Chapter 7

Strong Convergence Theorems

In this chapter, we prove convergence theorems for approximants of self-mappings and non-self mappings in Banach spaces. We also study a Halpern's type iteration process for approximation of fixed points of nonexpansive mappings in a Banach space with a uniformly Gâteaux differentiable norm.

7.1 Convergence of approximants of self-mappings

In this section, we study strong convergence of approximants of nonexpansive and asymptotically nonexpansive type self-mappings in Banach spaces.

First, we establish a fundamental strong convergence theorem for nonexpansive mappings in a Hilbert space.

Theorem 7.1.1 (Browder's convergence theorem) – *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let u be an element in C and $G_t : C \to C$, $t \in (0,1)$ the family of mappings defined by*

$$G_t x = (1-t)u + tTx, \quad x \in C.$$

Then the following hold:

(a) There is exactly one fixed point x_t of G_t, i.e.,

$$x_t = (1-t)u + tTx_t. \tag{7.1}$$

(b) The path $\{x_t\}$ converges strongly to Pu as $t \to 1$, where P is the metric projection mapping from C onto $F(T)$.

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Proof. (a) Note for each $t \in (0,1), G_t$ is a contraction mapping of C into itself. Hence G_t has a unique fixed point x_t in C.

(b) Because $F(T)$ is a nonempty closed convex subset of C, there exists an element $u_0 \in F(T)$ that is the nearest point of u. By boundedness of $\{x_t\}$, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup z \in C$. Write $x_{t_n} = x_n$. Because $x_n - Tx_n \to 0$, it follows that $z = Tz$. Indeed, for $z \neq Tz$

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - z\| &< \limsup_{n\to\infty} \|x_n - Tz\| \\ &\leq \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|Tx_n - Tz\|) \\ &\leq \limsup_{n\to\infty} \|x_n - z\|,\end{aligned}$$

a contradiction, because H has the Opial condition. Observe that

$$(1-t_n)x_n + t_n(x_n - Tx_n) = (1-t_n)u$$

and

$$(1-t_n)u_0 + t_n(u_0 - Tu_0) = (1-t_n)u_0.$$

Subtracting and taking the inner product of the difference with $x_n - u_0$, we get

$$\begin{aligned}(1-t_n)\langle x_n - u_0, x_n - u_0\rangle &+ t_n\langle Ux_n - Uu_0, x_n - u_0\rangle \\ &= (1-t_n)\langle u - u_0, x_n - u_0\rangle,\end{aligned}$$

where $U = I - T$. Because $U = I - T$ is monotone, $\langle Ux_n - Uu_0, x_n - u_0\rangle \geq 0$, it follows that

$$\|x_n - u_0\|^2 \leq \langle u - u_0, x_n - u_0\rangle \text{ for all } n \in \mathbb{N}.$$

Because $u_0 \in F(T)$ is the nearest point to u,

$$\langle u - u_0, z - u_0\rangle \leq 0,$$

which gives

$$\begin{aligned}\|x_n - u_0\|^2 &\leq \langle u - u_0, x_n - u_0\rangle \\ &= \langle u - u_0, x_n - z\rangle + \langle u - u_0, z - u_0\rangle \\ &\leq \langle u - u_0, x_n - z\rangle.\end{aligned}$$

Thus, from $x_n \rightharpoonup z$, we obtain $x_n \to u_0$ as $n \to \infty$. We show that $x_t \to u_0$ as $t \to 1$, i.e., u_0 is the only strong cluster point of $\{x_t\}$. Suppose, for contradiction, that $\{x_{t_{n'}}\}$ is another subsequence of $\{x_t\}$ such that $x_{t_{n'}} \to v \neq u_0$ as $n' \to \infty$. Set $x_{n'} := x_{t_{n'}}$. Because $x_{n'} - Tx_{n'} \to 0$, it follows that $v \in F(T)$. From (7.1), we have

$$x_t - Tx_t = (1-t)(u - Tx_t). \tag{7.2}$$

Because for $y \in F(T)$

$$\begin{aligned}\langle x_t - Tx_t, x_t - y\rangle &= \langle x_t - Ty + Ty - Tx_t, x_t - y\rangle \\ &= \|x_t - y\|^2 - \langle Tx_t - Ty, x_t - y\rangle \\ &\geq 0,\end{aligned}$$

this gives from (7.2) that $\langle u - Tx_t, x_t - y\rangle \geq 0$. Thus, $\langle x_t - u, x_t - y\rangle \leq 0$ for all $t \in (0,1)$ and $y \in F(T)$. It follows that

$$\langle u_0 - u, u_0 - v\rangle \leq 0 \text{ and } \langle v - u, v - u_0\rangle \leq 0,$$

which imply that $u_0 = v$, a contradiction. Therefore, $\{x_t\}$ converges strongly to Pu, where P is metric projection mapping from C onto $F(T)$. ∎

We now prove strong convergence of path $\{x_t\}$ in a more general situation.

Proposition 7.1.2 *Let C be a nonempty subset of a Banach space X and $T : C \to X$ a pseudocontractive mapping such that for some $u \in C$, the equation*

$$x = (1-t)u + tTx \tag{7.3}$$

has a unique solution x_t in C for each $t \in (0,1)$. If $F(T) \neq \emptyset$, there exists $j(x_t - v) \in J(x_t - v)$ such that

$$\langle x_t - u, j(x_t - v)\rangle \leq 0 \textit{ for all } v \in F(T) \textit{ and } t \in (0,1).$$

Proof. From (7.3) we have

$$x_t - Tx_t = (1-t)(u - Tx_t) \text{ for all } t \in (0,1).$$

For $y \in F(T)$, there exists $j(x_t - y) \in J(x_t - y)$ such that

$$\begin{aligned}\langle x_t - Tx_t, j(x_t - y)\rangle &= \langle x_t - Ty + Ty - Tx_t, j(x_t - y)\rangle \\ &= \|x_t - y\|^2 - \langle Tx_t - Ty, j(x_t - y)\rangle \\ &\geq 0,\end{aligned}$$

which implies that

$$\langle u - Tx_t, j(x_t - y)\rangle \geq 0.$$

It follows from (7.3) that

$$\langle x_t - u, j(x_t - y)\rangle \leq 0 \text{ for all } y \in F(T) \text{ and } t \in (0,1).$$ ∎

Theorem 7.1.3 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to X$ a demicontinuous pseudocontractive mapping such that for some $u \in C$, the equation defined by (7.3) has a unique solution x_t in C for each $t \in (0,1)$. If the path $\{x_t\}$ is bounded, then it converges strongly to a fixed point of T as $t \to 1$.*

Proof. Because $\{x_t\}$ is bounded, $\{Tx_t\}$ is bounded by (7.3) and

$$\|x_t - Tx_t\| \;=\; (1-t)\|u - Tx_t\| \le (1-t)\text{diam}(\{u - Tx_t\}) \to 0.$$

Because X is reflexive and $\{x_t\}$ is bounded, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup v$ as $t_n \to 1$. Write $x_{t_n} := x_n$. Because $(t^{-1} - 1)x_t = (t^{-1} - 1)u + Tx_t - x_t$, it follows that

$$\begin{aligned} \langle (t_n^{-1} - 1)x_n \;&-\; (t_m^{-1} - 1)x_m, J(x_n - x_m)\rangle \\ &= (t_n^{-1} - t_m^{-1})\langle u, J(x_n - x_m)\rangle \\ &\quad +\langle Tx_n - Tx_m - (x_n - x_m), J(x_n - x_m)\rangle \\ &\le (t_n^{-1} - t_m^{-1})\langle u, J(x_n - x_m)\rangle \text{ for all } n, m \in \mathbb{N}. \end{aligned}$$

Taking the limit as $m \to \infty$, we obtain

$$\langle (t_n^{-1} - 1)x_n, J(x_n - v)\rangle \le (t_n^{-1} - 1)\langle u, J(x_n - v)\rangle,$$

and thus,

$$\langle x_n - u, J(x_n - v)\rangle \le 0.$$

Hence

$$\|x_n - v\|^2 = \langle x_n - v, J(x_n - v)\rangle = \langle x_n - u, J(x_n - v)\rangle + \langle u - v, J(x_n - v)\rangle.$$

Therefore, $x_n \to v$ as $n \to \infty$. Because $Tx_n \to v$ by $x_n - Tx_n \to 0$, it follows from the demicontinuity of T that $v \in F(T)$.

We show that v is the only strong cluster point of $\{x_t\}$. Suppose, for contradiction, that $\{x_{t_{n'}}\}$ is another subsequence of $\{x_t\}$ such that $x_{t_{n'}} \to w\ (\neq v)$ as $t_{n'} \to 1$. It can be easily seen that $w = Tw$. Thus, from Proposition 7.1.2, we have

$$\langle x_{t_n} - u, J(x_n - w)\rangle \le 0 \text{ and } \langle x_{t_{n'}} - u, J(x_{t_{n'}} - v)\rangle \le 0$$

which imply that

$$\langle v - u, J(v - w)\rangle \le 0 \text{ and } \langle w - u, J(w - v)\rangle \le 0.$$

Hence

$$\|u - w\|^2 = \langle v - w, J(v - w)\rangle = \langle v - u, J(v - w)\rangle + \langle u - w, J(v - w)\rangle \le 0,$$

a contradiction. Therefore, $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$. ∎

Corollary 7.1.4 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to X$ a nonexpansive mapping such that for some $u \in C$, the equation (7.3) has a unique solution x_t in C for each $t \in (0, 1)$. If the path $\{x_t\}$ is bounded, then it converges strongly to a fixed point of T as $t \to 1$.*

Applying Theorem 7.1.3, we obtain

Theorem 7.1.5 *Let X be a reflexive Banach space with a weakly continuous duality mapping, C a nonempty closed convex bounded subset of X, u an element in C, and $T : C \to C$ a continuous pseudocontractive mapping. Then the following hold:*

(a) For each $t \in (0,1)$, there exists exactly one $x_t \in C$ such that

$$x_t = (1-t)u + tTx_t. \tag{7.4}$$

(b) $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$.

Proof. (a) For each $t \in (0,1)$, define $G_t : C \to C$ by

$$G_t x = (1-t)u + tTx, \ x \in C.$$

Then G_t is well defined because $u \in C$ and $T(C) \subset C$. Because for each $t \in (0,1)$, G_t is strongly pseudocontractive, it follows from Corollary 5.7.15 that G_t has exactly one fixed point $x_t \in C$.

(b) It follows from Theorem 7.1.3. ∎

Corollary 7.1.6 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$, C a nonempty closed convex bounded subset of X, and $T : C \to C$ a continuous pseudocontractive mapping. Then $F(T)$ is a sunny nonexpansive retract of C.*

Proof. For each $u \in C$, by Theorem 7.1.5, there is a unique path $\{x_t\}$ defined by (7.4) such that $\lim_{t\to 1} x_t = v \in F(T)$. Then there exists a mapping P from C onto $F(T)$ defined by $Pu = \lim_{t\to 1} x_t$, as u is an arbitrary element of C.

Because

$$\langle x_t - u, J(x_t - y)\rangle \le 0 \text{ for all } y \in F(T) \text{ and } t \in (0,1),$$

this implies that

$$\langle Pu - u, J(Pu - y)\rangle \le 0 \text{ for all } u \in C, y \in F(T).$$

Therefore, by Proposition 2.10.21, P is the sunny nonexpansive retraction from C onto $F(T)$. ∎

Next, we study a strong convergence theorem for the following more general class of mappings:

Definition 7.1.7 *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. Then T is said to be asymptotically pseudocontractive if for each $n \in \mathbb{N}$ and $x, y \in C$, there exist a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that $\langle T^n x - T^n y, j(x-y)\rangle \le k_n \|x-y\|^2$.*

We note that every asymptotically nonexpansive mapping is asymptotically pseudocontractive, but the converse is not true. In fact, if T is asymptotically nonexpansive with domain $Dom(T)$ and sequence $\{k_n\}$, then for each $n \in \mathbb{N}$ and $x, y \in Dom(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq \|T^n x - T^n y\|\|x-y\| \leq k_n\|x-y\|^2.$$

Theorem 7.1.8 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to C$ a demicontinuous asymptotically pseudocontractive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of nonnegative numbers in $(0,1)$ such that $t_n \to 1$ and $\lim_{n\to\infty}(k_n-1)/(1-t_n) = 0$. Let $\{x_n\}$ be a bounded sequence in C with $x_n - Tx_n \to 0$ such that*

$$x_n = (1-t_n)u + t_n T^n x_n \text{ for all } n \in \mathbb{N}. \tag{7.5}$$

If $I-T$ is demiclosed at zero, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From (7.5), we have

$$x_n - T^n x_n = (1-t_n)(u - T^n x_n) \text{ and } t_n(u - T^n x_n) = u - x_n.$$

Thus, whenever $y \in F(T)$, we have

$$\begin{aligned}
(1-t_n)\langle u - T^n x_n, J(x_n - y)\rangle &= \langle x_n - T^n x_n, J(x_n-y)\rangle \\
&= \langle x_n - y + y - T^n x_n, J(x_n - y)\rangle \\
&= \|x_n - y\|^2 - \langle T^n x_n - T^n y, J(x_n-y)\rangle \\
&\geq -(k_n - 1)\|x_n - y\|^2,
\end{aligned}$$

which yields

$$\langle x_n - u, J(x_n - y)\rangle \leq \frac{k_n - 1}{1-t_n}\|x_n - y\|^2 \leq \frac{k_n-1}{1-t_n}K \tag{7.6}$$

for some $K \geq 0$.

Because X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Because $I-T$ is demiclosed at zero, $v = Tv$. Hence

$$\begin{aligned}
\|x_{n_i} - v\|^2 &= \langle x_{n_i} - v, J(x_{n_i} - v)\rangle \\
&= \langle x_{n_i} - u, J(x_{n_i} - v)\rangle + \langle u - v, J(x_{n_i} - v)\rangle \\
&\leq \frac{k_{n_i} - 1}{1 - t_{n_i}}K + \langle u - v, J(x_{n_i} - v)\rangle.
\end{aligned}$$

From $J(x_{n_i} - v) \to^* 0$ and $(k_{n_i} - 1)/(1 - t_{n_i}) \to 0$, we get $x_{n_i} \to v$.

We now show that v is only strong cluster point of $\{x_n\}$. Suppose, for contradiction, that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ such that $x_{n_j} \to w \in C$. Because $x_{n_j} - Tx_{n_j} \to 0$, it follows that $Tx_{n_j} \to w$. By demicontinuity of T, we have that $Tx_{n_k} \rightharpoonup Tw$. Hence $Tw = w$. From (7.6), we have

$$\langle v-u, J(v-w)\rangle \leq 0 \text{ and } \langle w-u, J(w-v)\rangle \leq 0,$$

which imply that

$$\|v-w\|^2 = \langle v-w, J(u-w)\rangle = \langle v-u, J(v-w)\rangle + \langle u-w, J(v-w)\rangle \leq 0,$$

a contradiction. Therefore, $\{x_n\}$ converges strongly to a fixed point of T. ∎

Corollary 7.1.9 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of real numbers in $(0,1)$ such that $t_n \to 1$ and $\lim_{n\to\infty}(k_n-1)/(1-t_n) = 0$. Let $\{x_n\}$ be a bounded sequence in C with $x_n - Tx_n \to 0$ such that $x_n = (1-t_n)u + t_nT^nx_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

The following result is very useful for strong convergence of AFPS of self-mappings as well as non-self mappings.

Theorem 7.1.10 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, $T : C \to X$ a demicontinuous mapping with $F(T) \neq \emptyset$, and $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}^+$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\{x_n\}$ a bounded sequence in C such that $x_n - Tx_n \to 0$ as $n \to \infty$ and*

$$\langle x_n - Ax_n, J(x_n-p)\rangle \leq \alpha_n\|x_n-p\|^2 \text{ for all } n \in \mathbb{N} \text{ and } p \in F(T). \tag{7.7}$$

Suppose the set $M = \{x \in C : LIM_n\|x_n - x\|^2 = \inf_{y\in C} LIM\|x_n-y\|^2\}$ contains a fixed point of T, where LIM is a Banach limit. Then $\{x_n\}$ converges strongly to an element of $M \cap F(T)$.

Proof. By Theorem 2.9.11, M is a nonempty closed convex and bounded set. By assumption, T has a fixed point in M. Denote such a fixed point by v. It follows from Corollary 2.9.13 that

$$LIM_n\langle z, J(x_n-v)\rangle \leq 0 \text{ for all } x \in C.$$

In particular,

$$LIM_n\langle Av - v, J(x_n-v)\rangle \leq 0. \tag{7.8}$$

From (7.7), we obtain

$$LIM_n\langle x_n - Ax_n, J(x_n - v\rangle \leq 0. \tag{7.9}$$

Combining (7.8) and (7.9), we have

$$\begin{aligned} LIM_n\|x_n - v\|^2 &= LIM_n[\langle x_n - Ax_n, J(x_n - v)\rangle + \langle Ax_n - Av, J(x_n - v)\rangle \\ &\quad +\langle Av - v, J(x_n - v)\rangle] \\ &\leq kLIM_n\|x_n - v\|^2, \end{aligned}$$

i.e., $(1-k)LIM_n\|x_n - v\|^2 \leq 0$. Therefore, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges strongly to v. To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z$ as $j \to \infty$. Because $x_{n_j} - Tx_{n_j} \to 0$, it follows that $Tx_{n_j} \to z$. By demicontinuity of T, we have that $Tz = z$. From (7.7), we have

$$\langle v - Av, J(v - z)\rangle \leq 0 \text{ and } \langle z - Az, J(z - v)\rangle \leq 0.$$

Hence $z = v$. This proves that $\{x_n\}$ converges strongly to v. ∎

Corollary 7.1.11 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, and $T : C \to X$ a demicontinuous mapping with $F(T) \neq \emptyset$. Let u be an element in C, $\{\alpha_n\}$ a sequence in $\mathbb{R}^+$ with $\lim_{n\to\infty} \alpha_n = 0$, and $\{x_n\}$ a bounded sequence in C such that $x_n - Tx_n \to 0$ as $n \to \infty$ and*

$$\langle x_n - u, J(x_n - p)\rangle \leq \alpha_n\|x_n - p\|^2 \text{ for all } n \in \mathbb{N} \text{ and all } p \in F(T).$$

Suppose the set $M = \{x \in C : LIM_n\|x_n - x\|^2 = \inf_{y\in C} LIM\|x_n - y\|^2\}$ contains a fixed point of T, where LIM is a Banach limit. Then $\{x_n\}$ converges strongly to an element of $F(T)$.

We now prove a notable strong convergence theorem for nonexpansive mappings in a uniformly smooth Banach space.

Theorem 7.1.12 (Reich's convergence theorem) – *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, x an element in C, $T : C \to C$ a nonexpansive mapping, and $G_t : C \to C$, $t \in (0,1)$, the family of mappings defined by $G_t(x) = (1-t)x + tTG_t(x)$. If T has a fixed point, then for each $x \in C$, $\lim_{t\to 1} G_t(x)$ exists and is a fixed point of T.*

Proof. Let $\{t_n\}$ be a sequence of real numbers in $(0,1)$ such that $t_n \to 1$. Set $x_n := G_{t_n}(x)$. Because $F(T) \neq \emptyset$, it follows that $\{x_n\}$ is bounded and $x_n - Tx_n \to 0$ as $n \to \infty$. Then the set M defined by (2.32) is a nonempty closed convex bounded T-invariant subset of C (see Proposition 6.1.3). Note every uniformly smooth Banach space is reflexive and has normal structure. Hence every closed convex bounded set of X has fixed point property. Thus,

T has a fixed point in M. Observe that $\{x_n\}$ satisfies (7.7) with $\alpha_n = 0$ for all $n \in \mathbb{N}$ (see Proposition 7.1.2). It follows from Corollary 7.1.11 that $\{x_n\}$ converges strongly to an element of $F(T)$. ∎

Applying Corollary 7.1.11, we obtain

Theorem 7.1.13 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of real numbers in $(0,1)$ such that $t_n \to 1$ and $(k_n - 1)/(1 - t_n) \to 0$. Then the following hold:*

(a) There exists exactly one point $x_n \in C$ such that

$$x_n = (1 - t_n)u + t_n T^n x_n, \quad n \in \mathbb{N}.$$

(b) If $\{x_n\}$ is a bounded AFPS of T and $M = \{x \in C : LIM_n \|x_n - x\|^2 = \inf_{y \in C} LIM_n \|x_n - y\|^2\}$ contains a fixed point of T, then $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof. (a) Because $\lim_{n \to \infty} (k_n - 1)/(1 - t_n) = 0$, then there exists a sufficiently large natural number n_0 such that $k_n t_n < 1$ for all $n \geq n_0$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n x = (1 - t_n)u + t_n T^n x, \quad x \in C.$$

Because for each $n \geq n_0$, T_n is contraction, there exists exactly one fixed point $x_n \in C$ of T_n. We may assume that $x_n = u$ for all $n = 1, 2, \cdots, n_0 - 1$. Then

$$x_n = (1 - t_n)u + t_n T^n x_n \text{ for all } n \in \mathbb{N}.$$

(b) As in the proof of Theorem 7.1.8, it can be easily seen that $\{x_n\}$ satisfies the inequality (7.6). Note that M is a nonempty closed convex bounded set. Moreover, T has a fixed point in M by assumption.

Observe that
(i) (7.7) is satisfied with $\alpha_n = (k_n - 1)/(1 - t_n) \to 0$ as $n \to \infty$,
(ii) T has a fixed point in M,
(iii) $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Hence this part follows from Corollary 7.1.11. ∎

The following proposition shows that for a bounded AFPS, the set M satisfies the property (P) defined by (5.52).

Proposition 7.1.14 *Let C be a nonempty closed convex bounded subset of a reflexive Banach space X and $T : C \to C$ asymptotically nonexpansive. Let $\{x_n\}$ be an AFPS. Then the set M satisfies property (P).*

Proof. By Theorem 2.9.11, M is a nonempty closed convex bounded subset of C. Let $x \in M$. Because $\{T^m x\}$ is bounded in C, there exists a subsequence $\{T^{m_j} x\}$ of $\{T^m x\}$ such that $T^{m_j} x \rightharpoonup u \in C$. Let k_n be the Lipschitz constant of T^n. By w-lsc of the function $\varphi(z) = LIM_n \|x_n - z\|^2$, we have

$$\begin{aligned}
\varphi(u) &= \liminf_{j\to\infty} \varphi(T^{m_j} x) \\
&\le \limsup_{m\to\infty} \varphi(T^m x) \\
&= \limsup_{m\to\infty} (LIM_n \|x_n - T^m x\|^2) \\
&\le \limsup_{m\to\infty} (LIM_n (\|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{m-1} x_n - T^m x_n\| \\
&\quad + \|T^m x_n - T^m x\|)^2) \\
&\le \limsup_{m\to\infty} (LIM_n (k_m \|x_n - x\|))^2 \\
&= \varphi(x) = \inf_{z\in M} \varphi(z).
\end{aligned}$$

Thus, $u \in M$. Therefore, M has property (P). ∎

Applying Theorem 5.5.8 and Proposition 7.1.14, we obtain

Theorem 7.1.15 *Let C be a nonempty closed convex bounded subset of a uniformly smooth Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let $u \in C$ and $\{t_n\}$ a sequence in $(0,1)$ such that $t_n \to 1$ and $(k_n - 1)/(1 - t_n) \to 0$. Suppose the sequence $\{x_n\}$ defined by (7.5) is an AFPS of T. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Proposition 7.1.14, the set M has property (P). It follows from Theorem 5.5.8 that T has a fixed point in M. Therefore, by Theorem 7.1.13, $\{x_n\}$ converges strongly to an element of $F(T)$. ∎

7.2 Convergence of approximants of non-self mappings

In this section, we discuss strong convergence of approximants of non-self nonexpansive mappings.

The following theorem is an extension of Browder's strong convergence theorem for non-self nonexpansive mappings with unbounded domain.

Theorem 7.2.1 (Singh and Watson's convergence theorem) – *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a nonexpansive mapping such that $T(\partial C) \subseteq C$ and $T(C)$ is bounded. Let u be an element in C and define $G_t : C \to H$ by*

$$G_t x = (1-t)u + tTx, \quad x \in C \text{ and } t \in (0,1).$$

Let $x_t = G_t x_t$. Then $\{x_t\}$ converges strongly to v as $t \to 1$, where v is the fixed point of T closest to u.

Proof. Note $F(T)$ is nonempty by Theorem 5.2.25. Then for any $y \in F(T)$, we have

$$\|x_t - y\| \le \|u - y\| \text{ for all } t \in (0,1),$$

so $\{x_t\}$ is bounded. By boundedness of $\{Tx_t\}$, we obtain that

$$\|x_t - Tx_t\| \le (1-t) \sup_{t\in(0,1)} \|u - Tx_t\| \to 0 \text{ as } t \to 1.$$

Because H is reflexive, $\{x_t\}$ has a weakly convergent subsequence. Let $\{x_{t_n}\}$ be subsequence of $\{x_t\}$ such that $x_{t_n} \rightharpoonup z$ as $t_n \to 1$. Write $x_n = x_{t_n}$. Because $I - T$ is demiclosed at zero, $z \in F(T)$. Because $F(T)$ is a nonempty closed convex set in C by Corollary 5.2.29, there exists a unique point $v \in F(T)$ that is closest to u, i.e., $v \in F(T)$ is the nearest point projection of u. Now, for $y \in F(T)$, we have

$$\begin{aligned} \|x_t - u + t(u-y)\|^2 &= t^2\|Tx_t - y\|^2 \\ &\le t^2\|x_t - y\|^2 = t^2\|x_t - u + u - y\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|x_t - u\|^2 &+ t^2\|u-y\|^2 + 2t\langle x_t - u, u - y\rangle = \|x_t - u + t(u-y)\|^2 \\ &\le t^2(\|x_t - u\|^2 + \|u - y\|^2 + 2\langle x_t - u, u - y\rangle). \end{aligned}$$

It follows that

$$\|x_t - u\|^2 \le \frac{2t}{1+t}\langle x_t - u, y - u\rangle \le \langle x_t - u, y - u\rangle \le \|x_t - u\| \cdot \|y - u\|.$$

Hence $\|x_t - u\| \le \|y - u\|$. By w-lsc of the norm of H,

$$\|z - u\| \le \liminf_{n\to\infty} \|x_n - u\| \le \|y - u\| \text{ for all } y \in F(T).$$

But v is the nearest point projection of u. Therefore, $z = v$ is the unique element in $F(T)$ that is the nearest point projection of u. This shows that $x_n \rightharpoonup v$ as $n \to \infty$. It remains to show that the convergence is strong. Because

$$\|x_n - u\|^2 = \|x_n - v + v - u\|^2 = \|x_n - v\|^2 + \|u - v\|^2 + 2\langle x_n - v, v - u\rangle,$$

this implies that

$$\begin{aligned} \|x_n - v\|^2 &= \|x_n - u\|^2 - \|u - v\|^2 - 2\langle x_n - v, v - u\rangle \\ &\le -2\langle x_n - v, v - u\rangle \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{x_t\}$ converges strongly to v. ∎

We now establish a strong convergence theorem for non-self mappings in a Banach space.

Theorem 7.2.2 *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, u an element in C, and $T : C \to X$ a weakly inward nonexpansive mapping with $F(T) \neq \emptyset$. Suppose for $t \in (0,1)$, the contraction $G_t : C \to X$ defined by*

$$G_t x = (1-t)u + tTx, \ \ x \in C \tag{7.10}$$

has a unique fixed point $x_t \in C$. Then $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$.

Proof. Because $F(T)$ is nonempty, then $\{x_t\}$ is bounded. In fact, we have

$$\|x_t - v\| \leq \|u - v\| \text{ for all } v \in F(T) \text{ and } t \in (0,1).$$

We now show that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$. To this end, let $\{t_n\}$ be a sequence of real numbers in (0,1) such that $t_n \to 1$ as $n \to \infty$. Set $x_n := x_{t_n}$. Then we can define $\varphi : C \to [0,\infty)$ by $\varphi(x) = LIM_n\|x_n - x\|^2$. Then the set M defined by (2.32) is a nonempty closed convex bounded subset of C. Because

$$\|x_n - Tx_n\| = (1-t_n)\|Tx_n - u\| \to 0 \text{ as } n \to \infty, \tag{7.11}$$

it follows that for $x \in M$

$$\begin{aligned} \varphi(Tx) &= LIM_n\|x_n - Tx\|^2 \\ &\leq LIM_n\|Tx_n - Tx\|^2 \\ &\leq LIM_n\|x_n - x\|^2 = \varphi(x). \end{aligned} \tag{7.12}$$

By Theorem 2.9.11, M consists of one point, say z. We now show that this z is a fixed point of T. Because T is weakly inward, there are some $v_n \in C$ and $\lambda_n \geq 0$ such that

$$w_n := z + \lambda_n(v_n - z) \to Tz \ \text{ strongly}.$$

If $\lambda_n \leq 1$ for infinitely many n and for these n, then we have $w_n \in C$ and hence $Tz \in C$. Thus, we have $Tz = z$ by (7.12). So, we may assume $\lambda_n > 1$ for all sufficiently large n. We then write $v_n = r_n w_n + (1-r_n)z$, where $r_n = \lambda_n^{-1}$. Suppose $r_n \to 1$. Then $v_n \to Tz$ and hence $Tz \in C$. By (7.12), we have $Tz = z$. So, without loss of generality, we may assume $r_n \leq a < 1$. By Theorem 2.8.17, there exists a continuous increasing function $g = g_r : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|),$$

for all $x, y \in B_r[0]$ and $\lambda \in [0,1]$, where $B_r[0]$ (the closed ball centered at 0 and with radius r) is big enough so that $B_r[0]$ contains z and $\{w_n\}$. It follows that

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r[0]$ and $\lambda \in [0,1]$. Because $v_n \in C$, we obtain that

$$\begin{aligned}\varphi(z) &\leq \varphi(v_n)\\ &\leq r_n\varphi(w_n) + (1-r_n)\varphi(z) - r_n(1-r_n)g(\|w_n - z\|)\end{aligned}$$

and hence

$$\begin{aligned}(1-a)g(\|w_n - z\|) &\leq (1-r_n)g(\|w_n - z\|)\\ &\leq \varphi(w_n) - \varphi(z).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}(1-a)g(\|Tz - z\|) &\leq \varphi(Tz) - \varphi(z)\\ &\leq 0. \qquad (by\ (7.12))\end{aligned}$$

Therefore, $Tz = z$, i.e., z is a fixed point of T. Observe that

(i) $x_n - Tx_n \to 0$ by (7.11),

(ii) (7.7) is satisfied with $\alpha_n = 0$,

(iii) the set M contains a fixed point z of T.

By Corollary 7.1.11, we conclude that $\{x_t\}$ converges strongly to z as $t \to 1$. ▮

7.3 Convergence of Halpern iteration process

In Chapter 6, we have seen that the Mann and S-iteration processes are weakly convergent for nonexpansive mappings even in uniformly convex Banach spaces. The purpose of this section is to develop an iteration process so that it can generate a strongly convergent sequence in a Banach space.

Definition 7.3.1 *Let C be a nonempty convex subset of a linear space X and $T : C \to C$ a mapping. Let $u \in C$ and $\{\alpha_n\}$ a sequence in $[0,1]$. Then a sequence $\{x_n\}$ in C defined by*

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0 \end{cases} \tag{7.13}$$

is called the Halpern iteration.

We now prove the main convergence theorem of this section.

Theorem 7.3.2 *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in C$ and $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ that satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \ \textit{and} \ \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{7.14}$$

Suppose that $\{z_t\}$ *converges strongly to* $z \in F(T)$ *as* $t \to 1$*, where for* $t \in (0,1)$, z_t *is a unique element of* C *that satisfies* $z_t = (1-t)u + tTz_t$*. Then the sequence* $\{x_n\}$ *defined by (7.13) converges strongly to* z*.*

Proof. Because $F(T) \neq \emptyset$, it follows that $\{x_n\}$ and $\{Tx_n\}$ are bounded. Set $K := \sup\{\|u\| + \|Tx_n\| : n \in \mathbb{N}\}$. From (7.13), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}|(\|u\| + \|Tx_{n-1}\|) + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|x_n - x_{n-1}\|.\end{aligned}$$

Hence for $m, n \in \mathbb{N}$, we have

$$\begin{aligned}&\|x_{n+m+1} - x_{n+m}\| \\ &\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)K + \left(\prod_{k=m}^{n+m-1} (1-\alpha_{k+1})\right)\|x_{m+1} - x_m\| \\ &\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)K + exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1}\right)\|x_{m+1} - x_m\|.\end{aligned}$$

So the boundedness of $\{x_n\}$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ yield

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq \left(\sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|\right)K$$

for all $m \in \mathbb{N}$. Because $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Notice

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Tx_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Let LIM be a Banach limit. Then, we get

$$LIM_n\|x_n - Tz_t\|^2 \leq LIM_n\|x_n - z_t\|^2.$$

Because $t(x_n - Tz_t) = (x_n - z_t) - (1-t)(x_n - u)$, we have

$$\begin{aligned}t^2\|x_n - Tz_t\|^2 &\geq \|x_n - z_t\|^2 - 2(1-t)\langle x_n - u, J(x_n - z_t)\rangle \\ &= (2t-1)\|x_n - z_t\|^2 + 2(1-t)\langle u - z_t, J(x_n - z_t)\rangle\end{aligned}$$

for all $n \in \mathbb{N}$. These inequalities yield

$$\frac{1-t}{2} LIM_n\|x_n - z_t\|^2 \geq LIM_n\langle u - z_t, J(x_n - z_t)\rangle.$$

Letting t go to 1, we get

$$0 \geq LIM_n\langle u - z, J(x_n - z)\rangle,$$

because X has uniformly Gâteaux differentiable norm. Because $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} |\langle u - z, J(x_{n+1} - z)\rangle - \langle u - z, J(x_n - z)\rangle| = 0.$$

Hence by Proposition 2.9.7, we obtain

$$\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle \leq 0. \tag{7.15}$$

Because $(1 - \alpha_n)(Tx_n - z) = (x_{n+1} - z) - \alpha_n(u - z)$, we have

$$\|(1 - \alpha_n)(Tx_n - z)\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle,$$

it follows that

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)\|x_n - z\|^2 + 2(1 - (1 - \alpha_n))\langle u - z, J(x_{n+1} - z)\rangle$$

for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. From (7.15), there exists $n_0 \in \mathbb{N}$ such that

$$\langle u - z, J(x_n - z)\rangle \leq \varepsilon/2 \text{ for } \text{ all } n \geq n_0.$$

Then we have

$$\|x_{n+n_0} - z\|^2 \leq \left(\prod_{k=n_0}^{n+n_0-1}(1 - \alpha_k)\right)\|x_{n_0} - z\|^2 + \left(1 - \prod_{k=n_0}^{n+n_0-1}(1 - \alpha_k)\right)\varepsilon$$

for all $n \in \mathbb{N}$. By the condition $\sum_{k=0}^{\infty}\alpha_k = \infty$, we have

$$\limsup_{n\to\infty}\|x_n - z\|^2 = \limsup_{n\to\infty}\|x_{n+n_0} - z\|^2 \leq \varepsilon.$$

Therefore, $\{x_n\}$ converges strongly to z, because ε is an arbitrary positive real number. ∎

Corollary 7.3.3 *Let C be a nonempty closed convex subset of a uniformly smooth Banach space and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in C$ and $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ satisfying (7.14). Then the sequence $\{x_n\}$ defined by (7.13) converges strongly to a fixed point of T.*

Bibliographic Notes and Remarks

The main results presented in Section 7.1 are proved in Browder [29], Lim and Xu [98], Morales and Jung [114], Reich [126], and Takahashi and Ueda [158].

Theorem 7.2.1 is due to Singh and Watson [149]. The strong convergence of approximants of nonexpansive non-self mappings can be found in Jung and Kim [78], Xu [165], and Xu and Yin [168]. Theorem 7.3.2 follows from Shioji and Takahashi [144]. Such strong convergence results have been recently generalized by viscosity approximation method (see Moudafi [111], Xu [167]).

Exercises

7.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ a contraction mapping. Let $\{x_n\}$ be the sequence defined by the scheme

$$x_n = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f x_n,$$

where ε_n is a sequence $(0,1)$ with $\varepsilon_n \to 0$. Show that $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (I-f)\tilde{x}, \tilde{x}-x \rangle \leq 0 \text{ for all } x \in F(T).$$

7.2 Let H be a Hilbert space, C a closed convex subset of H, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ a contraction. Let $\{x_n\}$ be a sequence in C defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n) T x_n + \alpha_n f(x_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfies

$(H1)$ $\alpha_n \to 0$;

$(H2)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(H3)$ either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1$.

Show that under the hypotheses $(H1) \sim (H3)$, $x_n \to \tilde{x}$, where $\tilde{x}$ is the unique solution of the variational inequality:

$$\langle (I-f)\tilde{x}, \tilde{x}-x \rangle \leq 0 \text{ for all } x \in F(T).$$

7.3 Let C be a nonempty closed convex subset of a uniformly smooth Banach space X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If Π_C is the set of all contractions on C, show that the path $\{x_t\}$ defined by

$$x_t = t f x_t + (1-t) T x_t, \quad t \in (0,1), f \in \Pi_C,$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t \to 0^+} x_t, \quad f \in \Pi_C,$$

show that $Q(f)$ solves the variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-v) \rangle \leq 0, \quad f \in \Pi_C \text{ and } v \in F(T).$$

7.4 Let C be a nonempty closed convex subset of a Banach space X. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0, 1)$ and $T : C \to C$ a continuous pseudocontractive mapping. Show that

(a) for each $t \in (0, 1)$, there exists unique solution $x_t \in C$ of equation

$$x = tAx + (1 - t)Tx.$$

(b) Moreover, if v is a fixed point of T, then for each $t \in (0, 1)$, there exists $j(x_t - v) \in J(x_t - v)$ such that

$$\langle x_t - Ax_t, j(x_t - v)\rangle \leq 0;$$

(c) $\{x_t\}$ is bounded.

7.5 Let C be a nonempty closed convex subset of a Banach space X that has a uniformly Gâteaux differentiable norm and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. For a fixed $\delta \in (0, 1)$, define $S : C \to C$ by

$$Sx := (1 - \delta)x + \delta Tx$$

for all $x \in C$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of C that satisfies

$$z_t = tu + (1 - t)Tz_t$$

for arbitrary $u \in C$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ that satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For arbitrary $x_0 \in C$, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n.$$

Show that $\{x_n\}$ converges strongly to a fixed point of T.