x,y)} = \operatorname{ext}(c,x)\}.$$

We shall use the potential transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ for costs on edges $e = (x,y) \in E$, where $\varepsilon\colon X \to R$ is an arbitrary function on the vertex set X. In [24], it is noted that the potential transformation does not change the value and the optimal strategies of players in cyclic games.

Theorem 19. *Let (G, X_A, X_B, c) be an arbitrary network with the properties described in Section 14.1. Then there exists the value $\overline{v}(x)$, $x \in X$ and the function $\varepsilon\colon X \to R$, which determines the potential transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ for costs on edges $e = (x,y) \in E$, such that the following properties hold*

(a) $\overline{v}(x) = \operatorname{ext}(c', x)$ *for* $x \in X$,
(b) $\overline{v}(x) = \overline{v}(y)$ *for* $x \in X_A \cup X_B$ *and* $y \in \operatorname{VEXT}(c', x)$,
(c) $\overline{v}(x) \geq \overline{v}(y)$ *for* $x \in X_A$ *and* $y \in X_G(x)$,
(d) $\overline{v}(x) \leq \overline{v}(y)$ *for* $x \in X_B$ *and* $y \in X_G(x)$,
(e) $\max\limits_{e\in E} |c'_e| \leq 2|X| \max\limits_{e\in E} |c_e|$.

The values $\overline{v}(x), x \in X$ on network (G, X_A, X_B, c) are determined uniquely and the optimal strategies of players can be found in the following way: fix the arbitrary strategies $s_A^*\colon X_A \to X$ and $s_B^*\colon X_B \to X$ such that $s_A^*(x) \in \operatorname{VEXT}(c', x)$ for $x \in X_A$ and $s_B^*(x) \in \operatorname{VEXT}(c', x)$ for $x \in X_B$.

The proof of Theorem 19 is given in [24].

Further, we shall use Theorem 19 in the case of the ergodic network (G, X_1, X_2, c), i.e., we shall use the following corollary.

Corollary 10. *Let (G, X_A, X_B, c) be an ergodic network. Then there exists the value $\overline{v}$ and the function $\varepsilon\colon X \to R$, which determines the potential transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ for costs of edges $e = (x,y) \in E$ such that $\overline{v} = \operatorname{ext}(c', x)$ for $x \in X$. The optimal strategies of players can be found as follows: fix arbitrary strategies $s_A^*\colon X_A \to X$ and $s_B^*\colon X_B \to X$ such that $s_A^*(x) \in \operatorname{VEXT}(c', x)$ for $x \in X_A$ and $s_B^*(x) \in \operatorname{VEXT}(c', x)$ for $x \in X_B$.*

14.4 The Reduction of Cyclic Games to Ergodic Ones

Let us consider an arbitrary network (G, X_A, X_B, c) with the given starting position $x_0 \in X$, which determines a cyclic game. In [34, 39], it is shown that this game can be reduced to a cyclic game on an auxiliary ergodic network $(G', W_A, W_B, \overline{c})$, $G' = (W, E')$ in which the value $\overline{v}(x_0)$ is preserving and $x_0 \in W = X \cup U \cup Z$.

The graph $G' = (W, E')$ is obtained from G if each edge $e = (x, y)$ is changed by a triple of edges $e^1 = (x, u), e^2 = (u, z), e^3 = (z, y)$ with the costs $\overline{c}_{e^1} = \overline{c}_{e^2} = \overline{c}_{e^3} = c_e$. Here $u \in U$, $z \in Z$ and $x, y \in X$; $W = X \cup U \cup Z$. In addition, in G' each vertex u is connected with x_0 by edge (u, x_0) with the cost $\overline{c}_{(u,x_0)} = M$ (M is a great value) and each vertex z is connected with x_0 by edge (z, x_0) with the cost $\overline{c}_{(z,x_0)} = -M$. In $(G', W_A, W_B, \overline{c})$, the sets W_A and W_B are defined as follows: $W_A = X_A \cup Z$; $W_B = X_B \cup U$.

It is easy to observe that this reduction can be done in linear time.

14.5 Polynomial Time Algorithm for Solving Ergodic Zero-Value Cyclic Games

Let us consider an ergodic zero-value cyclic game determined by the network (G, X_A, X_B, c, x_0), where $G = (X, E)$. Then according to Theorem 19, there exists the function $\varepsilon : X \to R$, which determines the potentional transformation $c'_{(x,y)} = c_{(x,y)} + \varepsilon(y) - \varepsilon(x)$ on edges $(x, y) \in E$ such that

$$\text{ext}(c, x) = 0, \ \forall x \in X. \tag{22}$$

This means that if x_f is a vertex of the cycle C_{s^*} determined by optimal strategies s_A^* and s_B^*, then the problem of finding the function $\varepsilon : X \to R$, which determines the canonic potentional transformation, is equivalent to the problem of finding the values $\varepsilon(x)$, $x \in X$ in max-min paths problem on G with the sink vertex x_f where $\varepsilon(x_f) = 0$.

So, in order to solve the zero-value cyclic game, we fix each time a vertex $x \in X$ as a sink vertex ($x_f = x$) and solve a max-min paths problem on G with the sink vertex x_f. If for the given $x_f = x$ the function $\varepsilon : X \to R$ obtained on the basis of Algorithm 12 determines the potentional transformation that satisfies (22), then we fix s_A^* and s_B^* such that $s_A^*(x) \in \text{VEXT}(c', x)$ for $x \in X_A$ and $s_B^*(x) \in \text{VEXT}(c', x)$ for $x \in X_B$. If for the given x the function $\varepsilon : X \to R$ does not satisfy (22), then we select another vertex $x \in X$ as a sink vertex and so on. This means that the optimal strategies of players in zero-value ergodic cyclic games can be found in time $O(|X|^4)$.

14.6 Polynomial Time Algorithm for Determining the Value and Optimal Strategies of Players in Ergodic Cyclic Games

Assume that the ergodic cyclic game is determined by the network (G, X_A, X_B, c, x_0), and the value of the game may be different from zero. The graph G is assumed to be strongly connected.

At first, we propose a polynomial time algorithm for determining the value of this game and optimal strategies of players in the case when vertex x_0 belongs to a max-min cycle induced by optimal strategies of players.

We consider an auxiliary dynamic c-game determined by an auxiliary network $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}, x_0, x'_0)$, where the graph $G = (X \cup \{x'_0\}, \overline{E})$ is obtained from G by adding a copy x'_0 of vertex x_0 together with copies $e' = (x, x'_0)$ of edges $e = (x, x_0) \in E$ with costs $\overline{c}_{e'} = c_e$; for the rest of the edges $e \in E$, costs $\overline{c}_e$ and c_e coincide. So, for x'_0 in $\overline{G}$, there are no leaving edges (x'_0, x).

It is evident that if the value $\overline{v}$ of the ergodic cyclic game on (G, X_A, X_B, c, x_0) is known, then the problem of finding the optimal strategies of players is equivalent to the problem of finding the min-max paths on network $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}', x_0, x'_0)$ with the cost function

$$\overline{c}'_e = \overline{c}_e - \overline{v}(x_0) \text{ for } e \in \overline{E}.$$

This means that if s^*_A and s^*_B are optimal strategies of players in the ergodic cyclic game on (G, X_A, X_B, c, x_0), then $\overline{s}_A(x) = s_A(x)$ for $x \in X_A \setminus \{x_0\}$; $\overline{s}_B(x) = s_B(x)$ for $x \in X_B \setminus \{x_0\}$ are optimal strategies of players in the dynamic c-game on $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}', x_0, x'_0)$, where $\overline{c}' = c - \overline{v}$.

In the analogous way as for acyclic l-game, here we may mention the following properties.

1. The value $\overline{v}(x_0)$ of the ergodic cyclic game on network (G, X_A, X_B, c, x_0) is nonnegative if and only if the value $v(x_0)$ of the dynamic c-game on network $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}, x_0, x'_0)$ is nonnegative; moreover $\overline{v}(x_0) = 0$ if and only if $v(x_0) = 0$.
2. If $M^1 = \min_{e \in E} c_e$ and $M^2 = \max_{e \in E} c_e$, then $M^1 \le \overline{v}(x_0) \le M^2$.
3. If in networks (G, X_A, X_B, c, x_0) and $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}, x_0, x'_0)$ cost functions $c : E \to R$ and $\overline{c} : \overline{E} \to R$ are changed by $c' = c + h$ and $\overline{c} = \overline{c}' + h$, respectively, then the values $\overline{v}(x_0)$ and $v(x_0)$ are changed by $\overline{v}'(x_0) = \overline{v}(x_0) + h$ and $v'(x_0) = v(x_0) + h$, respectively.
   On the basis of these properties, we will seek for the unknown value $\overline{v}(x_0) = v(x)$, which we denote by h, using the dichotomy method on segment $[M^1, M^2]$ such that at each step of this method, we will solve a dynamic c-game with network $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}^h, x_0, x'_0)$, where $\overline{c}^h = \overline{c} - h$. So, the main idea of the general step of the algorithm is the following. We make shift transformation

   $$\overline{c}^k = \overline{c} - h_k \text{ for } e \in E,$$

   where h_k is a midpoint of segment $[M^1_k, M^2_k]$ at step k. After that, we apply Algorithm 12 for the dynamic c-game on network $(\overline{G}, X_A, X_B \cup \{x'_0\}, \overline{c}^h, x_0, x'_0)$ and find $v_{h_k}(x_0)$. If $v_{h_k}(x_0) > 0$, then we fix segment $[M^1_{k+1}, M^2_{k+1}]$, where $M^1_{k+1} = M^1_k$ and $M^2_{k+1} = \frac{M^1_k + M^2_k}{2}$; otherwise we put $M^1_{k+1} = \frac{M^1_k + M^2_k}{2}$ and $M^2_{k+1} = M^2_k$. If $v_{h_k}(x_0) = 0$, then STOP.

Algorithm 14. Determining the Value of Ergodic Cyclic Game and Optimal Strategies of Players when the Vertex x_0 Belongs to Max-Min Cycle

Preliminary step (step 0): Find the value $v(x_0)$ and optimal strategies s_A^* and s_B^* of the dynamic c-game on $(\overline{G}, X_A, X_B \cup \{x_0'\}, c, \overline{c}, x_0, x_0')$ by using Algorithm 12. If $v(x_0) = 0$, then fix s_A^* and s_B^* and STOP; otherwise fix $M_1^1 = \min_{e\in E} \overline{c}_e$, $M_1^2 = \max_{e\in E} \overline{c}_e$ and $L = \max_{e\in E} |\overline{c}_e|$.

General step (step k, $k \geq 1$): Find $h_k = \frac{M_k^1 + M_k^2}{2}$ and make the shift transformation of edges costs

$$\overline{c}_e^k = \overline{c}_e - h_k \text{ for } e \in \overline{E}.$$

Solve the dynamic c-game on network $(\overline{G}, X_A, X_B \cup \{x_0'\}, \overline{c}^k, x_0, x_0')$ and find the value $v_k(x_0)$ and the optimal strategies s_A^*, s_B^*. If $v_k(x_0) = 0$, then fix the optimal strategies s_A^* and s_B^* and put $\overline{v}(x_0) = h_k$. If $|v_k(x_0)| \leq \frac{1}{4|X|^2 L}$, where $l = \max_{e\in E} |c_e|$, then fix s_A^* and s_B^*. After that, find $s_A^* = \overline{s}_A^*$, $s_A^* = \overline{s}_A^*$ in G and calculate the value $\overline{v}(x_0)$ of max-min mean cycle generated by s_A^* and s_B^* in G and STOP. If $v_k(x_0) > \frac{1}{4|X|^2 L}$, then fix $M_{k+1}^1 = M_k^1$, $M_{k+1}^2 = h_k$ and go to step $k+1$. If $v_k(x_0) < -\frac{1}{4|X|^2 L}$, then fix $M_{k+1}^1 = h_k$, $M_{k+1}^2 = M_k^2$ and go to step $k+1$.

Theorem 20. *Let a dynamic c-game determined by the network (G, X_A, X_B, c, x_0) be given with integer cost function $c : E \to R$. Then Algorithm 14 finds correctly the value $\overline{v}(x_0)$ and optimal strategies s_A^*, s_B^* of the ergodic cyclic game. The running time of the algorithm is $O(|X|^3 \log(L+1) + |X|^3 \log |X|)$, where $L = \max_{e\in E} |c_e|$.*

The proof of this theorem is identical to the proof of Theorem 16.

In the case when x_0 may not belong to max-min cycle determined by optimal strategies of players in cyclic game, we solve $|X|$ problems by fixing each time the starting position $x_0 = x$ for $x \in X$. Then at least for a position $x_0 = x \in X$, we obtain the value of cyclic game and the optimal strategies of players.

14.7 Polynomial Time Algorithm for Solving Cyclic Games Based on Reduction to Acyclic l-Games

On the basis of the obtained results, we can propose polynomial time algorithm for solving cyclic games.

We consider an acyclic game on the ergodic network (G, X_A, X_B, c, x_0) with the given starting position x_0. The graph $G = (X, E)$ is considered to be strongly connected and $X = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$. Assume that x_0 belongs to the cycle C_{s^*} determined by the optimal strategies of players s_A^* and s_B^*.

We construct an auxiliary acyclic graph $GT_r = (\overline{W}_r, \overline{E}_r)$, where

$$\overline{W}_r = \{w_0^0\} \cup W^1 \cup W^2 \cup \cdots \cup W^r, \; W^i \cap W^j = \emptyset, \, i \neq j;$$

$$W^i = \{w_0^i, w_1^i, \dots, w_{n-1}^i\},\ i = \overline{1, r};$$
$$\overline{E} = E^0 \cup E^1 \cup E^2 \cup \dots \cup E^{r-1};$$
$$E^i = \{(w_k^{i+1}, w_l^i) |\, (x_k, x_l) \in E\},\ i = \overline{1, r-1};$$
$$E^0 = \{(w_k^i, w_0^0) |\, (x_k, x_0) \in E,\ i = \overline{1, r}\}.$$

The vertex set W_r of GT_r is obtained from X if it is doubled r times and then a sink vertex w_0^0 is added. The edge subset $E^i \subseteq \overline{E}$ in GT_r connects the vertices of the set W^{i+1} and the vertices of the set W^i in the following way: if in G there exists an edge $(x_k, x_l) \in E$, then in GT_r we add the edge (w_k^{i+1}, w_l^i). The edge subset $E^0 \subseteq \overline{E}$ in GT_r connects the vertices $w_k^i \in W^1 \cup W^2 \cup \dots \cup W^r$ with the sink vertex w_0^0, i.e., if there exists an edge $(x_k, x_0) \in E$, then in GT_r we add the edges $(w_k^i, w_0^0) \in E^0$, $i = \overline{1, r}$.

After that, we define the acyclic network (GT_r', W_A, W_B, c'), $GT_r' = (W_r, E_r)$ where GT_r' is obtained from GT_r by deleting the vertices $w_k^i \in \overline{W}_r$ from which the vertex w_0^0 cannot be attainable. The sets W_A, W_B and the cost function c': $E_r \to R$ are defined as follows:

$$W_A = \{w_k^i \in W_0 |\, x_k \in X_A\},\ W_B = \{w_k^i \in W_0 |\, x_k \in X_B\};$$
$$c'_{(w_k^{i+1}, w_l^i)} = c_{(x_k, x_l)} \ \text{ if } \ (x_k, x_l) \in E \ \text{ and } \ (w_k^{i+1}, w_l^i) \in E^i;\ i = \overline{1, r-1};$$
$$c'_{(w_k^i, w_0^0)} = c_{(x_k, x_0)} \ \text{ if } \ (x_k, x_0) \in E \ \text{ and } \ (w_k^i, w_0^0) \in E^0; i = \overline{1, r}.$$

Now we consider the acyclic c-game on the acyclic network (GT_r', W_A, W_B, c') with the sink vertex w_0^0 and the starting position w_0^r.

Lemma 5. *Let $\overline{v} = \overline{v}(x_0)$ be a value of the ergodic cyclic game on G, and the number of edges of the max-min cycle in G is equal to r. Moreover, let $\overline{v}_r(w_0^r)$ be the value of the l-game on (GT_r', W_A, W_B, c') with the starting position w_0^r. Then $\overline{v}(x_0) = \overline{v}_r(w_0^r)$.*

Proof. It is evident that there exists a bijective mapping between the set of cycles with no more than r edges (which contains the vertex x_0) in G and the set of directed paths with no more than r edges from w_0^r to w_0^0 in GT_r'. Therefore $\overline{v}(x_0) = \overline{v}_r(w_0^r)$. ∎

On the basis of this lemma, we can propose the following algorithm for finding the optimal strategies of players in cyclic games.

Algorithm 15. Determining the Optimal Stationary Strategies of Players in Cyclic Games with the Known Vertex x_0 of Max-Min Cycle of the Network

We construct the acyclic networks (GT_r', W_A, W_B, c'), $r = 2, 3, \dots, n$, and for each of them solve l-game. In such a way, we find the values $\overline{v}_2(w_0^2)$, $\overline{v}_3(w_0^3), \dots, \overline{v}_n(w_0^n)$ for these l-games. Then we consecutively fix $\overline{v} = \overline{v}_2(w_0^2)$, $\overline{v}_3(w_0^3), \dots, \overline{v}_n(w_0^n)$ and each time solve the c-game on network

(G, X_A, X_B, c'), where $c' = c - \overline{v}$. Fixing each time the values $\varepsilon'(x_k) = v(x_k)$ for $x_k \in X$, we check if the following condition

$$\operatorname{ext}(c^r, x_k) = 0,\ \forall x_k \in X$$

is satisfied, where $c^r_{(x_k,x_l)} = c'_{(x_k,x_l)} + \varepsilon(x_l) - \varepsilon(x_k)$. We find r for which this condition holds and fix the respective maps s_A^* and s_B^* such that $s_A^*(x_k) \in \mathrm{VEXT}(c', x_k)$ for $x_k \in X_A$ and $s_B^*(x_k) \in \mathrm{VEXT}(c', x_k)$ for $x_k \in X_B$. So, s_A^* and s_B^* represent the optimal strategies of players in cyclic games on G.

Remark 8. Algorithm 15 finds the value $\overline{v}(x_0)$ and optimal strategies of players in time $O(|X|^5 \log L + 4|X|^3 \log |X|)$, because Algorithm 13 needs $O(|X|^4 \log L + 4|X|^2 \log |X|)$ elementary operations for solving acyclic l-game on network (GT'_r, W_A, W_B, c'), where $L = \max_{e \in E} |c_e|$.

In the general case, if the pertinence of x_0 to the max-min cycle is unknown, then we use the following algorithm.

Algorithm 16. Determining the Optimal Strategies of Players in Ergodic Cyclic Games (General Case)

Preliminary step (step 0): Fix $Y_1 = X$.

General step (step k): Select a vertex $y \in Y_1$, fix $x_0 = y$, and apply Algorithm 15. If $\operatorname{ext}(c^r, x) = 0,\ \forall x \in X$ for $r \in \{2, 3, \dots, n\}$, then fix $s_A^* \in \mathrm{VEXT}(c^k, x)$ for $x \in X_A$ and $s_B^* \in \mathrm{VEXT}(c^k, x)$ for $x \in X_B$ and STOP; otherwise put $Y_{k+1} = Y_k \setminus \{y\}$ and go to next step $k+1$.

Remark 9. Algorithm 16 finds the value $\overline{v}$ and optimal strategies of players in time $O(|X|^6 \log L + 4|X|^4 \log |X|)$, because in the worst case Algorithm 15 is repeated $|X|$ times.

15 Nash Equilibria Condition for Cyclic Games with p Players

The cyclic game with p players is determined by the network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0)$, where $G = (X, E)$ is a directed graph in which every vertex $x \in X$ has at least one leaving edge $e = (x, y) \in E$. A partition $X = X_1 \cup X_2 \cup \dots \cup X_p$ $(X_i \cap X_j = \emptyset,\ i \neq j)$ on the vertex set X is given, and p functions $c^1 : E \to R^1$; $c^2 : E \to R^1; \dots; c^p : E \to R^1$ on the edge set E are defined. The strategies of players

$$s_i : x \to y \in X_G(x) \text{ for } x \in X_i,\ i = \overline{1,p}$$

and the payoff functions $\overline{H}^i_{x_0} : S_1 \times S_2 \times \dots \times S_P \to R$, $i = \overline{1,p}$, in the cyclic game with p players are defined in analogous way as for the zero-sum cyclic game from Section 14. Denote by $G_s = (X, E_s)$ a subgraph of G, generated by fixed strategies $s_1, s_2, \dots, s_p$ of players $1, 2, \dots, p$. Then G_s contains a unique directed cycle C_s, which can be reached from the given starting position x_0 through the edges $e \in E_s$. The values $\overline{H}^i_{x_0}(s_1, s_2, \dots, s_p)$ are considered to be equal to mean edges costs of cycle C_s, i.e.,

$$\overline{H}^i_{x_0}(s_1, s_2, \dots, s_p) = \frac{1}{n(C_s)} \sum_{e \in E(C_s)} c^i_e,$$

where $n(C_s)$ is a number of edges of cycle C_s, and $E(C_s)$ is a set of edges of this cycle.

Intuitively, it is clear that for cyclic games with p players, Nash equilibria may not exist. An example, for which Nash equilibria in the cyclic game of two players (with maximum criteria) does not exist, is given in [24]. This example is related to a cyclic game on a complete bipartite graph $G = (X_1 \cup X_2, E)$ with the set of positions $X_1 = \{x_1, x_2, x_3\}$ of the first player and the set of positions $X_2 = \{y_1, y_2, y_3\}$ of the second player; $E = \{(x_i, y_j) | i = \overline{1,3},\ j = \overline{1,3}\}$. The cost functions of the players on edges (in both directions) are defined by the matrices

$$C^1 = \begin{pmatrix} 0\ 0\ 1 \\ \varepsilon\ 0\ 0 \\ 0\ \varepsilon\ 0 \end{pmatrix};\ C^2 = \begin{pmatrix} 1 & 0\ 0 \\ 0 & 1\ 0 \\ 1 - \varepsilon & 0\ 1 \end{pmatrix}$$

If ε is a small value (for example $\varepsilon = 0.1$), then Nash equilibria for such game does not exist.

Here we formulate a necessary and sufficient condition for existence of Nash equilibria in so called ergodic cyclic games with p players, which extend zero-sum ergodic cyclic games.

Definition 5. *Let $s_1^*, s_2^*, \dots, s_p^*$ be a solution in the sense of Nash for the cyclic game determined by the network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0)$, where $G = (X, E)$ is a strongly connected directed graph. We call this game as an ergodic cyclic game if $s_1^*, s_2^*, \dots, s_p^*$ represent the solution in the sense of Nash for the cyclic game on the network $(G, X_1, X_2, \dots, X_p,$ $c^1, c^2, \dots, c^p, x)$ with an arbitrary starting position $x \in X$ and*

$$\overline{H}^i_x(s_1^*, s_2^*, \dots, s_p^*) = \overline{H}^i_y(s_1^*, s_2^*, \dots, s_p^*),\ \forall x, y \in X,\ i = \overline{1,p}.$$

Theorem 21. *The dynamic c-game determined by the network $(G, X_1,$ $X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0)$, where $G = (X, E)$ is a strongly connected directed graph, is ergodic one if and only if on X there exist p real functions*

$$\varepsilon^1 : X \to R^1,\ \varepsilon^2 : X \to R^1, \dots, \varepsilon^p : X \to R^1,$$

and p values $\overline{v}^1, \overline{v}^2, \dots, \overline{v}^p$ such that the following conditions are satisfied:

(a) $\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} - \overline{v}^i \geq 0,\ \forall (x,y) \in E_i,\ i = \overline{1,p}$;

(b) $\min\limits_{y \in X_G(x)} \{\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} - \overline{v}^i\} = 0,\ \forall x \in X_i,\ i = \overline{1,p}$;

(c) the subgraph $\overline{G}^0 = (X, \overline{E}^0)$ *generated by the edge set* $\overline{E}^0 = E_1^0 \cup E_2^0 \cup \cup \dots \cup E_p^0$, $E_i^0 = \{e = (x, y) \in E_i | \varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} - \overline{v}^i = 0\}$, $i = \overline{1,p}$, *has the property that it contains a connected subgraph* $\overline{G}^0 = (X, \overline{E}^0)$, *for which every vertex* $x \in X$ *has only one leaving edge* $e = (x, y) \in \overline{E}^0$ *and besides that*

$$\varepsilon^i(x) - \varepsilon^i(y) + c^i_{(x,y)} - \overline{v}^i = 0, \ \forall (x, y) \in \overline{E}^0, \ i = \overline{1,p}.$$

The optimal solution of the problem can be found by fixing the maps:

$$\begin{array}{c} s_1^* : x \to y \in X_{\overline{G}^0}(x) \text{ for } x \in X_1; \\ s_2^* : x \to y \in X_{\overline{G}^0}(x) \text{ for } x \in X_2; \\ ---------------- \\ s_p^* : x \to y \in X_{\overline{G}^0}(x) \text{ for } x \in X_p, \end{array}$$

where $X_{\overline{G}^0}(x) = \{y | (x, y) \in \overline{E}^0\}$.

Remark 10. The value $\overline{v}^i$, $i = \overline{1,p}$, coincides with the value of the payoff function $\overline{H}^i_x(s_1^*, s_2^*, \dots, s_p^*)$, $i = \overline{1,p}$. If $\overline{v}^i = 0$, then the ergodic cyclic game coincides with the dynamic c-game on the network $(G, X_1, X_2, \dots, X_p, c^1, c^2, \dots, c^p, x_0, x_0)$.

Some extension of cyclic games for stochastic cases has been considered in [13, 25, 26].

16 On Determining Pareto Optima for Cyclic Games with p Players

To determine Pareto solution for the cyclic game with p players, we can use the continuous model from Section 14.2 and extend it for the multiobjective case of the problem in the following way:
to minimize the vector function

$$\overline{H}(\alpha) = (\overline{H}^1(\alpha), \overline{H}^2(\alpha), \dots, \overline{H}^p(\alpha))$$

on subject

$$\begin{cases} \sum\limits_{e \in E^-(x)} \alpha_e - \sum\limits_{e \in E^+(x)} \alpha_e = 0, \ \forall x \in X; \\ \sum\limits_{e \in E} \alpha_e = 1; \\ \alpha_e \geq 0, \ e \in E, \end{cases}$$

where

$$\overline{H}^i(\alpha) = \textstyle\sum_{e \in E} c^i_e \alpha_e, \ i = \overline{1,p};$$
$$E^-(x) = \{e = (y, x) | (y, x) \in E\}; \ E^+(x) = \{e = (x, y) | (x, y) \in E\}.$$

Pareto optima for this multicriterion problem can be found by using the approach from [7–9,16,60]. Solutions of this continuous problem will correspond with solutions of the discrete multicriterion problem on a given strongly connected graph $G = (X, E)$ with cost functions $c^i : E \to R$, $i = \overline{1,p}$.

Note that Pareto solution for the cyclic game with p players on G does not depend on partition $X = X_1 \cup X_2 \cup \cdots \cup X_p$.

17 A General Approach for Algorithmic Solutions of Discrete Optimal Control Problems and its Game-Theoretic Extension

In this section, we study the control models for which the object function is defined algorithmically. We show that such statement of the control models allows us to extend the algorithms from Sections 1–7 for a larger class of problems.

17.1 General Optimal Control Model

Let L be a dynamical system with the set of states $X \subseteq R^n$ where at every moment of time $t = 0, 1, 2, \ldots,$ the state of L is $x(t) \in X$, $x(t) = \big(x_1(t), x_2(t), \ldots, x_n(t)\big) \in R^n$. The dynamics of the system L is described as follows

$$x(t+1) = g_t\big(x(t), u(t)\big),\ t = 0, 1, 2, \ldots, \tag{23}$$

where

$$x(0) = x_0 \tag{24}$$

is the starting point of the system L, and $u(t) = \big(u_1(t), u_2(t), \ldots, u_m(t)\big) \in R^m$ represents the vector of control parameters. For vectors of control parameters $u(t)$, $t = 0, 1, 2, \ldots,$ the admissible sets $U_t\big(x(t)\big)$ are given, i.e.,

$$u(t) \in U_t\big(x(t)\big),\ t = 0, 1, 2, \ldots. \tag{25}$$

We assume that the vector functions

$$g_t\big(x(t), u(t)\big) = \Big(g_t^1\big(x(t), u(t)\big), g_t^2\big(x(t), u(t)\big), \ldots, g_t^n\big(x(t), u(t)\big)\Big)$$

are determined uniquely by $x(t)$ and $u(t)$ at every moment of time $t = 0, 1, 2, \ldots.$ So, $x(t+1)$ is determined uniquely by $x(t)$ and $u(t)$.

Let

$$x(0), x(1), \ldots, x(t), \ldots \tag{26}$$

be a process generated according to (23)–(25) with given vectors of control parameters $u(t), t = 0, 1, 2, \ldots.$

For each state $x(t)$, $t = 0, 1, 2, \ldots$ of process (26), we define the numerical determination $F_t\big(x(t)\big)$ by using the following recursive formula

$$F_{t+1}\big(x(t+1)\big) = f_t\Big(x(t), u(t), F_t\big(x(t)\big)\Big),\ t = 0, 1, 2 \ldots$$

where

$$F_0\big(x(0)\big) = F_0$$

is a given representation of the starting state $x(0)$ of the system L; $f_t(\cdot, \cdot, \cdot)$, $t = 0, 1, 2, \ldots$ are arbitrary functions. In this model, $F_t(x(t))$ expresses the cost of system's passage from x_0 to $x(t)$. In the following, we distinguish two optimization problems:

Problem 6. For a given T, determine vectors of control parameters $u(0)$, $u(1), \ldots, u(T-1)$, which satisfy the conditions

$$\begin{gathered} x(t+1) = g_t\big(x(t), u(t)\big),\ t = 0, 1, 2, \ldots, T-1; \\ x(0) = x_0,\ x(T) = x_f, \\ u(t) \in U_t\big(x(t)\big),\ t = 0, 1, 2, \ldots, T-1; \\ F_{t+1}\big(x(t+1)\big) = f_t\Big(x(t), u(t), F_t\big(x(t)\big)\Big), t = 0, 1, 2, \ldots, T-1; \\ F_0\big(x(0)\big) = F_0 \end{gathered} \tag{27}$$

and minimize the object function

$$I_{x_0 x(T)}\big(u(t)\big) = F_T\big(x(T)\big) \tag{28}$$

Problem 7. For given T_1 and T_2, determine $T \in [T_1, T_2]$ and a control sequence $u(0), u(1), \ldots, u(T-1)$, which satisfy condition (27) and minimize objective function (28).

Remark 11. It is obvious that the optimal solution of Problem 7 can be obtained by its reducing to Problem 6 fixing the parameter $T = T_1, T = T_1 + 1, \ldots, T = T_2$. By choosing the optimal value of the solutions of the problems of type 6 with $T = T_1$, $T = T_1 + 1, \ldots, T = T_2$, we obtain the solution of the Problem 7 with $T \in [T_1, T_2]$.

It is easy to observe that a large class of dynamic optimization problems can be represented as a problem mentioned above. As example, if

$$f_t\Big(x(t), u(t), F_t\big(x(t)\big)\Big) = F_t\big(x(t)\big) + c_t\big(x(t), u(t)\big),$$

where

$$F_0(x_0) = 0$$

and $c_t\big(x(t), u(t)\big)$ represents the cost of the system's passage from the state $x(t)$ to the state $x(t+1)$, then we obtain the discrete control problems with integral-time, which are introduced and treated in [1–3,41–45,47]. Some classes of control problems from [2, 4] may be obtained if

$$f_t\Big(x(t), u(t), F_t\big(x(t)\big)\Big) = F_t\big(x(t)\big) \cdot c_t\big(x(t), u(t)\big),\ t = 1, 2, \ldots,$$

where

$$F_0(x_0) = 1$$

or

$$f_t\Big(x(t), u(t), F_t\big(x(t)\big)\Big) = \max\{F_t\big(x(t)\big), c_t\big(x(t), u(t)\big)\},$$

where

$$F_0(x_0) = 0.$$

We propose a general scheme based on dynamic programming for solving these problems.

17.2 Algorithm for Determining Optimal Solution of the Problem with Fixed Starting and Final States

We propose a general procedure for determining the optimal solutions of the formulated problems in the case when $f_t(x, u, F),\ t = 0, 1, 2, \ldots,$ are non-decreasing functions with respect to the third argument, i.e., with respect to F. So, we shall consider that for the fixed x and u, the functions $f_t(x, u, F),\ t = 0, 1, 2, \ldots$ satisfy the condition

$$f_t(x, u, F') \le f_t(x, u, F'') \quad \text{if} \quad F' \le F''. \tag{29}$$

Then the following algorithm determines the optimal solution of Problem 6.

Algorithm 17. Determining the Solution of General Optimal Control Problem

1. Set $F_0^*\big(x(0)\big) = F_0$; $F_t^*\big(x(t)\big) = \infty$; $x(t) \in X$, $t = 1, 2, \ldots$; $X_0 = \{x_0\}$.
2. For $t = 0, 1, 2, \ldots, T-1$ determine:

$$X_{t+1} = \{x(t+1) \in X \mid x(t+1) = g_t\big(x(t), u(t)\big),$$
$$x(t) \in X_t, u(t) \in U_t\big(x(t)\big)\}$$

   and for every $x(t+1) \in X_{t+1}$ determine

$$F_{t+1}^*\big(x(t+1)\big) = \min\Big\{f_t\Big(x(t), u(t), F_t^*\big(x(t)\big)\Big) | x(t+1) = g_t(x(t), u(t)),$$
$$x(t) \in X_t, u(t) \in U_t\big(x(t)\big)\Big\};$$

3. Find the sequence

$$x_T = x^*(T),\ x^*(T-1), x^*(T-2),\ \ldots,\ x^*(1), x^*(0) = x_0$$

   and

$$u^*(T-1),\ u^*(T-2),\ \ldots,\ u^*(1),\ u^*(0),$$

   which satisfy the conditions

$$F^*_{T-\tau}\big(x^*(T-1)\big) \ = f_{T-\tau-1}\Big(x^*(T-\tau-1), u^*(T-\tau-1), F^*_{T-\tau-1}\big(x(T-\tau-1)\big)\Big),\ \tau = 0,1,2,\ldots,T.$$

Then $u^*(0), u^*(1), u^*(2), \ldots, u^*(T-1)$ represent the optimal solution of Problem 6.

Theorem 22. *If $f_t(x,u,F),\ t = 0,1,2,\ldots,T$ are nondecreasing functions with respect to the third argument F, i.e., the functions $f_t(x,u,F), t = 0,1,2,\ldots,T$ satisfy condition (29), then the algorithm determines the optimal solution of Problem 6. Moreover, an arbitrary leading part $x^*(0)$,$x^*(1), \ldots, x^*(k)$ of the optimal trajectory $x^*(0), x^*(1), \ldots, x^*(k), \ldots, x^*(T)$ is again an optimal one.*

Proof. We prove the theorem by using the induction principle on the number of stages T. In the case $T \leq 1$, the theorem is evident. We consider that the theorem holds for $T \leq k$ and let us prove it for $T = k+1$.

Assume toward contradiction that $u^*(0), u^*(1), \ldots, u^*(T-2), u^*(T-1)$ is not an optimal solution of Problem 6 and $u'(0), u'(1), \ldots, u'(T-2), u'(T-1)$ is an optimal solution of Problem 6, which differs from $u^*(0), u^*(1), \ldots, u^*(T-2), u^*(T-1)$. Then $u'(0), u'(1), \ldots, u'(T-2), u'(T-1)$ generate a trajectory $x_0 = x'(0), x'(1), \ldots, x'(T) = x_T$ with corresponding numerical evaluations of states

$$F'_{t+1}\big(x'(t+1)\big) = f_t\Big(x'(t), u'(t), F'_t\big(x'(t)\big)\Big),\ t = 0,1,2,\ldots,T-1;$$

where $F'_0\big(x'(0)\big) = F_0$ and

$$F'_T\big(x'(T)\big) < F^*_T\big(x'(T)\big), \tag{30}$$

because $x'(T) = x^*(T)$. According to the induction principle for Problem 6 with $T-1$ stages the algorithm finds the optimal solution. So, for arbitrary $x(T-1) \in X$, we obtain the optimal evaluations $F^*_{T-1}\big(x(T-1)\big)$ for $x(T-1) \in\ X$. Therefore

$$F^*_{T-1}\big(x'(T-1)\big) \leq F'_{T-1}\big(x'(T-1)\big).$$

According to the algorithm

$$\begin{aligned} &f_{T-1}\Big(x^*(T-1), u^*(T-1), F^*_{T-1}\big(x^*(T-1)\big)\Big) \\ &\leq f_{T-1}\Big(x'(T-1), u'(T-1), F^*_{T-1}\big(x'(T-1)\big)\Big). \end{aligned} \tag{31}$$

Because $f_t(F,x,u),\ t = 0,1,2,\ldots$ are nondecreasing functions with respect to F, then

$$\begin{aligned} &f_{T-1}\Big(x'(T-1), u'(T-1), F^*_{T-1}\big(x'(T-1)\big)\Big) \\ &\leq f_{T-1}\Big(x'(T-1), u'(T-1), F'_{T-1}\big(x'(T-1)\big)\Big). \end{aligned} \tag{32}$$

Using (31) and (32), we obtain

$$\begin{aligned} F_T^*\big(x(T)\big) &= f_{T-1}\Big(x^*(T-1), u^*(T-1), F_{T-1}^*\big(x^*(T-1)\big)\Big) \\ &\le f_{T-1}\Big(x'(T-1), u'(T-1), F_{T-1}^*\big(x'(T-1)\big)\Big) \\ &\le f_{T-1}\Big(x'(T-1), u'(T-1), F_{T-1}'\big(x'(T-1)\big)\Big) = F_T'\big(x(T)\big), \end{aligned}$$

i.e.,

$$F_T^*\big(x(T)\big) \le F_T'\big(x(T)\big),$$

which is contrary to (30). So the algorithm finds the optimal solution of Problem 6 with $T = k + 1$. ∎

Theorem 23. *Let X and $U_t(x)$, $x \in X, t = 0, 1, 2, \dots, T-1$, be the finite sets. Then the algorithm uses at most $M \cdot |X| \cdot T$ elementary operations (excluding the operations for calculation of the values of functions $f_t(F, x, u)$ for given F, x and u), where*

$$M = \max_{x \in X,\, t=0,1,2,\dots,T-1} |U_t(x)|.$$

Proof. It is sufficient to prove that at step t, the algorithm uses no more than $M \cdot |X|$ elementary operations. Indeed, for finding the value $F_{t+1}\big(x(t+1)\big)$ for $x(t+1) \in X$, it is necessary to use $\sum_{x \in X} |U_t(x)|$ operation. Because $\sum_{x \in X} |U_t(x)| \le |X| \cdot M$, then at step t the algorithm uses no more than $|X| \cdot M$ elementary operations. So in general the algorithm uses no more than $M \cdot |X| \cdot T$ elementary operations. ∎

17.3 The Discrete Optimal Control Problem on Network

Let L be a dynamical system with a finite set of states X, and at every discrete moment of time $t = 0, 1, 2, \dots$, the state of the system L is $x(t) \in X$. Note that here we associate $x(t)$ with an abstract element (in Sections 17.1 and 17.2 $x(t)$ represents a vector from R^n). Two states x_0 and x_f are chosen in X, where x_0 is a starting state of the system L, $x_0 = x(0)$, and x_f is the final state of the system, i.e., x_f is the state in which the system must be brought. We consider the optimal control problem, when the dynamics of the system is described by a directed graph of transactions $G = (X, E)$ with given costs $c_e(t)$ on edges $e \in E$, i.e., we consider the control problem from Section 3.1. So, we are seeking for the sequence of the system's passages $(x(0), x(1)), (x(1), x(2)), \dots, (x(T-1), x(T)) \in E$ (which transfers the system L from the state $x_0 = x(0)$ to the state $x_f = x(T)$ with minimal integral-time cost) by a trajectory $x_0 = x(0), x(1), x(2), \dots, x(T) = x_f$. We will discuss two variants of the problem:

(1) the number of the stages (time T) is fixed;
(2) T is unknown and it must be determined.

It is easy to observe that for solving these problems, we can use the algorithm from Section 17.2. We put

$$F_0\big(x(0)\big) = 0$$

and

$$F_{t+1}\big(x(t+1)\big) = F_t\big(x(t)\big) + c_{\big(x(t),x(t+1)\big)}(t) \text{ for } \big(x(t), x(t+1)\big) \in E.$$

So we obtain the algorithm, which is based on dynamic programming techniques. The running time for solving this problem in case 1 by using Algorithm 17 is $O(n^2T)$.

A more general optimal control model on network is obtained if to each edge $e \in E$ at given moment of time t we associate a function $f_{e_t}(x(t), F_t(x(t)))$ that depends on state $x(t)$ and on numerical evaluation $F_t(x(t))$ of this state. Here $f_{e_t}(x(t), F_t(x(t)))$ has the same sense as $f_t(x(t), u(t), F_t(x(t)))$ in the previous model, where $u(t) = e_t$, i.e., $f_{e_t}(x(t), F_t(x(t))) = f_t(x(t), u(t), F_t(x(t)))$. For the given trajectory of system passages

$$x(0), x(1), \ldots, x(t), x(t+1)$$

the following recursive formulae

$$F_{t+1}(x(t+1)) = f_{e_t}(x(t), F_t(x(t))),\ t = 0, 1, 2, \ldots$$

for determining numerical evaluations of the states are given, where

$$F_0(x(0)) = F_0$$

is considered to be known.

In this model, we seek for a trajectory

$$x(0), x(1), \ldots, x(T-1), x(T) = x_f$$

which transfers the system L from the starting state x_0 to the final state x_f with minimal $F_T(x_f)$.

If $f_{e_t}(x(t), F_t(x(t)))$, $\forall e \in E$, $t = 1, 2, \ldots$, are increasing functions, then the control problem on network can be solved by using Algorithm 17.

17.4 The Game Control Model with p Players

Now we extend the control model using the concept of noncooperative games.

We assume that the dynamics of the system L is controlled by p players:

$$x(t+1) = g_t\big(x(t), u^1(t), u^2(t), \ldots, u^p(t)\big),\ t = 0, 1, 2, \ldots, \tag{33}$$

where

$$x(0) = x_0$$

is a given starting point of L, and $u^i(t)$ is a vector of control parameters of player i. For each player $i \in \{1, 2, \dots, p\}$, the admissible sets $U_t^i(x(t)), t = 1, 2, \dots,$ for vectors of control parameters $u^i(t)$, are given. Additionally, a numerical determination $F_t^i(x(t))$ of the state $x(t)$ at the time moment t for player i is defined according to the following recursive formula

$$F_{t+1}^i(x(t+1)) = f_t^i(x(t), u^1(t), u^2(t), \dots, u^p(t), F_t^i(x(t))), \quad t = 0, 1, 2, \dots, \tag{34}$$

where

$$F_0^i(x(0)) = F_0^i \tag{35}$$

are given representations of the starting state x_0 of the system L for player i; $f_t(\cdot, \cdot, \dots, \cdot)$, $t = 0, 1, 2, \dots,$ $i = 1, 2, \dots, p$, are arbitrary functions. Here $F_t^i(x(t))$ expresses the cost of the system's passage from the starting state x_0 to the state $x(t)$ for player i by a trajectory $x_0, x(1), x(2), \dots, x(t)$, determined by a fixed set of vectors of control parameters $u^1(t), u^2(t), \dots, u^p(t)$, $t = 0, 1, 2, \dots$.

In this model, we assume that players choose vectors of control parameters in order to achieve the final state x_f from the starting state at the moment of time $T(x_f)$, where

$$T_1 \le T(x_f) \le T_2.$$

Moreover, each player has to minimize his own cost of system's passage to x_f

$$I_{x_0 x_f}^i(u^1, u^2, \dots, u^p) = F_{T(x_f)}^i(x_f).$$

Note that for given $u^1(t), u^2(t), \dots, u^p(t)$, the cost of the system's passage from x_0 to x_f can be calculated on the basis of (34)–(35) if the corresponding trajectory

$$x_0, x(1), x(2), \dots, x(t), \dots$$

passes through the final state x_f. If for given $u^1(t), u^2(t), \dots, u^r(t)$ the trajectory $x_0, x(1), x(2), \dots, x(t), \dots$ does not pass through the state x_f, then we put

$$I_{x_0 x_f}^i(u^1, u^2, \dots, u^p) = \infty.$$

In this model, we are seeking for a Nash equilibrium. So, we consider the problem of finding $u^{1*}(t), u^{2*}(t), \dots, u^{p*}(t)$, for which the following condition is satisfied:

$$I_{x_0 x_f}^i(u^{1*}(t), u^{2*}(t), \dots, u^{i-1*}(t), u^{i*}(t), u^{i+1*}(t), \dots, u^{p*}(t))$$

$$\le I_{x_0 x_f}^i(u^{1*}(t), u^{2*}(t), \dots, u^{i-1*}(t), u^{i}(t), u^{i+1*}(t), \dots, u^{p*}(t)), \ i = \overline{1, p}.$$

In the following, we will assume that for the considered control problem the alternate players control condition (see Section 2) is satisfied. This will allow us to regard our problem as the game control problem on networks.

17.5 The Game Control Problem on Network and an Algorithm for Its Solving

In this section, we consider the game control problem on networks and propose an algorithm for determining the optimal strategies of players in the case when the structure of the network corresponds with T-partite directed graph.

General Statement of the Game Control Problem on Network

Let $G = (X, E)$ be a finite directed graph that describes the dynamics of the system L. So, an arbitrary directed edge $e = (x, y) \in E$ expresses the possibility of the dynamical system to pass from the state $x = x(t)$ to the state $y = x(t+1)$ at every moment of time $t = 0, 1, 2, \ldots$. Two states $x_0 = x(0)$ and x_f, which correspond with the starting and the final states of L, respectively, are distinguished in G. It is known that the system L has to reach the final state at the time moment $T(x_f)$ such that $T_1 \le T(x_f) \le T_2$; if $T_2 = T_1 = T$, then the system will reach the final state at the time T.

Assume that the vertex set X of G is divided into p subsets $X = X_1 \cup X_2 \cup \cup \cdots \cup X_p$ $(X_i \cap X_j = \emptyset,\ i \ne j)$, where vertices $x \in X_i$ are regarded as positions of player i, $i = 1, 2, \ldots, p$.

On G, we consider the following dynamic game. The game starts at the position $x_0 = x(0)$, for which the starting numerical representations

$$F_0^1(x_0) = F_0^1, F_0^2(x_0) = F_0^2, \ldots F_0^p(x_0) = F_0^p$$

of players $1, 2, \ldots, p$ are given. These quantities express the values of payoff functions of players $1, 2, \ldots, p$ at the time moment $t = 0$. If $x_0 \in X_{i_0}$, then the move is done by player i_0. This means that the system L is transferred from the state $x_0 = x(0)$ to the state $x_1 = x(1)$ such that $e_0 = (x(0), x(1)) \in E$. After that at the time moment $t = 1$, the values $F_1^i(x(1))$, $i = \overline{1,p}$ are calculated according to the following formula:

$$F_1^i(x(1)) = f_{e_0}^i(x(0), F_0^i(x(0))),\ i = \overline{1,p},$$

where $f_{e_0}^i$, $i = \overline{1,p}$, are arbitrary given functions associated to the edge e_0. Note that in G for each edge $e \in E$ at every time step the functions $f_{e_t}^1(\cdot,\cdot),\ f_{e_t}^2(\cdot,\cdot), \ldots, f_{e_t}^p(\cdot,\cdot)$ are considered to be given.

If at time step 1 position $x(1) \in X_{i_1}$ then the move is done by player i_1. This means that player i_1 transfers the system L from the state $x(1)$ to another state $x(2)$ such that $e_1 = (x(1), x(2)) \in E$. After that, the values

$$F_2^i(x(2)) = f_{e_1}^i(x(1), F_1^i(x(1))),\ i = \overline{1,p}$$

are calculated, and so on.

In general if $x(t) \in X_{i_t}$, then the move is done by player i_t, i.e., the system L is transferred from the state $x(t)$ to the state $x(t+1)$ such that $e_t = (x(t), x(t+1)) \in E$. At time step $t+1$, the quantities

$$F^i_{t+1}(x(t+1)) = f^i_{e_t}(x(t), F^i_t(x(t))),\ i = \overline{1,p}$$

are determined.

As soon as the final state is reached, i.e., $x(t+1) = x_f$, the game is over and the values of payoff functions of players are equal to $F^1_{T(x_f)}(x_f)$, $F^2_{T(x_f)}(x_f), \dots, F^p_{T(x_f)}(x_f)$, respectively, where $T(x_f) = t+1$. In this dynamic game, each player i has the aim to minimize his own payoff function $F^i_{T(x_f)}(x_f)$ such that $T_1 \le T(x_f) \le T_2$.

More strictly, the game control problem on G can be formulated as follows. We define the nonstationary strategies of players as maps:

$$u^i : (x,t) \to (y, t+1) \in X(x) \times \{t+1\} \text{ for } x \in X_i \setminus \{x_f\},$$
$$t = 0, 1, 2, \dots,\ i = \overline{1,p},$$

where $X(x) = \{y \in X \mid (x,y) \in E\}$. Here (x,t) has the same meaning as the notation $x(t)$, i.e., $(x,t) = x(t)$.

For any set of nonstationary strategies $u^1, u^2, \dots, u^p$ of players we define the quantities:

$$I^1_{x_0x_f}(u^1, u^2, \dots, u^p), I^2_{x_0x_f}(u^1, u^2, \dots, u^p), \dots, I^p_{x_0x_f}(u^1, u^2, \dots, u^p)$$

in the following way:

Let $u^1, u^2, \dots, u^p$ be an arbitrary set of strategies. Then either u^1,$u^2, \dots, u^p$ generate in G a finite trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from x_0 to x_f, where $T(x_f)$ represents the time moment when x_f is reached, or $u^1, u^2, \dots, u^p$ generate an infinite trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(t), \dots,$$

which does not pass through x_f, i.e., $T(x_f) = \infty$. If the state x_f is reached at the finite moment of time $T(x_f)$ and $T_1 \le T(x_f) \le T_2$, then we put

$$I^i_{x_0x_f}(u^1, u^2, \dots, u^p) = F^i_{T(x_f)}(x_f),\ i = \overline{1,p}$$

where $F^i_{T(x_f)}(x_f)$ is calculated recursively by using the following formula:

$$F^i_{t+1}(x(t+1)) = f^i_{(x(t),x(t+1))}(x(t), F^i_t(x(t))),\ t = \overline{0, T(x_f)-1};$$
$$F^i_0(x(0)) = 0;$$
$$x(t+1) = u^i(x(t)),\ t = \overline{0, T(x_f)-1}.$$

If the state x_f cannot be reached at the finite moment of time, then we set

$$I^i_{x_0x_f}(u^1, u^2, \dots, u^p) = \infty,\ i = \overline{1,p}.$$

Thus we regard the problem of finding the nonstationary strategies u^{1*}, $u^{2*}, \ldots, u^{p*}$, for which the following condition is satisfied:

$$I^i_{x_0 x_f}(u^{1*}, u^{2*}, \ldots, u^{i-1*}, u^{i*}, u^{i+1*}, \ldots, u^{p*})$$
$$\leq I^i_{x_0 x_f}(u^{1*}, u^{2*}, \ldots, u^{i-1*}, u^{i}, u^{i+1*}, \ldots, u^{p*}), \forall u^i,\ i = \overline{1,p}.$$

So, we consider the problem of finding the solution in the sense of Nash.

It is easy to observe that if

$$F^i_{t+1}(x(t+1)) = F^i_t(x(t)) + c^i_{(x(t),x(t+1))}(t)$$

for $(x(t), x(t+1)) \in E$, $i = \overline{1,p}$, then we obtain the problem from [4, 38].

The Game Control Problem on T-Partite Networks and Algorithm for Its Solving

Now we show that if G has a structure of $T+1$-partite graph, then Nash equilibria for game control problem exists. Moreover, we propose an algorithm for finding optimal strategies of players.

So, we assume that G has a structure of $(T+1)$-partite graph: the vertex set X in G is divided into $T+1$ nonempty sets: $X = Z_0 \cup Z_1 \cup \cdots \cup Z_T$, $Z_i \cap Z_j = \emptyset, i \neq j$; the edge set E in G is divided into T nonempty sets: $E = E_0 \cup E_1 \cup \cdots \cup E_{T-1}$, $E_i \cap E_j = \emptyset, i \neq j$, such that each edge $e = (x, y) \in E_t$ starts in $x \in Z_t$ and enters $y \in Z_{t+1}$, $t = \overline{0, T-1}$. On G, we consider the problem with $T_1 = T_2 = T$. So, $x_0 \in Z_0$ and $x_f \in Z_T$. Moreover, we assume that each set Z_t represents a position set for one of the players $i \in \{1, 2, \ldots, p\}$. So, for each Z_t there exists $i_t \in \{1, 2, \ldots, p\}$ such that $Z_t \subseteq X_{i_t}$, where $X = X_1 \cup X_2 \cup \cdots \cup X_p$ and X_i is a set of positions of player i.

In this case, for game control problem it is possible to extend Algorithm 17 if for every $e_t \in E$, $t = 0, 1, 2, \ldots$, the functions $f^i_{e_t}(x, F^i)$ are nondecreasing with respect to F^i. The values of payoff functions $I^i_{x_0 x_f}(u^1, u^2, \ldots, u^p) = F^i_t(x(t))$ can be found by using the following procedure:

Preliminary step (Step 0): For starting position $x(0) = x_0$, set $F^i_0(x(0)) = F^i_0$, $i = \overline{1,p}$;

General step (Step t, $t \geq 0$): Assume that at the time moment t, the position set Z_t is controlled by player $i_t \in \{1, 2, \ldots, p\}$, i.e., $Z_t \subseteq X_{i_t}$. Then for an arbitrary state $x(t+1) \in Z_{t+1}$, find a vertex $x'(t) \in X_t$ such that

$$f^{i_t}_{(x'(t),x(t+1))}\Big(x'(t), F^i_t\big(x'(t)\big)\Big)$$
$$= \min_{x(t) \in X^-(x(t+1))} \Big\{ f^{i_t}_{(x(t),x(t+1))}\Big(x(t), F^{i_t}_t\big(x(t)\big)\Big)\Big\},$$

where $X^-(x(t+1)) = \{x(t) \in Z_t \mid (x(t), x(t+1)) \in E_t\}$. Then calculate

$$F^i_{t+1}\big(x(t+1)\big) = f^i_{(x'(t),x(t+1))}\Big(x'(t), F^i_t\big(x'(t)\big)\Big),\ i = \overline{1,p}.$$

If $t < T-1$, then go to the next step; otherwise STOP.

If $F_t^i(x(t))$ are known for every $x(t) \in X$, then $u^1, u^2, \ldots, u^p$ can be found starting from the end position x_f by fixing each time $u^{i_k}(x(t)) = x(t+1)$, for which

$$F_{t+1}^{i_k}(x(t+1)) = f_{(x(t),x(t+1))}^{i_k}(x(t), F_t^{i_k}(x(t))) \text{ if } x(t) \in X^-(x(t+1)) \cap X_{i_k}.$$

For fixed $u^1, u^2, \ldots, u^{i-1}, u^{i+1}, \ldots, u^p$, the proposed procedure becomes Algorithm 17 for the control problem with respect to u^i. Therefore on the basis of results from Section 17.2, we obtain the following theorem.

Theorem 24. *If in $T+1$-partite graph $G = (X, E)$, there exists a directed path from $x_0 \in Z_0$ to $x_f \in Z_T$, then for the game control problem on G, there exists Nash equilibrium.*

Perhaps the proposed algorithm and Theorem 24 can be extended for game control problem on an arbitrary acyclic directed graph. This may involve existence of Nash equilibria for the game control problem on an arbitrary network and for the game control problem in general if the alternate players' control condition holds.

The proposed approach allows us to determine the optimal nonstationary strategies of players in dynamical games from [43], but do not allow us to determine the optimal strategies of players for dynamical games from [3].

A similar multicriterion control problem with Pareto optimality principle can be formulated and dynamic programming techniques for its solving can be developed.

17.6 Multicriterion Discrete Control Problem: Pareto Optimum

In this section, we extend the control model from Section 17.1 using the concept of cooperative games.

General Statement of the Problem

We assume that the dynamics of the system L is controlled by p players, who coordinate their actions using the common vector of control parameters $u(t)$. So the dynamics of the system L is described according to (23)–(25).

Let $x(0), x(1), \ldots, x(t), \ldots$ be a process generated according to (23)–(25) with the giving vector of control parameter $u(t)$, $t = 0, 1, 2, \ldots$. For each state we define the quantities $F_t^i(x(t))$, $i = 1, 2, \ldots, p$ in the following way:

$$F_{t+1}^i(x(t+1)) = f_t^i(x(t), u(t), F_t^i(x(t))) \tag{36}$$

where

$$F_0^i(x(0)) = F_0^i, \ i = 1, 2, \ldots, p \tag{37}$$

are given representations of the starting state $x(0)$ of the system L; $f_t^i(x(t), u(t), F_t^i(x(t)))$, $t = 0, 1, 2, \dots$ are arbitrary functions. So, $F_t^i(x(t))$ expresses the cost of the system's passage from the state $x(0)$ to the state $x(t)$ for player i.

In this model, we assume that players choose vectors of control parameters in order to achieve the final state x_f from the starting state x_0 at the moment of time $T(x_f)$, where $T_1 \le T(x_f) \le T_2$.

For the given $u(t)$, the cost of the system's passage from x_0 to x_f for player i is calculated on the basis of (23)–(25), (36), (37), and we put

$$I^i_{x_0 x_f}(u(t)) = F^i_{T(x_f)}(x_f),$$

if the trajectory passes through x_f at the time moment $T(x_f)$, such that $T_1 \le T(x_f) \le T_2$; otherwise we put

$$I^i_{x_0 x_f}(u(t)) = \infty.$$

We consider the problem of finding Pareto solution $u^*(t)$, i.e., there is no other vector $u(t)$, for which

$$\Big(I^1_{x_0 x_f}(u(t)), I^2_{x_0 x_f}(u(t)) \dots, I^p_{x_0 x_f}(u(t))\Big)$$
$$\le \Big(I^1_{x_0 x_f}(u^*(t)), I^2_{x_0 x_f}(u^*(t)) \dots, I^p_{x_0 x_f}(u^*(t))\Big)$$

and for any $i_0 \in \{1, 2, \dots, p\}$

$$I^{i_0}_{x_0 x_f}(u(t)) < I^{i_0}_{x_0 x_f}(u^*(t)).$$

Multicriterion Problem on Network and Algorithm for Its Solving on *T*-Partite Networks

We formulate the multicriterion control model on network in general form on the basis of the control model from Section 3.

Let $G = (X, E)$ be a directed graph of transactions for the dynamical system L with the given starting state $x_0 \in X$ and the final state $x_f \in X$. In addition, for the state x_0, starting representations $F_0^1(x_0) = F_0^1$, $F_0^2(x_0) = F_0^2, \dots, F_0^p(x_0) = F_0^p$ are given, which express the payoff functions of players at the time-moment $t = 0$. We define the control u^* on G as a map

$$u : (x, t) \to (y, t+1) \in X_G(x) \times \{t+1\} \text{ for } x \in X \setminus \{x_f\},\ t = 1, 2, \dots.$$

For an arbitrary control u, we define the quantities:

$$I^1_{x_0 x_f}(u),\ I^2_{x_0 x_f}(u),\ \dots, I^p_{x_0 x_f}(u)$$

in the following way.

Let

$$x_0 = x(0), x(1), x(2), \ldots, x(T(x_f)) = x_f$$

be a trajectory from x_0 to x_f generated by control u, where $T(x_f)$ is a time-moment, when the state x_f is reached. Then we put

$$I^i_{x_0 x_f}(u) = F^i_{T(x_f)}(x_f) \text{ if } T_1 \le T(x_f) \le T_2,\ i = \overline{1,p},$$

where $F^i_t(x(t))$ are calculated recursively by using the following formula

$$F^i_{t+1}(x(t+1)) = f^i_{(x(t),x(t+1))}(x(t), F^i_t(x(t))),\ t = \overline{0, T(x_f)-1};$$

$$F^i_0(x(0)) = F^i_0,$$

where $f^1_e(\cdot,\cdot)$, $f^2_e(\cdot,\cdot), \ldots, f^p_e(\cdot,\cdot)$ are arbitrary functions. If $T(x_f) \notin [T_1, T_2]$, then we put

$$I^i(u) = \infty,\ i = \overline{1,p}.$$

We regard the problem of finding Pareto solution u^*.

In the following, let us show that if the graph G has a structure of $(T+1)$-partite graph and $T_1 = T_2 = T$, then the algorithm from Section 17.2 can be extended for the multicriterion control problem on network.

So, assume that the vertex set X is represented as $X = Z_0 \cup Z_1 \cup \cdots \cup Z_T$, $Z_i \cap Z_j = \emptyset$, $i \ne j$, and the edge set E is divided into T nonempty subsets $E = E_0 \cup E_1 \cup \cdots \cup E_{T-1}$ such that an arbitrary edge $e = (y,z) \in E_t$ begins in $y \in Z_T$ and enters $z \in Z_{t+1}$, $t = \overline{0, T-1}$.

In this case, for the nondecreasing function $f^i_e(\cdot,\cdot)$ with respect to the second argument, the values $I^i(u) = F^i_t(x_t)$ can be calculated by using the following procedure.

Preliminary step (Step 0): For the starting position $x(0) = x_0$, set $F^i_0(x(0)) = F^i_0$, $i = \overline{1,p}$; for any $x \in X \setminus \{x_0\}$ put $F^i_t(x(t)) = \infty$, $i = \overline{1,p}$, $t = \overline{1,T}$.

General step (Step t, $t \ge 0$): For an arbitrary state $x(t+1) \in X_{t+1}$, find a vertex $x'(t) \in X_t$ such that there is no other vertex $x(t) \in X_t \setminus \{x_f\}$ for which

$$\begin{aligned}
&\Big(f^1_{(x(t),x(t+1))}(x(t), F^1_t(x(t))), f^2_{(x(t),x(t+1))}(x(t), F^2_t(x(t))) \\
&\qquad\qquad\qquad\qquad ,\ldots, f^p_{(x(t),x(t+1))}(x(t), F^p_t(x(t)))\Big) \\
&\le \Big(f^1_{(x'(t),x(t+1))}(x'(t), F^1_t(x'(t))), f^2_{(x'(t),x(t+1))}(x'(t), F^2_t(x'(t))) \\
&\qquad\qquad\qquad\qquad ,\ldots, f^p_{(x'(t),x(t+1))}(x'(t), F^p_t(x'(t)))\Big)
\end{aligned}$$

and

$$f^{i_0}_{(x(t),x(t+1))}(x(t), F^{i_0}_t(x(t))) < f^{i_0}_{(x'(t),x(t+1))}(x'(t), F^{i_0}_t(x'(t)))$$

for any $i_0 \in \{1, 2, \ldots, p\}$.
Then calculate

$$F^i_{t+1}(x(t+1)) = f^i_{(x'(t),x(t+1))}(x'(t), F^i_t(x'(t))),\ i = \overline{1,p}.$$

If $t < T - 1$, then go to the next step; otherwise STOP.

If $F^i_t(x(t))$ are known for every vertex $x(t) \in X$, then Pareto optimum u^* can be found starting from the end position x_f by fixing each time $u^*(x(t)) = x(t+1)$ for which

$$F^i_{t+1}(x(t+1)) = f^i_{(x(t),x(t+1))}(x(t), F^i_t(x(t))),\ i = \overline{1,p}.$$

18 The Game-Theoretic Approach for Dynamic Flow Problems on Networks

The game-theoretic approach we have used can be developed for the more general dynamic models such as minimum cost dynamic flow problems on networks. In the following, we can see that the optimal control problem on the network from Section 3 represents the particular case of the considered minimum cost flow problem on dynamic networks studied in [17–21,25,48,56]. This particular case is obtained for single source–single sink uncapacitated problems when the cost functions on edges do not depend on the amount of flow but depend only on time.

The minimum cost dynamic flow problem has a large implementation for many practical problems: product distribution, scheduling planning, telecommunication, transportation, communication, and management problems can be formulated and solved as minimum cost flow problems. The minimum cost flow problem on networks can be used for studying and solving the distribution problem, the synthesis problem of communication networks, or the allocation problem.

At first, we formulate the single-commodity case of the problem. Let a dynamic network $N = (X, E, u, \tau, d, \varphi)$ be given that consists of the directed graph $G = (X, E)$ with the set of vertices X and the set of edges E, the capacity function u: $E \times \mathtt{T} \to R_+$, the transit time function τ_e: $E \to R_+$, the demand function d: $X \times \mathtt{T} \to R$, and the nonlinear cost function φ: $E \times R_+ \times \mathtt{T} \to R_+$, where $\mathtt{T} = \{0, 1, 2, \ldots, T\}$. We consider that all the flow is dumped into the network at time 0. In order for the flow to exist, we require that $\sum_{t\in\mathtt{T}} \sum_{x\in X} d_x(t) = 0$.

Without loosing generality, we assume that in the network there is only one source $x_0 \in X$ and one sink $x_f \in X$ and there are no edges entering the source or leaving the sink. All other vertices $x \in X$, for which $d_x(t) = 0, \forall t \in \mathtt{T}$, are intermediary ones. In the case of many sources and sinks, the considered problem can be reduced to the initial one by introducing an additional artificial source and an additional artificial sink as well as edges leading from the new source to true sources and from true sinks to the new sink. The transit times of these new edges are zero, and the capacities of edges connecting the artificial source with all other sources are bounded by the demands of these sources; the capacities of edges connecting all other sinks with the artificial sink are bounded by the demands of these sinks.

We consider the discrete time model, in which all times are integral and bounded by horizon T. Time is measured in discrete steps, so that if one unit of flow leaves node x at time t on arc $e = (x, y)$, one unit of flow arrives at node y at time $t + \tau_e$, where τ_e is the transit time of arc e.

A feasible dynamic flow on N is a function α: $E \times \mathtt{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e\in E^+(x)\\ t-\tau_e\geq 0}} \alpha_e(t-\tau_e) - \sum_{e\in E^-(x)} \alpha_e(t) = d_x(t),\ \forall\, t\in \mathtt{T},\ \forall\, x\in X;$$

$$0 \leq \alpha_e(t) \leq u_e(t), \quad \forall\, t\in\mathtt{T},\ \forall\, e\in E;$$

$$\alpha_e(t) = 0,\ \forall\, e\in E,\ t = \overline{T-\tau_e+1, T};$$

where $E^+(x) = \{(y,x)\,|\,(y,x)\in E\}, \quad E^-(x) = \{(x,y)\,|\,(x,y)\in E\}$.

Here the function α defines the value $\alpha_e(t)$ of flow entering edge e at time t. It is easy to observe that the flow does not enter edge e at time t if it has to leave the edge after time T; this is ensured by the last condition.

To model transit costs, which may change over time, we define the cost function $\varphi_e(\alpha_e(t), t)$ with the meaning that flow of value $\xi = \alpha_e(t)$ entering edge e at time t will incur a transit cost of $\varphi_e(\xi, t)$.

The total cost $c(\alpha)$ of dynamic flow is defined as follows:

$$c(\alpha) = \sum_{t\in\mathtt{T}} \sum_{e\in E} \varphi_e(\alpha_e(t), t).$$

The dynamic minimum-cost flow problem consists in finding a feasible flow that minimizes this objective function.

In order to describe the game-theoretic approach for the considered problem, we shall use the multicommodity version of the optimal dynamic flow problem. Such a problem consists of shipping a given set of commodities from their respective sources to their sinks through a network in order to optimize the given criterion so that the total flow going through edges does not exceed their capacities. The minimum cost multicommodity dynamic flow problem asks for a feasible flow over time with given time horizon, satisfying all supplies and demands with minimum cost. This dynamic problem is considered on directed networks with a set of commodities, time-varying capacities of edges, fixed transit times on arcs, and a given time horizon. We assume that cost functions, defined on edges, are nonlinear and depend on time and flow, and the demand function also depends on time.

So, we consider a directed network $N = (X, E, K, w, u, \tau, d, \varphi)$ with set of vertices X, set of edges E, and set of commodities K that must be routed through the same network. Each edge $e \in E$ has a nonnegative time-varying capacity $w_e^k(t)$, which bounds the amount of flow of each commodity $k \in K$ allowed on each arc $e \in E$ in every moment of time $t \in \mathtt{T}$. We also consider that every arc $e \in E$ has a nonnegative time-varying capacity for all commodities, which is known as the mutual capacity $u_e(t)$. Moreover, each edge $e \in E$ has an associated nonnegative transit time τ_e, which determines the amount of time it takes for flow to travel from the tail to the head of that edge. The underlying network also consists of demand function d: $X \times K \times \mathtt{T} \to R$ and cost function φ: $E \times R_+ \times K \times \mathtt{T} \to R_+$, where $\mathtt{T} = \{0, 1, 2, \ldots, T\}$. All assumptions made above hold and in this case of the problem.

A feasible dynamic flow on N is a function α: $E \times K \times \mathtt{T} \to R_+$ that satisfies the following conditions:

$$\sum_{\substack{e \in E^+(x) \\ t-\tau_e \geq 0}} \alpha_e^k(t-\tau_e) - \sum_{e \in E^-(x)} \alpha_e^k(t) = d_x^k(t),\ \forall t \in \mathtt{T},\ \forall x \in X,\ \forall k \in K;$$

$$\sum_{k \in K} \alpha_e^k(t) \leq u_e(t),\ \forall t \in \mathtt{T},\ \forall e \in E;$$

$$0 \leq \alpha_e^k(t) \leq w_e^k(t),\ \ \forall t \in \mathtt{T},\ \forall e \in E,\ \forall k \in K;$$

$$\alpha_e^k(t) = 0,\ \forall e \in E,\ t = \overline{T - \tau_e + 1, T},\ \forall k \in K.$$

The total cost of the dynamic multicommodity flow is defined as follows:

$$c(\alpha) = \sum_{t \in \mathtt{T}} \sum_{e \in E} \varphi_e(\alpha_e^1(t), \alpha_e^2(t), \ldots, \alpha_e^k(t), t).$$

The object of the minimum cost multicommodity flow problem is to find a feasible flow that minimizes this objective function.

It is important to notice that in many practical cases, cost functions are presented in the following form:

$$\varphi_e(\alpha_e^1(t), \alpha_e^2(t), \ldots, \alpha_e^k(t), t) = \sum_{k \in K} \varphi_e^k(\alpha_e^k(t), t).$$

To develop algorithms for solving such kind of problems, we have used the special dynamic programming techniques based on time-expanded networks [19, 23, 48, 55] together with classic optimization methods [57, 63].

If we associate to each commodity a player, we can regard this problem as a game problem, where players interact between them, and the choices of one player influence the choices of the others. Control decisions are made by each player according to its own individual performance objectives and depending on the choices of the other players. The game theory fits perfectly in the realm of such a problem, and an equilibrium or stable operating point of the system has to be found.

Game theoretic models are widely employed in the context of flow control, routing, virtual path bandwidth allocation, and pricing in modem networking.

Flow problems in multimedia applications (teleconferencing, digital libraries) over high-speed broadband networks can serve as a good example of this. The problem of providing bandwidth that will be shared by many users ([1, 6]) is also a very important problem. As it is typical for games in such a problem, the interaction among the users on their individual strategies has to be imposed. The game theoretic approach can also be applied in a problem of power control in radio systems.

19 Pareto–Nash Equilibria for Multiobjective Games

In this section, we consider multiobjective games, which generalize noncooperative ones [32, 53, 54] and Pareto multicriterion problems [58–60]. The payoff functions of players in such games are presented as vector functions, where players intend to optimize them in the sense of Pareto on their sets of strategies. At the same time in our game model it is assumed that players are interested to preserve Nash optimality principle when they interact between them on the set of situations. Such statement of the game leads to a new equilibria notion that we call Pareto–Nash equilibria. Such concept can be used for multiobjective control problems, and algorithms for their solving can be derived.

19.1 Problem Formulation

The multiobjective game with p players is denoted by $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$, where X_i is the set of strategies of player i, $i = \overline{1,p}$, and $\overline{F}_i = (F_i^1, F_i^2, \ldots, F_i^{r_i})$ is the vector payoff function of player i, defined on the set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$\overline{F}_i \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^{r_i}, \ i = \overline{1,p}.$$

Each component F_i^k of $\overline{F}_i$ corresponds with a partial criterion of player i and represents a real function defined on set of situations $X = X_1 \times X_2 \times \cdots \times X_p$:

$$F_i^k \ : \ X_1 \times X_2 \times \cdots \times X_p \to R^1, \ k = \overline{1,r_i}, \ i = \overline{1,p}.$$

We call the solution of the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ Pareto–Nash equilibrium and define it in the following way.

Definition 6. *The situation $x^* = (x_1^*, x_2^*, \ldots, x_p^*) \in X$ is called Pareto–Nash equilibrium for the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ if for every $i \in \{1, 2, \ldots, p\}$, the strategy x_i^* represents Pareto solution for the following multicriterion problem:*

$$\max_{x_i \in X_i} \to \overline{f}^{\,i}_{x^*}(x_i) = (f^{i1}_{x^*}(x_i), f^{i2}_{x^*}(x_i), \ldots, f^{ir_i}_{x^*}(x_i)),$$

where

$$f^{ik}_{x^*}(x_i) = F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*), \ k = \overline{1,r_i}, \ i = \overline{1,p}.$$

This definition generalizes well-known Nash equilibria notion for classic noncooperative games (single-objective games) and Pareto optimum for multicriterion problems. If $r_i = 1$, $i = \overline{1,p}$, then $\overline{G}$ becomes classic noncooperative game, where x^* represents Nash equilibria solution; in the case $p = 1$, the game $\overline{G}$ becomes Pareto multicriterion problem, where x^* is Pareto solution.

An important special class of multiobjective games represents zero-sum games of two players. This class is obtained from the general case of the multiobjective game $\overline{G} = (X_1, X_2, \dots, X_p, \overline{F}_1, \overline{F}_2, \dots, \overline{F}_p)$ when $p = 2$, $r_1 = r_2 = r$ and $\overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2)$.

Zero-sum multiobjective game is denoted by $\overline{G} = (X_1, X_2, \overline{F})$, where $\overline{F}(x_1, x_2) = \overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2)$. Pareto–Nash equilibrium for this game corresponds with saddle point $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$ for the following max-min multiobjective problem:

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} \to \overline{F}(x_1, x_2) = (F^1(x_1, x_2), F^2(x_1, x_2), \dots, F^r(x_1, x_2)). \quad (38)$$

Strictly we define the saddle point $x^* = (x_1^*, x_2^*) \in X_1 \times X_2$ for zero-sum multiobjective problem (38) in the following way.

Definition 7. *The situation $(x_1^*, x_2^*) \in X_1 \times X_2$ is called the saddle point for max-min multiobjective problem (38) (i.e., for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$) if x_1^* is Pareto solution for multicriterion problem:*

$$\max_{x_1 \in X_1} \to \overline{F}(x_1, x_2^*) = (F^1(x_1, x_2^*), F^2(x_1, x_2^*), \dots, F^r(x_1, x_2^*)),$$

and x_2^* is Pareto solution for multicriterion problem:

$$\min_{x_2 \in X_2} \to \overline{F}(x_1^*, x_2) = (F^1(x_1^*, x_2), F^2(x_1^*, x_2), \dots, F^r(x_1^*, x_2)).$$

If $r = 1$, this notion corresponds with classic saddle point notion for min-max problems, i.e., we obtain saddle point notion for classic zero-sum games of two players.

In this section, we show that theorems of J. Nash [53] and J. Neumann [52,54] related to classic noncooperative games can be extended for our multiobjective case of games. Moreover, we show that all results related to discrete multiobjective games, especially matrix games, can be developed in analogous way as for classic ones. Algorithms for determining the optimal strategies of players in considered games will be developed.

19.2 The Main Results

First we formulate the main theorem, which represents an extension of the Nash theorem for our multiobjective version of the game.

Theorem 25. *Let $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ be a multiobjective game, where $X_1, X_2, \ldots, X_p$ are convex compact sets and $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p$ represent continuous vector payoff functions. Moreover, let us assume that for every $i \in \{1, 2, \ldots, p\}$, each component $F_i^k(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$, $k \in \{1, 2, \ldots, r_i\}$, of the vector function $\overline{F}_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$ represents a concave function with respect to x_i on X_i for fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p$. Then for the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ there exists Pareto–Nash equilibria situation $x^* = (x_1^*, x_2^*, \ldots, x_p^*) \in X_1 \times X_2 \times \cdots \times X_p$.*

Proof. Let $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr_p}$ be an arbitrary set of real numbers that satisfy the following condition

$$\begin{cases} \sum_{k=1}^{r_i} \alpha_{ik} = 1, \; i = \overline{1,p}; \\ \alpha_{ik} > 0, \qquad k = \overline{1,r_i}, \;\; i = \overline{1,p}. \end{cases} \tag{39}$$

We consider an auxiliary noncooperative game (single-objective game) $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$, where

$$f_i(x_1, x_2, \ldots, x_p) = \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1, x_2, \ldots, x_p), \;\; i = \overline{1,p}.$$

It is evident that $f_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$ for every $i \in \{1, 2, \ldots, p\}$ represents a continuous and concave function with respect to x_i on X_i for fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p$ because $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr_p}$ satisfy condition (39) and $F_i^k(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p)$ is a continuous and concave function with respect to x_i on X_i for fixed $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p$, $k = \overline{1,r_i}$, $i = \overline{1,p}$.

According to Nash theorem [53] for the noncooperative game $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$, there exists Nash equilibria situation $x^* = (x_1^*, x_2^*, \ldots, x_p^*)$, i.e.

$$f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*)$$

$$\leq f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_p^*), \;\; \forall x_i \in X_i, \;\; i = \overline{1,p}.$$

Let us show that $x^* = (x_1^*, x_2^*, \ldots, x_p^*)$ is Pareto–Nash equilibria solution for the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$. Indeed, for every $x_i \in X_i$ we have

$$\sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*)$$
$$= f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*)$$

$$\leq f_i(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_p^*)$$
$$= \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_p^*)$$
$$\forall x_i \in X_i,\ i = \overline{1,p}.$$

So,

$$\sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*)$$
$$\leq \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_p^*),\ \forall x_i \in X_i,\ i = \overline{1,p}, \tag{40}$$

for given $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr_p}$ which satisfy (39).

Taking in account that the functions $f_{x^*}^{i_k} = F_i^k(x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_p^*)$, $k = \overline{1, r_i}$, are concave functions with respect to x_i on convex set X_i and $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik}$ satisfy the condition $\sum_{k=1}^{r_i} \alpha_{ik} = 1$, $\alpha_{ik} > 0$, $k = \overline{1, r_i}$, then according to the theorem from [16] (see also [7–9]), condition (40) implies that x_i^* is Pareto solution for the following multicriterion problem:

$$\max_{x_i \in X_i} \rightarrow \overline{f}^i_{x^*}(x_i) = (f_{x^*}^{i1}(x_i), f_{x^*}^{i2}(x_i), \ldots, f_{x^*}^{ir_i}(x_i)),\ i \in \{1, 2, \ldots, p\}.$$

This means that $x^* = (x_1^*, x_2^*, \ldots, x_p^*)$ is Pareto–Nash equilibria solution for the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$. ■

So, if conditions of Theorem 25 are satisfied, then Pareto–Nash equilibria solution for the multiobjective game can be found by using the following algorithm.

Algorithm 18. Determining Pareto–Nash Equilibria of Multiobjective Game

1. Fix an arbitrary set of real numbers $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr_p}$, which satisfy condition (39);
2. Form the single objective game $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$, where
$$f_i(x_1, x_2, \ldots, x_p) = \sum_{k=1}^{r_i} \alpha_{ik} F_i^k(x_1, x_2, \ldots, x_p),\ i = \overline{1,p};$$
3. Find Nash equilibria $x^* = (x_1^*, x_2^*, \ldots, x_p^*)$ for noncooperative game $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$ and fix x^* as a Pareto–Nash equilibria solution for multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$.

Remark 12. Algorithm 18 finds only one of the solutions for the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$. In order to find all solutions in Pareto–Nash sense, it is necessary to apply algorithm 18 for every $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1r_1}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r_2}, \ldots, \alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{pr_p}$ that satisfy (39) and then to form the union of all obtained solutions.

Note that the proof of Theorem 25 is based on reduction of the multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F_1}, \overline{F_2}, \ldots, \overline{F_p})$ to an auxiliary one $G = (X_1, X_2, \ldots, X_p, f_1, f_2, \ldots, f_p)$ for which Nash theorem from [53] can be applied. In order to reduce the multiobjective game $\overline{G}$ to an auxiliary one, G linear convolution criteria for vector payoff functions in the proof of Theorem 25 have been used. A similar reduction of the multiobjective game to a classic one can be used applying the convolution criteria the standard procedure.

For zero-sum multiobjective game of two players, the following theorem holds.

Theorem 26. *Let $\overline{G} = (X_1, X_2, \overline{F})$ be a zero-sum multiobjective game of two players, where X_1, X_2 are convex compact sets and $\overline{F}(x_1, x_2)$ is a continuous vector function on $X_1 \times X_2$. Moreover, let us assume that each component $F^k(x_1, x_2)$, $k \in \{1, 2, \ldots, r\}$, of $\overline{F}(x_1, x_2)$ for fixed $x_1 \in X_1$ represents a convex function with respect to x_2 on X_2, and for every fixed $x_2 \in X_2$ it is a concave function with respect to x_1 on X_1. Then for the zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$, there exists saddle point $x^* = (x_1^*, x_2^*) \in \in X_1 \times X_2$, i.e., x_1^* is Pareto solution for multicriterion problem:*

$$\max_{x_1 \in X_1} \rightarrow \overline{F}(x_1, x_2^*) = (F^1(x_1, x_2^*), F^2(x_1, x_2^*), \ldots, F^r(x_1, x_2^*))$$

and x_2^* is Pareto solution for multicriterion problem:

$$\min_{x_2 \in X_2} \rightarrow \overline{F}(x_1^*, x_2) = (F^1(x_1^*, x_2), F^2(x_1^*, x_2), \ldots, F^r(x_1^*, x_2)).$$

Proof. The proof of Theorem 26 can be obtained as a corollary from Theorem 25 if we regard our zero-sum game as a game of two players of form $\overline{G} = (X_1, X_2, \overline{F}_1(x_1, x_2), \overline{F}_2(x_1, x_2)$, where $\overline{F}_2(x_1, x_2) = -\overline{F}_1(x_1, x_2) = \overline{F}(x_1, x_2)$.

The proof of Theorem 26 can be obtained also by reducing our zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$ to a classic single-objective case $G = (X_1, X_2, f)$ and applying Neumann theorem from [54], where

$$f(x_1, x_2) = \sum_{k=1}^{r} \alpha_k F^k(x_1, x_2)$$

and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are arbitrary real numbers, such that

$$\sum_{k=1}^{r} \alpha_k = 1; \;\; \alpha_k > 0, \;\; k = \overline{1, r}.$$

It is easy to show that if $x^* = (x_1^*, x_2^*)$ is a saddle point for the zero-sum game $G = (X_1, X_2, f)$, then $x^* = (x_1^*, x_2^*)$ represents a saddle point for the zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$. ■

So, if conditions of Theorem 26 are satisfied, then a solution of zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$ can be found by using the following algorithm.

Algorithm 19. Determining the Saddle Point of Payoff Functions in Zero-Sum Multiobjective Game

1. Fix an arbitrary set of real numbers $\alpha_1, \alpha_2, \dots, \alpha_r$, such that
$$\sum_{k=1}^{r} \alpha_k = 1;\ \alpha_k > 0,\ k = \overline{1,r};$$
2. Form the zero-sum game $G = (X_1, X_2, f)$, where
$$f(x_1, x_2) = \sum_{k=1}^{r} \alpha_k F^k(x_1, x_2).$$
3. Find a saddle point $x^* = (x_1^*, x_2^*)$ for the single-objective zero-sum game $G = (X_1, X_2, f)$. Then fix $x^* = (x_1^*, x_2^*)$ as a saddle point for zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$.

Remark 13. Algorithm 19 finds only a solution for the given zero-sum multiobjective game $\overline{G} = (X_1, X_2, \overline{F})$. In order to find all saddle points, it is necessary to apply Algorithm 19 for every $\alpha_1, \alpha_2, \dots, \alpha_r$ satisfying conditions $\sum_{k=1}^{r} \alpha_k = 1;\ \alpha_k > 0,\ k = \overline{1,r}$, and then to form the union of obtained solutions.

Note, that for reducing the zero-sum multiobjective games to classic ones can be used the convolution criteria from [16, 60].

19.3 Discrete and Matrix Multiobjective Games

Discrete multiobjective games are determined by the discrete structure of sets of strategies $X_1, X_2, \dots, X_p$. If $X_1, X_2, \dots, X_p$ are finite sets, then we may consider $X_i = J_i,\ J_i = \{1, 2, \dots, q_i\},\ i = \overline{1,p}$. In this case, the multiobjective game is determined by vectors
$$\overline{F}_i = (F_i^1, F_i^2, \dots, F_i^{r_i}),\ i = \overline{1,p},$$
where each component $F_i^k,\ k = \overline{1,r_i}$, represents p-dimensional matrix of size $q_1 \times q_2 \times \dots \times q_p$.

If $p = 2$, then we have bimatrix multiobjective game, and if $F_2 = -F_1$, then we obtain a matrix multiobjective one. In an analogous way as for single objective matrix games, here we can interpret the strategies $j_i \in J_i$, $i = \overline{1,p}$, of players as pure strategies.

It is evident that for such matrix multiobjective games, Pareto–Nash equilibria may not exist because Nash equilibria may not exist for bimatrix and matrix games in pure strategies. But to each finite discrete multiobjective game, we can associate a continuous multiobjective game $\overline{\overline{G}} = (Y_1, Y_2, \ldots, Y_p, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p)$ by introducing mixed strategies $y_i = (y_{i1}, y_{i2}, \ldots, y_{ir_i}) \in Y_i$ of player i and vector payoff functions $\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p$, which we define in the following way:

$$Y_i = \{y_i = (y_{i1}, y_{i2}, \ldots, y_{ir_i}) \in R^{r_i} \Big| \sum_{j=1}^{r_i} y_{ij} = 1, y_{ij} \geq 0, j = \overline{1, r_i}\};$$

$$\overline{f}_i = (f_i^1, f_i^2, \ldots, f_i^{r_i}),$$

where

$$f_i^k(y_{11}, y_{12}, \ldots, y_{1r_1}, y_{21}, y_{22}, \ldots, y_{2r_2}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pr_p})s$$
$$= \sum_{j_1=1}^{r_1} \sum_{j_2=1}^{r_2} \cdots \sum_{j_p=1}^{r_p} F^k(j_1, j_2, \ldots, j_p) y_{ij_1} y_{ij_2} \ldots y_{ij_p};\ k = \overline{1, r_i},\ i = \overline{1,p}.$$

It is easy to observe that for an auxiliary multiobjective game $\overline{\overline{G}} = (Y_1, Y_2, \ldots, Y_p, \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_p)$, conditions of Theorem 25 are satisfied and therefore Pareto–Nash equilibria $y^* = (y^*_{11}, y^*_{12}, \ldots, y^*_{1r_1}, y^*_{21}, y^*_{22}, \ldots, y^*_{2r_2}, \ldots, y^*_{p1}, y^*_{p2}, \ldots, y^*_{pr_p})$ exist.

In the case of matrix games, the auxiliary zero-sum multiobjective game of two players is defined as follows: $\overline{\overline{G}} = (Y_1, Y_2, \overline{f})$;

$$Y_1 = \{y_1 = (y_{11}, y_{12}, \ldots, y_{1r}) \in R^r \Big| \sum_{j=1}^{r} y_{1j} = 1, y_{1j} \geq 0, j = \overline{1, r}\};$$

$$Y_2 = \{y_2 = (y_{21}, y_{22}, \ldots, y_{2r}) \in R^r \Big| \sum_{j=1}^{r} y_{2j} = 1, y_{2j} \geq 0, j = \overline{1, r}\};$$

$$\overline{f} = (f^1, f^2, \ldots, f^r),$$

$$f^k(y_{11}, y_{12}, \ldots, y_{1r}, y_{21}, y_{22}, \ldots, y_{2r}) = \sum_{j_1=1}^{r} \sum_{j_2=1}^{r} F^k(j_1, j_2) y_{1j_1} y_{2j_2};\ k = \overline{1, r}.$$

The game $\overline{\overline{G}} = (Y_1, Y_2, \overline{f})$ satisfies conditions of Theorem 26 and therefore saddle point $y^* = (y_1^*, y_2^*) \in Y_1 \times Y_2$ exists.

So, the results related to discrete and matrix game can be extended for multiobjective case of the game and can be interpreted in an analogous way as for single-objective games. In order to solve these associated multiobjective games, Algorithms 18 and 19 can be applied.

19.4 Some Comments and Interpretation of Multiobjective Games

The considered multiobjective games extend classic ones and represent a combination of cooperative and noncooperative games. Indeed, the player i in multiobjective game $\overline{G} = (X_1, X_2, \ldots, X_p, \overline{F}_1, \overline{F}_2, \ldots, \overline{F}_p)$ can be regarded as a union of r_i subplayers with payoff functions $F_i^1, F_i^2, \ldots, F_i^{r_i}$, respectively. So, the game $\overline{G}$ represents a game with p coalitions $1, 2, \ldots, p$, which interact between them on the set of situations $X_1 \times X_2 \times \cdots \times X_p$.

The introduced Pareto–Nash equilibria notion uses the concept of cooperative games because according to this notion, subplayers of the same coalitions should optimize in the sense of Pareto their vector functions $\overline{F}$ on the set of strategies X_i. On the other hand, Pareto–Nash equilibria notion takes into account also the concept of noncooperative games because coalitions interact between them on the set of situations $X_1 \times X_2 \times \cdots \times X_p$ and are interested to preserve Nash equilibria between coalitions.

The obtained results allow us to describe a class of multiobjective games for which Pareto–Nash equilibria exist. Moreover, a suitable algorithm for finding Pareto–Nash equilibria is proposed.

20 Conclusion

The considered control models generalize classic ones and comprise a large class of practical and theoretical problems. A general concept of the game-theoretical approach for control problems with integral-time cost criterion by a trajectory with given starting and final states is described. The classification of necessary and sufficient conditions for the existence of Nash equilibria and Pareto optima in the considered game control models is obtained. The dynamic programming techniques for such class of problems is developed, and polynomial time algorithms for determining Nash equilibria and Pareto optima are elaborated. Efficient algorithms are derived for the dynamic c-game on network and the game control problem in positional form. The obtained results can be used in general decision-making system.

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A Military Application of Viability: Winning Cones, Differential Inclusions, and Lanchester Type Models for Combat

George Isac[1] and Alain Gosselin[2]

[1] Department of Mathematics, Royal Military College of Canada, STN Forces, Kingston, Ontario, Canada
isac-g@rmc.ca

[2] Department of Mathematics, Royal Military College of Canada, Kingston, Ontario, Canada K7K 7B4
gosselin-a@rmc.ca

Abstract During the First World War, F.W. Lanchester published his book *Aircraft in Warfare: The Dawn of the Fourth Arm* [31] in which he proposed several mathematical models based on differential equations to describe combat situations. Since then, his work has been extensively modified to represent a variety of competitions, ranging from isolated battles to entire wars.

There exists a class of mathematical models known under the name of *differential Lanchester type models*. Such models have been studied from different points of view by many authors in hundreds of papers and unpublished reports. We note that Lanchester type models are used in the planning of optimal strategies, supply, and tactics.

In our first paper on the subject [27], we studied Lanchester type models from a viability standpoint through the introduction of the new notion of *winning cone*. We have also considered a variation on optimal control that we call *Optimal Control by Viability*. Although the subject was mentioned, the difficulties and well-known problems associated with Lanchester coefficients was not considered in this first part.

Herein, we turn our attention to these coefficients and, to overcome this obstacle and facilitate the application of such models, we will introduce the notion of *Lanchester type differential inclusions* through the replacement of the classic coefficients by *intervals*. We will show how viability theory for set-valued mappings can be applied to determine viability conditions for the *winning cone*.

In the last section, we will again consider *Optimal Control by Viability*, but in the set-valued case represented by differential inclusions.

Key words: viability, nonlinear analysis, optimal control, Lanchester, differential inclusions, set-valued analysis

1 Introduction

The *Lanchester theory of combat* owes its name and origin to F.W. Lanchester. During the First World War, in 1916, he published his book *Aircraft in Warfare: The Dawn of the Fourth Arm* [31] in which he first introduced the use of differential equations to the mathematical modelling of combat. Since then, these models have been studied extensively, and the results have been published in hundreds of papers. We refer our reader to our article *Viability theory and differential Lanchester type models for combat. Differential systems* [27] for further details. There is also a detailed synthesis of the work done prior to 1980 written by Taylor [41].

Prior to our latest article on the subject, the research work carried out in this field relied on classic analysis. In our previous work, we used the tools of viability and optimal control to provide a new insight into the subject. However, we are left to deal with the difficulty posed by the *Lanchester coefficients*. The problem associated with them is illustrated in the many attempts to apply the models to historical battles [9, 11–13, 16–18, 20, 23].

The complications arise in the evaluation of these ever elusive *Lanchester coefficients*. They are the expression of the ability of one force to inflict damage on its opponent (see *The Lanchester Attrition-Rate Coefficient* by Bonder [7] and followed up by Barfoot [6]). The problem is multifold. The nature of combat being as it is, the collection of data is at best incomplete and imprecise. Additionally, these coefficients vary through time and conditions in a manner hard to quantify.

The analysis of combat through Lanchester type models has yielded laws on the progression of combat such as the *Linear Law*, the *Square Law* [31], and the *Logarithmic Law* [37] (see Taylor [41] for further details). The validity of this analysis with respect to the coefficients as well as the laws of combat it introduced is often criticized [2, 7, 21, 22, 26, 29]. More recently, Hembold [25] presented a very *à propos* paper on this issue highlighting what he called the *Constant Fallacy*. Validation of the models have repeatedly failed to prove the correctness of these laws of combat and/or stumbled over the evaluation of the coefficients.

Lanchester type models have found applications in numerous fields such as economy [36], biology, and evolution theory [19]. There is some interesting work carried out utilizing Nash equilibrium strategies [40] in the context of armament race and control [32, 38] as well as some publications studying Lanchester models from a dynamical systems angle [14]. There is now an increased interest in the application of Lanchester type models to economy and, more generally, to problems dealing with a competitive element.

Consequently in this publication, to overcome the difficulties that arise from the Lanchester coefficients, we will replace these single-valued coefficients by intervals in the sense given by Moore [33] as well as Alefeld and Herzberger [1]. These intervals are easier to estimate and provide a better approximation of the incertitude inherent in the reality they represent. The effect of the substitution is to transform the models made of differential equations into differential inclusions; see Aubin and Celina [3] for an excellent presentation on inclusions. In light of this change, we reexamine the Lanchester type models presented to verify their correct transfer to proper inclusions and study the existence of solutions.

One of the major reasons for interest in Lanchester type models of combat resides in the analysis of their evolution in time, giving us the possibility to attempt a prediction of the outcome. With that in mind, our next step is to bring into play the tools provided by set-valued analysis [4] and viability theory to these transformed models where we examine the existence of viable solutions to differential inclusions.

As a last step, we take the analysis of the Lanchester type differential inclusions to its natural progression into the domain of optimal control. In this last section, we will introduce the notion of *optimal control by viability* for the set-valued case and explore its application to the Lanchester theory of combat.

2 Preliminaries

Throughout this work, we will denote by $(H, \langle \cdot\ , \cdot \rangle)$ an arbitrary Hilbert space and by K a closed convex cone in H. Unless otherwise stated, the Hilbert space under consideration will be the Euclidean space $(R^n, \langle \cdot\ , \cdot \rangle)$. We recall that a convex set $\Omega \subseteq H$ is a subset of H such that

$$x, y \in \Omega \Longrightarrow (1-\lambda)x + \lambda y \in \Omega, \qquad \text{for any } \lambda \text{ such that } 0 < \lambda < 1.$$

A closed convex cone $K \subseteq H$ is a closed subset having the following properties:

1. $K + K \subseteq K$,
2. $\lambda K \subseteq K, \qquad \lambda \in \mathbf{R}_+$.

If in addition, K satisfies the property that $K \cap (-K) = 0$, then we say that K is a *pointed cone.*

The study of viability will require the application of the notion of contingent cone of which we now give a definition.

Definition 1 (`Contingent cone`). *Let X be a Hilbert space, $K \subset X$ a non-empty subset, and $B = \{x \in X \mid \|x\| \leq 1\}$. We say that the subset*

$$T_K(x) = \bigcap_{\epsilon>0} \ \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left(\frac{1}{h}(K - x) + \epsilon B \right)$$

is the contingent cone *(or the* Bouligand's contingent cone [8]*) to K at the point $x \in K$.*

In the case when K is convex, we call $T_K(x)$ the *tangent* cone to K at x and its representation is $T_K(x) = \overline{\bigcup_{h>0} \frac{1}{h}(K - x)}$. Because this article studies the viability of dynamical systems using convex cones as the viable subset, we give the following characterization of $T_K(x)$.

Theorem 1. *Given a convex cone K, a subset of a Hilbert space, $x \in K$, and $T_K(x)$ the contingent cone at x satisfying Definition (1), then*

$$K + \mathbf{R}x = K - \mathbf{R}_+ x \subset T_K(x)$$

furthermore

$$\overline{K - \mathbf{R}_+ x} = T_K(x). \tag{1}$$

Proof. We refer the interested reader to the proof in our *AJMAA* article [27]. ■

From Section 7 onward, the right-hand side of the differential equations will be transformed from single-valued mappings to set-valued mappings. This transformation has an important effect on the notion of continuity. In the single-valued setting, continuity of a function $f : X \to Y$ is equivalently characterized by:

1. for any neighbourhood $N(f(x))$ of $f(x)$, there exists a neighbourhood $N(x)$ of x such that $f(N(x)) \subseteq N(f(X))$ (commonly recited as: $\forall \epsilon, \exists \delta \ldots$); or
2. for any net convergent to x, the net $\{f(x_i)\}_{i \in I}$ is convergent to $f(x)$.

However, in set-valued settings, these two characterizations of continuity no longer represent the same property. We refer to the set-valued equivalent of (1) as *upper semi-continuity* and to that of (2) as *lower semi-continuity* and we give the following definitions. In the following, $P(Y) := \{U \mid U \subseteq Y, U \neq \emptyset\}$ and τ_Y is the topology defined on Y.

Definition 2 (`Set-valued mapping`)**.** *Given X and Y two nonempty sets, a mapping that assigns a nonempty subset of Y to the elements of X is a* set-valued mapping*:*

$$F : X \to P(Y)$$

Definition 3 (`Upper Semi-continuity`)**.** *Let X, Y be topological spaces and consider the set-valued mapping $F : X \to P(Y)$. The mapping F is* upper semi-continuous *(u.s.c.) at $x_0 \in X$ if and only if for any $G \in \tau_Y \mid G \supseteq F(x_0)$, there exists $V(x_0)$, a neighbourhood of x_0, such that $x \in V(x_0) \Longrightarrow G \supseteq F(x)$.*

Definition 4 (`Lower Semi-continuity`)**.** *Let X, Y be topological spaces and consider the set-valued mapping $F : X \to P(Y)$. The mapping F is* lower semi-continuous *(l.s.c) at $x_0 \in X$ if and only if for any $G \in \tau_Y \mid G \cap F(x_0) \neq \emptyset$, there exists $V(x_0)$, a neighbourhood of x_0, such that $x \in V(x_0) \Longrightarrow F(x) \cap G \neq \emptyset$.*

If F is both lower semi-continuous and upper semi-continuous, we say that F is continuous.

The set-valued mappings considered in this article are those generated through the application of *Interval Analysis*. To that effect, we now introduce some definitions and properties that are related to the subject and have been introduced by Moore [33] and Alefeld and Herzberger [1].

Definition 5 (`Real interval`). *We call A, a subset of $\mathbf{R}$, a closed real interval (or interval when there is no doubt as to the meaning) if it has the form:*

$$A = [a_1, a_2] = \left\{x \in \mathbf{R} \,\middle|\, a_1 \le x \le a_2, \text{ where } a_1, a_2 \in \mathbf{R}\right\}.$$

It is of interest to note that the single value $x \in \mathbf{R}$ can be expressed as the interval $[x, x] \subset \mathbf{R}$. An n-dimension interval is an n-tuple of intervals: $(A_1, A_2, \ldots, A_n)$. We will denote the set of all real intervals by $I(\mathbf{R})$ or, in the n-dimension case, $I(\mathbf{R}^n)$. The four common binary operations on the set of real numbers have the following definition on $I(\mathbf{R})$.

Definition 6 (`Interval binary operations`). *Consider $*$ to be one of the common four binary operation on $\mathbf{R}$, i.e. $* \in \{+, -, \cdot, \div\}$. For $A, B \in I(\mathbf{R})$, the operations are defined as:*

$$A * B \equiv \left\{x = a * b \,\middle|\, a \in A, b \in B\right\}.$$

Clearly, we must have that $0 \notin B$ in the case of the division. As these operations are continuous functions on compact sets, the result is again a real interval and, where $A = [a_1, a_2]$ and $B = [b_1, b_2]$, can be computed explicitly as:

$$\begin{aligned} A + B &= [a_1 + a_2, b_1 + b_2] \\ A - B &= [a_1 - b_2, a_2 - b_1] = A + [-1, -1] \cdot B \\ A \cdot B &= [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}] \\ A \div B &= [a_1, a_2] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right]. \end{aligned}$$

The algebraic properties of interval arithmetic include commutativity, associativity, and the existence of neutral elements. There is no inverse for either addition or multiplication but we have: $0 \in A - A$ and $1 \in A \div A$. Unless A is a singleton or $bc \le 0$, $\forall b \in B$, and $c \in C$, we can only rely on subdistributivity: $A \cdot (B + C) \subseteq AB + AC$.

As we will be modifying single-valued mappings through the introduction of intervals in replacement of the coefficients, we need to define the following:

Definition 7 (`United extension`). *Let M_1 and M_2 be arbitrary sets and let $f : M_1 \to M_2$ be an arbitrary single-valued mapping. Then, the set-valued mapping defined by:*

$$\overline{f} : P(M_1) \to P(M_2)$$
$$X \to \left\{ f(x) \,\middle|\, x \in X \right\}, X \in P(M_1)$$

is called the united extension *of f.*

Definition 8 (`Inclusion monotonicity`). *An interval valued function F of several variables $X_1, X_2, \ldots, X_n$ is* inclusion monotonic *if*

$$(Y_i \subseteq X_i, i = 1, \ldots, n) \Rightarrow f\left(Y_1, \ldots, Y_n\right) \subseteq f\left(X_1, \ldots . X_n\right)$$

Once the right-hand side (RHS) of the Lanchester type models have become set-valued mappings, we will have entered the realm of differential inclusions where one typically searches for a satisfactory selection.

Definition 9 (`Selection`). *Let $F : X \to P(Y)$ be a set-valued mapping. We say that the single-valued mapping $f : X \to Y$ is a* selection *for F if and only if, for any $x \in X$, we have $f(x) \in F(x)$.*

The problem then becomes to determine the conditions under which there exists a selection that also satisfies some stated regularity conditions (i.e., continuity, Lipschitz, etc.).

3 Differential Lanchester Type Models

Since the publication of Lanchester's book in 1916, numerous variations based on the original ideas have been proposed, see for instance the list presented by Taylor [41]. All these models are based on a modelling that takes roots in defining the rates of variation in strength of opposing forces. Our recent article [27] provides more detailed descriptions of the various models under study. We simply include the list here for quick and easy reference.

3.1 Various Lanchester Type Models

Aimed Fire

This model is the original one introduced in 1914 by Lanchester [30]:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 \end{cases}$$

where $a, b > 0$.

Area Fire

This model is also one of the first ones introduced by Lanchester [30] and is given by:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$.

Brackney

In 1959, Brackney [10] introduced a model that is based on a combination of the *Aimed Fire* model in 3.1 and the *Area Fire* model of Section 3.1. His model is expressed by

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \qquad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

where $a, b > 0$.

Peterson

In an attempt to explore the initial stage of battle, Peterson [37] introduced this model:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_2 \end{cases}$$

where $a, b > 0$.

Morse and Kimball

Morse and Kimball [34] put forth the hypothesis that losses from both combat and related operations contributed to the whole. The model is as follows:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2 - \beta x_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 - \alpha x_2 \end{cases}$$

where $a, b, \alpha, \beta > 0$.

Coleman

The model presented by Coleman [15] added the effect of reinforcements.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1 - bx_2 + R_{x_1}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -cx_1 - dx_2 + R_{x_2} \end{cases}$$

where $a, b, c, d > 0$ and R_{x_1}, R_{x_2} can be either positive or negative and are generally considered to be step functions.

Hembold

In 1964, Hembold [24], to model the inefficiency of scale, introduced two mappings that are a function of the ratio of the opposing forces numbers. The resulting model is

$$\begin{cases} \dfrac{dx_1}{dt} = -ag(\dfrac{x_1}{x_2})x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bh(\dfrac{x_2}{x_1})x_1 \end{cases}$$

where $a, b > 0$ while $g(.), h(.) \geq 0$ and $g(1) = h(1) = 1$.

Weiss

When Weiss [42] introduced his models, he approached the issue of the effect that scale has on the rates from a vulnerability point of view. His model became

$$\begin{cases} \dfrac{dx_1}{dt} = -a(\dfrac{x_1}{x_2})^{1-W} x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b(\dfrac{x_2}{x_1})^{1-W} x_1 \end{cases}$$

where $a, b > 0$.

Schreiber

Schreiber [39] was interested in a model that looked at command and control. Thus, he put forth a model where the *efficiency of command* came to play through the introduction of command efficiency constants $e_{x_1}, e_{x_2} \in [0, 1]$.

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left\{\dfrac{x_1 x_2}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\left\{\dfrac{x_1 x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\} \end{cases}$$

where $a, b > 0$, $x_{1,0}$ and $x_{2,0}$ represent the initial strengths, and $e_{x_1}, e_{x_2} \in [0, 1]$.

4 Viable Solutions for a Differential System

Let us first provide the tools necessary for a brief overview of viability analysis of the differential systems. As in [27], we will define a closed subset of state space, K, of the system to represent the set of *acceptable* status of our combat equations. That closed subset will be considered *viable* under the differential system if for every initial $x_0 \in K$, there exists at least one solution to the system starting at that point and remaining in K for some time. More formally, using the definitions given by Aubin [5]:

Definition 10 (`Viable function`). *Let K be a subset of a finite dimensional vector space X. We shall say that a function $x(\cdot)$ from $[0,T]$ to X is viable in K on $[0,T]$ if $\forall t \in [0,T]$, $x(t) \in K$.*

Consider the following differential equation for $f : U \to X$, $U \subset X$.

$$d(x(t))/dt = f(x(t)),\ x(0) = x_0 \in U. \tag{2}$$

Definition 11 (`Viability and Invariance`). *Let K be a subset of U. We shall say that K is* locally viable *under f if for any initial state x_0 of K, there exist $T > 0$ and a viable solution on $[0,T]$ to differential equation (2) starting at x_0. It is said to be (globally)* viable *under f if we can always take $T = \infty$.*

The subset K is said to be invariant *under f if for any initial state x_0 of K, all solutions to the differential equation (2) are viable in K.*

Because we are concerned with Lanchester type combat models, given $K \in \mathbf{R}^n$ and a mapping $x(t) : \mathbf{R}_+ \to \mathbf{R}^n$, we will say that $x(t)$ is viable in K whenever $x(t) \in K$, $\forall t \in \mathbf{R}_+$.

To verify the existence of solutions that are viable within a subset U, we use the Nagumo theorem [35].

Theorem 2 (`Nagumo`). *Let U be a closed subset of a Hilbert space H and f be a continuous map from U to H, $f : U \to H$, such that*

$$\forall x \in U, f(x) \in T_U(x). \tag{3}$$

Then for all $x_0 \in U$, there exists $T > 0$ such that Equation (2) has a viable trajectory on $[0,T]$.

5 Winning Cones

In our previous article, we introduced the concept of *winning cone*. This cone provides us with a useful tool as at all points within such a cone, one component is *superior* to the other.

Let E be a real vector space and $K_\alpha, K_\beta \subset E$ be closed convex cones. Let "$\leq$" be the ordering defined by the convex cone $K_\leq \subset E$, i.e. $x \leq y \Leftrightarrow y - x \in K_\leq$.

Definition 12 (`Winning cone`). *Consider the vector space $E \times E$. We say that the closed convex pointed cone K_0 is a* winning cone *if it has the following properties:*

1. $K_o \subset K_\alpha \times K_\beta$*; and*
2. $(x, y) \in K_0 \Rightarrow x \geq y$*, where "$\leq$" is the ordering defined by $K_\leq$.*

In our case, the analysis of Lanchester type models of combat, we will have $E = K_\beta = \mathbf{R}$, $K_\alpha = K_\leq = \mathbf{R}_+$, such that whenever $(x_1, x_2) \in K_0$, we have that x_1 dominates his opponent x_2. As a result, if a solution remains in such a *winning cone*, victory is assured for one of the opposing forces.

To such a purpose, we defined four *winning* cones, see Figure 1, where the first three provide a set in which combat power of one opponent always dominates the other, and the last cone provides a set within which one force can but improve its ratio over its opponent. For each of these cones K, we determine the tangent cone $T_K(x)$ associated with every $x = (x_1, x_2) \in K$ and the conditions it imposes in $\mathbf{R}^2$ on the dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x), \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = f_2(x) \end{cases}$$

where $f(x) = (f_1(x), f_2(x))$.

The cones selected are

1. $K_1 := \{x \in \mathbf{R}^2 \mid (x_1 \geq x_2) \wedge (x_2 \geq 0)\}$,
2. $K_2 := \{x \in \mathbf{R}^2 \mid x_1 \geq |x_2|\}$,
3. $K_3 := \{x \in \mathbf{R}^2 \mid x_1 \geq x_2 \wedge x_1 \geq 0\}$,
4. $K_4 := \left\{x \in \mathbf{R}^2 \mid x_1 \geq \frac{x_1(0)}{x_2(0)} x_2 \text{ and } x_1 \geq 0\right\}$.

After determining the tangent cones at every point for each cone, we have been able to obtain conditions on the RHS of the differential systems that guarantee viability of the solution. For easy reference, the results obtained in [27] are reproduced here in Table 1.

In other words, for a solution to remain viable in a given cone, all the associated conditions listed in the third column of the table must be met.

6 Winning Cones for Differential Lanchester Type Models

Using the results provided in Table 1, for each model we can translate these into a set of conditions on the coefficients to obtain viability for the respective *winning cones*. We provide an example of the process, taken from [27], for the simple case of the *Aimed Fire* model and display the conclusions for the remainder in Table 2.

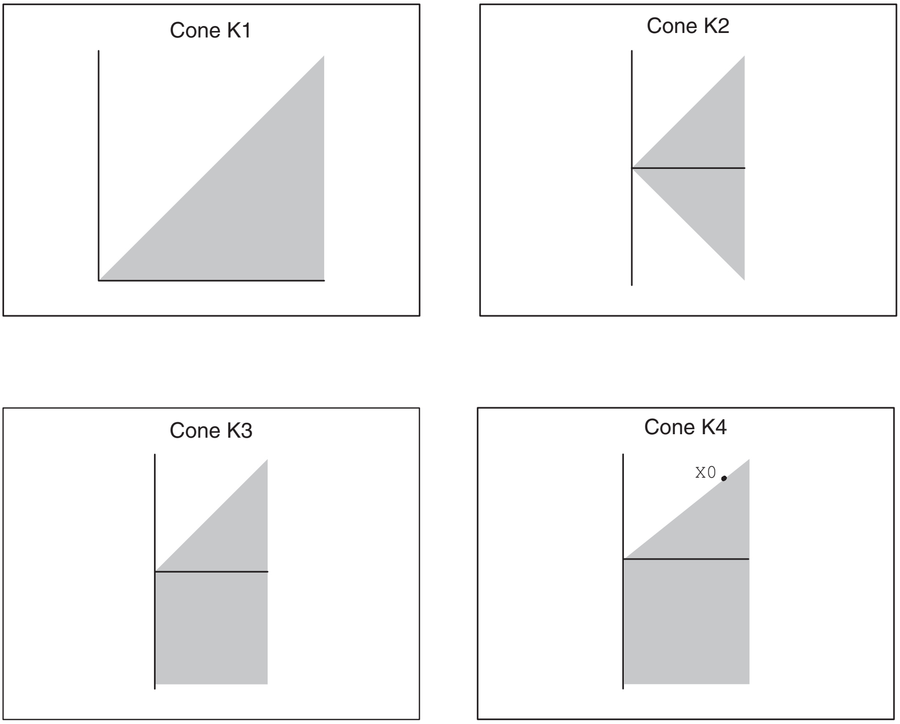

Figure 1. Viability cones

Table 1. Tangent Cones at $x \in K$

Winning cone (K)	**Points in** K	**Conditions on** $f(x)$
K_1	$x \in int(K_1)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_1$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_2 = 0,\ x_1 > 0$	$f_2(x) \geq 0$
K_2	$x \in int(K_2)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_2$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_1 = -x_2,\ x_1 > 0$	$f_1(x) \geq -f_2(x)$
K_3	$x \in int(K_3)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_3$
	$x_1 = x_2,\ x_1 > 0$	$f_1(x) \geq f_2(x)$
	$x_1 = 0,\ x_2 < 0$	$f_1(x) \geq 0$
K_4	$x \in int(K_4)$	$f(x) \in \mathbf{R}^2$
	$x = (0,0)$	$f(x) \in K_4$
	$x_1 = \frac{x_1(0)}{x_2(0)} x_2,\ x_1 > 0$	$f_1(x) \geq \frac{x_1(0)}{x_2(0)} f_2(x)$
	$x_1 = 0,\ x_2 < 0$	$f_1(x) \geq 0$

Table 2. Viability Conditions for Constant Coefficients Models

Model	K_1	K_2
Aimed Fire	None	$a = b$
Area Fire	$a \leq b$	$a \leq b$
Brackney (Modified)	$a \leq b$	$a \leq b$
Peterson	$a \leq b$	$a \leq b$
Morse and Kimball	None	$\lvert a-b \rvert \frac{x_1(0)}{x_2(0)}$
Coleman	Not Practical	$\lvert b-c \rvert \leq d-a$ $\lvert R_{x_2} \rvert \leq R_{x_1}$
Hembold	$a \leq b$	$a = b$
Schreiber	$\frac{a}{b} \leq \frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}$	Same as K_1
Model	K_3	K_4
Aimed Fire	$a \leq b$	$\frac{a}{b} \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2$
Area Fire	$a \leq b$	$a \leq \frac{x_1(0)}{x_2(0)} b$
Brackney (Modified)	$a \leq b$	$a \leq \frac{x_1(0)}{x_2(0)} b$
Peterson	$a \leq b$	$a \leq \frac{x_1(0)}{x_2(0)} b$
Morse and Kimball	$a - b \leq \alpha - \beta$	$a + \beta \leq \frac{x_1(0)}{x_2(0)} (b - \alpha)$
Coleman	$a + b \leq c + d$ $R_{x_2} \leq R_{x_1}$	$a + b \leq \frac{x_1(0)}{x_2(0)} (c + d)$
Hembold	$a \leq b$	$a \leq b$
Schreiber	Same as K_1	$\frac{a}{b} \leq \frac{x_{1,0}(x_{1,0}-e_{x_2}(x_{1,0}-x_1))}{x_{2,0}(x_{2,0}-e_{x_1}(x_{2,0}-x_2))}$

6.1 Aimed Fire, Winning Cone K_1

As we recall, from Section 3.1, the model introduced in 1914 is expressed mathematically by

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 \end{cases}$$

where $a, b > 0$. To link the models with the differential equation given earlier, the right-hand side of equation (2) is defined by

$$f(x) = \begin{cases} f_1(x) = -ax_2 \\ f_2(x) = -bx_1 \end{cases}$$

where $x = (x_1, x_2)$. From the tangent cones defined in Table 1, it is clear that for points belonging to the interior of the cone, $x \in int(K_1)$, any values for a and b are viable. Similarly, as $f(0) = 0$ for any $a, b > 0$ we have $f(x) \in K_1$ for any choice of coefficients. It remains to examine the coefficients required to meet the conditions of the Nagumo theorem at the cone's boundaries. For the *upper* boundary, we must have

$$\begin{aligned} f_1(x) &\geq f_2(x) \\ -ax_2 &\geq -bx_1 \\ -ax_1 &\geq -bx_1, && x_1 = x_2 \\ -a &\geq -b, && x_1 > 0 \\ a &\leq b. \end{aligned} \tag{4}$$

The restrictions on the coefficients for the *lower* boundary become

$$\begin{aligned} f_2(x) &\geq 0 \\ -bx_1 &\geq 0 \\ -b(1) &\geq 0, && \text{choose } x = (1,0) \text{ on boundary} \\ -b &\geq 0 \\ b &\leq 0 \end{aligned} \tag{5}$$

but as the model requires $b > 0$, it implies that there is no viable solution to equation (2) that remains viable once it enters cone K_1.

6.2 Aimed Fire, Winning Cone K_3

It is obvious through the similarities with the previous cone that the only restrictions additional to those imposed by the model are those generated by the *upper* and *lower* boundaries. In the case of the former, the results are also identical and given by equation (4), and for the latter, using Table 1 we have

$$\begin{aligned} f_1(x) &\geq 0 \\ -ax_2 &\geq 0 \\ -a &\leq 0, && x_2 < 0 \\ a &\geq 0. \end{aligned} \tag{6}$$

The restrictions imposed by equation (6) are more relaxed than that imposed by the model. As a consequence, only the additional inequality of equation (4) is required to consider. For a solution to the *Aimed Fire* model to remain viable once it enters K_3, its coefficients must be such that $a \leq b$.

6.3 Aimed Fire, Winning Cone K_4

From the inspection of Table 1, the only differences between this cone and K_3 are at the "Upper" boundary. From the corresponding entry in the table, we have

$$\begin{aligned}
f_1(x) &\geq \frac{x_1(0)}{x_2(0)} f_2(x) \\
-ax_2 &\geq \frac{x_1(0)}{x_2(0)}(-b)x_1 \\
-ax_2 &\geq \frac{x_1(0)}{x_2(0)}(-b)\frac{x_1(0)}{x_2(0)}x_2, && x_1 = \frac{x_1(0)}{x_2(0)}x_2 \\
-a &\geq \left(\frac{x_1(0)}{x_2(0)}\right)^2 (-b), && x_2 > 0 \\
a &\leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 b. && (7)
\end{aligned}$$

The restrictions on the coefficients imposed by the model are more restrictive than the boundary conditions with the exception of (7). As such, to make K_4 viable, the coefficients must meet $\frac{a}{b} \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2$ and $a, b > 0$.

6.4 Variable Coefficients

We have seen so far how viability can be used to establish conditions on the Lanchester coefficients to render possible one side's victory. In the Lanchester theory, each of these models has an equivalent one using variable coefficients. These models are similar to those introduced so far, but each attrition coefficient is replaced by a continuous function of time. For instance, the *Aimed Fire* model becomes

$$\begin{cases} \dfrac{dx_1}{dt} = -a(t)x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b(t)x_1 \end{cases}$$

.

Clearly these models prove more flexibility and the constant coefficients versions become a special case where $a(t)$ and $b(t)$ are constant functions. When we examined the tangent cones at interior points for each of our winning

Table 3. Viability Conditions for Variable Coefficients Models

Model	K_1 **to** K_3	**Gaining cone** K_4
Aimed Fire	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Area Fire	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Brackney (Modified)	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Peterson	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Morse and Kimball	$a(t) + \beta(t) \leq b(t) + \alpha(t)$	$a(t) + \beta(t) \leq \frac{x_1(0)}{x_2(0)} (b(t) + \alpha(t)$
Coleman	$(a(t) + b(t)) - (c(t) + d(t)) \leq \frac{R_{x_1}(t) + R_{x_2}(t)}{x_1(t)}$	$(a(t) + b(t)) - (c(t) + d(t)) \leq \frac{x_1(0)(R_{x_1}(t) + R_{x_2}(t))}{x_2(0) x_1(t)}$
Hembold	$a(t) \leq b(t)$	$a(t) \leq \frac{x_1(0)}{x_2(0)} b(t)$
Schreiber	$\frac{a(t)}{b(t)} \leq \frac{(1-e_{x_2})x_1(0) + e_{x_2} x_1(t)}{(1-e_{x_1})x_2(0) + e_{x_1} x_1(t)}$	$\frac{a(t)}{b(t)} \leq \frac{x_1(0)}{x_2(0)} \frac{(1-e_{x_2})x_1(0) + e_{x_2} x_1(t)}{(1-e_{x_1})x_2(0) + e_{x_1} x_1(t)}$

cones, we established that they were the complete space $\mathbf{R}^2$ (see Table 1). As such, the study of viability for variable coefficients need only be concerned by the behaviour at the boundaries of the *winning cones*. Furthermore, as we are concerned with actual war scenarios, we restrict our study to the "Upper" boundary and, using a method similar to that above, we get the results shown in Table 3 replicated from [27].

Viability, thus obtained, clearly depends on the *instantaneous* abilities of the forces at play when they reach the critical states that the boundaries represent. This idea is the basis of what we coined *optimal control by viability* and that we will now discuss.

6.5 Optimal Control by Viability, Differential Systems

Life, and in particular military operations, is such that our control over its evolution is often effected in a discrete manner. In the models introduced, it takes shape in the selection of weapons used, the commitment of reserve, or the resting of some forces to bring their effectiveness back up, for example.

In such cases, optimal control is not about the optimization of a function but about taking actions to maintain our progress toward an objective. With the use of our winning cones, this is expressed by the idea of maintaining viability and brings to play the idea of *optimal control by viability*, introduced in [27].

In the case of Lanchester type models discussed so far, the idea is to use the conditions for viability that have been obtained to determine the *moments* when they will be breached. This process generates a set of states where control must be applied to maintain viability. Of course, the constant coefficients case does not provide the modelling of the commander's ability to affect efficiency and, as such, we will concern ourselves with the variable coefficients models.

Let us examine the *Aimed Fire* model as an example. We are interested in time

$$\inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle|\, x_1(t) = x_2(t) \right\} \right\}.$$

which represent the next moment when a new control scheme must be employed. Table 4 expresses this for all models presented.

We can see how the study of the models through *optimal control by viability* produces excellent tools to assist in the decision-making process and provides means by which to study the various impacts of command decisions.

Considering a battle is a dynamical process, the coefficients in a Lanchester type model may change and, consequently, the condition $f(x) \in T_U(x)$ may not be satisfied for any $x \in U$ (see Theorem 2). In this case, the solution of the system

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t)) \\ x(0) = x_0 \in U, \end{cases}$$

may not be viable with respect to U. However, by adding to $f(x)$ a continuous function $\varphi(x)$, we can obtain from the given system a differential system

Table 4. Change of Control Scheme

Model	Decision time
Aimed Fire	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Area Fire	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Brackney (Modified)	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Peterson	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Morse and Kimball	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) + \beta(t) > b(t) + \alpha(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Coleman	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, (a(t) + b(t)) - (c(t) + d(t)) > \frac{R_{x_1}(t) + R_{x_2}(t)}{x_1(t)} \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Hembold	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, a(t) > b(t) \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$
Schreiber	$inf \left\{ \left\{ t \in \mathbf{R}_+ \,\middle\|\, \frac{a(t)}{b(t)} > \frac{(1-e_{x_2})x_1(0)+e_{x_2}x_1(t)}{(1-e_{x_1})x_2(0)+e_{x_1}x_1(t)} \right\} \cap \left\{ t \in \mathbf{R}_+ \,\middle\|\, x_1(t) = x_2(t) \right\} \right\}$

satisfying the assumption of Nagumo's theorem. It is well to find a military interpretation of the term $\varphi(x)$.

Theorem 3. *Let U be a closed convex subset of a Hilbert space H and $f : U \to H$ a continuous map. There exists at least a continuous map $\varphi : U \to H$ such that $f(x) + \varphi(x) \in T_U(x)$, for any $x \in U$. Moreover, for any $x_0 \in U$, there exists $T > 0$ and a viable solution (with respect to U) $x : [0,T] \to H$ for the systems $\frac{dx}{dt} = f(x(t)) + \varphi(x(t))$.*

Proof. Let $h : U \to H$ be an arbitrary continuous mapping. We define $\varphi : U \to H$ by $\varphi(x) = P_U[x+h(x)]-x-f(x)$, where $P_U[\cdot]$ is the metric projection onto U. We then have

$$f(x) + \varphi(x) = P_U[x + h(x)] - x.$$

Because U is a closed convex set, we know that, in this case

$$T_U(x) = \overline{\bigcup_{\lambda>0} \frac{1}{\lambda}(U - x)}.$$

Consequently, for any $x \in U$ we have $P_U[x + h(x)] - x \in T_U(x)$ and the conclusion of the theorem follows from Nagumo's theorem (Theorem 2). ∎

Remark 1. If we consider a battle as a sequential process and the map f is defined with the Lanchester type coefficients estimated at a moment t_n, we can take as a map h the map f defined with the Lanchester type coefficients estimated as a moment t_m with $m < n$. Another possible selection for the map h is $h(x) = -\alpha f(x)$, with α a small real positive number. In this case, we have that $f(x)+\varphi(x) = P_U[x-\alpha f(x)]-x$, and applying theorem 16 of [28] we obtain that the solution of the differential system, defined with $f(x) + \varphi(x)$, will be viable (with respect to U) for any t in its maximal interval of definition.

In the above theorem, we can replace the projection P_U by an arbitrary retraction $\mathcal{R} : H \to U$. This fact is important because in practical problems we can select an appropriate retraction.

Applying Theorem 3, we can obtain an interesting optimal control by viability for Lanchester type models, considering as the set U any winning cone defined in our paper [27].

In the cases studied so far, the system was defined through the use of single-valued mappings and $\frac{dx}{dt} = f(x(t))$. As expressed throughout this article and in the literature on the subject in general, one of the major problems of the above models relates to the imprecision and the lack of information on the Lanchester coefficients. In the next section and for the remainder of this publication, we will replace these coefficients by intervals, as introduced by Moore [33] and further studied by Alefeld and Herzberger [1]. This replacement of coefficients in $f(x(t))$ will change it to a set-valued mapping $F(x(t))$ (for a detailed study, see Aubin and Frankowska [4]), thus transforming the differential equation in an inclusion in the form

$$\frac{dx}{dt} \in F(x(t)). \tag{8}$$

7 Differential Inclusions of Lanchester Type Models

When we replace the Lanchester coefficients by intervals, the right-hand sides of our models become set-valued mappings. We will show in the following (Section 8) that for such extensions of continuous functions, the set-valued mappings generated are upper semi-continuous. This property coupled with the fact that their images are compact and convex will allow us to apply some viability tests to the associated differential inclusion in a fashion similar to that employed on differential systems above.

However, as a first step, we must verify that the replacement of the coefficients with intervals yield proper differential inclusions (i.e., that the models can be written in the form of equation (8)) and we now proceed to do so.

7.1 Introduction

Before considering the interval extensions of the Lanchester models, there is a requirement to set some building blocks. The notation presented here is not intended to be comprehensive.

The space of all real coefficients matrices with m lines and n columns is noted $M_{m,n}(\mathbf{R})$. The extension of these matrices to the realm of Interval Analysis replaces the coefficients with intervals in the form $[a, b]$, $a, b \in \mathbf{R}$; the set of all such intervals on the real axis is $I(\mathbf{R})$. In a fashion similar to matrices with real coefficients, the space of all interval coefficients matrices with m lines and n columns is noted $M_{m,n}(I(\mathbf{R}))$.

The transformation of coefficients into intervals has the effect of transforming single-valued functions into set-valued mappings with the result that differential equations become differential inclusions. Through this process, it will be useful to view elements of $M_{m,n}(I(\mathbf{R}))$ as sets in $M_{m,n}(\mathbf{R})$. To see how this is possible, consider the matrix $\mathbf{A} \in M_{m,n}(I(\mathbf{R}))$ where, as discussed above, each coefficient is an interval, $a_{i,j} \in I(\mathbf{R})$. Define the set U as

$$U = \{B \in M_{m,n}(\mathbf{R}) | b_{i,j} \in a_{i,j},\, 1 \le i \le m,\, 1 \le j \le n\},$$

clearly $U \subset M_{m,n}(\mathbf{R})$. We will write $B \in \mathbf{A}$ whenever $b_{i,j} \in a_{i,j}$, $1 \le i \le m$, $1 \le j \le n$. It is easy to see that

$$B \in \mathbf{A} \iff B \in U$$

and therefore, elements of $M_{m,n}(I(\mathbf{R}))$ can be considered as closed sets in finite dimensional space $M_{m,n}(\mathbf{R})$. Another way of looking at this is that elements of $M_{m,n}(I(\mathbf{R}))$ represent *hypercubes* in $I(\mathbf{R})^{m+n}$.

Remark 2. It is important to note that although some matrix notation is used, the set-valued mappings defining the differential inclusions are not linear. This comes from the manner by which some of these matrices are defined. See for example equation (9).

We will now proceed to show that the replacement of fixed coefficients by intervals in the various Lanchester type models does indeed yield a proper differential inclusion.

7.2 Aimed Fire

The single-valued differential equation is given by:

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1 \end{cases}$$

Let

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X_0,$$

the system of equations can then be rewritten as:

$$\frac{dX}{dt} = -AX, \qquad X(0) = X_0.$$

Interval Case

Consider A to be a matrix of interval, i.e., $A \in M_{2,2}(I(\mathbf{R}))$ and $X \in M_{2,1}(\mathbf{R})$, thus

$$AX : \mathbf{R}^2 \to M_{2,1}(I(\mathbf{R}))$$

and from the relationship between $M_{2,1}(I(\mathbf{R}))$ and $M_{2,1}(\mathbf{R})$, given $A \in M_{2,1}(I(\mathbf{R}))$, then $A \subset M_{2,1}(\mathbf{R})$ (see Section 7.1), we come to the conclusion that $AX \subset M_{2,1}(\mathbf{R})$. We can therefore write the differential inclusion

$$\frac{dX}{dt} \in -AX, \qquad X(0) = X_0.$$

7.3 Area Fire

The model, as before, is

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

Consider the following function

$$f : \mathbf{R}^2 \to \mathbf{R}$$
$$(x, y) \to x$$

and let

$$\begin{aligned} F(X) : M_{2,1}(\mathbf{R}) &\to M_{2,2}(\mathbf{R}) \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\to \begin{bmatrix} f(x_2, x_1) & 0 \\ 0 & f(x_1, x_2) \end{bmatrix} \end{aligned} \tag{9}$$

with

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X_0.$$

As a result

$$AF(X) = \begin{bmatrix} 0 & af(x_1, x_2) \\ bf(x_2, x_1) & 0 \end{bmatrix}$$
$$AF(X)X = \begin{bmatrix} af(x_1, x_2)x_2 \\ bf(x_2, x_1)x_1 \end{bmatrix} = \begin{bmatrix} ax_1x_2 \\ bx_1x_2 \end{bmatrix}$$

and

$$\frac{dX}{dt} = -AF(X)X.$$

Interval Case

Let us consider the case where $A \in M_{2,2}(I(\mathbf{R}))$. As a consequence

$$AF(X) \in M_{2,2}(I(\mathbf{R}))$$
$$-AF(X)X \in M_{2,1}(I(\mathbf{R}))$$
$$-AF(X)X \subset M_{2,1}(\mathbf{R})$$

The above yields the differential inclusion

$$\frac{dX}{dt} \in -AF(X)X, \qquad X(0) = X_0.$$

7.4 Brackney

This model, introduced by Brackney [10, 41], is a combination of *Aimed Fire* in Section 7.2 and *Area Fire* in Section 7.3.

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_2, \qquad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_1x_2 \end{cases}$$

Consider the following two functions

$$\begin{aligned} g(x,y) : \mathbf{R}^2 &\to \mathbf{R} \\ (x,y) &\to 1 \\ h(x,y) : \mathbf{R}^2 &\to \mathbf{R} \\ (x,y) &\to y \end{aligned}$$

let

$$\begin{aligned} F(X) : M_{2,1}(\mathbf{R}) &\to M_{2,2}(\mathbf{R}) \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\to \begin{bmatrix} h(x_1,x_2) & 0 \\ 0 & g(x_1,x_2) \end{bmatrix} \end{aligned}$$

and define the matrix

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$

As a result,

$$\begin{aligned} AF(X) &= \begin{bmatrix} 0 & ag(x_1,x_2) \\ bh(x_1,x_2) & 0 \end{bmatrix} \\ AF(X)X &= \begin{bmatrix} ag(x_1,x_2)x_2 \\ bh(x_1,x_2)x_1 \end{bmatrix} = \begin{bmatrix} ax_2 \\ bx_1x_2 \end{bmatrix} \end{aligned}$$

and

$$\frac{dX}{dt} = -AF(X)X.$$

Interval Case

To effect the transition toward Interval Analysis, the constants are once again replaced by intervals, i.e., matrix $A \in M_{2,2}(I(\mathbf{R}))$. As such, we come to the realization that

$$\begin{aligned} AF(X) &\in M_{2,2}(I(\mathbf{R})) \\ -AF(X)X &\in M_{2,1}(I(\mathbf{R})) \\ -AF(X)X &\subset M_{2,1}(\mathbf{R}) \end{aligned}$$

and, therefore

$$\frac{dX}{dt} \in -AF(X)X, \text{ as } X \in M_{2,1}(\mathbf{R}),$$

transforming the problem into a differential inclusion.

7.5 Peterson

The mathematical representation suggested by R. Peterson is

$$\begin{cases} \dfrac{dx_1}{dt} = -ax_1, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -bx_2 \end{cases}$$

Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The system can then be written as

$$\frac{dX}{dt} = -AX, \qquad X(0) = X_0.$$

Interval Case

This is identical to Section 7.2 where the matrix of real coefficients A is replaced by one of interval coefficients

$$A \in M_{2,2}(I(\mathbf{R}))$$

thus

$$\begin{aligned} &AX \in M_{2,1}(I(\mathbf{R})) \\ \Rightarrow\ &AX \subset M_{2,1}(\mathbf{R}) \end{aligned}$$

and the system is transformed to the inclusion

$$\frac{dX}{dt} \in -AX, \qquad X(0) = X_0.$$

7.6 Hembold and Weiss

The generic case proposed by Hembold is given by

$$\begin{cases} \dfrac{dx_1}{dt} = -a\, g\left(\dfrac{x_1}{x_2}\right) x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\, h\left(\dfrac{x_2}{x_1}\right) x_1 \end{cases}$$

however, the following special case was considered:

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left(\dfrac{x_1}{x_2}\right)^c x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\left(\dfrac{x_2}{x_1}\right)^c x_1 \end{cases}$$

It is interesting to note that a value of $c = 0$ yields the *Aimed Fire* model of Section 7.2 and that $c = 1$ gives the model presented by Peterson (Section 7.5). To obtain a linear model as in Section 7.3 where $\frac{dx}{dy} = constant$, the value of c can be set to $\frac{1}{2}$. As a result, Hembold's model is a generalization that encompasses some of the earlier models.

The combat model presented by Weiss [42] is a variation on the above. When the forces meeting in battle are of substantially different size, the larger one suffers from inefficiencies caused by its numbers. To take this phenomenon into account, Weiss suggested

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left(\dfrac{x_1}{x_2}\right)^{1-W} x_2, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\left(\dfrac{x_2}{x_1}\right)^{1-W} x_1 \end{cases}$$

where $W \in [0, 1]$ is a constant.

Lets refocus on the general Hembold type model. Let

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\begin{aligned} f_1(x_1, x_2) : \mathbf{R}^2 &\to \mathbf{R} \\ (x_1, x_2) &\to h\left(\frac{x_2}{x_1}\right) \end{aligned}$$

$$\begin{aligned} f_2(x_1, x_2) : \mathbf{R}^2 &\to \mathbf{R} \\ (x_1, x_2) &\to g\left(\frac{x_1}{x_2}\right) \end{aligned}$$

Using these functions, consider

$$\begin{aligned} F(X) : M_{2,1}(\mathbf{R}) &\to M_{2,2}(\mathbf{R}) \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\to \begin{bmatrix} f_1(x_1, x_2) & 0 \\ 0 & f_2(x_1, x_2) \end{bmatrix} \end{aligned}$$

which yields the results

$$AF(X) = \begin{bmatrix} 0 & af_2(x_1, x_2) \\ bf_1(x_1, x_2) & 0 \end{bmatrix}$$

$$AF(X)X = \begin{bmatrix} af_2(x_1, x_2)x_2 \\ bf_1(x_1, x_2)x_1 \end{bmatrix} = \begin{bmatrix} a\, g\left(\frac{x_1}{x_2}\right) x_2 \\ b\, h\left(\frac{x_2}{x_1}\right) x_1 \end{bmatrix} \in M_{2,1}(\mathbf{R}).$$

The Hembold model can therefore be written as

$$\frac{dX}{dt} = -AF(X)X, \qquad X(0) = X_0.$$

Interval Case

For the interval case, we substitute a real interval coefficient matrix for the real matrix:

$$A \in M_{2,2}(I(\mathbf{R}))$$

and, as before

$$\begin{aligned} &F(X) \in M_{2,2}(I(\mathbf{R})) \\ \Rightarrow &AF(X) \in M_{2,2}(I(\mathbf{R})) \\ \Rightarrow &AF(X)X : M_{2,2}(I(\mathbf{R})) \times M_{2,1}(\mathbf{R}) \\ \Rightarrow &AF(X)X \in M_{2,1}(I(\mathbf{R})) \\ \Rightarrow &AF(X)X \subset M_{2,1}(\mathbf{R}). \end{aligned}$$

We can therefore write

$$\frac{dX}{dt} \in -AF(X)X, \qquad X(0) = X_0, \text{ where } A \in M_{2,2}(I(\mathbf{R}))$$

a proper differential inclusion.

Note 1. Clearly, this is also valid for Weiss' model.

7.7 Schreiber

This model, as seen in Section 3.1, is given by:

$$\begin{cases} \dfrac{dx_1}{dt} = -a\left\{\dfrac{x_1x_2}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}, \quad x(0) = x_0 \\ \dfrac{dx_2}{dt} = -b\left\{\dfrac{x_1x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\} \end{cases}$$

where $a, b > 0$ and $e_{x_1}, e_{x_2} \in [0, 1]$. Let

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and consider

$$\begin{aligned} g(x_1, x_2) : \mathbf{R}^2 &\to \mathbf{R} \\ (x_1, x_2) &\to \frac{x_1}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)} \\ h(x_1, x_2) : \mathbf{R}^2 &\to \mathbf{R} \\ (x_1, x_2) &\to \frac{x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2).} \end{aligned}$$

We now define $F(X)$ as:

$$\begin{aligned} F(X) : M_{2,1}(\mathbf{R}) &\to M_{2,2}(\mathbf{R}) \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\to \begin{bmatrix} h(x_1, x_2) & 0 \\ 0 & g(x_1, x_2 \end{bmatrix} \end{aligned}$$

which results in

$$AF(X) = \begin{bmatrix} 0 & ag(x_1, x_2) \\ bh(x_1, x_2) & 0 \end{bmatrix}$$

and

$$AF(X)X = \begin{bmatrix} ag(x_1, x_2)x_2 \\ bh(x_1, x_2)x_1 \end{bmatrix}.$$

Schreiber's model can then be written as:

$$\frac{dX}{dt} = AF(X)X, \ X(0) = X_0.$$

Interval Case

Just as before, replacing the real coefficients with elements of $I(\mathbf{R})$, we obtain:

$$\begin{aligned} &A \in M_{2,2}(I(\mathbf{R})) \\ \Rightarrow &AF(X) \in M_{2,2}(I(\mathbf{R})) \\ \Rightarrow &AF(X)X \in M_{2,1}(I(\mathbf{R})) \\ \Rightarrow &AF(X)X \subset M_{2,1}(\mathbf{R}), \end{aligned}$$

which yields the differential inclusion:

$$\frac{dX}{dt} \in AF(X)X, \ X(0) = X_0.$$

We have now established that all our models transform properly into differential inclusions of the form of (8) and now proceed to examine these new models from a Viability perspective.

8 Viable Solutions for Differential Inclusions

In Sections 4, 5, and 6, we studied differential systems to determine the necessary conditions to ensure their viability with respect to our *winning cones*. We will again analyze our models to determine such conditions and, as we will be dealing with differential inclusions, we will therefore look for conditions that guarantee the existence of a selection that remains viable.

Probably the most famous selection theorem is *Michael's selection theorem;* it links lower semi-continuity to the existence of a continuous selection.

Theorem 4 (`Michael's selection theorem`). *Let (X, d) be a metric space, Y a Banach space, and F a set-valued mapping from X into the closed convex subsets of Y. If F is lower semi-continuous, then there exists $f : X \to Y$ a continuous selection for F.*

Of course, this theorem does not provide all the necessary tools to verify the viability of the Lanchester differential inclusions. To that end, we will use a set-valued equivalent of Nagumo's theorem (Theorem 2). Prior to introducing this theorem, we will first consider the continuity of our interval extensions in the sense applicable to set-valued mappings.

8.1 Continuity of Interval Extensions

To further study the differential inclusions generated by the Interval extensions of the Lanchester type models, we need to consider the *continuity* of these set-valued mappings in the sense presented in Section 2.

Theorem 5 (`United extensions continuity`). *Consider the function $f : X \to Y$, using the real coefficients $\omega_1, \omega_2, \ldots, \omega_n, n \in \mathbb{N}$ and noted by $f(x; \omega)$ or $f_\omega(x)$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbf{R}^n$. Let $\overline{f}$ be its* united interval extension*:*

$$\overline{f} : X \times \Omega \to P(Y)$$

where $\Omega \subseteq \mathbf{R}^n$ is closed and bounded. Equivalently $\overline{f}(x) = \left\{ f(x; \omega) \,\middle|\, \omega \in \Omega \right\}$.

If $f(x; \omega)$ is continuous for any given $\omega \in \Omega$, then $\overline{f}$ is lower semi-continuous*. Furthermore, if f is continuous when considered from $X \times \Omega \to Y$, then $\overline{f}$ is also* upper semi-continuous*.*

Proof. Let us first prove the lower semi-continuity of $\overline{f}$. Consider an element $x_0 \in X$, any open set G such that $\overline{f}(x_0) \cap G \neq \emptyset$, and the element $y \in \overline{f}(x_0) \cap G$. Then, $y = f(x_0; \omega_y)$ for some $\omega_y \in \Omega$ and, from the continuity of f, $V(x_0) = f_{\omega_y}^{-1}(G)$ is an open neighbourhood of x_0. Because $\forall\, x \in V(x_0)$, we have $f(x; \omega_y) \in G$ and that, from the definition of the united extension, $f(x; \omega_y) \in \overline{f}(x)$, the set $\overline{f}(x_0) \cap G$ contains $f(x; \omega_y)$ and is nonempty proving that $\overline{f}$ is lower semi-continuous at x_0. Because this is so for any arbitrary $x_0 \in X$, $\overline{f}$ is lower semi-continuous.

To prove the upper semi-continuity of $\overline{f}$ within the setting of the theorem, consider an element $x_0 \in X$ and any open set G such that $\overline{f}(x_0) \subseteq G$. For any $\omega \in \Omega$, $f(x_0;\omega) \in \overline{f}(x_0) \subseteq G$ and the continuity of $f : X \times \Omega \to Y$ implies that there exist $V_\omega(x_0) \subseteq X$ and $W(\omega) \subseteq \Omega$ open neighbourhoods of x_0 and ω, respectively, such that $x \in V_\omega, \mu \in W(\omega) \Rightarrow f(x;\mu) \in G$. Clearly,

$$\Omega \subseteq \bigcup_{\omega \in \Omega} W(\omega).$$

Additionally, Ω is a closed and bounded subset of $\mathbf{R}^n$ and therefore compact. The compacity of Ω implies that there exists a finite subcover of $\{W(\omega_i)\}_{i=i,\ldots,n}$ such that

$$\Omega \subseteq \bigcup_{i=i}^{n} W(\omega_i).$$

Consider $V(x_0) = \bigcap_{i=i}^{n} V_{\omega_i}(x_0)$. As it is a finite intersection of open sets, it is open. Furthermore, as $x_0 \in V_{\omega_i}(x_0) \Rightarrow x_0 \in V(x_0)$, $V(x_0)$ is an open neighbourhood of x_0 and, as $\{W(\omega_i)\}_{i=i,\ldots,n}$ is cover for Ω, $x \in V(x_0) \Rightarrow f(x;\omega) \in G, \forall \omega \in \Omega$. This last result is equivalent to $\overline{f}(x) \subseteq G$ and proves the upper semi-continuity of $\overline{f}$ at x_0. As this is true for any $x_0 \in X$, $\overline{f}$ is upper semi-continuous. ∎

And the following corollary is then a natural conclusion.

Corollary 1. *The* right-hand side *of the Lanchester type differential inclusions are both lower and upper semi-continuous.*

With these continuous properties of the Lanchester differential inclusions, we are now equipped to consider their viability.

8.2 Viability

Based on the work presented by Aubin and Cellina [3, 5], a natural extension of the Nagumo Theorem to differential inclusions on which we will base our viability analysis is given by:

Theorem 6 (`Viability for differential inclusions`)**.** *Consider a non-trivial upper semi-continuous set-valued map F with compact convex images from X to X and a* closed *subset $K \subset Dom(F)$. If*

$$\forall x \in K,\ F(x) \cap T_K(x) \neq \emptyset$$

then for any initial state $x_0 \in K$, there exist a positive T and a solution on $[0,T]$ to differential inclusion (8) *starting from x_0, viable in K, and satisfying*

$$\begin{cases} \textit{either } T = \infty \\ \quad \textit{or } \ T < \infty \textit{ and } \limsup\limits_{\substack{t \to T \\ t < T}} \|x(t)\| = \infty \end{cases}$$

Table 5. Tangent Cones to Winning Cones

Winning cone (K)	**Points in** K	**Tangent cone** $T_K(x)$
K_1	$x \in int(K_1)$	$\mathbf{R}^2$
	$x = (0, 0)$	K_1
	$x_1 = x_2,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq x_2\}$
	$x_2 = 0,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_2 \geq 0\}$
K_2	$x \in int(K_2)$	$\mathbf{R}^2$
	$x = (0, 0)$	K_2
	$x_1 = x_2,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq x_2\}$
	$x_2 = 0,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq -x_2\}$
K_3	$x \in int(K_3)$	$\mathbf{R}^2$
	$x = (0, 0)$	K_3
	$x_1 = x_2,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq x_2\}$
	$x_2 = 0,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq 0\}$
K_4	$x \in int(K_4)$	$\mathbf{R}^2$
	$x = (0, 0)$	K_4
	$x_1 = x_2,\ x_1 > 0$	$\left\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq \frac{x_1(0)}{x_2(0)} x_2\right\}$
	$x_2 = 0,\ x_1 > 0$	$\{(x_1, x_2) \in \mathbf{R}^2 \vert x_1 \geq 0\}$

As in the case of Lanchester type models presented, we have that $\|x(t)\| < \infty$, $t \in (0, +\infty)$, we avoid the case where $T < \infty$.

The *winning cones* that we now consider are those previously introduced in Section 5. For these, Table 5 shows the tangent cones for each $x \in K$ (the results are extracted from [27]).

The problem therefore consists of establishing conditions on the intervals that will guarantee the existence of a viable selection, i.e., a "winnable solution." From the Viability Theorem 6, because the right-hand side of the Lanchester inclusions are upper semi-continuous and have compact convex images, we are left with establishing conditions that yield $F(x) \cap T_k(x)$ to be nonempty. For this to happen, the information from Table 5 translates into the conditions in Table 6 for the right-hand side of the Lanchester inclusions.

With that information, let us have a look at what that means for our models. As before, the first model under consideration is the *Aimed Fire* model.

Aimed Fire

From Table 6 with $A, B \in I(\mathbf{R})$ and using the notation $A = [\underline{A}, \overline{A}]$ for the upper and lower bounds, for K_1 to be viable, we need to have

Table 6. Set-Valued Conditions Based on Tangent Cones at $x \in K$

Winning cone (K)	**Points in K**	**Conditions on $F(x)$**
K_1	$x \in int(K_1)$	$F(X) \subset I(\mathbf{R}^2), F(X) \neq \emptyset$
	$x = (0,0)$	$F(X) \cap K_1 \neq \emptyset$
	$x_1 = x_2,\ x_1 > 0$	$\overline{F_1(X)} \geq \underline{F_2(X)}$
	$x_2 = 0,\ x_1 > 0$	$\overline{F_2(X)} \geq 0$
K_2	$x \in int(K_2)$	$F(X) \subset I(\mathbf{R}^2), F(X) \neq \emptyset$
	$x = (0,0)$	$F(X) \cap K_2 \neq \emptyset$
	$x_1 = x_2,\ x_1 > 0$	$\overline{F_1(X)} \geq \underline{F_2(X)}$
	$x_1 = -x_2,\ x_1 > 0$	$\overline{F_1(X)} \geq \underline{-F_2(X)}$
K_3	$x \in int(K_3)$	$F(X) \subset I(\mathbf{R}^2), F(X) \neq \emptyset$
	$x = (0,0)$	$F(X) \cap K_3 \neq \emptyset$
	$x_1 = x_2,\ x_1 > 0$	$\overline{F_1(X)} \geq \underline{F_2(X)}$
	$x_1 = 0,\ x_2 < 0$	$\overline{F_1(X)} \geq 0$
K_4	$x \in int(K_4)$	$F(X) \subset I(\mathbf{R}^2), F(X) \neq \emptyset$
	$x = (0,0)$	$F(X) \cap K_4 \neq \emptyset$
	$x_1 = \frac{x_1(0)}{x_2(0)} x_2,\ x_1 > 0$	$\overline{F_1(X)} \geq \frac{x_1(0)}{x_2(0)} \underline{F_2(X)}$
	$x_1 = 0,\ x_2 < 0$	$\overline{F_1(X)} \geq 0$

$$
\begin{aligned}
\overline{-Ax_2} &\geq \underline{-Bx_1} \\
\underline{Ax_2} &\leq \overline{Bx_1} \\
\underline{A}x_2 &\leq \overline{B}x_1, && x_1, x_2 \text{ point intervals} \\
\underline{A} &\leq \overline{B}, && x_1 = x_2 > 0.
\end{aligned}
$$

In addition, at $X = (0,0)$ we get $F(X) = (0,0) \in K_1$, and therefore we obtain $F(0,0) \cap T_K(0,0) \neq \emptyset$. As for the lower boundary,

$$
\begin{aligned}
\overline{-Bx_1} &\geq 0 \\
\underline{Bx_1} &\leq 0 \\
\underline{B} &\leq 0, && x_1 > 0
\end{aligned}
$$

which is impossible as we have, from the model, $\underline{B} > 0$. There are therefore no conditions that will allow a viable selection for this inclusion in K_1.

In the case of the viability in K_2, the only variation from the previous is at the lower boundary where we have

$$\begin{aligned}
\overline{-Ax_2} &\geq \underline{-(-Bx_1)} \\
\overline{-Ax_2} &\geq \underline{Bx_1} \\
\overline{Ax_1} &\geq \underline{Bx_1}, & x_1 = -x_2 \\
\overline{A} &\geq \underline{B}, & x_1 > 0
\end{aligned}$$

When we combine together all the conditions, for K_2 to be viable it is required that $A \cap B \neq \emptyset$.

The next winning cone under consideration is K_3 where, at the upper boundary, we have the same conditions as K_1 and, at the lower boundary, we require

$$\begin{aligned}
\overline{-Ax_2} &\geq 0 \\
\underline{Ax_2} &\leq 0 \\
\overline{A} &\geq 0, & x_2 < 0 \\
\overline{A} &\geq \underline{B}, & x_1 > 0
\end{aligned}$$

which does not add any additional constraints on the interval coefficients. Overall, for viability in K_3 it is required that $\underline{A} \leq \overline{B}$.

Finally, for this model, we need to analyze the requirements at the upper boundary of K_4 where

$$\begin{aligned}
\overline{-Ax_2} &\geq \frac{x_1(0)}{x_2(0)} \underline{-Bx_1} \\
\underline{Ax_2} &\leq \frac{x_1(0)}{x_2(0)} \overline{Bx_1} \\
\underline{A}x_2 &\leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 \overline{B}x_2, & x_1 = \frac{x_1(0)}{x_2(0)} x_2 \\
\underline{A} &\leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 \overline{B}, & x_2 > 0
\end{aligned}$$

As no other restrictions are required by the other boundaries, for viability in K_4 it is required that $\underline{A} \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 \overline{B}$.

Area Fire

The next model is the *Area Fire* model. Again, from the table data we require for K_1 at the upper boundary

$$\begin{aligned}
\overline{-Ax_1x_2} &\geq \underline{-Bx_1x_2} \\
\underline{Ax_1x_2} &\leq \overline{Bx_1x_2} \\
\underline{A}x_1x_2 &\leq \overline{B}x_1x_2 \\
\underline{A} &\leq \overline{B}, & x_1 = x_2 > 0
\end{aligned}$$

and, considering the lower boundary

$$\begin{aligned} \overline{-Bx_1x_2} &\geq 0 \\ 0 &\geq 0, & x_2 = 0 \end{aligned}$$

and the overall conditions are then $\underline{A} \leq \overline{B}$ for K_1. In the case of K_2, the conditions, at the lower boundary, are

$$\begin{aligned} \overline{-Ax_1x_2} &\geq \underline{-(-Bx_1x_2)} \\ \overline{Ax_1x_1} &\geq \underline{-Bx_1x_1}, & x_1 = -x_2 \\ \overline{A} &\geq \underline{-B}, & x_1 > 0 \end{aligned}$$

which, given the initial conditions on the interval coefficients, is always true and the overall conditions for K_2 are the same as for K_1. Similarly for K_3, at the lower boundary we need

$$\begin{aligned} \overline{-Ax_1x_2} &\geq 0 \\ 0 &\geq 0, & x_2 = 0 \end{aligned}$$

and the conditions are again identical to that for K_1. The last cone is K_4 where the upper boundary requires

$$\begin{aligned} \overline{-Ax_1x_2} &\geq \frac{x_1(0)}{x_2(0)}\underline{-Bx_1x_2} \\ \underline{Ax_1x_2} &\leq \frac{x_1(0)}{x_2(0)}\overline{Bx_1x_2} \\ \underline{A} &\leq \frac{x_1(0)}{x_2(0)}\overline{B}, & x_1, x_2 > 0 \end{aligned}$$

and as the other boundaries do not add any other conditions, viability in K_4 hinges on $\underline{A} \leq \frac{x_1(0)}{x_2(0)}\overline{B}$.

Brackney

Beginning with the upper boundary of K_1

$$\begin{aligned} \overline{-Ax_2} &\geq \underline{-Bx_1x_2} \\ \underline{Ax_2} &\leq \overline{Bx_1x_2} \\ \underline{A} &\leq \overline{B}x_1, & x_2 > 0 \\ \underline{A} &\leq \overline{B}, & \text{modified Brackney} \end{aligned}$$

while at its lower boundary

$$\begin{aligned} \overline{-Bx_1x_2} &\geq 0 \\ 0 &\geq 0, & x_2 = 0 \end{aligned}$$

which means that the only condition on the interval coefficients for K_1 is $\underline{A} \leq \overline{B}$. Investigating the lower boundary of K_2 as it is the only difference with the above,

$$\begin{aligned}
\overline{-Ax_2} &\geq \underline{-(-Bx_1x_2)} & \\
\underline{Ax_2} &\leq \overline{-Bx_1x_2} & \\
\underline{Ax_1} &\leq \overline{B}x_1x_1, & x_1 = -x_2 \\
\underline{-A} &\leq \overline{B}, & x_1 > 0 \\
-\overline{A} &\leq \overline{B}, & \text{modified Brackney}
\end{aligned}$$

which is always true as $\overline{A}, \overline{B} > 0$. As a result, the cumulative conditions for K_2 are $\underline{A} \leq \overline{B}$. At the lower boundary of K_3

$$\begin{aligned}
\overline{-Ax_2} &\geq 0 & \\
\underline{Ax_2} &\leq 0 & \\
\underline{A}x_2 &\leq 0 & \\
\overline{A} &\geq 0, & x_2 < 0
\end{aligned}$$

which does not further restrict the interval coefficients so, for K_3, it is required that $\underline{A} \leq \overline{B}$. Considering the fourth winning cone's upper boundary

$$\begin{aligned}
\overline{-Ax_2} &\geq \frac{x_1(0)}{x_2(0)}\underline{-Bx_1x_2} & \\
\underline{Ax_2} &\leq \frac{x_1(0)}{x_2(0)}\overline{Bx_1x_2} & \\
\underline{A} &\leq \frac{x_1(0)}{x_2(0)}\overline{B}x_1, & x_2 > 0 \\
\underline{A} &\leq \frac{x_1(0)}{x_2(0)}\overline{B}, & \text{modified Brackney}
\end{aligned}$$

and as the other boundaries do not impose further restrictions (see K_3 above), we need to have $\underline{A} \leq \frac{x_1(0)}{x_2(0)}\overline{B}$ for viability with K_4.

Peterson

We now examine the Peterson differential inclusion model, and, as with all other models, we begin with the upper boundary of K_1:

$$\begin{aligned}
\overline{-Ax_1} &\geq \underline{-Bx_2} & \\
\underline{Ax_1} &\leq \overline{Bx_2} & \\
\underline{A} &\leq \overline{B}, & x_1 = x_2 > 0
\end{aligned}$$

and continue with the lower boundary

$$\begin{aligned} \overline{-Bx_2} &\geq 0 \\ 0 &\geq 0, & \qquad x_2 = 0 \end{aligned}$$

giving the overall condition for K_1 to be $\underline{A} \leq \overline{B}$. The condition for K_2 is also the same as the lower boundary generates no additional restrictions:

$$\begin{aligned} \overline{-Ax_1} &\geq \underline{-(-Bx_2)} \\ \overline{-Ax_1} &\leq \underline{-Bx_1}, & \qquad x_1 = -x_2 \\ \underline{Ax_1} &\leq \overline{Bx_1} \\ \underline{A} &\leq \overline{B}, & \qquad x_1 > 0. \end{aligned}$$

The following winning cone, K_3, requires also the same conditions as the previous two as can be seen by the examination of the requirements at its lower boundary:

$$\begin{aligned} \overline{-Ax_1} &\geq 0 \\ 0 &\geq 0, & \qquad x_1 = 0. \end{aligned}$$

Finally, at the upper boundary of K_4 we need

$$\begin{aligned} \overline{-Ax_1} &\geq \frac{x_1(0)}{x_2(0)}\underline{-Bx_2} \\ \underline{Ax_1} &\leq \overline{Bx_1}, & \qquad x_1 = \frac{x_1(0)}{x_2(0)}x_2 \\ \underline{A} &\leq \overline{B}, & \qquad x_1 > 0 \end{aligned}$$

which, once again, provides us with $\underline{A} \leq \overline{B}$ as the only condition on the interval coefficients for this winning cone.

Hembold

The second to last model under consideration is based on Hembold's model. In the case where the winning cone is K_1 and analyzing the requirements at the upper boundary, we get

$$\begin{aligned} \overline{-Ag(\frac{x_1}{x_2})x_2} &\geq \underline{-Bh(\frac{x_2}{x_1})x_1} \\ \underline{Ag(1)x_1} &\leq \overline{Bh(1)x_1}, & \qquad x_1 = x_2 \\ \underline{A} &\leq \overline{B}, & \qquad x_1 > 0,\ g(1) = h(1) = 1 \end{aligned}$$

and at the lower boundary

$$\overline{-Bh(\frac{x_2}{x_1})x_1} \geq 0$$
$$0 \geq 0, \qquad h(0) = 0$$

and the overall conditions for this cone are $\underline{A} \leq \overline{B}$. Turning our attention to K_2 at the lower boundary

$$\begin{aligned} \overline{-Ag(\frac{x_1}{x_2})x_2} &\geq \underline{-(-Bh(\frac{x_2}{x_1})x_1)} \\ \overline{Ag(\frac{x_1}{x_2})x_1} &\geq \underline{Bh(\frac{x_2}{x_1})x_1}, && x_1 = -x_2 \\ \overline{Ag(-1)x_1} &\geq \overline{Bh(-1)x_1} \\ \overline{A} &\geq \underline{B}, && g(-1) = h(-1) > 0,\ x_1 > 0 \end{aligned}$$

which means that for viability with K_2 for the Hembold model, it is required that $A \cap B \neq \emptyset$. The lower boundary of K_3 does not impose restrictions as

$$\overline{-Ag(\frac{x_1}{x_2})x_1} \geq 0$$
$$0 \geq 0, \qquad g(\frac{x_1}{x_2}) = 0$$

Finally for this model, the upper boundary of K_4 dictates that

$$\begin{aligned} \overline{-Ag(\frac{x_1}{x_2})x_2} &\geq \frac{x_1(0)}{x_2(0)} \underline{-Bh(\frac{x_2}{x_1})x_1} \\ \underline{Ag(\frac{x_1(0)}{x_2(0)})x_1} &\leq \frac{x_1(0)}{x_2(0)} \overline{Bh(\frac{x_2(0)}{x_1(0)})x_1}, && x_1 = \frac{x_1(0)}{x_2(0)}x_2 \\ \underline{A}g(\frac{x_1(0)}{x_2(0)}) &\leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 h(\frac{x_2(0)}{x_1(0)})\overline{B}, && x_1 > 0,\ g(1) = h(1) = 1 \end{aligned}$$

and are the only conditions on the coefficients using K_4 as the winning cone.

Schreiber

The last but not the least of our models. At the upper boundary of the winning cone K_1, we observe that we must have

$$\overline{-A\left\{\frac{x_1 x_2}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}} \geq \underline{-B\left\{\frac{x_1 x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\}}$$

$$\underline{A}\left\{\frac{1}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\} \leq \overline{B}\left\{\frac{1}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\}, \qquad x_1 = x_2 > 0$$

$$\underline{A} \leq \frac{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\overline{B}$$

At the lower boundary, perform a quick inspection while considering the fact that $x_2 = 0$ yields no additional conditions. Proceeding to the case where the winning cone is K_2, we examine the lower boundary to obtain

$$\overline{-A\left\{\frac{x_1x_2}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}} \geq \underline{B\left\{\frac{x_1x_2}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\}}$$

$$\overline{A\left\{\frac{x_1x_1}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\}} \geq \underline{-B\left\{\frac{x_1x_1}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\}}, \qquad x_1 = -x_2$$

$$\overline{A}\left\{\frac{1}{x_{1,0} - e_{x_2}(x_{1,0} - x_1)}\right\} \geq \underline{-B}\left\{\frac{1}{x_{2,0} - e_{x_1}(x_{2,0} - x_2)}\right\}, \quad x_1 = x_2 > 0$$

and, as the right-hand side is negative, it is always true. Putting it all together for K_2, the interval coefficients must be such that $\underline{A} \leq \frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\overline{B}$. Once more, a quick inspection of the lower boundary of K_3 while noting that $x_1 = 0$ shows that viability for this third cone has the same condition as the other two. It is also readily obvious that the condition for our final winning cone K_4 becomes $\underline{A} \leq \frac{x_1(0)}{x_2(0)}\frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\overline{B}$.

Compilation of Data

For each model, we have determined the conditions on the interval coefficients that make the respective winning cones viable. This type of analysis provides interesting information that can not only be valuable to the commander in the field but also can provide support in weapons design decision or means by which to study history, among other things. For easy reference, Table 7 presents all the results in a single location.

Table 7. Compilation of Conditions on Interval Coefficients

Model	K_1	K_2	K_3	K_4
Aimed Fire	None	$A \cap B \neq \emptyset$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 \overline{B}$
Area Fire	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \frac{x_1(0)}{x_2(0)}\overline{B}$
Brackney (Modified)	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \frac{x_1(0)}{x_2(0)}\overline{B}$
Peterson	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$	$\underline{A} \leq \overline{B}$
Hembold	$\underline{A} \leq \overline{B}$	$A \cap B \neq \emptyset$	$\underline{A} \leq \overline{B}$	$\underline{A}g(\frac{x_1(0)}{x_2(0)}) \leq \left(\frac{x_1(0)}{x_2(0)}\right)^2 h(\frac{x_2(0)}{x_1(0)})\overline{B}$
Schreiber	$\underline{A} \leq \frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\overline{B}$	see K_1	see K_1	$\underline{A} \leq \frac{x_1(0)}{x_2(0)} \cdot \frac{x_{1,0}-e_{x_2}(x_{1,0}-x_1)}{x_{2,0}-e_{x_1}(x_{2,0}-x_2)}\overline{B}$

9 Optimal Control by Viability of Lanchester Type Inclusions

In the work carried out so far, we have first analyzed the differential systems with the use of viability to give us a more reasonable approach given the uncertain nature of the Lanchester coefficients that come in play. At that point, we considered a form of *optimal control by viability*, Section 6.5, and saw how this idea can be used to provide analysis and decision-making tools. In the latter part, we have replaced the Lanchester coefficients with interval coefficients to express both the difficulty of giving them an accurate value as well as their natural sporadic variability. Analysis of these new models gave a new set of conditions on intervals to guarantee viability.

Although the replacement of these coefficients with intervals allows us to model the intrinsic variability of the Lanchester coefficients, we can still use different Lanchester intervals to reflect the current state of the troops involved. For instance, the relative combat effectiveness of rested troops over fatigued ones can be represented by different interval values. Furthermore, the different effectiveness of various weapons or types of forces can also be modelled through the various interval values. If there was only one battle to be fought, this last issue wouldn't be a consideration, however, having to fight on multiple fronts does not allow a commander to commit all forces at all areas at all times.

Given the above, this means that the interval coefficients that model such characteristics can be made to vary. This ability to impact on these coefficients represents a form of control and, as before, we can consider *optimal control by viability*. Once again, using the information generated in the previous section, we can establish the first time t when the conditions required for viability will first cease to be met and decide on the application of the tactical tools available that impact on the Lanchester intervals. These tactical options include such things as the commitment of fresh troops to the engagement or different weapons/forces on a particular front.

For example, let us consider the *Aimed Fire* model where the condition on viability is that $\underline{A} \leq \overline{B}$. This condition is one that needs to be met when the system reaches the boundary of the winning cone. As such, knowing that the current forces involved are such that the condition is not met, the commander can then select a tactical plan that will rectify this situation and, by monitoring the evolution of the system, apply this change in controls before "winnability" is lost.

By its discrete nature, this process of optimal control by viability is an excellent tool to assist in the decision-making process and can provide a means of assessing the various impacts of command decisions both on exercises and in history.

10 Conclusion

In this publication, we have seen how nonclassic analysis in the form of viability, interval analysis, set-valued analysis, differential inclusions, and optimal control can shed new light on an existing model. In the case of Lanchester type models, this transformation to Lanchester type differential inclusions allows them to overcome the difficulties inherent to the coefficients be it by their nature or from the "fog of war." These new models allow for variations that closer model reality and provide useful tools for the analysis and decision making processes in the form of *optimal control by viability.*

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Statics and Dynamics of Global Supply Chain Networks

Anna Nagurney,[1] Jose Cruz,[2] and Fuminori Toyasaki[3]

[1] Department of Finance and Operations Management, Isenberg School of Management, Amherst, Massachusetts, USA
nagurney@gbfin.umass.edu
[2] Department of Operations and Information Management, School of Business, University of Connecticut, Storrs, Connecticut, USA
jcruz@business.uconn.edu
[3] Faculty of Management, McGill University, Montreal, Quebec, Canada
fuminori.toyasaki@mcgill.ca

Abstract In this paper, we develop static and dynamic models of global supply chains as networks with three tiers of decision-makers: manufacturers, retailers, and consumers associated with the demand markets who compete within a tier but co-operate between tiers. The decision-makers may be based in the same or in different countries, may transact in different currencies, and are faced with different degrees of environmental concern. Moreover, we allow for electronic transactions in the form of electronic commerce between the decision-makers. The proposed *supernetwork* framework formalizes the modeling and theoretical analysis of such global supply chains and also enables the dynamic tracking of the evolution of the associated prices and product transactions (as well as the emissions) to the equilibrium state. Moreover, it measures the impacts on the environment associated with the behaviors of the decision-makers. Finally, we propose a discrete-time algorithm that allows for the discretization of the continuous time trajectories and that results in closed form expressions at each iteration.

Key words: global supply chains, networks, supernetworks, game theory, variational inequalities, projected dynamical systems, environmental management, multicriteria decision-making

1 Introduction

Growing competition and emphasis on efficiency and cost reduction, as well as the satisfaction of consumer demands, have brought new challenges for businesses in the global marketplace. At the same time that businesses and, in particular, supply chains have become increasingly globalized, criticism of globalization has increased, notably, from environmentalists on the basis that free trade may result in the growth of global pollution. In particular, some argue that free trade increases the scale of economic activity and, therefore, of accompanying pollution, and also that it may shift the production of the pollution-intensive goods from countries with strict environmental regulations toward those with lax ones. Others argue that environmental, health, and safety regulations are a form of protectionism. For example, countries may use a laborious and time-consuming regulatory process that is unevenly applied to international investors as a means of controlling access to domestic markets.

Indeed, the increase in environmental concerns is significantly influencing supply chains. Legal requirements and changing consumer preferences increasingly make suppliers, manufacturers, and distributors responsible for their products beyond their sales and delivery locations (cf. [7]). For example, recent legislation in the United States as well as abroad and, in particular, in Europe and in Japan, has refocused attention on recycling for the management of wastes and, specifically, that of electronic wastes (see, e.g., [2] and [32]). Massachusetts in 2000 banned cathode-ray tubes (CRTs) from landfills whereas Japan in 2001 enacted a law that requires retailers and manufacturers to bear some electronic waste collection and recycling cost of appliances (cf. [3, 4] and [33]). In addition, environmental pressure from consumers has, in part, affected the behavior of certain manufacturers so that they attempt to minimize their emissions, produce more environmentally friendly products, and/or establish sound recycling network systems (see, e.g., [7, 18], and [20]).

Moreover, according to [15], companies are being held accountable not only for their own performance but also for that of their suppliers, subcontractors, joint venture partners, distribution outlets, and even, ultimately, for the disposal of their products. Indeed, poor environmental performance at any stage of the supply chain process may damage a company's most important asset – its reputation.

On the other hand, innovations in technology and especially the availability of electronic commerce via the Internet in which the physical ordering of goods (and supplies) (and, in some cases, even delivery) is replaced by electronic orders, offers the potential for reducing risks associated with physical transportation due to potential threats and disruptions in supply chains as well as the possible reduction of pollution. Indeed, the introduction of electronic commerce (e-commerce) has unveiled new opportunities for the management of supply chain networks (cf. [28] and the references therein) and has had an immense effect on the manner in which businesses order goods and have them transported. According to [25], gains from electronic commerce could reach $450 billion a year by 2005, with consumer e-commerce in the United States alone expected to come close to the $108 billion predicted, despite a recession, terrorism, and war.

Many researchers have recently dealt with environmental risks in response to growing environmental concerns (see [5, 8, 39], and [38]). Furthermore, the importance of global issues in supply chain management and analysis has been emphasized in several papers (cf. [11, 24, 27]). Moreover, earlier surveys on global supply chain analysis indicate that the research interest is growing rapidly (see [10, 14], and [15]). The need to incorporate risk in supply chain decision-making and analysis is well-documented in the literature (see, e.g., [1, 15, 21, 40], and [43]). Nevertheless, the topic of supply chain modeling and analysis combined with environmental decision-making is fairly new and novel and, hence, methodological approaches that capture the operational as well as the financial aspects of such decision-making are sorely needed.

Frameworks for risk management in a global supply chain context with a focus on centralized decision-making and optimization have been proposed by [11, 19] (see also the references therein) and [15]. In this paper, in contrast, we build upon the recent work of [27] in the modeling of global supply chain networks with electronic commerce and that of [32] who introduced environmental criteria into a decentralized supply chain network.

In particular, in this paper, we develop both static and dynamic global supply chain network models with environmental decision-making handled as a multicriteria decision-making problem. In addition, we build upon our tradition of a network perspective to environmental management as described in the book on environmental networks by [12].

The paper is organized as follows. In Section 2, we present the static global supply chain network model with environmental decision-making, derive the optimality conditions for each set of network agents or decision-makers, and provide the finite-dimensional variational inequality formulation of the governing equilibrium conditions. In Section 3, we propose the projected dynamical system that describes the dynamic adjustment processes associated with the various decision-makers and demonstrate that the set of stationary points of this nonclassic dynamical system coincides with the set of solutions of the variational inequality problem (cf. [34] and [26]).

In Section 4, we provide qualitative properties of the equilibrium pattern and also provide, under suitable assumptions, existence and uniqueness results for the dynamic price and product transaction trajectories, from which the total emissions generated can also be obtained. In Section 5, we outline the computational procedure, which provides a time-discretization of the dynamic trajectories. We conclude the paper with a summary and suggestions for future research in Section 6.

2 The Global Supply Chain Network Equilibrium Model with Environmental Decision-Making

In this section, we develop the global supply chain network model and focus on the statics surrounding the equilibrium state. The model assumes that the manufacturers are involved in the production of a homogeneous product and considers L countries, with I manufacturers in each country, and J retailers, which are not country-specific but, rather, can be either physical or virtual, as in the case of electronic commerce. There are K demand markets for the homogeneous product in each country and H currencies in the global economy. We denote a typical country by l or $\hat{l}$, a typical manufacturer by i, and a typical retailer by j. A typical demand market, on the other hand, is denoted by k and a typical currency by h. We assume, for the sake of generality, that each manufacturer can transact directly in an electronic manner via the Internet with the consumers at the demand markets and can also conduct transactions with the retailers either physically or electronically in different currencies. Similarly, we assume that the demand for the product in a country can be associated with a particular currency. We let m refer to a mode of transaction with $m = 1$ denoting a physical transaction and $m = 2$ denoting an electronic transaction via the Internet. In addition, for the sake of flexibility, we assume that the consumers associated with the demand markets can transact with the retailers either physically or electronically. Of course, if either such a transaction is not feasible, then one may simply remove that possibility (or, analogously, assign a high associated transaction cost as described below) within the specific application.

The global supply chain *supernetwork* is now described and depicted graphically in Figure 1 (for other supernetwork structures that capture decision-making trade-offs regarding transportation versus telecommunication networks, see the book by [28]). The top tier of nodes consists of the manufacturers in the different countries, with manufacturer i in country l being referred to as manufacturer il and associated with node il. There are, hence, IL top-tiered nodes in the network. The middle tier of nodes consists of the retailers (which need not be country-specific) and who act as intermediaries

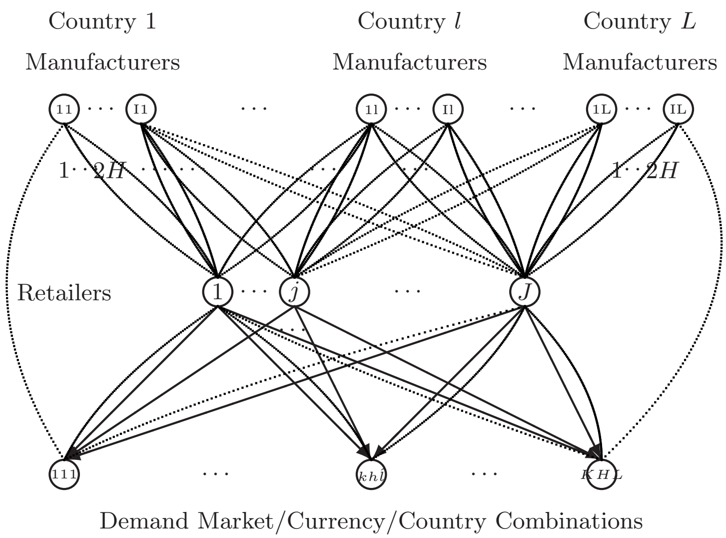

Figure 1. The structure of the global supply chain supernetwork

between the manufacturers and the demand markets, with a typical retailer j associated with node j in this (second) tier of nodes in the network. The bottom tier of nodes consists of the demand markets, with a typical demand market k in currency h and country $\hat{l}$ being associated with node $kh\hat{l}$ in the bottom tier of nodes. There are, as depicted in Figure 1, J middle (or second) tiered nodes corresponding with the retailers and KHL bottom (or third) tiered nodes in the global supply chain network.

We have identified the nodes in the global supply chain supernetwork and now we turn to the identification of the links joining the nodes in a given tier with those in the subsequent tier. We also associate the product transactions with the appropriate links that correspond with the flows on the links. We assume that each manufacturer i in country l can transact with a given retailer in either of the two modes and in any of the H available currencies, as represented, respectively, by the $2H$ links joining each top tier node with each middle tier node j; $j = 1, \dots, J$. The flow on the link joining node il with node j and corresponding with transacting via mode m is denoted by q^{il}_{jhm} and represents the nonnegative amount of the product transacted by manufacturer i in country l in currency h through retailer j via mode m. We further group all such transactions for all manufacturers in all countries into the column vector $Q^1 \in R^{2ILJH}_+$.

A manufacturer may also transact directly with the demand markets via the Internet. The flow on the link joining node il with node $kh\hat{l}$ is denoted by $q^{il}_{kh\hat{l}}$ and represents the amount of the product transacted in this manner between the manufacturer and demand market in a given country and currency. We group all such (electronic) transactions for all the manufacturers in all the countries into the column vector $Q^3 \in R^{ILKHL}_+$. For flexibility, we also group the product transactions associated with manufacturer i in country l into the column vector $q^{il} \in R^{2JH+KHL}_+$, and group these vectors for all manufacturers and countries into the vector $q \in R^{IL(2JH+KHL)}_+$.

From each retailer node j; $j = 1, \dots, J$, we then construct two links to each node khl, with the first such link denoting a physical transaction and the second such link an electronic transaction and with the respective flow on the link being denoted by q^j_{khlm} and corresponding with the amount of the product transacted between retailer j and demand market k in currency h and country l via mode m. The product transactions for all the retailers are then grouped into the column vector $Q^2 \in R^{2JKHL}_+$. Note that if a retailer is virtual, then we expect the transaction to take place electronically, although of course, the product itself may be delivered physically. Nevertheless, for the sake of generality, we allow for two modes of transaction between each manufacturer and retailer pair and each retailer demand market pair.

The notation for the prices is now given. Note that there will be prices associated with each of the tiers of nodes in the global supply chain supernetwork. Let ρ^{il}_{1jhm} denote the price associated with the product in currency h transacted between manufacturer il and retailer j via mode m and group these top tier prices into the column vector $\rho_1 \in R_+^{2ILJH}$. Let $\rho_{1kh\hat{l}}^{il}$, in turn, denote the price associated with manufacturer il and demand market k in currency h and country $\hat{l}$ and group all such prices into the column vector $\rho_{12} \in R_+^{ILKHL}$. Further, let ρ_{2khlm}^{j}, in turn, denote the price associated with retailer j and demand market k in currency h, country l, and mode m, and group all such prices into the column vector $\rho_2 \in R_+^{2JKHL}$. Also, let $\rho_{3kh\hat{l}}$ denote the price of the product at demand market k in currency h, and country $\hat{l}$, and group all such prices into the column vector $\rho_3 \in R_+^{KHL}$. Finally, we introduce the currency exchange rates: e_h; $h = 1, \ldots, H$, which are the exchange rates of respective currency h relative to the base currency. The exchange rates are exogenous and fixed in the model, whereas all the prices are endogenous.

We now turn to describing the behavior of the various global supply chain network decision-makers represented by the three tiers of nodes in Figure 1. The model is presented, for ease of exposition, for the case of a single homogeneous product. It can also handle multiple products through a replication of the links and added notation. We first focus on the manufacturers. We then turn to the retailers, and, finally, to the consumers at the demand markets.

2.1 The Behavior of the Manufacturers

We denote the transaction cost associated with manufacturer il transacting with retailer j for the product in currency h via mode m by c_{jhm}^{il} and assume that:

$$c_{jhm}^{il} = c_{jhm}^{il}(q_{jhm}^{il}), \quad \forall i, l, j, h, m, \tag{1}$$

that is, this cost can depend upon the volume of this transaction. In addition, we denote the transaction cost associated with manufacturer il transacting with demand market k in country $\hat{l}$ for the product in currency h (via the Internet) by $c_{kh\hat{l}}^{il}$ and assume that:

$$c_{kh\hat{l}}^{il}(q_{kh\hat{l}}^{il}), \quad \forall i, l, k, h, \hat{l}, \tag{2}$$

that is, this transaction cost also depends upon the volume of the transaction. These transaction cost functions are assumed to be convex and continuously differentiable. The transaction costs are assumed to be measured in the base currency. We note that in practice, transaction costs are not convex functions of the amount traded. Indeed, the costs for either buying or selling are likely to be concave. For example, a fixed charge for any nonzero trade is common, and there may be one or more breakpoints above which the transaction costs per share decrease. We consider a simple model that includes linear costs functions. By assuming that the cost functions are convex, it is crucial for determining the existence of equilibrium solution, which is presented in Section 5.

The total transaction costs incurred by manufacturer il are equal to the sum of all of his transaction costs associated with dealing with the distinct retailers and demand markets in the different currencies and countries. His revenue, in turn, is equal to the sum of the price (rate of return plus the rate of appreciation) that the manufacturer can obtain for the product times the total quantity sold of that product. Let now ρ^{il*}_{1jhm} denote the actual price charged by manufacturer il for the product transacted via mode m in currency h to retailer j (and that the retailer is willing to pay) and let $\rho^{il*}_{1kh\hat{l}}$, in turn, denote the price associated with manufacturer il transacting electronically with demand market $kh\hat{l}$. We later discuss how such prices are recovered.

We assume that each manufacturer seeks to maximize his profits. Also, we assume that the amount of the product produced by manufacturer il and denoted by q^{il} must be equal to the amount transacted with the subsequent tiers of nodes, that is,

$$\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} q^{il}_{jhm} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} q^{il}_{kh\hat{l}} = q^{il}, \quad \forall i, l. \tag{3}$$

In addition, we assume, as given, a production cost function for manufacturer il and denoted by f^{il}, which depends not only on the manufacturer's output (and transactions) but also on those of the other manufacturers. Hence, we may write (utilizing also (3)) that

$$f^{il} = f^{il}(q) = f^{il}(Q^1, Q^3), \quad \forall i, l. \tag{4}$$

Recall that the vector Q^1 represents all the product transactions between the top tier nodes and the middle tier nodes, and the vector Q^3 represents all the product transactions between the top tier nodes and the bottom tier nodes. The function f^{il} is assumed to be strictly convex and continuously differentiable.

We note that in economics, the cost function is concave when the production level is low and due to economy of scale. However, as the production level increases and exceeds the regular capacity, the cost function becomes convex. Here, we can assume that all the fixed costs have been paid, and so, can be treated as sunk cost. The cost function at low production level can be assumed linear. Thus, if the cost function is assumed linear at low production level, it is reasonable to assume that the production cost function is strictly convex.

We now construct the profit maximization problem facing manufacturer il. In particular, we can express the profit maximization problem facing manufacturer il as:

$$\begin{aligned}\max \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm} \times e_h)q^{il}_{jhm} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho^{il*}_{1kh\hat{l}} \times e_h)q^{il}_{kh\hat{l}} \\ - \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} c^{il}_{jhm}(q^{il}_{jhm}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c^{il}_{kh\hat{l}}(q^{il}_{kh\hat{l}}) - f^{il}(Q^1, Q^3),\end{aligned} \tag{5}$$

$$\text{s.t} \quad q^{il}_{jhm} \geq 0, \quad q^{il}_{kh\hat{l}} \geq 0, \quad \forall j, h, k, \hat{l}, m. \tag{6}$$

The first two terms in (5) represent the revenues whereas the subsequent three terms represent the costs faced by the manufacturer.

In addition to the criterion of profit maximization, we assume that each manufacturer is also concerned with environmental decision-making with such decision-making broadly defined as including the risks associated with his transactions. First, we consider the situation that a given manufacturer seeks to minimize the total amount of emissions associated with his production of the product as well as the total amount of emissions generated not only in the ultimate delivery of the product to the next tier of decision-makers (whether retailers or consumers at the demand markets). We assume that the emissions generated by manufacturer il in producing the product are given by the function ϵ^{il}, where

$$\epsilon^{il} = \epsilon^{il}(q^{il}), \quad \forall i, l, \tag{7}$$

whereas the emissions generated in transacting with retailer j for the product via mode m (which are currency independent) are given by a function ϵ^{il}_{jm}, such that

$$\epsilon^{il}_{jm} = \epsilon^{il}_{jm}(\sum_{h=1}^{H} q^{il}_{jhm}), \quad \forall i, l, j, m, \tag{8}$$

and, finally, the emissions generated and associated with the transaction with demand market k in country $\hat{l}$ is represented by a function $\epsilon^{il}_{k\hat{l}}$, where

$$\epsilon^{il}_{k\hat{l}} = \epsilon^{il}_{k\hat{l}}(\sum_{h=1}^{H} q^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}. \tag{9}$$

Note that (9) also does not depend on the currency used for the transaction. Indeed, emissions should not be currency-dependent but, rather, mode-dependent as well as dependent upon the nodes involved in the transaction.

Hence, the second criterion of each manufacturer il and reflecting the minimization of total emissions generated can be expressed mathematically as:

$$\min \epsilon^{il}(q^{il}) + \sum_{j=1}^{J} \sum_{m=1}^{2} \epsilon^{il}_{jm}(\sum_{h=1}^{H} q^{il}_{jhm}) + \sum_{k=1}^{K} \sum_{\hat{l}=1}^{L} \epsilon^{il}_{k\hat{l}}(\sum_{h=1}^{H} q^{il}_{kh\hat{l}}) \tag{10}$$

$$\text{s.t. } q^{il}_{jhm} \geq 0, \quad q^{il}_{kh\hat{l}} \geq 0, \quad \forall j, h, m, k, \hat{l}. \tag{11}$$

From this point on, we consider emission functions of specific form (cf. (7), (8), and (9)) given by

$$\epsilon^{il}(q^{il}) = \eta^{il} q^{il} = \eta^{il}(\sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{m=1}^{2} q^{il}_{jhm} + \sum_{k=1}^{K} \sum_{h=1}^{J} \sum_{\hat{l}=1}^{L} q^{il}_{kh\hat{l}}), \quad \forall i, l, \tag{12}$$

$$\epsilon^{il}_{jm}(\sum_{h=1}^{H} q^{il}_{jhm}) = \eta^{il}_{jm} \sum_{h=1}^{H} q^{il}_{jhm}, \quad \forall i, l, j, m, \tag{13}$$

$$\epsilon^{il}_{k\hat{l}}(\sum_{h=1}^{H} q^{il}_{kh\hat{l}}) = \eta^{il}_{k\hat{l}} \sum_{h=1}^{H} q^{il}_{kh\hat{l}}, \quad \forall i, l, k, \hat{l}, \tag{14}$$

where the η^{il}, η^{il}_{jm}, and $\eta^{il}_{k\hat{l}}$ terms are nonnegative and represent the amount of emissions generated per unit of product produced and transacted, respectively. Hence, here we explicitly allow the emissions generated to be distinct according to whether the transaction was conducted electronically or not. Thus, the manufacturer's decision-making problem concerning the emissions generated, in view of (3), (10), and (12)–(14), can be expressed as:

$$\begin{aligned} &\min \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\eta^{il}+\eta^{il}_{jm})q^{il}_{jhm} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\eta^{il}+\eta^{il}_{k\hat{l}})q^{il}_{kh\hat{l}}, \\ &\text{s.t. } (11). \end{aligned} \tag{15}$$

We also assume that each manufacturer is concerned with risk minimization and, as noted earlier, here we assume that the risk can also capture environmental risk, with such risk being interpreted broadly. Hence, for the sake of generality, we assume, as given, a risk function r^{il}, for manufacturer il, which is assumed to be continuous and convex, and a function of not only the product transactions associated with the particular manufacturer but also of those of the other manufacturers. Thus, we assume that

$$r^{il} = r^{il}(Q^1, Q^3), \quad \forall i, l. \tag{16}$$

The third criterion of manufacturer il can be expressed as:

$$\begin{aligned} &\min r^{il}(Q^1, Q^3), \\ &\text{s.t. } q^{il}_{jhm} \geq 0, \quad \forall j, h, m \\ &\qquad q^{il}_{kh\hat{l}} \geq 0, \quad \forall k, h, \hat{l}. \end{aligned} \tag{17}$$

The risk function may be distinct for each manufacturer/country combination and can assume whatever form is necessary, provided the above stated assumptions are satisfied.

2.1.1 A Manufacturer's Multicriteria Decision-Making Problem

We now discuss how to construct a value function associated with the criteria. In particular, we assume that manufacturer il assigns nonnegative weights as follows: the weight α^{il} is associated with the emission criterion (15), the weight ω^{il} is associated with the risk criterion (17), with the weight associated with profit maximization (cf. (5)) serving as the numeraire and being set equal to 1. Thus, we can construct a value function for each manufacturer (cf. [9, 16, 41], and [22]) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for manufacturer il is transformed into:

$$\max \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho_{1jhm}^{il*} \times e_h)q_{jhm}^{il} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho_{1kh\hat{l}}^{il*} \times e_h)q_{kh\hat{l}}^{il}$$
$$-\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} c_{jhm}^{il}(q_{jhm}^{il}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c_{kh\hat{l}}^{il}(q_{kh\hat{l}}^{il}) - f^{il}(Q^1, Q^3)$$
$$-\alpha^{il}(\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\eta^{il}+\eta_{jm}^{il})q_{jhm}^{il} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\eta^{il}+\eta_{k\hat{l}}^{il})q_{kh\hat{l}}^{il})$$
$$-\omega^{il}r^{il}(Q^1, Q^3), \tag{18}$$

s.t. the nonnegativity assumption on all the variables.

2.1.2 The Optimality Conditions of the Manufacturers

We assume that the manufacturers compete in a noncooperative fashion following [36, 37]. Hence, each manufacturer seeks to determine his optimal strategies, that is, the product transactions, given those of the other manufacturers. The optimality conditions of all manufacturers i; $i = 1, \ldots, I$; in all countries: l; $l = 1, \ldots, L$, simultaneously, under the above assumptions (see also [6, 17, 26, 29]), can be compactly expressed as a variational inequality problem given by: determine $(Q^{1*}, Q^{3*}) \in \mathcal{K}^1$, satisfying

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial f^{il}(Q^{1*}, Q^{3*})}{\partial q_{jhm}^{il}} + \frac{\partial c_{jhm}^{il}(q_{jhm}^{il*})}{\partial q_{jhm}^{il}} + \omega^{il}\frac{\partial r^{il}(Q^{1*}, Q^{3*})}{\partial q_{jhm}^{il}}\right.$$
$$\left. + \ \alpha^{il}(\eta^{il}+\eta_{jm}^{il}) - \rho_{1jhm}^{il*} \times e_h\right] \times [q_{jhm}^{il} - q_{jhm}^{il*}]$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*}, Q^{3*})}{\partial q_{kh\hat{l}}^{il}} + \frac{\partial c_{kh\hat{l}}^{il}(q_{kh\hat{l}}^{il*})}{\partial q_{kh\hat{l}}^{il}} + \omega^{il}\frac{\partial r^{il}(Q^{1*}, Q^{3*})}{\partial q_{kh\hat{l}}^{il}}\right.$$
$$\left. + \ \alpha^{il}(\eta^{il}+\eta_{k\hat{l}}^{il}) - \rho_{1kh\hat{l}}^{il*} \times e_h\right] \times [q_{kh\hat{l}}^{il} - q_{kh\hat{l}}^{il*}] \geq 0, \quad \forall (Q^1, Q^3) \in \mathcal{K}^1, \tag{19}$$

where the feasible set $\mathcal{K}^1 \equiv \{(Q^1, Q^3) | (Q^1, Q^3) \in R_+^{IL(2JH+KHL)}\}$.

The inequality (19), which is a variational inequality (cf. [26]), has a nice economic interpretation. In particular, from the first term we can infer that, if there is a positive shipment of the product transacted either in a classic manner or via the Internet from a manufacturer to a retailer, then the sum of the marginal cost of production, the marginal cost of transacting, the weighted marginal risk, and what can be interpreted as the marginal cost of emissions, $\alpha^{il}(\eta^{il} + \eta_{jm}^{il})$, must be equal to the price that the retailer is willing to pay for the product. If that sum, in turn, exceeds the price, then there will be zero volume of flow of the product thus transacted. The second term in (19) has a similar interpretation; in particular, there will be a positive volume of flow of the product from a manufacturer to a demand market if the sum of the marginal cost of production of the manufacturer, the marginal cost of transacting via the Internet for the manufacturer with the consumers, the weighted marginal risk, and the marginal cost of emissions, $\alpha^{il}(\eta^{il} + \eta^{il}_{k\hat{l}})$, is equal to the price the consumers are willing to pay for the product at the demand market.

2.2 The Behavior of the Retailers

The retailers (cf. Figure 1), in turn, are involved in transactions both with the manufacturers in the different countries, as well as with the ultimate consumers associated with the demand markets and represented by the bottom tier of nodes in the network.

A retailer j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of handling and storing the product plus the cost associated with transacting in the different currencies. We denote such a cost faced by retailer j by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} q^{il}_{jhm}$, that is, the handling/conversion cost of a retailer is a function of how much he has obtained of the product from the various manufacturers in the different countries and what currency the transactions took place in and in what transaction mode. For the sake of generality, however, we allow the function to depend also on the amounts of the product held and transacted by other retailers and, therefore, we may write:

$$c_j = c_j(Q^1), \quad \forall j. \tag{20}$$

The handling cost is measured in the base currency.

The retailers, which can be either physical or virtual, also have associated transaction costs in regards to transacting with the manufacturers, which we assume can be dependent on the type of currency as well as on the manufacturer and country. We denote the transaction cost associated with retailer j transacting with manufacturer il associated with currency h and mode m by $\hat{c}^{il}_{jhm}$ and we assume that it is of the form

$$\hat{c}^{il}_{jhm} = \hat{c}^{il}_{jhm}(q^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{21}$$

that is, such a transaction cost depends on the volume of the transaction. In addition, we assume that a retailer j also incurs a transaction cost $c^{j}_{kh\hat{l}m}$ associated with transacting with demand market khl via mode m, where

$$c^{j}_{kh\hat{l}m} = c^{j}_{kh\hat{l}m}(q^{j}_{kh\hat{l}m}), \quad \forall j, k, h, \hat{l}, m. \tag{22}$$

Hence, the transaction costs given in (22) can vary according to the retailer/currency/country combination and are a function of the volume of the product transacted. We assume that the cost functions (20)–(22) are convex and continuously differentiable and are measured in the base currency.

The actual price charged for the product by retailer j is denoted by $\rho^{j*}_{2kh\hat{l}m}$ and is associated with transacting with consumers at demand market k in currency h and country l via mode m. Subsequently, we discuss how such prices are arrived at. We assume that the retailers are also profit-maximizers, with the criterion of profit maximization for retailer j given by:

$$\max \sum_{k=1}^{k}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}(\rho^{j*}_{2kh\hat{l}m} \times e_h)q^j_{kh\hat{l}m} - c_j(Q^1)$$

$$-\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{c}^{il}_{jhm}(q^{il}_{jhm}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{m=1}^{2}\sum_{\hat{l}=1}^{L}c^j_{kh\hat{l}m}(q^j_{kh\hat{l}m})$$

$$-\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm} \times e_h)q^{il}_{jhm} \tag{23}$$

$$\text{s.t.}\quad q^{il}_{jhm} \geq 0, \quad q^j_{kh\hat{l}m} \geq 0, \quad \forall i, \hat{l}, h, m, \tag{24}$$

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} q^j_{kh\hat{l}m} \leq \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} q^{il}_{jhm}. \tag{25}$$

Objective function (23) expresses that the difference between the revenues (given by the first term) minus the handling cost, the two sets of transaction costs, and the payout to the manufacturers (given by the fifth term) should be maximized. The objective function in (23) is concave in its variables under the above posed assumptions. Constraint (25) guarantees that a retailer does not transact more of the product with the demand markets than he has in his possession.

We now turn to describing the criteria associated with a retailer's environmental decision-making similar to that developed above for a given manufacturer. Hence, we allow the retailers to also be faced with multiple criteria.

In particular, we assume that retailer j seeks to minimize the emissions associated with his transactions with the manufacturers, that is, he also is faced with the following problem:

$$\min \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}(\eta^{il} + \eta^{il}_{jm})q^{il}_{jhm} \tag{26}$$

$$\text{s.t.}\quad q^{il}_{jhm} \geq 0, \quad \forall i, l, h, m.$$

Note that we do not consider a retailer's decision-making concerning emissions generated to involve the demand markets (as, in a sense, this may be viewed as discriminatory). Below we describe how environmental decision-making is captured at the demand market level.

Moreover, each retailer seeks to also minimize the risk associated with obtaining the product from the manufacturers and transacting with the various demand markets, which we assume to also include a general form of environmental risk.

Hence, each retailer j is faced with his own individual risk denoted by r^j with the function being assumed to be continuous and convex and dependent on the transactions to and from all the retailers, that is,

$$r^j = r^j(Q^1, Q^2), \quad \forall j. \tag{27}$$

The third criterion or retailer j can be expressed as:

$$\begin{aligned} &\min\ r^j(Q^1, Q^2) \\ &\text{s.t. } q^{il}_{jhm} \geq 0, \quad \forall i, l, h, m \\ &\qquad q^j_{kh\hat{l}} \geq 0, \quad \forall k, h, \hat{l}. \end{aligned} \tag{28}$$

2.2.1 A Retailer's Multicriteria Decision-Making Problem

We now demonstrate (akin to the above construction for a given manufacturer) how the multiple criteria faced by a retailer can be transformed into a single optimization problem using, again, the concept of a value function.

In particular, we assume that retailer j associates a nonnegative weight β_j with the emission generation criterion (26), a weight ϑ_j with the risk criterion (28), and a weight equal to 1 with profit maximization (cf. (23)) (see also the discussion concerning the manufacturers above), yielding the following multicriteria decision-making problem:

$$\begin{aligned} \max \sum_{k=1}^{k}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} &(\rho^{j*}_{2kh\hat{l}m} \times e_h) q^j_{kh\hat{l}m} - c_j(Q^1) \\ &- \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} \hat{c}^{il}_{jhm}(q^{il}_{jhm}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{m=1}^{2}\sum_{\hat{l}=1}^{L} c^j_{kh\hat{l}m}(q^j_{kh\hat{l}m}) \\ &- \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} (\rho^{il*}_{1jhm} \times e_h) q^{il}_{jhm} \\ &- \beta_j (\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} (\eta^{il} + \eta^{il}_{jm}) q^{il}_{jhm}) - \vartheta_j r^j(Q^1, Q^2) \end{aligned} \tag{29}$$

$$\text{s.t. nonnegativity assumption on the variables and (25).} \tag{30}$$

2.2.2 Optimality Conditions of the Retailers

Here we assume that the retailers can also compete in a noncooperative manner with the governing optimality/equilibrium concept being that of Nash.

The optimality conditions for all retailers, simultaneously, under the above stated assumptions, can, hence, be expressed as the variational inequality problem: determine $(Q^{1*}, Q^{2*}, \gamma^*) \in \mathcal{K}^2$, such that

$$\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q^{il}_{jhm}} + \vartheta_j \frac{\partial r^j(Q^{1*}, Q^{2*})}{\partial q^{il}_{jhm}} + \beta_j(\eta^{il} + \eta^{il}_{jm}) \right.$$
$$\left. + \rho^{il*}_{1jhm} \times e_h + \frac{\partial \hat{c}^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}} - \gamma^*_j \right] \times \left[q^{il}_{jhm} - q^{il*}_{jhm}\right]$$
$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\frac{\partial c^j_{kh\hat{l}m}(q^{j*}_{kh\hat{l}m})}{\partial q^j_{kh\hat{l}m}} + \vartheta_j \frac{\partial r^j(Q^{1*}, Q^{2*})}{\partial q^j_{kh\hat{l}m}} - \rho^{j*}_{2kh\hat{l}m} \times e_h \right.$$
$$\left. + \gamma^*_j \right] \times \left[q^j_{kh\hat{l}m} - q^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} q^{il*}_{jhm} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} q^{j*}_{kh\hat{l}m}\right] \times \left[\gamma_j - \gamma^*_j\right] \geq 0,$$
$$\forall (Q^1, Q^2, \gamma) \in \mathcal{K}^2, \qquad (31)$$

where the feasible set $\mathcal{K}^2 \equiv \{(Q^1, Q^2, \gamma) \in R_+^{2ILJH+2JKHL+J}\}$.

Here γ_j denotes the Lagrange multiplier associated with constraint (25) (see [6]), and γ is the J-dimensional column vector of Lagrange multipliers of all the retailers with γ^* denoting the vector of optimal multipliers. Note that γ^*_j serves as the market clearing price for the product at retailer j (as can be seen from the last term in (31)). In particular, its value is positive if the product transactions from the retailer to all the demand markets in the countries and in the various currencies is precisely equal to the product transactions to the retailer from all the manufacturers in all the countries transacted in the different currencies (and modes).

We now highlight the economic interpretation of the retailers' optimality conditions. The first term in (31) states that if there is a positive amount of product transacted between a manufacturer/retailer pair via mode m and currency h, that is, $q^{il*}_{jhm} > 0$, then the shadow price at the retailer, γ^*_j, is equal to the price charged for the product plus the various marginal costs and the associated weighted marginal risk emission. In addition, the second term in (31) shows that, if consumers at demand market $kh\hat{l}$ purchase the product from a particular retailer j transacted through mode m, which means that, if the $q^{j*}_{kh\hat{l}m}$ is positive, then the price charged by retailer j, $\rho^{j*}_{2kh\hat{l}m}$, is equal to γ^*_j plus the marginal transaction costs in dealing with the demand market and the weighted marginal costs for the risk that he has to bear.

2.3 The Equilibrium Conditions at the Demand Markets

We now describe the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the price charged for product by the manufacturer and by the retailers but also their transaction costs associated with obtaining the product. We also describe how their environmental decision-making is captured.

We let $\hat{c}^j_{kh\hat{l}m}$ denote the transaction cost associated with consumers obtaining the product at demand market k in currency h and in country $\hat{l}$ via mode m from retailer j and recall that $q^j_{kh\hat{l}m}$ is the amount of the product transacted thus. We assume that the transaction cost function is continuous and of the general form:

$$\hat{c}^j_{kh\hat{l}m} = \hat{c}^j_{kh\hat{l}m}(Q^2), \quad \forall j, k, h, \hat{l}, m. \tag{32}$$

Furthermore, let $\hat{c}^{il}_{kh\hat{l}}$ denote the transaction cost associated with consumers obtaining the product at demand market k in currency h and in country $\hat{l}$ transacted electronically from manufacturer il, where we assume that the transaction cost is continuous and of the general form:

$$\hat{c}^{il}_{kh\hat{l}} = \hat{c}^{il}_{kh\hat{l}}(Q^3), \quad \forall i, l, k, h, \hat{l}. \tag{33}$$

Hence, the transaction cost associated with transacting directly with manufacturers is of a form of the same level of generality as the transaction costs associated with transacting with the retailers. Note that the above functional forms can capture congestion on the networks. Indeed, we allow for the transaction cost (from the perspective of consumers) to depend not only on the flow of the product from a manufacturer or from the retailer in the currency to the country (and mode) but also on the other product transactions in the other currencies and between other manufacturers and/or retailers and demand markets. The transaction cost functions above are assumed to be measured in the base currency.

Denote now the demand for the product at demand market k in currency h in country $\hat{l}$ by $d_{kh\hat{l}}$ and assume, as given, the continuous demand functions:

$$d_{kh\hat{l}} = d_{kh\hat{l}}(\rho_3), \quad \forall k, h, \hat{l}. \tag{34}$$

Thus, according to (34), the demand for the product at a demand market in a currency and country depends, in general, not only on the price of the product at that demand market (and currency and country) but also on the prices of the product at the other demand markets (and in other countries and currencies). Consequently, consumers at a demand market, in a sense, also compete with consumers at other demand markets.

The consumers take the price charged by the retailer, which was denoted by $\rho^{j*}_{2kh\hat{l}m}$ for retailer j, demand market k, currency h, and country $\hat{l}$ transacted via mode m, the price charged by manufacturer il, which was denoted by $\rho^{il*}_{1kh\hat{l}}$, and the rate of appreciation in the currency, plus the transaction costs, in making their consumption decisions. In addition, we assume that the consumers are also multicriteria decision-makers and weight the emissions associated with their transactions accordingly.

2.3.1 The Multicriteria Equilibrium Conditions for the Demand Markets

Let $\eta^j_{k\hat{l}m}$ denote the amount of emissions generated per unit of product transacted between retailer j and demand market k in country $\hat{l}$ via mode m and assume that this term is nonnegative for each $k, \hat{l}, m, j$. We assume that consumers at a demand market perceive the emissions generated through their transactions (and purchases) in an individual fashion with the nonnegative weight $\delta_{kh\hat{l}}$ associated with the total emissions generated through consumer transactions at demand market $kh\hat{l}$. This term may also be viewed as a monetary conversion factor associated with the per unit emissions generated. See also [32].

The equilibrium conditions for the consumers at demand market $kh\hat{l}$, thus, take the form: for all retailers: $j = 1, \ldots, J$ and all modes m; $m = 1, 2$:

$$\rho^{j*}_{2kh\hat{l}m} \times e_h + \hat{c}^j_{kh\hat{l}m}(Q^{2*}) + \delta_{kh\hat{l}}\eta^j_{k\hat{l}m} \begin{cases} = \rho^*_{3kh\hat{l}}, \text{ if } & q^{j*}_{kh\hat{l}m} > 0 \\ \geq \rho^*_{3kh\hat{l}}, \text{ if } & q^{j*}_{kh\hat{l}m} = 0, \end{cases} \tag{35}$$

and for all manufacturers il; $i = 1, \ldots, I$ and $l = 1, \ldots, L$:

$$\rho^{il*}_{1kh\hat{l}} \times e_h + \hat{c}^{il}_{kh\hat{l}}(Q^{3*}) + \delta_{kh\hat{l}}(\eta^{il} + \eta^{il}_{k\hat{l}}) \begin{cases} = \rho^*_{3kh\hat{l}}, \text{ if } & q^{il*}_{kh\hat{l}} > 0 \\ \geq \rho^*_{3kh\hat{l}}, \text{ if } & q^{il*}_{kh\hat{l}} = 0. \end{cases} \tag{36}$$

In addition, we must have that

$$d_{kh\hat{l}}(\rho^*_3) \begin{cases} = \sum_{j=1}^{J}\sum_{m=1}^{2} q^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\hat{l}}, \text{ if } & \rho^*_{3kh\hat{l}} > 0 \\ \leq \sum_{j=1}^{J}\sum_{m=1}^{2} q^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\hat{l}}, \text{ if } & \rho^*_{3kh\hat{l}} = 0. \end{cases} \tag{37}$$

Condition (35) states that consumers at demand market $kh\hat{l}$ will purchase the product from retailer j transacted via mode m, if the price charged by the retailer for the product and the appreciation rate for the currency plus the transaction cost (from the perspective of the consumer) and the weighted emission generation term does not exceed the price that the consumers are willing to pay for the product in that currency and country, i.e., $\rho^*_{3kh\hat{l}}$. Note that, according to (35), if the transaction costs are identically equal to zero, as is the weighted emission generation term, then the price faced by the consumers at a demand market is the price charged by the retailer for the particular product and currency and mode in the country plus the rate of appreciation in the currency. Condition (36) state the analogue, but for the case of electronic transactions with the manufacturers.

Condition (37), on the other hand, states that, if the price the consumers are willing to pay for the product at a demand market/currency/country is positive, then the quantity of the product transacted at the demand market/currency/country is precisely equal to the demand.

In equilibrium, conditions (35), (36), and (37) will have to hold for all demand markets in all countries, currencies, and modes. Hence, these equilibrium conditions can be expressed also as a variational inequality analogous to those in (19) and (31) and given by: determine $(Q^{2*}, Q^{3*}, \rho_3^*) \in R_+^{(IL+2J+1)KHL}$, such that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{\hat{L}}\sum_{m=1}^{2}\left[\rho_{2kh\hat{l}m}^{j*} \times e_h + \hat{c}_{kh\hat{l}m}^{j}(Q^{2*}) + \delta_{kh\hat{l}}\eta_{k\hat{l}m}^{j} - \rho_{3kh\hat{l}}^{*}\right] \times \left[q_{kh\hat{l}m}^{j} - q_{kh\hat{l}m}^{j*}\right]$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho_{1kh\hat{l}}^{il*} \times e_h + \hat{c}_{kh\hat{l}}^{il}(Q^{3*}) + \delta_{kh\hat{l}}(\eta^{il} + \eta_{k\hat{l}}^{il}) - \rho_{3kh\hat{l}}^{*}\right] \times \left[q_{kh\hat{l}}^{il} - q_{kh\hat{l}}^{il*}\right]$$
$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2} q_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L} q_{kh\hat{l}}^{il*} - d_{kh\hat{l}}(\rho_3^*)\right] \times \left[\rho_{3kh\hat{l}} - \rho_{3kh\hat{l}}^{*}\right] \geq 0, \ \forall (Q^2, Q^3, \rho_3) \in R_+^{(IL+2J+1)KHL}. \quad (38)$$

2.4 The Equilibrium Conditions for the Global Supply Chain Network

In equilibrium, the product transactions between the manufacturers in the different countries with the retailers must coincide with those that the retailers actually accept. In addition, the amounts of the product that are obtained by the consumers in the different countries and currencies must be equal to the amounts that the retailers and the manufacturers actually provide. Hence, although there may be competition between decision-makers at the same tier of nodes of the supply chain supernetwork, there must be, in a sense, cooperation between decision-makers associated with distinct tiers of nodes. Thus, in equilibrium, the prices and product transactions must satisfy the sum of the optimality conditions (19) and (31) and (38). We make these relationships rigorous through the subsequent definition and variational inequality derivation below.

Definition 1 (Global Supply Chain Network Equilibrium). *The equilibrium state of the supply chain supernetwork is one where the product transactions between the tiers of the network coincide and the product transactions and prices satisfy the sum of conditions (19), (31), and (38).*

The equilibrium state is equivalent to the following:

Theorem 1 (Variational Inequality Formulation). *The equilibrium conditions governing the global supply chain supernetwork according to Definition 1 are equivalent to the solution of the variational inequality given by: determine* $(Q^{1*}, Q^{2*}, Q^{3*}, \gamma^*, \rho_3^*) \in \mathcal{K}$, *satisfying:*

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}}+\frac{\partial c^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}}+\frac{\partial c_j(Q^{1*})}{\partial q^{il}_{jhm}}\right.$$
$$+\frac{\partial \hat{c}^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}}+\omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}}+\vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{il}_{jhm}}$$
$$\left.+(\alpha^{il}+\beta_j)(\eta^{il}+\eta^{il}_{jm})-\gamma^*_j\right]\times\left[q^{il}_{jhm}-q^{il*}_{jhm}\right]$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}}+\frac{\partial c^{il}_{kh\hat{l}}(q^{il*}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}}+\hat{c}^{il}_{kh\hat{l}}(Q^{3*})\right.$$
$$\left.+\omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}}+(\alpha^{il}+\delta_{kh\hat{l}})(\eta^{il}+\eta^{il}_{k\hat{l}})-\rho^*_{3kh\hat{l}}\right]$$
$$\times\left[q^{il}_{kh\hat{l}}-q^{il*}_{kh\hat{l}}\right]$$
$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\frac{\partial c^{j}_{khl}(q^{j*}_{kh\hat{l}m})}{\partial q_{kh\hat{l}m}{}^{j}}+\hat{c}^{j}_{kh\hat{l}m}(Q^{2*})+\vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{j}_{kh\hat{l}m}}\right.$$
$$\left.+\delta_{kh\hat{l}}\eta^{j}_{k\hat{l}m}+\gamma^*_j-\rho^*_{3kh\hat{l}}\right]\times\left[q^{j}_{kh\hat{l}m}-q^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}q^{il*}_{jhm}-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}q^{j*}_{kh\hat{l}m}\right]\times\left[\gamma_j-\gamma^*_j\right]$$
$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2}q^{j*}_{kh\hat{l}m}+\sum_{i=1}^{I}\sum_{l=1}^{L}q^{il*}_{kh\hat{l}}-d_{kh\hat{l}}(\rho^*_3)\right]$$
$$\times\left[\rho_{3kh\hat{l}}-\rho^*_{3kh\hat{l}}\right]\geq 0,\quad\forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in\mathcal{K},\tag{39}$$

where $\mathcal{K}\equiv\{\mathcal{K}^1\times\mathcal{K}^2\times\mathcal{K}^3\}$, *where* $\mathcal{K}^3\equiv\{\rho_3|\rho_3\in R^{KHL}_{+}\}$.

Proof. We first establish that the equilibrium conditions imply variational inequality (39). Indeed, summation of inequalities (19), (31), and (38), after algebraic simplifications, yields variational inequality (39).

We now establish the converse, that is, that a solution to variational inequality (39) satisfies the sum of conditions (19), (31), and (38), and is, hence, an equilibrium.

To inequality (39), add the term $-\rho^{il*}_{1jhm}\times e_h+\rho^{il*}_{1jhm}\times e_h$ to the term in the first set of brackets (preceding the first multiplication sign). Similarly, add the terms $-\rho^{il*}_{1kh\hat{l}} \times e_h + \rho^{il*}_{1kh\hat{l}} \times e_h$ to the term in brackets preceding the second multiplication sign and $-\rho^{j*}_{2kh\hat{l}m} \times e_h + \rho^{j*}_{2kh\hat{l}m} \times e_h$ to the term in brackets preceding the third multiplication sign in (39). The addition of such terms does not change (39) as the value of these terms is zero and yields:

$$
\begin{aligned}
&\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}} + \frac{\partial c_j(Q^{1*})}{\partial q^{il}_{jhm}}\right.\\
&\qquad + \frac{\partial \hat{c}^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}} + (\alpha^{il}+\beta_j)(\eta^{il}+\eta^{il}_{jm})\\
&\qquad + \omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}} + \vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{il}_{jhm}} - \gamma^*_j\\
&\qquad \left. - \rho^{il*}_{1jhm} \times e_h + \rho^{il*}_{1jhm} \times e_h\right] \times \left[q^{il}_{jhm} - q^{il*}_{jhm}\right]\\
&+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(q^{il*}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}} + \hat{c}^{il}_{kh\hat{l}}(Q^{3*})\right.\\
&\qquad + (\alpha^{il}+\delta_{kh\hat{l}})(\eta^{il}+\eta^{il}_{k\hat{l}}) + \omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}} - \rho^*_{3kh\hat{l}}\\
&\qquad \left. - \rho^{il*}_{1kh\hat{l}} \times e_h + \rho^{il*}_{1kh\hat{l}} \times e_h\right] \times \left[q^{il}_{kh\hat{l}} - q^{il*}_{kh\hat{l}}\right]\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\frac{\partial c^{j}_{khl}(q^{j*}_{kh\hat{l}m})}{\partial q^{j}_{kh\hat{l}m}} + \gamma^*_j + \hat{c}^{j}_{kh\hat{l}m}(Q^{2*}) + \vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{j}_{kh\hat{l}m}}\right.\\
&\qquad \left. + \delta_{kh\hat{l}}\eta^{j}_{k\hat{l}m} - \rho^*_{3kh\hat{l}} - \rho^{j*}_{2kh\hat{l}m} \times e_h + \rho^{j*}_{2kh\hat{l}m} + e^*_h\right]\\
&\qquad \times \left[q^{j}_{kh\hat{l}m} - q^{j*}_{kh\hat{l}m}\right]\\
&+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} q^{il*}_{jhm} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{\hat{L}}\sum_{m=1}^{2} q^{j*}_{kh\hat{l}m}\right] \times \left[\gamma_j - \gamma^*_j\right]\\
&+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2} q^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\hat{l}} - d_{kh\hat{l}}(\rho^*_3)\right]\\
&\qquad \times \left[\rho_{3kh\hat{l}} - \rho^*_{3kh\hat{l}}\right] \geq 0, \quad \forall (Q^1,Q^2,Q^3,\gamma,\rho_3) \in \mathcal{K}, \qquad (40)
\end{aligned}
$$

which, in turn, can be rewritten as:

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}}+\frac{\partial c^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}}+\omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{jhm}}\right.$$
$$\left.+\alpha^{il}(\eta^{il}+\eta^{il}_{jm})-\rho^{il*}_{1jhm}\times e_h\right]\times\left[q^{il}_{jhm}-q^{il*}_{jhm}\right]$$
$$+\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q^{il}_{jhm}}+\rho^{il*}_{1jhm}\times e_h+\frac{\partial\hat{c}^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}}\right.$$
$$\left.+\vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{il}_{jhm}}+\beta_j(\eta^{il}+\eta^{il}_{jm})-\gamma^*_j\right]$$
$$\times\left[q^{il}_{jhm}-q^{il*}_{jhm}\right]$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}}+\frac{\partial c^{il}_{kh\hat{l}}(q^{il*}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}}+\omega^{il}\frac{\partial r^{il}(Q^{1*},Q^{3*})}{\partial q^{il}_{kh\hat{l}}}\right.$$
$$\left.+\alpha^{il}(\eta^{il}+\eta^{il}_{k\hat{l}})-\rho^{il*}_{1kh\hat{l}}\times e_h\right]\times\left[q^{il}_{kh\hat{l}}-q^{il*}_{kh\hat{l}}\right]$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho^{il*}_{1kh\hat{l}}\times e_h+\hat{c}^{il}_{kh\hat{l}}(Q^{3*})+\delta_{kh\hat{l}}(\eta^{il}+\eta^{il}_{k\hat{l}})-\rho^*_{3kh\hat{l}}\right]$$
$$\times\left[q^{il}_{kh\hat{l}}-q^{il*}_{kh\hat{l}}\right]$$
$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\frac{\partial c^{j}_{khl}(q^{j*}_{kh\hat{l}m})}{\partial q^{j}_{kh\hat{l}m}}+\vartheta_j\frac{\partial r^{j}(Q^{1*},Q^{2*})}{\partial q^{j}_{kh\hat{l}m}}-\rho^{j*}_{2kh\hat{l}m}\times e_h+\gamma^*_j\right]$$
$$\times\left[q^{j}_{kh\hat{l}m}-q^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\rho^{j*}_{2kh\hat{l}m}\times e_h+\hat{c}^{j}_{kh\hat{l}m}(Q^{2*})+\delta_{kh\hat{l}}\eta^{j}_{k\hat{l}m}-\rho^*_{3kh\hat{l}}\right]$$
$$\times\left[q^{j}_{kh\hat{l}m}-q^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}q^{il*}_{jhm}-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}q^{j*}_{kh\hat{l}m}\right]\times\left[\gamma_j-\gamma^*_j\right]$$
$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2}q^{j*}_{kh\hat{l}m}+\sum_{i=1}^{I}\sum_{l=1}^{L}q^{il*}_{kh\hat{l}}-d_{kh\hat{l}}(\rho^*_3)\right]$$
$$\times\left[\rho_{3kh\hat{l}}-\rho^*_{3kh\hat{l}}\right]\geq 0. \tag{41}$$

But inequality (41) is equivalent to the sum of conditions (19), (31), and (38), and, hence, the product and price pattern is an equilibrium according to Definition 1. ■

We now put variational inequality (39) into standard form that will be utilized in the subsequent sections. For additional background on variational inequalities and their applications, see the book by [26]. For other applications of supernetworks along with the variational inequality formulations of the governing equilibrium conditions, see the book by [28].

In particular, we have that variational inequality (39) can be expressed as:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{42}$$

where $X \equiv (Q^1, Q^2, Q^3, \gamma, \rho_3)$ and $F(X) \equiv (F_{iljhm}, F_{ilkh\hat{l}}, F_{jkh\hat{l}m}, F_j, F_{kh\hat{l}})$ ($i = 1, \ldots, I$; $\hat{l} = l = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$), and the specific components of F are given by the functional terms preceding the multiplication signs in (39), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

We now describe how to recover the prices associated with the first two tiers of nodes in the global supply chain network. Clearly, the components of the vector ρ_3^* are obtained directly from the solution of variational inequality (39). In order to recover the second tier prices associated with the retailers and the appreciation rates, one can (after solving variational inequality (39) for the particular numerical problem) *either* (cf. (35)) set $\rho^{j*}_{2kh\hat{l}m} \times e_h = \rho^*_{3kh\hat{l}} - \hat{c}^j_{kh\hat{l}m}(Q^{2*}) - \delta_{kh\hat{l}}\eta^j_{k\hat{l}m}$, for any $j, k, h, \hat{l}, m$ such that $q^{j*}_{kh\hat{l}m} > 0$, *or* (cf. (28)) for any $q^{j*}_{kh\hat{l}m} > 0$, set $\rho^{j*}_{2kh\hat{l}m} \times e_h = \frac{\partial c^j_{kh\hat{l}m}(q^{j*}_{kh\hat{l}m})}{\partial q^j_{kh\hat{l}m}} + \vartheta_j \frac{\partial r^j(Q^{1*}, Q^{2*})}{\partial q^j_{kh\hat{l}m}} + \gamma^*_j$.

Similarly, from (31) we can infer that the top tier prices comprising the vector ρ_1^* can be recovered (once the variational inequality (39) is solved with particular data) thus: for any i, l, j, h, m, such that $q^{il*}_{jhm} > 0$, set (cf. (19)) $\rho^{il*}_{1jhm} \times e_h = \frac{\partial f^{il}(Q^{1*}, Q^{3*})}{\partial q^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(q^{il*}_{jhm})}{\partial q^{il}_{jhm}} + \omega^{il} \frac{\partial r^{il}(Q^{1*}, Q^{3*})}{\partial q^{il}_{jhm}} + \alpha^{il}(\eta^{il} + \eta^{il}_{jm})$.

Similarly (cf. (36)), set $\rho^{il*}_{1kh\hat{l}} \times e_h = \rho^*_{3kh\hat{l}} - \hat{c}^{il}_{kh\hat{l}}(Q^{3*}) - \delta_{kh\hat{l}}(\eta^{il} + \eta^{il}_{k\hat{l}})$, for any $i, l, k, h, \hat{l}$ such that $q^{il*}_{kh\hat{l}} > 0$, *or* (cf. (18)) for any $q^{il*}_{kh\hat{l}} > 0$, set $\rho^{il*}_{1kh\hat{l}} \times e_h = \frac{\partial f^{il}(Q^{1*}, Q^{3*})}{\partial q^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(q^{il*}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}} + \omega^{il} \frac{\partial r^{il}(Q^{1*}, Q^{3*})}{\partial q^{il}_{kh\hat{l}}} + \alpha^{il}(\eta^{il} + \eta^{il}_{k\hat{l}})$.

With the pricing mechanism described above, it is straightforward to verify that a solution of variational inequality (39) also satisfies the optimality conditions (19) and (31) as well as the equilibrium conditions (35)–(37) (see also (38)).

3 The Dynamic Global Supply Chain Network Model

In this section, we turn to the development of a dynamic global supply chain network model whose set of stationary points coincides with the set of solutions of the variational inequality problem (39) governing the static global supply chain network equilibrium model developed in Section 2. In particular, we propose a dynamical system, which is nonclassic, and termed a *projected dynamical system* (cf. [34]) that governs the behavior of the global supply chain supernetwork presented in Section 2. The proposed dynamic adjustment processes describe the disequilibrium dynamics as the various global supply chain decision-makers adjust their product transactions between the tiers and the prices associated with the different tiers adjust as well.

3.1 Demand Market Price Dynamics

The rate of change of the price $\rho_{3kh\hat{l}}$, denoted by $\dot{\rho}_{3kh\hat{l}}$, is assumed to be equal to the difference between the demand for the product at the demand market and currency and country and the amount of the product actually available there. Moreover, if the demand for the product at the demand market (and currency and country) at an instant in time exceeds the amount available from the various retailers and manufacturers, then the price will increase; if the amount available exceeds the demand at the price, then the price will decrease. We also have to make sure that the prices do not become negative. Therefore, the dynamics of the prices $\rho_{3kh\hat{l}}$, $\forall k, h, \hat{l}$, can be expressed in the following way:

$$\dot{\rho}_{3kh\hat{l}} = \begin{cases} d_{kh\hat{l}}(\rho_3) - \sum\limits_{j=1}^{J} \sum\limits_{m=1}^{2} q^j_{kh\hat{l}m} - \sum\limits_{i=1}^{I} \sum\limits_{l=1}^{L} q^{il}_{kh\hat{l}}, \\ \qquad\qquad \text{if} \quad \rho_{3kh\hat{l}} > 0 \\ \max\{0, d_{kh\hat{l}}(\rho_3) - \sum\limits_{j=1}^{J} \sum_{m=1}^{2} q^j_{kh\hat{l}m} - \sum\limits_{i=1}^{I} \sum\limits_{l=1}^{L} q^{il}_{kh\hat{l}}\}, \\ \qquad\qquad \text{if} \quad \rho_{3kh\hat{l}} = 0. \end{cases} \tag{43}$$

3.2 The Dynamics of the Product Transactions Between the Retailers and the Demand Markets

We assume that the rate of change of the product transaction between retailer j and demand market k, country $\hat{l}$, and transacting in currency h via mode m and denoted by $\dot{q}^j_{kh\hat{l}m}$ is equal to the difference between the price consumers at this particular demand market/country/currency combination are willing to pay for the product minus the price charged by the retailer and the various transaction costs and weighted marginal risk and weighted emissions generated. Here we also have to guarantee that the product transactions will not become negative. Thus, the rate of change of the product transactions between a retailer and a demand market in a country and currency via a mode can be written as: $\forall j, k, h, \hat{l}, m$:

$$\dot{q}^j_{kh\hat{l}m} = \begin{cases} \rho_{3kh\hat{l}} - \frac{\partial c^j_{kh\hat{l}m}(q^j_{kh\hat{l}m})}{\partial q^j_{kh\hat{l}m}} - \vartheta_j \frac{\partial r^j(Q^1,Q^2)}{\partial q^j_{kh\hat{l}m}} - \hat{c}^j_{kh\hat{l}m}(Q^2) \\ -\delta_{kh\hat{l}} \eta^j_{k\hat{l}m} - \gamma_j, \quad \text{if} \quad q^j_{kh\hat{l}m} > 0 \\ \max\{0, \rho_{3kh\hat{l}} - \frac{\partial c^j_{kh\hat{l}m}(q^j_{kh\hat{l}m})}{\partial q^j_{kh\hat{l}m}} - \vartheta_j \frac{\partial r^j(Q^1,Q^2)}{\partial q^j_{kh\hat{l}m}} - \hat{c}^j_{kh\hat{l}m}(Q^2) \\ \quad -\delta_{kh\hat{l}} \eta^j_{k\hat{l}m} - \gamma_j\}, \quad \text{if} \quad q^j_{kh\hat{l}m} = 0. \end{cases} \tag{44}$$

3.3 The Dynamics of the Prices at the Retailers

The prices at the retailers, whether they are physical or virtual, must reflect supply and demand conditions as well. In particular, we let $\dot{\gamma}_j$ denote the rate of change in the market clearing price associated with retailer j, and we propose the following dynamic adjustment for retailer j:

$$\dot{\gamma}_j = \begin{cases} \sum_{k=1}^K \sum_{h=1}^H \sum_{\hat{l}=1}^L \sum_{m=1}^2 q^j_{kh\hat{l}m} \\ -\sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 q^{il}_{jhm}, \quad \text{if} \quad \gamma_j > 0 \\ \max\{0, \sum_{k=1}^K \sum_{h=1}^H \sum_{\hat{l}=1}^L \sum_{m=1}^2 q^j_{kh\hat{l}m} \\ \quad \sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 q^{il}_{jhm}\}, \quad \text{if} \quad \gamma_j = 0. \end{cases} \tag{45}$$

Hence, if there is excess supply of the product at a retailer, then the price will decrease at that retailer; if there is excess demand, then the price will increase. Here we also guarantee that these prices do not become negative.

3.4 Dynamics of the Product Transactions Between Manufacturers and Retailers

The dynamics of the product transactions between manufacturers in the countries and the retailers in the different currencies and modes are now described. Note that in order for a transaction between nodes in these two tiers to take place, there must be agreement between the pair of decision-makers. Toward that end, we let $\dot{q}^{il}_{jhm}$ denote the rate of change of the product transaction between manufacturer il and retailer j transacted via mode m and in currency h and we have that for every i, l, j, h, m:

$$\dot{q}^{il}_{jhm} = \begin{cases} \gamma_j - \frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}} - \frac{\partial c^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} - \frac{\partial c_j(Q^1)}{\partial q^{il}_{jhm}} - \frac{\partial \hat{c}^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} \\ -\omega^{il} \frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}} - \vartheta_j \frac{\partial r^j(Q^1,Q^2)}{\partial q^{il}_{jhm}} - (\alpha^{il} + \beta_j)(\eta^{il} + \eta^{il}_{jm}), \\ \qquad \text{if} \quad q^{il}_{jhm} > 0 \\ \max\{0, \gamma_j - \frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}} - \frac{\partial c^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} - \frac{\partial c_j(Q^1)}{\partial q^{il}_{jhm}} \\ \quad - \frac{\partial \hat{c}^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} - \omega^{il} \frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}} - \vartheta_j \frac{\partial r^j(Q^1,Q^2)}{\partial q^{il}_{jhm}} \\ \quad -(\alpha^{il} + \beta_j)(\eta^{il} + \eta^{il}_{jm})\}, \\ \qquad \text{if} \quad q^{il}_{jhm} = 0. \end{cases} \tag{46}$$

Hence, the transaction between a manufacturer in a country and a retailer via a mode and in a currency will increase if the price that the retailer is willing to pay the manufacturer exceeds the various marginal costs plus the weighted marginal risks and emissions generated. Moreover, we guarantee that such a transaction never becomes negative.

3.5 The Dynamics of the Product Transactions Between Manufacturers and Demand Markets

The rate of change of the product transactions between a manufacturer in a country and demand market/currency/country pair is assumed to be equal to the price the consumers are willing to pay minus the various costs, including marginal ones, that the manufacturer incurs when transacting with the demand market in a country and currency and the weighted emissions generated and the weighted marginal risk. We denote this rate of change by $\dot{q}^{il}_{kh\hat{l}}$, and, mathematically, express it in the following way, $\forall i, l, k, h, \hat{l}$:

$$\dot{q}^{il}_{kh\hat{l}} = \begin{cases} \rho_{3kh\hat{l}} - \frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}} - \frac{\partial c^{il}_{kh\hat{l}}(q^{il}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}} - \hat{c}^{il}_{kh\hat{l}}(Q^3) \\ - \omega^{il} \frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}} - (\alpha^{il} + \delta_{kh\hat{l}})(\eta^{il} + \eta^{il}_{k\hat{l}}), \\ \qquad \text{if} \quad q^{il}_{kh\hat{l}} > 0 \\ \max\{0, \rho_{3kh\hat{l}} - \frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}} - \frac{\partial c^{il}_{kh\hat{l}}(q^{il}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}} - \hat{c}^{il}_{kh\hat{l}}(Q^3) \\ - \omega^{il} \frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}} - (\alpha^{il} + \delta_{kh\hat{l}})(\eta^{il} + \eta^{il}_{k\hat{l}})\}, \\ \qquad \text{if} \quad q^{il}_{kh\hat{l}} = 0. \end{cases} \tag{47}$$

Note that (47) guarantees that the volume of product transacted will not take on a negative value.

3.6 The Projected Dynamical System

Consider now a dynamical system in which the product transactions between manufacturers in the countries and the retailers evolve according to (46); the product transactions between manufacturers and demand markets in the various countries and associated with different currencies evolve according to (47); the product transactions between retailers and the demand market/country/currency combinations evolve according to (44); the prices at the retailers evolve according to (45); and the prices at the demand markets evolve according to (43). Let X and $F(X)$ be as defined following (39) and recall also the feasible set $\mathcal{K}$ as defined following (39). Then the dynamic model described by (43)–(47) can be rewritten as a *projected dynamical system* (see [13] and [34]) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{48}$$

where $\Pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X and $X_0 = (Q^{10}, Q^{20}, Q^{30}, \gamma^0, \rho_3^0)$ is the initial point corresponding with the initial product flow and price pattern. Note that as the feasible set $\mathcal{K}$ is simply the nonnegative orthant, the projection operation takes on a very simple form as revealed through (43)–(47).

The trajectory of (48) describes the dynamic evolution of and the dynamic interactions among the product transactions and prices. The dynamical system (48) is nonclassic as it has a discontinuous right-hand side due to the projection operation. Such dynamical systems were introduced by [13].

Importantly, we have the following result, which is immediate from [13]:

Theorem 2. *The set of stationary points of the projected dynamical system (48) coincides with the set of solutions of the variational inequality problem (39) and is, thus, according to Definition 1, a global supply chain network equilibrium. Hence, a vector X^* satisfying $0 = \Pi_{\mathcal{K}}(X^*, -F(X^*))$ also satisfies variational inequality (39).*

4 Qualitative Properties

In this section, we provide some qualitative properties of the solution to variational inequality (39). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (42)) that enters the variational inequality of interest here. Finally, we establish that the trajectories of the projected dynamical system (48) are well-defined under reasonable assumptions.

Because the feasible set is not compact, we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\begin{aligned}\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \gamma, \rho_3) | 0 \le Q^1 \le b_1;\ 0 \le Q^2 \le b_2;\ 0 \le Q^3 \le b_3;\\ 0 \le \gamma \le b_4;\ 0 \le \rho_3 \le b_5\},\end{aligned} \tag{49}$$

where $b = (b_1, b_2, b_3, b_4, b_5) \ge 0$ and $Q^1 \le b_1; Q^2 \le b_2; Q^3 \le b_3; \gamma \le b_4; \rho_3 \le b_5$ means that $q^{il}_{jhm} \le b_1$; $q^{j}_{kh\hat{l}m} \le b_2$; $q^{il}_{kh\hat{l}m} \le b_3$; $\gamma_j \le b_4$; and $\rho_{3khl} \le b_5$ for all $i, l, j, k, h, \hat{l}, m$. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{2ILJH+2JKHL+ILKHL+J+KHL}$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \ge 0, \quad \forall X^b \in \mathcal{K}_b, \tag{50}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, as $\mathcal{K}_b$ is compact and F is continuous. Following [23] (see also Theorem 1.5 in [26]), we then have:

Theorem 3. *Variational inequality (39) admits a solution if and only if there exists a $b > 0$, such that variational inequality (50) admits a solution in $\mathcal{K}_b$ with*

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad Q^{3b} < b_3, \quad \gamma^b < b_4, \quad \rho_3^b < b_5. \tag{51}$$

Theorem 4 (Existence). *Suppose that there exist positive constants M, N, R, with $R > 0$, such that:*

$$\frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} + \frac{\partial c_j(Q^1)}{\partial q^{il}_{jhm}} + \frac{\partial \hat{c}^{il}_{jhm}(q^{il}_{jhm})}{\partial q^{il}_{jhm}} + \omega^{il}\frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{jhm}}$$

$$+ \vartheta_j \frac{\partial r^j(Q^{1*},Q^{2*})}{\partial q^{il}_{jhm}} + (\alpha^{il} + \beta_j)(\eta^{il} + \eta^{il}_{jm}) \geq M,$$

$$\forall Q^1 \text{ with } q^{il}_{jhm} \geq N, \forall i,l,j,h,m, \tag{52}$$

$$\frac{\partial f^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(q^{il}_{kh\hat{l}})}{\partial q^{il}_{kh\hat{l}}} + \hat{c}^{il}_{kh\hat{l}}(Q^3) + \omega^{il}\frac{\partial r^{il}(Q^1,Q^3)}{\partial q^{il}_{kh\hat{l}}}$$

$$+ (\alpha^{il} + \delta_{kh\hat{l}})(\eta^{il} + \eta^{il}_{k\hat{l}}) \geq M, \quad \forall Q^3 \text{ with } q^{il}_{kh\hat{l}} \geq N, \quad \forall i,l,k,h,\hat{l}, \tag{53}$$

$$\frac{\partial c^j_{khl}(q^j_{kh\hat{l}m})}{\partial q^j_{kh\hat{l}m}} + \hat{c}^j_{kh\hat{l}m}(Q^2) + \vartheta_j \frac{\partial r^j(Q^1,Q^2)}{\partial q^j_{kh\hat{l}m}} + \delta_{kh\hat{l}}\eta^j_{k\hat{l}m} \geq M,$$

$$\forall Q^2 \text{ with } q^j_{kh\hat{l}m} \geq N, \quad \forall j,k,h,\hat{l},m, \tag{54}$$

$$d_{kh\hat{l}}(\rho_3) \leq N, \quad \forall \rho_3 \text{ with } \rho_{3kh\hat{l}} > R, \quad \forall k,h,\hat{l}. \tag{55}$$

Then variational inequality (39); equivalently, variational inequality (42), admits at least one solution.

Proof. Follows using analogous arguments as the proof of existence for Proposition 1 in [35]. ■

Assumptions (52), (53), and (54) are reasonable from an economics perspective, as when the product transaction between a manufacturer in a country and a retailer or a manufacturer and a demand market in a country (and currency) or a retailer and demand market is large, we can expect the corresponding sum of the associated marginal costs of production, transaction, handling, and the weighted marginal risks and emissions generated to exceed a positive lower bound. Moreover, in the case where the demand price of the product in a currency and country at a demand market is high (cf. (55)), we can expect that the demand for the product at the demand market will not exceed a positive bound.

We now establish additional qualitative properties both of the function F that enters the variational inequality problem (cf. (39) and (42)), as well as uniqueness of the equilibrium pattern. Because the proofs of Theorems 5 and 6 below are similar to the analogous proofs in [30], they are omitted here.

Additional background on the properties established below can be found in the books by [26] and [28].

We first recall the concept of *additive production cost*, which was introduced by [42] in the stability analysis of dynamic spatial oligopolies, and has also been utilized in the qualitative analysis of supply chain networks by [29].

Definition 2 (Additive Production Cost). *We term a production cost an additive production cost if for manufacturer il, the production cost f^{il} is of the following form:*

$$f^{il}(q) = f^{il1}(q^{il}) + f^{il2}(\bar{q}^{il}), \tag{56}$$

where f^{il1} is the internal production cost that depends solely on the manufacturer's own output level, and $f^{il2}(\bar{q}^{il})$ is the interdependent part of the production cost that is a function of all the other manufacturer's output levels $\bar{q}^{il} = (q^{11}, \ldots, q^{il-1}, q^{il+1}, \ldots, q^{il})$ and reflects the impact of the other manufacturers' production patterns on manufacturer il's cost.

Using the assumption of additive production costs, as well as several additional assumptions, we now state the following:

Theorem 5 (Monotonicity). *Suppose that the production cost functions f^{il}; $i = 1, \ldots, I$, the risk functions r^{il}; $i = 1, \ldots, I$; $l = 1, \ldots, L$, and r^j; $j = 1, \ldots, J$, are convex and that the c^{il}_{jhm}, $c^{il}_{kh\hat{l}}$, c_j, $\hat{c}^{il}_{jhm}$, and $c^j_{kh\hat{l}m}$ functions are convex; the $\hat{c}^j_{kh\hat{l}m}$ and $\hat{c}^{il}_{kh\hat{l}}$ functions are monotone increasing, and the $d_{kh\hat{l}}$ functions are monotone decreasing functions, for all $i, l, j, h, k, \hat{l}, m$. Assume also that the production cost functions are additive for all manufacturers il according to Definition 2. Then the vector function F that enters the variational inequality (42) is monotone, that is,*

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{57}$$

Monotonicity plays a role in the qualitative analysis of variational inequality problems similar to that played by convexity in the context of optimization problems. Under slightly stronger conditions, we obtain the following sharper result.

Theorem 6 (Strict Monotonicity). *Assume all the conditions of Theorem 5. In addition, suppose that one of the families of convex functions c^{il}_{jhm}; $i = 1, \ldots, I$; $l = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$; $c^{il}_{kh\hat{l}}$; $i = 1, \ldots, I$; $l = 1, \ldots, L$; $k = 1, \ldots, K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$, c_j; $j = 1, \ldots, J$; $\hat{c}^{il}_{jhm}$; $i = 1, \ldots, I$; $l = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$, and $c^j_{kh\hat{l}m}$; $j = 1, \ldots, J$; $k = 1, \ldots, K$; $h = 1, \ldots, H$, $\hat{l} = 1, \ldots, L$; $m = 1, 2$, is a family of strictly convex functions. Suppose also that $\hat{c}^j_{kh\hat{l}m}$; $j = 1, \ldots, J$; $k = 1, \ldots, K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$; $m = 1, 2$; $\hat{c}^{il}_{kh\hat{l}}$; $i = 1, \ldots, I$; $l = 1, \ldots, L$;*

$k = 1, \ldots, K$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$ and -$d_{kh\hat{l}}$, $k = 1, \ldots, K$,$h = 1, \ldots, H$; $\hat{l} = 1, \ldots, \hat{L}$, are strictly monotone. Then, the vector function F that enters the variational inequality (42) is strictly monotone, with respect to $(Q^1, Q^2, Q^3, \gamma, \rho_3)$, that is, for any two X', X'' with $(Q^{1'}, Q^{2'}, Q^{3'}, \gamma^{'}, \rho_3') \neq (Q^{1''}, Q^{2''}, Q^{3''}, \gamma^{''}, \rho_3{''})$

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle > 0. \tag{58}$$

Theorem 7 (Uniqueness). *Assuming the conditions of Theorem 6, there must be a unique equilibrium product pattern (Q^{1*}, Q^{2*}, Q^{3*}) and a unique demand price vector ρ_3^* satisfying the equilibrium conditions of the global supply chain network. In other words, if the variational inequality (39) admits a solution, then that is the only solution in (Q^1, Q^2, Q^3, ρ_3).*

Proof. Under the strict monotonicity result of Theorem 6, uniqueness follows from the standard variational inequality theory (cf. [23]). ■

Theorem 8 (Lipschitz Continuity). *The function that enters the variational inequality problem (41) is Lipschitz continuous, that is,*

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } L > 0, \tag{59}$$

under the following conditions:
(i). f^{il}, r^{il}, r^j, c^{il}_{jhm}, $c^{il}_{kh\hat{l}}$, c_j, $\hat{c}^{il}_{jhm}$, $c^j_{kh\hat{l}m}$ have bounded second-order derivatives, for all $i, l, \hat{l}, j, h, k, m$;
(ii). $\hat{c}^j_{kh\hat{l}m}$, $\hat{c}^{il}_{kh\hat{l}}$ and $d_{kh\hat{l}}$ have bounded first-order derivatives for all $j, k, h, l, \hat{l}, m$.

Proof. The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (39). ■

Theorem 9 (Existence and Uniqueness). *Assume the conditions of Theorem 8. Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (48).*

Note that Theorem 9, unlike Theorems 4 and 7, is concerned with the existence of a unique trajectory. Theorems 4 and 7, on the other hand, are concerned with the existence and uniqueness of an equilibrium pattern. Hence, according to Theorem 9, the disequilibrium dynamics of the global supply chain network are well-defined. Also, for completeness, we now provide a stability result (see [42]). First we recall the following:

Definition 3 (Stability of the System). *The system defined by (48) is stable if, for every X_0 and every equilibrium point X^*, the Euclidean distance $\|X^* - X_0(t)\|$ is a monotone nonincreasing function of time t.*

We now provide a stability result.

Theorem 10 (Stability of the Global Supply Chain Network). *Assume the conditions of Theorem 5. Then the dynamical system (48) underlying the global supply chain network is stable.*

Proof. Under the assumptions of Theorem 5, $F(X)$ is monotone and, hence, the conclusion follows directly from Theorem 4.1 of [42]. ∎

In the next section, we propose a discrete-time algorithm, the Euler method, which will track the dynamic trajectories until a stationary state is reached; equivalently, until an equilibrium point is reached satisfying Definition 1.

5 The Euler Method

In this section, we consider the computation of a stationary of (48). The algorithm that we propose is the Euler-type method, which is induced by the general iterative scheme of [13]. It has been applied to-date to solve a plethora of dynamic network models (see, e.g., [34] and [28]). The algorithm not only provides a discretization of the continuous time trajectory defined by (48) but also yields a stationary, that is, an equilibrium point that satisfies variational inequality (39).

The Euler Method

Step 0: Initialization
Set $X^0 = (Q^{10}, Q^{20}, Q^{30}, \gamma^0, \rho_3^0) \in \mathcal{K}$. Let $\mathcal{T}$ denote an iteration counter and set $\mathcal{T} = 1$. Set the sequence $\{a_\mathcal{T}\}$ so that $\sum_{\mathcal{T}=1}^{\infty} a_\mathcal{T}$, $a_\mathcal{T} > 0$, $a_\mathcal{T} \to 0$, as $\mathcal{T} \to \infty$ (which is a requirement for convergence).

Step 1: Computation
Compute $X^\mathcal{T} = (Q^{1\mathcal{T}}, Q^{2\mathcal{T}}, Q^{3\mathcal{T}}, \gamma^\mathcal{T}, \rho_3^\mathcal{T}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\langle X^\mathcal{T} + a_\mathcal{T} F(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1}, X - X^\mathcal{T} \rangle \geq 0, \qquad \forall X \in \mathcal{K}. \tag{60}$$

Step 2: Convergence Verification
If $|X^\mathcal{T} - X^{\mathcal{T}-1}| \leq \epsilon$, with $\epsilon > 0$, a prespecified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

Convergence results for the Euler method can be found in [13]. See the book by [28] for applications of this algorithm to other supernetwork problems in the context of dynamic supply chains and financial networks with intermediation.

Variational inequality subproblem (60) can be solved explicitly and in closed form. This is due to the simplicity of the feasible set $\mathcal{K}$ as formulated above. For completeness, and also to illustrate the simplicity of the proposed computational procedure in the context of the global supply chain network model, we provide the explicit formulae for the computation of the $Q^{1\mathcal{T}}$, the $Q^{2\mathcal{T}}$, the $Q^{3\mathcal{T}}$, the $\gamma^\mathcal{T}$, and the $\rho_3^\mathcal{T}$ below.

5.1 Computation of the Product Transactions

In particular, compute, at iteration $\mathcal{T}$, the $q_{jhm}^{il\mathcal{T}}$s according to:

$$q_{jhm}^{il\mathcal{T}} = \max\Bigg\{0, q_{jhm}^{il\mathcal{T}-1} - a_{\mathcal{T}}(\frac{\partial f^{il}(Q^{1\mathcal{T}-1}, Q^{3\mathcal{T}-1})}{\partial q_{jhm}^{il}} + \frac{\partial c_{jhm}^{il}(q_{jhm}^{il\mathcal{T}-1})}{\partial q_{jhm}^{il}}$$
$$+\frac{\partial c_j(Q^{1\mathcal{T}-1})}{\partial q_{jhm}^{il}} + \frac{\partial \hat{c}_{jhm}^{il}(q_{jhm}^{il\mathcal{T}-1})}{\partial q_{jhm}^{il}} + \omega^{il}\frac{\partial r^{il}(Q^{1\mathcal{T}-1}, Q^{3\mathcal{T}-1})}{\partial q_{jhm}^{il}}$$
$$+ \vartheta_j\frac{\partial r^j(Q^{1\mathcal{T}-1}, Q^{2\mathcal{T}-1})}{\partial q_{jhm}^{il}} + (\alpha^{il} + \beta_j)(\eta^{il} + \eta_{jm}^{il}) - \gamma_j^{\mathcal{T}-1})\Bigg\}$$
$$\forall i, l, j, h, m; \quad (61)$$

the $q_{kh\hat{l}}^{il\mathcal{T}}$s according to:

$$q_{kh\hat{l}}^{il\mathcal{T}} = \max\Bigg\{0, q_{kh\hat{l}}^{il\mathcal{T}-1} - a_{\mathcal{T}}(\frac{\partial f^{il}(Q^{1\mathcal{T}-1}, Q^{3\mathcal{T}-1})}{\partial q_{kh\hat{l}}^{il}} + \frac{\partial c_{kh\hat{l}}^{il}(q_{kh\hat{l}}^{il\mathcal{T}-1})}{\partial q_{kh\hat{l}}^{il}}$$
$$+\hat{c}_{kh\hat{l}}^{il}(Q^{3\mathcal{T}-1}) + \omega^{il}\frac{\partial r^{il}(Q^{1\mathcal{T}-1}, Q^{3\mathcal{T}-1})}{\partial q_{kh\hat{l}}^{il}}$$
$$-(\alpha^{il} + \delta_{kh\hat{l}})(\eta^{il} + \eta_{k\hat{l}}^{il}) - \rho_{3kh\hat{l}}^{\mathcal{T}-1})\Bigg\}, \quad \forall i, l, k, h, \hat{l}, \quad (62)$$

and the $q_{kh\hat{l}m}^{j\mathcal{T}}$s, according to:

$$q_{kh\hat{l}m}^{j\mathcal{T}} = \max\Bigg\{0, q_{kh\hat{l}m}^{j\mathcal{T}-1} - a_{\mathcal{T}}(\frac{\partial c_{kh\hat{l}m}^{j}(q_{kh\hat{l}m}^{j\mathcal{T}-1})}{\partial q_{kh\hat{l}m}^{j}} + \vartheta_j\frac{\partial r^j(Q^{1\mathcal{T}-1}, Q^{2\mathcal{T}-1})}{\partial q_{kh\hat{l}m}^{j}}$$
$$+\hat{c}_{kh\hat{l}m}^{j}(Q^{2\mathcal{T}-1}) + \delta_{kh\hat{l}}\eta_{k\hat{l}m}^{j} + \gamma_j^{\mathcal{T}-1} - \rho_{3kh\hat{l}}^{\mathcal{T}-1})\Bigg\},$$
$$\forall j, k, h, \hat{l}, m. \quad (63)$$

5.2 Computation of the Prices

At iteration $\mathcal{T}$, compute the $\gamma_j^{\mathcal{T}}$s according to:

$$\gamma_j^{\mathcal{T}} = \max\Bigg\{0, \gamma_j^{\mathcal{T}-1} - a_{\mathcal{T}}(\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} q_{jhm}^{il\mathcal{T}-1} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} q_{kh\hat{l}m}^{j\mathcal{T}-1})\Bigg\},$$
$$\forall j, \quad (64)$$

whereas the $\rho^{\mathcal{T}}_{3kh\hat{l}}$s are computed explicitly and in closed form according to:

$$\rho^{\mathcal{T}}_{3kh\hat{l}} = \max \left\{ 0, \rho^{\mathcal{T}-1}_{3kh\hat{l}} - a_{\mathcal{T}}(\sum_{j=1}^{J}\sum_{m=1}^{2} q^{j\mathcal{T}-1}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} q^{il}_{kh\hat{l}} - d_{kh\hat{l}}(\rho^{\mathcal{T}-1}_{3})) \right\}, \quad \forall k, h, \hat{l}. \tag{65}$$

Hence, at a given iteration, all the product transactions and prices can be solved explicitly and in closed form using the above simple formulae. Note that these computations can be done simultaneously, that is, in parallel. The algorithm also can be interpreted as a discrete-time adjustment process in which the product transactions between tiers adjust as well as the prices at the tiers until the equilibrium state is reached. Convergence conditions for the algorithm can be found in [13] and [34].

Note that one may recover the total emissions generated by a particular manufacturer in a country by simply computing the expression (15), with the product transactions at their equilibrium values, and with the summation over all manufacturers in all countries yielding the total number of emissions generated by all the manufacturers. The total amount of emissions generated by the consumers, in turn, in their transactions (cf. (35)) can be obtained by computing the expression: $\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\eta^{j}_{k\hat{l}m}\sum_{h=1}^{H} q^{j*}_{kh\hat{l}m}$.

6 Conclusions and Directions for Future Research

In this paper, we have proposed a framework for the formulation, analysis, and computation of solutions to global supply chain network problems with multicriteria decision-makers and environmental concerns in the presence of electronic commerce. In particular, we have proposed a global supply chain supernetwork consisting of three tiers of decision-makers: the manufacturers who are located in different countries and can trade in different currencies, the retailers, who can be either physical or virtual and need not be country specific, and the consumers associated with the demand markets in different countries who can transact in different currencies. We allowed for both physical and electronic transactions in the form of electronic commerce between manufacturers and retailers and between retailers and the consumers at the demand markets. Moreover, consumers can also obtain the products directly from the manufacturers through e-commerce. We presented both static and dynamic versions of the global supply chain network model with environmental decision-making and linked the equilibrium points of the former with the stationary points of the latter.

This framework generalizes the recent work of [32] in supply chain supernetworks and environmental criteria to the global dimension and to include also explicit risk minimization, which is of a form sufficiently general to also capture environmental risks associated with the various transactions. Theoretical results were obtained along with a proposed discrete-time algorithm for the discretization of the continuous time product transaction and price trajectories. Finally, we demonstrated how the total amount of emissions generated can also be recovered from the equilibrium solution.

Future research may include the incorporation of a variety of policy instruments as well as applying the algorithm to concrete numerical examples.

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Game Theory Models and Their Applications in Inventory Management and Supply Chain

Altannar Chinchuluun,[1] Athanasia Karakitsiou,[2] and Athanasia Mavrommati[2]

[1] Center for Applied Optimization, Department of Industrial and Systems Engineering, University of Florida, Gainesville, Florida, USA
altannar@ufl.edu

[2] Decision Support Systems Laboratory, Department of Production Engineering, Technical University of Crete, Chania, Greece
{atkar,atmavr}@verenike.ergasya.tuc.gr

Abstract Analysis of supply chain politics can benefit from applying game-theory concepts extensively. Game theory tries to enlighten the interactions between individuals or groups of people whose goals are opposed conflicting, or at least partially competing. In this chapter, we review classic game theoretical approaches to modeling and solving certain problems in supply chain management. Both noncooperative and cooperative models are discussed and solution procedures are presented in single-period and multiperiod settings. As used here, a "game" is a metaphor for any interaction among the decision makers in a supply chain.

Key words: inventory, supply chain, noncooperative games, cooperative games, Nash equilibrium, Stackelberg game, transferable utility, core, bargaining, biform games

1 Introduction

Inventory management of physical goods and other products or elements is an integral part of logistic systems common to all sectors of the economy including industry, agriculture, and defense. In a perfectly predictable economy, inventory may be needed in order to take advantage of the economic feature of a particular technology, to synchronize human tasks, or to regulate production process to meet the changing demands. When uncertainty is present, inventories are used as a protection against risk of stock being out.

The existence of inventory in a system generally implies the existence of an organized complex system involving inflow, accumulation, and outflow of some commodities, goods, items, or products. In business, for example, the inflow of goods is generated through procurement, purchase, or production.

The outflow is generated through demand for the goods. Finally, the difference between the rate of outflow and the rate of inflow generates inventory of goods.

The regulation and control of inventory must proceed within the context of this organized system. Rather than being interpreted as idle resources, inventories should be regarded an essential element, the study of which may provide insight in the aggregate operation of the system. The scientific analysis of inventory systems defines the degree of interrelationship between inflow, accumulation, and outflow, and identifies economic control methods for operating such systems.

Traditionally, inventory problems are concerned with a single decision maker, who makes the decisions on the ordered or produced quantity under certain assumptions on the demand, the planning horizon, etc., which the decision maker faces. Although such models capture important aspects of inventory problems, they totally ignore the decisions made by other competitors. In particular, most of such models assume that, if there are two or more products, they cannot be substituted for each other. However, in many real situations this is not true. A customer who cannot find a specific product at one retailer might decide to switch to another retailer who sells the same or a similar product.

It is a fact that, in many production-inventory-transportation problems, one can observe the existence of several decision makers with competitive objectives. In order to have an inventory model, which is able to adequately describe such situations, game theory method should be used. Single-period news-vendor models have typically been used for analyzing such situations [90].

In the current chapter, we are concerned with game theoretic approaches to modeling and solving certain problems in supply chain analysis. The remainder of the paper is organized as follows: Sections 2, 3, and 4 present basic concepts we use throughout the paper. Section 5 presents the application of noncooperative games in inventory management, and in Section 6 their application to supply chain coordination is presented. Section 7 is devoted to cooperative inventory games. Finally, new developments in Game Theory such as bargaining game and biform games, with applications to supply chain, are introduced in Section 8.

2 Basic Concepts in Game Theory

Game theory is a mathematical theory of decision making by participants in conflicting or cooperating situations. Its goal is to explain, or to provide a normative guide for, rational behavior of individuals confronted with strategic decisions or involved in social interaction. The theory is concerned with optimal strategic behavior, equilibrium situations, stable outcomes, bargaining, coalition formation, equitable allocations, and similar concepts related to resolving group differences. Game theory has a profound influence on methodologies of many different branches of sciences, especially those of economics, operations research, and management sciences.

Traditionally, game theory can be divided into two branches: *noncooperative* and *cooperative* game theory. *Noncooperative* game theory uses the notion of a *strategic equilibrium* or simply *equilibrium* to determine rational outcomes of a game. Numerous equilibrium concepts have been proposed in the literature (see [85] for an overview). Some widely used concepts are *dominant strategy*, *Nash equilibrium*, and *subgame perfect equilibrium*.

Nash Equilibrium: Strategies chosen by all players are said to be in Nash equilibrium if no player can benefit by unilaterally changing her strategy. Nash [54, 56] proved that every finite game has at least one Nash equilibrium.

Dominant strategy is one that achieves the highest payoff no matter what the strategies of other players are. In other words, one that is optimal in all circumstances. If strategies are dominant, they also constitute a Nash equilibrium, however, the opposite is not necessarily true.

Subgame perfect equilibrium: Strategies in extensive form are in subgame perfect equilibrium if the strategies constitute a Nash equilibrium at every desicion point.

In *cooperative game theory*, groups of players are taken as primitives and binding agreements can be made between players, which can form coalitions. In such a game, a utility is created when two or more players cooperate and form a coalition. Cooperative game theory can then determine a solution concept that must satisfy a set of assumptions (called axioms). The most important of them are

Pareto optimality: The total utility allocated to the players must be equal to the total utility of the game.

Individual rationality: The utility allocated in each player should be higher than the utility she gains by acting without the coalition.

Kick-back: The utility allocated to a player must always be non-negative.

Monotonicity: If the overall utility increases, the allocation to a player should be higher.

There are several excellent books [5, 42, 51, 62, 71] on the subject, and the reader should turn to them for further details.

3 The Classic Newsboy Problem

The classic newsboy problem is a one-period model in which a firm must choose an inventory level x at a cost c per unit for the perishable product it sells prior to knowing the true level of demand for it. When the demand is realized, the goods are sold at a price r per unit, which is usually assumed to be fixed. Demand is denoted by the random variable w with cumulative distribution $F(W) = P(W \leq w)$, which is assumed to have a continuous density $f(w) = \partial F(W)/\partial w$. Moreover if it is assumed that f is strictly positive on some interval, then F is strictly increasing and therefore it has an inverse function F_w^{-1}. As there is no initial inventory, the quantity ordered by the firm is the total amount available for sale; the firm's sale is the smallest amount between the demand and the inventory level. Excess demand given by $(w-x)^+$ is costly because it results in lost sales. It is therefore penalized by a shortage cost per unit p. Excess inventory, given by $(x-w)^+$, is costly as well because the salvage value s is lower than the cost of procuring inventory. The firm's profit is therefore:

$$\pi = \begin{cases} (r-c)x - (w-x)p \text{ if } x \leq w, \\ (r-c)w + (s-c)x \ \text{ if } x > w. \end{cases} \tag{1}$$

The firm wants to choose an inventory level x to maximize the expected profit.

$$E(\pi) = rE\min\{w, x\} + sE(x-w)^+ - cx - pE(w-x)^+. \tag{2}$$

Equating marginal revenues with marginal costs yields the optimal inventory x^* as the implicit solution of the equation

$$F(x^*) = \frac{r-c+s}{r+p-s} \Rightarrow x^* = F_w^{-1}\left(\frac{r-c+s}{r+p-s}\right). \tag{3}$$

The assumption that F is strictly increasing implies that x^* is unique. If there are no shortage costs and the salvage value is zero, then

$$F(x^*) = \frac{r-c}{r} \Rightarrow x^* = F_w^{-1}\left(\frac{r-c}{r}\right). \tag{4}$$

For a survey on the news-vendor problem and several of its extensions, see [40].

4 The Competitive Newsboy Model

In the competitive newsboy model, substitution often takes place between different products sold by different retailers when the products have stochastic demands. In such a situation, each retailer's profit depends not only on her own order quantity but also on her competitors' order. In other words, if a customer finds the shelves empty at the first firm she visits, she does not necessarily give up but may travel to another firm in order to satisfy her demand. The actual substitution between any two retailers takes place according to a substitution rate that depends on their products and other factors such as location.

The simplest competitive model has two retailers i and j; each one of them faces a demand w_i and w_j, respectively. Therefore $w = w_i + w_j$ is the industry demand. This allocation of the initial demand to each firm follows some specific splitting rules. If there exists excess demand $(w_i - x_i)^+$ at firm i, then the same proportion of the excess demand should be met by the inventory of firm j. That is, a reallocation of the initial demand at firm i occurs. Hence, the actual demand the firm j faces is

$$R_j = w_j + \beta_i(w_i - x_i)^+, \tag{5}$$

where $\beta_i \in [0, 1]$ is the substitution rate at which i's excess demand is allocated to firm j.

If x_i and x_j denote the firms's inventory levels respectively, then the expected profit for the firm j is

$$E[\pi_j(x_i, x_j)] = rE\min\{x_j, R_j\} + sE(x_j - R_j)^+ - pE(R_j - x_j)^+ - cx_j. \tag{6}$$

Parlar [63] is perhaps the first author to treat an inventory problem using game theory. She examines an extension of the classic newsboy problems in which two retailers (players) sell substitutable products. She modeled the two-product single-period problem as a two-person nonzero-sum game and showed that there exists a unique Nash equilibrium. In her two-player model, substitution occurs with a certain probability.

5 Noncooperative Solution

Noncooperative solution deals with how rational individuals interact with one another in an effort to achieve their own goals. The emphasis is on the strategies of players and the consequences of interaction on payoffs. The purpose is to make predictions on the outcome. The solution concepts that are commonly used are the Nash equilibrium introduced by J.F Nash [54] and the Stackelberg equilibrium introduced by the economist von Stackelberg [89].

5.1 Nash Equilibrium

Single-Period Model Formulation

A *Nash equilibrium* recommends a strategy to each player that the player cannot improve upon unilaterally, that is, given that the other players follow the recommendation. Because the other players are also rational, it is reasonable for each player to expect opponents to follow the recommendation as well. A vector $x^* = (x_i^*)_{i \in N} \in X$ is a Nash equilibrium if and only if for all $i \in N$

$$\pi_i(x_i^*, x_{-i}^*) \geq \pi_i(x_i, x_{-i}^*) \quad \forall x_i \in X. \tag{7}$$

In a Nash equilibrium, each player is doing the best she can do given the strategies of the other players, x_{-i}, i.e., player i has no incentive to deviate from x_i^* when all other players play x_{-i}^*.

Player's i *best response* (function) is the strategy x_i^* that maximizes the player's i payoff. That is

$$x_i^*(x_{-i}) = \arg \max_{x_i} \pi_i(x_i, x_{-i}). \tag{8}$$

The best response function is uniquely defined by the first-order condition if π_i is quasi-concave in x_i. The Nash equilibrium assumes no one of the players has the power to dominate the decision process.

When there is no cooperation between the firms and if both firms are "rational," one of the possible strategies they may adopt is the Nash strategy. A pair of inventory levels (x_i, x_j) $(i, j = 1, 2)$ is a *Nash equilibrium* if neither firm can improve its expected profit by altering its inventory, that is

$$\begin{aligned} E[\pi_i(x_i^*, x_j^*)] &\geq E[\pi_i(x_i, x_j^*)] \ \forall \ \ x_i \geq 0 \ \ (i, j = 1, 2) \\ E[\pi_i(x_i^*, x_j^*)] &\geq E[\pi_i(x_i^*, x_j)] \ \forall \ \ x_j \geq 0 \ \ (i, j = 1, 2). \end{aligned} \tag{9}$$

Equation (9) implies that, given the player's j Nash solution x_j^*, player i will not do better if she does not play her Nash solution x_i^*. That is, given the x_j^* , x_i^* maximizes player's objective function, and vice versa. Therefore, the best response for each player will be

$$x_i^*(x_j) = F_{R_i}^{-1}\left(\frac{r_i - c_i + s_i}{r_i + p_i - s_i}\right) \quad (i, j = 1, 2). \tag{10}$$

The best response function can be found by optimizing each player's expected profit function w.r.t the player's own order quantity, provided that $E[\pi_i]$ is continuously differentiable in x_i and it is concave for every x_j. Taking together the best response function of each player, we obtain a best response mapping $R^2 \to R^2$ (see Figure 1). Obviously if x_i^* is a best response to $x_j^*, \forall \ \ (i, j = 1, 2)$, then the outcome (x_i^*, x_j^*) is a *Nash equilibrium*. Parlar [63, Lemma 1-2, pp. 403–04] has proved that the slope of the best response

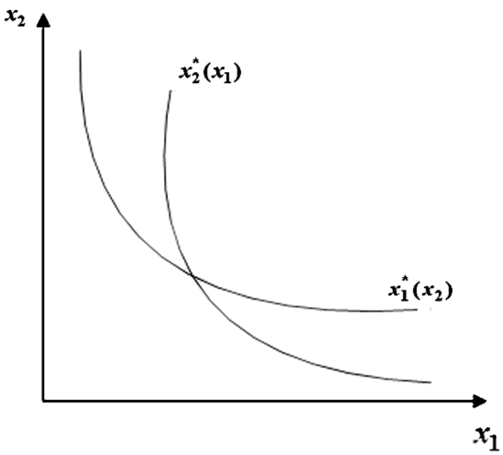

Figure 1. Best response functions in an inventory game

functions is negative, which implies that each player response is monotonically decreasing in the other player's strategy. Indeed, according to [63], the player's Nash solution is a unique point (x_i^*, x_j^*) obtained by solving a system of best responses:

$$x_1^*(x_2^*) = F_{R_1^*}^{-1}\left(\frac{r_1 - c_1 + s_1}{r_1 + p_1 - s_1}\right) \tag{11}$$

$$x_2^*(x_1^*) = F_{R_2^*}^{-1}\left(\frac{r_2 - c_2 + s_2}{r_2 + p_2 - s_2}\right) \tag{12}$$

where $R_i^* = w_i + \beta_i(w_j - x_j^*)^+, i, j = 1, 2$.

Historically, most researchers establish the existence of an equilibrium based on the study of the concavity or quasi-concavity of profit function. Dasgupta and Maskin [27], Parlar [63], Mahajan and van Ryzin [48], Netessine *et al.* [59], among others establish the existence of a Nash equilibrium based on the two above-mentioned properties of the profit function.

However, the existence of a Nash equilibrium for a general case can be established by employing the result of the supermodular game. A function $f(x_1, x_2)$ is supermodular if $f(x_1, x_2) + f(y_1, y_2) \geq f(x_1, y_2) + f(y_1, x_2)$, for all $(x_1, x_2) \geq (y_1, y_2)$. Notice that supermodularity is a weaker condition than concavity, see [95] for a detailed discussion. If the profits are supermodular, then the best response mapping is increased in the other player's strategy. When the best response function has such a monotonicity property, the existence of a Nash equilibrium could be established. The theory of supermodular games is a relatively recent development introduced and advanced by Topkis [84]. See [12, 16, 46, 58, 60] for its application to the competitive news-vendor problem.

As an extension of the model in [63], Wang and Parlar [92] studied the three-product single-period problem. Lippman and McCardle [46] also study an extension of the classic news-vendor problem in which the salvage value of excess inventory and penalty for unmet demand are assumed to be zero. Under this assumption, they examine the equilibrium inventory levels and the rules to reallocate excess demand. They provide conditions under which a Nash equilibrium exists for the case with two or more news-vendors. They examine both the two-firm game and a game with an arbitrary number of players. In their models, initial industry demand is allocated among the players according to a prespecified "splitting rule." This initial allocation may be either deterministic or stochastic. For the two-firm game, they establish the existence of a pure-strategy Nash equilibrium and show that the equilibrium is unique when the initial allocation is deterministic and strictly increasing in the total industry demand for each player. They have proved that competition can lead to higher inventories.

Mahajan and van Ryzin [48] study a model with n retailers that provides substitutable goods, assuming that the demand process is a stochastic sequence of heterogeneous consumers who choose dynamically from the available goods (or choose not to purchase) based on a utility maximization criterion. They demonstrate that an equilibrium exists and show that it is unique for a symmetric game. Their results are similar to [46].

Recent extensions of these models include the work by Rudi and Netessine [75]. They analyze a problem similar to [63] but for an arbitrary number of products. Given mild parametric assumptions, they establish the existence of, and characterize, a unique, globally stable Nash equilibrium. On the other hand, with the substitution structure of their model, they conclude that, under competition, some firms may stock less than under centralization. Under the long-run average payoff criterion, the nonlinear programming formulation developed by Filar *et al.* [33] can be used to compute Nash strategies. If the discounted payoff criterion is considered, then Nonlinear Program (NLP) due to Raghavan and Filar [72] is available. Chand *et al.* [21] and Drezner *et al.* consider the case where the substitution between products take place in a EOQ model.

Multiperiod Model Formulation

Because inventory models used in the literature often involve inventory replenishment decisions that are made over an infinite period of time, multiperiod games should be a logical extension of these inventory models. In the analysis of substitutable product inventory problem over infinite horizon, concepts of *sequential games*, introduced by [38], are used. Two retailers of different products who compete for the substitutable demand of these products are the players of the game. Each player's decision sequence influences the evolution of the process and affects the streams of rewards to all players. A sequential game is said to have a *myopic* solution if its data can be used easily to specify a one-period game such that infinite repetition of a Nash equilibrium of the one-period game comprises an equilibrium for the sequential game.

The mathematical formulation considered is a nonzero-sum game because what is earned (or lost) by one retailer may not be the loss (or earning) of the other retailer although what is earned or lost by each retailer depends on both strategies, not the strategy taken by just that retailer. Demand distributions of the products and the substitution rates are known by both players. So, being aware of all of the parameters and the strategies that can be employed by the opponent, each retailer tries to find out the best strategy as a reply to the opponent. Because the retailers somehow agree (although they do take their actions independently in a strictly competitive environment, they know all the parameters that would affect their decisions) on a pair of strategies, called Nash strategies in the context of nonzero-sum games, this pair is said to be an equilibrium point. Unilateral deviations of either of the players from her Nash strategy do not improve her expected payoff.

Specifically, in a multiple-period setting, we consider two retailers that simultaneously make inventory replenishment decisions at the beginning of each period using a periodic review base-stock policy. If one retailer experiences a stock-out, a portion of the customers who are not satisfied will switch to the other retailer. Leftover inventory at the end of the period is carried over to the next period, incurring inventory holding cost.

At the beginning of each period t, $(t = 1, 2, \ldots, n)$, two retailers review their inventories and simultaneously make replenishment decisions. Let w_i^t denote the exogenously given (random) demand for the product of retailer i in period t. Product i is sold for r_i per unit, $(i = 1, 2)$. Ordering cost is a linear function of the order quantity x_i^t for product i in period t. c_i, which satisfies $0 < c_i < r_i$, is the ordering cost per unit of product i. Let I_i^t be the inventory levels of the retailer's i, at the beginning of period t. Orders are delivered instantaneously so that $z_i^t = I_i^t + x_i^t$ are the inventory levels just after the orders are replenished. p_i is the unit lost sale cost, and h_i is the inventory holding cost per unit of product i per period. Substitution rates are given as the probabilities that a customer switches from one type of product to the other when the product demanded is sold out. β_i is the substitution rate at which i's excess demand is allocate to firm j. Further, the actual demand for retailer i depends on the beginning inventory of retailer j in period t I_j^t as well as on her own beginning inventory level at period t, I_i^t. That is,

$$R_i^t = w_i^t + \beta_j (w_j^t - z_j^t)^+ \quad i, j = 1, 2 \quad t = 1, 2, \ldots, n \tag{13}$$

The inventory balance equations are

$$I_i^{t+1} = [z_i^t - w_i^t - \beta_j (w_j^t - z_j^t)^+]^+ \quad i, j = 1, 2, \quad t = 1, 2, \ldots, n \tag{14}$$

Note that if retailer j cannot satisfy demand w_j^t fully, then the remaining demand $[w_j^t - z_j^t]^+$ switches to retailer i or vice versa. By suppressing subscript t, i.e., considering the order-up-to-levels as $z_i = I_i + x_i$, $i = 1, 2$, when the order-up-to-levels (z_1, z_2) are chosen by the two retailers in a singe period

$$E[\pi_i(z_1, z_2)] = r_i E \min\{R_i, z_i\} - h_i E(z_i - R_i)^+ - p_i E(R_i - z_i)^+ - c_i x_i \tag{15}$$

is the one-period expected profit for retailer i.

Because future payoffs are in general worth less today, it is reasonable to look at discounted payoffs. Suppose that each retailer starts with initial inventories (I_1^1, I_2^2) respectively, the expected discounted profit of retailer i for the remaining period until the end of the planning horizon is given by:

$$E[\pi_i] = E \sum_{t=1}^{\infty} \delta_i^{t-1} \left[r_i \min\{z_i^t, R_i^t\} - h_i (z_i^t - R_i^t)^+ - p_i (R_i^t - z_i^t)^+ - c_i x_i^t \right]. \tag{16}$$

The discount factor is assumed stationary and will be denoted by δ, $0 < \delta < 1$. By using manipulations proposed by Heyman and Sobel in [38], the objective function can be converted to:

$$E[\pi_i] = c_i x_i^1 + \sum_{t=1}^{\infty} \delta_i^{t-1} G_i^t(z_i^t), \quad i = 1, 2 \tag{17}$$

where $G_i^t(z_i^t)$ is the single-period objective function. If we assume that demand is stationary and independently distributed among periods, i.e., $w_i = w_i^t$, we obtain that $G_i^t(z_i^t) = G_i(z_i)$, furthermore if we assume that the inventory policy is stationary as well, i.e., $z_i^t + z_i, t = 1, \ldots, n$, then each retailer could solve the problem under consideration as a sequence of the solution to a single-period game, which is

$$z_i^* = F_{R_i^*}^{-1} = \left(\frac{r_i - c_i}{r_i + h_i + p_i - c_i \delta_i} \right) \quad i = 1, 2. \tag{18}$$

For a complete analysis, see Netessine *et al.*[59].

Avsarand and Baykal-Gürsoy [4] analyzed the substitutable product inventory problem using the concepts of stochastic game theory. It is assumed that there are two substitutable products that are sold by different retailers and the demand for each product is random. Game theoretic nature of this problem is the result of substitution between products. Because retailers compete for the substitutable demand, ordering decision of each retailer depends on the ordering decision of the other retailer. Under the discounted payoff criterion, this problem is formulated as a two-person nonzero-sum stochastic game. In the case of linear ordering cost, it is shown that there exists a Nash equilibrium characterized by a pair of stationary base-stock strategies for the infinite horizon problem. This is the unique Nash equilibrium within the class of stationary base-stock strategies.

In addition, more elaborate models capture some effects that are not present in static games. Netessine *et al.* [59] consider the case where when a product is out of stock, the customer often faces a choice of either placing a backorder or turning to a competitor selling a similar product. They consider the four alternative backordering scenarios and formulate each problem as a stochastic dynamic game. They proved that a stationary base-stock inventory policy is a Nash equilibrium of the game and hence it can be found by considering an appropriate static game.

van Mieghem and Dada [86] study a two-period game with capacity choice in the first period and production decision under the capacity constraint in the second period.

5.2 Stackelberg Equilibrium

Stackelberg equilibrium assumes that there is a player who has powerful position and dominates in the desicion process, *the leader*, and the other players, *the followers*, given that they are rational, are free to choose their optimal strategies given their knowledge of the leader's decision. If player i is the leader, she will choose her optimal strategy x_i^*, and the followers's best response x_{-i}^* will be

$$x_{-i}^*(x_i^*) = \{x_{-i}^* | \pi_{-i}(x_i, x_{-i}^*) \geq \pi_{-i}(x_i, x_{-i}\} \tag{19}$$

To find an equilibrium of a Stackelberg game, which is often called the Stackelberg equilibrium, we need to solve a dynamic multiperiod problem via backwards induction.

In a Stackelberg game, one firm, called leader, makes an order first, then the other firm, called follower, makes her order. Because the follower makes her decision after the leader announces hers, the Stackelberg solution will be located on the reaction curve of the follower's defined by equation:

$$x_2^*(x_1) = F_{R_2}^{-1}\left(\frac{r_2 - c_2 + s_2}{r_2 + p_2 - q_2}\right) \tag{20}$$

which means that the follower will always choose her order quantity x_2 to maximize her expected profit for each value of x_1. Intuitively, the leader chooses the best possible point on the follower's best response function; i.e., she tries to solve the following bilevel programming model [63]:

$$\max \quad E[\pi_1(x_1, x_2)] \tag{21}$$

$$\text{where} \quad x_2 \quad \text{solves}$$

$$\frac{\partial E[\pi_2(x_1, x_2)]}{\partial x_2} = 0. \tag{22}$$

Whereas the existence of a Stackelberg equilibrium is easy to demonstrate given the continues payoff function, uniqueness may be considerably harder to demonstrate [18].

Raju and Zhang [69] analyze the Stackelberg game in which one of the retailers is dominant and capable of unilaterally setting a retail price that will be adopted by all other retailers.

Lariviere and Porteus [43] consider a simple supply-chain contract in which a manufacturer sells to a retailer facing a news-vendor problem. The Stackelberg game they set up assumes that first the supplier establishes the wholesale price and then the news-vendor chooses an order quantity, the long contract parameter is the wholesale price. They show that the manufacturer's profit and sales quantity increase with market size, but the resulting wholesale price depends on how the market grows. Anand *et al.* [1] extend the Stackelberg equilibrium concept into multiple periods.

See Netessine and Rudi [57] for a Stackelberg game.

6 Supply Chain Coordination

In another line of research, there exists a large body of research that addresses echelon inventory system with the stationary stochastic demand and fixed lead time. Many of them use the following two-echelon gaming structure: a "manufacturer" wholesales a product to a $n \geq 1$ "retailers," who in turn retail it to the consumer. The literature on competitive supply chain inventory management recognizes that supply chain is usually operated by independent agents with individual preferences and possibly conflicting objectives.

Total expected supply chain profit will be maximized if all decisions are made by a single decision maker with access to all available information. This is referred as the *optimal case* or *first-best case* and is often associated with *centralized control*. Under centralized control, a system manager needs to know how to design a mechanism to optimize the performance of the whole supply chain.

However, in reality, no single agent has control over the entire supply chain, and hence no agent has the power to optimize the supply chain, and each player has his own incentives and state of information. This is referred as a *decentralized control* structure. Under decentralized control, each player needs to know how to behave in order to maximize his profit. In order to increase the total profit of a decentralized supply chain and improve the performance of the players, one strategy is to form contracts among players by modifying their payoffs. The main purpose of a supply chain contract is to overcome an inefficiency known as *double marginalization* [79]. This is because without coordination, the supplier and the retailer only have the incentive to optimize their own profit margin, and their collective decision is always less efficient than what could have been achieved by the system-optimal. Thus, the aim of a coordination contract is to provide the incentive for both players to implement the system-optimal solution, which results in higher total profits for the collective whole. Some contracts provide a means to bring the total profit resulting from decentralized control to the centralized optimal profit. This is referred to as *channel coordination*. Generally speaking, channel coordination may be achieved by three steps: First, determine the optimal solution under centralized control. Next, under decentralized control, apply game theory to determine how the players will behave when they each seek to maximize their own profits, and whether a Nash equilibrium exists. Finally, if the decentralized and centralized solutions differ, investigate how to modify the players' profit so that the new decentralized solution matches the centralized solution.

Consider the case of a supply chain that consists of two echelons: the first echelon is the supplier, usually the manufacturer, and the second echelon consists of two retailers. At the beginning of the period, the retailers place orders x_i and sell them to the customer at a unit price r_i. Supplier produces the product with unit production cost k and supplies x_i units to retailers at a price c_i. It is also assumed that supplier has infinite production capacity. The demand w_i during the period at each retailer is random but distributions are known. Customers encountering a stock-out at retailer i visit retailer j, $(i, j = 1, 2)$ with probability β_{ij} before leaving the system.

Thus, the total demand faced by retailer i is

$$R_i = w_i + \beta_{ij}(w_j - x_j)^+. \tag{23}$$

At the end of the season, the holding cost h_i or shortage cost p_i is incurred depending on whether there is unsold stock or a stock-out.

Assuming the whole supply chain is in centralized control, in order to maximize the total profit, what are optimal orders for both retailers?

In centralized control, a stock-out penalty incurred only when customers leave the system unsatisfied. This includes customers who visit only one retailer and leave unsatisfied and customers who visit both retailers and leave unsatisfied. In the latter case, the amount of the penalty incurred is assumed to be the stock-out penalty cost of the retailer visited first by the customer. The total profit is maximized if supplier should only provide what is needed by the two retailers, i.e., supplier does not face any shortage or holding cost. The expected total profit of the system $E[\pi(x_i, x_j)]$ is

$$E[\pi(x_1, x_2)] = E\left[\sum_{i=1}^{2} r_i \min\{y_i, R_i\} - \sum_{i=1}^{2} h_i(x_i - R_i)^+ - \sum_{i=1}^{2} p_i E(R_i - x_i)^+ - k\sum_{i=1}^{2} x_i\right], \tag{24}$$

which only depends on the retailers's sales quantity. We consider the whole supply chain as an entity, and the money flow within the system is not involved. Therefore, the optimal solution (x_1^*, x_2^*) does not depend on the wholesale prices, c_1 and c_2. Actually, supplier's price decision creates only a transfer payment among firms so it does not influence supply chain's profit.

Because equation (24) is concave, the optimal order quantity of both retailers can be found by solving the system of equations:

$$\begin{aligned} \frac{\partial E[\pi(x_1, x_2)]}{\partial x_1} &= 0 \\ \frac{\partial E[\pi(x_1, x_2)]}{\partial x_2} &= 0 \end{aligned} \tag{25}$$

If the supply chain is under decentralized control, each retailer tries to maximize his own profit. Therefore retailer $i \quad (i = 1, 2)$ profit will be

$$E[\pi_i(x_1, x_2)] = r_i E \min\{x_i, R_j\} - h_i E(x_i - R_i)^+ - p_i E(R_i - x_i)^+ - c_i x_i. \tag{26}$$

Because the decision of one retailer affects the total demand at the other retailer, a game arises as the two retailers make their ordering decisions. Based on the previously presented we know that the Nash equilibrium can be obtained by solving of best responses:

$$x_1^*(x_2^*) = F_{R_1^*}^{-1}\left(\frac{r_1 - c_1}{r_1 + p_1 + h_1}\right) \tag{27}$$

$$x_2^*(x_1^*) = F_{R_2^*}^{-1}\left(\frac{r_2 - c_2}{r_2 + p_2 + h_2}\right) \tag{28}$$

$$\text{where} \quad R_i^* = w_i + \beta_{ij}(w_j - x_j^*)^+, i, j = 1, 2$$

and depends on the wholesale prices c_i. If the chain is not coordinated, each retailer selfishly optimizes its own profit. Hence decentralized decision making may introduce inefficiency in the supply chain as a Nash equilibrium may not be *Pareto optimal*, see [14] for a use of Pareto optimality in the supply chain analysis.

The supply chain coordination can be obtained by determining the wholesale price c_i so as to make the optimal solution (x_1^*, x_2^*) obtained by (25) a Nash equilibrium, that is (x_1^*, x_2^*) must satisfy:

$$\begin{aligned} \frac{\partial E[\pi(x_1, x_2)]}{\partial x_1}|_{x_1=x_1^*, x_2=x_2^*} = 0 \\ \frac{\partial E[\pi(x_1, x_2)]}{\partial x_2}|_{x_1=x_1^*, x_2=x_2^*} = 0. \end{aligned} \tag{29}$$

The coordination mechanism modifies each decision maker's objective so that these modified objectives and the total objective of the supply chain yield to the same optimal solution. The mechanism that is mainly used for coordination in the supply chain is a contract. A contract is an argument between two parties. Most supply chain contracts include only two parties usually a supplier and a retailer, but these simplifications allow for studying optimal contracts. Different models of Supply Chain contracts have been developed in the literature. They include the quantity discounts [93], the backup agreements [31], the buy back or return policies [30], the quantity flexibility (QF) contracts [82], the incentive mechanisms [44], and the revenue sharing (RS) contracts [15].

Anupindi and Bassok [2] studied a model with one manufacture and two retailers. They consider two systems: one competitive, where they make independent decisions and stock inventories separately, and one where they cooperate to centralize stocks at a single location. They show that there exits a threshold level for the "market search" above which manufacturer loses, and that for high level of market search, even total supply chain's profit may decrease upon centralization. Market search is measured as the fraction of customers who due to a stock-out at their retailer search for the good at the other. In addition, they show that manufacturer could benefit, in either system, by offering a contract with a holding cost subsidy.

Cachon and Zipkin [19] investigate a two-stage (supplier and retailers) serial supply chain with stationary stochastic demand, fixed transportation time over an infinite horizon, and complete backordering. Both firms incur holding costs and a backorder penalty per unit of time for each unit that is backordered at the retailer; the supplier is not charged for its own backorders, it is only charged when units are backordered at the retailer. That fee reflects the supplier's desire to maintain an adequate stock of its product at the retailer. They compare the base-stock policies chosen under the competitive regime to those selected so to minimize total supply chain costs. Furthermore, they use a linear contract between the supplier and the retailer to modify the payoff of the players and make the total profit close to the global optimum. The model proposed by Cachon [11] is also a two-echelon serial supply chain with stochastic consumer demand. But when a customer arrives at the retailer and the retailer has no stock, a lost sale occurs. As in [19], both firms are concerned about the availability of inventory at the retailer, but in this model stock-outs create opportunity costs rather than backorder penalties.

Wang *et al.* [91] extend Cachon and Zipkin's model to a one-supplier and n-retailers situation. If there exist multiple retailers, the supply from a supplier might not satisfy the demand of multiple retailers. The problem is how to design the distributing scheme of the supplier, and this makes models of supply chain systems more complex. In order to guarantee optimal cooperation in the system, several Nash equilibrium contracts are designed in echelon inventory games and local inventory games.

Coordination technique proposed by Lee and Wang [44] assumes a two stage supply chain with stationary demand, with holding and backorder costs and fixed lead time. Furthermore, assume that supplier cares only about his inventory. The nonlinear transfer payments proposed by Lee and Wang uses the nonlinear transfer payment proposed by Clark and Scarf [24] but this type of payment leads to a Nash equilibrium for the decentralized supply chain. Using similar assumptions, Chen [22] studied a four-stages supply chain where players try to minimize total supply chain costs. The coordination scheme he proposed is linear transfer payments based on accounting inventory and backorders level, where stage is accounting inventory is the actual inventory that could fill its orders at stage $i + 1$ immediately. Porteus [68] proposed a incentive scheme that is a combination of the above two, called responsibility token.

In contrast, Klastorin *et al.* [41], to coordinate a two-echelon distribution system, use price discounts contract. The supplier, in order to influence the buyer's behavior, offers a price discount to any retailer who places an order that coincides with the beginning of retailer's cycle. They show that under specific conditions, this policy can lead to more efficient supply chain management, and present a method for determining the optimal price discount in the decentralized supply chain. For excellent reviews on supply chain coordination and contracts, see Tsay *et al.* [83] and Cachon [13].

6.1 Capacity Allocation in Supply Chain

In many situations, a single supplier provides products to several retailers. If retailers orders are uncertain and capacity is costly, the supplier may not be willing to have capacity that is high enough to cover all orders at any point in time. When the total order from retailers exceeds the supplier's capacity, then he must allocate it among retailers based on some sort of rules. In such a case, the two retailers compete for both supply and demand, and a game called *allocation game* or *shortage game* as is refereed by Lee *et al.* [45] arises.

Three allocation rules are commonly used: *proportional*, where the retailer receives a proportion of the avaliable capacity as a percentages of his order to the total orders; *linear*, where the retailer receives his order minus the difference between total order and capacity divided by the number of retailers; and *uniform*, where the supplier equally divides the avaliable capacity among retailers [17]. However, when the supplier's capacity is finite, the Nash equilibrium exists only under certain conditions.

Assuming that the avaliable capacity of the supplier is κ, suppose that retailer i makes an order $x_i < \kappa, (i = 1, 2)$, then retailer $j, (j = 1, 2)$ can react by making an order either $x_j \leq \kappa - x_i$, where because total orders do not exceed κ, each retailer gets exactly what she orders, or by making an order $x_j > \kappa - x_i$, then the capacity is apportioned by any allocation rule such that $\bar{x}_1 + \bar{x}_2 = \kappa$. Note that what the retailer i gets, $\bar{x}_i (i = 1, 2)$, differs from what he orders. Further assume that a pair of (x_1^*, x_2^*) is the unique Nash equilibrium that solves the news-vendor problem faced by retailers; if $x_1^* + x_2^* \leq \kappa$, then there exists a unique Nash equilibrium allocation as neither retailer has a profitable unileteral deviation.

Now, consider the case where $x_1^* + x_2^* > \kappa$, let $\hat{x}_1 \in x_2^*(x_2)$ for which it holds that $\hat{x}_1 + \bar{x}_2 = \kappa$ and $\hat{x}_2 \in x_2^*(x_1)$ such that $\bar{x}_1 + \hat{x}_2 = \kappa$, then there exists a Nash equilibrium if and only if there exist a pair of allocations $(\bar{x}_1, \bar{x}_2) \in (\hat{x}_1, \hat{x}_2)$ such that $\bar{x}_1$ is the optimal solution to the problem:

$$\begin{aligned} \max \qquad & E[\pi_1(\bar{x}, \kappa - \bar{x})] \\ \text{s.t } \max\{\kappa - \bar{x}_2, 0\} & \leq \bar{x} \leq \kappa \end{aligned} \tag{30}$$

and $\bar{x}_2$ solves the problem:

$$\begin{aligned} \max \qquad & E[\pi_2(\bar{x}, \kappa - \bar{x})] \\ \text{s.t } \max\{\kappa - \bar{x}_1, 0\} & \leq \bar{x} \leq \kappa. \end{aligned} \tag{31}$$

See Dai [26] for a detailed analysis and proofs, and [16, 45] for application of shortages game in the supply chain.

7 Cooperative Games

The subject of cooperative games was first introduced by von Newmann and Mörgestern [88]. Cooperative game theory assumes that binding agreements can be made between players on the advantage of the whole system. One of the main questions is whether the cooperation is stable, i.e., there is an allocation of the total benefit of the system among the players such that no group of players would like to leave the system. Cooperative game theory offers the concept of core as a direct answer to that question. For a long time, cooperative game theory did not enjoy as much attention in inventory management literature as noncooperative game theory. Papers employing cooperative game theory to study inventory problems had been scarce but are becoming more popular. The vast majority of them model the system in a news-vendor setting. In a news-vendor environment, retailers can increase their total profit if they decide to cooperate. The basic cooperation rules that could appear are (1) cooperative players might switch their excess inventory, if any, to anyone who has excess demand so that the latter can save in lost sales penalty cost, and (2) retailers might give a joint order and use this quantity to satisfy the total demand they are faced with. The allocation rules for this cooperation should be based on three criteria, namely nonemptiness of the core, computational ease, and justifiability [36].

If there are $n > 2$ players in the game, then there might be cooperation between some, but not necessarily all, of the players. We can ask which coalitions of players are likely to form and what are the relative bargaining strengths of the coalitions that do form. Label the players $1, 2, \ldots, n$. A coalition of players, S, is then a subset of $N = (1, 2, \ldots, n)$. Let $v(S)$ denote the maximum value $v(S)$ that coalition S can guarantee itself by coordinating the strategies of its members, no matter what the other players do. This is the called the *characteristic function*. By convention, we take $v(\emptyset) = 0$. The worst eventuality is that the rest of the players unite and form a single opposing coalition $T = N - S$. This is then a 2-person noncooperative game and we can calculate the maximum payoff that S can ensure for itself. In such a case, the cooperation is worthwhile, that is, any two groups, which act together, will get no less than that when they act independently. In other words, the property of superadditivity holds, i.e.,

$$v(S \cup T) \geq v(S) + v(T). \tag{32}$$

The distribution of individual rewards will affect whether any coalition is likely to form. Each individual will tend to join the coalition that offers her the greatest reward. Therefore, the game in such a form should provide an indication of how the joint maximum payoff $v(N)$ should be shared among the N players. *An imputation* for an n-person game, with characteristic function v, is defined as a distribution vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ satisfying:

$$\sum_{i}^{n} x_i = v(N) \quad \& \quad x_i \geq v(i) \quad \forall \quad i \in N \tag{33}$$

with x_i being the payoff to player i.

In other words, if x is an indication of how the joint payoff $v(N)$ is distributed among the players and if a player i is rational, then she is willing to join a coalition if and only if she gets no less than the amount she can get by acting independently. The first condition is often referred to as group rationality and the second condition as individual rationality.

Let $E(v)$ be the set of imputations, and $\mathbf{x}, \mathbf{y} \in E(v)$. We say that $\mathbf{y}$ dominates $\mathbf{x}$ over S if

$$y_i > x_i \quad \forall \quad i \in S \quad \& \quad \sum_{i \in S} y_i \le v(S). \tag{34}$$

In other words, an imputation is dominated if it is dominated via some coalition $S \subseteq N$. Members of the dominating coalition S benefit from forming S and leaving the grand coalition.

The core of a game with characteristic function v is the set, $C(v)$, of all imputations that are not dominated for any coalition. Therefore, an imputation $\mathbf{x}$ is in a core if and only if:

$$\sum_{i}^{n} x_i = v(N) \;\; \& \;\; \sum_{i \in S} x_i \ge v(S) \;\; \forall \;\; S \subseteq N. \tag{35}$$

The core collects undominated imputations. The core is a set-valued solution concept for cooperative games, as it can select multiple payoff vectors. Nonemptiness of the core means that there exists at least one allocation of the joint profits among the players such that no group of players has an incentive to leave. A game is balanced if it has a nonempty core (see Bondareva [9], Shapley [77]), and it is called totally balanced if each subgame $(S, v_{|S})$ is balanced, where $v_{|s}(T) = v(T)$ for all $T \subseteq S$. A subgame is any part of a game that remains to be played after a series of moves and it starts at a point where both players know all the moves that have been made up to that point [25].

Perhaps the first paper employing cooperative games in inventory management is Wang and Parlar [92]. In a model of inventory competition with fixed prices, they use cooperative game theory in one of its original uses: they start with a noncooperative game, then suppose that the players can cooperate on strategy choices with and without Transferable Utility.

What follows is based mainly on Slikker *et al.* [78]. A general cooperative news-vendor situation is characterized by a set of retailers N, the stochastic demand W_i for the good at retailer $i \in N$, furthermore c_i and r_i denote the prices that retailers pay to producer and the customers pay to the retailers, respectively. If several companies cooperate they can, after the realization of demand is known, transship goods. t_{ij} represents the cost of transshipping one unit from i to j, $i, j \in N$ and $t_{ij} \ge 0$. Let X^S be a collection of possible order vector of coalition S retailers defined by:

$$X^S = \left\{ x \in \mathbb{R}^N | x_i^S = 0 \;\; \forall \;\; i \in N \setminus S \;\; \text{and} \;\; x_i^S \ge 0 \;\; \forall \;\; i \in S \right\} \tag{36}$$

and suppose that coalition S has order vector $x^S \in X^S$ and they face demand vector $w^S \in \mathbb{R}^N$ with $w^S = 0$ for all $i \in N \setminus S$. If after the realization of demand, A_{ij}^S is the amount of products that are transshipped from retailer i to retailer j, the amount that is not transshipped is represented by A_{ij}^S for $i = j$. A reallocation matrix of x^S is then

$$A^S = \{A^S \in \mathbb{R}_+^{N\times N} | A_{ij}^S = 0 \ \text{ if } \ i \notin S \ \text{ or } \ j \notin S$$
$$\sum_{j\in S} A_{ij}^S = x_i^S \ \ \forall \ \ i \in S\} \tag{37}$$

The profit of the coalition S is

$$\pi^S(x^s, w^s) = \sum_{j\in S} r_j \min\left\{\sum_{i\in S} A_{ij}^S, x_j^S\right\} - \sum_{i\in S}\sum_{j\in S} A_{ij}^S t_{ij} - \sum_{i\in S} c_i x_i^S \tag{38}$$

The expected profit of coalition S depends on their order quantity vector x^S and the stochastic demand faced by each retailer, W^S, that is

$$\bar{\pi}^S(x^S, W^S) = E\left[\pi^S(x^S, W^S)\right] \tag{39}$$

and the associated game is defined by

$$v(S) = \max_{x\in X^S} \bar{\pi}^S(x, W^S) \ \ \forall \ \ S \subseteq N. \tag{40}$$

Slikker *et al.* [78] proved that there exist coalitions and there exist a reallocation matrix A^{*S} and an order quantity x^{*S} that maximizes the expected profit of coalition formation, as well as that the above cooperative news-vendor game has a nonempty core. Hartman *et al.* [37] and Müller *et al.* [49] consider the game in the above-mentioned setting, except that retailers identically single price c and r and in which the value of the group of retailers is their optimal profit if they jointly determine an order size without taking into account the transshipment cost. Both of the above-mentioned papers use the core to show that there is always a cost allocation scheme such that news-vendors will prefer to pool their inventory. The cost game they consider is

$$c(S) = (r - c)E[W^S] - v(S) \ \ \forall \ \ S \subseteq N. \tag{41}$$

Hartman *et al.* [37] prove that this game has a nonempty core under certain assumptions about the demand distribution, and Müller*et al.* [49] come up with a more powerful result, namely that the core of the news-vendor games are nonempty regardless of the distribution of the random demands.

7.1 Shapley Value

A solution concept that selects precisely one payoff vector for every cooperative game is the Shapley value. Used on the marginal contributions of all players in the game (N, v), the Shapley value [76], $\Phi(v) = (\Phi_i(v))_{i\in N}$ is defined by:

$$\Phi_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N| - 1 - |S|)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right) \tag{42}$$

So far, applications of Shapley value in inventory management are rather scarce, an exception is the paper of Robinson [73], who reexamines the allocation rules proposed by [34] in continuous review single-period inventory model and in terms of Shapley value, and the discussion presented in Granot and Sosic [35].

The Shapley value means that each player should be paid according to how valuable her cooperation is for the other players. In general, the Shapley value need not generate a core element. Hence, it may not be a reasonable prediction of the outcome of a game; because it is not in the core, there exists some subset of players that can deviate and improve their payoffs.

8 Bargaining Theory

In recent years, there is a trend in supply chain literature that considers the use of bargaining theoretic models to expand the view of negotiation and coordination in the supply chain. In this chapter, we present modeling paradigm for supply chain coordination using the notion of bargaining.

Bargaining theory helps to explore the relationship between the expected outcome from direct negotiation. Nash defines a *bargaining problem* as the situation in which two individuals have the opportunity to collaborate for mutual benefits in more than one way [55]. In other words, the bargaining problem arises in situations where there are gains Π from collaboration and is defined as the corresponding attempt to resolve a bargaining situation, i.e., to determine the particular form of cooperation and the corresponding division (π_1, π_2) of the bargaining surplus Π.

In a *bargaining game*, two or more players, who have competitive preferences, negotiate to follow a common mixed strategy in order to conclude with an outcome that is *fair* and *satisfactory* for all of them, that is how to divide bargaining surplus (gains) created from collaboration. It is assumed that both players are rational, self-interested, and risk neutral (expected value maximizers) with complete information. The possible outcome of the agreement depends on the negotiation power of each player.

A bargaining set $\mathcal{B}$ is a set of outcomes that can be jointly achieved by the players, $\mathcal{B} = \{(\pi_1, \pi_2) \in \mathbb{R}^2 : \pi_1 + \pi_2 \leq \Pi \ \text{ and } \ \pi_i \geq 0\}$. The players either reach an agreement $(\pi_1, \pi_2) \in \mathcal{B}$, or fail to reach agreement, in which case the disagreement event $D = (d_1, d_2)$ occurs and each gets nothing. An outcome is Pareto-efficient if it dominated over all possible outcomes, i.e., if no outcome exists that is strictly preferred by one player and not less preferred by any other player.

Given the bargaining set, a solution to the bargaining problem is concerned with the question of which outcome will eventually prevail, i.e., a solution is a rule that picks out one element of the bargaining set. Apparently, two different approaches of solutions to the bargaining problem exist in bargaining theory: (1) *axiomatic (cooperative game) approach*, which requires that the resulting solution should possess a list of axioms, and (2) *strategic (noncooperative game) approach*, in which the outcome is predicted by the notion of subgame perfect equilibrium.

8.1 Axiomatic Solution

The cooperative bargaining process was initiated by J. Nash [55]. In case of cooperative bargaining, the outcomes of negotiation are often described in terms of utilities; the notion of *utility* that satisfies the assumptions von Neumann–Morgenstern is used to quantify individual preferences. Consequently, for each player there is a function, called *utility function* u, which represents and scales her preference over the bargaining set. If $\bar{\pi}_i, \hat{\pi}_i \in \mathcal{B}$, and if for a player $i(i = 1, 2)$, $u_i(\bar{\pi}_i) \geq u_i(\hat{\pi}_i)$, then we can conclude that the outcome $\bar{\pi}_i$ is preferred to outcome $\hat{\pi}_i$ for the player i. Such a utility function is not unique, that is if u_i is a utility function, then the function $v_i = au_i + \beta$ is also a utility function for real numbers a, β and $a > 0$.

He started with a class of problems for which the bargaining set is convex and compact, and for which free disposal is allowed. A bargaining problem, as stated by Nash is concerned with the set of utility pairs, $\mathcal{P} \in \mathbb{R}^2$, that can be derived from the bargaining set $\mathcal{B}$, $\mathcal{P} = \{(p_1, p_2) \in \mathbb{R}^2 : (p_1, p_2) = [u_1(\pi_1), u_2(\pi_2)] \quad (\pi_1, \pi_2) \in \mathcal{B}\}$, where $\mathcal{P}$ is convex, compact, nonempty set, and a pair of utilities $\mathcal{D} = (\delta_1, \delta_2) = (u_1(d_1), u_2(d_2)) \in \mathcal{P}$ a vector on $\mathbb{R}^2$, which is assigned to be the disagreement point. Only if these requirements are satisfied the bargaining problem $\langle \mathcal{P}, \mathcal{D} \rangle$ can properly be called a Nash bargaining problem.

Nash did not build his solution around what the bargainer is doing. He tried to answer the question, "What would a good solution look like?" He came up with a short list of sensible-sounding conditions that a bargaining solution should satisfy. The nice thing about having a set of conditions to start with is that they will limit the set of solutions that you might consider.

> *"Rather than solve the two-person bargaining game by analyzing the bargaining process, one can attack the problem axiomatically by stating general properties that 'any reasonable solution' should possess. By specifying enough such properties one excludes all but one solution" [55].*

Nash proved that a solution to the bargaining problem $\langle \mathcal{P}, \mathcal{D} \rangle$ is a function $\phi(\cdot)$, also known as *arbitration function* that assigns a single outcome $(p_1, p_2) \in \mathcal{P}$ to every bargaining problem $\langle \mathcal{P}, \mathcal{D} \rangle$. Nash proposes that a bargaining solution should satisfy four conditions.

Pareto efficiency. Suppose $(p_1, p_2) = \phi(\mathcal{P}, \mathcal{D})$ is the solution to the bargaining problem $\langle \mathcal{P}, \mathcal{D} \rangle$, and a pair $(\hat{p}_1, \hat{p}_2) \in \mathcal{P}$ then should hold that $(p_1, p_2) > (\hat{p}_1, \hat{p}_2)$. This condition basically says that there is no feasible point $(\hat{p}_1, \hat{p}_2)$ that is Pareto superior to the solution.

Independence of linear transformation. If $v_i = r_i u_i + c_i$, for $i = 1, 2$ and $r_1 > 0$ is a linear transformation of the utility function u_i that generates $\mathcal{P}$, then v_i generates $\mathcal{P}' = \{(r_1 p_1 + c_1, r_2 p_2 + c_2) \in \mathbb{R}^2 : (p_1, p_2) \in \mathcal{P}\}$. Because v_i represents the same preference as u_i if both are applied to the same bargaining set $\mathcal{B}$, the bargaining problem $\langle \mathcal{P}', \mathcal{D}' \rangle$ represents the same bargaining problem with $\langle \mathcal{P}, \mathcal{D} \rangle$ if $\mathcal{D}' = (r_1\delta_1 + c_1, r_2\delta_2 + c_2)$, which is easy to check. Thus a solution to $\phi(\mathcal{P}', \mathcal{D}') = r\phi(\mathcal{P}, \mathcal{D}) + c$. This condition says that if you transform all the elements in $\langle \mathcal{P}, \mathcal{D} \rangle$, you will also transform the solution.

Symmetry. If the bargaining problem $\langle \mathcal{P}, \mathcal{D} \rangle$ is symmetric, and if $\delta_1 = \delta_2$, then $(p_1, p_2) = \phi(\mathcal{S}, \mathcal{D}) \Rightarrow p_1 = p_2$. In symmetric situations, both players get the same.

Independence of irrelevant alternatives. If $\langle \mathcal{P}, \mathcal{D} \rangle$ and $\langle \mathcal{P}', \mathcal{D}' \rangle$ are bargaining problems with $\mathcal{P} \subset \mathcal{P}'$ and $\phi(\mathcal{P}', \mathcal{D}') \in \mathcal{P}$, then $\phi(\mathcal{P}, \mathcal{D}) = \phi(\mathcal{P}', \mathcal{D}')$. This axiom states that the bargain solution does not depend on other available outcomes that the player had the opportunity to choose but did not. See [61] for details and proofs.

The solution that satisfies these four properties is unique and is characterized by the payoff pair $(p_, p_2)$, which maximizes the product of the player's benefits from cooperation, the so-called Nash product.

$$\phi(\mathcal{P}, \mathcal{D}) = \arg \max_{(p_1,p_2) \geq (\delta_1,\delta_2) \in \mathcal{P}} (p_1 - \delta_1)(p_2 - \delta_2). \tag{43}$$

If the symmetric axiom is ignored, the bargaining solution comes to depend on the bargaining powers of the two players, this is the generalized or asymmetric Nash bargaining solution,

$$\phi(\mathcal{P}, \mathcal{D}) = \arg \max_{(p_1,p_2) \geq (\delta_1,\delta_2) \in \mathcal{P}} (p_1 - \delta_1)^{\alpha}(p_2 - \delta_2)^{\beta} \tag{44}$$

where $\alpha, \beta,\ \alpha + \beta = 1$ represents the negotiation power of each player. Among the factors that affect negotiation power are their utility, their risk preference, and their position on the market.

The Nash bargaining solution can be extended to apply in the case with n players, and it can be shown that the unique bargaining solution that satisfies the axioms is the function that satisfies:

$$\phi(\mathcal{P}, \mathcal{D}) = \arg \max_{(p_1,p_2) \geq (\delta_1,\delta_2) \in \mathcal{P}} \prod_{i=1}^{n} (p_i - \delta_i). \tag{45}$$

Kalai and Smorodinsky [39] replace the rather controversial axiom of Independence of irrelevant alternatives with alternative, which they refer to as the *axiom of monotonicity*. Let $p_i^m(\mathcal{P}) = \max\{p_i : pi \in \mathcal{P}\}$ be the maximum that player i could attain (for $i = 1, 2$) in a bargaining situation $\langle \mathcal{P}, \mathcal{D} \rangle$ given that the players are individually rational. The payoff combination defined in this way is called *ideal point*. The Kalai–Smorodinsky solution requires then that if the ideal point belongs to bargaining games $\langle \mathcal{P}, \mathcal{D} \rangle$ and $\langle \mathcal{P}', \mathcal{D}' \rangle \in \mathcal{B}$ and if $\mathcal{P}' \subset \mathcal{P}$, then player i will receive at least as much as in $\langle \mathcal{P}, \mathcal{D} \rangle$ as in $\langle \mathcal{P}', \mathcal{D}' \rangle$. The Kalai–Smorodinsky solution is then a unique function that selects the maximum element in $\mathcal{P}$ on the line that joins the disagreement point (δ_1, δ_2) with the ideal point. For details and proofs, see [53].

To apply the Nash bargaining problem to the supply chain analysis, consider a supply chain with one supplier and one retailer. At the beginning of the period, the retailer places orders x and sells them to the customer at a unit price r. Supplier produces the product with unit production cost k and supplies x units to retailers at a price c. It is also assumed that supplier has infinite production capacity. The demand w faced by the retailer during the period is random but distribution is known. She also faces a unit holding, and shortage costs, denoted by h and p, respectively. Let $\tilde{\pi}_s$ and $\tilde{\pi}_r$ be the supplier's and retailer's profit, respectively, $\pi_s = E[\tilde{\pi}_s]$ and $\pi_r = E[\tilde{\pi}_r]$ their expected profits with:

$$\pi_s = cx - kx = (c - k)x \tag{46}$$

and

$$\pi_r = rE\min\{w, x\} - cx - L(x) \tag{47}$$

where $L(x) = hE(x - w)^+ + pE(w - x)^+$.

In addition, assume that the supply chain makes a positive expected profit Π^C that is greater than the disagreement points and therefore the rational players will always prefer to participate in the game. Furthermore, we assume that disagreement point for supplier is $d_s = kx$ and the retailer's disagreement point is $d_r = L(x)$.

The solution refers to the resulting payoff allocation that each of the players agrees upon. Given that both players are risk neutral, the necessary Pareto efficiency condition ensures the negotiated quantity is always the one that coordinates the whole chain, i.e., $x = x^C$ (where x^C is the coordinating quantity). In other words, this bargaining formulation gives us channel coordination for free [52]. Because the negotiated quantity is always x^C, then $d_s = kx^C$ and the retailers disagreement point is $d_r = L(x^C)$, similar assumptions have been employed in [52]. Suppose that the supplier and retailer negotiate to split the total expected profits of the system. Consequently, the bargaining set can be written as $\mathcal{B} = \{(\pi_s, \pi_r) \in \mathbb{R}^2 : \pi_s + \pi_r \leq \Pi^C, \text{ and } \pi_s, \pi_r \geq 0\}$, which is assumed to be a convex and compact set, in addition the formulation of disagreement points guarantees that it is nonempty. The corresponding Nash bargaining problem is $\mathcal{P} = \{(p_s, p_r) \in \mathbb{R}^2 : (p_s, p_r) = [E(u_s(\pi_s), E(u_r(\pi_r)] \ \ (\pi_s \pi_r) \in \mathcal{B}\}$, where $\mathcal{P}$ is a convex, compact, nonempty set, and $\mathcal{D} = (\delta_s, \delta_r) = (u_s(d_s), u_r(d_r)) \in \mathcal{P}$. Applying the Nash solution concept results in the two players maximizing the following expression

$$\max_{(p_s, p_r) \geq (\delta_s, \delta_r) \in \mathcal{P}} (p_s - \delta_s)(\Pi^C - p_s - \delta_r). \tag{48}$$

Taking the derivative with respect to p_s and p_r and equating to zero, we get respectively:

$$p_s = \frac{\Pi^C - \delta_r + \delta_s}{2} \tag{49}$$

$$p_r = \frac{\Pi^C - \delta_s + \delta_r}{2}. \tag{50}$$

Nagarajan and Bassok [52] consider a cooperative, multilateral bargaining game similar to that where n suppliers are selling complementary components to an assembler. They propose a three-stage game: First (stage 3) the suppliers form coalitions, second (stage 2) the coalitions compete for a position in the bargaining sequence, third (stage 1) the coalitions negotiate with the assembler on the wholesale price and the supply quantity. They show that each player's payoff is a function of the player s negotiation power, the negotiation sequence, and the coalitional structure.

Chae and Heidhues [20] study the effects of integration among downstream local distributors on the entry of upstream producers in a bargaining theoretic framework. They modeled both price formation and the entry of upstream producers in an input market. Using a bargaining solution that generalizes the Nash solution, they showed that a higher degree of concentration among downstream distributors reduces incentives to enter the upstream production industry. The reason is that higher concentration among downstream distributors reduces the bargaining power of upstream producers.

8.2 Strategic Approach

Whereas the cooperative approach is static, in the sense that only the outcome is analyzed without taking into account the bargaining procedure, the strategic approach to bargaining theory, initiated by Ståhl [80] and Rubinstein [74], is more concerned with these situations and analyzes exactly the bargaining procedures, in the attempt to find theoretical predictions of what agreement, if any, will be reached by the bargainers.

To this end, we now present the model developed by Rubinstein [74], in which the procedure is modeled explicitly as a game in real time. Here we think of bargaining as a sequential game. That is, there is a well-defined sequence of moves, and players have preferences over the time of agreement as well as the terms of agreement. There are two players $i, i = 1, 2$, whose task is to divide a single surplus of size 1. Each player is concerned only about the share of the surplus that she receives and prefers to receive more rather than less. Time proceeds without end as $t = 0, 1, 2, \ldots$.

The procedure is as follows. At $t = 0$, one player, say player 1, makes an offer, (π_1, π_2), where π_1 is player 1's share and p_2 is player 2's share where

$\pi_1 + \pi_2 \leq 1$, which is either accepted or rejected. If player 2 accepts the offer, the game ends and the surplus is divided accordingly. If player 2 rejects, she makes a counter offer at period $t+1$, which is either accepted or rejected with counter offer from 1 and so on. If no offer is ever accepted, the payoffs are 0. To simplify matters, we assume that both players have linear utility functions $u_1 = \pi_1$ and $u_2 = \pi_2$. Player i's utility for getting a share π_i of the surplus at time t is equal to $u_i = \pi_i \theta_i^t$, where $\theta \in [0, 1]$ is a fixed discount factor, and it is used to translate expected utility in any given future into present value terms. Rubinstein [74] proved that there is a unique subgame-perfect equilibrium in this game, based on playing the following strategies in every period:

1. Player 1 proposes an offer: $\left(\pi_1 = \frac{1-\theta_2}{1-\theta_1\theta_2}, \pi_2 = \frac{\theta_2(1-\theta_1)}{1-\theta_1\theta_2}\right)$ and accepts player 2's offer if and only if $\pi_1 \geq \frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2}$.
2. Player 2 proposes an offer: $\left(\pi_1 = \frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2}, \pi_2 = \frac{1-\theta_1}{1-\theta_1\theta_2}\right)$ and accepts player 1's offer if and only if $\pi_2 \geq \frac{\theta_2(1-\theta_1)}{1-\theta_1\theta_2}$.

Extentions of the Rubistein's model include the case where there is possibility for the negotiation to break down [7] and the influnce of a outside option in the negotiation proposed and implemented in different settings by [8, 50, 67]. The main assumption of these models is that a player can choose to decide to leave a negotiation if there is an outside deal that can optimize her objective.

To apply the noncooperative bargaining game, Ertogal and Wu [32] consider a bargaining situation between a supplier and a retailer (buyer) who negotiate to split certain system surplus, say π . The supplier and the retailer are to make several offers and counter offers before settling on a final agreement. Before entering negotiation, the supplier and retailer each have recallable outside options W_s and W_b, respectively. It is assumed that $\pi \geq W_s + W_r$, otherwise at least one of the players would have no incentive to enter the negotiation.

The sequence of events in our bargaining game is as follows:

1. With equal probability $\frac{p}{2}$, one of the two players proposes an offer that splits the system surplus π into certain amounts.
2. The other player may either:
   (a) accept the offer (the negotiation ends), or
   (b) reject the offer and wait for the next round.
3. With a certain probability $(1 - p)$, the negotiation breaks down and the players take their corresponding outside options.
4. If the negotiation continues, the game restarts from step 1.

They assume that in subgame perfect equilibrium there is an infinite number of solutions leading to gains ranging from m_b to M_b for the buyer, and m_r to M_r for the supplier, where:

$M_s(M_r)$: The maximum share the supplier (the retailer) could receive in a subgame perfect equilibrium for any subgame initiated with the supplier's (the retailer's) offer.

$m_s(m_r)$: The minimum share the supplier (the retailer) could receive in a subgame perfect equilibrium for any subgame initiated with the supplier's (the retailer's) offer.

They formulate and solve the negotiation-sequencing problem as a network flow problem. They proved that the following system of equations defines the subgame perfect equilibrium for the bargaining between the two players:

$$M_s = \pi - \left[\frac{p}{2}\left[(1-p)W_r + \frac{p}{2}(\pi - M_s + m_r) + \pi - M_s\right] + (1-p)W_r\right] \tag{51}$$

$$m_s = \pi - \left[\frac{p}{2}\left[(1-p)W_r + \frac{p}{2}(\pi - m_s + M_r) + \pi - m_s\right] + (1-p)W_r\right] \tag{52}$$

$$M_r = \pi - \left[\frac{p}{2}\left[(1-p)W_s + \frac{p}{2}(\pi - M_r + m_s) + \pi - M_r\right] + (1-p)W_s\right] \tag{53}$$

$$m_r = \pi - \left[\frac{p}{2}\left[(1-p)W_s + \frac{p}{2}(\pi - m_r + M_s) + \pi - m_r\right] + (1-p)W_s\right] \tag{54}$$

and the unique subgame perfect equilibrium strategies of the players are given as follows:

1. If the supplier is the offering party, she will ask for $X_s = \pi - W_r - \frac{p^2}{2(2-p)}(\pi - W_s - W_r)$ share of the surplus and leave $\pi - X_s$ to the retailer.
2. If the retailer is the offering party, she will ask for $X_r = \pi - W_s - \frac{p^2}{2(2-p)}(\pi - W_s - W_r)$ share of the surplus and leave $\pi - X_r$ to the supplier.

This result has important implications in that the bargaining game will end in one iteration when one of the two players initiates the negotiation with the perfect equilibrium offer. They further show that there is a first-mover advantage in this game, but the advantage diminishes as the probability of breakdown approaches zero. Wu [94] expands the model to analyze the trade-off between direct and intermediated exchanges.

Bernstein and Marx [6] address the problem of supply chain performance when one supplier sells to multiple competing retailers and who have bargaining power. They model a retailer's bargaining power through its ability to set a reservation profit level below which it will not participate in the supply chain. They also allow endogenously chosen reservation profit levels for the buyers that may depend on the retailer's opportunities within the supply chain, rather than taking those reservation profit levels as fixed and dependent only on outside opportunities. The retailers may compete in terms of the prices they charge or in terms of the amount of inventory they carry. Their results indicate that supply chain performance is not maximized, or it is maximized conditional on the number of retailers that offer the supplier's product, but some retailers are excluded from trade. They conclude that in equilibrium, retailer's choices of reservation profit levels may induce the supplier to trade only with a strict subset of the retailers, even when all retailers must be included in order for channel profit to be maximized.

de Fontenay and Gans [28] analyze vertical integration in the case of upstream competition in which they demonstrate that vertical integration can alter the joint payoff of integrating parties in ex post bargaining.

Van Mieghem [87] and Chod and Rudi [23] consider settings in which two firms trade capacity after receiving demand information. In [23], the authors consider two independent firms that invest in resources such as capacity or inventory based on imperfect market forecasts. After investment decisions are made, the firms update their forecasts of the market conditions and have the option to trade. Although the negotiation of this trade is formulated in a cooperative fashion, the firms do not cooperate in the investment stage. The problem is formulated as a noncooperative bargaining game, and the existence and uniqueness of an embedded Nash bargaining solution is proved. In Van Mieghem's work [87], after demand revelation, the manufacturer may purchase some of the excess capacity of the subcontractor. She formulates this problem as a noncooperative stochastic investment game. Her results indicate that all decentralization costs are eliminated only when the bargaining parameters depend on demand realization.

8.3 Biform Games

A biform game is a hybrid noncooperative/cooperative game model designed for modeling business interactions. It can be thought of as a noncooperative game with cooperative games as outcomes, and those cooperative games lead to specific payoff. The biform game was first formalized by Brandenburger and Stuart [10]. Hence the noncooperative solution concept of Nash equilibrium extends naturally to the biform game.

To define a biform game, consider a set of players N, indexed by $i = 1, \ldots, n$, and for each player $i \in N$, a finite set X_i of strategies. At the noncooperative stage (first stage), players make decision among their strategies, this game can be analyzed just like any other noncooperative game. Competition is then modeled by a cooperative game (second stage) in which the characteristic value function depends on the chosen actions. The core of this cooperative game is employed to determine the outcome of the game. Even though the core of such game is nonempty, it may yield to a range of outcomes, rather than a unique outcome; as a result, it is not immediately clear what value each player can expect. In such cases, it is necessary to describe each player's preferences over intervals. In a biform game, these preferences are represented by the numbers α_i for each player i. Each player then expects to earn in each possible cooperative game a weighted average of the minimum and maximum values in the core, with α_i being the weight. The parameter α_i can also be interpreted as an index player's i in her bargaining power.

Brandenburger and Stuart [10] proposed the biform game in which players make strategic investments, and then they play a cooperative game determined by their investments. The biform game formulation is employed in Plambeck and Taylor [66] where two independent, price-setting original equipment manufacturers (OEMs) are investing in innovations. They allow OEMs to outsource their productions to an independent contract manufacturer (CM); a bargaining game is employed to model the negotiations among (OEMs) and (CM). They show that the bargaining outcome induces the CM to invest in the system-optimal capacity level and to allocate capacity optimally among the OEMs. A subsequent paper [65] considers the situation where a manufacturer writes quantity flexibility contracts with two buyers. Then, the buyers invest in innovation, and the manufacturer builds capacity. Without renegotiation, quantity flexibility is necessary for the client capacity allocation, but reduces incentives for investment. Typically, allowing renegotiation reduces the flexibility in an optimal contract and increases the total expected profit.

He models the problem as a multivariate, multidimensional, competitive news-vendor problem. He argues that *ex ante* contracts may be too expensive or impossible to enforce, while the supplier's investments (in quality, IT infrastructure, and technology innovation) may be noncontractible.

In the application of the biform game to the news-vendor problem, ordering decisions of different retailers are made competitively whereas allocation decisions take place cooperatively.

In a recent article, Rudi *et al.* [70] consider a two-retailer model with transshipment of stock. They aim to find prices for which the joint decentralization profit achieves the centralized system profit.

Anupindi *et al.* [3] use a hybrid noncooperative/cooperative model to formulate a game where multiple retailers stock at their own locations as well as at several centralized warehouses. In the noncooperative stage, retailers make stocking decisions, for this stage they develop conditions for the existence of a pure Nash equilibrium. In the cooperative stage, retailers use cooperative game theory to characterize possible opportunities for cooperation, similar to Müller *et al.* [49].

Granot and Sosic [35] analyze a similar problem, they consider a network of retailers with stochastic demands: Each chooses its inventory level but in this models retailers are able to hold any inventory left from one period to the other, then demand is realized; and the retailers bargain cooperatively over the transshipment of excess inventory to meet excess demand. Their model has three stages: decision about the order quantity, decision about how much inventory to share with others, and finally the transshipment stage.

Stuart [81] provides a model of the competitive news-vendor problem in which there is price competition following the inventory decisions. The price competition is modeled by considering the core of the induced cooperative game. She shows that with no uncertainty, the inventory decision is equivalent to the capacity decision in Cournot competition. With uncertainty, the analysis again reduces to Cournot competition if the demand uncertainty is characterized by an appropriately constructed, expected demand curve.

Unpublished manuscripts and papers in the bibliography are available through CiteSeer, the Autonomous Citation Indexing and Scientific Literature Digital Library, at `http://citeseer.ist.psu.edu` and science-specific search engine Scirus at `http://www.scirus.com/srsapp/`.

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