Chapter 7
Measurable Functions on Monotone Measure Spaces

7.1 Measurable Functions

In this chapter, let $(X, \mathbf{F})$ be a measurable space, μ: $\mathbf{F} \to [0, \infty]$ be a monotone measure (or continuous, semicontinuous monotone measure), and $\mathbf{B}$ be the Borel field on $(-\infty, \infty)$.

Definition 7.1. A real-valued function f: $X \to (-\infty, \infty)$ on X is $\mathbf{F}$-*measurable* (or simply "*measurable*" when there is no confusion) iff

$$f^{-1}(B) = \{x|f(x) \in B\} \in \mathbf{F}$$

for any Borel set $B \in$ B. The set of all $\mathbf{F}$-measurable functions is denoted by $\mathbf{G}$.

Theorem 7.1. *If* $f: X \to (-\infty, \infty)$ *is a real-valued function, then the following statements are equivalent:*

(1) f is measurable;
(2) $\{x|f(x) \geq \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$;
(3) $\{x|f(x) > \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$;
(4) $\{x|f(x) \leq \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$;
(5) $\{x|f(x) < \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$.

Proof. (1) ⇒ (2): $\{x|f(x) \geq \alpha\} = f^{-1}([\alpha, \infty))$, and $[\alpha, \infty)$ is a Borel set.
(2) ⇒ (1): If $\{x|f(x) \geq \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$, then $f^{-1}(B) \in \mathbf{F}$ for any $B \in \{[\alpha, \infty)|\alpha \in (-\infty, \infty)\}$. Denoting $\mathbf{A} = \{B|f^{-1}(B) \in \mathbf{F}\}$, $\mathbf{C} = \{[\alpha, \infty)|\alpha \in (-\infty, \infty)\}$, we have $\mathbf{A} \supset \mathbf{C}$. Given any $B \in \mathbf{A}$, it follows that $\overline{B} \in \mathbf{A}$, since $f^{-1}(\overline{B}) = \overline{f^{-1}(B)} \in \mathbf{F}$; that is, $\mathbf{A}$ is closed under the formation of complements. Similarly, given any $\{B_n\} \subset \mathbf{A}$, it follows that $\bigcup_{n=1}^{\infty} B \in \mathbf{A}$ since $f^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \mathbf{F}$, that is, $\mathbf{A}$ is closed under the formation of countable unions. Hence, $\mathbf{A}$ is a σ-algebra, and consequently, $\mathbf{A} \supset \mathbf{F}(\mathbf{C}) = \mathbf{B}$. This shows that f is a measurable function.

The proof of the rest is similar. □

Corollary 7.1. *If f is a measurable function, then* $\{x|f(x) = \alpha\} \in \mathbf{F}$ *for any* $\alpha \in (-\infty, \infty)$.

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Definition 7.2. Let R^n be the n-dimensional Euclidean space and let

$$\mathbf{S}^{(n)} = \left\{ \prod_{i-1}^{n} [a_i, b_i) | -\infty < a_i \leq b_i < \infty, i = 1, 2, \cdots, n \right\}$$

The σ-algebra $\mathbf{B}^{(n)} = \mathbf{F}(\mathbf{S}^{(n)})$ is called the *Borel field* on R^{n} and the sets in $\mathbf{B}^{(n)}$ are called (*n-dimensional*) *Borel sets. A function f*: $\mathrm{R}^n \to R$ is called an (*n-ary*) *Borel function* iff it is a measurable function on the measurable space $(R^n, \ \mathbf{B}^{(n)})$.

Theorem 7.2. *Let* $f_1,\ldots, f_n$ *be measurable functions. If* g*:* $R^n \to R$ *is a Borel function, then* $g(f_1,\ldots, f_n)$ *is a measurable function.*

Proof. For any Borel set $B \subset (-\infty, \infty)$,

$$\begin{aligned}[g(f_1, \ldots, f_n)]^{-1}(B) &= \{x | g(f_1(x), \ldots, f_n(x)) \in B\} \\ &= \{x | (f_1(x), \ldots, f_n(x)) \in g^{-1}(B)\}.\end{aligned}$$

Since, for any $E = \prod_{i=1}^{n} [a_i, b_i) \in \mathbf{S}^{(n)}$,

$$\{x | (f_1(x), \cdots, f_n(x)) \in E\} = \bigcap_{i=1}^{n} \{x | f_i(x) \in [a_i, b_i)\} \in \mathbf{F}$$

by applying the method similarly used in the proof of Theorem 7.1, we have

$$\{x | (f_1(x), \cdots, f_n(x)) \in F\} \in \ \mathbf{F}$$

for any $F \in \mathbf{B}^{(n)}$. As g is a Borel function, $g^{-1}(B) \in \mathbf{B}^{(n)}$ for any Borel set $B \subset (-\infty, \infty)$. Thus, we have

$$\{x | (f_1(x), \cdots, f_n(x)) \in g^{-1}(B)\} \in \ \mathbf{F}$$

for any Borel set B $\subset (-\infty, \infty)$. Hence, $g(f_1,\ldots, f_n)$ is measurable. □

As a special case of Theorem 7.2, if f_1 and f_2 are measurable, and $\alpha \in (-\infty, \infty)$ is a constant, then $\alpha f_1, f_1 + f_2, f_1 - f_2, |f_1|, f_1 \cdot f_2, |f_1|^{\alpha}, f_1 \vee f_2, f_1 \wedge f_2$, and the constant α, all of these are measurable (this can also be proven directly). Furthermore, we have

$$\{x | f_1(x) = f_2(x)\} = \{x | f_1(x) - f_2(x) = 0\} \in \ \mathbf{F}.$$

Theorem 7.3. *If* $\{f_n\}$ *is a sequence of measurable functions, and*

$$h(x) = \sup_n \{ f_n(x)\},$$

$$g(x) = \inf_n \{ f_n(x)\},$$

for any $x \in X$*, then h and g are measurable.*

Proof. By using Theorem 7.1, for any $\alpha \in (-\infty, \infty)$,

$$\{x|h(x) > \alpha\} = \{x|\sup_n\{f_n(x)\} > \alpha\} = \bigcup_{n=1}^{\infty}\{x|f_n(x) > \alpha\} \in \mathbf{F}$$

and

$$\{x|g(x) \geq \alpha\} = \{x|\inf_n\{f_n(x)\} \geq \alpha\} = \bigcap_{n=1}^{\infty}\{x|f_n(x) \geq \alpha\} \in \mathbf{F}\ .$$

Thus, h and g are measurable. □

Corollary 7.2. *If* $\{f_n\}$ *is a sequence of measurable functions, and*

$$\bar{f}(x) = \overline{\lim_n} f_n(x),$$

$$\underline{f}(x) = \underline{\lim_n}\, f_n(x),$$

then $\bar{f}$ *and* $\underline{f}$ *are measurable. Furthermore, if* $\lim_n f_n$ *exists, then, it is measurable as well.*

Proof. Since

$$\bar{f}(x) = \inf_m \sup_{n \geq m}\{f_n(x)\} \quad \text{and} \quad \underline{f}(x) = \sup_m \inf_{n \geq m}\{f_n(x)\},$$

the conclusions issue from the above theorem. □

In this chapter, we consider only measurable functions that are nonnegative, and symbols $f, f_1, f_2, \ldots, f_n, \ldots$ are used to indicate nonnegative measurable functions. The class of all nonnegative measurable functions is denoted by **G**. Most results hereafter can be generalized, without any essential difficulty, to the case in which the measurable functions are extended real-valued.

7.2 “Almost” and “Pseudo-Almost”

The definition of a measurable function on a continuous monotone measure (or semicontinuous monotone measure) space $(X, \mathbf{F}, \mu)$ is identical with classical measure theory, and, consequently, it does not relate to the set function μ; however, aspects of the set function must be considered when properties of measurable functions are discussed. For example, if f is a measurable function on a finite monotone measure space, what is the meaning of the statement “f is equal to zero almost everywhere”?

In probability theory, the statement “a random variable ξ is equal to 0 with probability 1” is equivalent to the statement “a random variable ξ is not equal to 0 with probability 0," because the probability measures possess additivity; that is, if p is a probability measure, then

$$p(E) + p(\overline{E}) = p(E \cup \overline{E}) = 1$$

for any event E. Since the monotone measures generally lose the additivity, the concept "almost everywhere" splits naturally into two different concepts, "almost everywhere" and "pseudo-almost everywhere" on monotone measure space, as indicated in the following definition.

Definition 7.3. Let $A \in \mathbf{F}$, and let P be a proposition with respect to points in A. If there exists $E \in \mathbf{F}$ with $\mu(E) = 0$ such that P is true on $A - E$, then we say "P is *almost everywhere* true on A." If there exists $F \in \mathbf{F}$ with $\mu(A - F) = \mu(A)$ such that P is true on $A - F$, then we say "P is *pseudo-almost everywhere* true on A."

We denote "almost everywhere" and "pseudo-almost everywhere" by "a.e." and "p.a.e.," respectively, and denote "$\{f_n\}$ converges to f a.e." (or "$\{f_n\}$ converges to f p.a.e.") by "$f_n \xrightarrow{\text{a.e.}} f$" (or "$f_n \xrightarrow{\text{p.a.e.}} f$", respectively).

Example 7.1. Let $X = \{0,1\}$, $\mathbf{F} = \mathbf{P}(X)$, and

$$\mu(E) = \begin{cases} 1 & \text{if } E \neq \emptyset \\ 0 & \text{if } E = \emptyset \end{cases}$$

for any $E \in \mathbf{F}$. If we define a measurable function sequence on $(X, \mathbf{F}, \mu)$ as follows:

$$f_n(x) = \begin{cases} 1 - 1/n & \text{if } x = 1 \\ 1/n & \text{if } x = 0, \end{cases} \qquad n = 1, 2, \ldots,$$

then both $f_n \xrightarrow{\text{p.a.e.}} 0$ and $f_n \xrightarrow{\text{p.a.e.}} 1$, but neither $f_n \xrightarrow{\text{a.e.}} 0$ nor $f_n \xrightarrow{\text{a.e.}} 1$.

Example 7.2. Let $X = \{0, 1\}$, $\mathbf{F} = \mathbf{P}(X)$, and

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E \neq X \end{cases}$$

for any $E \in \mathbf{F}$. For the measurable function sequence given in Example 7.1, we have both $f_n \xrightarrow{\text{a.e.}} 0$ and $f_n \xrightarrow{\text{a.e.}} 1$, but neither $f_n \xrightarrow{\text{p.a.e.}} 0$ nor $f_n \xrightarrow{\text{p.a.e.}} 1$.

From these examples, we observe that this case is vastly different from that in the classical measure theory: If both $f_n \xrightarrow{\text{a.e.}} f$ and $f_n \xrightarrow{\text{a.e.}} f'$, then $f = f'$ a.e. But now, in Example 7.1 and Example 7.2, the limit functions 1 and 0 are not equal everywhere, of course—they are neither equal a.e. nor equal p.a.e.

We should note that the situations of the concepts "a.e." and "p.a.e." are not quite symmetric. In fact, if a proposition P is true a.e. on $A \in \mathbf{F}$, then it is also true a.e. on any subset of A that belongs to **F**; but such a statement is not always valid when we replace “a.e.” with “p.a.e.,” as the following example shows.

Example 7.3. Let $X = \{a,b,c\}$, $\mathbf{F} = \mathbf{P}(X)$, μ be given by

$$\mu(E) = \begin{cases} |E| & \text{if } E \neq \{a,b\} \\ 3 & \text{if } E = \{a,b\} \end{cases}$$

for any $E \in \mathbf{F}$, and

$$f(x) = \begin{cases} 0 & \text{if } x \in \{a,b\} \\ 1 & \text{if } x = c. \end{cases}$$

It is easy to verify that μ is a monotone measure, and

$$\mu(\{x|\, f(x) = 0, x \in X\}) = \mu(\{a,b\}) = 3 = \mu(X).$$

So, $f = 0$ on X p.a.e. But

$$\mu(\{x|\, f(x) = 0, x \in \{a, \quad c\}\}) = \mu(\{a\}) = 1 \neq \mu(\{a, \quad c\}) = 2.$$

So, the statement “$f = 0$ on $\{a, c\}$ p.a.e.” is not true.

The other related concepts, such as “almost uniform convergence” and “convergence in measure” for measurable function sequences, split on monotone measure spaces as well.

Definition 7.4. Let $A \in \mathbf{F}, f \in \mathbf{G}$, $\{f_n\} \subset \mathbf{G}$. If there exists $\{E_k\} \subset \mathbf{F}$ with $\lim_k \mu(E_k) = 0$ such that $\{f_n\}$ converges to f on $A - E_k$ uniformly for any fixed $k = 1, 2, \ldots$, then we say that $\{f_n\}$ converges to f on A *almost uniformly* and denote it by $f_n \xrightarrow{\text{a.u.}} f$. If there exists $\{F_k\} \subset \mathbf{F}$ with $\lim_k \mu(A - F_k) = \mu(A)$ such that $\{f_n\}$ converges to f on $A - \mathrm{F}_k$ uniformly for any fixed $k = 1, 2, \ldots$, then we say that $\{f_n\}$ converges to f on A *pseudo-almost uniformly* and denote it by $f_n \xrightarrow{\text{p.a.u.}} f$.

Definition 7.5. Let $A \in \mathbf{F}, f \in \mathbf{G}$, and $\{f_\mathrm{n}\} \subset \mathbf{G}$. If

$$\lim_n \mu(\{x||f_n(x) - f(x)| \geq \varepsilon\} \cap A) = 0$$

for any given $\varepsilon > 0$, then we say that $\{f_n\}$ converges *in* μ (or, converges *in measure* if there is no confusion) to f on A, and denote it by $f_n \xrightarrow{\mu} f$ on A. If

$$\lim_n \mu(\{x||\, f_n(x) - f(x)| < \varepsilon\} \cap A) = \mu(A)$$

for any given $\varepsilon > 0$, then we say $\{f_\mathrm{n}\}$ converges *pseudo-in* μ (or, converges *pseudo-in measure*) to f on A, and denote it by $f_n \xrightarrow{\text{p.}\mu} f$ on A.

In the above three definitions, when $A = X$, we can omit “on A” from the statements.

Example 7.4. Let $X = [0, \infty)$, $\mathbf{F} = \mathbf{B}_+$, and μ be the Lebesgue measure, where $\mathbf{B}_+$ is the class of all Borel sets in $[0, \infty)$. If we take $f_n(x) = x/n$, $n = 1, 2, \ldots$, and $f(x) = 0$ for any $x \in X$, then we have

$$f_n \xrightarrow{\text{p.a.u.}} f,$$

but $\{f_n\}$ does not converge to f on X almost uniformly. Also, we have

$$f_n \xrightarrow{\text{p.}\mu} f,$$

but $\{f_n\}$ does not converge in μ to f on X.

7.3 Relation Among Convergences of Measurable Function Sequence

The new concepts introduced in Section 7.2 complicate the relation among the several convergences of the measurable function sequence on a continuous monotone measure space or a semicontinuous monotone measure space. If only three concepts (a.e. convergence, a.u. convergence, and convergence in measure) are considered in classical measure theory, we should discuss six implication relations among them. Three of these relations are described by Egoroff's theorem, Lebesgue's theorem, and Riesz's theorem. But now, in monotone measure theory, since each convergence concept splits into two, there are 30 implication relations we should discuss. Using the structural characteristics of set function, which are introduced in Chapter 6, we examine the most important relations in this section.

Theorem 7.4. *For any $A \in \mathbf{F}$ and any proposition P with respect to the points in A, P is true on A p.a.e. whenever P is true on A a.e. if and only if μ is null-additive.*

Proof. Sufficiency: Let μ be null-additive. If P is true on A a.e. then there exists $E \in \mathbf{F}$ with $\mu(E) = 0$ such that $P(x)$ is true for any $x \in A - E$. By null-additivity and Theorem 6.2 we have $\mu(A - E) = \mu(A)$. So P is true on A p.a.e.

Necessity: For any $A \in \mathbf{F}$, $E \in \mathbf{F}$ with $\mu(E) = 0$, take "$x \in A - E$" as a proposition $P(x)$. Obviously, P is true on A a.e. If it implies that P is true on A p.a.e., then there exists $F \in \mathbf{F}$ with $\mu(A - F) = \mu(A)$ such that $P(x)$ is true for any $x \in A - F$. That is, $x \in A - F$ implies $x \in A - E$ and therefore

$$A - E \supset A - F.$$

By the monotonicity of μ we have

$$\mu(A - E) = \mu(A)$$

and from Theorem 6.2 we know that μ is null-additive. □

Corollary 7.3. *Let* $A \in \mathbf{F}, f \in \mathbf{G}$, $\{f_n\} \subset \mathbf{G}$, *and* μ *be null-additive. If* $f_n \xrightarrow{\text{a.e.}} f$ *on* A, *then* $f_n \xrightarrow{\text{p.a.e.}} f$ *on* A.

Theorem 7.5. *Let* $A \in \mathbf{F}, f \in \mathbf{G}$, $\{f_n\} \subset \mathbf{G}$, *and* μ *be autocontinuous from below. If* $f_n \xrightarrow{\text{a.}\mu\text{.}} f$ *on* A, *then* $f_n \xrightarrow{\text{p.a.e.}} f$ *on* A.

Proof. If $f_n \xrightarrow{\text{a.u.}} f$ on A, then there exists $\{E_k\} \subset \mathbf{F}$ with $\lim_k \mu(E_k) = 0$ such that $\{f_n\}$ converges to f *on* $A - E_k$ uniformly for any $k = 1, 2, \ldots$. Since μ is autocontinuous from below, we have $\lim_k \mu(A - E_k) = \mu(A)$ and consequently, $f_n \xrightarrow{\text{p.a.u.}} f$ on A. □

Theorem 7.6. *For any* $A \in \mathbf{F}$, *and for any* $f \in \mathbf{G}$ and $\{f_n\} \subset \mathbf{G}$, $f_n \xrightarrow{\text{p.}\mu} f$ *on* A *whenever* $f_n \xrightarrow{\mu} f$ *on* A *if and only if* μ *is autocontinuous from below.*

Proof. Sufficiency: Let μ be autocontinuous from below. If $f_n \xrightarrow{\mu} f$ on A, then for any given $\varepsilon > 0$, we have

$$\lim_n \mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\} \cap A) = 0.$$

Since μ is autocontinuous from below, we have

$$\begin{aligned} \lim_n \mu(\{x \mid |f_n(x) - f(x)| < \varepsilon\} \cap A) &= \lim_n \mu(A - \{x \mid |f_n(x) - f(x)| \geq \varepsilon\} \cap A)) \\ &= \mu(A). \end{aligned}$$

So, $f_n \xrightarrow{\text{p.}\mu} f$ on A.

Necessity: For any $A \in \mathbf{F}$ and any $\{B_n\} \subset \mathbf{F}$ with $\lim_n \mu(B_n) = 0$, we define a measurable function sequence $\{f_n\}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \overline{B}_n \\ 1 & \text{if } x \in B_n \end{cases}$$

for any $n = 1, 2, \ldots$. It is easy to see that $f_n \xrightarrow{\mu} 0$ on A. If it implies $f_n \xrightarrow{\text{p.}\mu} 0$ on A, then for $\varepsilon = 1 > 0$, we have

$$\lim_n \mu(\{x \mid |f_n(x)| < 1\} \cap A) = \mu(A).$$

As

$$\{x \mid |f_n(x)| < 1\} \cap A = \overline{B}_n \cap A = A - B_n,$$

so

$$\lim_n \mu(A - B_n) = \mu(A).$$

This shows that μ is autocontinuous from below. □

The validity of Theorems 7.4–7.6 is independent of the continuity of μ.

The following is a generalization of Egoroff's theorem from classical measure space to monotone measure space.

Theorem 7.7. *Let μ be a continuous monotone measure, $A \in \mathbf{F}$,* and $\mu(A) < \infty$. *If $f_n \to f$ on A everywhere, then both $f_n \xrightarrow{\text{a.u.}} f$ and $f_n \xrightarrow{\text{p.a.u.}} f$ on A.*

Proof. There is no loss of generality in assuming that $A = X$ and μ is finite.

If we denote

$$E_n^m = \bigcap_{i=n}^{\infty} \{x \| f_i(x) - f(x)| < 1/m\},$$

for any $m = 1, 2, \ldots$, then $E_1{}^m \subset E_2{}^m \subset, \ldots$. The set of all those points that are such that $\{f_n(x)\}$ converges to $f(x)$ is

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_n^m.$$

If $f_n \to f$ everywhere, then $\bigcup_{n=1}^{\infty} E_n^m = X$ for any $m = 1, 2, \ldots$. That is, $E_n^m \nearrow X$ as $n \to \infty$, and, therefore, $\overline{E_n^m} \searrow \emptyset$ as $n \to \infty$, for any fixed $m = 1, 2, \ldots$. Given $\varepsilon > 0$ arbitrarily, by using the continuity from above and the finiteness of μ, there exists n_1 such that $\mu(\overline{E_{n_1}^1}) < \varepsilon/2$; for this n_1, there exists n_2 such that

$$\mu(\overline{E_{n_1}^1} \cup \overline{E_{n_2}^2}) < \varepsilon/2 + \varepsilon/2^2 = \tfrac{3}{4}\varepsilon;$$

and so on. Generally, there exists $n_1, n_2, \ldots n_k$, such that $\mu(\bigcup_{i=1}^{k} \overline{E_{n_i}^i}) < \sum_{i=1}^{k} \varepsilon/2^i = (1 - 1/2^k)\varepsilon < \varepsilon$. Hence, we obtain a number sequence $\{n_i\}$ and a set sequence $\{\overline{E_{n_i}^i}\}$. By using the continuity from below of μ, we know that

$$\mu\left(\bigcup_{i=1}^{\infty} \overline{E_{n_i}^i}\right) \leq \varepsilon.$$

Now, we just need to prove that $\{f_n\}$ converges to f on $\cap_{i=1}^{\infty} E_{n_i}^i$ uniformly. For any given $\delta > 0$, we take $i_0 > 1/\delta$. If $x \in \cap_{i=1}^{\infty} E_{n_i}^i$, then, since $x \in E_{n_{i_0}}^{i_0}$, we have

$$|f_i(x) - f(x)| < 1/i_0 < \delta$$

whenever $i \geq n_{i_0}$. Thus, we have proved that $f_n \xrightarrow{\text{a.u.}} f$.

In a similar way, we can prove that $f_n \xrightarrow{\text{p.a.u.}} f$ on A. □

The following example shows that the result in Theorem 7.7 may not be true when $\mu(A) = \infty$.

Example 7.5. Let monotone measure space $(X, \mathbf{F}, \mu)$ and functions, $f, f_1, f_2, \ldots$ be the same as in Example 7.4. We have $\mu(X) = \infty$ and $f_n \to f$ on X everywhere. However, as pointed out in Example 7.4, $\{f_n\}$ does not converge to f on X almost uniformly.

Corollary 7.4. *Let μ be a continuous monotone measure, $A \in \mathbf{F}$, $\mu(A) < \infty$, and μ be null-additive. If $f_n \xrightarrow{\text{a.e.}} f$ on A, then both $f_n \xrightarrow{\text{a.u.}} f$ and $f_n \xrightarrow{\text{p.a.u.}} f$ on A.*

The following theorem gives an inverse conclusion of Corollary 7.4.

Theorem 7.8. *Let $A \in \mathbf{F}$. If $f_n \xrightarrow{\text{a.u.}} f$ (or $f_n \xrightarrow{\text{p.a.u.}} f$) on A; then $f_n \xrightarrow{\text{a.e.}} f$ (or $f_n \xrightarrow{\text{p.a.e.}} f$, respectively) on A.*

Proof. If $f_n \xrightarrow{\text{a.u.}} f$ on A, then there exists $\{E_k\} \subset \mathbf{F}$ with $\lim_k \mu(E_k) = 0$ such that $\{f_n\}$ converges to f on $A - E_k$ (even uniformly) for any $k = 1, 2, \ldots$. Take $E = \cap_{k=1}^{\infty} E_k$. Since $E \subset E_k$ for every k, by the monotonicity of μ, we have $\mu(E) = 0$. Thus, for any $x \in A - E$, there exists some E_k such that $x \in A - E_k$, and therefore, $\{f_n(x)\}$ converges to $f(x)$. This shows $f_n \xrightarrow{\text{a.e.}} f$ on A.

The proof that $f_n \xrightarrow{\text{p.a.e.}} f$ on A is similar. □

The validity of Theorem 7.8 is also independent of the continuity of μ.

Now, we give two forms of generalization on semicontinuous monotone measure spaces for Lebesgue's theorem in classical measure theory.

Theorem 7.9. *Let $A \in \mathbf{F}$. If $f_n \xrightarrow{\text{a.e.}} f$ on A, μ is continuous from above, and $\mu(A) < \infty$, then $f_n \xrightarrow{\mu} f$ on A; if $f_n \xrightarrow{\text{p.a.e.}} f$ on A and μ is continuous from below, then $f_n \xrightarrow{\text{p.}\mu} f$ on A.*

Proof. We only prove the second conclusion; the proof of the first one is similar.

If $f_n \xrightarrow{\text{p.a.e.}} f$ on A, then there exists $B \in \mathbf{F}$ with $B \subset A$ and $\mu(B) = \mu(A)$ such that for any $x \in B$, $\lim_n f_n(x) = f(x)$. Thus, for any given $\varepsilon > 0$ and $x \in B$, there exists $N(x)$ such that

$$|f_n(x) - f(x)| < \varepsilon$$

whenever $n \geq N(x)$. If we write

$$A_k = \{x | N(x) \leq k\} \cap B,$$

then

$$A_k \nearrow \bigcup_{k=1}^{\infty} A_k = B$$

Since

$$\{x \| f_n(x) - f(x)| < \varepsilon\} \cap A \supset A_n,$$

we have

$$B \supset \{x \| f_n(x) - f(x)| < \varepsilon\} \cap A \cap B \supset A_n \cap B = A_n \nearrow B$$

and, therefore,

$$\lim_n \mu(\{x||f_n(x) - f(x)| < \varepsilon\} \cap A \cap B) = \mu(B).$$

Consequently,

$$\begin{aligned}\mu(A) &\geq \lim_n \mu(\{x||f_n(x) - f(x)| < \varepsilon\} \cap A) \\ &\geq \lim_n \mu(\{x||f_n(x) - f(x)| < \varepsilon\} \cap A \cap B) \\ &= \mu(B) \\ &= \mu(A).\end{aligned}$$

This shows that $f_n \xrightarrow{\text{p.}\mu} f$ on A. □

The next theorem gives inverse conclusions to the above theorem. These conclusions generalize Riesz's theorem.

Theorem 7.10. *Let $A \in \mathbf{F}$, μ be a lower semicontinuous monotone measure that is autocontinuous from above. If $f_n \xrightarrow{\mu} f$ on A, then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that both $\{f_{n_i}\} \xrightarrow{\text{a.e.}} f$ and $\{f_{n_i}\} \xrightarrow{\text{p.a.e.}} f$ on A.*

Proof. We may assume $A = X$ without any loss of generality. If $f_n \xrightarrow{\mu} f$, then we have

$$\lim_n \mu(\{x||f_n(x) - f(x)| \geq 1/k\}) = 0$$

for any $k = 1, 2, \ldots$. So, there exists n_k such that

$$\mu(\{x||f_{n_k}(x) - f(x)| \geq 1/k\}) < 1/k.$$

We may assume $n_{k+1} > n_k$ for any $k = 1, 2, \ldots$. If we write

$$E_k = \{x||f_{n_k}(x) - f(x)| \geq 1/k\},$$

then

$$\lim_k \mu(E_k) = 0.$$

Since μ is autocontinuous from above, by Theorem 6.10 there exists some subsequence $\{E_{k_i}\}$ of $\{E_k\}$ such that

$$\mu(\overline{\lim_i}\, \mu(E_{k_i}) = 0.$$

Now we shall prove that $f_{n_{k_i}}$ converges to f on $X - \overline{\lim_i}\ E_{k_i}$. In fact, for any $x \in X - \overline{\lim_i}\ E_{k_i}$, since $x \in \cup_{j=1}^{\infty} \cap_{i=j}^{\infty} \overline{E_{k_i}}$, there exists $j(x)$ such that $x \in \cap_{i=j(x)}^{\infty} \overline{E_{k_i}}$, namely,

$$|f_{n_{k_i}}(x) - f(x)| < 1/k_i,$$

for every $i \geq j(x)$. Thus, for any given $\varepsilon > 0$, taking i_0 such that $1/k_{n_0} < \varepsilon$, we have

$$|f_{n_{k_i}}(x) - f(x)| < \frac{1}{k_i} \leq \frac{1}{k_{i_0}} < \varepsilon$$

whenever $i \geq j(x) \vee i_0$. This shows that

$$f_{n_{k_i}} \xrightarrow{\text{a.e.}} f.$$

As μ is null-additive, by Theorem 7.4 we have

$$f_{n_{k_i}} \xrightarrow{\text{p.a.e.}} f$$

as well. □

Corollary 7.5. *Let $A \in \mathbf{F}$, μ be a continuous monotone measure that is autocontinuous from below, and $\mu(A) < \infty$. If $f_n \xrightarrow{\mu} f$ on A then there exists some subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that both $f_{n_i} \xrightarrow{\text{a.e.}} f$ and $f_{n_i} \xrightarrow{\text{p.a.e.}} f$ on A.*

Proof. Since the autocontinuity from below is equivalent to the autocontinuity from above when μ is a finite continuous monotone measure, if we regard A as X, the conclusion follows from Theorem 7.10. □

Theorem 7.11. *Let $A \in \mathbf{F}$. If $f_n \xrightarrow{\text{a.u.}} f$ (or $f_n \xrightarrow{\text{p.a.u.}} f$) on A, then $f_n \xrightarrow{\mu} f$ (or $f_n \xrightarrow{\text{p.}\mu} f$, respectively) on A.*

Proof. If $f_n \xrightarrow{\text{a.u.}} f$ on A, then for any $\varepsilon > 0$ and $\delta > 0$ there exist $E \in \mathbf{F}$ with $\mu(E) < \delta$ and n_0 such that

$$|f_n(x) - f(x)| < \varepsilon$$

whenever $x \in A - E$ and $n \geq n_0$. So we have

$$\mu(\{x \,|\, |f_n(x) - f(x)| \geq \varepsilon\} \cap A) \leq \mu(E \cap A) \leq \mu(E) < \delta$$

for any $n \geq n_0$. This shows that $f_n \xrightarrow{\mu} f$ on A.

In a similar way, we can prove that $f_n \xrightarrow{\text{p.a.u.}} f$ on A implies $f_n \xrightarrow{p.\mu} f$ on A. □

7.4 Convergences of Measurable Function Sequence on Possibility Measure Spaces

Let π be a possibility measure on a measurable space $(X, \mathbf{F})$, where $\mathbf{F} = \mathbf{P}(X)$. We call $(X, \mathbf{P}(X), \pi)$ a *possibility measure space*. Since π is a finite upper semicontinuous monotone measure that is uniformly autocontinuous, the previous discussion in Chapters 6 and 7 works for the possibility measure space, assuming that we replace $f_n \xrightarrow{\mu} f$ with $f_n \xrightarrow{\pi} f$. Furthermore, taking advantage of the maxitivity of possibility measures, we can obtain rather interesting results.

Theorem 7.12. *Let $A \subset X$. Then, $f_n \xrightarrow{\pi} f$ on A is equivalent to $f_n \xrightarrow{\text{a.u.}} f$ on A.*

Proof. There is no loss of generality in assuming that $A = X$. The fact that $f_n \xrightarrow{\text{a.u.}} f$ implies $f_n \xrightarrow{\pi} f$ is guaranteed by Theorem 7.10 since possibility measure π is continuous from below as well as autocontinuous. Hence, we only need to prove that $f_n \xrightarrow{\pi} f$ implies $f_n \xrightarrow{\text{a.u.}} f$.

If $f_n \xrightarrow{\pi} f$, then for any positive integer i we have

$$\pi(\{x \| f_n(x) - f(x)| \geq 1/i\}) \to 0$$

as $n \to \infty$. That is, for any positive integer k, there exists n_{ik} such that

$$\pi(\{x \| f_n(x) - f(x)| \geq 1/i\}) < 1/k$$

as $n \geq n_{ik}$. Taking

$$E_k = \bigcup_{i=1}^{\infty} \bigcup_{n \geq n_{ik}} \{x \| f_n(x) - f(x)| \geq 1/i\},$$

we have

$$\pi(E_k) = \sup_{i \geq 1, n \geq n_{ik}} \pi(\{x \| f_n(x) - f(x)| \geq 1/i\}) \leq 1/k.$$

Now, we show that $\{f_n\}$ converges to f uniformly on $\overline{E}_k$. For any $\varepsilon > 0$, take i such that $1/i < \varepsilon$. If $x \notin E_k$, then $x \in \{x \| f_n(x) - f(x)| < 1/i\}$ for any $n \geq n_{ik}$; that is, there exists $n_0 = n_{ik}$ such that

$$| f_n(x) - f(x)| < 1/i < \varepsilon,$$

where $n \geq n_0$. The proof is now complete. □

By using Theorem 7.8 we immediately obtain the following corollary.

Corollary 7.6. *Let $A \subset X$. Then, $f_n \xrightarrow{\pi} f$ on A implies $f_n \xrightarrow{\text{a.e.}} f$ on A.*

Since, in general, a possibility measure is not continuous from above, we cannot get the inverse proposition of Corollary 7.6 by using Theorem 7.9. This is shown by the following counterexample.

Example 7.6. Let $X = (0, 1]$ and let a possibility measure π be defined as

$$\pi(E) = \begin{cases} 1 & \text{if } E \neq \emptyset \\ 0 & \text{if } E = \emptyset. \end{cases}$$

We take

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (1/n,\quad 1] \\ 1 & \text{otherwise.} \end{cases}$$

Then, $f_n \to 0$ everywhere on X, but $\{f_n\}$ does not converge to zero in measure π. In fact, taking $\varepsilon = 1/2$, we have

$$\pi(\{x|f_n(x) > 1/2\}) = \pi((0, 1/n]) = 1$$

for any n.

Theorem 7.13. *Let $A \subset X$. Then, $f_n \xrightarrow{\text{a.u.}} f$ on A is equivalent to $|f_n - f| \wedge p \to 0$ on A uniformly, where p is the possibility profile of π.*

Proof. As in the proof of Theorem 7.12, we can assume that $A = X$.

Suppose $f_n \xrightarrow{\text{a.u.}} f$. Then, for any given $\varepsilon > 0$, there exists a set $E \subset X$ with $\pi(E) < \varepsilon$ such that $|f_n - f| \to 0$ on $\overline{E}$ uniformly; that is, there exists $n(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon$ for any $x \notin E$ whenever $n \geq n(\varepsilon)$. Since $\pi(E) < \varepsilon$ implies $p(x) < \varepsilon$ for any $x \in E$, we have

$$|f_n(x) - f(x)| \wedge p(x) < \varepsilon$$

whenever $n \geq n(\varepsilon)$. This shows that

$$|f_n - f| \wedge p \to 0$$

uniformly.

Conversely, suppose $|f_n - f| \wedge p \to 0$ uniformly. Then, for any given positive integer k there exists n_k such that

$$|f_n(x) - f(x)| \wedge p(x) < 1/k$$

for any x whenever $n > n_k$. Denoting $E_k = \cup_{n \geq n_k}\{x||f_n(x) - f(x)| \geq 1/k\}$, we have $p(x) < 1/k$ for any E_k. If we take $F_i = \cup_{k \geq i} E_k$, then

$$\pi(F_i) = \sup_{x \in F_i} p(x) \leq 1/i.$$

Now, we show that $f_n \to f$ uniformly on $\overline{F}_i$ for each $i = 1, 2, \ldots$. Given an arbitrary $\varepsilon > 0$, take $k \geq i$ such that $1/k < \varepsilon$. For any $x \notin F_i$, we have $x \notin E_k$ and, therefore, $x \notin \{x \mid |f_n(x) - f(x)| \geq 1/k\}$ whenever $n \geq n_k$. That is, $|f_n(x) - f(x)| < 1/k < \varepsilon$ whenever $n \geq n_k$. The proof is now complete. □

Summing up the results presented in this section, we can characterize the relations among several convergences of a measurable function sequence on possibility measure spaces as follows:

$$f_n \xrightarrow{\pi} f \quad \Leftrightarrow \quad f_n \xrightarrow{\text{a.u.}} f \quad \Leftrightarrow \quad |f_n - f| \wedge p \xrightarrow{\text{u.}} 0 \quad \Rightarrow f_n \xrightarrow{\text{a.e.}} f,$$

where the symbol "$\xrightarrow{\text{u.}}$" means "converge uniformly."

The concepts of pseudo-convergences of a function sequence are unimportant on the possibility measure space.

Notes

7.1. The paper by Wang [1984] contains early discussions on convergences of measurable function sequences on monotone measure (fuzzy measure) spaces. After introducing the concept of "pseudo-almost," Wang [1985a] derived more results regarding the relationship among several types of convergences of measurable function sequences on the basis of the concepts of pseudo-autocontinuity and converse autocontinuity.

7.2. Some results presented in this chapter were generalized to fuzzy σ-algebra by Qiao [1990].

7.3. The convergences of measurable function sequences on possibility measure spaces were studied by Wang [1987].

Exercises

7.1. Let $(X, \mathbf{F})$ be a measurable space and let f_1 and f_2 be measurable functions. Without using Theorem 7.2, prove that the following functions are measurable:

$$cf_1 (c \text{ is a constant}); f_1 - f_2; f_1 + f_2; f_1 \vee f_2; f_1 \wedge f_2; |f_1|; f_1^2; f_1 \cdot f_2.$$

7.2. Let f be a measurable function on $(X, \mathbf{F})$. Prove that

$$\{x \mid f(x) = \alpha\} \in \mathbf{F}$$

for any $\alpha \in (-\infty, \infty)$.

7.3. Let f be a function defined on $(X, \mathbf{F})$. If $\{x|f(x) = \alpha\} \in \mathbf{F}$ for any $\alpha \in (-\infty, \infty)$, can you correctly assert that f is measurable? If you can, give a proof; if you cannot, give an example to justify your conclusion.

7.4. Let $\{f_n\}$ be a sequence of measurable functions on $(X, \mathbf{F})$. Prove that

$$\{x|\overline{\lim_n} f_n(x) = \underline{\lim_n} f_n(x)\} \in \mathbf{F}.$$

7.5. Let $\mathbf{G}$ be the class of all nonnegative finite measurable functions on a monotone measure space $(X, \mathbf{F}, \mu)$. Both $\overset{\text{a.e.}}{=}$ (almost everywhere equality) and $\overset{\text{p.a.e.}}{=}$ (pseudo-almost everywhere equality) are binary relations on $\mathbf{G}$. Prove that these relations are reflexive and symmetric, but not transitive in general.

7.6. Prove that the relation $\overset{\text{a.e.}}{=}$ is an equivalence relation on $\mathbf{G}$ (see Exercise 7.5) if and only if μ is weakly null-additive (see Exercise 6.1).

7.7. Can you find a condition such that the statement "P is true on A p.a.e." implies the statement "P is true on A a.e."?

7.8. Construct an example of a measurable function f defined on a monotone measure space $(X, \mathbf{F}\ \mu)$ in which "$f \overset{\text{a.e.}}{=} 0$" is true, but "$f \overset{\text{p.a.e}}{=} 0$" is not true.

7.9. Construct an example of a semicontinuous fuzzy measure space $(X, \mathbf{F}, \mu)$ and a sequence of measurable functions $\{f_n\}$ such that $\{f_n\}$ converges to some measurable function f almost everywhere, but does not converge to f pseudo-almost uniformly.

7.10. Let $X = (0, 1]$, $\mathbf{F}$ be the class of all Borel sets in X, and $\mu = m^2$, where m is the Lebesgue measure. Assume we order all rational numbers in X as follows:

$$x_1 = 1, x_2 = 1/2, x_3 = 1/3, x_4 = 2/3, x_5 = 1/4, x_6 = 3/4, x_7 = 1/5, x_8 = 2/5,\ \ x_9 = 3/5, x_{10} = 4/5, x_{11} = 1/6, x_{12} = 5/6, x_{13} = 1/7, x_{14} = 2/7, \ldots.$$

Furthermore, we define a sequence of measurable functions $\{f_n\}$ on $(X, \mathbf{F}, \mu)$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x - x_n| < 1/(2n)^{1/2} \\ 0 & \text{otherwise} \end{cases}$$

for $n = 1, 2, \ldots$. Prove that:
(a) μ is autocontinuous;
(b) $f_n \xrightarrow{\mu} 0$;
(c) f_n does not converge to 0 at any point in X.
Can you find a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $\{f_{n_i}\}$ converges to zero both almost everywhere and pseudo-almost everywhere?

7.11. Prove that if μ is a finite continuous monotone measure and $f_n \to f$ everywhere, then $f_n \xrightarrow{\text{p.a.u.}} f$.

7.12. Prove that if μ is a finite and null-additive continuous monotone measure, then $f_n \xrightarrow{\text{a.e.}} f$ implies $f_n \xrightarrow{\text{p.a.u.}} f$.