Chapter 37
Option Pricing and Hedging Performance Under Stochastic Volatility and Stochastic Interest Rates

Gurdip Bakshi, Charles Cao, and Zhiwu Chen

Abstract Recent studies have extended the Black–Scholes model to incorporate either stochastic interest rates or stochastic volatility. But, there is not yet any comprehensive empirical study demonstrating whether and by how much each generalized feature will improve option pricing and hedging performance. This paper fills this gap by first developing an implementable option model in closed-form that admits both stochastic volatility and stochastic interest rates and that is parsimonious in the number of parameters. The model includes many known ones as special cases. Based on the model, both delta-neutral and single-instrument minimum-variance hedging strategies are derived analytically. Using S&P 500 option prices, we then compare the pricing and hedging performance of this model with that of three existing ones that respectively allow for (i) constant volatility and constant interest rates (the Black–Scholes), (ii) constant volatility but stochastic interest rates, and (iii) stochastic volatility but constant interest rates. Overall, incorporating stochastic volatility and stochastic interest rates produces the best performance in pricing and hedging, with the remaining pricing and hedging errors no longer systematically related to contract features. The second performer in the horse-race is the stochastic volatility model, followed by the stochastic interest rates model and then by the Black–Scholes.

Keywords Stock option pricing • Hedge ratios • Hedging • Pricing performance • Hedging performance

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37.1 Introduction

Option pricing has, in the last two decades, witnessed an explosion of new models that each relax some of the restrictive assumptions underlying the seminal Black–Scholes (1973) model. In doing so, most of the focus has been on the counterfactual constant-volatility and constant-interest-rates assumptions. For example, Merton's (1973) option pricing model allows interest rates to be stochastic but keeps a constant volatility for the underlying asset, while Amin and Jarrow (1992) develop a similar model where, unlike in Merton's, interest rate risk is also priced. A second class of option models admits stochastic conditional volatility for the underlying asset, but maintains the constant-interest-rates assumption. These include the Cox and Ross (1976) constant-elasticity-of-variance model and the stochastic volatility models of Bailey and Stulz (1989), Bates (1996b, 2000), Heston (1993), Hull and White (1987a), Scott (1987), Stein and Stein (1991), and Wiggins (1987). Recently, Bakshi and Chen (1997) and Scott (1997) have developed closed-form equity option formulas that admit both stochastic volatility and stochastic interest rates.[1] Their efforts have, in some sense, helped reach the ultimate possibility of completely relaxing the Black–Scholes assumptions of constant volatility and constant interest rates. As a practical matter, these sufficiently general pricing formulas should in principle result in significant improvement in pricing and hedging performance over the Black–Scholes model. While option pricing theory has made such impressive progress, the empirical front is nonetheless far behind.[2] Will incorporating these general

[1] Amin and Ng (1993), Bailey and Stulz (1989), and Heston (1993) also incorporate both stochastic volatility and stochastic interest rates, but their option pricing formulas are not given in closed form, which makes applications difficult. Consequently, comparative statics and hedge ratios are difficult to obtain in their cases.

[2] There have been a few empirical studies that investigate the pricing, but not the hedging, performance of versions of the stochastic volatility model, relative to the Black–Scholes model. These include Bates (1996b, 2000), Dumas et al. (1998), Madan et al. (1998), Nandi (1996), and Rubinstein (1985). In Bates' work, currency and equity index options data are respectively used to test a stochastic volatility model with Poisson jumps included. Nandi does investigate the pricing and hedging

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features improve both pricing and hedging effectiveness? If so, by how much? Can these relaxed assumptions help resolve the well known empirical biases associated with the Black–Scholes formula, such as the volatility smiles [e.g., Rubinstein (1985, 1994)]? – These empirical questions must be answered before the potential of the general models can be fully realized in practical applications.

In this paper, we first develop a practically implementable version of the general equity option pricing models in Bakshi and Chen (1997) and Scott (1997), that admits stochastic interest rates and stochastic volatility, yet resembles to the extent possible the Black–Scholes model in its implementability. We present procedures for applying the resulting model to price and hedge option-like derivative products. Next, we conduct a complete analysis of the relative empirical performance, in both pricing and hedging, of the four classes of models that respectively allow for (i) constant volatility and constant interest rates (*the BS model*), (ii) constant volatility but stochastic interest rates (*the SI model*), (iii) stochastic volatility but constant interest rates (*the SV model*), and (iv) stochastic volatility and stochastic interest rates (*the SVSI model*). As the SVSI model has all the other three models nested, one should expect its static pricing and dynamic hedging performance to surpass that of the other classes. But, this performance improvement must come at the cost of potentially more complex implementation steps. In this sense, conducting such a horse-race study can at least offer a clear picture of possible tradeoffs between costs and benefits that each model may present.

Specifically, the SVSI option pricing formula is expressed in terms of the underlying stock price, the stock's volatility and the short-term interest rate. The spot volatility and the short interest rate are each assumed to follow a Markov mean-reverting square-root process. Consequently, seven structural parameters need to be estimated as input to the model. These parameters can be estimated using the Generalized Method of Moments (GMM) of Hansen (1982), as is done in, for instance, Andersen and Lund (1997), Chan et al. (1992), and Day and Lewis (1997). Or, they can be backed out from the pricing model itself by using observed option prices, as is similarly done for the BS model both in the existing literature and in Wall Street practice.

In our empirical investigation, we will adopt this implied parameter approach to implement the four models. In this regard, it is important to realize that the BS model is implemented as if the spot volatility and the spot interest rates were assumed to be time-varying within the model, that is, the spot volatility is backed out from option prices each day and used, together with the current yield curve, to price the following day's options. The SI and the SV models are implemented with a similarly internally inconsistent treatment, though to a lesser degree. Since this implementation is how one would expect each model to be applied, we chose to follow this convention in order to give the alternatives to the standard BS model the "toughest hurdle." Clearly, such a treatment works in the strongest favor of the BS model and is especially biased against the SVSI model.

Based on 38,749 S&P 500 call (and put) option prices for the sample period from June 1988 to May 1991, our empirical investigation leads to the following conclusions. First, on the basis of two out-of-sample pricing error measures, the SVSI model is found to perform slightly better than the SV model, while they both perform substantially better than the SI (the third-place performer) and the BS model. That is, when volatility is kept constant, allowing interest rates to vary stochastically can produce respectable pricing improvement over the BS model. However, in the presence of stochastic volatility, doing so no longer seems to improve pricing performance much further. Thus, modeling stochastic volatility is far more important than stochastic interest rates, at least for the purpose of pricing options. It is nonetheless encouraging to know that based on our sample both the SVSI and the SV models typically reduce the BS model's pricing errors by more than half, whereas the SI model helps reduce the BS pricing errors by 20% or more. While all four models inherit moneyness- and maturity-related pricing biases, the severity of these types of bias is increasingly reduced by the SI, the SV, and the SVSI models. In other words, the SVSI model produces pricing errors that are the least moneyness- or maturity-related. This conclusion is also confirmed when the Rubinstein (1985) implied-volatility-smile diagnostic is adopted to examine each model.

Two types of hedging strategy are employed in this study to gauge the relative hedging effectiveness. The first type is the conventional delta-neutral hedge, in which as many distinct hedging instruments as the number of risk sources affecting the hedging target's value are used so as to make the net position completely risk-immunized (locally). Take the SVSI model as an example. The call option value is driven by three risk sources: the underlying price shocks, volatility shocks, and shocks to interest rates. Accordingly, we employ the underlying stock, a different call option, and a position in a discount bond to create a delta-neutral hedge for a target call option. That closed-form expressions are derived for each hedge ratio is of great value for devising hedging strategies analytically. Similarly, for the SV model, we only need to rely on the underlying stock and an option contract to design a delta-neutral hedge. Based on the delta-neutral hedging errors, the same performance ranking of the four models obtains as that determined by their static pricing

performance of Heston's stochastic volatility model, but he focuses exclusively on a single-instrument minimum-variance hedge that involves only the S&P 500 futures. As will be clear shortly, we address in this paper both the pricing and the hedging effectiveness issues from different perspectives and for four distinct classes of option models.

performance, except that now the SVSI and the SV models, and the SI and the BS models, are respectively pairwise virtually indistinguishable. This reenforces the view that adding stochastic interest rates may not affect performance much. However, it is found that the average hedging errors by the SVSI and the SV models are typically less than one third of the corresponding BS model's hedging errors. Furthermore, reducing the frequency of hedge rebalancing does not tend to reduce the SV and the SVSI models' hedging effectiveness, whereas the BS and the SI models' hedging errors are often doubled when rebalancing frequency changes from daily to once every 5 days. Therefore, after stochastic volatility is controlled for, the frequency of hedge rebalancing will have relatively little impact on hedging effectiveness. This finding is in accord with Galai's (1983a) results that in any hedging scheme it is probably more important to control for stochastic volatility than for discrete hedging [see Hull and White (1987b) for a similar, simulation-based result for currency options].

To see how the models perform under different hedging schemes, we also look at minimum-variance hedges involving only a position in the underlying asset. As argued by Ross (1995), the need for this type of hedges may arise in contexts where a perfect delta-neutral hedge may not be feasible, either because some of the underlying risks are not traded or even reflected in any traded financial instruments, or because model misspecifications and transaction costs render it undesirable to use as many instruments to create a perfect hedge. In the present context, both volatility risk and interest rate risk are, of course, traded and hence can, as indicated above, be controlled for by employing an option and a bond. But, a point can be made that it is sometimes more preferable to adopt a single-instrument minimum-variance hedge. To study this type of hedges, we again calculate the absolute and the dollar-value hedging errors for each model. Results from this exercise indicate that the SV model performs the best among all four, while the BS and the SV models outperform their respective stochastic-interest-rates counterparts, the SI and the SVSI models. Therefore, under the single-instrument hedges, incorporating stochastic interest rates actually worsens hedging performance. It is also true that hedging errors under this type of hedges are always significantly higher than those under the conventional delta-neutral hedges, for each given moneyness and maturity option category. Thus, whenever possible, including more instruments in a hedge will in general produce better hedging effectiveness.

While our discussion is mainly focused on results obtained using the entire sample period and under specific model implementation designs, robustness of these empirical results is also checked by examining alternative implementation designs, different subperiods as well as option transaction price data. Especially, given the popularity of the "implied-volatility matrix" method among practitioners, we will also implement each of the four models, and compare their pricing and hedging performance, by using only option contracts from a given moneyness-maturity category. It turns out that this alternative implementation scheme does not change the rankings of the four models.

The rest of the paper proceeds as follows. Section 37.2 develops the SVSI option pricing formula. It discusses issues pertaining to the implementation of the formula and derives the hedge ratios analytically. Section 37.3 provides a description of the S&P 500 option data. In Sect. 37.4 we evaluate the static pricing and the dynamic hedging performance of the four models. Concluding remarks are offered in Sect. 37.5.

37.2 The Option Pricing Model

Consider an economy in which the instantaneous interest rate at time t, denoted $R(t)$, follows a Markov diffusion process:

$$dR(t) = [\theta_R - \bar{\kappa}_R\, R(t)]\, dt + \sigma_R \sqrt{R(t)} d\omega_R(t) \quad t \in [0, T], \tag{37.1}$$

where $\bar{\kappa}_R$ regulates the speed at which the interest rate adjusts to its long-run stationary value $\frac{\theta_R}{\bar{\kappa}_R}$, and $\omega_R = \{\omega_R(t) : t \in [0, T]\}$ is a standard Brownian motion.[3] This single-factor interest rate structure of Cox et al. (1985) is adopted as it requires the estimation of only three structural parameters. Adding more factors to the term structure model will of course lead to more plausible formulas for bond prices, but it can make the resulting option formula harder to implement.

Take a generic non-dividend-paying stock whose price dynamics are described by

$$\frac{dS(t)}{S(t)} = \mu(S, t)dt + \sqrt{V(t)}\, d\omega_S(t) \quad t \in [0, T], \tag{37.2}$$

where $\mu(S, t)$, which is left unspecified, is the instantaneous expected return, and ω_S a standard Brownian motion. The instantaneous stock return variance, $V(t)$, is assumed to follow a Markov process:

$$dV(t) = [\theta_v - \bar{\kappa}_v\, V(t)]\, dt + \sigma_v \sqrt{V(t)} d\omega_v(t) \quad t \in [0, T], \tag{37.3}$$

where again ω_v is a standard Brownian motion and the structural parameters have the usual interpretation. We refer to

[3] Here we follow a common practice to assume from the outset a structure for the underlying price and rate processes, rather than derive them from a full-blown general equilibrium. See Bates (1996a), Heston (1993), Melino and Turnbull (1990, 1995), and Scott (1987, 1997). The simple structure assumed in this section can, however, be derived from the general equilibrium model of Bakshi and Chen (1997).

$V(t)$ as the spot volatility or, simply, volatility. This process is also frugal in the number of parameters to be estimated and is similar to the one in Heston (1993). Letting ρ denote the correlation coefficient between ω_S and ω_v, the covariance between changes in $S(t)$ and in $V(t)$ is $Cov_t\,[dS(t), dV(t)] = \rho\sigma_S\sigma_v S(t)\,V(t)\,dt$, which can take either sign and is time-varying. According to Bakshi et al. (1997, 2000a, b), Bakshi and Chen (1997), Bates (1996a), Cao and Huang (2008), and Rubinstein (1985), this additional feature is important for explaining the skewness and kurtosis related biases associated with the BS formula. Finally, for ease of presentation, assume that the equity-related shocks and the interest rate shocks are uncorrelated:[4] $Cov_t(d\omega_S, d\omega_R) = Cov_t(d\omega_v, d\omega_R) = 0$.

By a result from Harrison and Kreps (1979), there are no free-lunches in the economy if and only if there exists an equivalent martingale measure with which one can value claims as if the economy were risk-neutral. For instance, the time-t price $B(t,\tau)$ of a zero-coupon bond that pays \$1 in τ periods can be determined via

$$B(t,\tau) = E_Q\left\{\exp\left(-\int_t^{t+\tau} R(s)ds\right)\right\}, \qquad (37.4)$$

where E_Q denotes the expectation with respect to an equivalent martingale measure and conditional on the information generated by $R(t)$ and $V(t)$. Assume that the factor risk premiums for $R(t)$ and $V(t)$ are respectively given by $\lambda_R\,R(t)$ and $\lambda_v\,V(t)$, for two constants λ_R and λ_v. Bakshi and Chen (1997) provide a general equilibrium model in which risk premiums have precisely this form and in which the interest rate and stock price processes are as assumed here. Under this assumption, we obtain the risk-neutralized processes for $R(t)$ and $V(t)$ below:

$$dR(t) = [\theta_R - \kappa_R\,R(t)]\,dt + \sigma_R\sqrt{R(t)}\,d\omega_R(t) \qquad (37.5)$$

$$dV(t) = [\theta_v - \kappa_v\,V(t)]\,dt + \sigma_v\sqrt{V(t)}\,d\omega_v(t), \qquad (37.6)$$

where $\kappa_R \equiv \bar{\kappa}_R + \lambda_R$ and $\kappa_v \equiv \bar{\kappa}_v + \lambda_v$. The risk-neutralized stock price process becomes

$$\frac{dS(t)}{S(t)} = R(t)\,dt + \sqrt{V(t)}\,d\omega_S(t), \qquad (37.7)$$

That is, under the martingale measure, the stock should earn no more and no less than the risk-free rate. With these adjustments, we solve the conditional expectation in (37.4) and obtain the familiar bond price equation below:

$$B(t,\tau) = \exp\left[-\varphi(\tau) - \varrho(\tau)R(t)\right], \qquad (37.8)$$

where $\varphi(\tau) = \frac{\theta_R}{\sigma_R^2}\left\{(\varsigma - \kappa_R)\,\tau + 2\ln\left[1 - \frac{(1-e^{-\varsigma\tau})(\varsigma-\kappa_R)}{2\varsigma}\right]\right\}$, $\varrho(\tau) = \frac{2(1-e^{-\varsigma\tau})}{2\varsigma-[\varsigma-\kappa_R](1-e^{-\varsigma\tau})}$ and $\varsigma \equiv \sqrt{\kappa_R^2 + 2\sigma_R^2}$. See Cox et al. (1985) for an analysis of this class of term structure models.

37.2.1 Pricing Formula for European Options

Now, consider a European call option written on the stock, with strike price K and term-to-expiration τ. Let its time-t price be denoted by $C(t,\tau)$. As (S, R, V) form a joint Markov process, the price $C(t,\tau)$ must be a function of $S(t)$, $R(t)$ and $V(t)$ (in addition to τ). By a standard argument, the option price must solve

$$\frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + R\,S\,\frac{\partial C}{\partial S} + \rho\sigma_v VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma_v^2 V\frac{\partial^2 C}{\partial V^2}$$
$$+ [\theta_v - \kappa_v\,V]\,\frac{\partial C}{\partial V} + \frac{1}{2}\sigma_R^2 R\frac{\partial^2 C}{\partial R^2}$$
$$+ [\theta_R - \kappa_R\,R]\,\frac{\partial C}{\partial R} - \frac{\partial C}{\partial \tau} - R\,C = 0, \qquad (37.9)$$

subject to $C(t+\tau, 0) = \max\{S(t+\tau) - K, 0\}$. In the Appendix it is shown that

$$C(t,\tau) = S(t)\Pi_1(t,\tau; S,R,V) - KB(t,\tau)\Pi_2(t,\tau,S,R,V), \qquad (37.10)$$

where the risk-neutral probabilities, Π_1 and Π_2, are recovered from inverting the respective characteristic functions [see Heston (1993), and Scott (1997) for similar treatments]:

[4] This assumption on the correlation between stock returns and interest rates is somewhat severe and likely counterfactual. To gauge the potential impact of this assumption on the resulting option model's performance, we initially adopted the following stock price dynamics:

$$\frac{dS(t)}{S(t)} = \mu(S,t)dt + \sqrt{V(t)}\,d\omega_S(t) + \sigma_{S,R}\,\sqrt{R(t)}d\omega_R(t) \quad t \in [0,T],$$

with the rest of the stochastic structure remaining the same as given above. Under this more realistic structure, the covariance between stock price changes and interest rate shocks is $Cov_t\,[dS(t), dR(t)] = \sigma_{S,R}\sigma_R\,R(t)S(t)\,dt$, so bond market innovations can be transmitted to the stock market and vice versa. The obtained closed-form option pricing formula under this scenario would have one more parameter $\sigma_{S,R}$ than the one presented shortly, but when we implemented this slightly more general model, we found its pricing and hedging performance to be indistinguishable from that of the SVSI model studied in this paper. For this reason, we chose to present the more parsimonious SVSI model derived under the stock price process in (37.2). We could also make both the drift and the diffusion terms of $V(t)$ a linear function of $R(t)$ and $\omega_R(t)$. In such cases, the stock returns, volatility and interest rates would all be correlated with each other (at least globally), and we could still derive the desired equity option valuation formula. But, that would again make the resulting formula more complex while not improving its performance.

$$\Pi_j\left(t,\tau;S(t),R(t),V(t)\right)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln[K]}f_j(t,\tau,S(t),R(t),V(t);\phi)}{i\phi}\right]d\phi, \tag{37.11}$$

for $j=1,2$. The characteristic functions f_j are respectively given by

$$\begin{aligned}
f_1(t,\tau)=\exp\Bigg\{&-\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1-\frac{[\xi_R-\kappa_R](1-e^{-\xi_R\tau})}{2\xi_R}\right)+[\xi_R-\kappa_R]\tau\right]\\
&-\frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1-\frac{[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}{2\xi_v}\right)\right]\\
&-\frac{\theta_v}{\sigma_v^2}[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v)]\,\tau+i\phi\ln[S(t)]+\frac{2i\phi(1-e^{-\xi_R\tau})}{2\xi_R-[\xi_R-\kappa_R](1-e^{-\xi_R\tau})}R(t)\\
&+\frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})}{2\xi_v-[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}V(t)\Bigg\},
\end{aligned} \tag{37.12}$$

and,

$$\begin{aligned}
f_2(t,\tau)=\exp\Bigg\{&-\frac{\theta_R}{\sigma_R^2}\left[2\ln\left(1-\frac{[\xi_R^*-\kappa_R](1-e^{-\xi_R^*\tau})}{2\xi_R^*}\right)+[\xi_R^*-\kappa_R]\tau\right]\\
&-\frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1-\frac{[\xi_v^*+i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}{2\xi_v^*}\right)+\left[\xi_v^*-\kappa_v+i\phi\rho\sigma_v\right]\tau\right]\\
&+i\phi\ln[S(t)]-\ln[B(t,\tau)]+\frac{2(i\phi-1)(1-e^{-\xi_R^*\tau})}{2\xi_R^*-[\xi_R^*-\kappa_R](1-e^{-\xi_R^*\tau})}R(t)\\
&+\frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})}{2\xi_v^*-[\xi_v^*-\kappa_v+i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}V(t)\Bigg\},
\end{aligned} \tag{37.13}$$

where $\xi_R=\sqrt{\kappa_R^2-2\sigma_R^2\,i\phi}$, $\xi_v=\sqrt{[\kappa_v-(1+i\phi)\rho\sigma_v]^2}$ $\sqrt{-i\phi(i\phi+1)\sigma_v^2}$, $\xi_R^*=\sqrt{\kappa_R^2-2\sigma_R^2\,(i\phi-1)}$, and $\xi_v^*=\sqrt{[\kappa_v-i\phi\rho\sigma_v]^2-i\phi(i\phi-1)\sigma_v^2}$. The price of a European put on the same stock can be determined from the put-call parity.

The option valuation model in (37.10) has several distinctive features. First, it applies to cases with stochastically-varying interest rates and volatility. It contains as special cases most existing models, such as the SV models, the SI models, and clearly the BS model. Second, as mentioned earlier, it allows for a flexible correlation structure between the stock return and its volatility, as opposed to the perfect correlation assumed in, for instance, Heston's (1993) model. Furthermore, the volatility risk premium is time-varying and state-dependent. This is a departure from Hull and White (1987), Scott (1987), Stein and Stein (1991), and Wiggins (1987) where the volatility risk premium is either a constant or zero. Third, when compared to the general models in Bakshi and Chen (1997) and Scott (1997), the formula in (37.10) is parsimonious in the number of parameters; Especially, it is given only as a function of identifiable variables such that all parameters can be estimated based on available financial market data.

The pricing formula in (37.10) applies to European equity options. But, in reality most of the traded option contracts are American in nature. While it is beyond the scope of the present paper to derive a model for American options, it is nevertheless possible to capture the first-order effect of early exercise in the following manner. For options with early exercise potential, compute the Barone-Adesi and Whaley (1987) or Kim (1990) early-exercise premium, treating it as if the stock volatility and the yield-curve were time-invariant. Adding this early-exercise adjustment component to the European option price in (37.10) should deliver a reasonable approximation of the corresponding American option price [e.g., Bates (1996b)].

37.2.2 Hedging and Hedge Ratios

One appealing feature of a closed-form option pricing formula, such as the one in (37.10), is the possibility of deriving comparative statics and hedge ratios analytically. In the present context, there are three sources of stochastic variations over time, price risk $S(t)$, volatility risk $V(t)$ and interest rate risk $R(t)$. Consequently, there are three deltas:

$$\Delta_S(t,\tau;K) \equiv \frac{\partial C(t,\tau)}{\partial S} = \Pi_1 > 0 \tag{37.14}$$

$$\Delta_V(t,\tau;K) \equiv \frac{\partial C(t,\tau)}{\partial V} = S(t)\frac{\partial \Pi_1}{\partial V} - KB(t,\tau)\frac{\partial \Pi_2}{\partial V} > 0 \tag{37.15}$$

$$\Delta_R(t,\tau;K) \equiv \frac{\partial C(t,\tau)}{\partial R} = S(t)\frac{\partial \Pi_1}{\partial R} - KB(t,\tau)\left\{\frac{\partial \Pi_2}{\partial R} - \varrho(\tau)\Pi_2\right\} > 0, \tag{37.16}$$

where, for $g = V, R$ and $j = 1, 2$,

$$\frac{\partial \Pi_j}{\partial g} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[(i\phi)^{-1} e^{-i\,\phi\,\ln[K]}\,\frac{\partial f_j}{\partial g}\right] d\phi. \tag{37.17}$$

The second-order partial derivatives with respect to these variables are provided in the Appendix.

As $V(t)$ and $R(t)$ are both stochastic in our model, these deltas will in general differ from their Black–Scholes counterpart. To see how they may differ, let's resort to an example in which we set $R(t) = 6.27\%$, $S(t) = 270$, $\sqrt{V(t)} = 22.12\%$, $\kappa_R = 0.481$, $\theta_R = 0.037$, $\sigma_R = 0.043$, $\kappa_v = 1.072$, $\theta_v = 0.041$, $\sigma_v = 0.284$, and $\rho = -0.60$. These values are backed out from the S&P 500 option prices as of July 5, 1988. Fix K = \$270 and $\tau = 45$ days. Let Δ_S be as given in (37.14) for the SVSI model and Δ_S^{bs} its BS counterpart, with Δ_S^{bs} calculated using the same implied volatility. Figure 37.1 plots the difference between Δ_S and Δ_S^{bs}, across different spot price levels and different correlation values. The correlation coefficient ρ is chosen to be the focus as it is known to play a crucial role in determining the skewness of the stock return distribution. When ρ is respectively at -0.50 and -1.0 (see the □-curve and the ○-curve), the difference between the deltas is W-shaped, and it reaches the highest value when the option is at the money. The reverse is true when ρ is positive. Thus, Δ_S is generally different from Δ_S^{bs}. Analogous difference patterns emerge when the other option

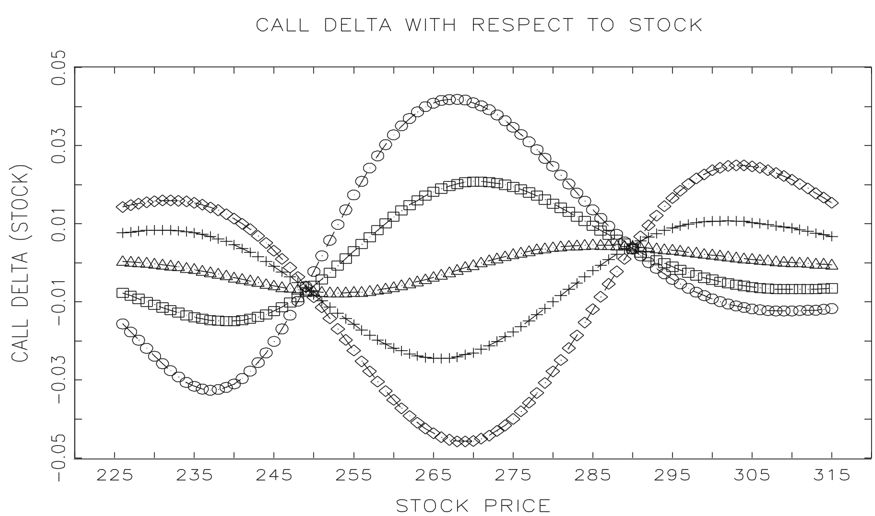

Fig. 37.1 The ○-curve, the □-curve, the △-curve, the +-curve, and the ◇-curve respectively plot the difference between the SVSI call option delta (with respect to stock) and its Black–Scholes counterpart, as ρ varies from $-1.0, -0.50, 0, 0.50,$ to 1.0. The structural parameter values used in the computation of the delta in (37.14) are backed out using Procedure B described in Sect. 37.2.3 and correspond to the calendar date July 5, 1988. The values of the structural parameters are: $\kappa_R = 0.4811$, $\theta_R = 0.0370$, $\sigma_R = 0.0429$, $\kappa_v = 1.072$, $\theta_v = 0.0409$, $\sigma_v = 0.284$, $\rho = -0.60$. The initial (time-t) $R = 0.062733$, $\sqrt{V} = 22.12\%$, $B(t, 0.1232) = 0.99163$. The strike price is fixed at \$270 and the term-to-expiration of the option is 45 days

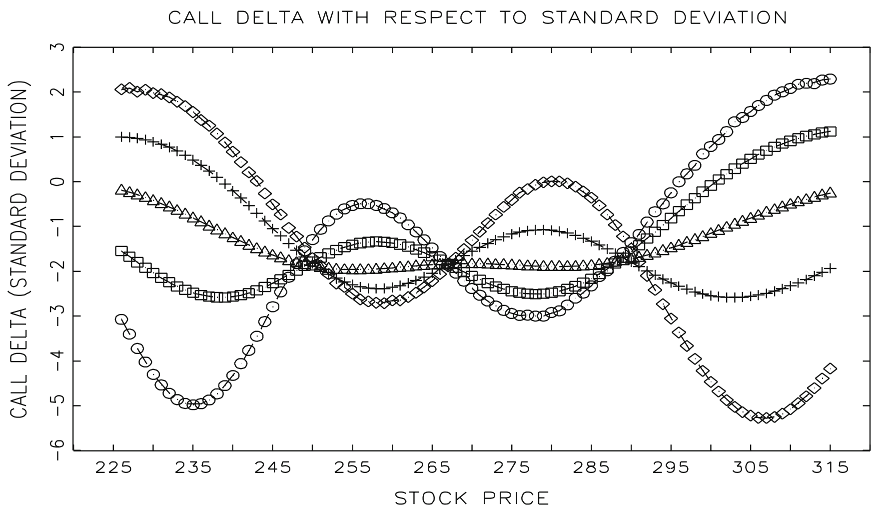

Fig. 37.2 The ○-curve, the □-curve, the △-curve, the +-curve, and the ◇-curve respectively plot the difference between the SVSI call option delta (with respect to the standard deviation) and the Black–Scholes counterpart, as ρ varies from $-1.0, -0.50, 0, 0.50$, to 1.0. The strike price is fixed at \$270 and the term-to-expiration of the option is 45 days. All computations are based on the parameter values given in the note to Fig. 37.1

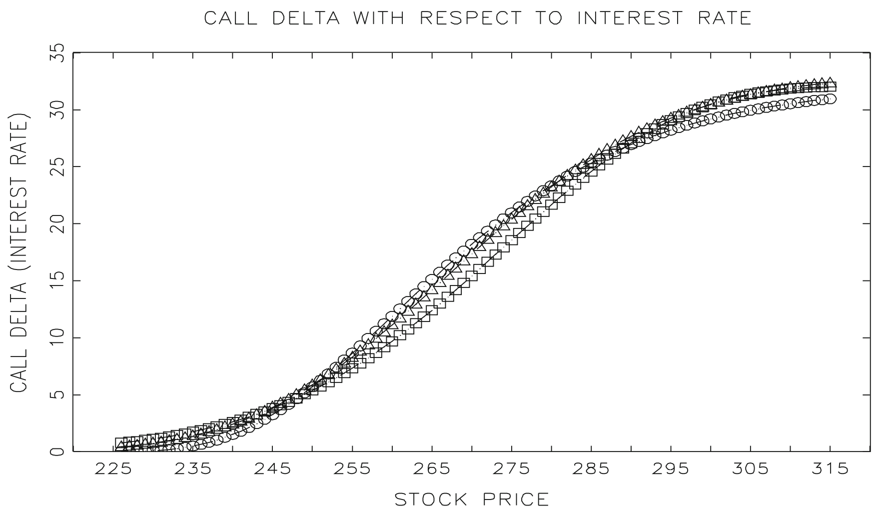

Fig. 37.3 The ○-curve, the □-curve, the △-curve, the +-curve, and the ◇-curve respectively plot the difference between the SVSI call option delta (with respect to the spot interest rate) and the Black–Scholes counterpart, as ρ varies from $-1.0, -0.50, 0, 0.50$, to 1.0. The strike price is fixed at \$270 and the term-to-expiration of the option is 45 days. All computations are based upon the parameter values reported in the note to Fig. 37.1

deltas are compared with their respective BS counterpart. From Figs. 37.2 and 37.3, one can observe the following. (i) The volatility hedge ratio Δ_V from the SVSI model is, at each spot price, lower than its BS counterpart (except for deep in-the-money options when $\rho < 0$, and for deep out-of-the-money options when $\rho > 0$).[5] (ii) The interest-rate delta, Δ_R, and its BS counterpart, Δ_R^{bs}, are almost not different from each other for slightly out-of-the-money options, but can be dramatically different for at-the-money options as well as for sufficiently deep in-the-money or deep out-of-the-money calls. For example, pick $\rho = -1.0$. When S = \$315, we have $\Delta_R = 30.94$ and $\Delta_R^{bs} = 32.35$; When S = \$226, we have $\Delta_R = 0.003$ and $\Delta_R^{bs} = 0.430$. (iii) As expected, out-of-the-money options are overall less sensitive to changes in the spot interest rate, regardless of the model used. In summary, if a portfolio manager/trader relies, in an environment with stochastic interest rates and stochastic volatility, on the BS model to design a hedge for option positions, the manager/trader will likely fail.

Analytical expressions for the deltas are useful for constructing hedges based on an option formula. Below, we present two types of hedges by using the SVSI model as an example.

37.2.2.1 Delta-Neutral Hedges

To demonstrate how the deltas may be used to construct a delta-neutral hedge, consider an example in which a financial institution intends to hedge a short position in a call option with τ periods to expiration and strike price K. In the stochastic interest rate-stochastic volatility environment, a perfectly delta-neutral hedge can be achieved by taking

[5] In making such a comparison, one should apply sufficient caution. In the BS model, the volatility delta is only a comparative static, not a hedge ratio, as volatility is assumed to be constant. In the context of the SVSI model, however, Δ_V is time-varying hedge ratio as volatility is stochastic. This distinction also applies to the case of the interest-rate delta Δ_R.

a long position in the replicating portfolio of the call. As three traded assets are needed to control the three sources of uncertainty, the replicating portfolio will involve a position in (i) some $X_S(t)$ shares of the underlying stock (to control for the $S(t)$ risk), (ii) some $X_B(t)$ units of a τ-period discount bond (to control for the $R(t)$ risk), and (iii) some $X_C(t)$ units of another call option with strike price $\bar{K}$ (or any option on the stock with a different maturity) in order to control for the volatility risk $V(t)$. Denote the time-t price of the replicating portfolio by $G(t)$: $G(t) = X_0(t) + X_S(t)\,S(t) + X_B(t)\,B(t,\tau) + X_C(t)\,C(t,\tau;\bar{K})$, where $X_0(t)$ denotes the amount put into the instantaneously-maturing risk-free bond and it serves as a residual "cash position." Deriving the dynamics for $G(t)$ and comparing them with those of $C(t,\tau;K)$, we find the following solution for the delta-neutral hedge:

$$X_C(t) = \frac{\Delta_V(t,\tau;K)}{\Delta_V(t,\tau;\bar{K})} \tag{37.18}$$

$$X_S(t) = \Delta_S(t,\tau;K) - \Delta_S(t,\tau;\bar{K})\,X_C(t) \tag{37.19}$$

$$X_B(t) = \frac{1}{B(t,\tau)\,\varrho(\tau)}\left\{\Delta_R(t,\tau;\bar{K})\,X_C(t) - \Delta_R(t,\tau;K)\right\} \tag{37.20}$$

and the residual amount put into the instantaneously-maturing bond is

$$X_0(t) = C(t,\tau;K) - X_S(t)\,S(t) - X_C(t)\,C(t,\tau;\bar{K}) - X_B(t)\,B(t,\tau), \tag{37.21}$$

where all the primitive deltas, Δ_S, Δ_R and Δ_V, are as determined in equations (37.14)–(37.16). Like the option prices, these hedge ratios all depend on the values taken by $S(t)$, $V(t)$ and $R(t)$ and those by the structural parameters. Such a hedge created using the general option pricing model should in principle perform better than using the BS model. In the latter case, only the underlying price uncertainty is controlled for, but not the uncertainties associated with volatility and interest rate fluctuations.

In theory this delta-neutral hedge requires continuous rebalancing to reflect the changing market conditions. In practice, of course, only discrete rebalancing is possible. To derive a hedging effectiveness measure, suppose that portfolio rebalancing takes place at intervals of length Δt. Then, precisely as described above, at time t short the call option, go long in (i) $X_S(t)$ shares of the underlying asset, (ii) $X_B(t)$ units of the τ–period bond, and (iii) $X_C(t)$ contracts of a call option with the same term-to-expiration but a different strike price $\bar{K}$, and invest the residual, X_0, in an instantaneously maturing riskfree bond. After the next interval, compute the hedging error according to

$$\begin{aligned} H(t+\Delta t) = {} & X_0 e^{R(t)\Delta t} + X_S(t)S(t+\Delta t) \\ & + X_B(t)B(t+\Delta t,\tau-\Delta t) \\ & + X_C(t)C(t+\Delta t,\tau-\Delta t;\bar{K}) \\ & - C(t+\Delta t,\tau-\Delta t;K). \end{aligned} \tag{37.22}$$

Then, at time $t+\Delta t$, reconstruct the self-financed portfolio, repeat the hedging error calculation at time $t+2\Delta t$, and so on. Record the hedging errors $H(t+j\Delta t)$, for $j = 1,\cdots,J \equiv \frac{\tau-t}{\Delta t}$. Finally, compute the average absolute hedging error as a function of rebalancing frequency Δt: $H(\Delta t) = \frac{1}{J}\sum_{j=1}^{J} \mid H(t+j\Delta t) \mid$, and the average dollar-value hedging error: $\bar{H}(\Delta t) = \frac{1}{J}\sum_{j=1}^{J} H(t+j\Delta t)$.

In comparison, if one relies on the BS model to construct a delta-neutral hedge, the hedging error measures can be similarly defined as in (37.22), except that $X_B(t)$ and $X_C(t)$ must be restricted to zero and $X_S(t)$ must be the BS delta. Likewise, if the SI model is applied, the only change is to set $X_C(t)$ to zero with Δ_S and Δ_R determined by the SI model; In the case of the SV model, set $X_B(t) = 0$ and let Δ_S and Δ_V be as determined in the SV model. The Appendix provides in closed form a SI option pricing formula and a SV option formula.

37.2.2.2 Single-Instrument Minimum-Variance Hedges

As discussed before, consideration of such factors as model misspecification and transaction costs may render it more practical to use only the underlying asset of the target option as the hedging instrument. Under this single-instrument constraint, a standard design is to choose a position in the underlying stock so as to minimize the variance of instantaneous changes in the value of the hedge. Letting $X_S(t)$ again be the number of shares of the stock to be purchased, solving the standard minimum-variance hedging problem under the SVSI model gives

$$X_S(t) = \frac{Cov_t\,[dS(t), dC(t,\tau)]}{Var\,[dS(t)]} = \Delta_S + \rho\sigma_v \frac{\Delta_V(t,\tau)}{S(t)}, \tag{37.23}$$

and the resulting residual cash position for the replicating portfolio is

$$X_0(t) = C(t,\tau) - X_S(t)S(t). \tag{37.24}$$

This minimum-variance hedge solution is quite intuitive, as it says that if stock volatility is deterministic (i.e., $\sigma_v = 0$), or if stock returns are not correlated with volatility changes (i.e., $\rho = 0$), one only needs to long $\Delta_S(t)$ shares of the stock and no other adjustment is necessary. However, if volatility is stochastic and correlated with stock returns, the position to be taken in the stock must control not only for the direct impact of underlying stock price changes on the target option value, but also for the indirect impact of that part of volatility changes which is correlated with stock price fluctuations. This effect is reflected in the last term in (37.23), which shows that the additional number of shares needed besides Δ_S is increasing in ρ (assuming $\sigma_v > 0$).

As for the previous case, suppose that the target call is shorted and that $X_S(t)$ shares are bought and $X_0(t)$ dollars are put into the instantaneous risk-free bond, at time t. The combined position is a self-financed portfolio. At time $t+\Delta t$, the hedging error of this minimum-variance hedge is calculated as

$$H(t + \Delta t) = X_S(t)S(t + \Delta t) + X_0(t)e^{R(t)\Delta t} \\ -C(t + \Delta t, \tau - \Delta t). \qquad (37.25)$$

Unlike in Nandi (1996) where he uses the remaining variance of the hedge as a hedging effectiveness gauge, we compute, based on the entire sample period, the average absolute and the average dollar hedging errors to measure the effectiveness of the hedge.

Minimum-variance hedging errors under the SV model as well as under the SI model can be similarly determined accounting for their modeling differences. In the case of the SV model, there is still an adjustment term for the single stock position as in (37.23). But, for the SI model, the corresponding $X_S(t)$ is the same as its Δ_S. For the BS model, this single-instrument minimum-variance hedge is the same as the delta-neutral hedge. Both types of hedging strategy will be examined under each of the four alternative models.

37.2.3 Implementation

In addition to the strike price and the term-to-expiration (which are specified in the contract), the SVSI pricing formula in (37.10) requires the following values as input:

- The spot stock price. If the stock pays dividends, the stock price must be adjusted by the present value of future dividends;
- The spot volatility;
- The spot interest rate;
- The matching τ-period yield-to-maturity (or the bond price);
- The seven structural parameters: $\{\kappa_R, \theta_R, \sigma_R, \kappa_v, \theta_v, \sigma_v, \rho\}$.

For computing the price of a European option, we offer two alternative two-step procedures below. One can implement these steps on any personal computer:

Procedure A:
Step 1. Obtain a time-series each for the short rate, the stock return, and the stock volatility. Jointly estimate the structural parameters, $\{\kappa_R, \theta_R, \sigma_R, \kappa_v, \theta_v, \sigma_v, \rho\}$, using Hansen's (1982) GMM.
Step 2. Determine the risk-neutral probabilities, Π_1 and Π_2, from the characteristic functions in (37.12) and (37.13). Substitute (i) the two probabilities, (ii) the stock price, and (iii) the yield-to-maturity, into (37.10) to compute the option price.

While offering an econometrically rigorous method to estimate the structural parameters, Step 1 in Procedure A may not be as practical or convenient, because of its requirement on historical data. A further difficulty with this approach is its dependence on the measurement of stock volatility. In implementing the BS model, practitioners predominantly use the implied volatility from the model itself, rather than relying on historical data. This practice has not only reduced data requirement dramatically but also resulted in significant performance improvement [e.g., Bates (2000), and Melino and Turnbull (1990, 1995)]. Clearly, one can also follow this practice to implement the SVSI model:

Procedure B:
Step 1. Collect N option prices on the same stock and taken from the same point in time (or same day), for any $N \geq 8$. Let $\hat{C}_n(t, \tau_n, K_n)$ be the observed price, and $C_n(t, \tau_n, K_n)$ the model price as determined by (37.10) with $S(t)$ and $R(t)$ taken from the market, for the nth option with τ_n periods to expiration and strike price K_n and for each $n = 1, \ldots, N$. Clearly, the difference between $\hat{C}_n$ and C_n is a function of the values taken by $V(t)$ and by $\Phi \equiv \{\kappa_R, \theta_R, \sigma_R, \kappa_v, \theta_v, \sigma_v, \rho\}$. Define

$$\epsilon_n[V(t), \Phi] \equiv \hat{C}_n(t, \tau_n, K_n) - C_n(t, \tau_n, K_n), \qquad (37.26)$$

for each n. Then, find $V(t)$ and parameter vector Φ (a total of eight), so as to minimize the sum of squared errors:

$$\sum_{n=1}^{N} \mid \epsilon_n[V(t), \Phi] \mid^2 . \qquad (37.27)$$

The result from this step is an estimate of the implied spot variance and seven structural parameter values, for date t. See Bates (1996b, 2000), Day and Lewis (1997), Dumas et al. (1998), Longstaff (1995), Madan et al. (1998), and Nandi (1996) where they adopt this technique for similar purposes.

Step 2. Based on the estimate from the first step, follow Step 2 of Procedure A to compute date-$(t+1)$'s option prices on the same stock.

In the existing literature, the performance of a new option pricing model is often judged relative to that of the BS model when the latter is implemented using the model's own implied volatility and the time-varying interest rates. Since volatility and interest rates in the BS are assumed to be constant over time, this internally inconsistent practice will clearly and significantly bias the application results in favor of the BS model. But, as this is the current standard in judging performance, we will follow Procedure B to implement the SVSI model and similar procedures to implement the BS, the SV, and the SI models. Then, the models will be ranked relative to each other according to their performance so determined.

37.3 Data Description

For all the tests to follow, we use, based on the following considerations, S&P 500 call option prices as the basis. First, options written on this index are the most actively traded European-style contracts. Recall that like the BS model, formula (37.10) applies to European options. Second, the daily dividend distributions are available for the index (from the S&P 500 Information Bulletin). Harvey and Whaley (1992a, b), for instance, emphasize that critical pricing errors can result when dividends are omitted from empirical tests of any option valuation model. Furthermore, S&P 500 options and options on S&P 500 futures have been the focus of many existing empirical investigations including, among others, Bates (2000), Dumas et al. (1998), Madan et al. (1998), Nandi (1996), and Rubinstein (1994). Finally, we also used S&P 500 put option prices to estimate the pricing and hedging errors of all four models and found the results to be similar, both qualitatively and quantitatively, to those reported in the paper. To save space, we chose to focus on the results based on the call option prices.

The sample period extends from June 1, 1988 through May 31, 1991. The intradaily transaction prices and bid-ask quotes for S&P 500 options are obtained from the Berkeley Option Database. Note that the recorded S&P 500 index values are not the daily closing index levels. Rather, they were the corresponding index levels at the moment when the recorded option transaction took place or when an option price quote was recorded. Thus, there is no non-synchronous price issue here, except that the S&P 500 index level itself may contain stale component stock prices at each point in time.

The data on the daily Treasury-bill bid and ask discounts with maturities up to 1 year are hand-collected from the *Wall Street Journal* and provided to us by Hyuk Choe and Steve Freund. By convention, the average of the bid and ask Treasury bill discounts is used and converted to an annualized interest rate. Careful attention is given to this construction since Treasury bills mature on Thursdays while index options expire on the third Friday of the month. In such cases, we utilize the two Treasury-bill rates straddling the option's expiration date to obtain the interest rate of that maturity, which is done for each contract and each day in the sample. The Treasury bill rate with 30-days to maturity is the surrogate used for the short rate in (37.1) [and in the determination of the probabilities in (37.10)].

For European options, the spot stock price must be adjusted for discrete dividends. For each option contract with τ periods to expiration from time t, we first obtain the present value of the daily dividends $D(t)$ by computing

$$\bar{D}(t,\tau) \equiv \sum_{s=1}^{\tau-t} e^{-R(t,s)s} D(t+s), \tag{37.28}$$

where $R(t,s)$ is the s-period yield-to-maturity. This procedure is repeated for all option maturities and for each day in our sample. In the next step, we subtract the present value of future dividends from the time-t index level, in order to obtain the dividend-exclusive S&P 500 spot index series that is later used as input into the option models.

Several exclusion filters are applied to construct the option price data set. First, option prices that are time-stamped later than 3:00 PM Central Daytime are eliminated. This ensures that the spot price is recorded synchronously with its option counterpart. Second, as options with less than 6 days to expiration may induce liquidity-related biases, they are excluded from the sample. Third, to mitigate the impact of price discreteness on option valuation, option prices lower than $\$\frac{3}{8}$ are not included. Finally, quote prices that are less than the intrinsic value of the option are taken out of the sample.

We divide the option data into several categories according to either moneyness or term to expiration. A call option is said to be at-the-money (ATM) if its $\frac{S}{K} \in (0.97, 1.03)$, where S is the spot price and K the strike; out-of-the-money (OTM) if $\frac{S}{K} \leq 0.97$; and in-the-money (ITM) if $\frac{S}{K} \geq 1.03$. A finer partition resulted in 9 moneyness categories. By the term to expiration, each option can be classified as [e.g., Rubinstein (1985)] (i) extremely short-term (<30 days); (ii) short-term (30–60 days); (iii) near-term (60–120 days); (iv) middle-maturity (120–180 days); and (v) long-term (>180 days). The proposed moneyness and term-to-expiration classifications resulted in 54 categories for which the empirical results will be reported.

Table 37.1 describes sample properties of the S&P 500 call option prices used in the tests. Summary statistics are

Table 37.1 Sample properties of S&P 500 index options. The reported numbers are respectively the average quoted bid-ask mid-point price and the number of observations. Each option contract is consolidated across moneyness and term-to-expiration categories. The sample period extends from June 1, 1988 through May 31, 1991 for a total of 38,749 calls. Daily information from the last quote of each option contract is used to obtain the summary statistics

Moneyness	Term-to-expiration (days)						
$\frac{S}{K}$	<30	30–60	60–90	90–120	120–180	≥180	Subtotal
<0.93	0.78	1.33	1.99	2.84	4.88	7.82	
	{23}	{246}	{266}	{431}	{1080}	{1538}	{3584}
0.93–0.95	1.02	1.91	3.30	5.08	8.14	12.86	
	{121}	{595}	{267}	{ 319}	{ 596}	{ 646}	{2544}
0.95–0.97	1.35	3.05	5.35	7.45	10.87	15.91	
	{488}	{1012}	{316}	{351}	{670}	{628}	{3465}
0.97–0.99	2.47	5.53	8.23	10.83	14.19	19.33	
	{838}	{1020}	{312}	{336}	{676}	{706}	{3888}
0.99–1.01	5.27	8.99	11.96	14.55	17.95	23.20	
	{776}	{954}	{285}	{308}	{629}	{631}	{3583}
1.01–1.03	9.65	13.17	15.99	18.84	22.06	27.74	
	{752}	{906}	{276}	{283}	{607}	{597}	{3421}
1.03–1.05	14.79	17.80	20.80	23.36	26.39	31.91	
	{675}	{844}	{241}	{264}	{542}	{501}	{3067}
1.05–1.07	20.20	22.63	25.83	27.83	30.69	35.70	
	{620}	{760}	{224}	{242}	{449}	{473}	{2818}
≥1.07	41.23	42.28	47.50	49.27	51.34	59.82	
	{2143}	{2350}	{1284}	{1355}	{2184}	{3063}	{12379}
Subtotal	{6436}	{8687}	{3471}	{3889}	{7483}	{8783}	{38749}

S denotes the spot S&P 500 Index level and K is the exercise price.

reported for the average bid-ask mid-point price and the total number of observations, for each moneyness-maturity category. Note that there is a total of 38,749 call price observations, with deep in-the-money and at-the-money options respectively taking up 32% and 28% of the total sample, and that the average call price ranges from \$0.78 for extremely short-term, deep out-of-the money options to \$59.82 for long-term, deep in-the-money options.

37.4 Empirical Tests

This section examines the relative empirical performance of the four models. The analysis is intended to present a complete picture of what each generalization of the benchmark BS model can really buy in terms of performance improvement and whether each generalization produces a worthy tradeoff between benefits and costs. We will pursue this analysis by using three yardsticks: (i) the size of the out-of-sample cross-sectional pricing errors (*static performance*); (ii) the size of model-based hedging errors (*dynamic performance*); and (iii) the existence of systematic biases across strike prices or across maturities (i.e., *does the implied volatility still smile*?).

Based on Procedure B of Sect. 37.2.3, Table 37.2 reports the summary statistics for the daily estimated structural parameters and the implied spot standard deviation, respectively for the SVSI, the SV, the SI and the BS models. Take the SVSI model as an example. Over the entire sample period 06:1988–05:1991, $\kappa_v = 0.906$, $\theta_v = 0.042$, and $\sigma_v = 0.414$. These estimates imply a long-run mean of 21.53% for the volatility process. The implicit (average) half-life for variance mean-reversion is 9.18 months. These estimates are similar in magnitude to those reported in Bates (1996b, 2000) for S&P 500 futures options. The estimated parameters for the (risk-neutralized) short rate process are also reasonable and comparable to those in Chan et al. (1992). The presented correlation estimate for ρ is -0.763. The average implied-standard-deviation is 19.27%. As seen from the reported standard errors in Table 37.2, for each given model the daily parameter and spot volatility estimates are quite stable from subperiod to subperiod. Histogram-based inferences (not reported) indicate that the majority of the estimated values are centered around the mean.

In estimating the structural parameters and the implied volatility for a given day, we used all S&P 500 options collected in the sample for that day (regardless of maturity and moneyness). This is the treatment applied to the SI, the SV and the SVSI models. For the BS model, however, Whaley (1982) makes the point that ATM options may give an implied-volatility estimate which produces the best pricing and hedging results. Based on his justification, we used, for each given day, one ATM option that had at least 15 days to expiration to back out the BS model's implied-volatility value. This estimate was then used to determine the next

Table 37.2 Estimates of the structural parameters for stochastic interest rates (SI), stochastic volatility (SV). and stochastic volatility and stockastic interest rates (SVSI) models. Each day in the sample, the structural parameters of a given model are estimated by minimizing the sum of squared pricing errors between the market price and the model-determined price for each option. The daily average of the estimated parameters is reported first, followed by its standard error in parentheses. The average implied volatility obtained from inverting the Black–Scholes model (using a short-term at-the-money option) is respectively 18.47%, 17.72%, 17.41%, and 20.52% over the sample periods: 06:1988–05:1991, 06:1988–05:1989, 06:1989–05:1990, and 06:1990–05:1991

	SI	SV	SVSI	SI	SV	SVSI	SI	SV	SVSI	SI	SV	SVSI
Parameters	06:1988–05:1991			06:1988–05:1989			06:1989–05:1990			06:1990–05:1991		
κ_R	2.35		0.61	2.51		0.49	2.29		0.61	2.03		0.76
	(0.03)		(0.02)	(0.06)		(0.02)	(0.05)		(0.03)	(0.05)		(0.04)
θ_R	0.35		0.02	0.33		0.02	0.34		0.02	0.35		0.02
	(0.00)		(0.00)	(0.01)		(0.00)	(0.01)		(0.00)	(0.01)		(0.00)
σ_R	0.04		0.03	0.04		0.04	0.04		0.03	0.04		0.03
	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.01)		(0.00)
κ_v		1.10	0.91		1.29	1.16		1.05	0.78		0.94	0.76
		(0.02)	(0.03)		(0.05)	(0.07)		(0.03)	(0.05)		(0.04)	(0.04)
θ_v		0.04	0.04		0.04	0.04		0.04	0.04		0.04	0.04
		(0.00)	(0.01)		(0.00)	(0.00)		(0.00)	(0.00)		(0.00)	(0.00)
σ_v		0.38	0.41		0.30	0.35		0.42	0.44		0.43	0.46
		(0.00)	(0.00)		(0.01)	(0.01)		(0.01)	(0.01)		(0.00)	(0.00)
ρ		−0.64	−0.76		−0.54	−0.73		−0.62	−0.74		−0.76	−0.82
		(0.01)	(0.01)		(0.01)	(0.01)		(0.01)	(0.01)		(0.01)	(0.01)
$\sqrt{V(t)}$ (%)	18.14	19.02	19.27	17.22	18.41	18.50	17.43	17.61	18.10	20.14	21.19	21.47
	(0.13)	(0.15)	(0.15)	(0.15)	(0.17)	(0.18)	(0.13)	(0.18)	(0.21)	(0.25)	(0.35)	(0.35)

day's pricing and hedging errors of the BS model. See Bates (1996a) for a review of alternative approaches to estimating the BS model's implied volatility.

Observe in Table 37.2 that for the overall sample period, the average implied standard-deviation is 19.27% by the SVSI model, 19.02% by the SV, 18.14% by the SI, and 18.47% by the BS model, where the difference between the highest and the lowest is only 1.13%. For each subperiod the implied-volatility estimates are similarly close across the four models. This is somewhat surprising. It should however be recognized that this comparison is based only on the average estimates over a given period. When we examined the day-to-day time-series paths of the four models' implied-volatility estimates, we found the difference between two models' implied standard-deviations to be sometimes as high as 6%. Economically, option prices and hedge ratios are generally quite sensitive to the volatility input [see Figlewski (1989)]. Even small differences in the implied-volatility estimate can lead to significantly different pricing and hedging results.

37.4.1 Static Performance

To examine out-of-sample cross-sectional pricing performance for each model, we use *previous day's* option prices to back out the required parameter values and then use them as input to compute *current day's* model-based option prices. Next, subtract the model-determined price from the observed market price, to compute both the absolute pricing error and the percentage pricing error. This procedure is repeated for each call and each day in the sample, to obtain the average absolute and the average percentage pricing errors and their associated standard errors. These steps are separately followed for each of the BS, the SI, the SV and the SVSI models. The results from this exercise are reported in Table 37.3.

Let's first examine the relative performance in pricing OTM options. Overpricing of OTM options is often considered a critical problem for the BS model [e.g., McBeth and Merville (1979) and Rubinstein (1985)]. Panel A of Table 37.3 reports the absolute and the percentage pricing error estimates for OTM options. According to both error measures, the overall ranking of the four models is consistent with our priors: the SVSI model outperforms all others, followed by the SV, the SI and finally the BS model. For extremely short-term (<30 days) and extremely out-of-the-money ($\frac{S}{K} < 0.93$) options, for example, the average absolute pricing error by the SVSI model is \$0.23 versus \$0.53 by the BS, \$0.28 by the SI, and \$0.25 by the SV model. For this category, the BS model's absolute pricing error is cut by more than a half by each of the other three models. Fix the moneyness category at $\frac{S}{K} \in (0.93, 0.95)$. Then, for medium-term (120–180 days) options, the SVSI model produces an average absolute pricing error of \$0.44 versus \$1.38 by the BS, \$0.72 by the SI, and \$0.39 by the SV model. For short-term (30–60 days) calls, the absolute pricing errors are \$0.44 by the SVSI, \$0.48 by the SV, \$0.73 by the SI, and \$0.90 by the

Table 37.3 Out-of-sample pricing errors.
Panel A: Out-of-the-money options. For a given model, compute the price of each option using previous day's implied parameters and implied stock volatility. The reported percentage pricing error is the sample average of the market price minus the model price divided by the market price. The reported absolute pricing error is the sample average of the absolute difference between the market price and the model price for each call. The corresponding standard errors are recorded in parentheses. The sample period is 06:1988–05:1991, with a total of 38,749 call option prices

Moneyness		Percentage pricing error Term-to-expiration (days)						Absolute pricing error Term-to-expiration (days)					
$\frac{S}{K}$	Model	<30	30–60	60–90	90–120	120–180	≥180	<30	30–60	60–90	90–120	120–180	≥180
<0.93	BS	−65.99	−86.80	−62.45	−57.63	−47.71	−33.72	0.53	1.00	1.14	1.50	1.96	2.36
		(12.02)	(4.51)	(2.96)	(2.92)	(1.37)	(1.05)	(0.10)	(0.04)	(0.05)	(0.06)	(0.05)	(0.06)
	SI	−24.53	−58.13	−40.04	−28.43	−16.70	−3.92	0.24	0.66	0.72	0.80	0.91	0.96
		(6.59)	(3.81)	(2.60)	(1.67)	(0.95)	(0.63)	(0.04)	(0.03)	(0.04)	(0.03)	(0.05)	(0.05)
	SV	−22.08	−30.38	−12.43	−4.02	0.89	6.08	0.25	0.44	0.34	0.33	0.43	0.62
		(6.90)	(3.07)	(1.54)	(0.90)	(0.47)	(0.39)	(0.04)	(0.03)	(0.02)	(0.02)	(0.04)	(0.05)
	SVSI	−16.29	−21.96	−5.68	−1.68	0.92	0.18	0.23	0.38	0.29	0.33	0.46	0.66
		(7.79)	(2.64)	(1.40)	(0.93)	(0.51)	(0.64)	(0.05)	(0.02)	(0.02)	(0.02)	(0.04)	(0.04)
0.93–0.95	BS	−53.68	−54.50	−33.82	−21.88	−16.43	−11.25	0.56	0.90	1.05	1.24	1.38	1.80
		(5.31)	(2.08)	(1.79)	(1.25)	(0.61)	(0.56)	(0.04)	(0.03)	(0.04)	(0.06)	(0.04)	(0.06)
	SI	−42.06	−49.30	−32.22	−15.78	−10.18	−5.91	0.42	0.77	0.92	0.83	0.85	0.98
		(5.32)	(2.18)	(2.07)	(1.07)	(0.55)	(0.43)	(0.03)	(0.02)	(0.05)	(0.05)	(0.03)	(0.05)
	SV	−25.68	−26.16	−8.83	−3.39	−0.55	1.23	0.40	0.48	0.35	0.39	0.39	0.52
		(4.61)	(1.43)	(0.81)	(0.61)	(0.30)	(0.24)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
	SVSI	−22.50	−18.85	−4.84	−2.39	0.66	0.71	0.38	0.44	0.31	0.42	0.44	0.58
		(4.53)	(1.43)	(0.85)	(0.74)	(0.32)	(0.26)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.03)
0.95–0.97	BS	−36.61	−28.83	−16.21	−9.91	−7.75	−5.77	0.55	0.81	0.87	1.03	1.05	1.44
		(2.33)	(0.93)	(0.95)	(0.84)	(0.41)	(0.45)	(0.03)	(0.02)	(0.04)	(0.05)	(0.04)	(0.06)
	SI	−35.83	−30.09	−18.97	−7.44	−5.70	−3.62	0.51	0.81	0.92	0.69	0.79	0.86
		(2.45)	(1.09)	(1.30)	(0.68)	(0.41)	(0.32)	(0.04)	(0.02)	(0.05)	(0.04)	(0.03)	(0.04)
	SV	−23.68	−16.94	−5.63	−1.63	−0.26	0.56	0.45	0.51	0.40	0.40	0.41	0.49
		(2.06)	(0.68)	(0.58)	(0.42)	(0.22)	(0.20)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
	SVSI	−16.90	−13.53	−3.59	−1.80	0.05	0.30	0.42	0.49	0.38	0.47	0.45	0.56
		(2.01)	(0.72)	(0.60)	(0.51)	(0.23)	(0.21)	(0.03)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)

BS model. Clearly, the performance improvement is significant for each moneyness and maturity category in Panel A, from the BS to the SI, to the SV, and to the SVSI model. This pricing performance ranking of the four models can also be seen using the average percentage pricing errors, as given in the same table. Here, the SVSI model produces percentage pricing errors that are the lowest in magnitude. As an example, take OTM options with term-to-expiration of 30–60 days and with $\frac{S}{K} \in (0.93, 0.95)$. In this category the BS, the SI, the SV, and the SVSI models respectively have average percentage pricing errors of −54.50%, −46.20%, −26.16%, and −18.85%. For long-term options with $\frac{S}{K} \in (0.93, 0.95)$ and with $\frac{S}{K} \in (0.95, 0.97)$, the SVSI model results in a percentage pricing error that is as low as 0.71% and 0.30%, respectively.

For ATM calls, recall that the BS model's implied-volatility input is backed out from the (previous day's) short-term ATM options, which should give the BS model a relative advantage in pricing ATM options. In contrast, the implied spot variance for the other models is obtained by minimizing the sum of squared errors for all options of the previous day. Thus, for ATM options, one would expect the BS model to perform relatively better. As seen from Panel B of Table 37.3, except for the shortest-term ATM calls, the SVSI model typically generates the lowest absolute and percentage pricing errors (especially for longer-term options), followed by the SV, by the SI and finally by the BS model. For the shortest-term options with $\frac{S}{K} \in (0.97, 0.99)$ and $\frac{S}{K} \in (0.99, 1.01)$, the BS and the SI models perform somewhat better than the other two.

Panel C of Table 37.3 reports the average absolute and percentage pricing errors of ITM calls by all four models. While the previous ranking of the models based on OTM and ATM options is preserved by Panel C, it can be noted that the average percentage pricing error is below 1.0% for 12 out of the 18 categories in the case of the SVSI model, for 8 out of the 18 categories in the case of the SV model, for 3 categories out of 18 for the SI model, and for none of the 18 categories

Table 37.3 Out-of-sample pricing errors.
Panel B: At the money options

Moneyness $\frac{S}{K}$	Model	Percentage pricing error, Term-to-expiration (days) <30	30–60	60–90	90–120	120–180	≥180	Absolute pricing error, Term-to-expiration (days) <30	30–60	60–90	90–120	120–180	≥180
0.97–0.99	BS	−19.94	−10.16	−4.83	−2.66	−2.21	−1.42	0.51	0.66	0.63	0.77	0.82	1.18
		(1.03)	(0.49)	(0.58)	(0.56)	(0.30)	(0.35)	(0.02)	(0.02)	(0.03)	(0.05)	(0.03)	(0.05)
	SI	−22.85	−11.38	−7.76	−1.69	−2.29	−1.76	0.53	0.68	0.71	0.58	0.67	0.74
		(1.16)	(0.59)	(0.70)	(0.46)	(0.30)	(0.23)	(0.02)	(0.02)	(0.04)	(0.03)	(0.03)	(0.03)
	SV	−18.93	−8.37	−2.76	−0.44	−0.16	−0.04	0.50	0.52	0.37	0.37	0.40	0.48
		(1.01)	(0.40)	(0.37)	(0.28)	(0.17)	(0.15)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
	SVSI	−17.10	−7.74	−2.36	−1.04	−0.42	−0.29	0.49	0.53	0.38	0.46	0.44	0.52
		(1.05)	(0.45)	(0.39)	(0.37)	(0.17)	(0.15)	(0.02)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
0.99–1.01	BS	−4.42	−1.29	1.14	1.44	1.06	0.86	0.44	0.57	0.54	0.74	0.77	0.97
		(0.47)	(0.30)	(0.39)	(0.45)	(0.24)	(0.26)	(0.01)	(0.02)	(0.03)	(0.04)	(0.03)	(0.04)
	SI	−5.97	−2.28	−1.08	1.81	0.29	−0.63	0.50	0.63	0.59	0.68	0.70	0.75
		(0.57)	(0.38)	(0.48)	(0.39)	(0.25)	(0.22)	(0.02)	(0.02)	(0.04)	(0.05)	(0.03)	(0.04)
	SV	−7.93	−3.72	−0.76	0.51	0.13	0.05	0.50	0.50	0.38	0.43	0.42	0.51
		(0.47)	(0.26)	(0.27)	(0.25)	(0.14)	(0.13)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
	SVSI	−8.22	−3.93	−0.74	−0.40	−0.31	−0.25	0.51	0.55	0.42	0.51	0.46	0.59
		(0.49)	(0.29)	(0.30)	(0.31)	(0.15)	(0.15)	(0.02)	(0.02)	(0.03)	(0.03)	(0.02)	(0.03)
1.01–1.03	BS	2.43	3.01	4.15	3.76	3.36	2.47	0.46	0.63	0.77	0.96	1.01	1.42
		(0.23)	(0.19)	(0.29)	(0.38)	(0.19)	(0.28)	(0.01)	(0.02)	(0.04)	(0.05)	(0.03)	(0.05)
	SI	1.65	2.29	2.08	3.52	1.76	−0.34	0.47	0.65	0.70	0.88	0.76	0.81
		(0.27)	(0.23)	(0.38)	(0.30)	(0.19)	(0.20)	(0.02)	(0.02)	(0.04)	(0.05)	(0.03)	(0.04)
	SV	−0.99	−0.69	0.46	1.04	0.38	−0.21	0.39	0.42	0.37	0.44	0.42	0.49
		(0.23)	(0.17)	(0.22)	(0.20)	(0.11)	(0.11)	(0.01)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
	SVSI	−1.49	−1.16	0.12	0.27	−0.19	−0.55	0.41	0.46	0.39	0.48	0.46	0.54
		(0.23)	(0.18)	(0.24)	(0.24)	(0.12)	(0.11)	(0.02)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)

in the case of the BS model. The pricing improvement by the SV and the SVSI models over the BS and the SI is quite substantial for ITM options, especially for long-term options.

Some patterns of mispricing can, however, be noted across all moneyness-maturity categories. First, all four models produce negative percentage pricing errors for options with moneyness $\frac{S}{K} \leq 0.99$, and positive percentage pricing errors for options with $\frac{S}{K} \geq 1.03$, subject to their time-to-expiration not exceeding 120 days. This means that the models systematically overprice OTM call options while underprice ITM calls. But the magnitude of such mispricing varies dramatically across the models, with the BS producing the strongest and the SVSI model the weakest systematic biases. Next, according to the absolute pricing error measure, the SV model seems to perform slightly better than the SVSI in pricing calls with more than 90 days to expiration. This pattern is, however, not supported by the percentage pricing errors reported in Table 37.3, possibly because for these relatively long-term calls the two models produce pricing errors that have mixed signs, in which case taking the average absolute value of the pricing errors can sometimes distort the picture. According to the percentage pricing errors, the SVSI model does slightly better than the SV in pricing those longer-term options. Finally, for the BS model, its absolute pricing error has a U-shaped relationship (i.e., "smile") with moneyness, and the magnitude of its percentage pricing error increases as the call goes from deep in the money to deep out of the money, regardless of time to expiration. These patterns are reduced by each relaxation of the BS model assumptions.

37.4.2 Dynamic Hedging Performance

Recall that in implementing a hedge using any of the four models, we follow three basic steps. First, based on Procedure B of Sect. 37.2.3, estimate the structural parameters and spot variance by using day 1's option prices. Next, on day 2, use previous day's parameter and spot volatility estimates and current day's spot price and interest rates, to construct the desired hedge as given in Sect. 37.2.2. Finally, rely on either equation (37.22) or equation (37.25) to calculate the

Table 37.3 Out-of-sample pricing errors.
Panel C: In the money options

Moneyness		Percentage pricing error Term-to-expiration (days)						Absolute pricing error Term-to-expiration (days)					
$\frac{S}{K}$	Model	<30	30–60	60–90	90–120	120–180	≥180	<30	30–60	60–90	90–120	120–180	≥180
1.03–1.05	BS	3.69	4.45	5.31	4.76	4.38	2.98	0.59	0.85	1.14	1.20	1.29	1.40
		(0.13)	(0.14)	(0.24)	(0.29)	(0.17)	(0.24)	(0.02)	(0.03)	(0.05)	(0.07)	(0.04)	(0.06)
	SI	3.37	3.83	3.64	4.08	2.51	0.17	0.57	0.82	0.92	1.04	0.90	0.83
		(0.15)	(0.18)	(0.28)	(0.25)	(0.17)	(0.19)	(0.02)	(0.03)	(0.05)	(0.06)	(0.04)	(0.04)
	SV	1.27	0.79	1.09	1.11	0.43	−0.20	0.38	0.42	0.42	0.43	0.42	0.50
		(0.13)	(0.13)	(0.17)	(0.15)	(0.10)	(0.11)	(0.01)	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)
	SVSI	0.84	0.32	0.83	0.29	−0.03	−0.41	0.37	0.45	0.42	0.49	0.45	0.54
		(0.13)	(0.14)	(0.19)	(0.20)	(0.10)	(0.11)	(0.01)	(0.02)	(0.03)	(0.03)	(0.02)	(0.03)
1.05–1.07	BS	3.37	4.54	5.57	5.08	4.82	4.27	0.70	1.06	1.46	1.47	1.56	1.73
		(0.10)	(0.11)	(0.22)	(0.27)	(0.14)	(0.22)	(0.02)	(0.02)	(0.05)	(0.06)	(0.04)	(0.06)
	SI	3.28	4.02	4.08	4.47	2.65	0.59	0.69	0.97	1.13	1.29	0.99	0.88
		(0.12)	(0.14)	(0.24)	(0.22)	(0.15)	(0.18)	(0.02)	(0.03)	(0.06)	(0.06)	(0.04)	(0.05)
	SV	1.82	1.41	1.47	1.44	0.54	−0.40	0.45	0.46	0.53	0.50	0.45	0.57
		(0.09)	(0.09)	(0.16)	(0.14)	(0.09)	(0.11)	(0.02)	(0.02)	(0.03)	(0.03)	(0.02)	(0.03)
	SVSI	1.59	1.12	1.35	0.83	0.17	−0.52	0.42	0.46	0.52	0.51	0.44	0.58
		(0.09)	(0.10)	(0.17)	(0.17)	(0.09)	(0.11)	(0.01)	(0.02)	(0.03)	(0.04)	(0.02)	(0.03)
>1.07	BS	1.79	2.65	2.96	3.10	3.36	2.61	0.60	0.95	1.22	1.35	1.56	1.58
		(0.04)	(0.05)	(0.07)	(0.08)	(0.05)	(0.05)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.02)
	SI	1.86	2.50	2.14	2.45	1.63	−0.76	0.59	0.89	0.88	1.04	0.86	1.09
		(0.05)	(0.06)	(0.07)	(0.08)	(0.05)	(0.05)	(0.01)	(0.02)	(0.02)	(0.03)	(0.02)	(0.02)
	SV	1.36	1.33	1.06	0.92	0.45	−0.64	0.50	0.55	0.52	0.49	0.42	0.64
		(0.13)	(0.13)	(0.17)	(0.15)	(0.10)	(0.11)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
	SVSI	1.29	1.26	1.18	0.81	0.40	−0.37	0.46	0.55	0.58	0.52	0.44	0.57
		(0.03)	(0.03)	(0.04)	(0.04)	(0.03)	(0.03)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)

hedging error as of day 3. We then compute both the average absolute and the average dollar hedging errors of all call options in a given moneyness-maturity category, to gauge the relative hedging performance of each model.

It should be recognized that in both the delta-neutral and the minimum-variance hedging exercises conducted in the two subsections below, the spot S&P 500 index, rather than an S&P 500 futures contract, is used in place of the "spot asset" for the hedges devised in Sect. 37.2.2. This is done out of two considerations. First, the spot S&P 500 and the immediate-expiration-month S&P 500 futures price generally have a correlation coefficient close to one. This means that whether the spot index or the futures price is used in the hedging exercises, the qualitative as well as the quantitative conclusions are most likely the same. In other words, if it is demonstrated using the spot index that one model results in better hedging performance than another, the same hedging performance ranking of the two models will likely be achieved by using an S&P 500 futures contract. After all, our main interest here lies in the relative performance of the models. Second, when a futures contract is used in constructing a hedge, a futures pricing formula has to be adopted. That will introduce another dimension of model misspecification (due to stochastic interest rates), which can in turn produce a compounded effect on the hedging results. For these reasons, using the spot index may lead to a cleaner comparison among the four option models.

37.4.2.1 Effectiveness of Delta-Neutral Hedges

Observe that the construction and the execution of the hedging strategy in (37.22) requires, in the cases of the SV and the SVSI models, (i) the availability of prices for four time-matched target and hedging-instrumental options: $C(t, \tau; K)$, $C(t, \tau; \bar{K})$, $C(t + \Delta t, \tau - \Delta t; K)$, $C(t + \Delta t, \tau - \Delta t; \bar{K})$ and (ii) the computation of Δ_S, Δ_V and Δ_R for the target and the instrumental option. Due to this requirement, it is important to match as closely as possible the time points at which the target and the instrumental option prices were respectively taken, in order to ensure that the hedge ratios are properly determined. For this reason, we use as hedging instruments only options whose prices on both the hedge construction day and the following liquidation day were quoted no more than 15 s apart from the times when the respective prices for the target option were quoted. This requirement makes the overall sample for the hedging exercise smaller than that used for the preceding pricing exercise, but it nonetheless guarantees that the deltas for the target and instrumental options on the same day are computed based on the same spot price. The remaining sample contains 15,041 matched pairs when hedging revision occurs at 1-day intervals, and 11,704 matched pairs when rebalancing takes place at 5-day intervals. In addition, we partition the target options into three maturity classes: less than 60 days, 60–180 days, and greater than 180 days, and report hedging results accordingly.

In theory, a call option with any expiration date and any strike price can be chosen as a hedging instrument for any given target option. In practice, however, different choices can mean different hedging effectiveness, even for the same option pricing model. Out of this consideration, we employ as a hedging instrument the call option which has the same expiration date as the target option and whose strike price is the closest, but not identical, to the target option's.

Table 37.4 presents delta-neutral hedging results for the four models. Several patterns emerge from this table. First, the BS model produces the worst hedging performance by most measures, the SI shows noticeable improvement according to the average dollar hedging errors (especially in the 5-day hedging revision categories) but not so according to the average absolute hedging errors, while the SV and the SVSI models have average absolute and average dollar hedging errors that are typically one-third of the corresponding BS hedging errors, or lower. The improvement by the SV and the SVSI is thus remarkable. Second, as portfolio adjustment frequency decreases from daily to once every 5 days, hedging effectiveness deteriorates, regardless of the model used. The deterioration is especially apparent for OTM and ATM options with $\frac{S}{K} \leq 1.05$. It should however be noted with emphasis that for both the SV and the SVSI models, their hedging effectiveness is relatively stable, whether the hedges are rebalanced each day or once every 5 days. For the BS and the SI models, such a change in revision frequency can mean doubling their hedging errors. This finding is strong evidence in support of the SV and the SVSI models for hedging.

Third, the BS model-based delta-neutral hedging strategy always overhedges a target call option, as its average dollar hedging error is negative for each moneyness-maturity category and at either frequency of portfolio rebalancing. In contrast, the dollar hedging errors based on the SV and the SVSI models are more random and can take either sign. Therefore, the BS formula has a systematic hedging bias pattern, whereas the SV and the SVSI do not.

Fourth, the SVSI model is indistinguishable from the SV according to their absolute hedging errors, but is slightly better than the latter when judged using their average dollar hedging errors. Similarly, the SI model has worse hedging performance than the BS according to their absolute hedging error values, but the reverse is true according to their dollar hedging errors. This phenomenon exists possibly because with stochastic interest rates there are larger hedging errors of opposite signs, so that when added together, these errors cancel out, but the sum of their absolute values is nonetheless large.

Finally, no matter which model is used, there do not appear to be moneyness- or maturity-related bias patterns in the hedging errors. In other words, hedging errors do not seem to "smile" across exercise prices or times to expiration, as pricing errors do. This is a striking disparity between pricing and hedging results.

37.4.2.2 Effectiveness of Single-Instrument Minimum-Variance Hedges

If one is, for reasons given before, constrained to using only the underlying stock to hedge a target call option, dimensions of uncertainty that move the target option value but are uncorrelated with the underlying stock price cannot be hedged by any position in the stock and will necessarily be uncontrolled for in such a single-instrument minimum-variance hedge. Based on the sample option data, the average absolute and the average dollar hedging errors, with either a daily or a 5-day rebalancing frequency, are given in Table 37.5 for each of the four models and each of the moneyness-maturity categories. With this type of hedges, the relative performance of the models is no longer clear-cut. For OTM options with $\frac{S}{K} \leq 0.97$, the SV model has, regardless of the hedging error measure used and the hedge revision frequency adopted, the lowest hedging errors, followed by the SVSI, then by the BS, and lastly by the SI model. For ATM options, the hedging performance by the BS and the SV models is almost indistinguishable, but still better, by a small margin, than that by both the SI and the SVSI models, whereas the latter two models' performance is also indistinguishable. Finally, for ITM options, the BS model has the best hedging performance, followed by the SV, the SVSI, and then by the SI model. Having said the above, it should nonetheless be noted that for virtually all cases in Table 37.5 the hedging error differences among the BS, the SV and the SVSI models are economically insignificant because of their low magnitude. Only the SI model's performance appears to be significantly poorer than the others'.

The fact that the SI model performs worse than the BS and that the SVSI model performs slightly worse than the

Table 37.4 Delta-neutral hedging errors
Panel A: Out-of-the-money options. For each call option, calculate the hedging error, which is the difference between the market price of the call and the replicating portfolio. The average dollar hedging error and the average absolute hedging error are reported for each model. The standard errors are given in parentheses. The sample period is 06:1988–05:1991. In calculating the hedging errors generated with daily (once every 5 days) hedge rebalancing, 15,041 (11,704) observations are used

		Dollar hedging error						Absolute hedging error					
		1-day revision			5-day revision			1-day revision			5-day revision		
		Term-to-expiration (days)						Term-to-expiration (days)					
Moneyness $\frac{S}{K}$	Model	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180
<0.93	BS	NA	−0.06	−0.04	NA	−0.33	−0.21	NA	0.37	0.45	NA	0.91	0.87
			(0.03)	(0.02)		(0.09)	(0.06)		(0.01)	(0.01)		(0.06)	(0.04)
	SI		−0.08	−0.05		−0.40	−0.36		0.35	0.40		0.69	0.81
			(0.02)	(0.02)		(0.05)	(0.06)		(0.01)	(0.01)		(0.03)	(0.04)
	SV		0.02	0.01		0.03	0.03		0.15	0.14		0.17	0.21
			(0.02)	(0.02)		(0.02)	(0.02)		(0.01)	(0.01)		(0.02)	(0.02)
	SVSI		0.02	0.00		0.03	0.00		0.16	0.14		0.17	0.22
			(0.02)	(0.02)		(0.02)	(0.03)		(0.01)	(0.01)		(0.01)	(0.02)
0.93–0.95	BS	NA	−0.06	−0.01	NA	−0.24	−0.02	NA	0.32	0.46	NA	0.79	0.82
			(0.02)	(0.03)		(0.07)	(0.07)		(0.01)	(0.02)		(0.04)	(0.05)
	SI		−0.08	0.00		−0.29	−0.22		0.33	0.43		0.67	0.66
			(0.02)	(0.04)		(0.05)	(0.07)		(0.01)	(0.02)		(0.03)	(0.05)
	SV		−0.01	−0.01		−0.00	0.00		0.12	0.23		0.13	0.18
			(0.01)	(0.02)		(0.01)	(0.03)		(0.01)	(0.01)		(0.01)	(0.02)
	SVSI		−0.01	−0.01		−0.00	0.00		0.12	0.24		0.13	0.18
			(0.01)	(0.02)		(0.01)	(0.03)		(0.01)	(0.02)		(0.01)	(0.02)
0.95–0.97	BS	−0.08	−0.06	−0.01	−0.55	−0.21	−0.12	0.23	0.33	0.45	0.66	0.77	0.85
		(0.06)	(0.02)	(0.03)	(0.16)	(0.06)	(0.08)	(0.04)	(0.02)	(0.03)	(0.13)	(0.04)	(0.05)
	SI	−0.06	−0.06	−0.06	−0.22	−0.31	−0.34	0.27	0.34	0.41	0.61	0.71	0.89
		(0.02)	(0.02)	(0.04)	(0.05)	(0.05)	(0.09)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.06)
	SV	0.03	−0.01	−0.01	−0.03	−0.00	−0.02	0.10	0.13	0.19	0.10	0.14	0.26
		(0.03)	(0.01)	(0.02)	(0.03)	(0.01)	(0.03)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)
	SVSI	0.02	−0.01	−0.01	−0.02	−0.01	−0.01	0.09	0.13	0.20	0.08	0.14	0.27
		(0.03)	(0.01)	(0.02)	(0.02)	(0.01)	(0.03)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)

SV suggests that adding stochastic interest rates to the option pricing framework actually make the single-instrument hedge's performance worse. This can be explained as follows. In the setup of the present paper, interest rate shocks are assumed to be independent of shocks to the stock price and/or to the stochastic volatility. Therefore, in the single-instrument minimum-variance hedges, there is no adjustment in the optimal position in the underlying stock to be taken. The hedging results in Table 37.5 have shown that if interest rate risk is not to be controlled by any position in the hedging instrument, then it is perhaps better to design a single-instrument hedge based on an option model that assumes no interest rate risk. Assuming interest rate risk in an option pricing model and yet not controlling for this risk in a hedge can make the hedging effectiveness worse.

In the case of the SV versus the BS model, the situation is somewhat different from the above. As volatility shocks are assumed to be correlated with stock price shocks, the position to be taken in the underlying stock (i.e., the hedging instrument) needs to be adjusted relative to the BS model-determined hedge, so that this single position not only helps contain the underlying stock's price risk but also neutralize that part of volatility risk which is related to stock price fluctuations [see equation (37.23)]. Thus, by rendering it possible to use the single hedging position to control for both stock price risk and volatility risk, introducing stochastic volatility into the BS framework helps improve the single-instrument hedging performance, albeit by a small amount. Nandi (1996) uses the remaining variance of a hedged position as a hedging effectiveness measure, according to which he finds the SV model performs better than the BS model. Our single-instrument hedging results are hence consistent with his, regarding the SV versus the BS model.

Table 37.4 Delta-neutral hedging errors.
Panel B: At the money options

Moneyness $\frac{S}{K}$	Model	Dollar hedging error 1-day revision: Term-to-expiration (days) <60	60–180	≥180	Dollar hedging error 5-day revision: <60	60–180	≥180	Absolute hedging error 1-day revision: <60	60–180	≥180	Absolute hedging error 5-day revision: <60	60–180	≥180
0.97–0.99	BS	−0.01	−0.04	−0.04	−0.36	−0.11	−0.21	0.34	0.34	0.46	0.54	0.75	0.89
		(0.05)	(0.02)	(0.03)	(0.08)	(0.05)	(0.07)	(0.03)	(0.01)	(0.02)	(0.06)	(0.03)	(0.05)
	SI	0.08	−0.05	−0.07	−0.30	−0.25	−0.44	0.29	0.36	0.41	0.73	0.70	0.79
		(0.01)	(0.02)	(0.02)	(0.05)	(0.05)	(0.08)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.06)
	SV	−0.03	0.01	0.00	−0.04	−0.00	−0.01	0.12	0.13	0.17	0.14	0.14	0.23
		(0.02)	(0.01)	(0.01)	(0.03)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
	SVSI	−0.02	0.01	0.00	0.02	0.00	−0.01	0.12	0.13	0.17	0.13	0.14	0.24
		(0.02)	(0.01)	(0.01)	(0.03)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
0.99–1.01	BS	−0.10	−0.02	−0.01	−0.43	−0.08	−0.10	0.37	0.37	0.47	0.80	0.77	0.77
		(0.01)	(0.02)	(0.03)	(0.09)	(0.05)	(0.07)	(0.01)	(0.01)	(0.02)	(0.06)	(0.03)	(0.05)
	SI	−0.08	−0.05	−0.03	−0.29	−0.24	−0.18	0.36	0.37	0.42	0.81	0.67	0.61
		(0.02)	(0.02)	(0.03)	(0.05)	(0.05)	(0.08)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
	SV	0.01	−0.00	−0.01	0.02	0.01	0.03	0.14	0.13	0.17	0.15	0.15	0.25
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)
	SVSI	0.01	0.00	−0.01	0.02	−0.00	0.04	0.14	0.13	0.17	0.16	0.15	0.25
		(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.03)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)
1.01–1.03	BS	−0.09	−0.02	−0.01	−0.40	−0.11	−0.09	0.40	0.39	0.46	0.82	0.75	0.82
		(0.03)	(0.02)	(0.03)	(0.08)	(0.05)	(0.07)	(0.02)	(0.01)	(0.02)	(0.05)	(0.03)	(0.05)
	SI	−0.09	−0.05	−0.07	−0.30	−0.25	−0.27	0.38	0.36	0.43	0.75	0.65	0.71
		(0.02)	(0.02)	(0.04)	(0.05)	(0.05)	(0.09)	(0.01)	(0.01)	(0.01)	(0.03)	(0.03)	(0.06)
	SV	0.00	−0.00	0.03	0.01	−0.01	0.05	0.13	0.14	0.17	0.13	0.17	0.24
		(0.03)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SVSI	0.00	−0.00	0.03	−0.00	−0.01	0.05	0.14	0.14	0.17	0.13	0.17	0.24
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)

It is useful to recall that all four models are implemented allowing both the spot volatility and the spot interest rates to vary from day to day, which is, except in the sole case of the SVSI model, not consistent with the models' assumptions. Given this practical *ad hoc* treatment, it may not come as a surprise that when only the underlying asset is used as the hedging instrument, the four models performed virtually indifferently, with the magnitude of their hedging error differences being generally small. As easily seen, if all four models were implemented in a way consistent with the respective model setups, the single-instrument hedges based on the SVSI model would for sure perform the best.

Comparing Tables 37.4 and 37.5, one can conclude that based on a given option model, the conventional delta-neutral hedges perform far better than their single-instrument counterparts, for every moneyness-maturity category. This is not surprising as the former type of hedges involves more hedging instruments (except under the BS model).

37.4.3 Regression Analysis of Option Pricing and Hedging Errors

So far we have examined pricing and hedging performance according to option moneyness-maturity categories. The purpose was to see whether the errors have clear moneyness- and maturity-related biases. By appealing to a regression analysis, we can more rigorously study the association of the errors with factors that are either contract-specific or market condition-dependent. Fix an option pricing model, and let $\epsilon_n(t)$ denote the nth call option's percentage pricing error on day t. Then, run the regression below for the entire sample:

$$\epsilon_n(t) = \beta_0 + \beta_1 \frac{S(t)}{K_n} + \beta_2\, \tau_n + \beta_3\, SPREAD_n(t) + \beta_5\, LAGVOL(t-1) + \beta_4\, SLOPE(t) + \eta_n(t), \tag{37.29}$$

Table 37.4 Delta-neutral hedging errors.
Panel C: In the money options

		Dollar hedging error						Absolute hedging error					
		1-day revision			5-day revision			1-day revision			5-day revision		
		Term-to-expiration (days)						Term-to-expiration (days)					
Moneyness $\frac{S}{K}$	Model	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180
1.03–1.05	BS	−0.06	−0.03	−0.05	−0.36	−0.09	−0.23	0.40	0.38	0.47	0.70	0.69	0.90
		(0.02)	(0.02)	(0.03)	(0.05)	(0.04)	(0.08)	(0.02)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
	SI	−0.09	−0.05	−0.07	−0.31	−0.19	−0.35	0.39	0.36	0.41	0.65	0.61	0.81
		(0.02)	(0.02)	(0.04)	(0.06)	(0.05)	(0.09)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.06)
	SV	0.01	0.01	−0.01	0.00	0.00	−0.03	0.15	0.13	0.15	0.17	0.15	0.24
		(0.03)	(0.01)	(0.02)	(0.01)	(0.01)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SVSI	0.01	0.00	−0.00	0.00	0.00	−0.03	0.15	0.12	0.16	0.16	0.14	0.25
		(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
1.05–1.07	BS	−0.05	−0.02	−0.06	−0.35	−0.06	−0.22	0.41	0.40	0.47	0.68	0.64	0.77
		(0.02)	(0.02)	(0.04)	(0.04)	(0.04)	(0.07)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.04)
	SI	−0.07	−0.04	−0.11	−0.26	−0.12	−0.56	0.40	0.37	0.44	0.57	0.51	0.59
		(0.02)	(0.02)	(0.04)	(0.04)	(0.04)	(0.08)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.01)
	SV	−0.00	−0.00	−0.01	−0.05	−0.02	0.00	0.16	0.13	0.18	0.18	0.15	0.22
		(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SVSI	−0.00	0.00	−0.00	−0.03	−0.02	0.00	0.15	0.12	0.17	0.17	0.15	0.22
		(0.01)	(0.01)	(0.02)	(0.01)	(0.00)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
>1.07	BS	−0.04	−0.03	−0.02	−0.15	−0.07	−0.10	0.36	0.39	0.48	0.51	0.58	0.72
		(0.01)	(0.00)	(0.01)	(0.02)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SI	−0.05	−0.04	−0.03	−0.18	0.08	−0.21	0.35	0.37	0.43	0.45	0.51	0.66
		(0.03)	(0.01)	(0.02)	(0.02)	(0.02)	(0.08)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SV	−0.01	−0.01	−0.00	−0.03	−0.01	0.01	0.15	0.14	0.20	0.17	0.18	0.27
		(0.01)	(0.00)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)	(0.01)	(0.01)	(0.00)	(0.00)	(0.00)
	SVSI	−0.00	−0.00	0.00	−0.01	0.00	0.01	0.15	0.14	0.20	0.16	0.17	0.27
		(0.00)	(0.00)	(0.01)	(0.01)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)

where K_n is the strike price of the call, τ_n the remaining time to expiration, and $SPREAD_n(t)$ the percentage bid-ask spread at date t of the call (constructed by computing $\frac{Ask-Bid}{0.5(Ask+Bid)}$), all of which are contract-specific variables. The variable, $LAGVOL(t-1)$, is the (annualized) standard deviation of the previous day's intraday S&P 500 returns computed over 5-min intervals, and it is included in the regression to see whether the previous day's volatility of the underlying may cause systematic pricing biases. The variable, $SLOPE(t)$, represents the yield differential between 1-year and 30-day Treasury bills. This variable can provide information on whether the single-factor Cox-Ingersoll-Ross (1985) term structure model assumed in the present paper is sufficient to make the resulting option formula capture all term structure-related effects on the S&P 500 index options. In some sense, the contract-specific variables help detect the existence of cross-sectional pricing biases, whereas $LAGVOL(t-1)$ and $SLOPE(t)$ serve to indicate whether the pricing errors over time are related to the dynamically changing market conditions. Similar regression analyses have been done for the BS pricing errors in, for example, Galai (1983b), George and Longstaff (1993), and Madan et al. (1998). For each given option model, the same regression as in (37.29) is also run for the conventional delta-neutral hedging errors, with $\epsilon_n(t)$ in (37.29) replaced by the dollar hedging error for the nth option on day t.

Table 37.6 reports the regression results based on the entire sample period, where the standard error for each coefficient estimate is adjusted according to the White (1980) heteroskedasticity-consistent estimator and is given in the parentheses. Let's first examine the pricing error regressions. For every option model, each independent variable has statistically significant explanatory power of the remaining pricing errors. That is, the pricing errors from each model have some moneyness, maturity, intraday volatility, bid-ask spread, and term structure related biases. The magnitude of each such bias, however, decreases from the BS to the SI, to the SV, and to the SVSI model. For instance, the BS percentage pricing errors will on the average be 2.29 points higher when the yield spread $SLOPE(t)$ increases by one point, whereas

Table 37.5 Single instrument hedging errors.
Panel A: Out-of-the-money options. For each call option, calculate the hedging error, which is the difference between the market price of the call and the replicating portfolio. The average dollar hedging error and the average absolute hedging error are reported for each model. The standard errors are shown in parentheses. The sample period is 06:1988–05:1991. In calculating the hedging errors generated with daily (once every 5 days) hedge rebalancing, 15,041 (11,704) observations are used

		Dollar hedging error						Absolute hedging error					
		1-day revision			5-day revision			1-day revision			5-day revision		
		Term-to-expiration (days)						Term-to-expiration (days)					
Moneyness $\frac{S}{K}$	Model	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180
<0.93	BS	NA	−0.06	−0.04	NA	−0.33	−0.21	NA	0.37	0.45	NA	0.91	0.87
			(0.03)	(0.02)		(0.09)	(0.06)		(0.01)	(0.01)		(0.06)	(0.04)
	SI		−0.09	−0.06		−0.51	−0.45		0.49	0.52		1.33	0.91
			(0.05)	(0.04)		(0.17)	(0.08)		(0.03)	(0.03)		(0.10)	(0.05)
	SV		−0.02	−0.05		−0.03	−0.09		0.30	0.39		0.75	0.75
			(0.03)	(0.02)		(0.08)	(0.05)		(0.02)	(0.02)		(0.05)	(0.04)
	SVSI		0.02	−0.04		0.11	−0.13		0.35	0.43		0.76	0.82
			(0.03)	(0.03)		(0.08)	(0.05)		(0.03)	(0.02)		(0.05)	(0.04)
0.93–0.95	BS	NA	−0.06	−0.01	NA	−0.24	−0.02	NA	0.32	0.46	NA	0.79	0.82
			(0.02)	(0.03)		(0.07)	(0.07)		(0.01)	(0.02)		(0.04)	(0.05)
	SI		−0.10	−0.00		−0.36	−0.38		0.35	0.52		0.97	0.72
			(0.03)	(0.04)		(0.10)	(0.08)		(0.02)	(0.03)		(0.06)	(0.06)
	SV		−0.06	0.00		−0.14	−0.00		0.33	0.43		0.79	0.73
			(0.02)	(0.03)		(0.07)	(0.06)		(0.02)	(0.02)		(0.04)	(0.04)
	SVSI		−0.04	−0.00		−0.17	−0.17		0.36	0.47		0.82	0.78
			(0.03)	(0.03)		(0.07)	(0.07)		(0.02)	(0.02)		(0.04)	(0.04)
0.95–0.97	BS	−0.08	−0.06	−0.01	−0.55	−0.21	−0.12	0.23	0.33	0.45	0.66	0.77	0.85
		(0.06)	(0.02)	(0.03)	(0.16)	(0.06)	(0.08)	(0.04)	(0.02)	(0.03)	(0.13)	(0.04)	(0.05)
	SI	−0.04	−0.09	0.04	−0.44	−0.38	−0.42	0.33	0.39	0.47	0.74	0.94	0.92
		(0.11)	(0.03)	(0.05)	(0.05)	(0.09)	(0.10)	(0.01)	(0.01)	(0.03)	(0.09)	(0.06)	(0.07)
	SV	−0.03	−0.05	−0.01	−0.06	−0.14	−0.16	0.21	0.32	0.42	0.48	0.77	0.74
		(0.06)	(0.02)	(0.03)	(0.16)	(0.06)	(0.07)	(0.04)	(0.02)	(0.02)	(0.09)	(0.04)	(0.04)
	SVSI	−0.04	−0.02	−0.00	−0.21	−0.14	−0.23	0.22	0.35	0.46	0.65	0.83	0.79
		(0.01)	(0.02)	(0.03)	(0.03)	(0.04)	(0.06)	(0.03)	(0.01)	(0.02)	(0.11)	(0.04)	(0.04)

the SV and the SVSI percentage errors will only be, respectively, 0.32 and 0.34 points higher in response. Thus, a higher yield spread on the term structure means higher pricing errors, regardless of the option model used. This points out that *a possible direction to further improve pricing performance is to include the yield spread as a second factor in the term structure model of interest rates*. Other noticeable patterns include the following. The BS pricing errors are decreasing, while the SI, the SV and the SVSI pricing errors are increasing, in both the option's time-to-expiration and the underlying stock's volatility on the previous day. The deeper in-the-money the call or the wider its bid-ask spread, the lower the SI's, the SV's and the SVSI model's mispricing. But, for the BS model, its mispricing increases with moneyness and decreases with bid-ask spread.

Even though all four models' pricing errors are significantly related to each independent variable, the collective explanatory power of these variables is not so impressive. The adjusted R^2 is 29% for the BS formula's pricing errors, 22% for the SI's, 12% for the SV's, and 7% for the SVSI model's. Therefore, while both the BS and the SI model have significant overall biases related to contract terms and market conditions (indicating systematic model misspecifications), the remaining pricing errors under the SV and the SVSI are not as significantly associated with these variables. About 93% of the SVSI model's pricing errors cannot be explained by these variables!

As reported in Table 37.6, delta-neutral hedging errors by the BS and the SI model tend to increase with the moneyness and the bid-ask spread of the target call, but decrease with the non-contract-specific yield spread and lagged stock volatility variables. Therefore, the two models are misspecified for hedging purposes and they lead to systematic hedging biases. But, overall, these variables can explain only 1%

Table 37.5 Single instrument hedging errors.
Panel B: At the money options

		Dollar hedging error						Absolute hedging error					
		1-day revision			5-day revision			1-day revision			5-day revision		
		Term-to-expiration (days)						Term-to-expiration (days)					
Moneyness $\frac{S}{K}$	Model	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180
0.97–0.99	BS	−0.01	−0.04	−0.04	−0.36	−0.11	−0.21	0.34	0.34	0.46	0.54	0.75	0.89
		(0.05)	(0.02)	(0.03)	(0.08)	(0.05)	(0.07)	(0.03)	(0.01)	(0.02)	(0.06)	(0.03)	(0.05)
	SI	0.00	−0.06	−0.06	−0.34	−0.11	−0.42	0.34	0.40	0.51	0.82	0.90	0.82
		(0.06)	(0.03)	(0.04)	(0.05)	(0.09)	(0.08)	(0.05)	(0.02)	(0.03)	(0.09)	(0.06)	(0.06)
	SV	−0.01	−0.05	−0.06	−0.20	−0.11	−0.19	0.35	0.35	0.44	0.56	0.81	0.84
		(0.06)	(0.02)	(0.03)	(0.12)	(0.06)	(0.07)	(0.04)	(0.01)	(0.02)	(0.07)	(0.03)	(0.04)
	SVSI	−0.03	−0.05	−0.03	−0.15	−0.16	−0.27	0.36	0.37	0.45	0.59	0.86	0.89
		(0.06)	(0.02)	(0.03)	(0.11)	(0.06)	(0.07)	(0.04)	(0.01)	(0.02)	(0.07)	(0.03)	(0.05)
0.99–1.01	BS	−0.10	−0.02	−0.01	−0.43	−0.08	−0.10	0.37	0.37	0.47	0.80	0.77	0.77
		(0.01)	(0.02)	(0.03)	(0.09)	(0.05)	(0.07)	(0.01)	(0.01)	(0.02)	(0.06)	(0.03)	(0.05)
	SI	−0.15	−0.04	0.03	−0.34	−0.18	−0.31	0.39	0.41	0.55	0.77	0.82	0.78
		(0.05)	(0.03)	(0.05)	(0.15)	(0.07)	(0.09)	(0.03)	(0.02)	(0.03)	(0.09)	(0.05)	(0.06)
	SV	−0.12	−0.02	−0.00	−0.22	−0.12	−0.06	0.38	0.37	0.45	0.78	0.79	0.69
		(0.04)	(0.02)	(0.03)	(0.11)	(0.05)	(0.06)	(0.03)	(0.01)	(0.02)	(0.06)	(0.03)	(0.04)
	SVSI	−0.04	−0.01	−0.00	−0.27	−0.13	−0.15	0.38	0.40	0.47	0.88	0.89	0.80
		(0.01)	(0.02)	(0.03)	(0.11)	(0.05)	(0.06)	(0.03)	(0.01)	(0.02)	(0.07)	(0.04)	(0.04)
1.01–1.03	BS	−0.09	−0.03	−0.01	−0.40	−0.11	−0.09	0.40	0.39	0.46	0.82	0.75	0.82
		(0.03)	(0.02)	(0.03)	(0.08)	(0.05)	(0.07)	(0.02)	(0.01)	(0.02)	(0.05)	(0.03)	(0.05)
	SI	−0.10	−0.05	−0.04	−0.43	−0.16	−0.34	0.41	0.41	0.51	0.86	0.83	0.89
		(0.04)	(0.03)	(0.05)	(0.11)	(0.07)	(0.10)	(0.02)	(0.01)	(0.03)	(0.07)	(0.05)	(0.07)
	SV	−0.06	−0.04	0.00	−0.21	−0.15	−0.03	0.38	0.39	0.43	0.84	0.81	0.72
		(0.03)	(0.02)	(0.03)	(0.09)	(0.05)	(0.06)	(0.02)	(0.02)	(0.02)	(0.05)	(0.03)	(0.04)
	SVSI	0.06	−0.04	−0.01	−0.29	−0.18	−0.17	0.41	0.42	0.45	0.91	0.85	0.76
		(0.03)	(0.02)	(0.03)	(0.08)	(0.05)	(0.06)	(0.02)	(0.01)	(0.02)	(0.05)	(0.03)	(0.04)

of the hedging errors by the two models. And, even more impressively, none of the included independent variables can explain any of the remaining hedging errors by the SV and the SVSI model, as their R^2 values are both zero.

Finally, when the dollar pricing errors are used to replace the percentage pricing errors or when the percentage hedging errors are employed to replace the dollar hedging errors in the above regressions, the sign of each resulting coefficient estimate and the magnitude of each R^2 value in Table 37.6 remain unchanged. Thus, the conclusions drawn from Table 37.6 are independent of the choice of the pricing or hedging error measure. Results from these exercises are not reported here but available upon request.

37.4.4 Robustness of Empirical Results

Using the entire sample period data, we have concluded that the evidence, based on both static performance and dynamic performance measures, is in favor of both the SVSI and the SV model. However, it is important to demonstrate that this conclusion still holds when alternative test designs and different sample periods are used. Below we briefly report results from two controlled experiments.

According to Rubinstein (1985), the volatility smile pattern and the nature of pricing biases are time period-dependent. To see whether our conclusion may be reversed, we separately examined the pricing and hedging performance of the models in three sub-periods: 06:1988–05:1989, 06:1989–05:1990, and 06:1990–05:1991. Each sub-period contains about 10,000 call option observations. As the results are similar for each subperiod, we provide the percentage pricing errors in Panel A and the absolute delta-neutral hedging errors in Panel B of Table 37.7, for the subperiod 06:1990–05:1991. It is seen that these results are qualitatively the same as those in Tables 37.3 and 37.4.

We examined the pricing and hedging error measures of each model when the structural parameters were not updated daily. Rather, retain the structural parameter values estimated

Table 37.5 Single instrument hedging errors.
Panel C: In the money options.

		Dollar hedging error						Absolute hedging error					
		1-day revision			5-day revision			1-day revision			5-day revision		
		Term-to-expiration (days)						Term-to-expiration (days)					
Moneyness $\frac{S}{K}$	Model	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180	<60	60–180	≥180
1.03–1.05	BS	−0.06	−0.03	−0.05	−0.36	−0.09	−0.23	0.40	0.38	0.47	0.70	0.69	0.90
		(0.02)	(0.02)	(0.03)	(0.05)	(0.04)	(0.08)	(0.02)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
	SI	−0.08	−0.03	−0.04	−0.47	−0.11	−0.52	0.43	0.40	0.48	0.84	0.77	0.97
		(0.03)	(0.03)	(0.05)	(0.08)	(0.06)	(0.09)	(0.02)	(0.02)	(0.03)	(0.05)	(0.04)	(0.07)
	SV	−0.04	−0.02	−0.07	−0.23	−0.09	−0.23	0.41	0.39	0.46	0.70	0.76	0.88
		(0.03)	(0.02)	(0.04)	(0.06)	(0.05)	(0.08)	(0.02)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
	SVSI	−0.05	−0.02	−0.04	−0.27	−0.14	−0.31	0.41	0.40	0.47	0.77	0.80	0.90
		(0.02)	(0.02)	(0.03)	(0.05)	(0.05)	(0.08)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
1.05–1.07	BS	−0.05	−0.02	−0.06	−0.35	−0.06	−0.22	0.41	0.40	0.47	0.68	0.64	0.77
		(0.02)	(0.02)	(0.04)	(0.04)	(0.04)	(0.07)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.04)
	SI	−0.07	−0.04	−0.07	−0.37	−0.09	−0.55	0.45	0.42	0.50	0.74	0.66	0.81
		(0.03)	(0.03)	(0.05)	(0.05)	(0.06)	(0.08)	(0.02)	(0.02)	(0.03)	(0.04)	(0.04)	(0.06)
	SV	−0.06	−0.03	−0.05	−0.32	−0.09	−0.10	0.43	0.40	0.44	0.71	0.68	0.69
		(0.02)	(0.02)	(0.04)	(0.04)	(0.04)	(0.07)	(0.02)	(0.01)	(0.03)	(0.02)	(0.03)	(0.04)
	SVSI	−0.05	−0.02	−0.02	−0.31	−0.09	−0.09	0.42	0.43	0.46	0.74	0.79	0.75
		(0.02)	(0.02)	(0.04)	(0.04)	(0.04)	(0.07)	(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.05)
>1.07	BS	−0.04	−0.03	−0.02	−0.15	−0.07	−0.10	0.36	0.39	0.48	0.51	0.58	0.72
		(0.01)	(0.00)	(0.01)	(0.02)	(0.02)	(0.03)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SI	−0.05	−0.04	−0.04	−0.17	−0.07	−0.26	0.40	0.41	0.47	0.61	0.66	0.79
		(0.01)	(0.01)	(0.02)	(0.03)	(0.03)	(0.03)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)
	SV	−0.04	−0.04	−0.02	−0.18	−0.14	−0.09	0.35	0.39	0.44	0.50	0.58	0.64
		(0.04)	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)
	SVSI	−0.04	−0.03	−0.01	−0.18	−0.14	−0.10	0.36	0.41	0.46	0.50	0.63	0.70
		(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)

Table 37.6 Regression analysis of pricing and hedging errors. The regression results below are based on the equation:
$\epsilon_n(t) = \beta_0 + \beta_1 \frac{S(t)}{K_n} + \beta_2 \tau_n + \beta_3 SPREAD_n(t) + \beta_4 SLOPE(t) + \beta_4 LAGVOL(t-1) + \eta_n(t)$,
where $\epsilon_n(t)$ denotes either the percentage pricing error or the dollar hedging error of the nth call on date-t; $\frac{S}{K_n}$ and τ_n respectively represent the moneyness and the term-to-expiration of the option contract; The variable $SPREAD_n(t)$ is the percentage bid-ask spread; $SLOPE(t)$ the yield differential between the 1-year and the 30-day Treasury bill rates; And $LAGVOL(t-1)$ the previous day's (annualized) standard deviation of S&P 500 index returns computed from 5-min intradaily returns. The standard errors, reported in parenthesis, are White's (1980) heteroskedastically consistent estimator. The sample period is 06:1988–05:1991 for a total of 38,749 observations.

	BS	SI	SV	SVSI	BS	SI	SV	SVSI
Coefficient	Percentage pricing errors				Hedging errors			
Constant	−0.05	0.28	0.24	0.11	−0.41	−0.30	0.00	−0.03
	(0.03)	(0.03)	(0.02)	(0.02)	(0.11)	(0.10)	(0.05)	(0.05)
$\frac{S}{K}$	0.22	−0.18	−0.20	−0.09	0.34	0.29	0.00	0.03
	(0.03)	(0.02)	(0.01)	(0.02)	(0.09)	(0.08)	(0.04)	(0.04)
τ	−0.04	0.04	0.08	0.05	0.03	0.08	0.00	0.00
	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.01)
SPREAD	−5.24	−4.48	−2.13	−1.57	2.26	1.04	0.32	0.34
	(0.12)	(0.11)	(0.07)	(0.08)	(0.48)	(0.32)	(0.23)	(0.23)
SLOPE	2.29	1.33	0.32	0.34	−2.09	−2.01	−0.39	−0.39
	(0.16)	(0.13)	(0.08)	(0.11)	(0.58)	(0.65)	(0.26)	(0.25)
LAGVOL	−0.16	0.12	0.06	0.04	−0.31	−0.51	−0.06	−0.05
	(0.02)	(0.02)	(0.01)	(0.01)	(0.07)	(0.05)	(0.02)	(0.02)
Adj. R^2	0.29	0.22	0.12	0.07	0.01	0.01	0.00	0.00

Table 37.7 Robustness analysis. Panel A: percentage pricing errors, 06:1990–05:1991. The reported percentage pricing error is the sample average of the market price minus the model price divided by the market price. The sample period is 06:1990–05:1991 for a total of 11,979 call options

Moneyness		Term-to-expiration (days)					
$\frac{S}{K}$	Model	<30	30–60	60–90	90–120	120–180	≥180
<0.93	BS	−76.51	−96.75	−74.73	−78.71	−61.36	−46.83
	SI	−26.88	−62.89	−38.89	−34.40	−18.68	−5.14
	SV	−25.05	−35.80	−12.59	−1.25	1.64	8.51
	SVSI	−20.56	−31.82	−8.15	−3.00	0.58	3.57
0.93–0.95	BS	−54.99	−59.06	−31.97	−24.29	−19.28	−12.36
	SI	−46.25	−46.77	−28.99	−10.62	−9.55	−5.37
	SV	−26.57	−25.32	−8.57	−1.60	−0.33	1.50
	SVSI	−28.25	−21.71	−6.26	−2.18	−0.04	0.79
0.95–0.97	BS	−34.72	−29.97	−16.71	−12.74	−10.27	−7.33
	SI	−31.85	−24.18	−16.79	−4.92	−5.35	−4.49
	SV	−20.09	−13.89	−5.54	−1.68	−0.56	0.61
	SVSI	−15.83	−13.08	−4.20	−3.29	−0.91	0.18
0.97–0.99	BS	−15.93	−10.36	−5.84	−3.22	−3.46	−2.14
	SI	−15.27	−7.45	−7.22	2.36	−2.05	−1.24
	SV	−13.09	−7.04	−3.64	0.77	−0.47	0.15
	SVSI	−12.38	−7.49	−3.37	−0.90	−0.83	−0.33
0.99–1.01	BS	−3.92	−1.23	0.62	1.54	0.65	1.99
	SI	−3.24	−0.09	−0.87	5.22	0.38	−0.09
	SV	−6.69	−3.43	−1.23	1.22	−0.10	0.14
	SVSI	−7.46	−4.17	−1.49	−0.33	−0.48	−0.27
1.01–1.03	BS	2.48	3.36	4.17	4.05	3.28	2.93
	SI	2.73	3.75	2.78	5.57	1.82	−0.60
	SV	−0.92	−0.56	0.24	1.44	0.19	−0.24
	SVSI	−1.41	−1.02	−0.09	−0.59	−0.16	−0.53
1.03–1.05	BS	3.93	4.86	5.35	5.57	4.62	3.41
	SI	3.95	4.92	4.08	6.62	2.61	−0.08
	SV	1.21	0.74	0.82	1.87	0.39	−0.17
	SVSI	0.85	0.19	0.48	0.79	0.08	−0.40
1.05–1.07	BS	3.69	5.07	5.84	6.36	5.12	4.93
	SI	3.82	5.08	4.67	6.55	2.87	1.29
	SV	1.83	1.52	1.49	2.07	0.51	−0.54
	SVSI	1.68	1.18	1.17	1.38	0.32	−0.57
>1.07	BS	1.99	2.98	3.58	4.35	4.00	3.59
	SI	2.44	2.90	2.77	4.08	1.93	−1.01
	SV	1.43	1.40	1.21	1.47	0.45	−0.69
	SVSI	1.38	1.30	1.18	1.18	0.46	−0.40

Table 37.7 Robustness analysis. Panel B: Absolute hedging errors (1 and 5 day), 06:1990–05:1991. The average absolute hedging error for each model is reported based on the sub-sample period 06:1990–05:1991 (with a total of 6,440 observations)

		1-day revision			5-day revision		
Moneyness		Term-to-expiration (days)					
$\frac{S}{K}$	Model	<60	60–180	>180	<60	60–180	>180
<0.93	BS	NA	0.42	0.48	NA	1.13	0.99
	SI		0.46	0.45		0.77	0.82
	SV		0.17	0.22		0.18	0.30
	SVSI		0.17	0.22		0.18	0.30
0.93–0.95	BS	NA	0.40	0.50	NA	0.97	0.95
	SI		0.45	0.47		0.73	0.75
	SV		0.13	0.25		0.15	0.30
	SVSI		0.13	0.25		0.15	0.30
0.95–0.97	BS	NA	0.37	0.44	NA	0.96	0.85
	SI		0.45	0.44		0.77	0.86
	SV		0.16	0.22		0.16	0.29
	SVSI		0.16	0.22		0.16	0.29
0.97–0.99	BS	0.39	0.42	0.47	0.72	0.95	0.97
	SI	0.33	0.45	0.41	0.66	0.74	0.76
	SV	0.14	0.17	0.17	0.16	0.17	0.24
	SVSI	0.14	0.17	0.16	0.15	0.17	0.23
0.99–1.01	BS	0.41	0.43	0.50	0.99	0.91	0.89
	SI	0.40	0.48	0.50	0.79	0.71	0.78
	SV	0.16	0.16	0.17	0.20	0.17	0.28
	SVSI	0.16	0.16	0.17	0.19	0.17	0.26
1.01–1.03	BS	0.40	0.46	0.47	0.99	0.89	0.83
	SI	0.45	0.44	0.45	0.74	0.71	0.73
	SV	0.17	0.17	0.18	0.19	0.20	0.25
	SVSI	0.17	0.17	0.17	0.19	0.20	0.25
1.03–1.05	BS	0.45	0.43	0.50	0.88	0.85	0.97
	SI	0.46	0.44	0.48	0.71	0.72	0.68
	SV	0.17	0.14	0.17	0.18	0.16	0.27
	SVSI	0.17	0.14	0.17	0.18	0.16	0.27
1.05–1.07	BS	0.46	0.47	0.51	0.73	0.78	0.77
	SI	0.47	0.45	0.50	0.61	0.67	0.68
	SV	0.18	0.14	0.22	0.19	0.16	0.24
	SVSI	0.17	0.14	0.21	0.19	0.16	0.22
>1.07	BS	0.41	0.45	0.53	0.62	0.70	0.81
	SI	0.38	0.46	0.50	0.48	0.64	0.75
	SV	0.17	0.15	0.22	0.18	0.19	0.32
	SVSI	0.16	0.15	0.21	0.18	0.18	0.31

from the options of the first day of each month and then, for the remainder of the month, use them as input to compute the corresponding model-based price for each traded option, except that the implied spot volatility is updated each day based on the previous day's option prices. The obtained absolute pricing errors for the subperiod 06:1990–05:1991 indicate that the performance ranking of the four models remains the same as before.

In addition, when we used only ATM (or only ITM or only OTM) option prices to back out each model's parameter values, the resulting pricing and hedging errors did not change the performance ranking of the models either. This means that even if one would estimate and use a matrix of implied volatilities (across moneynesses and maturities) to accordingly price and hedge options in different moneyness-maturity categories, it would still not change the fact that the SV and the SVSI models are better specified than the other two for pricing and hedging. Given that the implied-volatility matrix method has gained some popularity among practitioners, our results should be appealing. On the one hand, they suggest that with the SV and the SVSI models there is far less a need to engage in moneyness- and maturity-related fitting. On the other hand, if one is still interested in the matrix method, the SV and the SVSI models should be better model choices.

Early in the project we used only option transaction price data for the pricing and hedging estimations. But, that meant a far smaller data set, especially for the hedging estimations. Nonetheless, the results obtained from the transaction prices were similar to these presented and discussed in this paper.

37.5 Conclusions

We have developed and analyzed a simple option pricing model that admits both stochastic volatility and stochastic interest rates. It is shown that this closed-form pricing formula is practically implementable, leads to useful analytical hedge ratios, and contains many known option formulas as special cases. This last feature has made it relatively straightforward to conduct a comparative empirical study of the four classes of option pricing models.

According to the pricing and hedging performance measures, the SVSI and the SV models both perform much better than the SI and the BS models, as the former typically reduce the pricing and hedging errors of the latter by more than a half. These error reductions are also economically significant. Furthermore, the hedging errors by the SV and the SVSI models are relatively insensitive to the frequency of portfolio revision, whereas those of the SI and the BS models are sensitive. Given that both the SV and the SVSI models can be easily implemented on a personal computer, they should thus be better alternatives to the widely applied BS formula. A regression-based analysis of the pricing and hedging errors indicates that while the BS and the SI models show significant pricing biases related to moneyness, time-to-expiration, bid-ask spread, lagged stock volatility and interest rate term spread, pricing errors by the SV and the SVSI models are not as systematically related to either contract-specific or market-dependent variables. Overall, the results lend empirical support to the claim that incorporating stochastic interest rates and, especially, stochastic volatility, can both improve option pricing and hedging performance substantially and resolve some known empirical biases associated with the BS model.

The empirical issues and questions addressed in this paper can also be re-examined using data from individual stock options, American-style index options, options on futures, currency and commodity options, and so on. Eventually, the acceptability of option pricing models with the added features will be judged not only by its easy implementability or even its impressive pricing and hedging performance as demonstrated in this paper using European-style index calls, but also by its success or failure in pricing and hedging other types of options. These extensions are left for future research.

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Appendix 37A

Proof of the option pricing formula in (37.10). The valuation PDE in (37.9) can be re-written as:

$$\frac{1}{2}\frac{\partial^2 C}{\partial L^2} + \left(R - \frac{1}{2}V\right)\frac{\partial C}{\partial L} + \rho\sigma_v V \frac{\partial^2 C}{\partial L \partial V} + \frac{1}{2}\sigma_v^2 V \frac{\partial^2 C}{\partial V^2} + [\theta_v - \kappa_v V]\frac{\partial C}{\partial V} + \frac{1}{2}\sigma_R^2 R \frac{\partial^2 C}{\partial R^2} + [\theta_R - \kappa_R R]\frac{\partial C}{\partial R} - \frac{\partial C}{\partial \tau} - RC = 0, \tag{37.30}$$

where we have applied the transformation $L(t) \equiv \ln[S(t)]$. Inserting the conjectured solution in (37.10) into (37.30) produces the PDEs for the risk-neutralized probabilities, Π_j for $j = 1, 2$:

$$\frac{1}{2}\frac{\partial^2 \Pi_1}{\partial L^2} + \left(R + \frac{1}{2}V\right)\frac{\partial \Pi_1}{\partial L} + \rho\sigma_v V \frac{\partial^2 \Pi_1}{\partial L \partial V} + \frac{1}{2}\sigma_v^2 V \frac{\partial^2 \Pi_1}{\partial V^2} + [\theta_v - (\kappa_v - \rho\sigma_v)V]\frac{\partial \Pi_1}{\partial V} + \frac{1}{2}\sigma_R^2 R \frac{\partial^2 \Pi_1}{\partial R^2} + [\theta_R - \kappa_R R]\frac{\partial \Pi_1}{\partial R} - \frac{\partial \Pi_1}{\partial \tau} = 0, \tag{37.31}$$

and

$$\frac{1}{2}\frac{\partial^2 \Pi_2}{\partial L^2}+\left(R-\frac{1}{2}V\right)\frac{\partial \Pi_2}{\partial L}+\rho\sigma_v V\frac{\partial^2 \Pi_2}{\partial L\partial V}$$
$$+\frac{1}{2}\sigma_v^2 V\frac{\partial^2 \Pi_2}{\partial V^2}+[\theta_v-\kappa_v V]\frac{\partial \Pi_2}{\partial V}+\frac{1}{2}\sigma_R^2 R\frac{\partial^2 \Pi_2}{\partial R^2}$$
$$+\left[\theta_R-\left(\kappa_R-\frac{\sigma_R^2}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial R}\right)R\right]\frac{\partial \Pi_2}{\partial R}-\frac{\partial \Pi_2}{\partial \tau}=0. \tag{37.32}$$

Observe that (37.31) and (37.32) are the Fokker-Planck forward equations for probability functions. This implies that Π_1 and Π_2 must indeed be valid probability functions, with values bounded between 0 and 1. These PDEs must be separately solved subject to the terminal condition:

$$\Pi_j\,(t+\tau,0)=1_{L(t+\tau)\geq K}\quad j=1,2. \tag{37.33}$$

The corresponding characteristic functions for Π_1 and Π_2 will also satisfy similar PDEs:

$$\frac{1}{2}\frac{\partial^2 f_1}{\partial L^2}+\left(R+\frac{1}{2}V\right)\frac{\partial f_1}{\partial L}+\rho\sigma_v V\frac{\partial^2 f_1}{\partial L\partial V}$$
$$+\frac{1}{2}\sigma_v^2 V\frac{\partial^2 f_1}{\partial V^2}+[\theta_v-(\kappa_v-\rho\sigma_v)V]\frac{\partial f_1}{\partial V}$$
$$+\frac{1}{2}\sigma_R^2 R\frac{\partial^2 f_1}{\partial R^2}+[\theta_R-\kappa_R R]\frac{\partial f_1}{\partial R}-\frac{\partial f_1}{\partial \tau}=0, \tag{37.34}$$

and

$$\frac{1}{2}\frac{\partial^2 f_2}{\partial L^2}+\left(R-\frac{1}{2}V\right)\frac{\partial f_2}{\partial L}+\rho\sigma_v V\frac{\partial^2 f_2}{\partial L\partial V}$$
$$+\frac{1}{2}\sigma_v^2 V\frac{\partial^2 f_2}{\partial V^2}+[\theta_v-\kappa_v V]\frac{\partial f_2}{\partial V}+\frac{1}{2}\sigma_R^2 R\frac{\partial^2 f_2}{\partial R^2}$$
$$+\left[\theta_R-\left(\kappa_R-\frac{\sigma_R^2}{B(t,\tau)}\frac{\partial B(t,\tau)}{\partial R}\right)R\right]\frac{\partial f_2}{\partial R}-\frac{\partial f_2}{\partial \tau}=0, \tag{37.35}$$

with the boundary condition:

$$f_j\,(t+\tau,0;\phi)=e^{i\phi L(t+\tau)}\quad j=1,2. \tag{37.36}$$

Conjecture that the solution to the PDEs (37.34) and (37.35) is respectively given by

$$f_1\,(t,\tau,S(t),V(t),R(t);\phi)=\exp\{u_r(\tau)+u_v(\tau)$$
$$+x_r(\tau)\,R(t)+x_v(\tau)\,V(t)$$
$$+i\phi\ln[S(t)]\} \tag{37.37}$$

$$f_2\,(t,\tau,S(t),V(t),R(t);\phi)=\exp\{z_r(\tau)+z_v(\tau)$$
$$+y_r(\tau)\,R(t)+y_v(\tau)\,V(t)$$
$$+i\phi\ln[S(t)]-\ln[B(t,\tau)]\} \tag{37.38}$$

with $u_r(0)=u_v(0)=x_r(0)=x_v(0)=0$ and $z_r(0)=z_v(0)=y_r(0)=y_v(0)=0$. Solving the resulting system of differential equations and noting that $B(t+\tau,0)=1$ will respectively produce the desired characteristic functions in (37.12) and (37.13).

Both the constant interest rate–stochastic volatility and constant volatility–stochastic interest rate option pricing models are nested in (37.10). In the constant interest rate–stochastic volatility model, for instance, the partial derivatives with respect to R vanishes in (37.30). The general solution in (37.37)–(37.38) will still apply except that now $R(t)=R$ (a constant), $B(t,\tau)=e^{-R\tau}$, $x_r(\tau)=i\phi\tau$, $y_r(\tau)=(i\phi-1)\tau$, and $u_r(\tau)=z_r(\tau)=0$. The final characteristic functions $\hat{f}_j$ for the constant interest rate–stochastic volatility option model are respectively given by

$$\hat{f}_1=\exp\left\{-i\phi\ln[B(t,\tau)]-\frac{\theta_v}{\sigma_v^2}\left[2\ln\left(1-\frac{[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}{2\xi_v}\right)\right]\right.$$
$$\left.-\frac{\theta_v}{\sigma_v^2}[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v)]\,\tau+i\phi\ln[S(t)]+\frac{i\phi(i\phi+1)(1-e^{-\xi_v\tau})}{2\xi_v-[\xi_v-\kappa_v+(1+i\phi)\rho\sigma_v](1-e^{-\xi_v\tau})}V(t)\right\}, \tag{37.39}$$

and,

$$\hat{f}_2 = \exp\left\{-i\phi\ln[B(t,\tau)] - \frac{\theta_v}{\sigma_v^2}\left[2\ \ln\left(1-\frac{[\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}{2\xi_v^*}\right)\right]\right.$$
$$\left. - \frac{\theta_v}{\sigma_v^2}\left[\xi_v^* - \kappa_v + i\phi\rho\sigma_v\right]\tau + i\phi\ln[S(t)] + \frac{i\phi(i\phi-1)(1-e^{-\xi_v^*\tau})}{2\xi_v^* - [\xi_v^* - \kappa_v + i\phi\rho\sigma_v](1-e^{-\xi_v^*\tau})}V(t)\right\}, \tag{37.40}$$

Similarly, the constant volatility–stochastic interest rate option model obtains with $V(t) = V$ (a constant), $x_v(\tau) = \frac{1}{2}i\phi(1+i\phi)\tau$, $y_v(\tau) = \frac{1}{2}i\phi(i\phi-1)\tau$, and $u_v(\tau) = z_v(\tau) = 0$. The final characteristic functions $\tilde{f}_j$ for the stochastic interest rate–constant volatility model are:

$$\tilde{f}_1 = \exp\left\{\frac{1}{2}i\phi(1+i\phi)V\tau + i\phi\ln[S(t)]\right.$$
$$- \frac{\theta_R}{\sigma_R^2}\left[2\ \ln\left(1-\frac{[\xi_R - \kappa_R](1-e^{-\xi_R\tau})}{2\xi_R}\right) + [\xi_R - \kappa_R]\tau\right]$$
$$\left. + \frac{2i\phi(1-e^{-\xi_R\tau})}{2\xi_R - [\xi_R-\kappa_R](1-e^{-\xi_R\tau})}R(t)\right\}, \tag{37.41}$$

and,

$$\tilde{f}_2 = \exp\left\{\frac{1}{2}i\phi(i\phi-1)V\tau + i\phi\ln[S(t)] - \ln[B(t,\tau)]\right.$$
$$- \frac{\theta_R}{\sigma_R^2}\left[2\ \ln\left(1-\frac{[\xi_R^* - \kappa_R](1-e^{-\xi_R^*\tau})}{2\xi_R^*}\right) + [\xi_R^* - \kappa_R]\tau\right]$$
$$\left. + \frac{2(i\phi-1)(1-e^{-\xi_R^*\tau})}{2\xi_R^* - [\xi_R^*-\kappa_R](1-e^{-\xi_R^*\tau})}R(t)\right\}, \tag{37.42}$$

□

Expressions for the gamma measures. The various second-order partial derivatives of the call price in (37.10), which are commonly referred to as Gamma measures, are given below:

$$\Gamma_S \equiv \frac{\partial^2 C(t,\tau)}{\partial S^2} = \frac{\partial \Pi_1}{\partial S}$$
$$= \frac{1}{\pi}\int_0^\infty Re\left[(i\phi)^{-1}e^{-i\ \phi\ \ln[K]}\ f_1\frac{i\phi}{S}\right]d\phi. > 0. \tag{37.43}$$

$$\Gamma_V \equiv \frac{\partial^2 C(t,\tau)}{\partial V^2} = S(t)\frac{\partial^2 \Pi_1}{\partial V^2} - KB(t,\tau)\frac{\partial^2 \Pi_2}{\partial V^2} \tag{37.44}$$

$$\Gamma_R \equiv \frac{\partial^2 C(t,\tau)}{\partial R^2} = S(t)\frac{\partial^2 \Pi_1}{\partial R^2}$$
$$-KB(t,\tau)\left\{\frac{\partial^2 \Pi_2}{\partial R^2} - 2\varrho(\tau)\frac{\partial \Pi_2}{\partial R} + \varrho^2(\tau)\Pi_2\right\}. \tag{37.45}$$

$$\Gamma_{S,V} \equiv \frac{\partial^2 C(t,\tau)}{\partial S\partial V} = \frac{\partial \Pi_1}{\partial V}$$
$$= \frac{1}{\pi}\int_0^\infty Re\left[(i\phi)^{-1}e^{-i\ \phi\ \ln[K]}\ \frac{\partial f_1}{\partial V}\right]d\phi. \tag{37.46}$$

where for $g = V, R$ and $j = 1, 2$

$$\frac{\partial^2 \Pi_j}{\partial g^2} = \frac{1}{\pi}\int_0^\infty Re\left[(i\phi)^{-1}e^{-i\ \phi\ \ln[K]}\ \frac{\partial^2 f_j}{\partial g^2}\right]d\phi. \tag{37.47}$$

□