Chapter 4

Existence Theorems in Metric Spaces

In this chapter, we study asymptotic fixed point theorems for contraction mappings and for mappings that are more general than contraction mappings in metric spaces.

4.1 Contraction mappings and their generalizations

In this section, we establish a fundamental asymptotic fixed point theorem that is known as the "Banach contraction principle" and further we give its generalizations in metric spaces.

By an asymptotic fixed point theorem for the mapping T, we mean a theorem that guarantees the existence of a fixed point of T, if the iterative T^n possess certain properties. Before to establish the Banach contraction principle, we discuss some basic definitions and results:

Let (X, d) be a metric space and let $Lip(X)$ denote the class of mappings $T : X \to X$ such that

$$\sigma(T^n) = \sup\left\{\frac{d(T^n x, T^n y)}{d(x, y)} : x, y \in X, x \neq y\right\} < \infty$$

for all $n \in \mathbb{N}$.

Members of $Lip(X)$ are called *Lipschitzian mappings* and $\sigma(T^n)$ is the *Lipschitz constant* of T^n. Note that $\sigma(T) = 0$ if and only if T is constant on X. For two Lipschitzian mappings $T : X \to X$ and $S : X \to X$ such that $S(X) \subseteq Dom(T)$, we have

$$\sigma(T \circ S) \leq \sigma(T)\sigma(S).$$

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It is clear that the mapping $T \in Lip(X)$ if there exists a constant $L_n \geq 0$ such that

$$d(T^n x, T^n y) \leq L_n d(x,y) \text{ for all } x, y \in X \text{ and } n \in \mathbb{N}. \tag{4.1}$$

Moreover, the smallest constant L_n for which (4.1) holds is the *Lipschitz constant of* T^n. A Lipschitzian mapping $T : X \to X$ is said to be *uniformly L-Lipschitzian* if $L_n = L$ for all $n \in \mathbb{N}$. A Lipschitzian mapping is said to be *contraction* (*nonexpansive*) if $\sigma(T) < 1$ ($\sigma(T) = 1$).

The following result plays an important role in proving several existence theorems in metric spaces.

Proposition 4.1.1 *Let (X,d) be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a bounded below lower semicontinuous function. Suppose that $\{x_n\}$ is a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}) \textit{ for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Then $\{x_n\}$ converges to a point $v \in X$ and $d(x_n, v) \leq \varphi(x_n) - \varphi(v)$ for all $n \in \mathbb{N}_0$.

Proof. Because

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \quad n \in \mathbb{N}_0,$$

it follows that $\{\varphi(x_n)\}$ is a decreasing sequence. Moreover, for $m \in \mathbb{N}_0$

$$\begin{aligned} \sum_{n=0}^{m} d(x_n, x_{n+1}) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_m, x_{m+1}) \\ &\leq \varphi(x_0) - \varphi(x_{m+1}) \\ &\leq \varphi(x_0) - \inf_{n \in \mathbb{N}_0} \varphi(x_n). \end{aligned}$$

Letting $m \to \infty$, we have

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Because X is complete, there exists $v \in X$ such that $\lim_{n \to \infty} x_n = v$. Let $m, n \in \mathbb{N}_0$ with $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \varphi(x_n) - \varphi(x_m). \end{aligned}$$

Letting $m \to \infty$, we obtain

$$d(x_n, v) \leq \varphi(x_n) - \lim_{m \to \infty} \varphi(x_m) \leq \varphi(x_n) - \varphi(v) \text{ for all } n \in \mathbb{N}_0. \quad \blacksquare$$

We now begin with Caristi's fixed point theorem. To prove it, we need the following important result.

Theorem 4.1.2 *Let X be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a proper, bounded below and lower semicontinuous function. Suppose that, for each $u \in X$ with $\inf_{x\in X} \varphi(x) < \varphi(u)$, there exists a $v \in X$ such that*

$$u \neq v \quad \text{and} \quad d(u,v) \leq \varphi(u) - \varphi(v).$$

Then there exists an $x_0 \in X$ such that $\varphi(x_0) = \inf_{x\in X} \varphi(x)$.

Proof. Suppose that $\inf_{x\in X} \varphi(x) < \varphi(y)$ for every $y \in X$. Let $u_0 \in X$ with $\varphi(u_0) < \infty$. If $\inf_{x\in X} \varphi(x) = \varphi(u_0)$, then we are done. Otherwise $\inf_{x\in X} \varphi(x) < \varphi(u_0)$, and there exists a $u_1 \in X$ such that $u_0 \neq u_1$ and $d(u_0, u_1) \leq \varphi(u_0) - \varphi(u_1)$.

Define inductively a sequence $\{u_n\}$ in X, starting with u_0. Suppose $u_{n-1} \in X$ is known. Then choose $u_n \in S_n$, where

$$S_n := \{w \in X : d(u_{n-1}, w) \leq \varphi(u_{n-1}) - \varphi(w)\}$$

such that

$$\varphi(u_n) \leq \inf_{w\in S_n} \varphi(w) + \frac{1}{2}\{\varphi(u_{n-1}) - \inf_{w\in S_n} \varphi(w)\}. \tag{4.2}$$

Because $u_n \in S_n$, we get

$$d(u_{n-1}, u_n) \leq \varphi(u_{n-1}) - \varphi(u_n), \quad n \in \mathbb{N}.$$

Proposition 4.1.1 implies that $u_n \to v \in X$ and $d(u_{n-1}, v) \leq \varphi(u_{n-1}) - \varphi(v)$. By hypothesis, there exists a $z \in X$ such that $z \neq v$ and $d(v,z) \leq \varphi(v) - \varphi(z)$. Observe that

$$\begin{aligned}
\varphi(z) &\leq \varphi(v) - d(v,z) \\
&\leq \varphi(v) - d(v,z) + \varphi(u_{n-1}) - \varphi(v) - d(u_{n-1}, v) \\
&= \varphi(u_{n-1}) - [d(v,z) + d(u_{n-1}, v)] \\
&\leq \varphi(u_{n-1}) - d(u_{n-1}, z).
\end{aligned}$$

This implies that $z \in S_n$. It follows from (4.2) that

$$2\varphi(u_n) - \varphi(u_{n-1}) \leq \inf_{w\in S_n} \varphi(w) \leq \varphi(z).$$

Thus,

$$\varphi(z) < \varphi(v) \leq \lim_{n\to\infty} \varphi(u_n) \leq \varphi(z),$$

a contradiction. Therefore, there exists a point $x_0 \in X$ such that $\varphi(x_0) = \inf_{x\in X} \varphi(x)$. ∎

Theorem 4.1.3 (Caristi's fixed point theorem) – *Let X be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a proper, bounded below and lower semicontinuous function. Let $T : X \to X$ be a mapping such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \quad \textit{for all } x \in X. \tag{4.3}$$

Then there exists a point $v \in X$ such that $v = Tv$ and $\varphi(v) < \infty$.

Proof. Because φ is proper, there exists $u \in X$ such that $\varphi(u) < \infty$. Let

$$C = \{x \in X : d(u, x) \leq \varphi(u) - \varphi(x)\}.$$

Then C is a nonempty closed subset of X. We show that C is invariant under T. For each $x \in C$, we have

$$d(u, x) \leq \varphi(u) - \varphi(x)$$

and hence from (4.3), we have

$$\begin{aligned} \varphi(Tx) &\leq \varphi(x) - d(x, Tx) \\ &\leq \varphi(x) - d(x, Tx) + \varphi(u) - \varphi(x) - d(u, x) \\ &= \varphi(u) - [d(x, Tx) + d(u, x)] \\ &\leq \varphi(u) - d(u, Tx), \end{aligned}$$

and it follows that $Tx \in C$.

Suppose, for contradiction, that $x \neq Tx$ for all $x \in C$. Then, for each $x \in C$, there exists $w \in C$ such that

$$x \neq w \text{ and } d(x, w) \leq \varphi(x) - \varphi(w).$$

By Theorem 4.1.2, there exists an $x_0 \in C$ with $\varphi(x_0) = \inf_{x \in C} \varphi(x)$. Hence for such an $x_0 \in C$, we have

$$\begin{aligned} 0 < d(x_0, Tx_0) &\leq \varphi(x_0) - \varphi(Tx_0) && (\inf_{x \in C} \varphi(x) = \varphi(x_0) \leq \varphi(Tx_0)) \\ &\leq \varphi(Tx_0) - \varphi(Tx_0) \\ &= 0, \end{aligned}$$

a contradiction. ∎

Remark 4.1.4 *The fixed point of the mapping T in Theorem 4.1.3 need not be unique.*

We now state and prove the Banach contraction principle, which gives a unique fixed point of the mapping.

Theorem 4.1.5 (Banach's contraction principle) – *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with Lipschitz constant $k \in (0, 1)$. Then we have the following:*

(a) There exists a unique fixed point $v \in X$.

(b) For arbitrary $x_0 \in X$, *the Picard iteration process defined by*

$$x_{n+1} = Tx_n, \ n \in \mathbb{N}_0$$

converges to v.

(c) $d(x_n, v) \leq (1-k)^{-1} k^n d(x_0, x_1)$ *for all* $n \in \mathbb{N}_0$.

Proof. (a) Define the function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = (1-k)^{-1} d(x, Tx)$, $x \in X$. Hence φ is a continuous function. Because T is a contraction mapping,

$$d(Tx, T^2x) \leq kd(x, Tx), \quad x \in X, \tag{4.4}$$

which implies that

$$d(x, Tx) - kd(x, Tx) \leq d(x, Tx) - d(Tx, T^2x).$$

Hence

$$\begin{aligned} d(x, Tx) &\leq \frac{1}{1-k}[d(x, Tx) - d(Tx, T^2x)] \\ &= \varphi(x) - \varphi(Tx). \end{aligned} \tag{4.5}$$

Let x be an arbitrary element in X and define the sequence $\{x_n\}$ in X by

$$x_n = T^n x, \quad n \in \mathbb{N}_0.$$

From (4.5), we have

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \quad n \in \mathbb{N}_0,$$

and it follows from Proposition 4.1.1 that

$$\lim_{n \to \infty} x_n = v \in X$$

and

$$d(x_n, v) \leq \varphi(x_n), \quad n \in \mathbb{N}_0. \tag{4.6}$$

Because T is continuous and $x_{n+1} = Tx_n$, it follows that $v = Tv$. Suppose z is another fixed point of T. Then

$$0 < d(v, z) = d(Tv, Tz) \leq kd(v, z) < d(v, z),$$

a contradiction. Hence T has unique fixed point $v \in X$.

(b) It follows from part (a).

(c) From (4.4) we have that $\varphi(x_n) \leq k^n \varphi(x_0)$. This implies from (4.6) that $d(x_n, v) \leq k^n \varphi(x_0)$. ∎

Let us give some examples of contraction mappings.

Example 4.1.6 *Let* $X = [a, b]$ *and* $T : X \to X$ *a mapping such that* T *is differentiable at every* $x \in (a, b)$ *such that* $|T^{'}(x)| \leq k < 1$. *Then, by the mean value theorem, if* $x, y \in X$, *there is a point* ξ *between* x *and* y *such that*

$$Tx - Ty = T^{'}(\xi)(x - y).$$

Thus,

$$|Tx - Ty| = |T^{'}(\xi)|\ |x - y| \leq k|x - y|.$$

Therefore, T *is contraction and it has a unique fixed point.*

Example 4.1.7 *Let* $X = \mathbb{R}$ *and* $T : \mathbb{R} \to \mathbb{R}$ *a mapping defined by*

$$Tx = \frac{1}{2}x + 1, \quad x \in \mathbb{R}.$$

Then T *is contraction and* $F(T) = \{2\}$.

The following example shows that there exists a mapping that is not a contraction, but it has a unique fixed point.

Example 4.1.8 *Let* $X = [0, 1]$ *and* $T : [0, 1] \to [0, 1]$ *a mapping defined by*

$$Tx = 1 - x, \ x \in [0, 1].$$

Then T *has a unique fixed point* $1/2$, *but* T *is not a contraction.*

Let (X, d) be a metric space. Then a mapping $T : X \to X$ is said to be *contractive* if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X, \ x \neq y.$$

It is clear that the class of contractive mappings falls between the class of contraction mappings and that of nonexpansive mappings.

Observation

- A contractive mapping can have at most one fixed point.

The contractive mapping may not have a fixed point. It can be seen from the following example.

Example 4.1.9 *Let* X *be the space* c_0 *consisting of all real sequences* $x = \{x_i\}$ *with* $\lim_{i\to\infty} x_i = 0$ *and* $d(x, y) = \|x - y\| = \sup_{i\in\mathbb{N}} |x_i - y_i|, x = \{x_i\}, y = \{y_i\} \in c_0$. *Let* $B_X = \{x \in c_0 : \|x\| \leq 1\}$. *For each* $x \in B_X$, *define*

$$T(x_1, x_2, \cdots, x_i, \cdots) = (y_1, y_2, \cdots, y_i, \cdots),$$

where $y_1 = (1 + \|x\|)/2$ *and* $y_i = (1 - 1/2^{i+1})x_{i-1}$ *for* $i = 2, 3, \cdots$. *Note that* $|y_1| \leq 1$ *and* $|y_i| \leq |x_{i-1}| \leq 1$ *for all* $i = 2, 3, \cdots$. *Hence* $T : B_X \to B_X$.

Suppose x and y are two distinct points in B_X. Then

$$\begin{aligned}
\|Tx - Ty\| &= \sup\left\{\frac{\|x\| - \|y\|}{2}, \left(1 - \frac{1}{2^{i+1}}\right)|x_{i-1} - y_{i-1}| : i = 2, 3, \cdots\right\} \\
&\leq \sup\left\{\frac{\|x - y\|}{2}, \left(1 - \frac{1}{2^{i+1}}\right)|x_{i-1} - y_{i-1}| : i = 2, 3, \cdots\right\} \\
&< \|x - y\|.
\end{aligned}$$

Suppose that there is a point $v \in B_X$ such that $Tv = v$. Then $v_1 = (1 + \|v\|)/2 > 0$ and for $i \geq 2$

$$|v_i| = \left(1 - \frac{1}{2^{i+1}}\right)|v_{i-1}|.$$

This implies for all $i \geq 2$

$$\begin{aligned}
|v_i| &= \left(1 - \frac{1}{2^{i+1}}\right)|v_{i-1}| \\
&= \left(1 - \frac{1}{2^{i+1}}\right)\left(1 - \frac{1}{2^{i}}\right)|v_{i-2}| \\
&\cdots \\
&= \prod_{k=0}^{i-2}\left(1 - \frac{1}{2^{i+1-k}}\right)|v_1| \\
&\geq \left(1 - \sum_{k=0}^{i-2}\frac{1}{2^{i+1-k}}\right)|v_1| \\
&= \left(1 - \sum_{j=3}^{i+1}\frac{1}{2^j}\right)|v_1| > \frac{3}{4}|v_1|.
\end{aligned}$$

This is not possible, because $v_i \to 0$ as $i \to \infty$. Thus, T has no fixed point in B_X. ∎

We note that completeness and boundedness of a metric space do not ensure the existence of fixed points of contractive mappings. However, contractive mappings always have fixed points in compact metric spaces.

Theorem 4.1.10 *Let X be a compact metric space and $T : X \to X$ a contractive mapping. Then T has a unique fixed point v in X. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ of iterates converges to v.*

Proof. For each $x \in X$, define a function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = d(x, Tx)$. Then φ is continuous on X and by compactness of X, φ attains its minimum, say $\varphi(v)$, at $v \in X$. Then $\varphi(v) = \min_{x \in X} \varphi(x)$. If $v \neq Tv$, then

$$\varphi(Tv) = d(Tv, T^2 v) < d(v, Tv) = \varphi(v),$$

a contradiction. Hence $v = Tv$. Uniqueness of v follows from the contractive condition of T.

Now, let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Set $c_n := d(T^n x_0, v), n \in \mathbb{N}_0$. Because

$$c_{n+1} = d(T^{n+1}x_0, v) < d(T^n x_0, v) = c_n,$$

$\{c_n\}$ is a nonincreasing sequence in $\mathbb{R}^+$. Hence $\lim\limits_{n\to\infty} c_n$ exists. Suppose $\lim\limits_{n\to\infty} c_n = c \geq 0$. Assume that $c > 0$. Because X is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in X$. Observe that

$$0 < c = \lim_{i\to\infty} c_{n_i} = \lim_{i\to\infty} d(T^{n_i}x_0, v) = d(z, v),$$

i.e., $z \neq v$. Because T is contractive and continuous,

$$c = \lim_{i\to\infty} d(T^{n_i+1}x_0, v) = d(Tz, v) < d(z, v) = c,$$

a contradiction. Thus, $c = 0$, i.e., $z = v$. This means that every convergent subsequence of $\{T^n x_0\}$ must converge to v. Therefore, $\{T^n x_0\}$ converges to v. $\blacksquare$

The following example shows that in general, even in a Hilbert space for contractive mappings we cannot have that $T^n x \to x_0$ for every $x \in B_X$ and $x_0 = Tx_0$.

Example 4.1.11 *Let* $X = \ell_2 = \{(x_1, x_2, \cdots, x_i, \cdots) : x_i$ *real for each* $i \in \mathbb{N}$ *and* $\sum_{i=1}^\infty |x_i|^2 < \infty\}$ *and* $B_X = \{x \in X : \|x\|_2 = (\sum_{i=1}^\infty |x_i\|^2)^{1/2} \leq 1\}$. *Define a mapping* $T : B_X \to B_X$ *by*

$$Tx = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_i x_i \cdots), \quad x = (x_1, x_2, \cdots, x_i, \cdots) \in B_X,$$

where $\alpha_1 = 1$; $\alpha_i = (1 - 1/i^2)$, $i = 2, 3, \cdots$. *It is easy to see that* T *is contractive with fixed point* $(0, 0, \cdots, 0, \cdots)$.

Now, let $x = (1, 0, \cdots, 0, \cdots) \in B_X$, *then*

$$T^n x = (0, 0, \cdots, \prod_{i=1}^n \alpha_i, 0, \cdots) \textit{ for all } n \in \mathbb{N}.$$

Thus,

$$\|T^n x\| = \frac{n+2}{2(n+1)} \to \frac{1}{2} \textit{ as } n \to \infty,$$

and hence $T^n x \nrightarrow 0$.

We now consider some important generalizations of the Banach contraction principle in which the Lipschitz constant k is replaced by some real-valued control function.

Theorem 4.1.12 (Boyd and Wong's fixed point theorem) – *Let* X *be a complete metric space and* $T : X \to X$ *a mapping that satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)) \quad \textit{for all} \quad x, y \in X, \tag{4.7}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is upper semicontinuous function from the right (i.e.,* $\lambda_i \downarrow \lambda \geq 0 \Rightarrow \limsup_{i\to\infty} \psi(\lambda_i) \leq \psi(\lambda)$*) such that* $\psi(t) < t$ *for each* $t > 0$*. Then* T *has a unique fixed point* $v \in X$*. Moreover, for each* $x \in X$, $\lim_{n\to\infty} T^n x = v$*.*

Proof. Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x$, $n \in \mathbb{N}_0$. Set $d_n := d(x_n, x_{n+1})$. We divide the proof into three steps:

Step 1. $\lim_{n\to\infty} d_n = 0$.

Note

$$d_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \psi(d_n), \quad n \in \mathbb{N}_0.$$

Hence $\{d_n\}$ is monotonic decreasing and bounded below. Hence $\lim_{n\to\infty} d_n$ exists. Let $\lim_{n\to\infty} d_n = \delta \geq 0$. Assume that $\delta > 0$. By the right continuity of ψ,

$$\delta = \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} \psi(d_n) \leq \psi(\delta) < \delta,$$

so $\delta = 0$.

Step 2. $\{x_n\}$ *is Cauchy sequence.*

Assume that $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}_0$ such that $m_k > n_k \geq k$ and

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{for} \quad k = 0, 1, 2, \cdots .$$

Also, choosing m_k as small as possible, it may be assumed that

$$d(x_{m_k - 1}, x_{n_k}) < \varepsilon.$$

Hence for each $k \in \mathbb{N}_0$, we have

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k-1}, x_{m_k}) + \varepsilon \\ &= d_{m_k-1} + \varepsilon, \end{aligned}$$

and it follows from the fact $d_{m_k} \to 0$ that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. Observe that

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq d_{m_k} + \psi(d(x_{m_k}, x_{n_k})) + d_{n_k}. \end{aligned}$$

Letting $k \to \infty$ and using the upper semicontinuity of ψ from the right, we obtain

$$\varepsilon = \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \lim_{k\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \psi(\varepsilon),$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X.

Step 3. Existence and uniqueness of fixed points.

Because $\{x_n\}$ is Cauchy and X is complete, $\lim_{n\to\infty} x_n = v \in X$. By continuity of T, we have $v = Tv$. Uniqueness of v easily follows from condition (4.7). ∎

Let Φ denote the class of all mappings $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:

(i) φ is continuous,

(ii) $\varphi(t) < t$ for all $t > 0$.

As an immediate consequence of the Boyd-Wong's fixed point theorem, we have the following important result, which will be useful in establishing existence theorems concerning asymptotic contraction mappings.

Corollary 4.1.13 *Let X be a complete metric space and $T : X \to X$ a mapping that satisfies*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \text{ for all } x, y \in X,$$

where $\varphi \in \Phi$. Then T has a unique fixed point $v \in X$. Moreover, for each $x \in X$, $\lim_{n\to\infty} T^n x = v$.

We now introduce a wider class of mappings that we call "*asymptotic contractions.*"

Definition 4.1.14 *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an asymptotic contraction if for each $n \in \mathbb{N}$*

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \text{ for all } x, y \in X, \tag{4.8}$$

where $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d.

The following theorem shows that asymptotic contractions have unique fixed points.

Theorem 4.1.15 *Let X be a complete metric space and $T : X \to X$ a continuous asymptotic contraction for which the mappings φ_n in (4.8) are also continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $v \in X$ and for each $x \in X$, $\{T^n x\}$ converges to v.*

Proof. Because the sequence $\{\varphi_i\}$ is uniformly convergent, it follows that φ is continuous. For any $x, y \in X, x \neq y$, we have

$$\limsup_{n\to\infty} d(T^n x, T^n y) \leq \limsup_{n\to\infty} \varphi_n(d(x, y)) = \varphi(d(x, y)) < d(x, y).$$

If there exist $x, y \in X$ and $\varepsilon > 0$ such that $\limsup_{n\to\infty} d(T^n x, T^n y) = \varepsilon$, then there exists $k \in \mathbb{N}$ such that $\varphi(d(T^k x, T^k y)) < \varepsilon$ because φ is continuous, and $\varphi(\varepsilon) < \varepsilon$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty} d(T^n x, T^n y) &= \limsup_{n\to\infty} d(T^n(T^k x), T^n(T^k y) \\
&\leq \limsup_{n\to\infty} \varphi_n(d(T^k x, T^k y)) \\
&= \varphi(d(T^k x, T^k y)) < \varepsilon,
\end{aligned}$$

a contraction. Hence

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0 \text{ for any } x, y \in X. \tag{4.9}$$

Thus, all sequences of the Picard iterates defined by T, are equi-convergent and bounded.

Now let $z_0 \in X$ be arbitrary, $\{z_n\}$ be a sequence of Picard iterates of T at the point z_0, $C = \overline{\{z_n\}}$ and $F_n = \{x \in C : d(x, T^k x) \leq 1/n, k = 1, \cdots, n\}$. Because $\{z_n\}$ is bounded, C is bounded. It follows from (4.9) that F_n is nonempty. Because T is continuous, we have F_n is closed, for any n. Also, we have $F_{n+1} \subseteq F_n$. Let $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences such that $x_n, y_n \in F_n$. Let $\{n_j\}$ be a sequence of integers such that $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = \limsup\limits_{n\to\infty} d(x_n, y_n)$.

Observe that

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_{n_j}) &\leq \lim_{j\to\infty} (d(x_{n_j}, T^{n_j} x_{n_j}) + d(T^{n_j} x_{n_j}, T^{n_j} y_{n_j}) + d(y_{n_j}, T^{n_j} y_{n_j})) \\
&= \lim_{j\to\infty} \varphi_{n_j}(d(x_{n_j}, y_{n_j})) \\
&= \varphi(\lim_{j\to\infty} d(x_{n_j}, y_{n_j})),
\end{aligned}$$

and hence $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = \varphi(\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}))$, which implies that $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = 0$, because C is bounded. Thus, $\limsup\limits_{n\to\infty} d(x_n, y_n) = 0$ and hence $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. This implies that $\lim\limits_{n\to\infty} diam(F_n) = 0$. By the completeness of C, it follows that there exists $v \in X$ such that $\cap_{n=1}^{\infty} F_n = \{v\}$. Because $d(v, Tv) \leq 1/n$ for any n, we have $Tv = v$. From (4.9), we have $\lim\limits_{n\to\infty} d(T^n x, v) = 0$ for any $x \in X$. ∎

We now study an important generalization of the Boyd and Wong's fixed point theorem in which the control function φ is extended in a different direction. Interestingly, in the following result the continuity condition on φ is replaced by $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

Theorem 4.1.16 (Matkowski's fixed point theorem) – *Let X be a complete metric space and $T : X \to X$ a mapping that satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)) \textit{ for all } x, y \in X,$$

where $\psi : (0, \infty) \to (0, \infty)$ is nondecreasing and satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point $v \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} T^n x = v$.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0,\ n \in \mathbb{N}$. Observe that

$$0 \leq \limsup_{n\to\infty} d(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \psi^n(d(x_0, x_1)) = 0.$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Because $\psi^n(t) \to 0$ for $t > 0$, $\psi(s) < s$ for any $s > 0$. Because $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, given any $\varepsilon > 0$, it is possible to choose n such that

$$d(x_{n+1}, x_n) \le \varepsilon - \psi(\varepsilon).$$

Now for $z \in B_\varepsilon[x_n] = \{x \in X : d(x, x_n) \le \varepsilon\}$, we have

$$\begin{aligned} d(Tz, x_n) &\le d(Tz, Tx_n) + d(Tx_n, x_n) \\ &\le \psi(d(z, x_n)) + d(x_{n+1}, x_n) \\ &\le \psi(\varepsilon) + (\varepsilon - \psi(\varepsilon)) = \varepsilon. \end{aligned}$$

Therefore, $T : B_\varepsilon[x_n] \to B_\varepsilon[x_n]$ and it follows that $d(x_m, x_n) \le \varepsilon$ for all $m \ge n$. Hence $\{x_n\}$ is a Cauchy sequence. The conclusion of the proof follows as in Theorem 4.1.12. ∎

We now introduce the concept of nearly Lipschitzian mappings:

Let (X, d) be a metric space and fix a sequence $\{a_n\}$ in $\mathbb{R}^+$ with $a_n \to 0$. A mapping $T : X \to X$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \ge 0$ such that

$$d(T^n x, T^n y) \le k_n(d(x, y) + a_n) \text{ for all } x, y \in C. \tag{4.10}$$

The infimum of constants k_n for which (4.10) holds is denoted by $\eta(T^n)$ and called the *nearly Lipschitz constant.*

Notice that

$$\eta(T^n) = \sup\left\{\frac{d(T^n x, T^n y)}{d(x, y) + a_n} : x, y \in C, x \ne y\right\}.$$

A nearly Lipschitzian mapping T with sequence $\{(\eta(T^n), a_n)\}$ is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \ge 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \le 1$,

(iv) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \le k$ for all $n \in \mathbb{N}$,

(v) *nearly uniformly k-contraction* if $\eta(T^n) \le k < 1$ for all $n \in \mathbb{N}$.

Example 4.1.17 *Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$ and $T : X \to X$ a mapping defined by*

$$Tx = \begin{cases} 1/2 \text{ if } x \in [0, 1/2], \\ 0 \text{ if } x \in (1/2, 1]. \end{cases}$$

Thus, T is discontinuous and non-Lipschitzian. However, it is nearly nonexpansive mapping. Indeed, for a sequence $\{a_n\}$ with $a_1 = 1/2$ and $a_n \to 0$, we have

$$d(Tx, Ty) \le d(x, y) + a_1 \text{ for all } x, y \in X$$

and

$$d(T^n x, T^n y) \leq d(x,y) + a_n \text{ for all } x, y \in X \text{ and } n \geq 2,$$

because

$$T^n x = \frac{1}{2} \text{ for all } x \in [0,1] \text{ and } n \geq 2.$$

We now develop a technique for studying the existence and uniqueness of fixed points of nearly Lipschitzian mappings.

Theorem 4.1.18 *Let X be a complete metric space and $T : X \to X$ a continuous nearly Lipschitzian mapping with sequence $\{(\eta(T^n), a_n)\}$, i.e., for a fixed sequence $\{a_n\}$ in $\mathbb{R}^+$ with $a_n \to 0$ and for each $n \in \mathbb{N}$, there exists a constant $\eta(T^n) > 0$ such that*

$$d(T^n x, T^n y) \leq \eta(T^n)(d(x,y) + a_n) \quad \text{for all} \quad x, y \in X.$$

Suppose $\eta_\infty(T) = \limsup_{n\to\infty} [\eta(T^n)]^{1/n} < 1$. Then we have the following:

(a) T has a unique fixed point $v \in X$.

(b) For each $x \in X$, the sequence $\{T^n x\}$ converges to v.

(c) $d(T^n x, v) \leq \sum_{i=n}^{\infty} \eta(T^i)(d(x,Tx) + M)$ for all $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$.

Proof. (a) Fix $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. Set $d_n := d(x_n, x_{n+1})$. Hence

$$d_n = d(T^n x, T^{n+1} x) \leq \eta(T^n)(d(x,Tx) + a_n),$$

which implies that

$$\sum_{n=1}^{\infty} d_n \leq (d(x,Tx) + M) \sum_{n=1}^{\infty} \eta(T^n)$$

for some $M > 0$, because $\lim_{n\to\infty} a_n = 0$. By the Root Test for convergence of series, if $\eta_\infty(T) = \limsup_{n\to\infty} [\eta(T^n)]^{1/n} < 1$, then $\sum_{n=1}^{\infty} \eta(T^n) < \infty$. It follows that $\sum_{n=1}^{\infty} d_n < \infty$ and hence $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n\to\infty} x_n$ exists (say $v \in X$). By the continuity of T, v is fixed point of T. Let w be another fixed point T. Then

$$\begin{aligned} \infty = \sum_{n=1}^{\infty} d(v,w) &= \sum_{n=1}^{\infty} d(T^n v, T^n w) \leq \sum_{n=1}^{\infty} \eta(T^n)(d(v,w) + a_n) \\ &\leq (d(u,w) + M) \sum_{n=1}^{\infty} \eta(T^n) < \infty, \end{aligned}$$

a contradiction, hence T has a unique fixed point $v \in X$.

(b) It follows from part (a).

(c) If $m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &= d(T^n x, T^{n+m} x) \\ &\leq \sum_{i=n}^{n+m-1} d(T^i x, T^{i+1} x) \\ &\leq \sum_{i=n}^{n+m-1} \eta(T^i)(d(x, Tx) + a_i) \\ &\leq \sum_{i=n}^{n+m-1} \eta(T^i)(d(x, Tx) + M). \end{aligned}$$

Letting $m \to \infty$, we obtain

$$d(x_n, v) \leq \sum_{i=n}^{\infty} \eta(T^i)(d(x, Tx) + M). \qquad \blacksquare$$

Remark 4.1.19 *In the case of a nearly uniformly k-Lipschitzian mapping, we have*

$$\limsup_{n\to\infty}[\eta(T^n)]^{1/n} = \limsup_{n\to\infty}(k)^{1/n} = 1.$$

Therefore, the assumptions of Theorem 4.1.18 do not hold for nearly uniformly k-Lipschitzian mappings.

4.2 Multivalued mappings

Let A be a nonempty subset of a metric space X. For $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $CB(X)$ denote the set of nonempty closed bounded subsets of X and $\mathcal{K}(X)$ denote the set of nonempty compact subsets of X. It is clear that $\mathcal{K}(X)$ is included in $CB(X)$.

For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\},$$

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\} = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

Example 4.2.1 *Let* $X = \mathbb{R}$, $A = [1, 2]$ *and* $B = [2, 3]$. *Then*

$$\delta(A, B) = \sup_{a\in A} d(a, B) = 1 \text{ and } \delta(B, A) = \sup_{b\in B} d(b, A) = 1.$$

Hence $H(A, B) = \max\{\delta(A, B), \delta(B, A)\} = 1$.

Note that set distance δ is not symmetric. However, δ and H have the following properties:

Proposition 4.2.2 *Let (X, d) be a metric space. Let $A, B, C \in CB(X)$. Then we have the following:*

(a) $\delta(A, B) = 0 \Leftrightarrow A \subset B$.

(b) $B \subset C \Rightarrow \delta(A, C) \leq \delta(A, B)$.

(c) $\delta(A \cup B, C) = \max\{\delta(A, C), \delta(B, C)\}$.

(d) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.

Proof. (a) By the definition δ, we have

$$\begin{aligned} \delta(A, B) = 0 \quad &\Leftrightarrow \quad \sup_{x \in A} d(x, B) = 0 \\ &\Leftrightarrow \quad d(x, B) = 0 \text{ for all } x \in A. \end{aligned}$$

Because B is closed in X,

$$d(x, B) = 0 \Leftrightarrow x \in B.$$

Thus,

$$\delta(A, B) = 0 \Leftrightarrow A \subset B.$$

(b) Observe that

$$B \subset C \Rightarrow d(x, C) \leq d(x, B) \text{ for all } x \in X.$$

(c) Observe that

$$\delta(A \cup B, C) = \sup_{x \in A \cup B} d(x, C) = \max\{\sup_{x \in A} d(x, C), \sup_{x \in B} d(x, C)\}.$$

(d) Let $a \in A$, $b \in B$ and $c \in C$. Then

$$d(a, b) \leq d(a, c) + d(c, b),$$

which implies that

$$d(a, B) \leq d(a, c) + d(c, B)$$

and hence

$$d(a, B) \leq d(a, c) + \delta(C, B).$$

Because $c \in C$ is arbitrary, we have

$$d(a, B) \leq d(a, C) + \delta(C, B).$$

Similarly, because $a \in A$ is arbitrary, we have

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B). \quad \blacksquare$$

Proposition 4.2.3 *Let (X, d) be a metric space. Then H is a metric on $CB(X)$.*

Proof. By the definition of H, we have

$$H(A,B) \geq 0 \text{ and } H(A,B) = H(B,A).$$

Observe that

$$\begin{aligned} H(A,B)=0 &\Leftrightarrow \max\{\delta(A,B),\delta(B,A)\}=0 \\ &\Leftrightarrow \delta(A,B)=0 \quad \text{and} \quad \delta(B,A)=0 \\ &\Leftrightarrow A\subset B \quad \text{and} \quad B\subset A \\ &\Leftrightarrow A=B. \end{aligned}$$

Using Proposition 4.2.2, we obtain

$$\begin{aligned} H(A,B) &= \max\{\delta(A,B),\delta(B,A)\} \\ &\leq \max\{\delta(A,C)+\delta(C,B),\delta(B,C)+\delta(C,A)) \\ &\leq \max\{\delta(A,C),\delta(C,A)\}+\max\{\delta(B,C),\delta(C,B)\} \\ &= H(A,C)+H(C,B). \quad \blacksquare \end{aligned}$$

The metric H on $CB(X)$ is called the *Hausdorff metric*. The metric H depends on the metric d. It is easy to see that the completeness of (X,d) implies the completeness of $(CB(X),H)$ and $(\mathcal{K}(X),H)$.

Remark 4.2.4 *Let $A,B\in CB(X)$ and $a\in A$. Then for $\varepsilon>0$, there must exist a point $b\in B$ such that $d(a,b)\leq H(A,B)+\varepsilon$.*

The following proposition gives a characteristic property of the Hausdorff metric that will be used in Section 8.1.

Proposition 4.2.5 *Let X be a metric space. Then*

$$H(A\cup B,C\cup D)\leq \max\{H(A,C),H(B,D)\} \text{ for all } A,B,C,D\in CB(X).$$

Proof. Observe that

$$\begin{aligned} \delta(A\cup B,C\cup D) &= \max\{\delta(A,C\cup D),\delta(B,C\cup D)\} \\ &\leq \max\{\delta(A,C),\delta(B,D)\} \\ &\leq \max\{H(A,C),H(B,D)\}. \end{aligned}$$

Similarly, we have

$$\delta(C\cup D,A\cup B)\leq \max\{H(A,C),H(B,D)\}$$

By definition of H, we have

$$\begin{aligned} H(A\cup B,C\cup D) &= \max\{\delta(A\cup B,C\cup D),\delta(C\cup D,A\cup B)\} \\ &\leq \max\{H(A,C),H(B,D)\} \text{ for all } A,B,C,D\in CB(X). \quad \blacksquare \end{aligned}$$

Let $F(X)$ denote the family of nonempty closed subsets of a metric space (X,d). Then we have

Proposition 4.2.6 *Let C be a nonempty subset of a metric space (X,d). Suppose the mapping $T : C \to F(X)$ is an upper semicontinuous at $x_0 \in C$. Then the mapping $\varphi : C \to \mathbb{R}^+$ defined by $\varphi(x) = d(x,Tx)$, $x \in C$ is lower semicontinuous at x_0.*

Proof. Let $\varepsilon > 0$. By the upper semicontinuity of T at x_0, there exists $\delta > 0$ such that $y \in B_\delta[x_0] \cap C$ implies Ty lies in an $\varepsilon/4$-neighborhood of Tx_0, and moreover we may suppose $\delta \leq \varepsilon/4$. Select $u \in Ty$ such that

$$d(y,u) \leq d(y,Ty) + \frac{\varepsilon}{2}$$

and select $v \in Tx_0$ so that $d(u,v) \leq \varepsilon/4$. Then

$$\begin{aligned}
d(x_0,Tx_0) - \left[d(y,Ty) + \frac{\varepsilon}{2}\right] &\leq d(x_0,Tx_0) - d(y,u) \\
&\leq d(x_0,v) - d(y,u) \\
&\leq d(x_0,y) + d(y,u) + d(u,v) - d(y,u) \\
&\leq d(x_0,y) + d(u,v) \\
&\leq \delta + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}$$

and hence

$$d(x_0,Tx_0) \leq d(y,Ty) + \varepsilon.$$

Therefore, φ is lower semicontinuous at x_0. ∎

We now introduce the class of multivalued contraction mappings and obtain a fixed point theorem for this class of mappings:

Let T be a mapping from a metric space (X,d) into $CB(X)$. Then T is said to be *Lipschitzian* if there exists a constant $k > 0$ such that

$$H(Tx,Ty) \leq kd(x,y) \ \text{ for all } x,y \in X.$$

A multivalued Lipschitzian mapping T is said to be *contraction* (*nonexpansive*) if $k < 1$ $(k = 1)$. Let $F(T)$ denote the set of fixed points of T, i.e., $F(T) = \{x \in X : x \in Tx\}$.

Theorem 4.2.7 (Nadler's fixed point theorem) – *Let X be a complete metric space and $T : X \to CB(X)$ a contraction mapping. Then T has a fixed point in X.*

Proof. Let k, $0 < k < 1$ be the Lipschitz constant of T. Let $x_0 \in X$ and $x_1 \in Tx_0$. By Remark 4.2.4, there must exist $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq H(Tx_0,Tx_1) + k.$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) \leq H(Tx_1,Tx_2) + k^2.$$

Thus, there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + k^n \quad \text{for all } n \in \mathbb{N}.$$

Notice for each $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$ and so

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + k^n \\ &\leq kd(x_{n-1}, x_n) + k^n \\ &\leq k[kd(x_{n-2}, x_{n-1}) + k^{n-1}] + k^n \\ &\leq k^2 d(x_{n-2}, x_{n-1}) + 2k^n \\ &\cdots \\ &\leq k^n d(x_0, x_1) + nk^n. \end{aligned}$$

Because $\sum_{n=0}^{\infty} k^n < \infty$ and $\sum_{n=0}^{\infty} nk^n < \infty$, we have

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq d(x_0, x_1) \sum_{n=0}^{\infty} k^n + \sum_{n=0}^{\infty} nk^n < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Again, by the continuity of T,

$$\lim_{n\to\infty} H(Tx_n, Tv) = 0.$$

Because $x_{n+1} \in Tx_n$,

$$\lim_{n\to\infty} d(x_{n+1}, Tv) = 0,$$

which implies that $d(v, Tv) = 0$. Because Tv is closed, it follows that $v \in Tv$. ∎

Example 4.2.8 *Let $X = [0, 1]$ and $f : [0, 1] \to [0, 1]$ a mapping such that*

$$f(x) = \begin{cases} x/2 + 1/2, & 0 \leq x \leq 1/2, \\ -x/2 + 1, & 1/2 \leq x \leq 1. \end{cases}$$

Define $T : X \to 2^X$ by $Tx = \{f(x)\} \cup \{0\}$, $x \in X$. Then T is a multivalued contraction mapping with $F(T) = \{0, 2/3\}$.

Remark 4.2.9 *Example 4.2.8 shows that the fixed point of a multivalued contraction mapping is not necessarily unique.*

We now discuss a stability result (Theorem 4.2.11) for multivalued contraction mappings.

Proposition 4.2.10 *Let X be a complete metric space and let $S, T : X \to CB(X)$ be two contraction mappings each having Lipschitz constant $k < 1$, i.e.,*

$$H(Sx, Sy) \leq kd(x, y) \text{ and } H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then $H(F(S), F(T)) \leq (1 - k)^{-1} \sup_{x\in X} H(Sx, Tx)$.

Proof. Let $\varepsilon > 0$ and $c > 0$ be such that $c\sum_{n=1}^{\infty} nk^n < 1$. For $x_0 \in F(S)$, select $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \leq H(Sx_0, Tx_0) + \varepsilon.$$

Because $H(Tx_0, Tx_1) \leq kd(x_0, x_1)$, it is possible to select $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \frac{c\varepsilon k}{1-k} \\ &\leq kd(x_0, x_1) + \frac{c\varepsilon k}{1-k}. \end{aligned}$$

Define $\{x_n\}$ inductively by

$$x_{n+1} \in Tx_n \text{ and } d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}) + \frac{c\varepsilon k^n}{1-k}.$$

Set $\eta := c\varepsilon/(1-k)$. Observe that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq kd(x_n, x_{n-1}) + \eta k^n \\ &\leq k(kd(x_{n-1}, x_{n-2}) + \eta k^{n-1}) + \eta k^n \\ &\leq k^2 d(x_{n-1}, x_{n-2}) + 2\eta k^n \\ &\cdots \\ &\leq k^n d(x_0, x_1) + n\eta k^n. \end{aligned}$$

Because $\sum_{n=1}^{\infty} k^n < \infty$ and $\sum_{n=1}^{\infty} nk^n < \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X and it converges to some point $v \in X$. Because $\lim_{n\to\infty} H(Tx_n, Tv) = 0$ by continuity of T, it follows from $x_{n+1} \in Tx_n$ that $v \in F(T)$. Observe that

$$\begin{aligned} d(x_0, v) &\leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} k^n d(x_0, x_1) + \eta \sum_{n=0}^{\infty} nk^n \\ &\leq (1-k)^{-1} d(x_0, x_1) + \eta \sum_{n=0}^{\infty} nk^n \\ &\leq (1-k)^{-1}(d(x_0, x_1) + \varepsilon) \\ &\leq (1-k)^{-1}(H(Sx_0, Tx_0) + 2\varepsilon). \end{aligned}$$

Interchanging the roles of S and T, we conclude:

For each $y_0 \in F(T)$, there exist $y_1 \in Sy_0$ and $u \in F(S)$ such that

$$d(y_0, u) \leq (1-k)^{-1}(H(Sy_0, Ty_0) + 2\varepsilon).$$

Because $\varepsilon > 0$ is arbitrary, the conclusion follows. ∎

Theorem 4.2.11 *Let X be a complete metric space and let $T_n : X \to CB(X)$ $(n = 1, 2, \cdots)$ be contraction mappings each having Lipschitz constant $k < 1$, i.e.,*

$$H(T_n x, T_n y) \leq kd(x, y) \quad \text{for all } x, y \in X \text{ and } \; n \in \mathbb{N}.$$

If $\lim_{n\to\infty} H(T_n x, T_0 x) = 0$ uniformly for $x \in X$, then $\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0$.

Proof. Let $\varepsilon > 0$. Because $\lim_{n\to\infty} H(T_n x, T_0 x) = 0$ uniformly for $x \in X$, it is possible to select $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in X} H(T_n x, T_0 x) \leq (1-k)\varepsilon \text{ for all } n \geq n_0.$$

By Proposition 4.2.10, we have $H(F(T_n), F(T_0)) < \varepsilon$ for all $n \geq n_0$. ∎

Next, we extend Nadler's fixed point theorem for non-self multivalued mappings in a metric space. First, we define a metrically convex metric space.

Definition 4.2.12 *A metric space (X, d) is said to be metrically convex*[1] *if for any $x, y \in X$ with $x \neq y$, there exists $z \in X$, $x \neq y \neq z$ such that*

$$d(x, z) + d(z, y) = d(x, y).$$

We note that in such a space, each two points are the end points of at least one metric segment. This fact immediately yields a very useful lemma.

Lemma 4.2.13 *If C is a nonempty closed subset of a complete and metrically convex metric space (X, d), then for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Now we are in a position to establish a fundamental result on the existence of fixed points for non-self multivalued contraction mappings.

Theorem 4.2.14 (Assad and Kirk's fixed point theorem) – *Let (X, d) be a complete and metrically convex metric space, C a nonempty closed subset of X, and $T : C \to CB(X)$ a contraction mapping, i.e.,*

$$H(Tx, Ty) \leq k d(x, y) \text{ for all } x, y \in X,$$

where $k \in (0, 1)$. If $Tx \subset C$ for each $x \in \partial C$, then T has a fixed point in C.

Proof. We construct a sequence $\{p_n\}$ in C in the following way:

Let $p_0 \in C$ and $p_1' \in Tp_0$. If $p_1' \in C$, let $p_1 = p_1'$. Otherwise, select a point $p_1 \in \partial C$ such that

$$d(p_0, p_1) + d(p_1, p_1') = d(p_0, p_1').$$

Thus, $p_1 \in C$. By Remark 4.2.4, we may choose $p_2' \in Tp_1$ such that

$$d(p_1', p_2') \leq H(Tp_0, Tp_1) + k.$$

Now, if $p_2' \in C$, let $p_2' = p_2$, otherwise, let $p_2 \in \partial C$ such that

$$d(p_1, p_2) + d(p_2, p_2') = d(p_1, p_2').$$

Continuing in this manner, we obtain sequences $\{p_n\}$ and $\{p_n'\}$ such that for $n \in \mathbb{N}$,

[1] The concept of metric convexity was introduced by K. Menger in 1953.

(i) $p'_{n+1} \in Tp_n$;

(ii) $d(p'_{n+1}, p'_n) \leq H(Tp_n, Tp_{n-1}) + k^n$,

where $p'_{n+1} = p_{n+1}$, if $p'_{n+1} \in C$ or

$$d(p_n, p_{n+1}) + d(p_{n+1}, p'_{n+1}) = d(p_n, p'_{n+1}) \text{ if } p'_{n+1} \notin C \text{ and } p_{n+1} \in \partial C. \quad (4.11)$$

Now, set

$$\begin{array}{rcl} P: & = & \{p_i \in \{p_n\} : p_i = p'_i, i \in \mathbb{N}\}; \\ Q: & = & \{p_i \in \{p_n\} : p_i \neq p'_i, i \in \mathbb{N}\}. \end{array}$$

Observe that if $p_i \in Q$ for some i, then $p_{i+1} \in P$ be the boundary condition.

We wish to estimate the distance $d(p_n, p_{n+1})$ for $n \geq 2$. For this, we consider three cases:

Case I. $p_n \in P$ *and* $p_{n+1} \in P$.

In this case, we have

$$\begin{array}{rcl} d(p_n, p_{n+1}) = d(p'_n, p'_{n+1}) & \leq & H(Tp_n, Tp_{n-1}) + k^n \\ & \leq & kd(p_n, p_{n-1}) + k^n. \end{array}$$

Case II. $p_n \in P$ *and* $p_{n+1} \in Q$.

By (4.11), we have

$$\begin{array}{rcl} d(p_n, p_{n+1}) & \leq & d(p_n, p'_{n+1}) = d(p'_n, p'_{n+1}) \\ & \leq & H(Tp_{n-1}, Tp_n) + k^n \\ & \leq & kd(p_{n-1}, p_n) + k^n. \end{array}$$

Case III. $p_n \in Q$ *and* $p_{n+1} \in P$.

By the above observation, two consecutive terms of $\{p_n\}$ cannot be in Q, hence $p_{n-1} \in P$ and $p'_{n-1} = p_{n-1}$. Using this fact, we obtain

$$\begin{array}{rcl} d(p_n, p_{n+1}) & \leq & d(p_n, p'_n) + d(p'_n, p_{n+1}) \\ & = & d(p_n, p'{}_n) + d(p'_n, p'_{n+1}) \\ & \leq & d(p_n, p'_n) + H(Tp_{n-1}, Tp_n) + k^n \\ & \leq & d(p_n, p'_n) + \alpha d(p_{n-1}, p_n) + k^n \\ & < & d(p_{n-1}, p'_n) + k^n \\ & = & d(p'_{n-1}, p'_n) + k^n \\ & \leq & H(Tp_{n-2}, Tp_{n-1}) + k^{n-1} + k^n \\ & \leq & kd(p_{n-2}, p_{n-1}) + k^{n-1} + k^n. \end{array}$$

The only other possibility, $p_n \in Q,\ p_{n+1} \in Q$ cannot occur. Thus, for $n \geq 2$, we have

$$d(p_n, p_{n+1}) = \begin{cases} kd(p_n, p_{n-1}) + k^n, & \text{or} \\ kd(p_{n-2}, p_{n-1}) + k^n + k^{n-1}. \end{cases} \quad (4.12)$$

Set $\delta := k^{-1/2}\max\{d(p_0,p_1), d(p_1,p_2)\}$. We now prove that

$$d(p_n,p_{n+1}) \le k^{n/2}(\delta+n), \quad n \in \mathbb{N}. \tag{4.13}$$

For $n = 1$

$$d(p_1,p_2) \le k^{1/2}(\delta+1).$$

For $n = 2$, we use (4.12) and taking each case separately, we obtain

$$\begin{aligned} d(p_2,p_3) &\le kd(p_1,p_2)+k^2 \\ &\le kk^{1/2}(\delta+1)+k^2 \\ &\le k(\delta+2); \\ d(p_2,p_3) &\le kd(p_0,p_1)+k^2+k \\ &\le k(k^{1/2}\delta+k+1) \\ &\le k(\delta+2). \end{aligned}$$

Now assume that (4.13) holds for $1 \le n \le m$. Observe that for $m \ge 2$

$$\begin{aligned} d(p_{m+1},p_{m+2}) &\le kd(p_m,p_{m+1})+k^{m+1} \\ &\le k[k^{m/2}(\delta+m)]+k^{m+1} \\ &\le k^{(m+1)/2}(\delta+m)+k^{(m+1)/2}k^{(m+1)/2} \\ &\le k^{(m+1)/2}[\delta+(m+1)] \end{aligned}$$

or

$$\begin{aligned} d(p_{m+1},p_{m+2}) &\le kd(p_{m-1},p_m)+k^{m+1}+k^m \\ &\le k[k^{(m-1)/2}(\delta+m-1)]+k^{m+1}+k^m \\ &\le k^{(m+1)/2}(\delta+m-1)+k^{(m+1)/2}k^{(m+1)/2}+k^{(m+1)/2}k^{(m-1)/2} \\ &\le k^{(m+1)/2}(\delta+m-1)+k^{(m+1)/2}+k^{(m+1)/2} \\ &= k^{(m+1)/2}(\delta+m+1), \end{aligned}$$

and it follows that (4.13) is true for all $n \in \mathbb{N}$. Using (4.13) we obtain

$$d(p_n,p_m) \le \delta\sum_{i=m}^{\infty}(k^{1/2})^i + \sum_{i=m}^{\infty} i(k^{1/2})^i, \quad n > m \ge 1.$$

This means that $\{p_n\}$ is a Cauchy sequence. Because C is closed, $\{p_n\}$ converges to a point $z \in C$. By our choice of $\{p_n\}$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \in P$, i.e., $p_{n_i} = p'_{n_i}, \quad i = 1,2,\cdots$. Note $p'_{n_i} \in Tp_{n_i-1}$ for $i \in \mathbb{N}$ by (i) and $p_{n_i-1} \to z$ imply that $Tp_{n_i-1} \to Tz$ as $i \to \infty$ in the Hausdorff metric H. Because

$$d(p_{n_i},Tz) \le H(Tp_{n_i-1},Tz) \to 0 \text{ as } i \to \infty,$$

it follows that $d(z,Tz) = 0$. As Tz is closed, $z \in Tz$. ∎

4.3 Convexity structure and fixed points

Let C be a nonempty subset of a metric space X and $T : C \to C$ a mapping. Then a sequence $\{x_n\}$ in C is said to be an *approximating fixed point sequence* (in short AFPS) of T if $\lim_{n\to\infty} d(x_n,\ Tx_n) = 0$.

We have seen in the Banach contraction principle that every contraction mapping has an approximating fixed point sequence in a metric space. In fact, the Picard iterative sequence ($x_{n+1} = Tx_n, n \in \mathbb{N}$) is an approximating fixed point sequence of the contraction mapping T.

The following example shows that the Picard iterative sequence is not necessarily an approximating fixed point sequence of nonexpansive mappings.

Example 4.3.1 *Let $X = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ a mapping defined by*

$$Tx = -x \text{ for all } x \in \mathbb{R}.$$

Note that T is nonexpansive with $F(T) = \{0\}$. However for $x_0 > 0$, the iterative sequence of the Picard iteration process is

$$x_{n+1} = Tx_n = (-1)^n x_0, \quad n \in \mathbb{N}_0.$$

Hence $d(x_n,\ Tx_n) = |(-1)^{n-1} - (-1)^n| x_0 = 2x_0 \nrightarrow 0$ as $n \to \infty$.

The following Proposition 4.3.9 shows that the convexity structure has an important role in the existence of AFPS for nonexpansive mappings. We define convexity structure in a metric space.

Definition 4.3.2 *Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure*[2] *on X, if for all $x, y \in X$ and $\lambda \in [0,1]$ the following condition is satisfied:*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y) \text{ for all } u \in X. \tag{4.14}$$

A metric space X with convex structure is called a convex metric space.

A subset C of a convex metric space X is said to be *convex* if $W(x, y; \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. A convex metric space X is said to have *property (B)* if

$$d(W(u, x; \lambda), W(u, y; \lambda)) = (1-\lambda) d(x, y) \text{ for all } u, x, y \in X \text{ and } \lambda \in (0,1).$$

Example 4.3.3 *A normed space and each of its convex subsets are convex metric spaces with convexity structure $W(x, y; \lambda) = \lambda x + (1-\lambda) y$.*

[2] The convexity structure in a metric space was introduced by W. Takahashi in 1970.

Example 4.3.4 *Let X be a linear space that is also a metric space with the following properties:*

(i) $d(x, y) = d(x - y, 0)$ *for all* $x, y \in X$;

(ii) $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x, 0) + (1-\lambda)d(y, 0)$ *for all* $x, y \in X$ *and* $\lambda \in [0, 1]$.

Then X is a convex metric space.

Example 4.3.5 *A Fréchet space is not necessarily a convex metric space.*

The following propositions are very useful in various applications.

Proposition 4.3.6 *Let* $\{C_\alpha : \alpha \in \Lambda\}$ *be a family of convex subsets of a convex metric space X. Then* $\cap_{\alpha\in\Lambda} C_\alpha$ *is also a convex subset of X.*

Proposition 4.3.7 *The open balls* $B_r(x)$ *and the closed balls* $B_r[x]$ *in a convex metric space X are convex subsets of X.*

Proof. For $y, z \in B_r(x)$ and $\lambda \in [0, 1]$, there exists $W(y, z; \lambda) \in X$. Because X is a convex metric space,

$$\begin{aligned} d(x, W(y, z; \lambda)) &\leq \lambda d(x, y) + (1-\lambda)d(x, z) \\ &< \lambda r + (1-\lambda)r = r. \end{aligned}$$

Therefore, $W(y, z; \lambda) \in B_r(x)$. Similarly, $B_r[x]$ is a convex subset of X. ∎

Proposition 4.3.8 *Let X be a convex metric space. Then*

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) \text{ for all } x, y \in X \text{ and } \lambda \in [0, 1].$$

Proof. Because X is a convex metric space, we obtain

$$\begin{aligned} d(x, y) &\leq d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y) \\ &\leq \lambda d(x, x) + (1-\lambda)d(x, y) + \lambda d(x, y) + (1-\lambda)d(y, y) \\ &= \lambda d(x, y) + (1-\lambda)d(x, y) \\ &= d(x, y) \end{aligned}$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. Therefore,

$$d(x, y) = d(x, W(x, y; \lambda) + d(W(x, y; \lambda), y) \text{ for all } x, y \in X \text{ and } \lambda \in [0, 1].$$ ∎

We now apply the convexity structure defined in Definition 4.3.2 to obtain AFPS for nonexpansive mappings in a metric space. Note, similar results are also discussed in Chapter 5.

Proposition 4.3.9 *Let X be a complete convex metric space with property (B), C a nonempty closed convex subset of X, and* $T : C \to C$ *a nonexpansive mapping. Then we have the following:*

(a) For $u \in C$ *and* $t \in (0, 1)$, *there exists exactly one point* $x_t \in C$ *such that*

$$x_t = W(u, Tx_t; 1-t)$$

(b) If C is bounded, then $d(x_t, Tx_t) \to 0$ *as* $t \to 1$, *i.e., T has an AFPS.*

Proof. (a) For $t \in (0,1)$, consider the mapping $T_t : C \to C$ defined by

$$T_t x = W(u, Tx; 1-t).$$

By property (B), we have

$$d(T_t x, T_t y) = t d(Tx, Ty) \le t d(x, y) \text{ for all } x, y \in C.$$

By the Banach contraction principle, T_t has exactly one fixed point x_t in C. Therefore,

$$x_t = W(u, Tx_t; 1-t).$$

(b) By boundedness of C, we get

$$\begin{aligned} d(x_t, Tx_t) &= d(Tx_t, W(u, Tx_t; 1-t)) \\ &\le (1-t) d(Tx_t, u) \le (1-t)\ diam(C) \to 0 \text{ as } t \to 1. \end{aligned}$$ $\blacksquare$

Applying Proposition 4.3.9, we have

Theorem 4.3.10 *Let X be a complete convex metric space X with property (B), C a nonempty compact convex subset of X, and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof. By Proposition 4.3.9, there exists a sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{4.15}$$

Because C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to v \in C$. Hence from (4.15), we have $v = Tv$. $\blacksquare$

In Theorem 4.3.14, we will see that compactness can be dropped if C has normal structure. To see this, we extend the notion of normal structure in metric space X.

For $C \subset X$, we denote the following, which will be used throughout the remainder of this chapter:

$$\begin{aligned} r_x(C) &= \sup\{d(x,y) : y \in C\},\ x \in C, \\ r(C) &= \inf\{r_x(C) : x \in C\}, \\ \mathcal{Z}_C &= \{x \in C : r_x(C) = r(C)\}. \end{aligned}$$

A point $x_0 \in C$ is said to be a *diametral point* of C if

$$\sup\{d(x_0, y) : y \in C\} = diam(C).$$

A convex metric space X is said to have *normal structure* if for each closed convex bounded subset C of X that contains at least two points, there exists $x_0 \in C$ that is not a diametral point of C.

Example 4.3.11 *Every compact convex metric space has normal structure.*

A convex metric space X is said to have *property (C)* if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection. By Smulian's theorem, every weakly compact convex subset of a Banach space has property (C).

Using property (C), we have

Proposition 4.3.12 *If a convex metric space X has property (C), then $\mathcal{Z}_C$ is nonempty, closed, and convex.*

Proof. Let $C_n(x) = \{y \in C : d(x,y) \leq r(C) + 1/n\}$ for $n \in \mathbb{N}$ and $x \in X$. It is easily seen that the sets $C_n = \cap_{x\in X} C_n(x)$ form a decreasing sequence of nonempty closed convex sets, and hence $\cap_{n=1}^{\infty} C_n$ is nonempty closed convex by property (C). Because $\mathcal{Z}_C = \cap_{n=1}^{\infty} C_n$, the proof is complete. ∎

Proposition 4.3.13 *Let C be a nonempty compact subset of a convex metric space X and let D be the least closed convex set containing C. If $diam(C) > 0$, then there exists an element $x_0 \in D$ such that* $\sup\{d(x,x_0) : x \in C\} < diam(C)$.

Proof. Because C is compact, we may find $x_1, x_2 \in C$ such that $d(x_1,x_2) = diam(C)$. Let $C_0 \subset C$ be maximal so that $C_0 \supset \{x_1,x_2\}$ and $d(x,y) = 0$ or $diam(C)$ for all $x, y \in C_0$. It is easy to see that C_0 is finite. Let $C_0 = \{x_1, x_2, \cdots, x_n\}$. Because X is a convex metric space, we can define

$$\begin{aligned}
y_1 &= W(x_1, x_2; \frac{1}{2}),\\
y_2 &= W(x_3, y_1; \frac{1}{3}),\\
&\cdots\\
y_{n-2} &= W(x_{n-1}, y_{n-3}; \frac{1}{n-1}),\\
y_{n-1} &= W(x_n, y_{n-2}; \frac{1}{n}) = u.
\end{aligned}$$

Because C is compact, we can find $y_0 \in C$ such that

$$d(y_0, u) = \sup\{d(x,u) : x \in C\}.$$

From (4.14), we obtain

$$\begin{aligned}
d(y_0,u) &\leq \frac{1}{n}\, d(y_0,x_n) + \frac{n-1}{n}\, d(y_0,y_{n-2})\\
&\leq \frac{1}{n}\, d(y_0,x_n) + \frac{n-1}{n}\left(\frac{1}{n-1}\, d(y_0,x_{n-1}) + \frac{n-2}{n-1}\, d(y_0,y_{n-3})\right)\\
&= \frac{1}{n}\, d(y_0,x_n) + \frac{1}{n}\, d(y_0,x_{n-1}) + \frac{n-2}{n}\, d(y_0,y_{n-3})\\
&\cdots\\
&\leq \frac{1}{n}\sum_{k=1}^{n} d(y_0,x_k) \leq diam(C).
\end{aligned}$$

Therefore, if $d(y_0, u) = diam(C)$, then we must have $d(y_0, x_k) = diam(C) > 0$ for all $k = 1, 2, \cdots, n$. Hence $y_0 \in C_0$ by definition of C_0. But, then we must have $y_0 = x_k$ for some $k = 1, 2, \cdots, n$. This is a contradiction. Therefore,

$$\sup\{d(x, u) : x \in C\} = d(y_0, u) < diam(C). \quad \blacksquare$$

A closed convex subset C of a convex metric space X is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive $T : C \to C$ has a fixed point.

We now prove that every closed convex subset of a convex metric space has fixed point property for nonexpansive mappings under normal structure.

Theorem 4.3.14 *Let X be a convex metric space with property (C). Let C be a nonempty closed convex bounded subset of X with normal structure and T a nonexpansive mapping from C into itself. Then T has a fixed point in C.*

Proof. Let $\mathcal{F}$ be the family of all nonempty closed convex subsets of C, each of which is mapped into itself by T. By property (C) and Zorn's lemma, $\mathcal{F}$ has a minimal element C_0. We show that C_0 consists of a single point. Let $x \in \mathcal{Z}_{C_0}$. Then

$$d(Tx, Ty) \leq d(x, y) \leq r_x(C_0) \text{ for all } y \in C_0.$$

Hence $T(C_0)$ is contained in the ball $B = B_{r(C_0)}[Tx]$. Because $T(C_0 \cap B) \subset C_0 \cap B$, the minimality of C_0 implies that $C_0 \subset B$. Hence $r_{Tx}(C_0) \leq r(C_0)$. Because $r(C_0) \leq r_x(C_0)$ for all $x \in C_0$, we have $r_{Tx}(C_0) = r(C_0)$. Hence $Tx \in \mathcal{Z}_{C_0}$ and $T(\mathcal{Z}_{C_0}) \subset \mathcal{Z}_{C_0}$. By Proposition 4.3.12, $\mathcal{Z}_{C_0} \in \mathcal{F}$. If $z, w \in \mathcal{Z}_{C_0}$, then $d(z, w) \leq r_z(C_0) = r(C_0)$. Hence, by normal structure,

$$\delta(\mathcal{Z}_{C_0}) \leq r(C_0) < \delta(C_0).$$

Because this contradicts the minimality of C_0, $diam(C_0) = 0$ and C_0 consists of a single point. $\blacksquare$

4.4 Normal structure coefficient and fixed points

In this section, we discuss another convexity structure on metric space and the existence of fixed points of uniformly L-Lipschitzian mappings in a metric space with uniformly normal structure.

Let $\mathcal{F}(X)$ denote a nonempty family of subsets of a metric space (X, d). We say that $\mathcal{F}(X)$ defines a *convexity structure* on X if $\mathcal{F}(X)$ is stable by intersection and that $\mathcal{F}(X)$ has *property* (R) if any decreasing sequence $\{C_n\}$ of nonempty closed bounded subsets of X with $C_n \in \mathcal{F}(X)$ has nonvoid intersection.

A subset of X is said to be *admissible* if it is an intersection of closed balls. We denote by $\mathcal{A}(X)$ the family of all admissible subsets of X. It is obvious that

$\mathcal{A}(X)$ defines a convexity structure on X. In this section, any other convexity structure $\mathcal{F}(X)$ on X is always assumed to contain $\mathcal{A}(X)$.

For a bounded subset C of X, we define the admissible hull of C, denoted by $ad(C)$, as the intersection of all those admissible subsets of X that contain C, i.e.,

$$ad(C) = \cap\{B : C \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.$$

A basic property of admissible hull is given in the following proposition.

Proposition 4.4.1 *Let C be a bounded subset of a metric space X and $x \in X$. Then*

$$r_x(ad(C)) = r_x(C).$$

Proof. Suppose $r = r_x(ad(C)) > r_x(C)$. Then $C \subseteq B_{\overline{r}}[x]$ for any $\overline{r}$ with $r_x(C) < \overline{r} < r$. It follows that $ad(C) \subseteq B_{\overline{r}}[x]$. Hence

$$r_x(ad(C)) = \sup\{d(x, y) : y \in ad(C)\} \leq \overline{r} < r,$$

a contradiction. ∎

We introduce normal structure and uniformly normal structure with respect to convexity structure $\mathcal{F}(X)$ in a metric space X, respectively.

Definition 4.4.2 *A metric space (X, d) is said to have normal structure if there exists a convexity structure $\mathcal{F}(X)$ such that $r(C) < diam(C)$ for all $C \in \mathcal{F}(X)$ that is bounded and consists of more than one point. We say that $\mathcal{F}(X)$ is normal.*

Definition 4.4.3 *A metric space (X, d) is said to have uniformly normal structure if there exists a convexity structure $\mathcal{F}(X)$ such that $r(C) \leq \alpha \cdot diam(C)$ for some constant $\alpha \in (0, 1)$ and for all $C \in \mathcal{F}(X)$ that is bounded and consists of more than one point. We also say that $\mathcal{F}(X)$ is uniformly normal.*

We now define the normal structure coefficient of X (with respect to a given convexity structure $\mathcal{F}(X)$):

The number $N(X)$ is said to be the *normal structure coefficient* if

$$N(X) = \inf\left\{\frac{diam(C)}{R(C)}\right\},$$

where the infimum is taken over all bounded $C \in \mathcal{F}(X)$ with $diam(C) > 0$. It is easy to see that X has uniformly normal structure if and only if $N(X) > 1$.

The following theorem shows that every convexity structure with uniformly normal structure has property (R).

Theorem 4.4.4 *Let X be a complete metric space with a convexity structure $\mathcal{F}(X)$ that is uniformly normal. Let $\{C_n\}$ be a decreasing sequence of nonempty bounded subsets of X with $C_n \in \mathcal{F}(X)$. Then $\cap_{n=1}^{\infty}\overline{C}_n \neq \emptyset$.*

Proof. Without loss of generality, we may assume that $diam(C_n) > 0$ for all $n \in \mathbb{N}$. Let η be a real number with $\tilde{N}(X) < \eta < 1$, where $\tilde{N}(X) = N(X)^{-1}$. Define a sequence $\{x_{n,k}\}$ in X as follows:

For arbitrary $x_{n,1} \in C_n, n \in \mathbb{N}$, select $x_{n,k} \in ad(\{x_{m,k-1}\}_{m\geq n})$ such that

$$\sup\{d(x_{n,k}, x) : x \in ad(\{x_{m,k-1}\}_{m\geq n}) \leq \eta\ diam(ad\{x_{m,k-1}\}_{m\geq n}).$$

Set $A_{n,k} := ad(\{x_{m,k}\}_{m\geq n})$. Observe that

$$A_{n,k} \subseteq C_n \text{ for all } n, k \in \mathbb{N}$$

and for $m \geq n$,

$$\begin{aligned} d(x_{n,k}, x_{m,k}) &\leq \sup\{d(x_{n,k}, x) : x \in A_{n,k-1}\} \\ &\leq \eta\ diam(A_{n,k-1}) \\ &\leq \eta\ diam(\{x_{i,k-1}\}_{i\geq 1}). \end{aligned}$$

For $k \geq 2$, we have

$$\begin{aligned} diam(\{x_{n,k}\}) &\leq \eta\ diam(\{x_{n,k-1}\}) \\ &\leq \eta^2\ diam(\{x_{n,k-2}\}) \\ &\cdots \\ &\leq \eta^{k-1}\ diam(\{x_{n,1}\}) \\ &\leq \eta^{k-1}\ diam(C_1). \end{aligned}$$

Now we consider a subsequence $\{x_{n,n}\}$ of $\{x_{n,k}\}$. Then $\{x_{n,n}\}$ is Cauchy, because

$$d(x_{n,n}, x_{m,m}) \leq \eta^{n-1}\ diam(C_1) \text{ for } m \geq n.$$

Therefore, there exists an $x \in \cap_{n=1}^{\infty}\overline{C}_n$ such that $\{x_{n,n}\}$ converges to x, i.e., $\cap_{n=1}^{\infty}\overline{C}_n \neq \emptyset$. ∎

Corollary 4.4.5 *Let X be a complete bounded metric space and $\mathcal{F}(X)$ a convexity structure of X with uniformly normal structure. Then $\mathcal{F}(X)$ has property (R).*

We now introduce the property $(\mathcal{P})$ for metric spaces.

Definition 4.4.6 *A metric space (X, d) is said to have property $(\mathcal{P})$ if given any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \cap_{n=1}^{\infty} ad(\{z_j : j \geq n\})$ such that*

$$\limsup_{n\to\infty} d(z, x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, x_n).$$

Remark 4.4.7 *If X has property (R), then $\cap_{n=1}^{\infty} ad(\{z_j : j \geq n\}) \neq \emptyset$. Also, if X is a weakly compact convex subset of a normed space, then admissible hulls are closed convex and $\cap_{n=1}^{\infty} ad\{z_j : j \geq n\} \neq \emptyset$ by weak compactness of X, and that X possesses property $(\mathcal{P})$ follows directly from the w-lsc of the function $\limsup_{n\to\infty} \|x_n - x\|$.*

We establish the following key result that can be viewed as the metric space formulation of Theorem 3.4.20.

Theorem 4.4.8 *Let (X,d) be a complete bounded metric space with both property $(\mathcal{P})$ and uniformly normal structure. Let $N(X)$ be the normal structure coefficient with respect to the given convexity structure $\mathcal{F}(X)$. Then for any sequence $\{x_n\}$ in X and any constant $\alpha > \tilde{N}(X)$, there exists a point $z \in X$ satisfying the properties:*

(a) $d(z,y) \leq \limsup_{n\to\infty} d(x_n, y)$ *for all* $y \in X$,

(b) $\limsup_{n\to\infty} d(z, x_n) \leq \alpha \; diam(\{x_n\})$.

Proof. (a) For each $n \in \mathbb{N}$, let $A_n = ad(\{x_j : j \geq n\})$. Then $\{A_n\}$ is a decreasing sequence of admissible subsets of X and hence $A := \cap A_n \neq \emptyset$ by Corollary 4.4.5. We observe by Proposition 4.4.1 that

$$\begin{aligned} diam(A_n) &= \sup\{r_x(A_n) : x \in A_n\} \\ &= \sup_{x\in A_n} \sup_{j\geq n} d(x, x_j) \\ &= \sup_{j\geq n} \sup_{x\in A_n} d(x, x_j) = \sup_{j\geq n} r_{x_j}(A_n) \\ &= \sup_{j\geq n} \sup_{i\geq n} d(x_j, x_i) \\ &\leq \sup\{d(x_i, x_j) : i, j \in \mathbb{N}\} = diam(\{x_n\}). \end{aligned}$$

For any $z \in A$ and any $y \in X$, we have

$$\sup_{j\geq n} d(y, x_j) = r_y(A_n) \geq r_y(A) \geq d(y, z).$$

It follows that

$$d(y,z) \leq \limsup_{n\to\infty} d(y, x_n).$$

(b) Without loss of generality, we may assume that $diam(\{x_n\}) > 0$. Then for $\alpha > \tilde{N}(X)$, we choose $\varepsilon > 0$ so small that

$$\tilde{N}(X) diam(\{x_n\}) + \varepsilon \leq \alpha \; diam(\{x_n\}).$$

By definition, one can find *a* $z_n \in A_n$ such that

$$\begin{aligned} r_{z_n}(A_n) &< r(A_n) + \varepsilon \\ &\leq \tilde{N}(X) \; diam(A_n) + \varepsilon \\ &\leq \tilde{N}(X) diam(\{x_n\}) + \varepsilon \\ &\leq \alpha \; diam(\{x_n\}). \end{aligned}$$

Hence for each $i \geq 1$,

$$\limsup_{m\to\infty} d(z_i, x_m) \leq \alpha \; diam(\{x_i\}).$$

Now property $(\mathcal{P})$ yields a point $z \in \cap_{i=1}^{\infty} ad(\{z_n : n \geq i\})$ such that

$$\limsup_{m\to\infty} d(z, x_m) \leq \limsup_{i\to\infty} \limsup_{m\to\infty} d(z_i, x_m).$$

Thus, $z \in A$ and satisfies

$$\limsup_{m\to\infty} d(z, x_m) \leq \alpha \; diam(\{x_i\}). \quad \blacksquare$$

We now present the existence theorem for uniformly L-Lipschitzian mappings in a metric space.

Theorem 4.4.9 *Let (X, d) be a complete bounded metric space with both property $(\mathcal{P})$ and uniformly normal structure and $T : X \to X$ a uniformly L-Lipschitzian mapping with $L < \tilde{N}(X)^{-1/2}$. Then T has a fixed point in X.*

Proof. Choose a constant $\alpha, 1 > \alpha > \tilde{N}(X)$, such that $L < \alpha^{-1/2}$. Let $x_0 \in X$. By Theorem 4.4.8, we can inductively construct a sequence $\{x_m\}_{m=1}^{\infty}$ in X:

for each integer $m \geq 0$,

(a) $\limsup_{i\to\infty} d(x_{m+1}, T^i x_m) \leq \alpha \; diam(\{T^i x_m\})$;

(b) $d(x_{m+1}, y) \leq \limsup_{i\to\infty} d(T^i x_m, y)$ for all $y \in X$.

Set $r_m := \limsup_{i\to\infty} d(x_{m+1}, T^i x_m)$ and $h := \alpha L^2 < 1$. Note for each $i > j \geq 0$,

$$\begin{aligned} d(T^j x_m, T^i x_m) &\leq Ld(x_m, T^{i-j} x_m) \\ &\leq L \limsup_{n\to\infty} d(T^n x_{m-1}, T^{i-j} x_m) \qquad (by\ (b)) \\ &\leq L^2 \limsup_{n\to\infty} d(T^n x_{m-1}, x_m) \\ &\leq L^2 r_{m-1}. \end{aligned}$$

Observe that

$$\begin{aligned} r_m &= \limsup_{i\to\infty} d(x_{m+1}, T^i x_m) \\ &\leq \alpha \; diam(\{T^i x_m\}) \\ &\leq \alpha L^2 r_{m-1} = h r_{m-1} \\ &\cdots \\ &\leq h^m r_0. \end{aligned}$$

Hence for each integer $i \geq 0$,

$$\begin{aligned} d(x_{m+1}, x_m) &\leq d(x_{m+1}, T^i x_m) + d(x_m, T^i x_m) \\ &\leq d(x_{m+1}, T^i x_m) + \limsup_{j\to\infty} d(T^j x_{m-1}, T^i x_m) \\ &\leq d(x_{m+1}, T^i x_m) + L \limsup_{j\to\infty} d(T^j x_{m-1}, x_m) \\ &\leq d(x_{m+1}, T^i x_m) + L r_{m-1}, \end{aligned}$$

which implies that

$$d(x_{m+1}, x_m) \leq r_m + Lr_{m-1} \leq (h^m + Lh^{m-1})r_0.$$

This implies that $\{x_m\}$ is Cauchy. Let $\lim_{m\to\infty} x_m = v \in X$. Observe that

$$\begin{aligned} d(v, Tv) &\leq d(v, x_{m+1}) + d(x_{m+1}, T^i x_m) + d(T^i x_m, Tv) \\ &\leq d(z, x_{m+1}) + d(x_{m+1}, T^i x_m) + Ld(T^{i-1} x_m, v) \\ &\leq d(v, x_{m+1}) + d(x_{m+1}, T^i x_m) + Ld(T^{i-1} x_m, x_{m+1}) + Ld(x_{m+1}, v), \end{aligned}$$

which implies that

$$d(v, Tv) \leq (1 + L)d(v, x_{m+1}) + (1 + L)r_m \to 0 \text{ as } m \to \infty.$$

Therefore, v is a fixed point of T. ▮

4.5 Lifschitz's coefficient and fixed points

In this section, we give an existence theorem concerning uniformly L-Lipschitzian mappings in a metric space.

First, we define the Lifschitz's coefficient of a metric space:

Let (X, d) be a metric space. Then the Lifschitz's coefficient $\kappa(X)$ is a number defined by

$$\begin{aligned} \kappa(X) &= \sup\{\beta > 0 : \exists\ \alpha > 1 \text{ such that for all } x, y \in X, \text{ for all } r > 0, \\ &\quad [d(x, y) > r \Rightarrow \exists\ z \in X \text{ such that} B_{\alpha r}[x] \cap B_{\beta r}[y] \subseteq B_r[z]]\}. \end{aligned}$$

It is clear that $\kappa(X) \geq 1$ for any metric space X. For a strictly convex Banach space X, $\kappa(X) > 1$ and for a Hilbert space H, $\kappa(H) = \sqrt{2}$.

We are now in a position to prove a fundamental existence theorem for uniformly L-Lipschitzian mappings in a metric space with Lifschitz's coefficient $\kappa(X)$.

Theorem 4.5.1 *Let (X, d) be a bounded complete metric space and $T : X \to X$ a uniformly L-Lipschitzian mapping with $L < \kappa(X)$. Then T has a fixed point in X.*

Proof. If $\kappa(X) = 1$, then T is contraction and hence T has a unique fixed point. So, suppose $\kappa(X) > 1$. For $b \in (L, \kappa(X))$, there exists $a > 1$ such that

$$\forall\ u, v \in X,\ r > 0 \text{ with } d(x, y) > r \Rightarrow \exists z \in X :\ B_{br}[u] \cap B_{ar}[v] \subset B_r[z]. \quad (4.16)$$

For any $x \in X$, let

$$r(x) = \inf\{R > 0 :\ \text{there exists } y \in X \text{ such that } \limsup_{n\to\infty} d(T^n x, y) \leq R\}.$$

Observe that r is a lower semicontinuous and $r(x) = 0$ implies $x = Tx$.

Take $\lambda \in (0,1)$ such that $\gamma = \min\{\lambda a, \lambda b/L\} > 1$. We now show that there exists a sequence $\{y_m\}$ in X that satisfies the following:

$$r(y_{m+1}) \leq \lambda r(y_m) \text{ and } d(y_m, y_{m+1}) \leq (\lambda + \gamma) r(y_m) \text{ for all } m \in \mathbb{N}_0. \quad (4.17)$$

Indeed, consider an arbitrary point $y_0 \in X$ and assume that $y_0, y_1, \cdots, y_m$ are given. We now construct y_{m+1}. If $r(y_m) = 0$, then $y_{m+1} = y_m$. If $r(y_m) > 0$, then for a number $\lambda r(y_m)$, there exists $n \in \mathbb{N}$ such that

$$d(y_m, T^n y_m) > \lambda r(y_m).$$

From the definition of $r(y_m)$, there exists $x \in X$ such that

$$\limsup_{n\to\infty} d(y_m, T^n x) \leq r(y_m) < \gamma r(y_m).$$

Hence for $i > j$

$$d(T^i x, T^j y_m) \leq L\ d(T^{i-j} x, y_m),$$

which implies that

$$\limsup_{i\to\infty} d(T^i x, T^j y_m) \quad \leq \quad L\ \limsup_{i\to\infty} d(T^{i-j} x, y_m) \leq L\gamma\ r(m).$$

Because

$$B_{\gamma r(y_m)}[y_m] \cap B_{L\gamma r(y_m)}[T^n y_m] \subset B_{a\lambda r(y_m)}[y_m] \cap B_{b\lambda r(y_m)}[T^n y_m] = C,$$

the set C is contained in a closed ball centered at w with radius $\lambda r(y_m)$ (Condition (4.16)). Thus, $\limsup_{n\to\infty} d(T^n x, w) \leq \lambda r(y_m)$. Take $w = y_{m+1}$, and it follows from above that $\{y_m\}$ satisfies (4.17).

Note

$$r(y_{m+1}) \leq \lambda r(y_m) \leq \cdots \leq \lambda^{m+1} r(y_0) \to 0 \text{ as } m \to \infty$$

and

$$d(y_m, y_{m+1}) \leq (\lambda + \gamma) r(y_m) \to 0 \text{ as } m \to \infty.$$

Hence $\{y_m\}$ converges to $v \in M$. But because $r(v) = 0$, v is a fixed point of T. ∎

Bibliographic Notes and Remarks

Most of the results presented in Section 4.1 may be found in Goebel and Kirk [59], Khamsi and Kirk [85], Kirk and Sims [91], and Martin [106]. Theorem 4.1.16 first appeared in Matkowski [108] as a generalization of the result of Boyd and Wong [22], and it was recently extended for non-self mappings in Reich and

Zaslavski [133](see also recent results of Agarwal, O'Regan and their coworkers [1, 117]).

The notion of nearly non-Lipschitzian mappings was introduced by Sahu [137]. Some existence theorems for demicontinuous nearly contraction and nearly asymptotically nonexpansive mappings were also established in [137].

The fixed point theory of multivalued contraction self-mappings was first proved in Nadler [115]. It was extended for multivalued non-self contraction mappings by Assad and Kirk [4].

The results describes in Sections 4.3∼4.5 can be found in Lifschitz [97], Lim and Xu [99], and Takahashi [154].

Exercises

4.1 Let X be a complete metric space and $T : X \to X$ a mapping such that T^m is contraction for some $m \in \mathbb{N}$. Show that T has a unique fixed point.

4.2 Let (X, d) be a metric space and $T : X \to X$ a mapping. T is said to be a Zamfirescu mapping if there exist the real number a, b, and c satisfying $0 < a < 1, 0 < b, c < 1/2$ such that for each pair x, y in X, at least one of the following is true:

(Z_1) $d(Tx, Ty) \leq ad(x, y)$,

(Z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(Z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

If (X, d) is a complete metric space and $T : X \to X$ a Zamfirescu mapping, show that T has a unique fixed point $z \in X$ and for each $x \in X, \{T^n x\}$ converges to z.

4.3 Let T be a mapping from a complete metric space X into itself satisfying the condition:

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(y, Tx) + d(x, Ty)]$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that $a + 2b + 2c < 1$. Show that T has a unique fixed point $z \in X$ and for each $x \in X, \{T^n x\}$ converges to z.

4.4 Let T be a mapping from a complete metric space into itself. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, Tx) < \delta \Rightarrow T(B_\varepsilon[x]) \subset B_\varepsilon[x].$$

If $d(T^n x, T^{n+1} x) \to 0$ for some $x \in X$, show that the sequence $\{T^n x\}$ converges to z, which is a fixed point of T.

4.5 Let X be a complete metric space and $T : X \to X$ an expansion mapping, i.e., there exists constant $k > 1$ such that

$$d(Tx, Ty) \geq kd(x, y) \text{ for all } x, y \in X.$$

Assume that $T(X) = X$. Show that

(a) T is one to one,

(b) T has a unique fixed point $z \in X$ with $T^n x \to z$ as $n \to \infty$ for some $x \in X$.

4.6 Let (X, d) be a complete metric space and $T : X \to CB(X)$ a mapping. If α is a function from $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r \to t^+} \alpha(r) < t$ for every $t \in [0, \infty)$ and if

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, show that T has a fixed point in X.