Chapter 1

Fundamentals

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the book. Moreover, the results in this chapter may be found in most standard books on functional analysis.

1.1 Topological spaces

Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+ := [0, \infty)$ a function. Then d is called a *metric* on X if the following properties hold:

(d_1) $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called *distance between x and y*, and the ordered pair (X, d) is called *metric space.*

Example 1.1.1 *The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric d is called the usual metric for $\mathbb{R}$.*

For any $r > 0$ and an element x in a metric space (X, d), we define

$B_r(x) := \{y \in X : d(x, y) < r\}$, the *open ball* with center x and radius r;

$B_r[x] := \{y \in X : d(x, y) \leq r\}$, the *closed ball* with center x and radius r;

$\partial B_r(x) := \{y \in X : d(x, y) = r\}$, the *boundary of ball* with center x and radius r.

For a subset C of X and a point $x \in X$, the distance between x and C, denoted by $d(x, C)$, is defined as the smallest distance from x to elements of C. More precisely,

$$d(x, C) = \inf_{x \in C} d(x, y).$$

The number $\sup\{d(x, y) : x, y \in C\}$ is referred to as the *diameter of set C* and is denoted by $diam(C)$. If $diam(C)$ is finite, then C is said to be *bounded*, and

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if not, then C is said to be *unbounded*. In other words, C is bounded if there exists a sufficiently large ball that contains C.

Interior points and open set – Let G be a subset of a metric space (X, d). Then $x \in G$ is said to be an *interior of* G if there exists an $r > 0$ such that $B_r(x) \subset G$. The set G is said to be *open* if all its points are interior or is the empty set. The interior of set G is denoted by $int(G)$.

Observation

- $int(G) \subset G$ for any subset G of metric space X.
- For any open set $G \subset X$, $int(G) = G$.
- The empty set $\emptyset$ and entire space X are open.

Definition 1.1.2 *Let X be a nonempty set and τ a collection of subsets of X. Then τ is said to be a topology on X if the following conditions are satisfied:*

(i) $\emptyset \in \tau$ and $X \in \tau$,

(ii) τ is closed under arbitrary unions,

(iii) τ is closed under finite intersections.

The ordered pair (X, τ) is called topological space.

Observation

- The members of τ are called τ-*open sets* or simply *open sets*.

Definition 1.1.3 *A topological space is said to be metrizable if its topology can be obtained from a metric on the underlying space.*

Denoting the class of all open sets of a metric space (X, d) by τ_d, then we have

(1) $\emptyset$ and X are in τ_d,

(2) an arbitrary union of open sets is open,

(3) a finite intersection of open sets is open.

The class τ_d is called a *metric topology* on X.

Definition 1.1.4 *Let C be a subset of a topological space X. Then the interior of C is the union of all open subsets of C. It is denoted by $int(C)$.*

In other words, if $\{G_i : i \in \Lambda\}$ are all open subsets of C, then $int(C) = \cup_{i \in \Lambda}\{G_i : G_i \subset C\}$.

Observation

- $int(C)$ is open, because it is union of open sets.
- $int(C)$ is the largest open set of C.
- If G is an open subset of C, then $G \subset int(C) \subset C$.

Definition 1.1.5 *A set F in a topological space X whose complement $F^c = X - F$ is open is called a closed set.*

Theorem 1.1.6 *Let $\mathcal{C}$ be a collection of all closed sets in a topological space (X, τ). Then $\mathcal{C}$ has the following properties:*

(i) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$,

(ii) $\mathcal{C}$ is closed under arbitrary intersections,

(iii) $\mathcal{C}$ is closed under finite unions.

Definition 1.1.7 *Let C be a subset of a topological space X. Then the closure of C is the intersection of all closed supersets of C. The closure of C is denoted by $\overline{C}$.*

In other words, if $\{F_i : i \in \Lambda\}$ is a collection of all closed supersets of C in X, then $\overline{C} = \cap_{i\in\Lambda} F_i$.

Observation

- $\overline{C}$ is closed, because it is the intersection of closed sets.
- $\overline{C}$ is the smallest closed superset of C.
- If F is a closed subset of X containing C, then $C \subset \overline{C} \subset F$.

Theorem 1.1.8 *Let C be a subset of a topological space X. Then C is closed if and only if $C = \overline{C}$.*

Exterior points and boundary of sets – Let C be a subset of a topological space X. Then the *exterior* of C, written by $ext(C)$, is the interior of the complement of C, i.e., $ext(C) = int(C^c)$. The *boundary of* C is a set of points that do not belong to the interior or the exterior of C. The boundary of set C is denoted by $\partial(C)$. Obviously, $\partial(C) = \overline{C} \cap \overline{(X \setminus C)}$ is a closed set.

Proposition 1.1.9 *Let A and B be two subsets of a topological space X. Then the following properties hold:*

Properties of interiors	*Properties of closures*
$int(int(A)) = int(A)$	$\overline{(\overline{A})} = \overline{A}$
$int(A \cap B) = int(A) \cap int(B)$	$\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
$int(A \cup B) \supset int(A) \cup int(B)$	$\overline{(A \cup B)} = \overline{A} \cup \overline{B}$
$A \subset B \Rightarrow int(A) \subset int(B)$	$A \subset B \Rightarrow \overline{A} \subset \overline{B}$

Definition 1.1.10 *Let τ_1 and τ_2 be two topologies on a topological space X. Then τ_1 is said to be weaker than τ_2 if $\tau_1 \subset \tau_2$.*

Note that if τ_1 and τ_2 are two topologies on X such that $\tau = \tau_1 \cap \tau_2$. Then the topology τ is weaker than τ_1 and τ_2 both.

Theorem 1.1.11 *Let $\{\tau_i : i \in \Lambda\}$ be a collection of topologies on a topological space X. Then the intersection $\cap_{i\in\Lambda}\tau_i$ is also a topology on X.*

We now turn to the notion of a base for the topology τ.

Definition 1.1.12 *Let (X, τ) be a topological space. Then a subclass $\mathcal{B}$ of τ is said to be a base for τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.*

Observation

- Every topology has a base. In fact, we can take $\mathcal{B} = \tau$.
- In a metric space (X, d), collection of all open balls $B_r(x)$ ($x \in X$, $r > 0$) is a base for the metric topology.

Then, we have the following theorem:

Theorem 1.1.13 *Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for τ if and only if, for every $x \in X$ and every open set G containing x, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset G$.*

We now consider a base of open sets at a point.

Definition 1.1.14 *Let (X, τ) be a topological space and $x_0 \in X$. Then the collection $\mathcal{B}_{x_0} \subset \tau$ is called a base at a point x_0 if, for any open set G containing x_0, there exists $B \in \mathcal{B}_{x_0}$ such that $x_0 \in B \subset G$.*

Observation

- In the metric topology of a metric space (X, d), the collection of all $B_r(x_0)$, where r runs through the positive real numbers, constitutes a base at a point $x_0 \in X$.

Neighborhoods – Let X be a topological space and G an open set. Then G is called an *open neighborhood* of a point $x_0 \in X$ if $x_0 \in G$. The set G without x_0, i.e., $G \setminus \{x_0\}$, is called a *deleted open neighborhood* of a point $x_0 \in X$. A subset C of X is said to be a *neighborhood of a point* $x_0 \in X$ if there exists an open set $G \in \tau$ such that $x_0 \in G \subset C$.

Let (X, τ) be a topological space. Then a collection ν of neighborhoods of $x_0 \in X$ is said to be a *neighborhood base at a point* x_0 if every neighborhood of x_0 contains a member of ν.

A collection σ of subsets of a topological space (X, τ) is said to be a *subbase* for τ if $\sigma \subset \tau$ and every member of τ is a union of finite intersections of sets from σ. In other words, σ is a subbase for τ if $\sigma \subset \tau$ and for all $G \in \tau$ and $x \in G$, there are sets $U_1, U_2, \cdots, U_n$ in σ such that $x \in \cap_{i=1}^{n} U_i \subset G$.

Let (X, τ) be a topological space. Then X is said to be

1. a T_0-*space* if x and y are any two distinct points in X, then there exists an open set that contains one of them, but not the other;

2. a T_1-*space* if x and y are two distinct points in X, there exists an open set U containing x and not y, and there exists another open set V containing y, but not x;

3. a T_2-*space* or *Hausdorff topological space* if x and y are two distinct points in X, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

Observation

- Every Hausdorff space is a T_1-space.
- A topological space X is T_1-space if and only if every subset consisting of a single point is closed.
- Every metric space is a Hausdorff space.

A topological space (X, τ) is said to be *compact* if every open cover has a finite subcover, i.e., if whenever $X = \cup_{i \in \Lambda} G_i$, where G_i is an open set, then $X = \cup_{i \in \Lambda_0} G_i$ for some finite subset Λ_0 of Λ.

A subset C of a topological space (X, τ) is said to be *compact* if every open cover has finite open subcover, i.e., if whenever $C \subseteq \cup_{i \in \Lambda} G_i$, where G_i is an open set, then $C \subseteq \cup_{i \in \Lambda_0} G_i$ for some finite subset Λ_0 of Λ.

Observation

- Every finite set of a topological space is compact.
- Every closed subset of a compact space is compact.
- In a compact Hausdorff space, a set is compact if and only if it is closed.

Net – Let D be a nonempty set and $\preceq$ a relation on D. Then the ordered pair $(D, \preceq)$ is said to be *directed* if

(i) $\preceq$ is reflexive: $\alpha \preceq \alpha$ for all $\alpha \in D$;

(ii) $\preceq$ is transitive: whenever $\alpha \preceq \beta$ and $\beta \preceq y \Rightarrow \alpha \preceq y$ for all $\alpha, \beta, \gamma \in D$;

(iii) for any two elements α and β, there exists γ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Observation

- $(\mathbb{N}, \geq)$ is a directed set.
- If $X \neq \emptyset$, then $(P(X), \subseteq)$ and $(P(X), \supseteq)$ are directed sets, where $P(X)$ is the power set of X.
- Every lattice is a directed set.

A *net*, or a *generalized sequence* in a set X is a mapping S from a directed set D into X. The net $\{x_\alpha : \alpha \in D\}$ is simply written as $\{x_\alpha\}$.

Let $\{x_\alpha : \alpha \in D\}$ be a net in a set X and let E be another directed set. Then a net $\{x_{\alpha_\beta} : \beta \in E\}$ in X is said to be a *subnet* of $\{x_\alpha : \alpha \in D\}$ if it satisfies the following conditions:

(i) $\{x_{\alpha_\beta} : \beta \in E\} \subset \{x_\alpha : \alpha \in D\}$;

(ii) for any $\alpha_0 \in D$, there exists $\beta_0 \in E$ such that $\alpha_0 \preceq \alpha_\beta$ exists $\beta_0 \preceq \beta$.

A net $\{x_\alpha : \alpha \in D\}$ in a topological space X is said to *converge to the point x in X* if for every neighborhood U of x, there exists $\alpha_0 \in D$ such that $x_\alpha \in U$ whenever $\alpha \succeq \alpha_0$. In this case, we write

$$x_\alpha \to x, \text{ or } \lim_\alpha x_\alpha = x.$$

A point x in a topological space X is said to be a *cluster point* of a net $\{x_\alpha : \alpha \in D\}$ if for every neighborhood U of x and every $\alpha \in D$, there exists $\beta \in D$ such that $\beta \succeq \alpha$ and $x_\beta \in U$.

Theorem 1.1.15 *Let $\{x_\alpha\}_{\alpha \in D}$ be a net in a topological space X and let $x \in X$. Then x is a cluster point of the net $\{x_\alpha\}_{\alpha \in D}$ if and only if the net $\{x_\alpha\}_{\alpha \in D}$ has a subnet converging to x.*

In a metric space (X, d), a sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$, i.e., if given $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ in a metric space (X, d) is said to be *Cauchy* if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

Observation

- In a Hausdorff topological space, the limit of a net is unique.
- In a metric space, every convergent sequence is Cauchy.

A subset E of a directed set D is said to be *eventual* if there exists $\beta \in D$ such that for all $\alpha \in D$, $\alpha \preceq \beta$ implies that $\alpha \in E$. A net $S : D \to X$ is said to be *eventually* in a subset C of X if the set $S^{-1}(C)$ is an eventual subset of D. A net $\{x_\alpha\}$ in a set X is called a *universal net* if for each subset C of X, either $\{x_\alpha\}$ is eventually in C or $\{x_\alpha\}$ eventually in $X \setminus C$.

The following facts are important:

(a) Every net in a set has a universal subnet.

(b) If $f : X_1 \to X_2$ is a mapping and if $\{x_\alpha\}$ is a universal net in X_1, then $\{f(x_\alpha)\}$ is a universal net in X_2.

(c) If X is compact and if $\{x_\alpha\}$ is a universal net in X, then $\lim_\alpha x_\alpha$ exists.

We now state the following important result:

Theorem 1.1.16 *For a topological space (X, τ), the following statements are equivalent:*

(a) X is compact.

(b) For any collection of closed sets $\{F_i\}_{i\in\Lambda}$ having the finite intersection property (i.e., the intersection of any finite number of sets from the collection is nonempty), then $\bigcap_{i\in\Lambda} F_i \neq \emptyset$.

(c) Every net in X has a limit point (or, equivalently, every net has a convergent subnet).

(d) Every filter in X has a limit point (or, equivalently, every net has a convergent subfilter).

(e) Every ultrafilter in X is convergent.

We now turn our attention to the concept of continuity in topological spaces.

Definition 1.1.17 *Let (X, τ) and (Y, τ') be two topological spaces. Then a function $f : X \to Y$ is said to be continuous relative to τ and τ' (more precisely, $\tau - \tau'$ continuous) or simply continuous at a point $x \in X$ if for each $V \in \tau'$ with $f(x) \in V$, there exists $U \in \tau$ such that $x \in U$ and $f(U) \subset V$.*

The function f is called *continuous* if it is continuous at each point of X. Using the concept of net, we have the following result for continuity of a function in a topological space.

Theorem 1.1.18 *Let X and Y be two topological spaces and let f be a mapping from X into Y. Then f is continuous at a point x in X if and only if for every net $\{x_\alpha\}$ in X,*

$$x_\alpha \to x \Rightarrow f(x_\alpha) \to f(x).$$

Some other formulations for continuous functions are the following:

Theorem 1.1.19 *Let f be a function from a topological space (X, τ) into another topological space (Y, τ'). Then the following statements are equivalent:*

(1) f is continuous (i.e., $\tau - \tau'$ continuous).

(2) For each $V \in \tau', f^{-1}(V) \in \tau$.

(3) For each closed subset A of Y, $f^{-1}(A)$ is closed in X.

(4) For all $A \subset X, f(\overline{A}) \subset \overline{f(A)}$.

(5) There exists a subbase σ of τ' such that $f^{-1}(V) \in \tau$ for all $V \in \sigma$.

The following result shows that continuous image of a compact set is compact.

Theorem 1.1.20 *Let X and Y be two topological spaces and let $T : X \to Y$ be a continuous mapping. If $C \subseteq X$ is compact, then $T(C)$ is compact.*

The following result shows that there exists the smallest topology for which each member of $\{f_i : i \in \Lambda\}$ is continuous.

Theorem 1.1.21 *Let $\{(X_i, \tau_i) : i \in \Lambda\}$ be an indexed family of topological spaces, X any set, and $\{f_i : i \in \Lambda\}$ an indexed collection of functions such that for each $i \in \Lambda$, f_i is a function from X to X_i. Then there exists the smallest topology τ on X that makes each f_i continuous (i.e., $\tau - \tau_i$ continuous).*

Proof. Let $\sigma = \{f_i^{-1}(V_i) : V_i \subset X_i \text{ } \textit{is open in } \tau_i \ (i \in \Lambda)\}$ be a subbase for the topology τ given by

$$\tau = \{\cup_{F \in \mathcal{F}} \cap_{C \in F} C : \mathcal{F} \subset \overline{\sigma}\} \cup \{\emptyset, X\}, \tag{1.1}$$

where $\overline{\sigma}$ is the set of all finite subsets of σ. Thus, $G \subset X$ is open in τ if and only if for every $x \in G$, there are $i_1, i_2, \cdots, i_n \in \Lambda$ and $V_{i_1} \in \tau_{i_1}$, $V_{i_2} \in \tau_{i_2}, \cdots$, $V_{i_n} \in \tau_{i_n}$ such that $x \in \cap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k}) \subset G$. $\blacksquare$

Remark 1.1.22 *The topology τ on X defined by (1.1) making each f_i continuous ($\tau - \tau'$ continuous) is called the weak topology generated by $\mathcal{F}$ and is denoted by $\sigma(X, \mathcal{F})$.*

Product space – Let $X_1, X_2, \cdots, X_n$ be n arbitrary sets with the Cartesian product $X = X_1 \times X_2 \times \cdots \times X_n$. For each $i = 1, 2, \cdots, n$, define $\pi_i : X \to X_i$ by $\pi_i(x_1, x_2, \cdots, x_n) = x_i$. Then π_i is called the *projection on* X_i or *the* i^{th} *projection*. If $x \in X$, then $\pi_i(x)$ is called the i^{th} coordinate of x.

Theorem 1.1.23 *Let $\{(X_i, \tau_i) : i = 1, 2, \cdots, n\}$ be a collection of topological spaces and (X, τ) their topological product, i.e., $X = \prod_i X_i$ and $\tau = \bigcap_i \tau_i$. Then each projection π_i is continuous. Moreover, if Y is any topological space, then a function $f : Y \to X$ is continuous if and only if the mapping $\pi_i o f : Y \to X_i$ is continuous for all $i = 1, 2, \cdots, n$.*

Theorem 1.1.24 (Tychonoff's theorem) – *The Cartesian product X of an arbitrary collection $\{X_i\}_{i \in \Lambda}$ of compact spaces is compact (with respect to product topology).*

1.2 Normed spaces

A linear space or vector space X over the field $\mathbb{K}$ (the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1. $x + y = y + x$,
2. $(x + y) + z = x + (y + z)$,
3. there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$,
4. for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse* or *the negative of* x such that $x + (-x) = 0$,
5. $\alpha(x + y) = \alpha x + \alpha y$,
6. $(\alpha + \beta)x = \alpha x + \beta x$,
7. $(\alpha\beta)x = \alpha(\beta x)$,
8. $1 \cdot x = x$.

The elements of a vector space X are called *vectors*, and the elements of $\mathbb{K}$ are called *scalars*. In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Observation

- With the usual addition and multiplication, $\mathbb{R}$ and $\mathbb{C}$ are linear spaces over $\mathbb{R}$.
- $X = \{x = (a_1, a_2, \cdots) : a_i \in \mathbb{R}\}$ is a linear space.
- The set of solutions of a linear differential equation (and linear partial differential equation) is a linear space.

A subset S of a linear space X is a *linear subspace* (or a *subspace*) of X if S is itself a linear space, i.e., $\alpha x + \beta y \in S$ for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in S$.

If S is a subset of a linear space X, then the *linear span* of S is the intersection of all linear subspaces containing S. It is the smallest linear subspace of X containing S. The linear span of set S is denoted by $[S]$.

Given the points $x_1, x_2, \cdots, x_n$ of a linear space X, then the element

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n, \quad a_i \in \mathbb{K}$$

is called *linear combination* of $\{x_1, x_2, \cdots, x_n\}$.

Proposition 1.2.1 *Let S be a nonempty subset of a linear space X. Then the linear span of S is the set of all linear combinations of elements of S.*

A linear space X is said to be *finite-dimensional* if it is generated by the linear combination of a finite number of points (which are linearly independent). Otherwise, it is infinite-dimensional. The dimension of a linear space X is denoted by $dim(X)$.

Convex set – Let C be a subset of a linear space X. Then C is said to be *convex* if $(1-\lambda)x + \lambda y \in C$ for all $x, y \in C$ and all scalar $\lambda \in [0, 1]$.

By definition of convexity, we have the following fact:

Proposition 1.2.2 *Let C be a subset of a linear space X. Then C is convex if and only if $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \in C$ for any finite set $\{x_1, x_2, \cdots, x_n\} \subseteq C$ and any scalars $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.*

Convex hull – Let C be an arbitrary subset (not necessarily convex) of a linear space X. Then the *convex hull* of C in X is the intersection of all convex subsets of X containing C. It is denoted by $co(C)$. Hence

$$co(C) = \cap\{D \subseteq X : C \subseteq D,\ D\ is \text{ convex}\}.$$

Thus, $co(C)$ is the unique smallest convex set containing C. Clearly,

$$\begin{aligned} co(C) &= \left\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : x_i \in C, \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1\right\} \\ &= \text{the set of all convex combination of elements of } C. \end{aligned}$$

The *closure of convex hull* of C is denoted by $\overline{co(C)}$. Thus,

$$\overline{co(C)} = \overline{\left\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : x_i \in C, \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1\right\}}.$$

The *closed convex hull* of C in X is the intersection of all closed convex subsets of X containing C. It is denoted by $\overline{co}(C)$. Thus,

$$\overline{co}(C) = \cap\{D \subseteq X : C \subseteq D,\ D \text{ is closed and convex}\}.$$

One may easily see that closure of convex hull of C is closed convex hull of C, i.e., $\overline{co}(C) = \overline{co(C)}$.

Observation

- The empty set $\emptyset$ is convex.
- For two convex subsets C and D in a linear space X, we have
  (i) $C + D$ is convex,
  (ii) λC is convex for any scalar λ.
- Any translate $C + x_0$ of a convex set C is convex.
- If $\{C_i : i \in \Lambda\}$ is any family of convex sets in a linear space X, then $\cap_i C_i$ is convex.
- If C is a convex subset of a linear space X, then
  (i) the closure $\overline{C}$ and the interior $int(C)$ are convex,
  (ii) $co(C) = C$.
- If C is a subset of a linear space, $\overline{co}(C) = \overline{co(C)}$.
- In general, $\overline{co}(C) \neq co(\overline{C})$.

The vector space axioms only describe algebraic properties of the elements of the space: vector addition, scalar multiplication, and other combinations of these. For the topological concepts such as openness, closure, convergence, and completeness, we need a measure of distance in a space.

Definition 1.2.3 *Let X be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $f : X \to \mathbb{R}^+$ a function. Then f is said to be a norm if the following properties hold:*

(N_1) $f(x) = 0$ *if and only if* $x = 0$; *(strict positivity)*

(N_2) $f(\lambda x) = |\lambda| f(x)$ *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$; *(absolute homogeneity)*

(N_3) $f(x + y) \leq f(x) + f(y)$ *for all* $x, y \in X$. *(triangle inequality or subadditivity)*

The ordered pair (X, f) is called a normed space.

Observation

- $f(x) \geq 0$ for all $x \in X$.

- $|f(x) - f(y)| \leq f(x - y)$ and $|f(x) - f(y)| \leq f(x + y)$ for all $x, y \in X$.
- f is a continuous function, i.e., $x_n \to x \Rightarrow f(x_n) \to f(x)$.
- f is a convex function, i.e., $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$.
- Addition and scalar multiplication are jointly continuous, i.e., if $x_n \to x$ and $x_n \to y$, then $x_n + y_n \to x + y$ and if $x_n \to x$ and $\lambda_n \to \lambda$, then $\lambda_n x_n \to \lambda x$.

We use the notation $\|\cdot\|$ for norm. Then every normed space $(X, \|\cdot\|)$ is a metric space (X, d) with induced metric $d(x, y) = \|x-y\|$ and a topological space with the induced topology. It means that the induced metric $d(x, y) = \|x - y\|$ in turn, defines a topology on X, the norm topology.

Observation

- In every linear space X, we can easily define a function $\rho : X \times X \to \mathbb{R}^+$ by

$$\rho(x, y) = \begin{cases} 0 & if \quad x = y, \\ 1 & if \quad x \neq y, \end{cases} \tag{1.2}$$

which is a metric on X. It shows that every linear space (not necessarily normed space) is always a metric space.

At this stage, there arises a natural question:

Under what conditions will any metric on a linear space be a normed space? Such sufficient conditions are given in following proposition:

Proposition 1.2.4 *Let d be a metric on a linear space X. Then function $\|\cdot\| : X \to \mathbb{R}^+$ defined by*

$$\|x\| = d(x, 0) \ for\ all\ x \in X$$

is a norm if d satisfies the following conditions:

(d_1) d is homogeneous : $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$;

(d_2) d is translation invariant : $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$.

Remark 1.2.5 *The metric ρ defined by (1.2) is not homogeneous and the linear space X is a metric space under metric ρ, but not a normed space.*

The following example also demonstrates that a metric space is not necessarily a normed space.

Example 1.2.6 *Let X be a space of all complex sequences $\{x_i\}_{i=1}^{\infty}$ and $d(\cdot, \cdot)$ a metric on X defined by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|}, \quad x = \{x_i\},\ y = \{y_i\} \in X. \tag{1.3}$$

Then d is not a norm under the relation $d(x,y) = \|x-y\|$. In fact,

$$d(\lambda x, \lambda y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\lambda||x_i - y_i|}{1+|\lambda||x_i - y_i|} \neq |\lambda| \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1+|x_i - y_i|} = |\lambda| d(x,y),$$

i.e., d is not homogeneous.

Remark 1.2.7 *The metric d defined by (1.3) is bounded, because*

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1+|x_i - y_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

This metric is called a Fréchet metric for X.

We now consider some examples of normed spaces:

Example 1.2.8 *Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:*

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n; \\
\|x\|_p &= \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \text{ and } p \in (1, \infty); \\
\|x\|_\infty &= \max_{1 \leq i \leq n} |x_i| \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.
\end{aligned}$$

Remark 1.2.9 *(a) $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ is denoted by ℓ_p^n for all $1 \leq p < \infty$.*
(b) $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ is denoted by ℓ_∞^n.

Example 1.2.10 *Let $X = \ell_1$, the linear space whose elements consist of all absolutely convergent sequences $(x_1, x_2, \cdots, x_i, \cdots)$ of scalars (real or complex numbers), i.e.,*

$$\ell_1 = \left\{ x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty \right\}.$$

Then ℓ_1 is a normed space with the norm defined by $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 1.2.11 *Let $X = \ell_p$ $(1 < p < \infty)$, the linear space whose elements consist of all p-summable sequences $(x_1, x_2, \cdots, x_i, \cdots)$ of scalars, i.e.,*

$$\ell_p = \left\{ x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

Then ℓ_p is a normed space with the norm defined by $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 1.2.12 *Let $X = \ell_\infty$, the linear space whose elements consist of all bounded sequences $(x_1, x_2, \cdots, x_i, \cdots)$ of scalars, i.e.,*

$$\ell_\infty = \{x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \{x_i\}_{i=1}^\infty \text{ is bounded}\}.$$

Then ℓ_∞ is a normed space with the norm defined by $\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$.

Example 1.2.13 *Let $X = c$, the sequence space of all convergent sequences of scalars, i.e.,*

$$c = \{x : x = (x_1, x_2, \cdots, x_i, \cdots) \ \text{ and } \{x_i\}_{i=1}^\infty \text{ is } \ \text{convergent}\}.$$

Then c space is a normed space with the norm $\|\cdot\|_\infty$.

Example 1.2.14 *Let $X = c_0$, the sequence space of all convergent sequences of scalars with limit zero, i.e.,*

$$c_0 = \{x = (x_1, x_2, \cdots, x_i, \cdots) : \{x_i\}_{i=1}^\infty \text{ is convergent to zero}\}.$$

The c_0 space is a normed space with norm $\|\cdot\|_\infty$.

Example 1.2.15 *Let $X = c_{00}$, the sequence space defined by*

$$c_{00} = \{x = \{x_i\}_{i=1}^\infty \in \ell_\infty : \{x_i\}_{i=1}^\infty \text{ has only a finite number of nonzero terms}\}.$$

Then c_{00} space is a normed space with norm $\|\cdot\|_\infty$.

Observation

- $c_{00} \subset \ell_p \subset c_0 \subset c \subset \ell_\infty$ for all $1 \le p < \infty$.
- If $1 \le p < q \le \infty$, then $\ell_p \subset \ell_q$. In fact, let $x = (1, 1/2, \cdots, 1/n, \cdots)$, and we have

$$\sum_{i=1}^\infty |x_i| = \sum_{i=1}^\infty \frac{1}{i} = \infty, \text{ and } \sum_{i=1}^\infty |x_i|^2 = \sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6} < \infty.$$

Note that $x \in \ell_2$, but $x \notin \ell_1$. Hence an element of ℓ_2 is not necessarily an element of ℓ_1. But each element of ℓ_1 is an element of ℓ_2.

Example 1.2.16 *Let $X = L_p[a, b]$ $(1 \le p < \infty)$, the linear space of all equivalence classes of p-integrable functions on $[a, b]$. Then $L_p[a, b]$ space is a normed space with the norm defined by*

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} < \infty.$$

Example 1.2.17 *Let $X = L_\infty[a, b]$, the linear space of all equivalence classes of essentially bounded functions on $[a, b]$. Then $L_\infty[a, b]$ space is a normed space with the norm defined by*

$$\|f\|_\infty = ess\ sup|f(t)| < \infty.$$

Example 1.2.18 *Let $X = C[a,b]$, the set of all continuous scalar-valued functions and let "$+$" and "$\cdot$" be operations defined by*

$$\begin{aligned}(f+g)(t) &= f(t)+g(t) \text{ for all } f,g \in C[a,b];\\ (\lambda f)(t) &= \lambda f(t) \text{ for all } f \in C[a,b] \text{ and scalar } \lambda \in \mathbb{K}.\end{aligned}$$

Then $C[a,b]$ is a linear space and is also a normed space with the norms:

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \qquad 1 \leq p < \infty; \tag{1.4}$$

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|. \tag{1.5}$$

Observation

- The norm $\|\cdot\|_p$ defined by (1.4) on $C[a,b]$ is called a L_p-norm.
- The norm $\|\cdot\|_\infty$ defined by (1.5) on $C[a,b]$ is called a uniform convergence norm.

Equivalent norms – Let X be a linear space over $\mathbb{K}$ and let $\|\cdot\|^{'}$ and $\|\cdot\|^{''}$ be two norms on X. Then $\|\cdot\|^{'}$ is said to be *equivalent* to $\|\cdot\|^{''}$ (written as $\|\cdot\|^{'} \sim \|\cdot\|^{''}$) if there exist positive numbers a and b such that

$$a\|x\|^{'} \leq \|x\|^{''} \leq b\|x\|^{'} \text{ for all } x \in X,$$

or

$$a\|x\|^{''} \leq \|x\|^{'} \leq b\|x\|^{''} \text{ for all } x \in X.$$

Observation

- The relation $\sim$ is an equivalence relation on the set of all norms on X.
- In a finite-dimensional normed space X, all norms on X are equivalent.
- If $\|\cdot\|^{'}$ and $\|\cdot\|^{''}$ are equivalent norms on a linear space X, then a sequence $\{x_n\}$ that is convergent (Cauchy) with respect to $\|\cdot\|^{'}$ is also convergent (Cauchy) with respect to $\|\cdot\|^{''}$ and vice versa.
- If $\|\cdot\|^{'}$ and $\|\cdot\|^{''}$ are equivalent norms on a linear space X, then the class of open sets with respect to $\|\cdot\|^{'}$ is same as the class of open sets with respect to $\|\cdot\|^{''}$ and vice versa.

Seminorm – Let X be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Then a function $p : X \to \mathbb{R}^+$ is said to be a *seminorm* on X if (N_2) and (N_3) (see Definition 1.2.3) are satisfied. The ordered pair (X,p) is called *seminormed space*. Note that a seminorm p is a norm if $p(x) = 0 \Rightarrow x = 0$.

Example 1.2.19 *Let $X = \mathbb{R}^2$ and define $p : X \to \mathbb{R}^+$ by*

$$p(x) = p((x_1,x_2)) = |x_1|, \quad x \in X.$$

Then p is a seminorm, but not a norm, because $p(x_1,x_2) = 0$ implies that only the first component of x is zero, i.e., $x_1 = 0$.

We now consider the notion of topological linear spaces.

Definition 1.2.20 *A linear space X over $\mathbb{K}$ is said to be a topological linear space if on X, there exists a topology τ such that $X \times X$ and $\mathbb{K} \times X$ with the product topology have the property that vector addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{K} \times X \to X$ are continuous functions.*

In this case, τ is called a *linear topology* on X.

Definition 1.2.21 *A linear topology on a topological linear space X is said to be a locally convex topology if every neighborhood of 0 (the zero of X) includes a convex neighborhood of 0. Then X is called a locally convex topological space.*

Then we have the following interesting result.

Proposition 1.2.22 *If X is a locally convex topological linear space over $\mathbb{K}$, then a topology of X is determined by a family of seminorms $\{p_i\}_{i \in I}$.*

Inner product – Let X be a linear space over field $\mathbb{C}$. An *inner product* on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ with the following three properties:

(I_1) $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(I_2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where the bar denotes complex conjugation;

(I_3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$.

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space*. Sometimes, it is called a *pre-Hilbert space*. $\langle x, y \rangle$ is called inner product of two elements $x, y \in X$.

Example 1.2.23 *Let $X = \mathbb{R}^n$, the set of n-tuples of real numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \ for\ all\ x = (x_1, x_2, \cdots, x_n),\ y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$$

is an inner product on $\mathbb{R}^n$. $\mathbb{R}^n$ with this inner product is called real Euclidean n-space.

Example 1.2.24 *Let $X = \mathbb{C}^n$, the set of n-tuples of complex numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i \ for\ all\ x = (x_1, x_2, \cdots, x_n),\ y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$$

is an inner product on $\mathbb{C}^n$. $\mathbb{C}^n$ with this inner product is called a complex Euclidean n-space.

Example 1.2.25 *Let* $X = \ell_2$*, the set of all sequences of complex numbers* $(a_1, a_2, \cdots, a_i, \cdots)$ *with* $\sum_{i=1}^{\infty} |a_i|^2 < \infty$*. Then the function* $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{C}$ *defined by*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y}_i \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in X \tag{1.6}$$

is an inner product on ℓ_2*.*

We note that the series (1.6) converges by the Cauchy-Schwarz inequality (see Proposition 1.2.28).

Example 1.2.26 *Let* $X = C[a, b]$*, the linear space of all scalar-valued continuous functions on* $[a, b]$*. Then the function* $\langle \cdot, \cdot \rangle : C[a, b] \times C[a, b] \to \mathbb{C}$ *defined by*

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)} dt \text{ for all } f, g \in C[a, b] \tag{1.7}$$

is an inner product on $C[a, b]$*.*

We now give some interesting characterizations of linear spaces having inner products.

Proposition 1.2.27 *Let* X *be an inner product space. Then the function* $\| \cdot \| : X \to \mathbb{R}^+$ *defined by*

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in X$$

is a norm on X*.*

Proposition 1.2.28 (The Cauchy-Schwarz inequality) – *Let* X *be an inner product space. Then the following holds:*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \quad \text{for all } x, y \in X,$$

i.e.,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \text{for all } x, y \in X.$$

Proposition 1.2.29 (The parallelogram law) – *Let* X *be an inner product space. Then* $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *for all* $x, y \in X$*.*

Proposition 1.2.30 *The norm on a normed linear space* X *is given by an inner product if and only if the norm satisfies the parallelogram law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all} \quad x, y \in X.$$

Proposition 1.2.31 (The polarization identity) – *Let* X *be an inner product space. Then*

$$\langle x, y \rangle = \frac{1}{4}\left\{ \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right\} \text{ for all } x, y \in X.$$

Orthogonality of vectors – Let x and y be two vectors in an inner product space X. Then x and y are said to be *orthogonal* if $\langle x, y\rangle = 0$.

Remark 1.2.32 *If x and y are orthogonal, then we denote $x \perp y$ and we say "x is perpendicular to y."*

Proposition 1.2.33 *Let X be an inner product space and let $x, y \in X$ such that $x \perp y$. Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Observation

- $0 \perp x$ for all $x \in X$.
- $x \perp x$ if and only if $x = 0$.
- Every inner product space is a normed space.
- Every normed space is an inner product space if and only if its norm satisfies the parallelogram law.

Convergent sequence – A sequence $\{x_n\}$ in a normed space X is said to be *convergent to x* if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. In this case, we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

Observation

- $x_n \to x \Rightarrow \|x_n\| \to \|x\|$ (this fact can be easily shown by the continuity of norm). The converse of this fact is not true in general (see Theorem 2.2.13).
- The limit of convergent sequence is unique. To see it, suppose $x_n \to x$ and $x_n \to y$. Then $\|x - y\| \leq \|x_n - x\| + \|x_n - y\| \to 0$.

Cauchy sequence – A sequence $\{x_n\}$ in a normed space X is said to be *Cauchy* if $\lim\limits_{m,\, n\to\infty} \|x_m - x_n\| = 0$, i.e., for $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq n_o$.

Observation

- A sequence in $(\mathbb{R}, |\cdot|)$ is convergent if and only if it is Cauchy sequence.
- Every convergent sequence is a Cauchy, but the converse need not be true in general. In fact, if $x_n \to x$, then

  $$\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\| \to 0 \; as \; m, n \to \infty.$$

  Conversely, suppose $X = c_{00}$ is the linear space of finitely nonzero sequences $(x_1, x_2, \cdots, x_i, 0, \cdots)$ with the norm $\|x\| = \sup\limits_{i\in\mathbb{N}} |x_i|$. Let $\{x_n = (1, 1/2, 1/3, \cdots, 1/n, \cdots)\}$ be a sequence in X. Now

  $$\|x_n - x_m\| = \max\{1/n, 1/m\} \to 0 \;\; as \;\; m, n \to \infty,$$

  i.e., $\{x_n\}$ is a Cauchy sequence. Clearly, the limit x has infinitely nonzero elements. Thus, $x \notin X$. Therefore, a Cauchy sequence is not convergent in X.

- Every Cauchy sequence is bounded.
- Every Cauchy sequence is convergent if and only if it has a convergent subsequence.

Hilbert space and Banach space – A normed space $(X, \|\cdot\|)$ is said to be *complete if* it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.

A complete normed space (inner product space) is called a *Banach space* (*Hilbert space*).

Example 1.2.34 ℓ_p^n $(1 \leq p \leq \infty)$ *are (finite-dimensional) Banach spaces.*

Example 1.2.35 ℓ_p *and* $L_p[0,1], 1 \leq p \leq \infty$ *are (infinite-dimensional) Banach spaces.*

Example 1.2.36 *The linear space* $C[a,b]$ *of continuous functions on closed and bounded interval* $[a,b]$ *is a Banach space with the uniform convergence norm* $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$, *but an incomplete normed space with the norm*

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \ 1 \leq p < \infty.$$

Example 1.2.37 c_{00} *is not complete.*

Theorem 1.2.38 *Every finite-dimensional normed space is a Banach space.*

The topological property closedness has an important role in the construction of Banach spaces from its subspaces. A point x in a normed space X is said to be a *limit point* of a subset $C \subseteq X$ if there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} x_n = x$. Also *a* subset C of a normed space is said to be *closed* if it contains all of its limit points, i.e., $C = \overline{C}$.

Theorem 1.2.39 *A closed subspace of a Banach space is a Banach space.*

Theorem 1.2.40 *Let* C *be a subset of a normed space* X *and let* $x \in X$. *Then* $x \in \overline{C}$ *if and only if there exists a sequence* $\{x_n\}$ *in* C *such that* $\lim_{n\to\infty} x_n = x$.

Observation

- The subspaces c and c_0 are closed subspaces of ℓ_∞ (and hence are Banach spaces). The space c_{00} is only a subspace in c_0, but not closed in c_0 (and hence not in ℓ_∞). Therefore, c_{00} is not a Banach space.
- The subspace $C[a,b]$ is not closed in $L_p[a,b]$ for $1 \leq p < \infty$. Hence $C[a,b]$ is not a Banach space with the L_p-norm $\|\cdot\|_p$ $(1 \leq p < \infty)$ defined by (1.4).

We now give examples of Banach spaces that are not Hilbert spaces.

Example 1.2.41 *ℓ_p^n is a finite-dimensional Banach space that is not a Hilbert space for $p \neq 2$. Indeed, for $x = (1, 1, 0, 0, \cdots)$ and $y = (1, -1, 0, 0, \cdots)$, we have $x + y = (2, 0, 0, 0, \cdots)$ and $x - y = (0, 2, 0, 0, \cdots)$. Hence*

$$\begin{aligned} \|x\| &= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = (1^p + 1^p)^{1/p} = 2^{1/p}, \\ \|y\| &= (1^p + 1^p) = 2^{1/p}, \\ \|x+y\| &= (2^p)^{1/p} = 2, \\ \|x-y\| &= (2^p)^{1/p} = 2. \end{aligned}$$

If $p = 2$, then the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is satisfied, which shows that ℓ_2^n is a Hilbert space. If $p \neq 2$, then the parallelogram law is not satisfied. Therefore, ℓ_p^n is not a Hilbert space for $p \neq 2$.

The following example shows that there exists an infinite-dimensional Banach space that is not a Hilbert space.

Example 1.2.42 *Let $X = C[0, 2\pi]$, the space of all real-valued continuous functions on $[0, 2\pi]$ with "sup" norm. Then $(C[0, 2\pi], \|\cdot\|_\infty)$ is a Banach space, but $\|\cdot\|_\infty$ does not satisfy the parallelogram law. In fact, for $x(t) = \max\{\sin t, 0\}$, $y(t) = \min\{\sin t, 0\}$, we have*

$$\|x\|_\infty = 1, \|y\|_\infty = 1, \|x+y\|_\infty = 1, \|x-y\|_\infty = 1,$$

i.e., the parallelogram law:

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2$$

is not satisfied.

Remark 1.2.43 *$C[a, b]$ is an inner product space with the inner product defined by (1.7), but not a Hilbert space.*

Observation

- ℓ_2^n, ℓ_2, $L_2[a, b]$ are Hilbert spaces.
- ℓ_p^n, ℓ_p, $L_p[a, b]$ $(p \neq 2)$ are not Hilbert spaces.

We conclude this section with some important facts about the completeness property.

Definition 1.2.44 *A subset C of a normed space X is said to be complete if every Cauchy sequence in C converges to a point in C.*

Definition 1.2.45 *Let $\sum_{n=1}^{\infty} x_n$ be an infinite series of elements $x_1, x_2, \cdots, x_n, \cdots$ in a normed space X. Then the series $\sum_{n=1}^{\infty} x_n$ is said to converge to an element $x \in X$ if $\lim\limits_{n\to\infty} \|s_n - x\| = 0$, where $s_n = x_1 + x_2 + \cdots + x_n$ is n^{th} partial sum of series $\sum_{n=1}^{\infty} x_n$.*

Definition 1.2.46 *The series* $\sum_{n=1}^{\infty} x_n$ *in a normed space* X *is said to be absolutely convergent if* $\sum_{n=1}^{\infty} \|x_n\|$ *converges.*

The following result shows that completeness and closure are equivalent in a Banach space.

Theorem 1.2.47 *In a Banach space, a subset is complete if and only if it is closed.*

Remark 1.2.48 *Notice every normed space is closed, but not necessarily complete.*

Theorem 1.2.49 *A normed space* X *is a Banach space if and only if every absolutely convergent series of elements in* X *is convergent in* X*.*

Theorem 1.2.50 (Cantor's intersection theorem) – *A normed space* X *is a Banach space if and only if given any descending sequence* $\{F_n\}$ *of closed bounded subsets of* X*,*

$$\lim_{n\to\infty} diam(F_n) = 0 \Rightarrow \cap_{n=1}^{\infty} F_n \neq \emptyset. \tag{1.8}$$

Proof. Let X be a Banach space and $\{F_n\}$ a descending sequence of nonempty closed bounded subsets of X for which $\lim_{n\to\infty} diam(F_n) = 0$. For each n, select $x_n \in F_n$. Then given $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow diam(F_n) < \varepsilon$. If $m, n \geq n_0$, both x_n and x_m are in F_{n_0}, then $\|x_n - x_m\| \leq \varepsilon$. Hence $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. This shows that $x \in \overline{F}_n = F_n$ if $n \geq n_0$. Because the sequence $\{F_n\}$ is descending, $x \in \cap_{n=1}^{\infty} F_n$.

Conversely, suppose that the condition (1.8) holds. Suppose $\{x_n\} \subseteq X$ is a Cauchy sequence. For each $n \in \mathbb{N}$, let $F_n = \{x_n, x_{n+1}, \cdots\}$. Then $\{\overline{F}_n\}$ is a descending sequence of nonempty closed subsets of X for which $\lim_{n\to\infty} diam(\overline{F}_n) = 0$. By assumption, there exists a point $x \in \bigcap_{n=1}^{\infty} \overline{F}_n$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so large that

$$n \geq N \Rightarrow diam(\overline{F}_n) < \varepsilon.$$

Then clearly for such n we have that $\|x_n - x\| \leq \varepsilon$. Hence $\lim_{n\to\infty} x_n = x$. Therefore, X is complete. ∎

1.3 Dense set and separable space

A sequence $\{x_n\}$ in a normed space X is said to be a *(Schauder) basis* of X if each $x \in X$ has a unique expansion $x = \sum_{n=1}^{\infty} \alpha_n x_n$ for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$.

Observation

- $\{x_n\}$ is a basis of a normed space X if for each $x \in X$, there exists a unique sequence $\{\alpha_n\}$ of scalars such that $\lim_{n\to\infty} \|x - \sum_{i=1}^{n} \alpha_i x_i\| = 0$.
- The elements

$$e_n = (0, 0, 0, \cdots,\ \underset{\substack{\uparrow \\ n^{th}\text{ position}}}{1},\ 0, \cdots), \quad n \in \mathbb{N}$$

  from a basis for c_{00}, c_0 and ℓ_p $(1 \le p < \infty)$.

- $\{e_n\}_{n\in\mathbb{N}}$ is not a Schauder basis of ℓ_∞.
- The sequence $(\mathbf{1}, e_1, e_2, \cdots)$ is a basis for c, where $\mathbf{1} = (1, 1, 1, \cdots)$.

A subset C of a metric space (X, d) is said to be *dense* in X if $\overline{C} = X$. This means that C is dense in X if and only if $C \cap B_r(x) \neq \emptyset$ for all $x \in X$ and $r > 0$.

A metric space (X, d) is said to be *separable* if it contains a countable dense subset, i.e., there exists a countable set C in X such that $\overline{C} = X$.

Observation

- If X is a separable metric space, then $C \subset X$ is separable in the induced metric.
- A metric space X is separable if and only if there is a countable family $\{G_i\}$ of open sets such that for any open set $G \subset X$,

$$G = \cup_{G_i \subset G} G_i.$$

Next, we give some examples of separable and nonseparable spaces.

Example 1.3.1 *The space $\ell_p, 1 \le p < \infty$ is separable metric space.*

Example 1.3.2 *The ℓ_∞ space is not a separable space.*

Example 1.3.3 *The linear space X of all infinite sequences of real numbers with metric d defined by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$

$$x = (x_1, x_2, \cdots, x_i, \cdots), y = (y_1, y_2, \cdots, y_i, \cdots) \in X$$

is a separable complete metric space.

Theorem 1.3.4 *Every normed space with basis is separable.*

Theorem 1.3.5 *Every subset of a separable normed space is separable.*

Theorem 1.3.6 *Every finite-dimensional normed space is separable.*

Observation

- $\mathbb{R}$, $\mathbb{R}^n$, c, $C[0, 1]$, ℓ_p, L_p $(1 \le p < \infty)$ are separable normed spaces.
- ℓ_∞, L_∞ are not separable.

1.4 Linear operators

Let X and Y be two linear spaces over the same field $\mathbb{K}$ and $T : X \to Y$ an operator with domain $Dom(T)$ and range $R(T)$. Then T is said to be a *linear operator* if

(i) T is additive: $T(x+y) = Tx + Ty$ for all $x, y \in X$;

(ii) T is homogeneous: $T(\alpha x) = \alpha Tx$ for all $x \in X, \alpha \in \mathbb{K}$.

One may easily check that T is linear if and only if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \text{ for all } x, y \in X \text{ and } \alpha, \beta \in \mathbb{K}.$$

Otherwise, the operator is called *nonlinear*. The linear operator is called a *linear functional* if $Y = \mathbb{R}$.

Example 1.4.1 *Let $X = \mathbb{R}^n$, $Y = \mathbb{R}$, and $T : X \to \mathbb{R}$ an operator defined by*

$$Tx = \sum_{i=1}^{n} x_i y_i \ for\ all\ x = (x_1, x_2, \cdots, x_n),$$

where $y = (y_1, y_2, \cdots, y_n)$ is the fixed element in $\mathbb{R}^n$. Then T is a linear functional on $\mathbb{R}^n$.

Example 1.4.2 *Let $X = Y = \ell_2$ and $T : \ell_2 \to \ell_2$ an operator defined by*

$$Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots\right) \text{ for all } x = (x_1, x_2, x_3, \cdots, x_n, \cdots) \in \ell_2.$$

Then T is a linear operator on ℓ_2.

Example 1.4.3 *Let $X = C[a,b]$, the linear space of all continuous real-valued functions on closed bounded interval $[a,b]$. Then the operator $T : C[a,b] \to C[a,b]$ defined by*

$$T(f)t = \int_a^t f(u)du, \quad t \in [a,b]$$

is a linear operator.

Example 1.4.4 *Let $X = L_2[0,1]$, $Y = \mathbb{R}$ and $T : X \to \mathbb{R}$ an operator defined by*

$$Tx = \int_0^1 x(t)y(t)dt \ for\ all\ x \in L_2[0,1],$$

where y is a fixed element in $L_2[0,1]$. Then T is a linear functional on $L_2[0,1]$.

The following result is very useful for linear operators:

Proposition 1.4.5 *Let X and Y be two linear spaces over the same field $\mathbb{K}$ and $T : X \to Y$ a linear operator. Then we have the following:*

(a) $T(0) = 0$.

(b) $R(T) = \{y \in Y : y = Tx$ *for some* $x \in X\}$, *the range of* T *is a linear subspace of* Y.

(c) T *is one-one if and only if* $Tx = 0 \Rightarrow x = 0$.

(d) *If* T *is one-one operator, then* T^{-1} *exists on* $R(T)$ *and* $T^{-1} : R(T) \to X$ *is also a linear operator.*

(e) *If* $dim(Dom(T)) = n < \infty$ *and* T^{-1} *exists, then* $dim(R(T)) = dim(Dom(T))$.

Recall an operator T from a normed space X into another normed space Y is continuous if for any sequence $\{x_n\}$ in X with $x_n \to x \in X \Rightarrow Tx_n \to Tx$. The following Theorem 1.4.6 is very interesting because the continuity of any linear operator can be verified by only verifying $Tx_n \to 0$ for any sequence $\{x_n\} \subseteq X$ with $x_n \to 0$.

Theorem 1.4.6 *Let* X *and* Y *be two normed spaces and* $T : X \to Y$ *a linear operator. If* T *is continuous at a single point in* X, *then* T *is continuous throughout space* X.

Proof. Suppose T is continuous at a point $x_0 \in X$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x \in X$. By the linearity of T, we have

$$\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\|.$$

Because T is continuous at x_0,

$$\lim_{n\to\infty} (x_n - x + x_0) = x_0 \Rightarrow \lim_{n\to\infty} T(x_n - x + x_0) = Tx_0,$$

it follows that $\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\| \to 0$ as $n \to \infty$. Thus, T is a continuous operator at an arbitrary point $x \in X$. ∎

Boundedness of linear operator – Let X and Y be two normed spaces and $T : X \to Y$ a linear operator. Then T is said be *bounded* if there exists a constant $M > 0$ such that

$$\|Tx\| \leq M\|x\| \text{ for all } x \in X.$$

A linear functional $f : X \to \mathbb{R}$ is called *bounded* if there exists a constant $M > 0$ such that

$$|f(x)| \leq M\|x\| \text{ for all } x \in X.$$

We now present an example of a linear operator that is unbounded.

Example 1.4.7 *Let* $X = c_{00}$, *the linear space of finitely nonzero real sequences with "sup" norm and* $T : X \to \mathbb{R}$ *a functional defined by*

$$Tx = \sum_{i=1}^{n} ix_i \text{ for all } x = (x_1, x_2, \cdots, x_n, 0, 0, \cdots) \in X.$$

Then T *is clearly a linear functional, but it is unbounded.*

With this example, we remark that linearity of the operator does not imply boundedness. Hence we require additional assumption for boundedness of any linear operator. The following important result shows that such an additional assumption is continuity of the linear operator.

Theorem 1.4.8 *A linear operator on a normed space is bounded if and only if it is continuous.*

Proof. Let T be a bounded linear operator from a normed space X into another normed space Y. Then there exists a constant $M > 0$ such that

$$\|Tx\| \leq M\|x\| \text{ for all } x \in X.$$

Then if $x_n \to 0$, we have that

$$\|Tx_n\| \leq M\|x_n\| \to 0 \text{ as } n \to \infty,$$

and it follows that T is continuous at zero. By Theorem 1.4.6, we conclude that T is continuous on X.

Conversely, suppose T is continuous. We show that T is bounded. Suppose, for contradiction, that T is unbounded. Hence there exists a sequence $\{x_n\}$ in X such that

$$\|Tx_n\| > n\|x_n\| \text{ for all } n \in \mathbb{N}.$$

Because $T0 = 0$, this implies that $x_n \neq 0$. Set $y_n := x_n/(n\|x_n\|), n \in \mathbb{N}$. Then $\|y_n\| = \|x_n/(n\|x_n\|)\| = 1/n \to 0$, which implies that $\lim_{n\to\infty} y_n = 0$. Observe that

$$\|Ty_n\| = \|T\left(\frac{x_n}{n\|x_n\|}\right)\| = \frac{1}{n\|x_n\|}\|Tx_n\| > 1 \text{ for all } n \in \mathbb{N}$$

and hence $\{Ty_n\}$ does not converge to zero. This means that T is not continuous at zero, a contradiction. ∎

If the dimension of X is finite, it also forces the boundedness of a linear operator.

Theorem 1.4.9 *Let X and Y be two normed spaces. If X is a finite-dimensional normed space, then all linear operators $T : X \to Y$ are continuous (hence bounded).*

Remark 1.4.10 *Example 1.4.7 shows that the conclusion of Theorem 1.4.9 is not true in general (in infinite-dimensional normed spaces). Thus, linear operators may be discontinuous in infinite-dimensional normed spaces.*

1.5 Space of bounded linear operators

Let X and Y be two normed spaces. Given two bounded linear operators $T_1, T_2 : X \to Y$, we define

$$\begin{aligned} (T_1 + T_2)x &= T_1x + T_2x, \\ (\alpha T_1)x &= \alpha T_1 x \text{ for all } x \in X \text{ and } \alpha \in \mathbb{K}. \end{aligned}$$

We denote by $B(X,Y)$, the family of all bounded linear operators from X into Y. Then $B(X,Y)$ is a linear space. The space $B(X,Y)$ becomes a normed space by assigning a norm as below:

$$\begin{aligned} \|T\|_B &= \inf\{M : \|Tx\| \leq M\|x\|, x \in X\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0, x \in X\right\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| = 1\}. \end{aligned}$$

Theorem 1.5.1 *The normed space $B(X,Y)$ is a Banach space if Y is a Banach space.*

We now state an important result:

Theorem 1.5.2 (Uniform boundedness principle) – *Let X be a Banach space, Y a normed space, and $\{T_i\}_{i\in\Lambda} \subseteq B(X,Y)$ a family of bounded linear operators of X into Y such that $\{T_i x\}$ is bounded set in Y for each $x \in X$, i.e., for each $x \in X$, there exists $M_x > 0$ such that*

$$\|T_i x\| \leq M_x \textit{ for all } i \in \mathbb{N}.$$

Then $\{\|T_i\|_B\}$ is a bounded set in $\mathbb{R}^+$, i.e., T_i are uniformly bounded.

As an immediate consequence of Theorem 1.5.2 (uniform boundedness principle), we have

Theorem 1.5.3 *Let X and Y be two Banach spaces and $\{T_n\}$ a sequence in $B(X,Y)$. For each $x \in X$, let $\{T_n x\}$ converges to Tx. Then we have the following:*

(a) T is a bounded linear operator, i.e., $T \in B(X,Y)$;

(b) $\|T\|_B \leq \liminf_{n\to\infty} \|T_n\|_B$.

Proof. (a) Because each T_n is linear, it follows that

$$\begin{aligned} T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} T_n(\beta y) \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha Tx + \beta Ty \end{aligned}$$

for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$. Further, because the norm is continuous,

$$\lim_{n\to\infty} \|T_n x\| = \|Tx\| \text{ for all } x \in X,$$

it follows that $\{T_n x\}$ is a bounded set in Y. By the uniform boundedness principle, there exists a positive constant $M > 0$ such that $\sup_{n\in\mathbb{N}} \|T_n\|_B \leq M$. Thus,

$$\|T_n x\| \leq \|T_n\|_B \|x\| \leq M\|x\|.$$

Taking the limit as $n \to \infty$, we have

$$\|Tx\| \leq M\|x\|,$$

so T is bounded. Therefore, $T \in B(X, Y)$.

(b) Because

$$\|T_n x\| \quad \leq \quad \|T_n\|_B \|x\|,$$

this implies that

$$\liminf_{n\to\infty} \|T_n x\| \quad \leq \quad \liminf_{n\to\infty} \|T_n\|_B \|x\|.$$

Hence $\|Tx\| \leq \liminf_{n\to\infty} \|T_n\|_B \|x\|$. Thus, $\|T\|_B \leq \liminf_{n\to\infty} \|T_n\|_B$. ▮

Dual space – The space of all bounded linear functionals on a normed space X is called *the dual* of X and is denoted by X^*. Clearly, $X^* = B(X, \mathbb{R})$ and is a normed space with norm denoted and defined by

$$\|f\|_* = \sup\{|f(x)| : x \in S_X\}.$$

In view of Theorem 1.5.1, we have the following interesting result, which is very useful for the construction of Banach spaces from normed spaces.

Corollary 1.5.4 *The dual space $(X^*, \|\cdot\|_*)$ of a normed space X is always a Banach space.*

We now give basic dual spaces:

The dual of $\mathbb{R}^n$ – Let $\mathbb{R}^n$ be a normed space of vectors $x = (x_1, x_2, \cdots, x_n)$ with norm $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$. Then for $y = (y_1, y_2, \cdots, y_i, \cdots, y_n) \in \mathbb{R}^n$, any functional $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \sum_{i=1}^n x_i y_i, \qquad x = (x_1, x_2, \cdots, x_i, \cdots, x_n) \in \mathbb{R}^n$$

is linear. Further, from the Cauchy-Schwarz inequality,

$$|f(x)| = \left|\sum_{i=1}^n x_i y_i\right| \leq \left(\sum_{i=1}^n x_i^2\right)^{1/2} \left(\sum_{i=1}^n y_i^2\right)^{1/2} = \left(\sum_{i=1}^n y_i^2\right)^{1/2} \|x\|_2,$$

which shows that f is bounded with $\|f\|_* \leq (\sum_{i=1}^n y_i^2)^{1/2}$. However, because for $x = (y_1, y_2, \cdots, y_n)$ equality is achieved in the Cauchy-Schwarz inequality, we must in fact have $\|f\|_* = (\sum_{i=1}^n y_i^2)^{1/2}$.

Now, let j be any bounded linear functional on $X = \mathbb{R}^n$. Define the basis vectors e_i in $\mathbb{R}^n$ by

$$e_i = (0, 0, \cdots, \underset{\substack{\uparrow \\ i^{th} \text{ position}}}{1}, 0, \cdots, 0).$$

Suppose $j(e_i) = a_i$. Then for any $x = (x_1, x_2, \cdots, x_n)$, we have $x = \sum_{i=1}^n x_i e_i$. By the linearity of j, we have

$$j(x) = \sum_{i=1}^n j(e_i x_i) = \sum_{i=1}^n j(e_i) x_i = \sum_{i=1}^n a_i x_i.$$

Thus, the dual space X^* of $X = \mathbb{R}^n$ is itself $\mathbb{R}^n$ in the sense that the space X^* consists of all functionals of the form $f(x) = \sum_{i=1}^n a_i x_i$ and the norm on X^* is $\|f\|_* = (\sum_{i=1}^n |a_i|^2)^{1/2} = \|a\|$, where $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$.

The dual of $\boldsymbol{\ell_p, 1 \leq p < \infty}$ – For $1 \leq p < \infty$, the dual space of ℓ_p is ℓ_q $(1/p + 1/q = 1)$ in the sense that there is a one-one correspondence between elements $y \in \ell_q$ and bounded linear functionals f_y on ℓ_p such that

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i, \qquad x = \{x_i\}_{i=1}^{\infty} \in \ell_p,$$

where

$$y = \{y_i\}_{i=1}^{\infty} \in \ell_q$$

and

$$\|f_y\|_* = \|y\|_q = \begin{cases} (\sum_{i=1}^{\infty} |y_i|^q)^{1/q}, & \text{if } 1 < p < \infty, \\ \sup\limits_{i \in \mathbb{N}} |y_i| & \text{if } p = 1. \end{cases}$$

Observation

- The dual of ℓ_1 is ℓ_∞.
- The dual of ℓ_p is ℓ_q, $1 < p < \infty$ and $1/p + 1/q = 1$.
- The dual of ℓ_∞ is not ℓ_1.

The dual of $\mathbf{c_0}$ – The Banach space c_0 of all real sequences $x = \{x_i\}$ such that $\lim\limits_{i \to \infty} x_i = 0$ with norm $\|x\|_\infty = \sup\limits_{i \in \mathbb{N}} |x_i|$ is a subspace of ℓ_∞. The dual of c_0 is ℓ_1 in the usual sense that the bounded linear functionals on c_0 can be represented as

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i, \qquad x = \{x_i\}_{i=1}^{\infty} \in c_0,$$

where $y = \{y_i\}_{i=1}^{\infty} \in \ell_1$ and $\|f_y\|_* = \|y\|_1 = \sum_{i=1}^{\infty} |y_i|$.

The dual of $\mathbf{L_p[0,1], 1 \leq p < \infty}$ – For $1 \leq p < \infty$, the dual space of $L_p[0,1]$ is $L_q[0,1], (1/p + 1/q = 1)$ in the sense that there is one-one correspondence between elements $y \in L_q[0,1]$ and bounded linear functionals $f_y : L_p[0,1] \to \mathbb{R}$ such that

$$f_y(x) = \int_0^1 x(t)y(t)dt \text{ and } \|f_y\|_* = \|y\|_q.$$

We now state an important theorem in Hilbert space that is called the *Riesz representation theorem.* This theorem demonstrates that any bounded linear functional on a Hilbert space H can be represented as an inner product with a unique element in H.

Theorem 1.5.5 (Reisz representation theorem) – *Let H be a Hilbert space and $f \in H^*$. Then we have the following:*

(1) There exists a unique element $y_0 \in H$ such that $f(x) = \langle x, y_0 \rangle$ for each $x \in H$.

(2) Moreover, $\|f\|_* = \|y_0\|$.

Remark 1.5.6 *In a Hilbert space H, (distinct) bounded linear functionals f on H are generated by (distinct) elements y of the space H itself, i.e., there is one-one correspondence between $f \in H^*$ and $y \in H$. Therefore, $H^* = H$.*

1.6 Hahn-Banach theorem and applications

The Hahn-Banach theorem is one of the most important theorems in functional analysis. To state it, we need the following definitions:

Sublinear functional – Let X be a linear space and $p : X \to \mathbb{R}$ a functional. Then p is said to be a *sublinear functional* on X if

(i) p is subadditive: $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,

(ii) p is positive homogeneous: $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \geq 0$.

It is evident that every norm is a sublinear functional.

The sublinear functional p on X is called *convex functional* on X if $p(x) \geq 0$ for all $x \in X$. Obviously, every norm is a convex functional also.

Example 1.6.1 *Let $p : \ell_\infty \to \mathbb{R}$ be a functional defined by*

$$p(x) = \limsup_{n\to\infty} x_n \textit{ for all } x = (x_1, x_2, \cdots, x_n, \cdots) \in \ell_\infty.$$

Then p is a sublinear functional on ℓ_∞.

Extension mapping – Let C be a proper subset of a linear space X and f a mapping from C into another linear space Y. If there exists a mapping $F : X \to Y$ such that

$$F(x) = f(x), \quad x \in C,$$

then F is called an *extension of f*.

Example 1.6.2 *Let* $X = [0,1], C = [0,1)$ *and* $f : C \to \mathbb{R}$ *defined by*

$$f(x) = x^2, \quad x \in [0,1).$$

Then

$$F_1(x) = \begin{cases} f(x) & if\ x \in C, \\ 0 & if\ x = 1 \end{cases}$$

and

$$F_2(x) = \begin{cases} f(x) & if\ x \in C, \\ 1 & if\ x = 1 \end{cases}$$

are two extensions of f*, where* F_2 *is continuous, but* F_1 *is not.*

Simply, the Hahn-Banach theorem states that a bounded linear functional f defined only on a subspace C of a normed space X can be extended to a bounded linear functional F defined on the entire space and with norm equal to that of f on C, i.e.,

$$\|F\|_X = \|f\|_C = \sup_{x \in C} \frac{|f(x)|}{\|x\|}.$$

We now state the theorem without proof.

Theorem 1.6.3 (Hahn-Banach theorem) – *Let* C *be a subspace of a real linear space* X*,* p *a sublinear functional on* X*, and* f *a linear functional defined on* C *satisfying the condition:*

$$f(x) \le p(x)\ for\ all\ x \in C.$$

Then there exists a linear extension F *of* f *such that* $F(x) \le p(x)$ *for all* $x \in X$*.*

Corollary 1.6.4 *Let* C *be a subspace of a real normed space* X *and* f *a bounded linear functional on* C*. Then there exists a bounded linear functional* F *defined on* X *that is an extension of* f *such that* $\|F\|_* = \|f\|_C$*.*

Proof. Take $p(x) = \|f\|_C \|x\|, x \in X$. ∎

The following corollary gives the existence of nontrivial bounded linear functionals on an arbitrary normed space.

Corollary 1.6.5 *Let* x *be an element of a normed space* X*. Then there exists (nonzero)* $j \in X^*$ *such that* $j(x) = \|j\|_* \|x\|$ *and* $\|j\|_* = \|x\|$*.*

Corollary 1.6.6 *Let* x *be a nonzero element of a normed space* X*. Then there exists* $j \in X^*$ *such that* $j(x) = \|x\|$ *and* $\|j\|_* = 1$*.*

Corollary 1.6.7 *Let* X *be a normed space. Then for any* $x \in X$*,*

$$\|x\| = \sup_{\|j\|_* \le 1} |j(x)|.$$

Corollary 1.6.8 *If X is a normed space and $x_0 \in X$ such that $j(x_0) = 0$ for all $j \in X^*$, then $x_0 = 0$.*

Proof. Suppose $x_0 \neq 0$. By Corollary 1.6.6, there exists a functional $j \in X^*$ such that

$$j(x_0) = \|x_0\| \text{ and } \|j\|_* = 1.$$

This implies that $j(x_0) \neq 0$, which is a contradiction. Hence $j(x_0) = 0$ for all $j \in X^* \Rightarrow x_0 = 0$. $\blacksquare$

The following theorems are very useful in many applications.

Theorem 1.6.9 *Let C be a subspace of a normed space X and x_0 an element in X such that $d(x_0, C) = d > 0$. Then there exists a bounded linear functional $j \in X^*$ with norm 1 such that $j(x_0) = d$ and $j(x) = 0$ for all $x \in C$.*

Theorem 1.6.10 (Separability) – *If X^* is the dual space of a normed space X and X^* is separable, then X is also separable.*

Next, we discuss geometric forms of the Hahn-Banach theorem. We need the following:

Hyperplane – A subset H of a linear space X is said to be a *hyperplane* if there exists a linear functional $f \neq 0$ on X such that

$$H = \{x \in X : f(x) = \alpha\}, \quad \alpha \in \mathbb{R}.$$

$f(x) = \alpha$ is called the *equation of the hyperplane.*

Example 1.6.11 *Let $X = \mathbb{R}$, $f(x) = 3x$, $\alpha = 2$. Then the set*

$$H = \{x \in X : f(x) = \alpha\} = \{x \in X : 3x = 2\} = \{2/3\}.$$

Hence H is a hyperplane.

We have the following interesting result.

Proposition 1.6.12 *Let X be a topological linear space. Then the hyperplane $\{x \in X : f(x) = \alpha\}$ is closed if and only if f is continuous.*

Let $f(x) = \alpha$, $\alpha \in \mathbb{R}$, be the equation of hyperplane in a linear space X. Then we have the following:

(i) $\{x \in X : f(x) < \alpha\}$ and $\{x \in X : f(x) > \alpha\}$ are open half-spaces.

(ii) $\{x \in X : f(x) \leq \alpha\}$ and $\{x \in X : f(x) \geq \alpha\}$ are closed half-spaces.

It is easy to see that the boundary of each of the four half-spaces is just a hyperplane.

Remark 1.6.13 *In a topological linear space X, we have*

(i) open half-spaces are open sets,

(ii) the closed half-spaces are closed sets if and only if f is continuous, i.e., the hyperplane $\{x \in X : f(x) = \alpha\}$ is closed.

Let X be a linear space. We say that the hyperplane $\{x \in X : f(x) = \alpha\}$ *separates* two sets $A \subset X$ and $B \subset X$ if $f(x) \leq \alpha$ for all $x \in A$ and $f(x) \geq \alpha$ for all $x \in B$. We say that the hyperplane $\{x \in X : f(x) = \alpha\}$ *strictly separates* two sets $A \subset X$ and $B \subset X$ if $f(x) < \alpha$ for all $x \in A$ and $f(x) > \alpha$ for all $x \in B$.

Theorem 1.6.14 (Hahn-Banach separation theorem) – *Let X be a normed space and let $A \subset X, B \subset X$ be two nonempty disjoint convex sets. Suppose that A is open. Then there exists a closed hyperplane that separates A and B, i.e., there exist $j \in X^*$ and a number $\alpha \in \mathbb{R}$ such that*

$$j(x) > \alpha \text{ if } x \in A \text{ and } j(x) \leq \alpha \text{ if } x \in B.$$

Proposition 1.6.15 *Let C be a nonempty open convex subset of a normed space X. Then for $x_0 \in X$, $x_0 \notin C$, there exists $f \in X^*$ such that*

$$f(x) < \alpha \quad \text{for all } x \in C,$$

where $f(x_0) = \alpha$.

An immediate consequence of the separation theorem shows that $\overline{co}(C)$ is the intersection of all closed half-spaces containing C. Indeed,

Theorem 1.6.16 *Let C be a nonempty subset of a normed space X. Then*

$$\overline{co}(C) = \{x \in X : f(x) \leq \sup_{y \in C} f(y) \text{ for all } f \in X^*\}.$$

Theorem 1.6.17 *Let C be a nonempty closed convex subset of a normed space X. If x is not an element in C, there exists a continuous linear functional $j \in X^*$ such that*

$$j(x) < \inf\{j(y) : y \in C\}.$$

Theorem 1.6.18 (Hahn-Banach strictly separation theorem) – *Let A and B be two nonempty disjoint convex subsets of a normed space X. Suppose A is closed and B is compact. Then there exists a closed hyperplane that strictly separates A and B.*

Supporting hyperplane – Let C be a convex subset of a normed space X with $int(C) \neq \emptyset$ and $x_0 \in \partial C$. Then a nonzero functional $f \in X^*$ is said to be a *support functional for C at x_0* if $f(x) \leq f(x_0)$ for all $x \in C$. The corresponding hyperplane $\{x \in X : f(x) = f(x_0)\}$ is called a *supporting hyperplane* for C at x_0.

A point of C through which a supporting hyperplane passes is called *a point of support of C*.

Observation

- Any supporting hyperplane of a set C with nonempty interior is closed.
- An interior point of C cannot be a point of support.

We give some conditions on C under which a boundary point is a point of support.

Theorem 1.6.19 *Let C be a convex subset of a normed space X with $int(C) \neq \emptyset$. Then every boundary point of C is a point of support, i.e., for every $x_0 \in \partial C$, there exists an $f \in X^*$ such that $f \neq 0$ and $f(x_0) = \sup_{x \in C} f(x)$.*

1.7 Compactness

Let (X, d) be a metric space. Recall that a subset C of X is called *compact* if every open cover of C has a finite subcover. Equivalently, a subset C of X is compact if every sequence in C contains a convergent subsequence with a limit in C.

A subset C of X is said to be *totally bounded* if for each $\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, \cdots, x_n$ in X such that $C \subseteq \cup_{i=1}^{n} B_\varepsilon(x_i)$. The set $\{x_1, x_2, \cdots, x_n\}$ is called a finite ε-net.

Observation

- Every subset of a totally bounded set is totally bounded.
- Every totally bounded set is bounded, but a bounded set need not be totally bounded.

Proposition 1.7.1 *A subset of a compact metric space is compact if and only if it closed.*

Proposition 1.7.2 *Let X be a metric space. Then the following are equivalent:*

(a) X is compact.

(b) Every sequence in X has a convergent subsequence.

(c) X is complete and totally bounded.

Proposition 1.7.3 *Let C be a subset of a complete metric space X. Then we have the following:*

(a) C is compact if and only if C is closed and totally bounded.

(b) $\overline{C}$ is compact if and only if C is totally bounded.

Observation

- $X = (0, 1)$ with usual metric is totally bounded, but not compact.
- $X = \mathbb{R}$ with usual metric is complete. But it is not totally bounded and hence not compact.

A subset C of a topological space is said to be *relatively compact* if its closure is compact, i.e., $\overline{C}$ is compact. In particular, we have an interesting result:

Proposition 1.7.4 *Let C be a closed subset of a complete metric space. Then C is compact if and only if it is relatively compact.*

We now state the following fundamental theorems concerning compactness.

Theorem 1.7.5 (The Heine-Borel theorem) – *A subset C of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Corollary 1.7.6 *A set $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 1.7.7 (Weierstrass theorem) – *Let C be a nonempty compact subset of a metric space (X, d) and $f : C \to \mathbb{R}$ a continuous function. Then f attains its maximum and minimum, i.e., there exist $\underline{x}$, $\overline{x} \in C$ such that*

$$f(\underline{x}) = \inf_{x \in C} f(x) \text{ and } f(\overline{x}) = \sup_{x \in C} f(x).$$

Theorem 1.7.8 (Mazur's theorem) – *The closed convex hull $\overline{co}(C)$ of a compact set C of a Banach space is compact.*

Observation

- $\mathbb{R}^n$, $n \geq 1$ is not compact. However, every closed bonded subset of $\mathbb{R}^n$ is compact. For example, $C = [0, 1] \subset \mathbb{R}$ is compact, but $\mathbb{R}$ itself is not compact.
- $C[0, 1]$ and ℓ_2 are not compact.
- The subset $C = \{\{x_n\} \in \ell_2 : |x_n| \leq 1/n, n \in \mathbb{N}\}$ of ℓ_2 is compact.
- The closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ in infinite-dimensional normed space is not compact in the topology induced by norm (see Proposition 1.7.14).

Proposition 1.7.9 *A subset C of ℓ_p space is compact if C is bounded and for $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} |x_i|^p < \varepsilon^p$ for all $n \geq n_0$ and $x = \{x_i\}_{i=1}^{\infty} \in C$.*

Proposition 1.7.10 *Every compact subset of a normed space X is closed, but the converse may not be true.*

Observation

- $\mathbb{R}^n$ is closed.

Proposition 1.7.11 *Every compact subset of a normed space X is complete, but the converse may not be true.*

Proposition 1.7.12 *Every compact subset of a normed space is bounded, but the converse may not be true.*

Proposition 1.7.13 *Every compact subset of a normed space is separable.*

Proposition 1.7.14 *A closed and bounded subset of a normed space need not be compact.*

Proof. Let $X = \ell_2$. Then the unit ball $B_X = \{x \in \ell_2 : \|x\|_2 = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} \leq 1\}$ is closed and bounded. We now show that B_X is not compact. Let $\{x_n\}$ be a sequence in B_X defined by

$$x_n = (0, 0, \cdots, 1, 0, \cdots), \quad n \in \mathbb{N}.$$

$$\uparrow$$
$$n^{th} \text{ position}$$

Hence for $m \neq n$,

$$\|x_n - x_m\|_2 = \sqrt{2},$$

i.e., there is no convergent subsequence of $\{x_n\}$. Therefore, B_X is not totally bounded and hence it is not compact. ∎

Remark 1.7.15 *B_{ℓ_2} is compact in the weak topology (see Theorem 1.9.26).*

Proposition 1.7.16 *A normed space X is finite-dimensional if and only if every closed and bounded subset of X is compact.*

1.8 Reflexivity

Let $X_1, X_2, \cdots, X_m$ be m linear spaces over the same field $\mathbb{K}$. Then a functional $f : X_1 \times X_2 \times \cdots \times X_m \to \mathbb{R}$ is said to be an *m-linear (multilinear) functional* on $X = X_1 \times X_2 \times \cdots \times X_m$ if it is linear with respect to each of the variables separately. For $m = 2$, such a functional is called a *bilinear functional*.

Duality pairing - Given a normed space X and its dual X^*, we define the duality pairing as the functional $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{K}$ such that

$$\langle x, j \rangle = j(x) \quad \text{for all } x \in X \text{ and } j \in X^*.$$

The properties of duality pairing can be easily derived from the definition:

Proposition 1.8.1 *Let X^* be the dual of a normed space X. Then we have the following:*

(a) The duality pairing is a bilinear functional on $X \times X^*$:
(i) $\langle ax + by, j \rangle = a\langle x, j \rangle + b\langle y, j \rangle$ for all $x, y \in X; j \in X^*$ and $a, b \in \mathbb{K}$;
(ii) $\langle x, \alpha j_1 + \beta j_2 \rangle = \alpha \langle x, j_1 \rangle + \beta \langle y, j_2 \rangle$ for all $x \in X; j_1, j_2 \in X^*$; $\alpha, \beta \in \mathbb{K}$.

(b) $\langle x, j \rangle = 0$ for all $x \in X$ implies $j = 0$.

(c) $\langle x, j \rangle = 0$ for all $j \in X^*$ implies $x = 0$.

Natural embedding mapping - Let $(X, \|\cdot\|)$ be a normed space. Then $(X^*, \|\cdot\|_*)$ is a Banach space. Let $j \in X^*$. Hence for given $x \in X$, the equation

$$f_x(j) = \langle x, j \rangle$$

defines a functional f_x on the dual space X^*. The functional f_x is linear by Proposition 1.8.1. Moreover, for $j \in X^*$ we have

$$|f_x(j)| = |\langle x, j\rangle| \leq \|x\|\|j\|_*. \tag{1.9}$$

This shows that f_x is bounded and hence f_x is a bounded linear functional on X^*.

The space of all bounded linear functionals on X^* is denoted by X^{**} and is called the *second dual of* X. Then $f_x \in X^{**}$. Note that X^{**} is a Banach space. Let $\|\cdot\|_{**}$ denote a norm on X^{**}. From (1.9), we have

$$\|f_x\|_{**} \leq \|x\|.$$

By Corollary 1.6.5, there exists a nonzero functional $j \in X^*$ such that

$$\langle x, j\rangle = \|x\|\|j\|_* \text{ and } \|j\|_* = \|x\|.$$

This implies that $\|f_x\|_{**} = \|x\|$.

Define a mapping $\varphi : X \to X^{**}$ by $\varphi(x) = f_x$, $x \in X$. Then φ is called the *natural embedding mapping* from X into X^{**}. It has the following properties:

(i) φ is linear: $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ for all $x, y \in X, \alpha, \beta \in \mathbb{K}$;

(ii) $\varphi(x)$ is isometry: $\|\varphi(x)\| = \|x\|$ for all $x \in X$.

Generally, however, the natural embedding mapping φ from X into X^{**} is not onto. It means that there may be elements in X^{**} that cannot be represented by elements in X.

In the case when φ is onto, we have an important class of normed spaces.

Definition 1.8.2 *A normed space X is said to be reflexive if the natural embedding mapping $\varphi : X \to X^{**}$ is onto. In this case, we write $X \cong X^{**}$ or $X = X^{**}$.*

Observation

- $\mathbb{R}^n$ is reflexive. (In fact, every finite-dimensional Banach space is reflexive.)
- ℓ_p and L_p for $1 < p < \infty$ are reflexive Banach spaces.
- Every Hilbert space is a reflexive Banach space, i.e., $H^{**} = H$.
- ℓ_1, ℓ_∞, L_1 and L_∞ are not reflexive.
- c and c_0 are not reflexive Banach spaces.

We now state the following facts for the class of reflexive Banach spaces.

Proposition 1.8.3 *(a) Any reflexive normed space must be complete and, hence, is a Banach space.*

(b) A closed subspace of a reflexive Banach space is reflexive.

(c) The Cartesian product of two reflexive spaces is reflexive.

(d) The dual of a reflexive Banach space is reflexive.

Theorem 1.8.4 (James theorem) – *A Banach space X is reflexive if and only if for each $j \in S_{X^*}$, there exists $x \in S_X$ such that $j(x) = 1$.*

1.9 Weak topologies

Let X^* be the dual space of a Banach space X. The convergence of a sequence in a Banach space X is the usual *norm convergence* or *strong convergence*, i.e., $\{x_n\}$ in X converges to x if $\lim_{n\to\infty} \|x_n - x\| = 0$. This is related to the strong topology on X with neighborhood base $B_r(0) = \{x \in X : \|x\| < r\}, r > 0$ at the origin. There is also a weak topology on X generated by the bounded linear functionals on X. Indeed, $G \subset X$ is open in the weak topology (we say G is w-open) if and only if for every $x \in G$, there are bounded linear functionals $f_1, f_2, \cdots, f_n$ and positive real numbers $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ such that

$$\{y \in X : |f_i(x) - f_i(y)| < \varepsilon_i, i = 1, 2, \cdots, n\} \subset G.$$

Hence a subbase σ for the weak topology on X generated by a base of neighborhoods of $x_0 \in X$ is given by the following sets:

$$V(f_1, f_2 \cdots, f_n : \varepsilon) = \{x \in X : |\langle x - x_0, f_i\rangle| < \varepsilon, \ \text{ for every } \ i = 1, 2, \cdots, n\}.$$

In particular, a sequence $\{x_n\}$ in X converges to $x \in X$ for weak topology $\sigma(X, X^*)$ if and only if $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$.

Observation

- The weak topology is not metrizable if X is infinite-dimensional.
- Under the weak topology, the normed space X is a locally convex topological space.
- The weak topology of a normed space is a Hausdorff topology.

We are now in a position to define convergence, closedness, completeness, and compactness with respect to the weak topology.

Weakly convergent – A sequence $\{x_n\}$ in a normed space X is said to converge weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, we write $x_n \rightharpoonup x$ or weak-$\lim_{n\to\infty} x_n = x$.

Weakly closed – A subset C of a Banach space X is said to be a *weakly closed* if it is closed in the weak topology.

Weak Cauchy sequence – A sequence $\{x_n\}$ in a normed space X is said to be a *weak Cauchy* if for each $f \in X^*, \{f(x_n)\}$ is a Cauchy sequence in $\mathbb{K}$.

Weakly complete – A normed space X is said to be *weakly complete* if every weak Cauchy sequence in X converges weakly to some element in X.

Weakly compact – A subset C of a normed space X is said to be *weakly compact* if C is compact in the weak topology.

Schur property – A Banach space is said to satisfy *Schur property* if there exist weakly convergent sequences that are norm convergent.

Theorem 1.9.1 (Schur's theorem) – *In ℓ_1, weak and norm convergences of sequences coincide.*

We have the following basic properties of weakly convergent sequences in normed spaces:

Proposition 1.9.2 (Uniqueness of weak limit) – *Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$. Then $x = y$.*

Proof. Because $\{f(x_n)\}$ is a sequence of scalars such that $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$, it follows that $f(x) = f(y)$. This implies that $f(x - y) = 0$. Therefore, $x = y$ by Corollary 1.6.8. ∎

Proposition 1.9.3 (Strong convergence implies weak convergence) – *Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \to x$. Then $x_n \rightharpoonup x$.*

Proof. Because $x_n \to x$, $\|x_n - x\| \to 0$. Hence

$$|f(x_n) - f(x)| \leq \|f\|_* \|x_n - x\| \to 0 \text{ for all } f \in X^*.$$

Therefore, $x_n \rightharpoonup x$. ∎

The converse of Proposition 1.9.3 is not true in general. It can be seen from the following example:

Example 1.9.4 *Let $X = \ell_2$ and $\{x_n\}$ be a sequence in ℓ_2 such that*

$$x_n = (0, 0, 0, \cdots, \underset{\substack{\uparrow \\ n^{th} \text{ position}}}{1}, 0, \cdots), \qquad n \in \mathbb{N}.$$

For any $y = (y_1, y_2, \cdots, y_n, \cdots) \in X^* = \ell_2$, we have

$$(x_n, y) = y_n \to 0 \text{ as } n \to \infty.$$

Hence $x_n \rightharpoonup 0$ as $n \to \infty$. However, $\{x_n\}$ does not converge strongly because $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Therefore, a weakly convergent sequence need not be convergent in norm.

Theorem 1.9.5 (Weak convergence in ℓ_p space, $1 < p < \infty$) – *For $1 < p < \infty$, let*

$$x_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_i^n, \cdots) \in \ell_p, \quad n \in \mathbb{N}$$

and

$$x = (\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots) \in \ell_p.$$

Then $x_n \rightharpoonup x$ *if and only if*

(i) $\{x_n\}$ *is bounded, i.e.,* $\|x_n\| \le M$ *for all* $n \in \mathbb{N}$ *and for some* $M \ge 0$*;*

(ii) *for each* i*,* $\alpha_i^{(n)} \to \alpha_i$ *as* $n \to \infty$*.*

Theorem 1.9.6 *Let* X *be a finite-dimensional normed space. Then strong convergence is equivalent to weak convergence.*

Theorem 1.9.7 *Every reflexive normed space is weakly complete.*

Convergence of sequences in B(X, Y) – Let X and Y be two normed spaces. A sequence $\{T_n\}$ in $B(X, Y)$ is said to be

(i) *uniformly convergent to* $T \in B(X, Y)$ in the norm of $B(X, Y)$ if $\|T_n - T\|_B \to 0$ as $n \to \infty$, i.e., for $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\sup_{\|x\| \le 1} \|T_n x - Tx\| < \varepsilon$ for all $n \ge n_0$,

[uniform convergence of $\{\mathbf{T_n}\}$]

(ii) *strongly convergent to* $T \in B(X, Y)$ if $\lim_{n \to \infty} \|T_n x - Tx\| = 0$ for all $x \in X$,

[strong convergence of $\{\mathbf{T_n}\}$]

(iii) *weakly convergent to* $T \in B(X, Y)$ if $|f(T_n x) - f(Tx)| \to 0$ for all $x \in X$ and $f \in Y^*$.

[weak convergence of $\{\mathbf{T_n}\}$]

It follows immediately from the inequality

$$\|T_n x - Tx\| \le \|T_n - T\|_B \|x\|, \quad x \in X$$

that the uniform convergence implies strong convergence. It can be easily observed for the sequence of operators in $B(X, Y)$ that

uniform convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

We note that the converse is not true in general.

Weak* topology - We have seen that if τ is the norm topology of a normed space X, then the weak topology $\sigma(X, X^*)$ is a subset of the original norm topology τ. Let τ^* be the norm topology of X^* generated by the norm $\|\cdot\|_*$ (of X^*). Then there exists a topology denoted by $\sigma(X^*, X)$ on X^* such that $\sigma(X^*, X) \subset \tau^*$. The topology $\sigma(X^*, X)$ is called the *weak* topology* on X^*. Thus, we can speak about strong neighborhood, strongly closed, strongly bounded, weak convergence in $(X^*, \|\cdot\|_*)$ and weak* neighborhood, weak*ly closed, weak*ly bounded, weak*ly convergence in $(X^*, \sigma(X^*, X))$, respectively.

We now study some basic properties of the weak topology and weak* topology. We begin with a simple characterization for the convergence of sequences in the weak topologies.

Proposition 1.9.8 *Let X be a normed space and $\{f_n\}$ a sequence in X^*. Then we have the following:*

(a) $\{f_n\}$ converges strongly to f in the norm topology on X^* (denoted by $f_n \to f$) if

$$\|f_n - f\|_* \to 0.$$

(b) $\{f_n\}$ converges to f in the weak topology on X^* (denoted by $f_n \rightharpoonup f$) if

$$\langle f_n - f, g\rangle \to 0 \text{ for all } g \in X^{**}.$$

(c) $\{f_n\}$ converges to f in the weak topology on X^* (denoted by $f_n \to f$ weak*ly or $f_n \rightharpoonup^* f$) if*

$$\langle x, f_n - f\rangle \to 0 \text{ for all } x \in X.$$

On the other hand, the following result is an immediate consequence of Theorem 1.5.3.

Corollary 1.9.9 *Let C be a nonempty subset of a Banach space X. For each $f \in X^*$, let $f(C) = \cup_{x\in C}\langle x, f\rangle$ be a bounded set in $\mathbb{R}$. Then C is bounded.*

Proof. Set $X := X^*, Y := \mathbb{R}$, and $T_x(f) := \langle x, f\rangle$, $x \in C$. Then $T_x \in B(X^*, \mathbb{R})$. Because $f(C)$ is bounded, it follows that

$$\sup_{x\in C} |T_x(f)| = \sup_{x\in C} |\langle x, f\rangle| \leq K,$$

for some $K > 0$. By the uniform boundedness principle, there exists a constant $M > 0$ such that

$$\|T_x\| \leq M \text{ for all } x \in C.$$

This implies that

$$|\langle x, f\rangle| \quad = \quad |T_x(f)| \leq \|T_x\|\|f\|_* \leq M\|f\|_*.$$

By Corollary 1.6.7, we have

$$\|x\| \leq M \text{ for all } x \in C.$$

Therefore, C is bounded. ∎

Applying Corollary 1.9.9, we have

Theorem 1.9.10 *Let $\{x_n\}$ be a sequence in a Banach space X. Then we have the following:*

(a) $x_n \rightharpoonup x$ (in X) implies $\{x_n\}$ is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(b) $x_n \rightharpoonup x$ in X and $f_n \to f$ in X^* imply $f_n(x_n) \to f(x)$ in $\mathbb{R}$.

Proof. (a) Because $x_n \rightharpoonup x$, then $f(x_n) \to f(x)$ for all $f \in X^*$. Hence $\{f(x_n)\}$ is bounded for all $f \in X^*$. Thus, by Corollary 1.9.9, $\{x_n\}$ is bounded.

Moreover,

$$|\langle x_n, f\rangle| \le \|x_n\|\|f\|_*.$$

Taking liminf in the above inequality, we have

$$|\langle x, f\rangle| \le \liminf_{n\to\infty} \|x_n\|\|f\|_*.$$

By Corollary 1.6.7, we obtain

$$\|x\| = \sup_{\|f\|_*\le 1} |\langle x, f\rangle| \le \sup_{\|f\|_*\le 1} (\liminf_{n\to\infty} \|x_n\|\|f\|_*) \le \liminf_{n\to\infty} \|x_n\|.$$

(b) Because $x_n \rightharpoonup x$ in X, it follows that $\langle x_n - x, f\rangle = f(x_n) - f(x) \to 0$ and $\{x_n\}$ is bounded (by part (a)). Hence

$$\begin{aligned} |\langle x_n, f_n\rangle - \langle x, f\rangle| &\le |\langle x_n, f_n\rangle - \langle x_n, f\rangle| + |\langle x_n, f\rangle - \langle x, f\rangle| \\ &= |\langle x_n, f_n - f\rangle| + |\langle x_n - x, f\rangle| \\ &\le \|x_n\| \; \|f_n - f\|_* + |\langle x_n - x, f\rangle| \\ &\le M\|f_n - f\|_* + |\langle x_n - x, f\rangle| \to 0 \end{aligned}$$

for some constant $M > 0$. Therefore, $f_n(x_n) \to f(x)$. ∎

Observation

- Let $\{x_n\}$ be a sequence in a Banach space X with $x_n \rightharpoonup x \in X$ and $\{\alpha_n\}$ a sequence of scalars such that $\alpha_n \to \alpha$. Then $\{\alpha_n x_n\}$ converges weakly to αx.

Theorem 1.9.11 *Let X be a Banach space and $\{x_n\}$ a sequence in X such that $x_n \rightharpoonup x \in X$. Then there exists a sequence of convex combinations of $\{x_n\}$ that converges strongly to x, i.e., there exists convex combination $\{y_n\}$ such that*

$$y_n = \sum_{i=n}^{m} \lambda_i x_i, \ \text{where} \sum_{i=n}^{m} \lambda_i = 1 \text{ and } \lambda_i \ge 0, \quad n \le i \le m,$$

which converges strongly to x.

Corollary 1.9.12 *Let C a nonempty subset of a Banach space X and $\{x_n\}$ a sequence in C such that $x_n \rightharpoonup x \in X$. Then $x \in \overline{co}(C)$.*

The weak topology is weaker than the norm topology, and every w-closed set is also norm closed. The following result shows that for convex sets, the converse is also true.

Proposition 1.9.13 *Let C be a convex subset of a normed space X. Then C is weakly closed if and only if C is closed.*

The following proposition is a generalization of Theorem 1.7.8.

Proposition 1.9.14 *Let C be a weakly compact subset of a Banach space X. Then $\overline{co}(C)$ is also weakly compact.*

The following result is a direct consequence of the uniform bounded principle:

Proposition 1.9.15 *Let C be a weakly compact subset of a Banach space X. Then C is bounded.*

Theorem 1.9.16 (Eberlein-Smulian theorem) – *Let C be a weakly closed subset of a Banach space. Then the following are equivalent:*

(a) C is weakly compact.

(b) C is weakly sequentially compact, i.e., each sequence $\{x_n\}$ in C has a subsequence that converges weakly to a point in C.

Corollary 1.9.17 *Let C be a closed convex subset of a Banach space. Then the following are equivalent:*

(a) C is weakly compact.

(b) Each sequence $\{x_n\}$ in C has a subsequence that converges weakly to a point in C.

Proposition 1.9.18 *Any closed convex subset of a weakly compact set is itself weakly compact.*

Theorem 1.9.19 (Kakutani's theorem) – *Let X be a Banach space. Then X is reflexive if and only if the unit closed ball $B_X := \{x \in X : \|x\| \leq 1\}$ is weakly compact (i.e., B_X is compact in the weak topology of X).*

Using Proposition 1.9.13 and Kakutani's theorem, we obtain

Theorem 1.9.20 *Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset of X is weakly compact (compact in weak topology).*

Theorem 1.9.21 *Let C be a subset of a reflexive Banach space. Then*

C is weakly compact	$\Leftrightarrow$	*C is bounded*
(compactness in weak topology)		*(boundedness in strong topology)*

Theorem 1.9.22 *Let $\{x_n\}$ be a sequence in a weakly compact convex subset of a Banach space X and $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of $\{x_n\}$. Then $\overline{co}(\omega_w(\{x_n\})) = \cap_{n=1}^{\infty}\overline{co}(\{x_k\}_{k\geq n})$.*

Proof. Set $W := \omega_w(\{x_n\})$, $A_n := \overline{co}(\{x_k\}_{k\geq n})$, and $A := \cap_{n=1}^{\infty} A_n$. We now show that $\overline{co}(W) = A$. The inclusion $W \subset A$ (and hence $\overline{co}(W) \subset A$) is trivial.

Hence it suffices to prove that $A \subset \overline{co}(W)$. Suppose, for contradiction, that $x \in A \setminus \overline{co}(W)$. Then there exists $j \in X^*$ such that

$$\langle x, j\rangle > \sup\{\langle y, j\rangle : y \in \overline{co}(W)\} = \sup\{\langle y, j\rangle : y \in W\rangle. \tag{1.10}$$

Because $x \in A \subset A_n$,

$$\langle x, j\rangle \leq \sup\{\langle y, j\rangle : y \in A_n\} = \sup\{\langle x_k, j\rangle : k \geq n\}.$$

Therefore,

$$\langle x, j\rangle \leq \limsup_{n\to\infty}\langle x_n, j\rangle.$$

It follows from the Eberlein-Smulian theorem that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup x' \quad \text{and} \quad \langle x, j\rangle \leq \langle x', j\rangle.$$

Because $x' \in W$ by definition, this is a contradiction of (1.10). ∎

Corollary 1.9.23 *Let X be a Banach space and $\{x_n\}$ a sequence in X weakly convergent to z. Let $A_n = \overline{co}(\{x_k\}_{k\geq n})$. Then $\cap_{n=1}^{\infty} A_n = \{z\}$.*

Proposition 1.9.24 *Let $\{x_n\}$ be a bounded sequence in reflexive Banach space X and $A_n = \overline{co}(\{x_n\}_{k\geq n})$. If $\cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty}\overline{co(\{x_n, x_{n+1}, \cdots\})} = \{x\}$, then $x_n \rightharpoonup x$.*

Proposition 1.9.25 *Let $\{x_n\}$ be a weakly null sequence in a Banach space X and $\{j_n\}$ a bounded sequence in X^*. Then for each $\varepsilon > 0$, there exists an increasing sequence $\{n_k\}$ in $\mathbb{N}$ such that $|\langle x_{n_i}, j_{n_k}\rangle| < \varepsilon$ if $i \neq k$.*

Proof. Without loss of generality, we may assume that X is a separable space. We can assume that $\{j_n\}$ converges weak*ly to some $j \in B_{X^*}$. Given $\varepsilon > 0$, we find n_1 such that $|\langle x_n, j\rangle| < \varepsilon/2$ for all $n \geq n_1$. Next, having $n_1 < n_2 < \cdots < n_{k-1}$, we pick $n_k > n_{k-1}$ with $|\langle x_{n_k}, j_{n_i}\rangle| < \varepsilon$ and $|\langle x_{n_i}, j_{n_k} - j\rangle| < \varepsilon/2$ for all $i = 1, 2, \cdots, k-1$. Then $|\langle x_{n_i}, j_{n_k}\rangle| < \varepsilon$. ∎

We now list several properties that characterize reflexivity.

Theorem 1.9.26 *Let X be a Banach space. Then following statements are equivalent:*

(a) X is reflexive.

(b) B_X is weakly compact.

(c) Every bounded sequence in X in strong topology has a weakly convergent subsequence.

(d) For any $f \in X^*$, there exists $x \in B_X$ such that $f(x) = \|f\|_*$.

(e) *X^* is reflexive.*

(f) *$\sigma(X^*, X) = \sigma(X^*, X^{**})$, i.e., on X^* the weak topology and the weak* topology coincide.*

(g) *If $\{C_n\}$ is any descending sequence of nonempty closed convex bounded subsets of X, then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

(h) *For any closed convex bounded subset C of X and any $j \in X^*$, there exists $x \in C$ such that $\langle x, j\rangle = \sup\{\langle y, j\rangle : y \in C\}$.*

Finally, we give the fundamental result concerning the weak* topology.

Theorem 1.9.27 (Banach-Alaoglu's theorem) – *The unit ball B_{X^*} of the dual of a normed space X is compact in the weak* topology.*

1.10 Continuity of mappings

In this section, we discuss various forms of continuity of mappings with their properties.

Definition 1.10.1 *Let T be a mapping from a metric space (X, d) into another metric space (Y, ρ). Then T is said to be*

(i) *continuous at $x_0 \in X$ if $x_n \to x_0$ implies $Tx_n \to Tx_0$ in Y, i.e., for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, x_0) > 0$ such that $\rho(Tx_0, Ty) < \varepsilon$ whenever $d(x_0, y) < \delta$ for all $y \in X$,*

(ii) *uniformly continuous on X if for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\rho(Tx, Ty) < \varepsilon \text{ whenever } d(x, y) < \delta \text{ for all } x, y \in X.$$

Example 1.10.2 *Let $X = (0, 1]$ and $Y = \mathbb{R}$ and let X and Y have usual metric defined by absolute value. Then the mapping $T : X \to Y$ defined by $Tx = 1/x$ is continuous, but not uniformly continuous.*

Observation

- Every uniformly continuous mapping from X into Y is continuous at each point of X, but pointwise continuity does not necessary imply uniform continuity.

- Every uniformly continuous mapping T from a metric space X into another metric space Y maps a Cauchy sequence in X into a Cauchy sequence in Y.

Proposition 1.10.3 *Let T be a continuous mapping from a compact metric space (X, d) into another metric space (Y, ρ). Then T is uniformly continuous.*

A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy *Lipschitz condition* on X if there exists a constant $L > 0$ such that

$$\rho(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called *Lipschitz constant.* In this case, we say that T is an *L-Lipschitz mapping* or simply a *Lipschitzian mapping* with Lipschitz constant L. Otherwise, it is called *non-Lipschitzian mapping.* An L-Lipschitz mapping T is said to be *contraction* if $L < 1$ and *nonexpansive* if $L = 1$. The mapping T is said to be *contractive* if

$$\rho(Tx, Ty) < d(x, y) \text{ for all } x, y \in X, \ x \neq y.$$

Remark 1.10.4 *Every Lipschitz continuous mapping T from a metric space X into another metric space Y is uniformly continuous on X. Indeed, choose $\delta < \varepsilon/L$ (independent of x), and we get*

$$\rho(Tx, Tx_0) \leq Ld(x, x_0) < \varepsilon.$$

The following example shows that the distance functional $f(x) = d(x, C)$ is nonexpansive.

Example 1.10.5 *Let C be a nonempty subset of a normed space X. Then for each pair x, y in X*

$$|d(x, C) - d(y, C)| \leq \|x - y\|.$$

In particular, the function $x \mapsto d(x, C)$ is nonexpansive and hence uniformly continuous.

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition 1.10.6 *Let $T : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b). Suppose T' is continuous on $[a, b]$. Then T is a Lipschitz continuous function (and hence is uniformly continuous).*

Proof. By the Lagrange's theorem, we have

$$Ty - Tx = T'(c)(y - x) \text{ for all } a \leq x < y \leq b,$$

where $c \in (x, y) \subset [a, b]$. Because T' is continuous and interval $[a, b]$ is compact in $\mathbb{R}$, by Weierstrass theorem, there exists $x_0 \in [a, b]$ such that

$$L = |T'(x_0)| = \sup_{c \in [a,b]} |T'(c)|.$$

Thus, $|Tx - Ty| \leq L|x - y|$, which proves that T is Lipschitz continuous. ∎

The following example shows that there is a Lipschitzian mapping for which T' does not exist.

Example 1.10.7 *The function* $Tx = |x|,\ x \in [-1, 1]$ *satisfies Lipschitz condition with* $L = 1$, *i.e.,* $|Tx - Ty| \leq |x - y|$ *for all* $x, y \in [-1, 1]$. *Note* T *is not differentiable at zero.*

We now give an example of a non-Lipschitzian mapping that is continuous.

Example 1.10.8 *Let* $T : \left[-\frac{1}{\pi}, \frac{1}{\pi}\right] \to \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ *be a mapping defined by*

$$Tx = \begin{cases} 0 & \text{if} \quad x = 0, \\ \frac{x}{2}\sin(1/x) & \text{if} \quad x \neq 0. \end{cases}$$

Then T *is continuous, but not Lipschitz continuous.*

For linear mappings, the continuity condition can be restated in terms of uniform continuity.

Proposition 1.10.9 *Let* X *and* Y *be two normed spaces and* $T : X \to Y$ *a linear mapping. Then the following conditions are equivalent:*

(a) T *is continuous.*

(b) T *is Lipschitz function: there exists* $M > 0$ *such that* $\|Tx\| \leq M\|x\|$ *for all* $x \in X$.

(c) T *is uniformly continuous.*

Let X and Y be two Banach spaces and let T be a mapping from X into Y. Then the mapping T is said to be

1. *bounded* if C is bounded in X implies $T(C)$ is bounded;
2. *locally bounded* if each point in X has a bounded neighborhood U such that $T(U)$ is bounded;
3. *weakly continuous* if $x_n \rightharpoonup x$ in X implies $Tx_n \rightharpoonup Tx$ in Y;
4. *demicontinuous* if $x_n \to x$ in X implies $Tx_n \rightharpoonup Tx$ in Y;
5. *hemicontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ converging to x_0 along a line implies $Tx_n \rightharpoonup Tx_0$, i.e., $Tx_n = T(x_0 + t_n x) \rightharpoonup Tx_0$ as $t_n \to 0$ for all $x \in X$;
6. *closed* if $x_n \to x$ in X and $Tx_n \to y$ in Y imply $Tx = y$;
7. *weakly closed* if $x_n \rightharpoonup x \in X$ and $Tx_n \rightharpoonup y$ in Y imply $Tx = y$;
8. *demiclosed* if $x_n \rightharpoonup x$ in X and $Tx_n \to y$ in Y imply $Tx = y$;
9. *compact* if C is bounded implies $T(C)$ is relatively compact ($\overline{T(C)}$ is compact), i.e., for every bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has convergent subsequence in Y;

10. *completely continuous* if it is continuous and compact;
11. *demicompact* if any bounded sequence $\{x_n\}$ in X such that $\{x_n - Tx_n\}$ converges strongly has a convergent subsequence.

In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general.

Proposition 1.10.10 *Every continuous linear mapping $T : X \to Y$ is weakly continuous.*

Proposition 1.10.11 *Let X be a reflexive Banach space and Y a general Banach space. Then every weakly continuous mapping $T : X \to Y$ is bounded.*

Proposition 1.10.12 *A completely continuous mapping maps a weakly convergent sequence into a strongly convergent.*

Proposition 1.10.13 *Every linear mapping is hemicontinuous.*

Proof. Every linear and demicontinuous mapping is continuous. ∎

It is clear that every demicontinuous mapping is hemicontinuous, but the converse is not true.

Example 1.10.14 *Let $X = \mathbb{R}^2, Y = \mathbb{R}$, and $T : X \to Y$ a mapping defined by*

$$T(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Then T is hemicontinuous at $(0,0)$, but not demicontinuous at $(0,0)$.

Let X and Y be two sets. A *multivalued* T from X to Y, denoted by $T : X \to Y$, is a subset $T \subseteq X \times Y$. The inverse of $T : X \to Y$ is a multivalued function $T^{-1} : Y \to X$ defined by $(y,x) \in T^{-1}$ if and only if $(x,y) \in T$. The values of T are the sets $Tx = \{y \in Y : (x,y) \in T\}$; the *fibers* of T are the sets $T^{-1}(y) = \{x \in X : (x,y) \in T\}$ for $y \in Y$.

For $A \subset X$, the set

$$T(A) = \cup_{x\in A} Tx = \{y \in Y : T^{-1}(y) \cap A \neq \emptyset\}$$

is called the *image of* A under T; for $B \subset Y$, the set

$$T^{-1}(B) = \cup_{y\in B} T^{-1}(y) = \{x \in X : Tx \cap B \neq \emptyset\},$$

the image of B under T^{-1}, is called *inverse image* of B under T. A point of a set that is invariant under any transformation is called a *fixed point* of the transformation. A point $x_0 \in X$ is said to be a fixed point of T if $x_0 \in Tx_0$.

Let X and Y be two topological spaces. Then a multivalued function $T : X \to Y$ is said to be *upper semicontinuous* (*lower semicontinuous*) if the inverse image of a closed set (open set) is closed (open). A multivalued function is continuous if it is both upper and lower semicontinuous.

Finally, we conclude the chapter with the following important fixed point theorems.

Theorem 1.10.15 (Brouwer's fixed point theorem) – *Every continuous mapping from the unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Theorem 1.10.16 (Schauder's fixed point theorem) – *Let C be a nonempty closed convex bounded subset of a Banach space X. Then every continuous compact mapping $T : C \to C$ has a fixed point.*

Theorem 1.10.17 (Tychonoff's fixed point theorem) – *Let C be a nonempty compact convex subset of a locally convex topological linear space X and $T : C \to C$ a continuous mapping. Then T has a fixed point.*

Exercises

1.1 Let (X, d) be a metric space. Show that $\rho(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$ is also a metric space.

1.2 Give an example of a seminorm that is not a norm.

1.3 Let $\langle \cdot, \cdot \rangle$ be an inner product on a linear space X and $T : X \to X$ a one-one linear mapping. Let $\langle x, y \rangle_T = \langle Tx, Ty \rangle$ for all $x, y \in X$. Show that $\langle \cdot, \cdot \rangle_T$ is an inner product space.

1.4 Show that the space c_0 of all real sequences converging to 0 is a normed space with norm $\|x\| = \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty$.

1.5 Let c_{00} be a normed space with ℓ_p-norm $(1 \le p \le \infty)$ and $\{f_n\}$ a sequence of functional on c_{00} defined by $f_n(x) = nx_n$ for all $x = (x_1, x_2, \cdots, x_n, \cdots)$. Show that $f_n(x) \to 0$ for every $x \in c_{00}$, but $\|f_n\| = n$ for all n.

1.6 Show that the space ℓ_p $(1 < p < \infty)$ is reflexive, but ℓ_1 is not reflexive.

1.7 Let C be a nonempty closed convex subset of a normed space X and $\{x_n\}$ a sequence in C such that $x_n \rightharpoonup x$ in X. Show that $x \in C$.

1.8 Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \rightharpoonup x$. Show that $x \in \text{span}\, \{x_n\}$.

1.9 Let $\{x_n\}$ be a sequence in normed space X such that $x_n \rightharpoonup x$. Show that $\{x_n\}$ is bounded.

1.10 Let $X = c_{00}$ or c_0 with norm $\|\cdot\|_\infty$. Show that $x_n \rightharpoonup x$ in X if and only if $\{x_n\}$ is bounded in X and $x_{n,i} \to x_i$ as $n \to \infty$ for each $i = 1, 2, \cdots$.