Chapter 38
Application of the Characteristic Function in Financial Research

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Abstract In this chapter we introduce the application of the characteristic function in financial research. We consider the technique of the characteristic function useful for many option pricing models. Two option pricing models are derived in details based on the characteristic functions.

Keywords Characteristic function • Constant elasticity of variance • Option pricing • Stochastic volatility

38.1 Introduction

The characteristic function in nonprobabilistic contexts is called the Fourier transform. The characteristic function was used widely in applied physics (signal process, quantum mechanics).

The technique of the characteristic function is also useful for determining the option prices. In Heston (1993), the importance of the characteristic function was demonstrated to find a closed-form solution for options with stochastic volatility. It was subsequently considered by many authors, including Bates (1996), Bakshi and Chen (1997), Scott (1997), Carr and Madan (1999), among others. In addition, we consider the constant elasticity of variance (CEV) option pricing model (see Cox 1996; Schroder 1989; and Hsu et al. 2008) and options with stochastic volatility (see Heston 1993) based on the characteristic functions.

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38.2 The Characteristic Functions

In probability theory, the characteristic function of any random variable completely defines its probability distribution. On the real line it is given by the following formula, where X is any random variable with the distribution in question:

$$\begin{aligned}\varphi_X(t) &= E(e^{itX}) \\ &= E(\cos(tX)) + iE(\sin(tX)) \qquad (38.1)\end{aligned}$$

where t is a real number, I is the imaginary unit, and E denotes the expected value. The characteristic function is thus defined as the moment generating function but with the real argument s replaced by it; it has the advantage that it always exists because e^{itx} is bounded.

If F_X is the cumulative distribution function, then the characteristic function is given by the Riemann-Stieltjes integral

$$E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF_X(x). \qquad (38.2)$$

If there is a probability density function, f_X, this becomes

$$E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f_X(x) dx. \qquad (38.3)$$

If **X** is a vector-valued random variable, one takes the argument **t** to be a vector and $\mathbf{t}'\mathbf{X}$ to be an inner product. The characteristic functions for the common distributions are given in Table 38.1.

Besides, if F is a one-dimensional distribution function and f is corresponding characteristic function, then the cumulative distribution function F_X and its corresponding probability density function $\phi(x) = F'(x)$ can be retrieved via:

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} f(t) dt \qquad (38.4)$$

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Table 38.1 The characteristic functions of the specific probability functions

Distribution	Probability function	Interval	Characteristic function
Normal	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$-\infty < x < \infty$	$e^{-t^2/2}$
Uniform	1	$0 < x < 1$	$\frac{e^{it}-1}{it}$
Exponential	e^{-x}	$0 < x < \infty$	$\frac{1}{1-it}$
Chi-Squared	$\frac{x^{(\nu-2)/2} e^{-x/2}}{2^{\nu}\Gamma(\nu/2)}$	$0 < x < \infty$	$(1-2it)^{-\nu/2}$
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, 1, \ldots, n$	$(1-p+pe^{it})^n$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$	$x = 0, 1, \ldots$	$e^{\lambda(e^{it}-1)}$

$$F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{itx} f(-t) - e^{-itx} f(t)}{it} dt \tag{38.5}$$

or

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{itx} f(t)}{it}\right] dt. \tag{38.6}$$

If F_1 and F_2 have respective characteristic functions $\varphi_1(t)$ and $\varphi_2(t)$, then the convolution of F_1 and F_2 has characteristic function $\varphi_1(t)\varphi_2(t)$. Although convolution is essential to the study of sums of independent random variables, it is a complicated operation, and it is often simpler to study the products of the corresponding characteristic functions. Every probability distribution on $\mathbf{R}$ or on $\mathbf{R}^n$ has a characteristic function, because one is integrating a bounded function over a space whose measure is finite, and for every characteristic function there is exactly one probability distribution.

38.3 CEV Option Pricing Model

Cox (1996) has derived the renowned CEV option pricing model. The CEV option pricing model assumes that the stock price is governed by the diffusion process

$$dS = \mu S dt + \sigma S^{\beta/2} dz, \quad \beta < 2 \tag{38.7}$$

where dz is a Wiener process and σ is a positive constant. If $\beta = 2$, the stock prices are log-normally distributed as in Black–Scholes model (Black and Scholes 1973). In the Black–Scholes case, there is only one source of randomness – the stock price, which can be hedged with the stock. From the view of no-arbitrage, we can form a portfolio that grows in the risk-free rate. We then have the partial differential equation (PDE) subject to some boundary conditions. Similarly, we can get the option price derived from the CEV model by the way of Black-Scholes.

Now, we set up a portfolio Π containing the option being priced whose value is denoted by $U(S,t)$, a quantity $-\Delta$ of the stock $\left(\Delta = \frac{\partial U}{\partial S}\right)$. That is,

$$\Pi = U - \Delta S. \tag{38.8}$$

The small change of the portfolio in a time interval dt is given by

$$\begin{aligned} d\Pi &= \left\{\frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}(dS)^2 - \frac{\partial U}{\partial S}dS\right\} \\ &= \left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^{\beta}\right) dt. \end{aligned} \tag{38.9}$$

Since there is no diffusion term, the value of the arbitrage portfolio is certain. In order to preclude arbitrage, the payoff must equal $\Pi r dt$; that is,

$$\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^{\beta}\right) dt = \left(U - \frac{\partial U}{\partial S} S\right) r dt. \tag{38.10}$$

We have the PDE

$$\frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^{\beta} - rU = 0. \tag{38.11}$$

Hence, we have the derivative's price by solving Equation (38.11) and giving the boundary conditions. If $U(S,t)$ is a call option, $C_T = \max(S_T - K, 0)$, then

$$\begin{cases} \dfrac{\partial C}{\partial t} + rS\dfrac{\partial C}{\partial S} + \dfrac{1}{2}\dfrac{\partial^2 C}{\partial S^2}\sigma^2 S^{\beta} - rC = 0 \\ C_T = \max(S_T - K, 0) \end{cases} \tag{38.12}$$

where K is the strike price of the option.

Now, we can simplify Equation (38.11) by rewriting it in terms of the logarithm of the spot price; that is, $x = \ln(S)$, thus

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{1}{S}, \quad \frac{\partial C}{\partial S^2} = \frac{1}{S^2}\frac{\partial^2 C}{\partial x^2} - \frac{1}{S}\frac{\partial C}{\partial x}. \tag{38.13}$$

We transform the problem into

$$\begin{cases} \dfrac{\partial C}{\partial t} + \left(r - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial C}{\partial x} + \dfrac{1}{2}\dfrac{\partial^2 C}{\partial x^2}\sigma^2 e^{x(\beta-2)} - rC = 0 \\ C_T = \max(e^{x_T} - K, 0). \end{cases} \tag{38.14}$$

By analogy with the Black–Scholes formula, we guess a solution of the form:

$$C_t(x,t) = e^x P_1(x,t) - Ke^{-r(T-t)} P_2(x,t) \quad (38.15)$$

where the first term is the present value of the spot asset upon optimal exercise, and the second term is the present value of the strike-price payment. And we substitute the proposed solution Equation (38.15) into (38.14).

Substituting the proposed solution into the PDE shows that both $P_1(x,t)$ and $P_2(x,t)$ are satisfied

$$\frac{\partial P_1}{\partial t} + \left(r + \frac{1}{2}\sigma^2\right)\frac{\partial P_1}{\partial x} + \frac{1}{2}\frac{\partial^2 P_1}{\partial x^2}\sigma^2 e^{x(\beta-2)} = 0 \quad (38.16)$$

$$\frac{\partial P_2}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial P_2}{\partial x} + \frac{1}{2}\frac{\partial^2 P_2}{\partial x^2}\sigma^2 e^{x(\beta-2)} = 0, \quad (38.17)$$

Respectively, and subject to the terminal condition for $j = 1, 2$

$$\lim_{t\to 0} P_j(x,t;\ln[K]) = \begin{cases} 1, \text{ if } x_T > \ln[K] \\ 0, \text{ if } x_T \le \ln[K] \end{cases}. \quad (38.18)$$

We can apply the characteristic function of $P_j(x,t) = P_j(x_T \ge \ln[K]|x_t)$.

It can be taken the following form

$$P_j(x_t) = P_j(x_T \ge \ln[K]|x_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\varphi\ln[K]}}{i\varphi}\tilde{f}_j(x_{T,\varphi}|x_t)d\varphi. \quad (38.19)$$

Using the above formula, we can shift Equation (38.14) in Fourier space and define $\tau = T - t$, we have

$$-\frac{\partial \tilde{f}_1}{\partial \tau} + \left(r + \frac{1}{2}\sigma^2\right)\frac{\partial \tilde{f}_1}{\partial x} + \frac{1}{2}\frac{\partial^2 \tilde{f}_1}{\partial x^2}\sigma^2 e^{x(\beta-2)} = 0, \quad (38.20)$$

subject to $\tilde{f}_1(x_{\tau=0},\varphi) = e^{i\varphi x_{[\tau=0]}}$,

$$-\frac{\partial \tilde{f}_2}{\partial \tau} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \tilde{f}_2}{\partial x} + \frac{1}{2}\frac{\partial^2 \tilde{f}_2}{\partial x^2}\sigma^2 e^{x(\beta-2)} = 0, \quad (38.21)$$

subject to $\tilde{f}_2(x_{\tau=0},\varphi) = e^{i\varphi x_{[\tau=0]}}$.

We guess the solution $\tilde{f}_j(x_{\tau=0},\varphi|x_\tau) = e^{C_\tau^j + i\varphi x_\tau}$.

Thus, we have for $j = 1, 2$

$$\frac{\partial \tilde{f}_j}{\partial x} = i\phi e^{C_\tau^j + i\phi x}, \quad (38.22)$$

$$\frac{\partial^2 \tilde{f}_j}{\partial x^2} = -\phi^2 e^{C_\tau^j + i\phi x}, \quad (38.23)$$

$$\frac{\partial \tilde{f}_j}{\partial \tau} = e^{C_\tau^j + i\phi x}\frac{\partial C_\tau^j}{\partial \tau}. \quad (38.24)$$

Substituting the above Equations into (38.20) and (38.21), we have

$$\frac{\partial C_\tau^1}{\partial \tau} = ri\phi + \frac{1}{2}\sigma^2\phi[i - \phi e^{x(\beta-2)}], \quad (38.25)$$

$$C_{\tau=0}^1 = 0, \quad (38.26)$$

$$\frac{\partial C_\tau^2}{\partial \tau} = ri\phi - \frac{1}{2}\sigma^2\phi[i + \phi e^{x(\beta-2)}], \quad (38.27)$$

$$C_{\tau=0}^2 = 0. \quad (38.28)$$

They are all first-order ordinary differential equations with constant coefficient, and they can be solved using a single integration:

$$C_\tau^1 = ri\phi\tau + \frac{1}{2}\sigma^2\phi[i - \phi e^{x(\beta-2)}]\tau, \quad (38.29)$$

$$C_\tau^2 = ri\phi\tau - \frac{1}{2}\sigma^2\phi[i + \phi e^{x(\beta-2)}]\tau. \quad (38.30)$$

Hence, we have $\tilde{f}_j(x_{\tau=0},\varphi|x_\tau) = e^{C_\tau^j + i\varphi x_\tau}$ where C_τ^1 and C_τ^2 are given by above equations.

Finally, we can found the analytical form of probability functions P_j; that is,

$$P_j(x_{\tau=0} \ge \ln[K]|x_\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Re}\left[\frac{e^{-i\varphi\ln[K]}}{i\varphi}\tilde{f}_j(x_{T,\varphi}|x_t)\right]d\varphi \quad (38.31)$$

or

$$\mathrm{P_j}(x_{\tau=0} \ge \ln[K]|x_\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Im}\left[\frac{e^{-i\phi\ln[K]}}{i\phi}\tilde{f}_j(x_{T,\phi}|x_t)\right]d\phi. \quad (38.32)$$

38.4 Options with Stochastic Volatility

Heston (1993) used the characteristic functions to derive a closed-form solution for the price of a European call option on an asset with stochastic volatility.

We begin by assume that the spot asset at time t follows the diffusion

$$dS(t) = \mu S dt + \sqrt{v(t)} S dz_1(t), \tag{38.33}$$

where $z_1(t)$ is a Wiener process. If the volatility follows an Ornstein–Uhlenbeck process (Uhlenbeck and Ornstein 1930),

$$d\sqrt{v(t)} = -\beta\sqrt{v(t)}dt + \delta dz_2(t), \tag{38.34}$$

then Itô's Lemma (Itô 1944) shows that the variance follows the process

$$dv(t) = [\delta^2 - 2\beta v(t)]dt + 2\delta\sqrt{v(t)}dz_2(t), \tag{38.35}$$

where $z_2(t)$ has correlation ρ with $z_1(t)$. This can be written as the familiar square-root process

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)}dz_2(t). \tag{38.36}$$

For simplicity, we assume a constant interest rate r.

Therefore, the price at time t of a unit discount bond that matures at time $t + \tau$ is

$$P(t, t + \tau) = e^{-r\tau}. \tag{38.37}$$

In this case, there two random sources – the stock price and random change in volatility – which needs to be hedged to form a riskless portfolio. Thus, we set up a portfolio Π containing the option being priced the value of which is denoted by $U(S, v, t)$, a quantity $-\Delta$ of the stock and a quantity $-\Delta_1$ of another asset whose value U_1 depends on volatility. That is,

$$\Pi = U - \Delta S - \Delta_1 U_1 \tag{38.38}$$

The small change of the portfolio in a time interval dt is given by

$$\begin{aligned} d\Pi = &\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS(t)\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right\}dt \\ &-\Delta_1\left\{\frac{\partial U_1}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U_1}{\partial S^2} + \rho\sigma vS(t)\frac{\partial^2 U_1}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U_1}{\partial v^2}\right\}dt \\ &+\left\{\frac{\partial U}{\partial S} - \Delta_1\frac{\partial U_1}{\partial S} - \Delta\right\}dS \\ &+\left\{\frac{\partial U}{\partial v} - \Delta_1\frac{\partial U_1}{\partial v}\right\}dv. \end{aligned} \tag{38.39}$$

In order to make the portfolio instantaneously risk-free, we choose

$$\frac{\partial U}{\partial S} - \Delta_1\frac{\partial U_1}{\partial S} - \Delta = 0 \quad \text{and} \quad \frac{\partial U}{\partial v} - \Delta_1\frac{\partial U_1}{\partial v} = 0. \tag{38.40}$$

The portfolio grows the risk-free rate. We have

$$\begin{aligned} d\Pi = &\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS(t)\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right\}dt \\ &-\Delta_1\left\{\frac{\partial U_1}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U_1}{\partial S^2} + \rho\sigma vS(t)\frac{\partial^2 U_1}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U_1}{\partial v^2}\right\}dt \\ = &\, r\Pi dt \\ = &\, r\left(U - \Delta S - \Delta_1 U_1\right)dt. \end{aligned} \tag{38.41}$$

If we use some simple algebra to collect all U terms on the left-hand side and U_1 terms on the left-hand side, we get

$$\begin{aligned} &\frac{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U}{\partial S^2} + 2\rho\delta vS(t)\frac{\partial^2 U}{\partial v\partial S} + 2v\delta^2\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial v}} \\ &= \frac{\frac{\partial U_1}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U_1}{\partial S^2} + 2\rho\delta vS(t)\frac{\partial^2 U_1}{\partial v\partial S} + 2v\delta^2\frac{\partial^2 U_1}{\partial v^2} + rS\frac{\partial U_1}{\partial S} - rU_1}{\frac{\partial U_1}{\partial v}}. \end{aligned} \tag{38.42}$$

From the factor model, the two risk assets must satisfy the internal consistent relationship. Then the value of any asset $U(S, v, t)$ must satisfy the PDE

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}vS^2(t)\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS(t)\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} - rU \\ &= -\{\kappa[\theta - v(t)] - \lambda(S, v, t)\}\frac{\partial U}{\partial v}. \end{aligned} \tag{38.43}$$

The unspecified term $\lambda(S, v, t)$ represents the price of volatility risk, and must be independent of the particular asset. Conventionally, $\lambda(S, v, t)$ is called the market price of volatility risk. We note that in Breeden's (1979) consumption-based model,

$$\lambda(S, v, t)dt = \gamma Cov\left(dv, \frac{dC}{C}\right), \tag{38.44}$$

where $C(t)$ is the consumption rate and γ is the relative-risk aversion of an investor. And we note that in Cox et al. (1985) model,

$$dC(t) = \mu_c v(t) C dt + \sigma_c \sqrt{v(t)} C dz_3(t), \tag{38.45}$$

where consumption growth has constant correlation with the spot asset return. These two papers motivate us with the choice of $\lambda(S, v, t)$ to v, $\lambda(S, v, t) = \lambda v$.

A European call option with strike price K and maturing at time T satisfies Equation (38.43) subject to the following boundary conditions:

$$U(S, v, T) = \mathrm{Max}(0, \mathrm{S} - \mathrm{K}), \tag{38.46}$$

$$U(0, v, t) = 0, \tag{38.47}$$

$$\frac{\partial U}{\partial S}(\infty, v, t) = 1, \tag{38.48}$$

$$rS\frac{\partial U}{\partial S}(S, 0, t) + \delta^2 \frac{\partial U}{\partial v}(S, 0, t) - rU(S, 0, t) + \frac{\partial U}{\partial t}(S, 0, t) = 0, \tag{38.49}$$

$$U(S, \infty, t) = \mathrm{S}. \tag{38.50}$$

Now, we can simplify Equation (38.43) by rewriting them in terms of the logarithm of the spot price; that is, $x = \ln(S)$ and $V(x, v, t) = U(S, v, t)$.

$$\frac{\partial V}{\partial t} + \frac{1}{2} v \frac{\partial^2 V}{\partial x^2} + \rho\sigma v \frac{\partial^2 V}{\partial v \partial x} + \frac{1}{2} v \sigma^2 \frac{\partial^2 V}{\partial v^2} + \left(r - \frac{1}{2} v\right) \frac{\partial V}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial V}{\partial v} - rV = 0. \tag{38.51}$$

By analogy with the Black–Scholes formula, we can guess a solution of the form

$$C(S, v, t) = SP_1(x, v, t) - Ke^{-r(T-t)} P_2(x, v, t), \tag{38.52}$$

where the first term is the present value of the spot asset upon optimal exercise, and the second term is the present value of the strike-price payment. And we substitute the proposed solution into Equation (38.51).

For $P_2(x, v, t)$, we have

$$\frac{\partial P_2}{\partial t} + \frac{1}{2} v \frac{\partial^2 P_2}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_2}{\partial v \partial x} + \frac{1}{2} v \sigma^2 \frac{\partial^2 P_2}{\partial v^2} + \left(r - \frac{1}{2} v\right) \frac{\partial P_2}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial P_2}{\partial v} = 0. \tag{38.53}$$

For $P_1(x, v, t)$, we have

$$\frac{\partial P_1}{\partial t} + \frac{1}{2} v \frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v \frac{\partial^2 P_1}{\partial v \partial x} + \frac{1}{2} v \sigma^2 \frac{\partial^2 P_1}{\partial v^2} + \left(r + \frac{1}{2} v\right) \frac{\partial P_1}{\partial x} + [\kappa(\theta - v) + \rho\sigma v - \lambda v] \frac{\partial P_1}{\partial v} = 0. \tag{38.54}$$

Besides, both $P_1(x, v, t)$ and $P_2(x, v, t)$ are subject to the terminal condition

$$\lim_{t \to 0} P_j(x, v, t; \ln[K]) = \begin{cases} 1, \text{ if } x > \ln[K] \\ 0, \text{ if } x \le \ln[K] \end{cases}. \tag{38.55}$$

The probabilities of $P_1(x, v, t)$ and $P_2(x, v, t)$ are not immediately available in close form. We will show that their characteristic function satisfies Equation (38.51).

Suppose that we have two processes

$$dx(t) = \left[r - \frac{1}{2} v(t)\right] dt + \sqrt{v(t)} dW_1(t), \tag{38.56}$$

$$dv(t) = \{\kappa[\theta - v(t)] - \lambda v(t)\} dt + \sigma \sqrt{v(t)} dW_2(t), \tag{38.57}$$

$$\mathrm{cov}\,[dW_1(t), dW_2(t)] = \rho dt, \tag{38.58}$$

and a twice-differentiable function

$$f\,(x(t), v(t), t) = E\,[g(x(T), v(T)) | x(t) = x, v(t) = v]\,. \tag{38.59}$$

From Itô's Lemma we have

$$df = \left(\frac{1}{2} v \sigma^2 \frac{\partial^2 f}{\partial v^2} + \rho\sigma v \frac{\partial^2 f}{\partial v \partial x} + \frac{1}{2} v \frac{\partial^2 f}{\partial x^2} + \left(r - \frac{1}{2} v\right) \frac{\partial f}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}\right) dt + \left(r - \frac{1}{2} v\right) \frac{\partial f}{\partial x} dW_1 + \{\kappa[\theta - v] - \lambda v\} dW_2 \tag{38.60}$$

Besides, by integrated expectations, we know that $f(x(t), v(t), t)$ is a martingale, then the df coefficient must vanish; that is,

$$\frac{1}{2} v \sigma^2 \frac{\partial^2 f}{\partial v^2} + \rho\sigma v \frac{\partial^2 f}{\partial v \partial x} + \frac{1}{2} v \frac{\partial^2 f}{\partial x^2} + \left(r - \frac{1}{2} v\right) \frac{\partial f}{\partial x} + [\kappa(\theta - v) - \lambda v] \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0. \tag{38.61}$$

The final condition $f(x, v, T) = g(x, v)$ which depends on the choice of g. Choosing the $g(x, v) = e^{i\varphi x}$ the solution is the characteristic function, which is available in closed form.

In order to solve Equation (38.61) with above condition, we invert the time direction: $\tau = T - t$. It means that we have solved

$$\frac{1}{2}v\sigma^2\frac{\partial^2 f}{\partial v^2} + \rho\sigma v\frac{\partial^2 f}{\partial v\partial x} + \frac{1}{2}v\frac{\partial^2 f}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial f}{\partial x} + [\kappa(\theta - v) - \lambda v]\frac{\partial f}{\partial v} - \frac{\partial f}{\partial t} = 0 \quad (38.62)$$

subject to the initial condition:

$$f(x, v, 0) = e^{i\phi x}. \quad (38.63)$$

To solve the characteristic function explicitly, we guess the functional form to be

$$f(x, v, t) = e^{[C(\tau)+D(\tau)v]+i\varphi x]} \quad (38.64)$$

with $C(0) = 0$ and $D(0) = 0$.

By substituting the function form Equation (38.64) into (38.62) we have:

$$\frac{1}{2}v\sigma^2 D^2 f + \rho\sigma v i\phi D f - \frac{1}{2}v\phi^2 f + \left(r - \frac{1}{2}v\right)i\phi f + [\kappa(\theta - v) - \lambda v]\,Df - (C' + D'v)f = 0. \quad (38.65)$$

Therefore, we get the PDE,

$$\left(\frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D - \frac{1}{2}\phi^2 + \frac{1}{2}i\phi - (\kappa+\lambda)D - D'\right)vf + \left(ir\phi + \kappa\theta D - C'\right)f = 0, \quad (38.66)$$

to reduce it two ordinary differential equations, respectively,

$$D' = \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\varphi D - \frac{1}{2}\varphi^2 + \frac{1}{2}i\varphi - (\kappa+\lambda)D \quad (38.67)$$

and

$$C' = ir\phi + \kappa\theta D. \quad (38.68)$$

For Equation (38.67), we shall solve the Riccati differential equation

$$D' = \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\varphi D - \frac{1}{2}\varphi^2 + \frac{1}{2}i\varphi - (\kappa+\lambda)D \quad (38.69)$$

by using $D = -\frac{E'}{\frac{\sigma^2}{2}E}$.

It follows that $E'' - (\rho\sigma i\varphi - \kappa - \lambda)E' - \frac{1}{2}\sigma^2\left(\frac{1}{2}\varphi^2 + \frac{1}{2}i\varphi\right) = 0$.

Let $d = \sqrt{(\rho\sigma i\varphi - \kappa - \lambda)^2 + \sigma^2(\varphi^2 + i\varphi)}$, then the above equation has the general solution

$$E(\tau) = Ae^{y_1\tau} + Be^{y_2\tau}, \quad (38.70)$$

where $y_1 = \frac{(\rho\sigma i\varphi - \kappa - \lambda) + d}{2}$ and $y_2 = \frac{(\rho\sigma i\varphi - \kappa - \lambda) - d}{2}$.

A and B can get from the boundary conditions

$$\begin{cases} E(0) = A + B \\ Ay_1 + By_2 = 0 \end{cases}.$$

Hence, we obtain

$$E(\tau) = \frac{E(0)}{g-1}\left(ge^{y_1\tau} - e^{y_2\tau}\right),$$

$$E'(\tau) = \frac{E(0)}{g-1}\left(gy_1e^{y_1\tau} - y_2e^{y_2\tau}\right),$$

$$g = \frac{y_1}{y_2} = \frac{\rho\sigma i\phi - \kappa - \lambda - d}{\rho\sigma i\phi - \kappa - \lambda + d},$$

and

$$D(\tau) = -\frac{2}{\sigma^2}y_2\frac{e^{y_2\tau} - e^{y_1\tau}}{e^{y_2\tau} - ge^{y_1\tau}} = \frac{\kappa + \lambda + d - \rho\sigma i\phi}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right].$$

For Equation (38.68), the $C(\tau)$ can be solved integration merely,

$$\begin{aligned} C(\tau) &= ir\varphi\tau + \kappa\theta\int_\tau^0 -\frac{E'(s)}{\frac{\sigma^2}{2}E(s)}ds \\ &= ir\varphi\tau - \frac{2\kappa\theta}{\sigma^2}\int_\tau^0 \frac{E'(s)}{E(s)}ds \\ &= ir\varphi\tau - \frac{2\kappa\theta}{\sigma^2}\ln\frac{E(\tau)}{E(0)} \\ &= ir\varphi\tau + \frac{\kappa\theta}{\sigma^2}\left[(\kappa + \lambda + d - \rho\sigma i\varphi)\tau - 2\ln\left(\frac{1 - ge^{d\tau}}{1 - e^{d\tau}}\right)\right]. \end{aligned} \quad (38.71)$$

As a result, we can invert the characteristic function to get the desired probabilities.

$$P_j(x, v, t; \ln[K]) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\varphi\ln[K]}f_j(x, v, T; \varphi)}{i\varphi}\right]d\varphi. \quad (38.72)$$

38.5 Conclusion

In this chapter, we use characteristic functions to solve option pricing problems. The characteristic functions are widely used in solving differential equations and the inversion formula permits one to determine the underlying distribution function from the characteristic function. The use of the characteristic functions in finance will provide an effective and practical means of dealing with the option pricing.

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