Chapter 4
Special Areas of Generalized Measure Theory

4.1 An Overview

The term "generalized measure theory," as it is understood in this book, is delimited by two extremes—the classical theory of additive measures or signed additive measures at one extreme and the theory of general measures or signed general measures at the other extreme. There are of course many measure theories between these two extremes. They are based on measures that do not require additivity, but that are not fully general as well. Three major types of measures that are in this category are introduced in Chapter 3. They are monotone measures and their large subclasses: superadditive and subadditive measures. The purpose of this chapter is to further refine these large classes of measures by introducing their various subclasses. We focus on those subclasses that are well established in the literature.

In Section 4.2, we begin with an important family of measures that are referred to in the literature as *Choquet capacities of various orders*. Classes of measures captured by this family are significant as they are linearly ordered in terms of their interpretations and methodological capabilities. In some sense this family of measures is the core of generalized measure theory. Classes of measures in this family are benchmarks against which other classes of measures are compared in terms of their roles in generalized measure theory.

After introducing this important family of measures in Section 4.2, we return to classical measure theory and examine the various ways of how to generalize it. First, we introduce in Section 4.3 a simple generalization of classical measures via the so-called λ-measures. Next, we show in Section 4.4 that the class of λ-measures is a member of a broader class of measures that we call quasi-measures. Each member of this broader class of measures is connected to additive measures via a particular type of reversible transformation. After examining quasi-measures, we proceed in Section 4.5 to the strongest Choquet capacities (referred to as capacities of order ∞) and their dual measures (referred to *as alternating capacities of order* ∞). These pairs of dual measures, when normalized, form a basis for a well-developed and highly visible theory of uncertainty, which is usually referred to in the literature as the Dempster–Shafer theory. Another important and well-known theory of uncertainty, which is in

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some specific way connected with the Dempster–Shafer theory, is possibility theory. Nonadditive measures upon which possibility theory is based are introduced and examined in Section 4.6. Finally, some properties of finite monotone measures are presented in Section 4.7.

4.2 Choquet Capacities

Definition 4.1. Given a particular integer $k \geq 2$, a *Choquet capacity of order k* is a monotone measure μ on a measurable space (X, **F**) that satisfies the inequalities

$$\mu\left(\bigcup_{j=1}^{k} A_j\right) \geq \sum_{\substack{K \subseteq N_k \\ K \neq \emptyset}} (-1)^{|K|+1} \mu\left(\bigcap_{j \in K} A_j\right) \tag{4.1}$$

for all families of k sets in **F**, where $N_k = \{1, 2, \ldots, k\}$.

Since sets A_j in the inequalities (4.1) are not necessarily distinct, every Choquet capacity of order k is also of order $k' = k - 1, k - 2, \ldots, 2$. However, a capacity of order k may not be a capacity of any higher order ($k + 1$, $k + 2$, etc.). Hence, capacities of order 2, which satisfy the simple inequalities

$$\mu(A_1 \cup A_2) \geq \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2) \tag{4.2}$$

for all pairs of sets in **F**, are the most general capacities. The least general ones are those of order k for all $k \geq 2$. These are called *Choquet capacities of order* ∞ or *totally monotone measures*. They satisfy the inequalities

$$\begin{aligned} \mu(A_1 \cup A_2 \cup \cdots \cup A_k) \geq & \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + - \cdots \\ & + (-1)^{k+1} \mu(A_1 \cap A_2 \cap \cdots \cap A_k) \end{aligned} \tag{4.3}$$

for every $k \geq 2$ and every family of k sets in **F**.

It is trivial to see that the set of inequalities (4.2) contains all the inequalities required for superadditive measures in Definition 3.3 (when $A_1 \cap A_2 = \emptyset$), but contains additional inequalities (when $A_1 \cap A_2 \neq \emptyset$). Choquet capacities of order 2—the most general class of Choquet capacities—are thus a subclass of superadditive measures.

Definition 4.2. Given a particular integer $k \geq 2$, an *alternating Choquet capacity of order k* is a monotone measure μ on a measurable space (X, **F**) that satisfies for all families of k sets in **F** the inequalities

$$\mu\left(\bigcap_{j=1}^{k} A_j\right) \leq \sum_{\substack{K \subseteq N_k \\ K \neq \emptyset}} (-1)^{|K|+1} \mu\left(\bigcup_{j \in K} A_j\right). \tag{4.4}$$

It is clear that the requirements for alternating capacities of some order $k \geq 2$ are weaker than those of orders $k+1, k+2, \ldots$. *Alternating capacities of order* 2, which are required to satisfy the inequalities

$$\mu(A_1 \cap A_2) \leq \mu(A_1) + \mu(A_2) - \mu(A_1 \cup A_2) \tag{4.5}$$

for all pairs of sets in **F**, are thus the most general alternating capacities. On the other hand, *alternating Choquet capacities of order* ∞, which are defined by the inequalities

$$\begin{aligned} \mu(A_1 \cap A_2 \cap \cdots \cap A_k) \leq & \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cup A_j) + - \cdots \\ & + (-1)^{k+1} \mu(A_1 \cup A_2 \cup \cdots \cup A_k) \end{aligned} \tag{4.6}$$

for every $k \geq 2$ and every family of k sets in **F**, are the least general ones.

It is obvious that the set of inequalities (4.5) contains all the inequalities required in Definition 3.4 for subadditive measures (when $A_1 \cap A_2 = \emptyset$), but contains some additional inequalities (when $A_1 \cap A_2 \neq \emptyset$). Alternating Choquet capacities of order 2—the most general class of alternating Choquet capacities—are thus subadditive measures, but not the other way around.

Choquet capacities of order k are often referred to in the literature as *k-monotone measures* and, similarly, alternating Choquet capacities are often called *k-alternating measures*. These shorter names are adopted, by and large, in this book. For convenience, monotone measures that are not 2-monotone are often referred to as 1-*monotone measures*. Using this terminology the inclusion relationship among the introduced classes of k-monotone and k-alternating measures for $k \geq 1$is depicted in Fig. 4.1.

Theorem 4.1. *Let μ be a normalized 2-monotone measure on a measurable space* $(X, \mathbf{F})$. *Then the dual measure of μ, denoted by μ^*, is a normalized 2-alternating measure on* $(X, \mathbf{F})$.

Proof.

$$\begin{aligned} \mu^*(A_1 \cap A_2) &= 1 - \mu(\overline{A_1 \cap A_2}) \\ &= 1 - \mu(\overline{A_1} \cup \overline{A_2}) \\ &\leq 1 - \mu(\overline{A_1}) - \mu(\overline{A_2}) + \mu(\overline{A_1} \cap \overline{A_2}) \\ &= 1 - \mu(\overline{A_1}) + 1 - \mu(\overline{A_2}) - 1 + \mu(\overline{A_1} \cap \overline{A_2}) \\ &= 1 - \mu(\overline{A_1}) + 1 - \mu(\overline{A_2}) - 1 + \mu(\overline{A_1 \cup A_2}) \\ &= \mu^*(A_1) + \mu^*(A_2) - \mu^*(A_1 \cup A_2). \end{aligned}$$

□

This theorem can be easily generalized to normalized k-monotone measures for any $k \geq 2$. Observe, however, that the dual measure of a 1-monotone

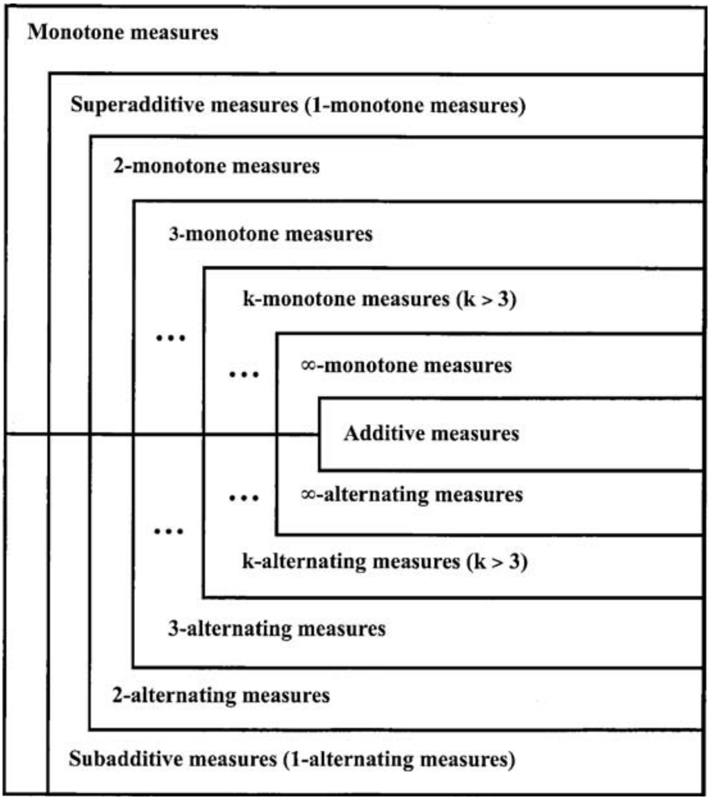

Fig. 4.1 Inclusion relationship among k-monotone and k-alternating measures for $k \geq 1$

measure that is superadditive is not necessarily subadditive, as is shown by the following counterexample.

Example 4.1. Let $X = \{a, b, c\}, \mathbf{F} = \mathbf{P}(X)$, and let μ be the 1-monotone measure on $(X, \mathbf{P}(X))$ defined in Table 4.1. This measure is clearly normalized and superadditive, but it is not 2-monotone due to the following two violations of the required inequalities (4.2):

$$\mu(X) = 1 < \mu(\{a, b\}) + \mu(\{b, c\} - \mu(\{b\}) = 1.4,$$
$$\mu(X) = 1 < \mu(\{a, c\}) + \mu(\{b, c\}) - \mu(\{c\}) = 1.1.$$

The dual measure of μ, denoted in Table 4.1 by μ^*, is not subadditive due to the following violation of the inequalities required for subadditive measures in Definition 3.4:

$$\mu(\{a, b\}) = 0.8 > \mu(\{a\}) + \mu(\{b\}) = 0.7,$$
$$\mu(\{a, c\}) = 1 > \mu(\{a\}) + \mu(\{c\}) = 0.6.$$

Table 4.1 Superadditive measure μ and its dual measure μ^* (Example 4.1)

	a b c	$\mu(A)$	$\mu^*(A)$
A:	0 0 0	0.0	0.0
	1 0 0	0.1	0.2
	0 1 0	0.0	0.5
	0 0 1	0.2	0.4
	1 1 0	0.6	0.8
	1 0 1	0.5	1.0
	0 1 1	0.8	0.9
	1 1 1	1.0	1.0

The whole family of k-monotone and k-alternating classes of measures plays an important role in generalized measure theory and, particularly, in its applications dealing with various types of uncertainty. Especially important are the classes of 2-monotone and 2-alternating measures, which are the most general classes in this family, and the classes of ∞-monotone and ∞-alternating measures. They represent important benchmarks from mathematical and computational points of view. These issues are discussed later in the book in various contexts.

Thus far, we have followed a top-down approach: we started by defining general measures and we proceeded to defining monotone measures, superadditive and subadditive measures, and, finally, k-monotone and k-alternating measures. In the rest of this chapter we switch to the complementary, bottom-up approach: we start with examining in detail some of the simplest generalizations of classical measures and we proceed then by enlarging the framework to discuss the various higher-level generalizations.

4.3 λ-Measures

Definition 4.3. A monotone measure μ satisfies the *λ-rule* (on **C**) iff there exists

$$\lambda \in \left(-\frac{1}{\sup \mu}, \infty\right) \cup \{0\},$$

where $\sup \mu = \sup_{E \in \mathbf{C}} \mu(E)$, such that

$$\mu(E \cup F) = \mu(E) + \mu(F) + \lambda \cdot \mu(E) \cdot \mu(F),$$

whenever

$$E \in \mathbf{C}, F \in \mathbf{C},\ \ E \cup F \in \mathbf{C}, \text{and}\ \ E \cap F = \emptyset.$$

μ satisfies the *finite λ-rule* (on **C**) iff there exists the above-mentioned λ such that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \begin{cases} \frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1+\lambda\cdot\mu(E_i)] - 1\right\}, & \text{as } \lambda \neq 0 \\ \sum_{i=1}^{n}\mu(E_i), & \text{as } \lambda = 0 \end{cases}$$

for any finite disjoint class $\{E_1, \ldots, E_n\}$ of sets in **C** whose union is also in **C**; μ satisfies the *σ-λ-rule* (on **C**) iff there exists the above-mentioned λ, such that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \begin{cases} \frac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1+\lambda\cdot\mu(E_i)] - 1\right\}, & \text{as } \lambda \neq 0, \\ \sum_{i=1}^{\infty}\mu(E_i), & \text{as } \lambda = 0, \end{cases}$$

for any disjoint sequence $\{E_n\}$ of sets in **C** whose union is also in **C**.

When $\lambda = 0$, the λ-rule, the finite λ-rule, or the σ-λ-rule is just the additivity, the finite additivity, or the σ-additivity, respectively.

Theorem 4.2. *If* **C** = **R** *is a ring and* μ *satisfies the* λ*-rule, then* μ *satisfies the finite* λ*-rule.*

Proof. The conclusion is obvious when $\lambda = 0$. Let $\lambda \neq 0$ and $\{E_1, \ldots, E_n\}$ be a disjoint class of sets in **R**. We use the mathematical induction to prove

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1+\lambda\cdot\mu(E_i)] - 1\right\}. \tag{4.7}$$

From the definition we know directly that (4.7) is true when $n = 2$. Now, suppose that (4.7) is true for $n = k-1$. We have

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{k} E_i\right) &= \mu\left(\left(\bigcup_{i=1}^{k-1} E_i\right) \cup E_k\right) \\
&= \mu\left(\bigcup_{i=1}^{k-1} E_i\right)[1+\lambda\cdot\mu(E_k)] + \mu(E_k) \\
&= \frac{1}{\lambda}\left\{\prod_{i=1}^{k-1}[1+\lambda\cdot\mu(E_i)] - 1\right\}\cdot[1+\lambda\cdot\mu(E_k)] + \mu(E_k) \\
&= \frac{1}{\lambda}\left\{\prod_{i=1}^{k}[1+\lambda\cdot\mu(E_i)] - [1+\lambda\cdot\mu(E_k)]\right\} + \mu(E_k) \\
&= \frac{1}{\lambda}\left\{\prod_{i=1}^{k}[1+\lambda\cdot\mu(E_i)] - [1+\lambda\cdot\mu(E_k)] + \lambda\cdot\mu(E_k)\right\} \\
&= \frac{1}{\lambda}\left\{\prod_{i=1}^{k}[1+\lambda\cdot\mu(E_i)] - 1\right\}.
\end{aligned}$$

That is, (4.7) is true for $n = k$. The proof is complete. □

In fact, Theorem 4.2 holds also when **C** is only a semiring. This is shown in Section 4.4, after introducing a new concept called *quasi-additivity*.

Example 4.2. Let $X = \{a, b\}$ and $\mathbf{C} = \mathbf{P}(X)$. If

$$\mu(E) = \begin{cases} 0 & E = \emptyset \\ 0.2 & E = \{a\} \\ 0.4 & E = \{b\} \\ 1 & E = X, \end{cases}$$

then μ satisfies the λ-rule, where $\lambda = 5$. Since **C** is a finite ring, μ satisfies the finite λ-rule and also the σ-λ-rule.

Definition 4.4. μ is called a *λ-measure* on **C** iff it satisfies the σ-λ-rule on **C** and there exists at least one set $E \in C$ such that $\mu(E) < \infty$.

Usually the λ-measure is denoted by g_λ. When **C** is a σ-algebra and $g_\lambda(X) = 1$, the λ-measure g_λ is also called a *Sugeno measure*. The set function given in Example 4.2 is a Sugeno measure.

Example 4.3. Let $X = \{x_1, x_2, \ldots\}$ be a countable set, **C** be the semiring consisting of all singletons of X and the empty set $\emptyset$, and $\{a_i\}$ be a sequence of nonnegative real numbers. Define $\mu(\{x_i\}) = a_i, i = 1, 2, \ldots,$ and $\mu(\emptyset) = 0$. Then μ is a λ-measure for any $\lambda \in (-1/\sup \mu, \infty) \cup \{0\}$, where $\sup \mu = \sup(\{a_i | i = 1, 2, \ldots\})$.

Theorem 4.3. *If g_λ is a λ-measure on a class* **C** *containing the empty set $\emptyset$, then $g_\lambda(\emptyset) = 0$, and g_λ satisfies the finite λ-rule.*

Proof. From Definition 4.4, there exists $E \in \mathbf{C}$ such that $g_\lambda(E) < \infty$. When $\lambda = 0, g_\lambda$ is a classical measure and therefore $g_\lambda(\emptyset) = 0$. Otherwise, $\lambda \neq 0$. Since $\{E, E_2, E_3, \ldots\}$, where $E_2 = E_3 = \cdots = \emptyset$ is a disjoint sequence of sets in **C** whose union is E, we have

$$g_\lambda(E) = \frac{1}{\lambda}\left\{\prod_{i=2}^{\infty}[1 + \lambda \cdot g_\lambda(E_i)] \cdot [1 + \lambda \cdot g_\lambda(E)] - 1\right\},$$

where $E_i = \emptyset$, and $i = 2, 3, \ldots$ That is,

$$1 + \lambda \cdot g_\lambda(E) = [1 + \lambda \cdot g_\lambda(E)] \cdot \left\{\prod_{i=2}^{\infty}[1 + \lambda \cdot g_\lambda(E_i)]\right\}.$$

Noting the fact that $\lambda \in (-1/\sup g_\lambda, \infty)$ and $g_\lambda(E) < \infty$, we know that

$$0 < 1 + \lambda \cdot g_\lambda(E) < \infty.$$

Thus, we have

$$\prod_{i=2}^{\infty}[1+\lambda\cdot g_\lambda(E_i)]=1$$

and therefore,

$$1+\lambda g_\lambda(\emptyset)=1.$$

Consequently, we have

$$g_\lambda(\emptyset)=0.$$

By using this result, the second conclusion is clear. □

Theorem 4.4. *If g_λ is a λ-measure on a semiring* $\mathbf{S}$*, then g_λ is monotone.*

Proof. When $\lambda=0$ we refer the monotonicity of classical measures (Section 2.2). Now, let $\lambda\neq 0$ and let $E\in\mathbf{S},\ F\in\mathbf{S}$, and $E\subset F$. Since $\mathbf{S}$ is a semiring, $F-E=\cup_{i=1}^{n}D_i,$ where $\{D_i\}$ is a finite disjoint class of sets in $\mathbf{S}$, and we have

$$\frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1+\lambda\cdot g_\lambda(D_1)-1\right\}\geq 0$$

in both cases where $\lambda>0$ and $\lambda<0$. By using Theorem 4.3, g_λ satisfies the finite λ-rule. So, we have

$$\begin{aligned}g_\lambda(F)&=g_\lambda(E\cup D_1\cup\cdots\cup D_n)\\&=\frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1+\lambda\cdot g_\lambda(D_1)][1+\lambda\cdot g_\lambda(E)]-1\right\}\\&=g_\lambda(E)+\frac{1}{\lambda}\left\{\prod_{i=1}^{n}[1+\lambda\cdot g_\lambda(D_1)]-1\right\}[1+\lambda\cdot g_\lambda(E)]\\&\geq g_\lambda(E).\end{aligned}$$

□

Though we can prove directly that any λ-measure on a semiring possesses the continuity now, it seems more convenient to show this fact after introducing a new concept called a *quasi-measure*. However, from Theorem 4.3, Theorem 4.4, and the fact that λ-measures are continuous, we know that any λ-measure on a semiring is a monotone measure.

Theorem 4.5. *Let g_λ be a λ-measure on a semiring* $\mathbf{S}$*. Then, it is subadditive when $\lambda<0$; it is superadditive when $\lambda>0$; and it is additive when $\lambda=0$.*

Proof. From Theorems 4.3 and 4.4, we know that μ satisfies the λ-rule and is monotone. The conclusion of this theorem can be obtained directly from Definition 4.3. □

By selecting the parameter λ appropriately, we can use a λ-measure to fit a given monotone measure approximately.

Theorem 4.6. *Let g_λ be a λ-measure on a ring* $\mathbf{R}$. *Then, for any $E \in \mathbf{R}$ and $F \in \mathbf{R}$,*

(1) $g_\lambda(E-F) = \dfrac{g_\lambda(E) - g_\lambda(E\cap F)}{1+\lambda\cdot g_\lambda(E\cap F)}$,

(2) $g_\lambda(E\cup F) = \dfrac{g_\lambda(E) + g_\lambda(F) - g_\lambda(E\cap F) + \lambda\cdot g_\lambda(E)\cdot g_\lambda(F)}{1+\lambda\cdot g_\lambda(E\cap F)}$.

Furthermore, if $\mathbf{R}$ *is an algebra and g_λ is normalized, then*

(3) $g_\lambda(\bar{E}) = \dfrac{1-g_\lambda(E)}{1+\lambda\cdot g_\lambda(E)}$.

Proof. From

$$\begin{aligned} g_\lambda(E) &= g_\lambda((E\cap F)\cup(E-F)) \\ &= g_\lambda(E\cap F) + g_\lambda(E-F)[1+\lambda\cdot g_\lambda(E\cap F)] \end{aligned}$$

we obtain (1). As to (2), we have

$$\begin{aligned} g_\lambda(E\cup F) &= g_\lambda(E\cup[F-(E\cap F]) \\ &= g_\lambda(E) + g_\lambda(F-(E\cap F))\cdot[1+\lambda\cdot g_\lambda(E)] \\ &= g_\lambda(E) + \frac{g_\lambda(F) - g_\lambda(E\cap F)}{1+\lambda\cdot g_\lambda(E\cap F)}\cdot[1+\lambda\cdot g_\lambda(E)] \\ &= \frac{g_\lambda(E) + g_\lambda(F) - g_\lambda(E\cap F) + \lambda\cdot g_\lambda(E)\cdot g_\lambda(F)}{1+\lambda\cdot g_\lambda(E\cap F)} \end{aligned}$$

Formula (3) is a direct result of (1) and the normalization of g_λ. □

How to construct a λ-measure on a semiring (or ring, algebra, σ-ring, σ-algebra, respectively) is a significant and interesting problem. If $X = \{x_1, \ldots, x_n\}$ is a finite set, $\mathbf{C}$ consists of X and all singletons of X, μ is defined on $\mathbf{C}$ such that $\mu(\{x_i\}) < \mu(X) < \infty$ for $i = 1, 2, \ldots, n$, and there are at least two points, x_{i_1} and x_{i_2}, satisfying $\mu(\{x_{i_j}\}) > 0$, $j = 1, 2$, then such a set function μ is always a λ-measure on $\mathbf{C}$ for some parameter λ. When $\mu(X) = \Sigma_{i=1}^n \mu(\{x_i\})$, $\lambda = 0$; otherwise, λ can be determined by the equation

$$\mu(X) = \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1+\lambda\cdot\mu(\{x_i\})) - 1\right]. \tag{4.8}$$

In fact, we have the following theorem.

Theorem 4.7. *Under the condition mentioned above, the equation*

$$1 + \lambda \cdot \mu(X) = \prod_{i=1}^{n} [1 + \lambda \cdot \mu(\{x_i\})]$$

determines the parameter λ uniquely:

(1) $\lambda > 0$ when $\sum_{i=1}^{n} \mu(\{x_i\}) < \mu(X)$;

(2) $\lambda = 0$ when $\sum_{i=1}^{n} \mu(\{x_i\}) = \mu(X)$;

(3) $-\frac{1}{\mu(X)} < \lambda < 0$ when $\sum_{i=1}^{n} \mu(\{x_i\}) > \mu(X)$.

Proof. Denote $\mu(X) = a, \mu(\{x_i\}) = a_i$ for $i = 1, 2, \ldots, n$, and $f_k(\lambda) = \prod_{i=1}^{k} (1 + a_i\lambda)$ for $k = 2, \ldots, n$. There is no loss of generality in assuming $a_1 > 0$ and $a_2 > 0$. From the given condition we know that $(1 + a_k\lambda) > 0$ for $k = 1, \ldots, n$ and any $\lambda \in (-1/a, \infty)$. Since

$$f_k(\lambda) = (1 + a_k\lambda) f_{k-1}(\lambda),$$

we have

$$f'_k(\lambda) = a_k \cdot f_{k-1}(\lambda) + (1 + a_k\lambda) f'_{k-1}(\lambda),$$

and

$$f''_k(\lambda) = 2a_k \cdot f'_{k-1}(\lambda) + (1 + a_k\lambda) f''_{k-1}(\lambda).$$

It is easy to see that, for any $k = 2, \ldots, n$ and any $\lambda \in (-1/a, \infty)$, if $f'_{k-1}(\lambda) > 0$ and $f''_{k-1} > 0$, then so are $f'_k(\lambda)$ and $f''_k(\lambda)$. Now, since

$$f'_2(\lambda) = a_1(1 + a_2\lambda) + a_2(1 + a_1\lambda) > 0$$

and

$$f''_2(\lambda) = 2a_1a_2 > 0,$$

we know that $f''_k(\lambda) > 0$. This means that the function $f_n(\lambda)$ is concave in $(-1/a, \infty)$. From the derivative of $f_n(\lambda)$,

$$f'_n(0) = \sum_{i=1}^{n} a_i.$$

Noting $\lim_{\lambda \to \infty} f_n(\lambda) = \infty$, we know that, if $\sum_{i=1}^{n} a_i < a$, the curve of $f_n(\lambda)$ has a unique intersection point with the line $f(\lambda) = 1 + a \cdot \lambda$ (illustrated in Fig. 4.2a) on some $\lambda > 0$. If $\sum_{i=1}^{n} a_i = a$, then the line $f(\lambda) = 1 + a \cdot \lambda$ is just a tangent of

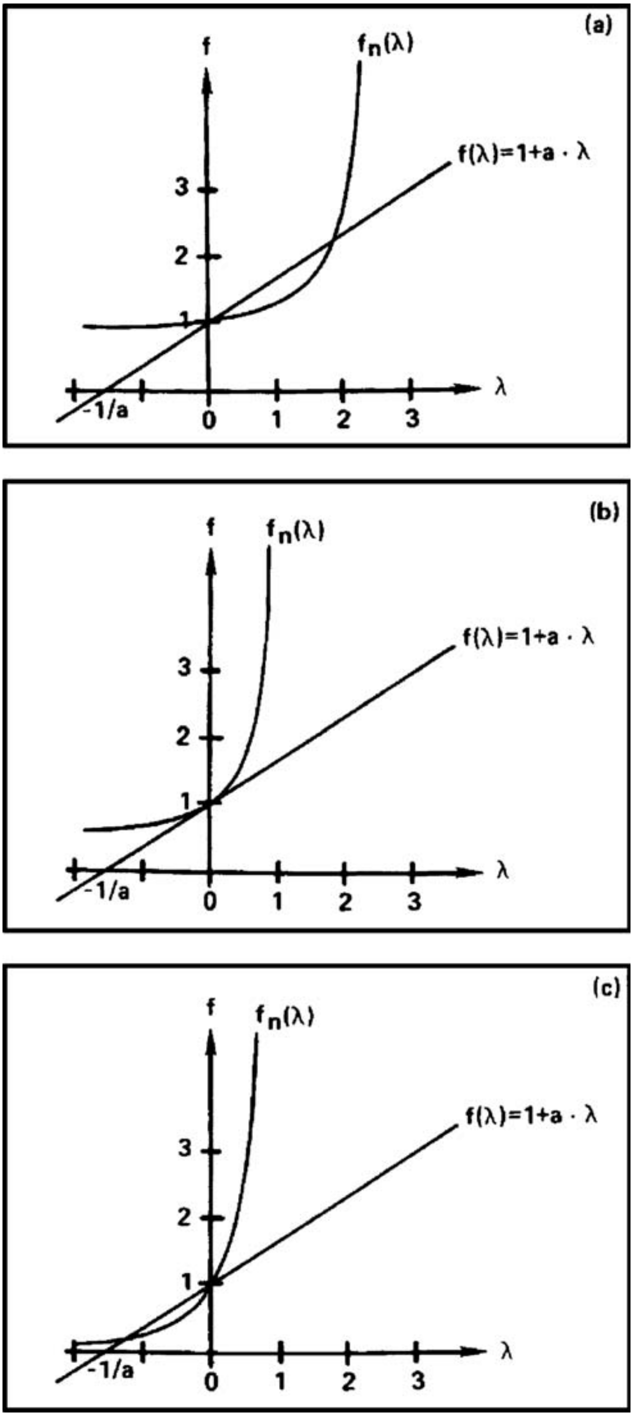

Fig. 4.2 The uniqueness of parameter λ

$f_n(\lambda)$ at point $\lambda = 0$ (illustrated in Fig. 4.2b), and therefore, the curve of $f_n(\lambda)$ has no intersection point anywhere else with the line $f(\lambda) = 1 + a \cdot \lambda$. If $\sum_{i=1}^{n} a_i > a$, since $f_n'(\lambda) > 0$, and $f(\lambda) = 1 + a \cdot \lambda \leq 0$ when $\lambda \leq -1/a$, the curve of $f_n(\lambda)$ must have a unique intersection point with the line $f(\lambda) = 1 + a \cdot \lambda$ on some $\lambda \in (-1/a, 0)$ (illustrated in Fig. 4.2c). Now, the proof is complete. □

If there is some x_i such that $\mu(\{x_i\}) = \mu(X)$, then Eq. (4.8) has infinitely many solutions (i.e., μ is a λ-measure for any $\lambda \in (-1/\mu(X), \infty)$) only when $\mu(\{x_j\}) = 0$ for all $j \neq i$; otherwise, it has no solution in $(-1/\mu(X), \infty)$.

After determining the value of λ, it is not difficult to extend this λ-measure from **C** onto the power set **P**(X) by using the finite λ-rule.

Example 4.4. Let $X = \{a, b, c\}$, $\mu(X) = 1$, $\mu(\{a\}) = \mu(\{b\}) = 0.2$, $\mu(\{c\}) = 0.1$. According to Theorem 4.7, μ is a λ-measure. Now we use (4.8) to determine the value of the parameter λ. From (4.8), we have

$$1 = \frac{(1 + 0.2\lambda)(1 + 0.2\lambda)(1 + 0.1\lambda) - 1}{\lambda},$$

which results in the quadratic equation,

$$0.004\lambda^2 + 0.08\lambda - 0.5 = 0.$$

Solving this equation, we have

$$\begin{aligned}\lambda &= \frac{-0.08 \pm (0.0064 + 0.008)^{1/2}}{0.008} \\ &= \frac{-0.08 \pm 0.12}{0.008} \\ &= 5 \text{ or } -25\end{aligned}.$$

Since $-25 < -1$, the unique feasible solution is $\lambda = 5$.

Now we turn to consider constructing a normalized λ-measure on the Borel field for a given $\lambda \in (-1, \infty)$. We already know that $\mathbf{S} = \{[a, b) | -\infty < a \leq b < \infty\}$ is a semiring. If $h(x)$ is a probability distribution function (left continuous) on $(-\infty, \infty)$, then we can define a set function ψ on **S** as follows:

$$\psi([a, b)) = \frac{h(b) - h(a)}{1 + \lambda \cdot h(a)}.$$

This set function ψ is continuous, and we can define

$$\psi(X) = \psi((-\infty, \infty)) = \lim_{a \to -\infty, b \to \infty} \psi([a, b)).$$

Since $\lim_{x \to \infty} h(x) = 0$ and $\lim_{x \to \infty} h(x) = 1$, we have

$$\psi(X) = 1.$$

Moreover, we can verify that such a set function ψ satisfies the λ-rule on **S**. In fact, for any $[a,b) \in \mathbf{S}$ and $[b,c) \in \mathbf{S}$, $[a,b) \cup [b,c) = [a,c) \in \mathbf{S}$ and

$$\begin{aligned}
\psi([a,b)) + \psi([b,c)) + \lambda \cdot \psi([a,b)) \cdot \psi([b,c)) \\
&= \psi([a,b)) + \psi([b,c)) \cdot [1 + \lambda \cdot \psi(a,b)] \\
&= \frac{h(b)-h(a)}{1+\lambda \cdot h(a)} + \frac{h(c)-h(b)}{1+\lambda \cdot h(b)} \cdot \left[1 + \lambda \frac{h(b)-h(a)}{1+\lambda \cdot h(a)}\right] \\
&= \frac{h(b)-h(a)}{1+\lambda \cdot h(a)} + \frac{[h(c)-h(b)] \cdot [1+\lambda \cdot h(b)]}{[1+\lambda \cdot h(b)] \cdot [1+\lambda \cdot h(a)]} \\
&= \frac{h(c)-h(a)}{1+\lambda \cdot h(a)} \\
&= \psi([a,c)).
\end{aligned}$$

It is possible, but rather difficult to verify that such a set function ψ satisfies the σ-λ-rule on **S** and to extend ψ onto the Borel field in a way similar to that used for classical measures. However, if we use the aid of the concept of a quasi-measure, which is introduced in the next section, this problem becomes quite easy to solve.

4.4 Quasi-Measures

Definition 4.5. Let $a \in (0,\infty]$. An extended real function $\theta : [0,a] \to [0,\infty]$ is called a *T-function* iff it is continuous, strictly increasing, and such that $\theta(0) = 0$ and $\theta^{-1}(\{\infty\}) = \emptyset$ or $\{\infty\}$, according to a being finite or not.

Definition 4.6. μ is called *quasi-additive* iff there exists a T-function θ, whose domain of definition contains the range of μ, such that the set function $\theta \circ \mu$ defined on **C** by

$$(\theta \circ \mu)(E) = \theta(\mu(E)), \text{ for any } E \in \mathbf{C} \;,$$

is additive; μ is called a *quasi-measure* iff there exists a T-function θ such that $\theta \circ \mu$ is a classical measure on **C**. The T-function θ is called the *proper T-function* of μ.

A normalized quasi-measure is called a *quasi-probability*.

Clearly, any classical measure is a quasi-measure with the identity function as its proper T-function.

Example 4.5. The monotone measure given in Example 3.4 is a quasi-measure. Its proper T-function is $\theta(y) = \sqrt{y}$, $y \in [0,1]$.

Theorem 4.8. *Any quasi-measure on a semiring is a quasi-additive monotone measure.*

Proof. Let μ be a quasi-measure on a semiring **S** and θ be its proper T-function. Since any classical measure on a semiring is additive, μ is quasi-additive.

Furthermore, θ^{-1} exists, and it is continuous, strictly increasing, and $\theta^{-1}(0) = 0$. So, $\mu = \theta^{-1} \circ (\theta \circ \mu)$ is continuous, monotone, and $\mu(\emptyset) = 0$. That is, μ is a monotone measure. □

Theorem 4.9. *If μ is a classical measure, then, for any T-function θ whose range contains the range of μ, $\theta^{-1} \circ \mu$ is a quasi-measure with θ as its proper T-function.*

Proof. Since $\theta \circ (\theta^{-1} \circ \mu) = \mu$, the conclusion of this theorem is clear. □

Theorem 4.10. *Let μ be quasi-additive on a ring* **R** *with $\mu(\emptyset) = 0$. If μ is either continuous from below on* **R***, or continuous from above at $\emptyset$ and finite, then μ is a quasi-measure on* **R**.

Proof. Since μ is quasi-additive, there exists a T-function θ such that $\theta \circ \mu$ is additive on **R**. The composition $\theta \circ \mu$ is either continuous from below on **R**, or continuous from above at $\emptyset$ and finite. So $\theta \circ \mu$ is a measure on **R** (Section 2.2, Theorem 2.32). That is, μ is a quasi-measure on **R**. □

Corollary 4.1. *Any quasi-additive monotone measure on a ring is a quasi-measure.*

Now, we return to solve the problems that are raised in Section 4.3.

Theorem 4.11. *Let $\lambda \neq 0$. Any λ-measure g_λ is a quasi-measure with*

$$\theta_\lambda(y) = \frac{\ln(1+\lambda y)}{k\lambda}, y \in [0, \sup g_\lambda],$$

as its proper T-function, where k is an arbitrary finite positive real number. Conversely, if μ is a classical measure, then $\theta_\lambda^{-1} \circ \mu$ is a λ-measure, where

$$\theta_\lambda^{-1}(x) = \frac{e^{k\lambda x} - 1}{\lambda}, \quad x \in [0, \infty],$$

and k is an arbitrary finite positive real number.

Proof. θ_λ is a T-function. Let $\{E_n\}$ be a disjoint sequence of sets in **C** whose union $\cup_{n=1}^{\infty} E_n$ is also in **C.** If g_λ is a λ-measure on **C** then it satisfies the σ-λ-rule and there exists $E_0 \in$ **C** such that $g_\lambda(E_0) < \infty$. Therefore, we have

$$\begin{aligned}(\theta_\lambda \circ g_\lambda)\Big(\bigcup_{n=1}^{\infty} E_n\Big) &= \frac{1}{k \cdot \lambda} \cdot \ \ln \left[1 + \lambda \cdot g_\lambda\Big(\bigcup_{n=1}^{\infty} E_n\Big)\right] \\ &= \frac{1}{k \cdot \lambda} \cdot \ \ln \left(1 + \left[\prod_{n=1}^{\infty}[1 + \lambda \cdot g_\lambda(E_n)]\right] - 1\right) \\ &= \frac{1}{k \cdot \lambda} \cdot \sum_{n=1}^{\infty} \ln[1 + \lambda \cdot g_\lambda(E_n)] \\ &= \sum_{n=1}^{\infty} \frac{\ln \ [1 + \lambda \cdot g_\lambda(E_n)]}{k \cdot \lambda} \\ &= \sum_{n=1}^{\infty} (\theta_\lambda \circ g_\lambda)(E_n),\end{aligned}$$

and $(\theta_\lambda \circ g_\lambda)(E_0) = \theta_\lambda(g_\lambda(E_0)) < \infty$. So $\theta_\lambda \circ g_\lambda$ is a classical measure on $\mathbf{C}$. Conversely, if μ is a classical measure on $\mathbf{C}$, then it is σ-additive, and there exists $E_0 \in \mathbf{C}$ such that $\mu(E_0) < \infty$. Therefore, we have

$$\begin{aligned}
(\theta_\lambda^{-1} \circ \mu)\left(\bigcup_{n=1}^{\infty} E_n\right) &= \theta_\lambda^{-1}\left[\sum_{n=1}^{\infty} \mu(E_n)\right] \\
&= \frac{\exp\left[k\lambda \sum_{n=1}^{\infty} \mu(E_n)\right] - 1}{\lambda} \\
&= \frac{\prod_{n=1}^{\infty} e^{k\lambda \cdot \mu(E_n)} - 1}{\lambda} \\
&= (1/\lambda)\left\{\prod_{n=1}^{\infty}[1 + \lambda \cdot \theta_\lambda^{-1}(\mu(E_n))] - 1\right\} \\
&= (1/\lambda)\left\{\prod_{n=1}^{\infty}[1 + \lambda \cdot (\theta_\lambda^{-1} \circ \mu)(E_n)] - 1\right\};
\end{aligned}$$

that is, $\theta_\lambda^{-1} \circ \mu$ satisfies the σ-λ–rule. Noting that $(\theta_\lambda^{-1} \circ \mu)(E_0) = \theta_\lambda^{-1}(\mu(E_0)) < \infty$, we conclude that $\theta_\lambda^{-1} \circ \mu$ is a λ-measure on $\mathbf{C}$. □

Example 4.6. Let $X = \{a, b\}, \mathbf{F} = \mathbf{P}(X), g_\lambda$ be defined by

$$g_\lambda(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.2 & \text{if } E = \{a\} \\ 0.4 & \text{if } E = \{b\} \\ 1 & \text{if } E = X. \end{cases}$$

Then g_λ is a λ-measure with a parameter $\lambda = 5$. If we take

$$\theta_\lambda(y) = \frac{\ln(1 + \lambda y)}{\ln(1 + \lambda)} = \frac{\ln(1 + 5y)}{\ln 6},$$

then we have

$$(\theta_\lambda \circ g_\lambda)(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.387 & \text{if } E = \{a\} \\ 0.613 & \text{if } E = \{b\} \\ 1 & \text{if } E = X. \end{cases}$$

$\theta_\lambda \circ g_\lambda$ is a probability measure.

Example 4.7. Let $X = \{a, b\}$, $\mathbf{F} = \mathbf{P}(X)$, and let g_λ be a λ-measure defined by

$$g_\lambda(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.5 & \text{if } E = \{a\} \\ 0.8 & \text{if } E = \{b\} \\ 1 & \text{if } E = X. \end{cases}$$

with $\lambda = -0.75$. If we take

$$\theta_\lambda(y) = \frac{\ln(1 - 0.75y)}{\ln 0.25},$$

then

$$(\theta_\lambda \circ g_\lambda)(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.34 & \text{if } E = \{a\} \\ 0.66 & \text{if } E = \{b\} \\ 1 & \text{if } E = X, \end{cases}$$

which is a probability measure.

In a similar way, we know that under the mapping θ_λ the λ-rule and the finite λ-rule become the additivity and the finite additivity, respectively. Conversely, under the mapping θ_λ^{-1} the additivity and the finite additivity become the λ-rule and the finite λ-rule, respectively. Recalling some relevant knowledge in classical measure theory, we have the following corollaries.

Corollary 4.2. *On a semiring, the λ-rule is equivalent to the finite λ-rule.*

Corollary 4.3. *Any λ-measure on a semiring is continuous.*

Corollary 4.4. *On a ring, the λ-rule together with continuity are equivalent to the σ-λ-rule. Thus, on a ring, any monotone measure that satisfies the λ-rule is a λ-measure.*

Similarly as in classical measure theory, a monotone measure on a semiring that satisfies the λ-rule (or, is quasi-additive) may not satisfy the σ-λ-rule (or, may not be a quasi-measure).

Corollary 4.5. *If g_λ is a normalized λ-measure on an algebra $\mathbf{R}$, then its dual measure μ, which is defined by*

$$\mu(E) = 1 - g_\lambda(\bar{E}) \text{ for any } E \in \mathbf{R},$$

is also a normalized λ-measure on $\mathbf{R}$, and the corresponding parameter is $\lambda' = -\lambda/(\lambda + 1)$.

Proof. Let $E \in \mathbf{R}, F \in \mathbf{R},$ and $E \cap F = \varnothing$. By using Theorem 4.6, we have

$$\begin{aligned}
&\mu(E) + \mu(F) - \frac{\lambda}{\lambda+1}\mu(E)\mu(F) \\
&= 1 - g_\lambda(\bar{E}) + 1 - g_\lambda(\bar{F}) - \frac{\lambda}{\lambda+1}[1 - g_\lambda(\bar{E})][1 - g_\lambda(\bar{F})] \\
&= \frac{(\lambda+1)g_\lambda(E)}{1+\lambda g_\lambda(E)} + \frac{(\lambda+1)g_\lambda(F)}{1+\lambda g_\lambda(F)} - \lambda\frac{(\lambda+1)g_\lambda(E)g_\lambda(F)}{[1+\lambda g_\lambda(E)][1+\lambda g_\lambda(F)]} \\
&= \frac{(\lambda+1)[g_\lambda(E) + g_\lambda(F) + \lambda g_\lambda(E)g_\lambda(F)]}{[1+\lambda g_\lambda(E)][1+\lambda g_\lambda(F)]} \\
&= \frac{(\lambda+1)g_\lambda(E\cup F)}{1+\lambda g_\lambda(E\cup F)} \\
&= 1 - g_\lambda(\overline{E\cup F}) \\
&= \mu(E\cup F).
\end{aligned}$$

Since μ is continuous, by Corollary 3.4, μ satisfies the σ-λ-rule with a parameter $\lambda' = -\lambda/(\lambda+1)$. So, noting that $\mu(X) = 1 - g_\lambda(\varnothing) = 1$, we know that μ is a normalized λ-measure on **R** with a parameter $\lambda' = -\lambda/(\lambda+1)$. □

As to the problem of constructing a λ-measure on the Borel field, we deal with it in Chapter 6.

4.5 Belief Measures and Plausibility Measures

In Section 4.4, a nonadditive measure is induced from a classical measure by a transformation of the range of the latter. In this section we attempt to construct a nonadditive measure in another way.

Definition 4.7. Let **P(P(**X**))** be the power set of **P(**X). If p is a discrete probability measure on (**P**(X), **P(P(**X**))**) with $p(\{\varnothing\}) = 0$, then the set function m: **P**(X) → [0, 1] determined by

$$m(E) = p(\{E\}) \text{for any } E \in \mathbf{P}(X)$$

is called a *basic probability assignment* on **P(**X).

Theorem 4.12. *A set function m:* **P**(X) → [0, 1] *is a basic probability assignment if and only if*

(1) $m(\varnothing) = 0$;

(2) $\sum_{E\in\mathbf{P}(X)} m(E) = 1.$

Proof. The necessity of these two conditions follows directly from Definition 4.7. As for their sufficiency, if we write

$$\mathbf{D}_n = \left\{E \middle| \tfrac{1}{n+1} < m(E) \le \tfrac{1}{n}\right\}, n = 1, 2, \ldots,$$

then every D_n is a finite class,

$$\mathbf{D} = \bigcup_{n=1}^{\infty} \mathbf{D}_n = \{E | m(E) > 0\}$$

is a countable class, and $\hat{\mathbf{S}} = \{\{E | E \in \mathbf{P}(X)\} \cup \{\varnothing\}$ is a semiring. Define

$$p(\{E\}) = \begin{cases} m(E) & \text{if } E \in \mathbf{D} \\ 0 & \text{otherwise} \end{cases}$$

for any $E \in \mathbf{P}(X)$ and $p(\{\varnothing\}) = 0$. Then, p is a probability measure on $\hat{\mathbf{S}}$ with $p(\{\varnothing\}) = 0$, which can be extended uniquely to a discrete probability measure on **(P**(X), **P(P**(X)**))** by the formula

$$p(\mathbf{E}) = \sum_{E \in \mathbf{E}} p(\{E\}).$$

for any $\mathbf{E} \in \mathbf{P}(\mathbf{P}(X))$. □

Definition 4.8. If m is a basic probability assignment on $\mathbf{P}(X)$, then the set function Bel: $\mathbf{P}(X) \to [0, 1]$ determined by the formula

$$\mathrm{Bel}(E) = \sum_{F \subset E} m(F) \ \forall E \in \mathbf{P}(X) \tag{4.9}$$

is called a *belief measure* on (X, $\mathbf{P}(X)$), or, more specifically, a belief measure induced from m.

Lemma 4.1. *If E is a nonempty finite set, then*

$$\sum_{F \subset E} (-1)^{|F|} = 0.$$

Proof. Let $E = \{x_1, \ldots, x_n\}$Then, we have

$$\{|F| | F \subset E\} = \{0, 1, \ldots, n\}$$

and

$$|\{F | |F| = i\}| = \binom{n}{i}, \ i = 0, 1, \ \ldots, n.$$

So, we have

$$\sum_{F\subset E}(-1)^{|F|}=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}=(1-1)^{n}=0.$$

□

Lemma 4.2. *If E is a finite set, $F\subset E$ and $F\neq E$, then*

$$\sum_{G|F\subset G\subset E}(-1)^{|G|}=0.$$

Proof. $E-F$ is a nonempty finite set. Using Lemma 4.1, we have

$$\sum_{G|F\subset G\subset E}(-1)^{|G|}=\sum_{D\subset E-F}(-1)^{|F\cup D|}=(-1)^{|F|}\sum_{D\subset E-F}(-1)^{|D|}=0.$$

□

Lemma 4.3. *Let X be finite, and λ and ν be finite set functions defined on* $\mathbf{P}(X)$. *Then we have*

$$\lambda(E)=\sum_{F\subset E}\nu(F)\ \ \forall E\in\mathbf{P}(X) \tag{4.10}$$

if and only if

$$\nu(E)=\sum_{F\subset E}(-1)^{|E-F|}\lambda(F)\ \ \forall E\in\mathbf{P}(X). \tag{4.11}$$

Proof. If (4.10) is true, then

$$\begin{aligned}\sum_{F\subset E}(-1)^{|E-F|}\lambda(F)&=(-1)^{|E|}\sum_{F\subset E}(-1)^{|F|}\lambda(F)\\&=(-1)^{|E|}\sum_{F\subset E}\left[(-1)^{|F|}\sum_{G\subset F}\nu(G)\right]\\&=(-1)^{|E|}\sum_{G\subset E}\left[\nu(G)\sum_{F|G\subset F\subset E}(-1)^{|F|}\right]\\&=(-1)^{|E|}\nu(E)(-1)^{|E|}\\&=\nu(E).\end{aligned}$$

Conversely, if (4.11) is true, then we have

$$\begin{aligned}
\sum_{F\subset E}\nu(F) &= \sum_{F\subset E}\sum_{G\subset F}(-1)^{|F-G|}\lambda(G)\\
&= \sum_{G\subset E}\left[(-1)^{|G|}\lambda(G)\sum_{F|G\subset F\subset E}(-1)^{|F|}\right]\\
&= (-1)^{|E|}\lambda(E)(-1)^{|E|}\\
&= \lambda(E).
\end{aligned}$$

□

Theorem 4.13. *If Bel is a belief measure on* $(X, \mathbf{P}(X))$*, then*

(BM1) $\mathrm{Bel}(\emptyset) = 0;$

(BM2) $\mathrm{Bel}(X) = 1;$

(BM3) $\mathrm{Bel}\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{I\subset\{1,\ldots,n\},I\neq\emptyset}(-1)^{|I|+1}\ \mathrm{Bel}\left(\bigcap_{i\in I} E_i\right),$

where $\{E_1, \ldots, E_n\}$ *is any finite subclass of* $\mathbf{P}(X)$*;*

(BM4) Bel *is continuous from above.*

Proof. From Theorem 4.12 and Definition 4.8, it is easy to see that (BM1) and (BM2) are true. To show that (BM3) holds, let us consider an arbitrary finite subclass $\{E_1, \ldots, E_n\}$, and set $I(F) = \{i|1 \leq i \leq n, F \subset E_i\}$, for any $F \in \mathbf{P}(X)$. Using Lemma 4.1, we have

$$\begin{aligned}
\sum_{I\subset\{1,\ldots,n\},I\neq\emptyset}(-1)^{|I|+1}\ \mathrm{Bel}\left(\bigcap_{i\in I}E_i\right) &= \sum_{I\subset\{1,\ldots,n\},I\neq\emptyset}\left[(-1)^{|I|+1}\sum_{F\subset\cap_{i\in I}E_i}m(F)\right]\\
&= \sum_{F|I(F)\neq\emptyset}\left[m(F)\sum_{I\subset I(F),I\neq\emptyset}(-1)^{|I|+1}\right]\\
&= \sum_{F|I(F)\neq\emptyset}\left[m(F)\left(1-\sum_{I\subset I(F)}(-1)^{|I|}\right)\right]\\
&= \sum_{F|I(F)\neq\emptyset}m(F)\\
&= \sum_{F\subset F_i\text{ for some }i}m(F)\\
&\leq \sum_{F\subset\cup_{i=1}^{n}E_i}m(F)\\
&= \mathrm{Bel}\left(\bigcup_{i=1}^{n}E_i\right)
\end{aligned}$$

As to (BM4), let E_i be a decreasing sequence of sets in $\mathbf{P}(X)$, and $\bigcap_{i=1}^{\infty} E_i = E$. From Theorem 4.12, we know there exists a countable class $\{D_n\} \subset \mathbf{P}(X)$, such that $m(F) = 0$ whenever $F \notin \{D_n\}$, and for any $\epsilon > 0$ there exists n_0 such that $\sum_{n>n_0} m(D_n) < \varepsilon$. Then, for each D_n, where $n \le n_0$, if $D_n \not\subset E$ (that is, $D_n - E \ne \emptyset$), there exists $i(n)$, such that $D_n \not\subset E_{i(n)}$. Let $i_0 = \max(i(1), \ldots, i(n_0))$. Then, if $D_n \not\subset E$, we have $D_n \not\subset E_{i0}$ for any $n \le n_0$. Hence,

$$
\begin{aligned}
\mathrm{Bel}(E) &= \sum_{F \subset E} m(F) \\
&= \sum_{D_n \subset E} m(D_n) \\
&\ge \sum_{D_n \subset E, n \le n_0} m(D_n) \\
&\ge \sum_{D_n \subset E_{i_0}, n \le n_0} m(D_n) \\
&\ge \sum_{D_n \subset E_{i_0}} m(D_n) - \sum_{n > n_0} m(D_n) \\
&> \sum_{F \subset E_{i_0}} m(F) - \varepsilon \\
&= \mathrm{Bel}(E_{i_0}) - \varepsilon.
\end{aligned}
$$

Noting that $\mathrm{Bel}(E) \le \mathrm{Bel}\{E_i\}$ for $i = 1, 2, \ldots$, and $\{\mathrm{Bel}(E_i)\}$ is decreasing with respect to i, we have $\mathrm{Bel}(E) = \lim_i \mathrm{Bel}(E_i)$. □

Observe that due to property (BM3), established for belief measures by Theorem 4.13, belief measures are ∞-monotone measures introduced in Section 4.2.

Theorem 4.14. *Any belief measure is monotone and superadditive.*

Proof. Let $E_1 \subset X, E_2 \subset X,$ and $E_1 \cap E_2 = \emptyset$. We have

$$
\begin{aligned}
\mathrm{Bel}(E_1 \cup E_2) &\ge \mathrm{Bel}(E_1) + \mathrm{Bel}(E_2) - \mathrm{Bel}(E_1 \cap E_2) \\
&= \mathrm{Bel}(E_1) + \mathrm{Bel}(E_2) \ge \max\{\mathrm{Bel}(E_1),\ \mathrm{Bel}(E_2)\}.
\end{aligned}
$$

From this inequality, it is easy to see that Bel is monotone and superadditive. □

From Theorems 4.13 and 4.14, we know that the belief measure is an upper semicontinuous monotone measure.

On a finite space, we can express a basic probability assignment by the belief measure induced from it.

Theorem 4.15. *Let X be finite. If a set function* μ: $\mathbf{P}(X) \to [0, 1]$ *satisfies the conditions*

(1) $\mu(\emptyset) = 1$;
(2) $\mu(X) = 1$;
(3) $\mu\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{I \subset \{1,\ldots n\}, I \neq \emptyset} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} E_i\right)$,

where $\{E_1, \ldots, E_n\}$ *is any finite subclass of* $\mathbf{P}(X)$, *then the set function m determined by*

$$m(E) = \sum_{F \subset E} (-1)^{|E-F|} \mu(F) \ \ \forall E \in \mathbf{P}(X), \tag{4.12}$$

is a basic probability assignment, and μ *is the belief measure induced from m. That is,*

$$\mu(E) = \ \mathrm{Bel}(E) = \sum_{F \subset E} m(F).$$

Proof. First, $m(\emptyset) = \sum_{F \subset \emptyset} (-1)^{|\emptyset - F|} \mu(F) = \mu(\emptyset) = 0$. Next, from (4.12) and Lemma 4.3, we have

$$\sum_{E \subset X} m(E) = \mu(X) = 1.$$

To prove that m is a basic probability assignment, we should show that $m(E) \geq 0$ for any $E \subset X$. Indeed, since X is finite, E is also finite, and we can write $E = \{x_1, \ldots, x_n\}$. If we denote $E_i = E - \{x_i\}$, then $E = \bigcup_{i=1}^{n} E_i$ and

$$\begin{aligned} m(E) &= \sum_{F \subset E} (-1)^{|E-F|} \mu(F) \\ &= \mu(E) - \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} E_i\right) \\ &= \mu\left(\bigcup_{i=1}^{n} E_i\right) - \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} E_i\right) \\ &\geq 0. \end{aligned}$$

The last conclusion in this theorem is a direct result of Lemma 4.3. □

Definition 4.9. If m is a basic probability assignment on $\mathbf{P}(X)$, then the set function Pl: $\mathbf{P}(X) \to [0, 1]$ determined by

$$\mathrm{P1}(E) = \sum_{F \cap E \neq \emptyset} m(F) \ \forall \ E \in \mathbf{P}(X) \tag{4.13}$$

is called a *plausibility measure* on $(X, \mathbf{P}(X))$, or, more exactly, a plausibility measure induced from m.

Theorem 4.16. *If* Bel *and* Pl *are the belief measure and plausibility measure, respectively, induced from the same basic probability assignment then*

$$\mathrm{Bel}(E) = 1 - \ \mathrm{P1}(\bar{E}) \tag{4.14}$$

and

$$\mathrm{Bel}(E) \leq \ \mathrm{P1}(E)$$

for any $E \subset X$.

Proof.

$$\begin{aligned}\mathrm{Bel}(E) &= \sum_{F \subset E} m(F) \\ &= \sum_{F \subset X} m(F) - \sum_{F \not\subset E} m(F) \\ &= 1 - \sum_{F \cap \bar{E} \neq \emptyset} m(F) \\ &= 1 - \ \mathrm{P1}(\bar{E}).\end{aligned}$$

The second conclusion can be obtained directly from Definitions 4.8 and 4.9.□

Theorem 4.17. *If* P1 *is a plausibility measure on* $(X, \mathbf{P}(X))$, *then*

(PMI) $\mathrm{P1}(\emptyset) = 0;$

(PM2) $\mathrm{P1}(X) = 1;$

(PM3) $\mathrm{P1}\left(\bigcap_{i=1}^{n} E_i\right) \leq \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \ \mathrm{P1}\left(\bigcup_{i \in I} E_i\right),$

where $\{E_1, \ldots, E_n\}$ *is any finite subclass of* $\mathbf{P}(X)$.

(PM4) P1 *is continuous from below.*

Proof. From Theorem 4.13 and Theorem 4.16, we can directly obtain (PM1), (PM2), and (PM4). As to (PM3), by using Lemma 4.1, we have

$$\begin{aligned}
\mathrm{P1}\left(\bigcap_{i=1}^{n} E_i\right) &= 1 - \mathrm{Bel}\left(\overline{\bigcap_{i=1}^{n} E_i}\right) \\
&= 1 - \mathrm{Bel}\left(\bigcup_{i=1}^{n} \bar{E}_i\right) \\
&\leq 1 - \sum_{I \subset \{1,\ldots,n\},\, I \neq \emptyset} (-1)^{|I|+1} \mathrm{Bel}\left(\bigcap_{i \in I} \bar{E}_i\right) \\
&= \sum_{I \subset \{1,\ldots,n\},\, I \neq \emptyset} (-1)^{|I|+1} \left[1 - \mathrm{Bel}\left(\bigcap_{i \in I} \bar{E}_i\right)\right] \\
&= \sum_{I \subset \{1,\ldots,n\},\, I \neq \emptyset} (-1)^{|I|+1} \left[1 - \mathrm{Bel}\left(\overline{\bigcup_{i \in I} E_i}\right)\right] \\
&= \sum_{I \subset \{1,\ldots,n\},\, I \neq \emptyset} (-1)^{|I|+1} \mathrm{Pl}\left(\bigcup_{i \in I} E_i\right).
\end{aligned}$$

□

Due to the property (PM3), which is established for plausibility measures by Theorem 4.17, plausibility measures are ∞-alternating measures introduced in Section 4.2.

Theorem 4.18. *Any plausibility measure is monotone and subadditive.*

Proof. $E \subset F \subset X$, then $\bar{F} \subset \bar{E} \subset X$. From Theorem 4.14 and Theorem 4.16, we have

$$\mathrm{P1}(E) = 1 - \mathrm{Bel}(\bar{E}) \leq 1 - \mathrm{Bel}(\bar{F}) = \mathrm{P1}(F)$$

As to subadditivity, if $E_1 \subset X$ and $E_2 \subset X$, then

$$\begin{aligned}
0 &\leq \mathrm{P1}(E_1 \cap E_2) \\
&\leq \mathrm{P1}(E_1) + \mathrm{P1}(E_2) - \mathrm{P1}(E_1 \cup E_2).
\end{aligned}$$

So $\mathrm{P1}(E_1 \cup E_2) \leq \mathrm{P1}(E_1) + \mathrm{P1}(E_2)$. □

From Theorem 4.17 and Theorem 4.18, we know that the plausibility measure is a lower semicontinuous monotone measure.

Theorem 4.19. *Any discrete probability measure p on $(X, \mathbf{P}(X))$ is both a belief measure and a plausibility measure. The corresponding basic probability assignment focuses on the singletons of $\mathbf{P}(X)$. Conversely, if m is a basic probability* *assignment focusing on the singletons of* $\mathbf{P}(X)$*, then the belief measure and the plausibility measure induced from m coincide, resulting in a discrete probability measure on* $(X, \mathbf{P}(X))$.

Proof. Since p is a discrete probability measure, there exists a countable set $\{x_1, x_2, \cdots\} \subset X$, such that

$$\sum_{i=1}^{\infty} p(\{x_i\}) = 1.$$

Let

$$m(E) = \begin{cases} p(E) & \text{if } E = \{x_i\} \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

for any $E \in \mathbf{P}(X)$. Then, m is a basic probability assignment, and

$$p(E) = \sum_{x_i \in E} p(\{x_i\}) = \sum_{F \subset E} m(F) = \sum_{F \cap E \neq \emptyset} m(F)$$

for any $E \in \mathbf{P}(X)$. That is, p is both a belief measure and a plausibility measure. Conversely, if a basic probability assignment m focuses only on the singletons of $\mathbf{P}(X)$, then, for any $E \in \mathbf{P}(X)$,

$$\text{Bel}(E) = \sum_{F \subset E} m(F) = \sum_{x \in E} m(\{x\}) = \sum_{F \cap E \neq \emptyset} m(F) = \text{Pl}(E).$$

So, Bel and Pl coincide, and it is easy to verify that they are σ-additive. Consequently, they are discrete probability measures on $(X, \mathbf{P}(X))$. □

Theorem 4.20. *Let* Bel *and* Pl *be the belief measure and the plausibility measure, respectively, induced from a basic probability assignment m. If* Bel *coincides with* Pl*, then m focuses only on singletons.*

Proof. If there exists $E \in \mathbf{P}(X)$ that is not a singleton of $\mathbf{P}(X)$ such that $m(E) > 0$, then, for any $x \in E$,

$$\text{Bel}(\{x\}) = m(\{x\}) < m(\{x\}) + m(E) \leq \sum_{F \cap \{x\} \neq \emptyset} m(F) = \text{Pl}(\{x\}).$$

This contradicts the coincidence of Bel and Pl. □

The Sugeno measures defined on the power set $\mathbf{P}(X)$ are special examples of belief measures and plausibility measures when X is countable.

Theorem 4.21. *Let X be countable, and* $g_\lambda (\lambda \neq 0)$ *be a Sugeno measure on* $(X, \mathbf{P}(X))$. *Then* g_λ *is a belief measure when* $\lambda > 0$*, and is a plausibility measure when* $\lambda < 0$.

Proof. Let $X = \{x_1, x_2, \ldots\}$. When $\lambda > 0$, we define m: $\mathbf{P}(X) \to [0, 1]$ by

$$m(E) = \begin{cases} \lambda^{|E|-1} \prod_{x_i \in E} g_\lambda(\{x_i\}) & \text{if } E \neq \varnothing \\ 0 & \text{if } E = \varnothing \end{cases}$$

for any $E \in \mathrm{P}(X)$. Obviously, $m(E) \geq 0$ for any $E \in \mathbf{P}(X)$. From Definition 4.3, we have

$$\begin{aligned} g_\lambda(E) &= \frac{1}{\lambda}\left[\prod_{x_i \in E}(1 + \lambda \cdot g_\lambda(\{x_i\})) - 1\right] \\ &= \frac{1}{\lambda}\sum_{F \subset E, F \neq \varnothing}\left[\lambda^{|F|} \cdot \prod_{x_i \in F} g_\lambda(\{x_i\})\right] \\ &= \sum_{F \subset E, F \neq \varnothing}\left[\lambda^{|F|-1} \cdot \prod_{x_i \in F} g_\lambda(\{x_i\})\right] \\ &= \sum_{F \subset E} m(F). \end{aligned}$$

Since $g_\lambda(X) = 1$, we have

$$\sum_{F \subset X} m(F) = 1.$$

Therefore, m is a basic probability assignment, and thus, g_λ is the belief measure induced from m. When $\lambda < 0$, we have $\lambda' = -\lambda/(\lambda + 1) > 0$. By using Corollary 4.5 and Theorem 4.16, we know that g_λ is a plausibility measure. □

4.6 Possibility Measures and Necessity Measures

Definition 4.10. A monotone measure μ is called *maxitive* on **C** iff

$$\mu\left(\bigcup_{t \in T} E_t\right) = \sup_{t \in T} \mu(E_t) \tag{4.15}$$

for any subclass $\{E_t | t \in T\}$ of **C** whose union is in **C**, where T is an arbitrary index set.

If **C** is a finite class, then the maxitivity of μ on **C** is equivalent to the simpler requirement that

$$\mu(E_i \cup E_2) = \mu(E_1) \vee \mu(E_2) \tag{4.16}$$

whenever $E_i \in \mathbf{C}$, $E_2 \in \mathbf{C}$, and $E_1 \cup E_2 \in \mathbf{C}$. Symbol $\vee$ denotes the maximum of $\mu(E_1)$ and $\mu(E_2)$.

Definition 4.11. A monotone measure μ is called a *generalized possibility measure* on $\mathbf{C}$ iff it is maxitive on $\mathbf{C}$ and there exists $E \in \mathbf{C}$ such that $\mu(E) < \infty$.

Usually, a generalized possibility measure is denoted by π.

Definition 4.12. If π is a generalized possibility measure defined on $\mathbf{P}(X)$, then the function f defined on X by

$$f(x) = \pi(\{x\}) \text{ for any } x \in X$$

is called its *possibility profile*.

Theorem 4.22. *Any generalized possibility measure* π (*on* $\mathbf{C}$) *is a lower semicontinuous monotone measure* (*on* $\mathbf{C}$).

Proof. According to the convention, when $T = \emptyset$ we have $\cup_{t\in T} E_t = \emptyset$ and $\sup_{t\in T} \mu(E_t) = 0$. So, if $\emptyset \in \mathrm{C}$, then $\pi(\emptyset) = 0$. Furthermore, if $E \in \mathbf{C}$, $F \in \mathbf{C}$, and $E \subset F$, then, by using maxitivity, we have

$$\pi(F) = \pi(E \cup F) = \pi(E) \vee \pi(F) \geq \pi(E).$$

At last, π is continuous from below. In fact, if $\{E_n\}$ is an increasing sequence of sets in $\mathbf{C}$ whose union E is also in $\mathbf{C}$, from the definition of the supremum, for any $\varepsilon > 0$, there exists n_0 such that

$$\pi(E_{n_0}) \geq \sup_n \pi(E_n) - \varepsilon = \pi(E) - \varepsilon.$$

Noting that π is monotone, we know that

$$\lim_n \pi(E_n) = \pi(E).$$

□

Definition 4.13. When a generalized possibility measure π defined on $\mathbf{P}(X)$ is normalized, it is called a *possibility measure*.

The following example shows that a possibility measure is not necessarily continuous from above.

Example 4.8. Let $X = (-\infty, \infty)$. A set function $\pi : \mathbf{P}(X) \to [0, 1]$ is defined by

$$\pi(E) = \begin{cases} 1 & \text{if } E \neq \emptyset, \\ 0 & \text{if } E = \emptyset \end{cases}$$

for any $E \in \mathbf{P}(X)$. Clearly, π is maxitive and $\pi(X) = 1$; therefore it is a possibility measure on $\mathbf{P}(X)$. But it is not continuous from above. In fact, if we take $E = (0,1/n)$, then $\{E_n\}$ is decreasing, and $\bigcap_{n=1}^{\infty} E_n = \emptyset$. We have $\pi(E_n) = 1$ for all $n = 1, 2, \ldots$, but $\pi(\emptyset) = 0$. So $\lim_n \pi(E_n) \neq \pi(\bigcap_{n=1}^{\infty} E_n)$.

Theorem 4.23. *If f is the possibility profile of a possibility measure π, then*

$$\sup_{x \in X} f(x) = 1. \tag{4.17}$$

Conversely, if a function $f : X \to [0, 1]$ *satisfies (4.17), then f can determine a possibility measure π uniquely, and f is the possibility profile of π.*

Proof. From (4.15), we have

$$\begin{aligned}\sup_{x \in X} f(x) &= \sup_{x \in X} \pi(\{\pi\}) \\ &= \pi\Big(\bigcup_{x \in X} \{x\}\Big) \\ &= \pi(X) \\ &= 1.\end{aligned}$$

Conversely, let

$$\pi(E) = \sup_{x \in E} f(x)$$

for any $E \in \mathbf{P}(X)$, then π is a possibility measure, and

$$\pi(\{x\}) = \sup_{x \in \{x\}} f(x) = f(x).$$

□

A similar result can be easily obtained for generalized possibility measures: Any function $f : X \to [0, \infty)$ uniquely determines a generalized possibility measure π on $\mathbf{P}(X)$ by

$$\pi(E) = \sup_{x \in E} f(x) \text{ for any } E \in \mathbf{P}(X).$$

Definition 4.14. A basic probability assignment is called *consonant* iff it focuses on a *nest* (that is, a class fully ordered by the inclusion relation of sets).

Theorem 4.24. *Let X be finite. Then any possibility measure is a plausibility measure, and the corresponding basic probability assignment is consonant. Conversely, the plausibility measure induced by a consonant basic probability assignment is a possibility measure.*

Proof. Let $X = \{x_1, \ldots, x_n\}$ and π be a possibility measure. There is no loss of generality in assuming

$$1 = \pi(\{x_1\}) \geq \pi(\{x_2\}) \geq \cdots \geq \pi(\{x_n\}).$$

Define a set function m on $\mathbf{P}(X)$ by

$$m(E) = \begin{cases} \pi(\{x_i\}) - \pi(\{x_{i+1}\}) & \text{if } E = F_i, i = 1, \ldots, n-1 \\ \pi(\{x_n\}) & \text{if } E = F_n \\ 0 & \text{otherwise,} \end{cases}$$

where $F_i = \{x_1, \ldots, x_i\}, i = 1, \ldots, n$. Then m is a basic probability assignment focusing on $\{F_1, \ldots, F_n\}$, which is a nest. The plausibility measure induced from this basic probability assignment m is just π. Conversely, let m be a basic probability assignment focusing on a nest $\{F_1, \ldots, F_k\}$ that satisfies $F_1 \subset F_2 \subset \cdots \subset F_k$ and Pl be the plausibility measure induced from m. For any $E_1 \in \mathbf{P}(X), E_2 \in \mathbf{P}(X)$, denote

$$j_0 = \min\{j | F_j \cap (E_1 \cup E_2) \neq \emptyset\},$$

and

$$j_{0i} = \min\{j | F_j \cap E_i \neq \emptyset\}, i = 1, 2.$$

Then we have

$$\begin{aligned} \mathrm{Pl}(E_1 \cup E_2) &= \sum_{F_j \cap (E_1 \cup E_2) \neq \emptyset} m(F_j) \\ &= \sum_{j \geq j_0} m(F_j) \\ &= \left[\sum_{j \geq j_{01}} m(F_j)\right] \vee \left[\sum_{j \geq j_{02}} m(F_j)\right] \\ &= \left[\sum_{F_j \cap E_1 \neq \emptyset} m(F_j)\right] \vee \left[\sum_{F_j \cap E_2 \neq \emptyset} m(F_j)\right] \\ &= \mathrm{Pl}(E_1) \vee \mathrm{Pl}(E_2). \end{aligned}$$

That is, Pl satisfies (4.16) on $\mathbf{P}(X)$. So, Pl is a possibility measure. □

Example 4.9. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, π be a possibility measure on $(X, \mathbf{P}(X))$ with a possibility profile $f(x) = \pi(\{x\}), x = x_1, \ldots, x_5$, as follows:

$$f(x_1) = 1, \; f(x_2) = 0.9, \; f(x_3) = 0.5, \; f(x_4) = 0.5, \; f(x_5) = 0.3.$$

The corresponding basic probability assignment m focuses on four subsets of $X: F_1 = \{x_1\}, F_2 = \{x_1, x_2\}, F_4 = \{x_1, x_2, x_3, x_4\}$, and $F_5 = X$, with $m(F_1) = 0.1, m(F_2) = 0.4, m(F_4) = 0.2$, and $m(F_5) = 0.3$. This is illustrated in Fig. 4.3. $\{F_1, F_2, F_4, F_5\}$ forms a nest. In this example, $m(F_3) = m(\{x_1, x_2, x_3\}) = 0$.

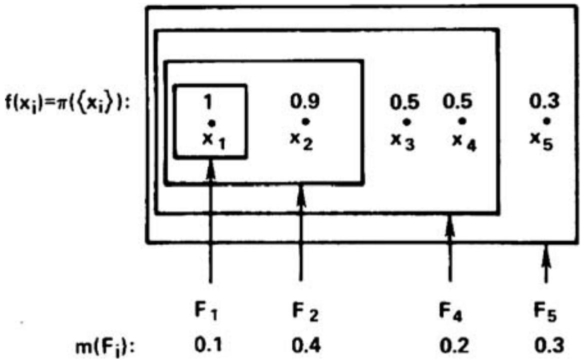

Fig. 4.3 A possibility profile on a finite space and the corresponding basic probability assignment

When X is not finite, a possibility measure on $\mathbf{P}(X)$ may not be a plausibility measure even when X is countable.

Example 4.10. Let X be the set of all rational numbers in [0, 1] and $f(x) = x, \forall x \in X$. X is a countable set. Define a set function π on $\mathbf{P}(X)$ as follows:

$$\pi(E) = \sup_{x \in E} f(x), \forall E \in \mathbf{P}(X).$$

Then, π is a possibility measure on $\mathbf{P}(X)$, but it is not a plausibility measure.

Definition 4.15. If π is a possibility measure on $\mathbf{P}(X)$, then its dual set function ν, which is defined by

$$\nu(E) = 1 - \pi(\bar{E}) \text{ for any } E \in \mathbf{P}(X)$$

is called a *necessity measure* (or *consonant belief measure*) on $\mathbf{P}(X)$.

Theorem 4.25. *A set function* $\nu : \mathbf{P}(X) \to [0, 1]$ *is a necessity measure if and only if it satisfies*

$$\nu\left(\bigcap_{t \in T} E_t\right) = \inf_{t \in T} \nu(E_t),$$

for any subclass $\{E_t | t \in T\}$ *of* $\mathbf{P}(X)$*, where* T *is an index set, and* $\nu(\emptyset) = 0$.

Proof. From Definitions 4.13 and 4.15, the conclusion is easy to obtain. □

Theorem 4.26. *Any necessity measure is an upper semicontinuous monotone measure. Moreover, if X is finite, then any necessity measure is a special example of belief measure and the corresponding basic probability assignment is consonant.*

Proof. The conclusion follows directly from Definition 4.15, Theorem 4.16, and Theorem 4.24. □

4.7 Properties of Finite Monotone Measures

In this section, we take a σ-ring **F** as the class **C**.

Theorem 4.27. *If μ is a finite monotone measure, then we have*

$$\lim_n \mu(E_n) = \mu(\lim_n E_n)$$

for any sequence $\{E_n\} \subset \mathbf{F}$ whose limit exists.

Proof. Let $\{E_n\}$ be a sequence of sets in **F** whose limit exists. Write $E = \lim_n E_n = \limsup_n E_n = \liminf_n E_n$. By applying the finiteness of μ, we have

$$\begin{aligned}\mu(E) = & \ \mu(\limsup_n E_n) = \lim_n \mu(\bigcup_{i=n}^{\infty} E_i) = \limsup_n \mu(\bigcup_{i=n}^{\infty} E_i) \\ \geq & \ \limsup_n \mu(E_n) \ \geq \liminf_n \ \mu \ (E_n) \\ \geq & \ \liminf_n \mu\left(\bigcap_{i=n}^{\infty} E_i\right) = \mu(\liminf_n E_n) = \mu(E)\end{aligned}$$

Therefore, $\lim_n \mu(E_n)$ exists and

$$\lim_n \mu(E_n) = \mu(E)$$

□

Definition 4.16. μ is *exhaustive* iff

$$\lim_n \mu(E_n) = 0$$

for any disjoint sequence $\{E_n\}$ of sets in **F**.

Theorem 4.28. *If μ is a finite upper semicontinuous monotone measure, then it is exhaustive.*

Proof. Let $\{E_n\}$ be a disjoint sequence of sets in **F**. If we write $F_n = \bigcup_{i=n}^{\infty} E_i$, then $\{F_n\}$ is a decreasing sequence of sets in **F**, and

$$\lim_n F_n = \bigcap_{n=1}^{\infty} F_n = \limsup_n E_n = \emptyset.$$

Since μ is a finite upper semicontinuous monotone measure, by using the finiteness and the continuity from above of μ, we have

$$\lim_n \mu(F_n) = \mu(\mathit{lim}_n F_n) = \mu(\emptyset) = 0.$$

Noting that

$$0 \leq \mu(E_n) \leq \mu(F_n),$$

we obtain

$$\lim_n \mu(E_n) = 0.$$

So, μ is exhaustive. □

Corollary 4.6. *Any finite monotone measure on a measurable space is exhaustive.*

Notes

4.1. The special nonadditive measures that are now called Choquet capacities were introduced by Gustave Choquet in the historical context outlined in Chapter 1. After their introduction [Choquet, 1953–54], they were virtually ignored for almost twenty years. They became a subject of interest of a small group of researchers in the early 1970s, primarily in the context of statistics. Among them, Peter Huber played an important role by recognizing that Choquet capacities are useful in developing *robust statistics* [Huber, 1972, 1973, 1981, Huber and Strassen, 1973]. Another researcher in this group, Anger [1971, 1977], focused more on further study of mathematical properties of Choquet capacities. It seems that the interest of these researchers in Choquet capacities was stimulated by an important earlier work of Dempster on upper and lower probabilities [Dempster, 1967a,b, 1968a,b]. Although Dempster was apparently not aware of Choquet capacities (at least he does not refer to the seminal paper by Choquet in his papers), the mathematical structure he developed for dealing with upper and lower probabilities is closely connected with Choquet capacities. It is well documented that Dempster's work on upper and lower probabilities also stimulated in the 1970s the development of *evidence theory*, which is based on ∞-monotone and ∞-alternating measures (Note 4.5). References to Choquet capacities in the literature have increased significantly since the late 1980s, primarily within the emerging areas of imprecise probabilities [Kyburg, 1987, Chateauneuf and Jaffray, 1989, De Campos and Bolanos, 1989, Wasserman and Kadane, 1990, 1992, Grabisch et al., 1995, Kadane and Wasserman, 1996, Walley, 1991].

4.2. The class of λ-measures was introduced and investigated by Sugeno [1974, 1977]. The fact that any λ-measure can be induced from a classical measure was shown by Wang [1981]. λ-measures were also investigated by Kruse [1980, 1982ab, 1983], Banon [1981], and Wierzchon [1982, 1983].

4.3. The concept of *quasi-measures* (often referred to in the literature as *pseudo-additive measures*) was introduced and investigated by Wang [1981]. Important examples of quasi-measures are special monotone measures that are called *decomposable measures*. These are normalized monotone measures, $\mu_\perp$, on measurable space $(X, \mathbf{C})$ that are semicontinuous from below and satisfy the property $\mu_\perp(A \cup B) = \perp[\mu_\perp(A), \mu_\perp(B)]$ for all $A, B, A \cup B \in \mathbf{C}$ such that $A \cap B = \emptyset$. Symbol $\perp$ denotes here a function from $[0,1]^2$ to [0,1] that qualifies as a triangular conorm (or t-conorm) [Klement, et al., 2000] and **C** is usually a σ-algebra. Since decomposable measures are not covered in this book, the following are some useful references for their study: [Dubois and Prade, 1982, Weber, 1984, Chateauneuf, 1996, Pap, 1997a,b, 1999, 2002b, Grabisch, 1997d].

4.4. In an early paper, Banon [1981] presents a comprehensive overview of the various types of monotone measures (defined on finite spaces) and discusses their classification. Lamata and Moral [1989] continue this discussion by introducing a classification of pairs of dual monotone measures. This classification is particularly significant in the area of imprecise probabilities, where one of the dual measures represents the lower probability and the other one the upper probability.

4.5 A theory based upon belief and plausibility measures was originated and developed by Shafer [1976]. Its emergence was motivated by previous work on lower and upper probabilities by Dempster [1967a,b, 1968a,b], as well as by Shafer's historical reflection upon the concept of probability [Shafer, 1978] and his critical examination of the Bayesian treatment of evidence [Shafer, 1981]. The theory is now usually referred to as the *Dempster–Shafer theory of evidence* (or just *evidence theory*). Although the seminal book by Shafer [1976] is still the best introduction to the theory (even though it is restricted to finite sets), several other books devoted to the theory, which are more up-to-date, are now available: [Guan and Bell, 1991–92, Kohlas and Monney, 1995, Kramosil, 2001, Yager et al., 1994]. There are too many articles dealing with the theory and its applications to be listed here, but most of them can be found in reference lists of the mentioned books and in two special journal issues devoted to the theory: *Intern. J. of Approximate Reasoning,* **31**(1–2), 2002, pp. 1–154, and *Intern. J. of Intelligent Systems*, **18**(1), 2003, pp. 1–148. The theory is well covered from different points of view in articles by Shafer [1979, 1981, 1982, 1990], Höhle [1982], Dubois and Prade [1985, 1986a], Walley [1987], Smets [1988, 1992, 2002], and Smets and Kennes [1994]. Possible ways of fuzzifying the theory are suggested by Höhle [1984], Dubois and Prade [1985], and Yen [1990]. Axiomatic characterizations of comparative belief structures, which are generalizations of comparative probability structures [Walley and Fine, 1979], were formulated by Wong, Yao, and Bollmann [1992].

4.6. A mathematical theory that is closely connected with Dempster-Shafer theory, but which is beyond the scope of this book, is the *theory of random sets*. Random sets were originally conceived in connection with stochastic geometry. They were proposed in the 1970s independently by two authors, Kendall [1973, 1974] and Matheron [1975]. The connection of random sets with belief measures is examined by Nguyen [1978] and Smets [1992], and it is also the subject of several articles in a book edited by Goutsias et al. [1997]. A recent book by Molchanov [2005] is currently the most comprehensive and up-to-date reference for the theory and applications of random sets. A good introduction to random sets was written by Nguyen [2006].

4.7. Possibility measures were introduced in several different contexts. In the late 1940s the British economist George Shackle introduced possibility measures indirectly, via monotone decreasing set functions that he called measures of *potential surprise* [Shackle, 1949]. He argued that these functions are essential in dealing with uncertainty in economics [Shackle 1955, 1961]. As shown by Klir [2002], measures of potential surprise can be reformulated in terms of monotone increasing measures—possibility measures. In the late 1970s possibility measures were introduced in two very different contexts: the context of fuzzy sets [Zadeh, 1978] and the context of plausibility measures [Shafer, 1976, 1987]. The literature on the theory based on possibility measures (and their dual necessity measures) is now very extensive. An early book by Dubois and Prade [1988] is a classic in this area. More recent developments in the theory are covered in a text by Kruse et al. [1994] and in monographs by Wolkenhauer [1998] and Borgelt and Kruse [2002]. Important sources are also edited books by De Cooman et al. [1995] and Yager [1982]. A sequence of three papers by De Cooman [1997] is perhaps the most comprehensive and general treatment of possibility theory. Thorough surveys of possibility theory with extensive bibliographies were written by Dubois et al. [1998, 2000].

4.8. An interesting connection between modal logic [Chellas, 1980; Hughes and Cresswell, 1996] and the various nonadditive measures is suggested in papers by Resconi et al. [1992, 1993]. Modal logic interpretation of belief and plausibility measures on finite sets is studied in detail by Harmanec et al. [1994] and Tsiporkova et al. [1999], and on infinite sets by Harmanec et al. [1996]. A modal logic interpretation of possibility theory is established in a paper by Klir and Harmanec [1994].

Exercises

4.1. Consider the monotone measures μ_i $(i = 1, 2, \ldots, 9)$ on $(X, \mathbf{P}(X))$, where $X = \{a, b, c\}$, which are defined in Table 4.2. Determine for each of these measures the following:

Table 4.2 Monotone measures in Exercises 4.1. and 4.2

	a	b	c	$\mu_1(A)$	$\mu_2(A)$	$\mu_3(A)$	$\mu_4(A)$	$\mu_5(A)$	$\mu_6(A)$	$\mu_7(A)$	$\mu_8(A)$	$\mu_9(A)$
A:	0	0	0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	1	0	0	0.0	0.2	0.4	0.2	0.0	1.0	0.2	0.3	0.2
	0	1	0	0.0	0.2	0.2	0.3	0.0	1.0	0.0	0.1	0.3
	0	0	1	0.0	0.2	0.0	0.4	0.0	1.0	0.0	0.3	0.4
	1	1	0	0.7	0.6	0.5	0.6	1.0	1.0	0.5	0.3	0.6
	1	0	1	0.8	0.6	0.6	0.6	1.0	1.0	0.2	0.6	0.7
	0	1	1	0.9	0.4	0.5	0.7	1.0	1.0	0.0	1.0	0.8
	1	1	1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

(a) Is the measure superadditive or subadditive?
(b) Is the measure 2-monotone or 2-alternating?
(c) Is the measure a belief measure or a plausibility measure?
(d) Is the measure a possibility measure or a necessity measure?

4.2. Determine the dual measures for each of the measures in Exercise 4.1, and answer for each of them the questions stated in Exercise 4.1.

4.3. Check for each of the following set functions whether it is a λ-measure. If the answer is affirmative, determine the parameter λ.

(a) $X = \{a, b\}, \mathbf{F} = \mathbf{P}(X)$, and μ is given by $\mu(\emptyset) = 0, \mu(\{a\}) = 1/2$, $\mu(\{b\}) = 3/4, \mu(X) = 1$.
(b) $X = \{a, b\}, \mathbf{F} = \mathbf{P}(X)$, and μ is given by $\mu(\emptyset) = 0, \mu(\{a\}) = 1/2$, $\mu(\{b\}) = 1/3, \mu(X) = 1$.
(c) $X = \{a, b, c\}, \mathbf{F} = \mathbf{P}(X)$, and μ is given by

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E = \emptyset \\ 1/2 & \text{otherwise} \end{cases}$$

for any $E \in \mathbf{F}$
(d) $X = \{a, b, c\}, \mathbf{F} = \mathbf{P}(X)$, and μ is given by

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{otherwise} \end{cases}$$

for any $E \in \mathbf{F}$.

4.4. Is any of the set functions defined in Exercise 4.3 a normalized λ-measure? For each that is a normalized λ-measure, determine the dual λ-measure as well as the value of the corresponding parameter λ.

4.5. Prove that the σ-λ-rule is equivalent to the continuity and the λ-rule for a nonnegative set function defined on a ring. Give an example to show that a similar conclusion need not be true on a semiring.

4.6. Let $X = \{x_1, x_2, x_3, x_4\}$, and $a_1 = 0.1, a_2 = 0.2, a_3 = 0.3, a_4 = 0.4$. Find the λ-measure, g_λ, defined on $(X, \mathbf{P}(X))$ and subject to $g_\lambda(\{x_i\}) = a_i$, $i = 1, 2, 3, 4$, for each of the following values of parameter λ:

(a) $\lambda = 5$; (b) $\lambda = 2$; (c) $\lambda = 1$; (d) $\lambda = 0$; (e) $\lambda = -1$; (f) $\lambda = -2$; (g) $\lambda = -2.4$.

Can you use $\lambda = -2.5$ or $\lambda = -5$ to find a λ-measure satisfying the above-mentioned requirement? Justify your answer.

4.7. Prove the following: If μ is a Dirac measure on $(X, \mathbf{F})$, then μ is a Sugeno measure for any $\lambda \in (-1/\sup\mu, \infty) \cup \{0\}$; conversely, if X is countable, $\mathbf{F} = \mathbf{P}(X)$, and μ is a Sugeno measure on $(X, \mathbf{F})$ for two different parameters λ and λ', then μ is a Dirac measure.

4.8. Let $X = \{a, b, c\}$ and $\mu(\{a\}) = 0.25, \mu(\{b\}) = \mu(\{c\}) = 0.625, \mu(X) = 1$. Viewing μ as a λ-measure, determine the value of the associated parameter λ.

4.9. Let $X = \{a, b\}, \mathbf{F} = \mathbf{P}(X)$, and let m be a measure on $(X, \mathbf{F})$ defined by

$$m(E) = \begin{cases} 1 & \text{if } E = X \\ 3/4 & \text{if } E = \{b\} \\ 1/4 & \text{if } E = \{a\} \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Find a quasi-measure μ by using $\theta(y) = \sqrt{y}, y \in [0, 1]$, as its proper T-function. Is there any other T-function (say θ') such that $\mu = \theta' \circ m$? If you find any such T-functions, what can you conclude from them?

4.10. Let $X = \{a_1, a_2\}$ and μ be a nonnegative set function of $\mathbf{P}(X)$. Show that if $0 = \mu(\emptyset) < \mu(a_i) < \mu(X) < \infty, i = 1, 2, \ldots$, then μ is a quasi-measure.

4.11. Let $X = \{a, b, c, d\}$ and let $m(\{a\}) = 0.4, m(\{b, c\}) = 0.1, m(\{a, c, d\}) = 0.3, m(X) = 0.2$ be a basic probability assignment. Determine the corresponding belief measure and plausibility measure.

4.12. Repeat Exercise 4.11 for each of the basic probability assignments given in Table 4.3, where subsets of X are defined by their characteristic functions.

4.13. Determine which basic probability assignments given in Table 4.3 are consonant.

4.14. Determine which basic probability assignments given in Table 4.3 induce a discrete probability measure on $(X, \mathbf{P}(X))$.

4.15. Given $X = \{a, b, c, d\}$,

$\text{Bel}(\emptyset) = \text{Bel}(\{b\}) = \text{Bel}(\{c\}) = \text{Bel}(\{d\}) = \text{Bel}(\{b, d\}) = \text{Bel}(\{c, d\}) = 0,$

$\text{Bel}(\{a\}) = \text{Bel}(\{a,b\}) = \text{Bel}(\{a,c\}) = \text{Bel}(\{a,d\}) = \text{Bel}(\{a,b,d\}) = 0.1,$

Table 4.3 Basic probability assignments employed in Exercises 4.12–4.14

a	b	c	d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0.2	0	0	0.2	0.2	0	0.05	0	0	0
0	0	1	0	0	0.4	0	0	0.2	0	0.05	0	0	0
0	0	1	1	0	0	0	0.1	0	0	0.05	0	0	0
0	1	0	0	0	0.5	0	0	0.3	1	0.05	0.2	0	0.9
0	1	0	1	0	0	0	0	0	0	0.05	0	0	0
0	1	1	0	0.3	0	0	0	0	0	0.05	0	0	0
0	1	1	1	0	0	0	0	0	0	0.05	0.5	0	0
1	0	0	0	0.1	0.1	0.2	0	0.3	0	0.05	0	0	0.1
1	0	0	1	0	0	0	0	0	0	0.05	0	0	0
1	0	1	0	0.1	0	0.3	0	0	0	0.05	0	0	0
1	0	1	1	0	0	0	0	0	0	0.1	0	0	0
1	1	0	0	0	0	0	0	0	0	0.1	0	1	0
1	1	0	1	0.2	0	0	0	0	0	0.1	0	0	0
1	1	1	0	0.1	0	0.4	0	0	0	0.1	0	0	0
1	1	1	1	0	0	0.1	0.7	0	0	0.1	0.3	0	0

Table 4.4 Possibility profiles employed in Exercises 4.16 and 4.17

	f_1	f_2	f_3	f_4	f_5	f_6
a	1	1	0	0.9	1	1
b	0.8	1	1	0	1	1
c	0.4	0.2	0.3	0	1	0
d	0.1	0.6	0.3	1	1	0

$\text{Bel}(\{b, c\}) = \text{Bel}(\{b, c, d\}) = 0.2, \text{Bel}(\{a, b, c\}) = 0.3, \text{Bel}(\{a, c, d\}) = 0.4,$ $\text{Bel}(X) = 1$, determine the corresponding basic probability assignment.

4.16. Let $X = \{a, b, c, d\}$. Use each of the possibility profiles given in Table 4.4 to determine the corresponding possibility measures and basic probability assignments.

4.17. Determine the dual necessity measure for each possibility measure obtained in Exercise 4.16.

4.18. Find an example that illustrates that a possibility measure defined on an infinite space need not be a plausibility measure.