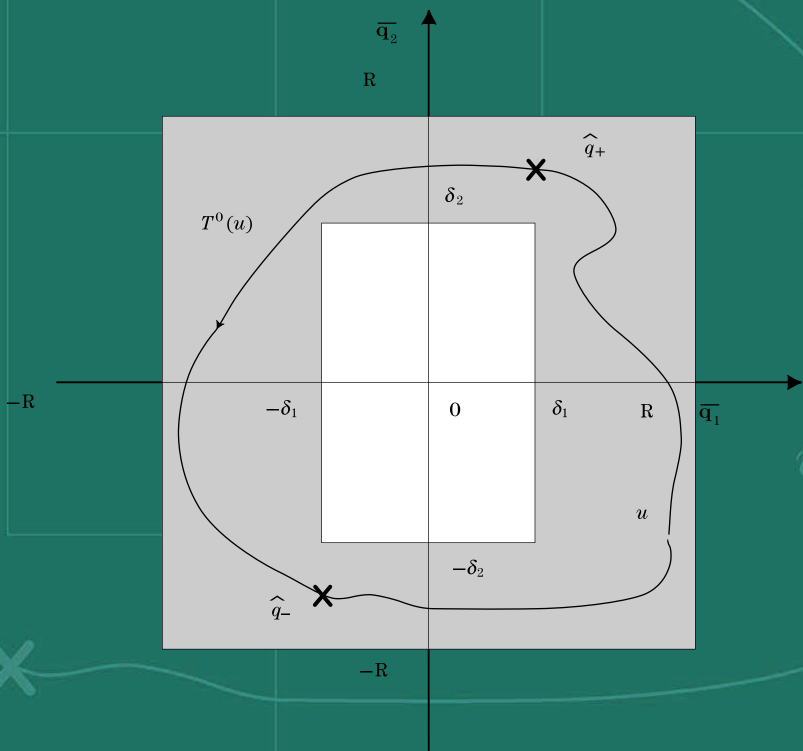
Topological Fixed Point Theory and Its Applications

Fixed Point Theory for Lipschitzian-type Mappings with Applications

Ravi P. Agarwal
Donal O'Regan
D.R. Sahu



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Dedicated to our daughters

Sheba Agarwal
Lorna Emily O'Regan
Gargi Sahu

Preface

Over the past few decades, fixed point theory of Lipschitzian and non-Lipschitzian mappings has been developed into an important field of study in both pure and applied mathematics. The main purpose of this book is to present many of the basic techniques and results of this theory. Of course, not all aspects of this theory could be included in this exposition.

The book contains eight chapters. The first chapter is devoted to some of the basic results of nonlinear functional analysis. The final section in this chapter deals with the classic results of fixed point theory. Our goal is to study nonlinear problems in Banach spaces. We remark here that it is hard to study these without the geometric properties of Banach spaces. As a result in Chapter 2, we discuss elements of convexity and smoothness of Banach spaces and properties of duality mappings. This chapter also includes many interesting results related to Banach limits, metric projection mappings, and retraction mappings. In Chapter 3, we consider normal structure coefficient, weak normal structure coefficient, and related coefficients. This includes the most recent work in the literature. Our treatment of the main subject in the book begins in Chapter 4. In this chapter, we consider the problem of existence of fixed points of Lipschitzian and non-Lipschitzian mappings in metric spaces. Chapter 5 is devoted to problems of existence of fixed points of nonexpansive, asymptotically nonexpansive, pseudocontractive mappings in Banach spaces. Most of the results are discussed in infinite-dimensional Banach spaces. The theory of iteration processes for computing fixed points of nonexpansive, asymptotically nonexpansive, pseudocontractive mappings is developed in Chapter 6. In Chapter 7, we prove strong convergence theorems for nonexpansive, pseudocontractive, and asymptotically pseudocontractive mappings in Banach spaces. Finally in Chapter 8, we discuss several applicable problems arising in different fields.

Each chapter in this book contains a brief introduction to describe the topic that is covered. Also, an exercise section is included in each chapter. Because the book is self-contained, the book should be of interest to graduate students and mathematicians interested in learning fundamental theorems about the theory of Lipschitzian and non-Lipschitzian mappings and fixed points.

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R.P. Agarwal
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Chapter 1

Fundamentals

The aim of this chapter is to introduce the basic concepts, notations, and elementary results that are used throughout the book. Moreover, the results in this chapter may be found in most standard books on functional analysis.

1.1 Topological spaces

Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+ := [0, \infty)$ a function. Then d is called a *metric* on X if the following properties hold:

(d_1) $d(x, y) = 0$ if and only if $x = y$ for some $x, y \in X$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called *distance between x and y*, and the ordered pair (X, d) is called *metric space*.

Example 1.1.1 *The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is a metric space. The metric d is called the usual metric for $\mathbb{R}$.*

For any $r > 0$ and an element x in a metric space (X, d), we define

$B_r(x) := \{y \in X : d(x, y) < r\}$, the *open ball* with center x and radius r;

$B_r[x] := \{y \in X : d(x, y) \leq r\}$, the *closed ball* with center x and radius r;

$\partial B_r(x) := \{y \in X : d(x, y) = r\}$, the *boundary of ball* with center x and radius r.

For a subset C of X and a point $x \in X$, the distance between x and C, denoted by $d(x, C)$, is defined as the smallest distance from x to elements of C. More precisely,

$$d(x, C) = \inf_{x \in C} d(x, y).$$

The number $\sup\{d(x, y) : x, y \in C\}$ is referred to as the *diameter of set C* and is denoted by $diam(C)$. If $diam(C)$ is finite, then C is said to be *bounded*, and

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if not, then C is said to be *unbounded*. In other words, C is bounded if there exists a sufficiently large ball that contains C.

Interior points and open set – Let G be a subset of a metric space (X, d). Then $x \in G$ is said to be an *interior of* G if there exists an $r > 0$ such that $B_r(x) \subset G$. The set G is said to be *open* if all its points are interior or is the empty set. The interior of set G is denoted by $int(G)$.

Observation

- $int(G) \subset G$ for any subset G of metric space X.
- For any open set $G \subset X$, $int(G) = G$.
- The empty set $\emptyset$ and entire space X are open.

Definition 1.1.2 *Let X be a nonempty set and τ a collection of subsets of X. Then τ is said to be a topology on X if the following conditions are satisfied:*

(i) $\emptyset \in \tau$ and $X \in \tau$,

(ii) τ is closed under arbitrary unions,

(iii) τ is closed under finite intersections.

The ordered pair (X, τ) is called topological space.

Observation

- The members of τ are called τ-*open sets* or simply *open sets*.

Definition 1.1.3 *A topological space is said to be metrizable if its topology can be obtained from a metric on the underlying space.*

Denoting the class of all open sets of a metric space (X, d) by τ_d, then we have

(1) $\emptyset$ and X are in τ_d,

(2) an arbitrary union of open sets is open,

(3) a finite intersection of open sets is open.

The class τ_d is called a *metric topology* on X.

Definition 1.1.4 *Let C be a subset of a topological space X. Then the interior of C is the union of all open subsets of C. It is denoted by $int(C)$.*

In other words, if $\{G_i : i \in \Lambda\}$ are all open subsets of C, then $int(C) = \cup_{i\in\Lambda}\{G_i : G_i \subset C\}$.

Observation

- $int(C)$ is open, because it is union of open sets.
- $int(C)$ is the largest open set of C.
- If G is an open subset of C, then $G \subset int(C) \subset C$.

Definition 1.1.5 *A set F in a topological space X whose complement $F^c = X - F$ is open is called a closed set.*

Theorem 1.1.6 *Let $\mathcal{C}$ be a collection of all closed sets in a topological space (X, τ). Then $\mathcal{C}$ has the following properties:*

(i) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$,

(ii) $\mathcal{C}$ is closed under arbitrary intersections,

(iii) $\mathcal{C}$ is closed under finite unions.

Definition 1.1.7 *Let C be a subset of a topological space X. Then the closure of C is the intersection of all closed supersets of C. The closure of C is denoted by $\overline{C}$.*

In other words, if $\{F_i : i \in \Lambda\}$ is a collection of all closed supersets of C in X, then $\overline{C} = \cap_{i\in\Lambda} F_i$.

Observation

- $\overline{C}$ is closed, because it is the intersection of closed sets.
- $\overline{C}$ is the smallest closed superset of C.
- If F is a closed subset of X containing C, then $C \subset \overline{C} \subset F$.

Theorem 1.1.8 *Let C be a subset of a topological space X. Then C is closed if and only if $C = \overline{C}$.*

Exterior points and boundary of sets – Let C be a subset of a topological space X. Then the *exterior* of C, written by $ext(C)$, is the interior of the complement of C, i.e., $ext(C) = int(C^c)$. The *boundary of* C is a set of points that do not belong to the interior or the exterior of C. The boundary of set C is denoted by $\partial(C)$. Obviously, $\partial(C) = \overline{C} \cap \overline{(X \setminus C)}$ is a closed set.

Proposition 1.1.9 *Let A and B be two subsets of a topological space X. Then the following properties hold:*

Properties of interiors	*Properties of closures*
$int(int(A)) = int(A)$	$\overline{(\overline{A})} = \overline{A}$
$int(A \cap B) = int(A) \cap int(B)$	$\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
$int(A \cup B) \supset int(A) \cup int(B)$	$\overline{(A \cup B)} = \overline{A} \cup \overline{B}$
$A \subset B \Rightarrow int(A) \subset int(B)$	$A \subset B \Rightarrow \overline{A} \subset \overline{B}$

Definition 1.1.10 *Let τ_1 and τ_2 be two topologies on a topological space X. Then τ_1 is said to be weaker than τ_2 if $\tau_1 \subset \tau_2$.*

Note that if τ_1 and τ_2 are two topologies on X such that $\tau = \tau_1 \cap \tau_2$. Then the topology τ is weaker than τ_1 and τ_2 both.

Theorem 1.1.11 *Let $\{\tau_i : i \in \Lambda\}$ be a collection of topologies on a topological space X. Then the intersection $\cap_{i\in\Lambda}\tau_i$ is also a topology on X.*

We now turn to the notion of a base for the topology τ.

Definition 1.1.12 *Let (X, τ) be a topological space. Then a subclass $\mathcal{B}$ of τ is said to be a base for τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.*

Observation

- Every topology has a base. In fact, we can take $\mathcal{B} = \tau$.
- In a metric space (X, d), collection of all open balls $B_r(x)$ ($x \in X$, $r > 0$) is a base for the metric topology.

Then, we have the following theorem:

Theorem 1.1.13 *Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for τ if and only if, for every $x \in X$ and every open set G containing x, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset G$.*

We now consider a base of open sets at a point.

Definition 1.1.14 *Let (X, τ) be a topological space and $x_0 \in X$. Then the collection $\mathcal{B}_{x_0} \subset \tau$ is called a base at a point x_0 if, for any open set G containing x_0, there exists $B \in \mathcal{B}_{x_0}$ such that $x_0 \in B \subset G$.*

Observation

- In the metric topology of a metric space (X, d), the collection of all $B_r(x_0)$, where r runs through the positive real numbers, constitutes a base at a point $x_0 \in X$.

Neighborhoods – Let X be a topological space and G an open set. Then G is called an *open neighborhood* of a point $x_0 \in X$ if $x_0 \in G$. The set G without x_0, i.e., $G \setminus \{x_0\}$, is called a *deleted open neighborhood* of a point $x_0 \in X$. A subset C of X is said to be a *neighborhood of a point* $x_0 \in X$ if there exists an open set $G \in \tau$ such that $x_0 \in G \subset C$.

Let (X, τ) be a topological space. Then a collection ν of neighborhoods of $x_0 \in X$ is said to be a *neighborhood base at a point* x_0 if every neighborhood of x_0 contains a member of ν.

A collection σ of subsets of a topological space (X, τ) is said to be a *subbase* for τ if $\sigma \subset \tau$ and every member of τ is a union of finite intersections of sets from σ. In other words, σ is a subbase for τ if $\sigma \subset \tau$ and for all $G \in \tau$ and $x \in G$, there are sets $U_1, U_2, \cdots, U_n$ in σ such that $x \in \cap_{i=1}^n U_i \subset G$.

Let (X, τ) be a topological space. Then X is said to be

1. a T_0-*space* if x and y are any two distinct points in X, then there exists an open set that contains one of them, but not the other;

2. a T_1-*space* if x and y are two distinct points in X, there exists an open set U containing x and not y, and there exists another open set V containing y, but not x;

3. a T_2-*space* or *Hausdorff topological space* if x and y are two distinct points in X, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

Observation

- Every Hausdorff space is a T_1-space.
- A topological space X is T_1-space if and only if every subset consisting of a single point is closed.
- Every metric space is a Hausdorff space.

A topological space (X, τ) is said to be *compact* if every open cover has a finite subcover, i.e., if whenever $X = \cup_{i \in \Lambda} G_i$, where G_i is an open set, then $X = \cup_{i \in \Lambda_0} G_i$ for some finite subset Λ_0 of Λ.

A subset C of a topological space (X, τ) is said to be *compact* if every open cover has finite open subcover, i.e., if whenever $C \subseteq \cup_{i \in \Lambda} G_i$, where G_i is an open set, then $C \subseteq \cup_{i \in \Lambda_0} G_i$ for some finite subset Λ_0 of Λ.

Observation

- Every finite set of a topological space is compact.
- Every closed subset of a compact space is compact.
- In a compact Hausdorff space, a set is compact if and only if it is closed.

Net – Let D be a nonempty set and $\preceq$ a relation on D. Then the ordered pair $(D, \preceq)$ is said to be *directed* if

(i) $\preceq$ is reflexive: $\alpha \preceq \alpha$ for all $\alpha \in D$;

(ii) $\preceq$ is transitive: whenever $\alpha \preceq \beta$ and $\beta \preceq y \Rightarrow \alpha \preceq y$ for all $\alpha, \beta, \gamma \in D$;

(iii) for any two elements α and β, there exists γ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Observation

- $(\mathbb{N}, \geq)$ is a directed set.
- If $X \neq \emptyset$, then $(P(X), \subseteq)$ and $(P(X), \supseteq)$ are directed sets, where $P(X)$ is the power set of X.
- Every lattice is a directed set.

A *net*, or a *generalized sequence* in a set X is a mapping S from a directed set D into X. The net $\{x_\alpha : \alpha \in D\}$ is simply written as $\{x_\alpha\}$.

Let $\{x_\alpha : \alpha \in D\}$ be a net in a set X and let E be another directed set. Then a net $\{x_{\alpha_\beta} : \beta \in E\}$ in X is said to be a *subnet* of $\{x_\alpha : \alpha \in D\}$ if it satisfies the following conditions:

(i) $\{x_{\alpha_\beta} : \beta \in E\} \subset \{x_\alpha : \alpha \in D\}$;

(ii) for any $\alpha_0 \in D$, there exists $\beta_0 \in E$ such that $\alpha_0 \preceq \alpha_\beta$ exists $\beta_0 \preceq \beta$.

A net $\{x_\alpha : \alpha \in D\}$ in a topological space X is said to *converge to the point x in X* if for every neighborhood U of x, there exists $\alpha_0 \in D$ such that $x_\alpha \in U$ whenever $\alpha \succeq \alpha_0$. In this case, we write

$$x_\alpha \to x, \text{ or } \lim_\alpha x_\alpha = x.$$

A point x in a topological space X is said to be a *cluster point* of a net $\{x_\alpha : \alpha \in D\}$ if for every neighborhood U of x and every $\alpha \in D$, there exists $\beta \in D$ such that $\beta \succeq \alpha$ and $x_\beta \in U$.

Theorem 1.1.15 *Let $\{x_\alpha\}_{\alpha\in D}$ be a net in a topological space X and let $x \in X$. Then x is a cluster point of the net $\{x_\alpha\}_{\alpha\in D}$ if and only if the net $\{x_\alpha\}_{\alpha\in D}$ has a subnet converging to x.*

In a metric space (X,d), a sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$, i.e., if given $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for all $n \geq n_0$. A sequence $\{x_n\}$ in a metric space (X,d) is said to be *Cauchy* if $\lim_{m,n\to\infty} d(x_n,x_m) = 0$. A metric space (X,d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

Observation

- In a Hausdorff topological space, the limit of a net is unique.
- In a metric space, every convergent sequence is Cauchy.

A subset E of a directed set D is said to be *eventual* if there exists $\beta \in D$ such that for all $\alpha \in D$, $\alpha \preceq \beta$ implies that $\alpha \in E$. A net $S : D \to X$ is said to be *eventually* in a subset C of X if the set $S^{-1}(C)$ is an eventual subset of D. A net $\{x_\alpha\}$ in a set X is called a *universal net* if for each subset C of X, either $\{x_\alpha\}$ is eventually in C or $\{x_\alpha\}$ eventually in $X \setminus C$.

The following facts are important:

(a) Every net in a set has a universal subnet.

(b) If $f : X_1 \to X_2$ is a mapping and if $\{x_\alpha\}$ is a universal net in X_1, then $\{f(x_\alpha)\}$ is a universal net in X_2.

(c) If X is compact and if $\{x_\alpha\}$ is a universal net in X, then $\lim_\alpha x_\alpha$ exists.

We now state the following important result:

Theorem 1.1.16 *For a topological space (X,τ), the following statements are equivalent:*

(a) X is compact.

(b) For any collection of closed sets $\{F_i\}_{i\in\Lambda}$ having the finite intersection property (i.e., the intersection of any finite number of sets from the collection is nonempty), then $\bigcap_{i\in\Lambda} F_i \neq \emptyset$.

(c) Every net in X has a limit point (or, equivalently, every net has a convergent subnet).

(d) Every filter in X has a limit point (or, equivalently, every net has a convergent subfilter).

(e) Every ultrafilter in X is convergent.

We now turn our attention to the concept of continuity in topological spaces.

Definition 1.1.17 *Let (X, τ) and (Y, τ') be two topological spaces. Then a function $f : X \to Y$ is said to be continuous relative to τ and τ' (more precisely, $\tau - \tau'$ continuous) or simply continuous at a point $x \in X$ if for each $V \in \tau'$ with $f(x) \in V$, there exists $U \in \tau$ such that $x \in U$ and $f(U) \subset V$.*

The function f is called *continuous* if it is continuous at each point of X. Using the concept of net, we have the following result for continuity of a function in a topological space.

Theorem 1.1.18 *Let X and Y be two topological spaces and let f be a mapping from X into Y. Then f is continuous at a point x in X if and only if for every net $\{x_\alpha\}$ in X,*

$$x_\alpha \to x \Rightarrow f(x_\alpha) \to f(x).$$

Some other formulations for continuous functions are the following:

Theorem 1.1.19 *Let f be a function from a topological space (X, τ) into another topological space (Y, τ'). Then the following statements are equivalent:*

(1) f is continuous (i.e., $\tau - \tau'$ continuous).

(2) For each $V \in \tau', f^{-1}(V) \in \tau$.

(3) For each closed subset A of Y, $f^{-1}(A)$ is closed in X.

(4) For all $A \subset X, f(\overline{A}) \subset \overline{f(A)}$.

(5) There exists a subbase σ of τ' such that $f^{-1}(V) \in \tau$ for all $V \in \sigma$.

The following result shows that continuous image of a compact set is compact.

Theorem 1.1.20 *Let X and Y be two topological spaces and let $T : X \to Y$ be a continuous mapping. If $C \subseteq X$ is compact, then $T(C)$ is compact.*

The following result shows that there exists the smallest topology for which each member of $\{f_i : i \in \Lambda\}$ is continuous.

Theorem 1.1.21 *Let $\{(X_i, \tau_i) : i \in \Lambda\}$ be an indexed family of topological spaces, X any set, and $\{f_i : i \in \Lambda\}$ an indexed collection of functions such that for each $i \in \Lambda$, f_i is a function from X to X_i. Then there exists the smallest topology τ on X that makes each f_i continuous (i.e., $\tau - \tau_i$ continuous).*

Proof. Let $\sigma = \{f_i^{-1}(V_i) : V_i \subset X_i \textit{ is open in } \tau_i \ (i \in \Lambda)\}$ be a subbase for the topology τ given by

$$\tau = \{\cup_{F\in\mathcal{F}} \cap_{C\in F} C : \mathcal{F} \subset \overline{\sigma}\} \cup \{\emptyset, X\}, \tag{1.1}$$

where $\overline{\sigma}$ is the set of all finite subsets of σ. Thus, $G \subset X$ is open in τ if and only if for every $x \in G$, there are $i_1, i_2, \cdots, i_n \in \Lambda$ and $V_{i_1} \in \tau_{i_1}$, $V_{i_2} \in \tau_{i_2}, \cdots$, $V_{i_n} \in \tau_{i_n}$ such that $x \in \cap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k}) \subset G$. ∎

Remark 1.1.22 *The topology τ on X defined by (1.1) making each f_i continuous ($\tau - \tau'$ continuous) is called the weak topology generated by $\mathcal{F}$ and is denoted by $\sigma(X, \mathcal{F})$.*

Product space – Let $X_1, X_2, \cdots, X_n$ be n arbitrary sets with the Cartesian product $X = X_1 \times X_2 \times \cdots \times X_n$. For each $i = 1, 2, \cdots, n$, define $\pi_i : X \to X_i$ by $\pi_i(x_1, x_2, \cdots, x_n) = x_i$. Then π_i is called the *projection on* X_i or *the* i^{th} *projection.* If $x \in X$, then $\pi_i(x)$ is called the i^{th} coordinate of x.

Theorem 1.1.23 *Let $\{(X_i, \tau_i) : i = 1, 2, \cdots, n\}$ be a collection of topological spaces and (X, τ) their topological product, i.e., $X = \prod_i X_i$ and $\tau = \bigcap_i \tau_i$. Then each projection π_i is continuous. Moreover, if Y is any topological space, then a function $f : Y \to X$ is continuous if and only if the mapping $\pi_i o f : Y \to X_i$ is continuous for all $i = 1, 2, \cdots, n$.*

Theorem 1.1.24 (Tychonoff's theorem) – *The Cartesian product X of an arbitrary collection $\{X_i\}_{i\in\Lambda}$ of compact spaces is compact (with respect to product topology).*

1.2 Normed spaces

A linear space or vector space X over the field $\mathbb{K}$ (the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1. $x + y = y + x$,
2. $(x + y) + z = x + (y + z)$,
3. there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$,
4. for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse* or *the negative of* x such that $x + (-x) = 0$,
5. $\alpha(x + y) = \alpha x + \alpha y$,
6. $(\alpha + \beta)x = \alpha x + \beta x$,
7. $(\alpha\beta)x = \alpha(\beta x)$,
8. $1 \cdot x = x$.

The elements of a vector space X are called *vectors*, and the elements of $\mathbb{K}$ are called *scalars*. In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Observation

- With the usual addition and multiplication, $\mathbb{R}$ and $\mathbb{C}$ are linear spaces over $\mathbb{R}$.
- $X = \{x = (a_1, a_2, \cdots) : a_i \in \mathbb{R}\}$ is a linear space.
- The set of solutions of a linear differential equation (and linear partial differential equation) is a linear space.

A subset S of a linear space X is a *linear subspace* (or a *subspace*) of X if S is itself a linear space, i.e., $\alpha x + \beta y \in S$ for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in S$.

If S is a subset of a linear space X, then the *linear span* of S is the intersection of all linear subspaces containing S. It is the smallest linear subspace of X containing S. The linear span of set S is denoted by $[S]$.

Given the points $x_1, x_2, \cdots, x_n$ of a linear space X, then the element

$$x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n, \quad a_i \in \mathbb{K}$$

is called *linear combination* of $\{x_1, x_2, \cdots, x_n\}$.

Proposition 1.2.1 *Let S be a nonempty subset of a linear space X. Then the linear span of S is the set of all linear combinations of elements of S.*

A linear space X is said to be *finite-dimensional* if it is generated by the linear combination of a finite number of points (which are linearly independent). Otherwise, it is infinite-dimensional. The dimension of a linear space X is denoted by $dim(X)$.

Convex set – Let C be a subset of a linear space X. Then C is said to be *convex* if $(1-\lambda)x + \lambda y \in C$ for all $x, y \in C$ and all scalar $\lambda \in [0, 1]$.

By definition of convexity, we have the following fact:

Proposition 1.2.2 *Let C be a subset of a linear space X. Then C is convex if and only if $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \in C$ for any finite set $\{x_1, x_2, \cdots, x_n\} \subseteq C$ and any scalars $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.*

Convex hull – Let C be an arbitrary subset (not necessarily convex) of a linear space X. Then the *convex hull* of C in X is the intersection of all convex subsets of X containing C. It is denoted by $co(C)$. Hence

$$co(C) = \cap\{D \subseteq X : C \subseteq D,\ D \ is \text{ convex}\}.$$

Thus, $co(C)$ is the unique smallest convex set containing C. Clearly,

$$\begin{aligned} co(C) &= \left\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : x_i \in C, \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1\right\} \\ &= \text{the set of all convex combination of elements of } C. \end{aligned}$$

The *closure of convex hull* of C is denoted by $\overline{co(C)}$. Thus,

$$\overline{co(C)} = \overline{\left\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : x_i \in C, \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1\right\}}.$$

The *closed convex hull* of C in X is the intersection of all closed convex subsets of X containing C. It is denoted by $\overline{co}(C)$. Thus,

$$\overline{co}(C) = \cap\{D \subseteq X : C \subseteq D,\ D \text{ is closed and convex}\}.$$

One may easily see that closure of convex hull of C is closed convex hull of C, i.e., $\overline{co}(C) = \overline{co(C)}$.

Observation

- The empty set $\emptyset$ is convex.
- For two convex subsets C and D in a linear space X, we have
  (i) $C + D$ is convex,
  (ii) λC is convex for any scalar λ.
- Any translate $C + x_0$ of a convex set C is convex.
- If $\{C_i : i \in \Lambda\}$ is any family of convex sets in a linear space X, then $\cap_i C_i$ is convex.
- If C is a convex subset of a linear space X, then
  (i) the closure $\overline{C}$ and the interior $int(C)$ are convex,
  (ii) $co(C) = C$.
- If C is a subset of a linear space, $\overline{co}(C) = \overline{co(C)}$.
- In general, $\overline{co}(C) \neq co(\overline{C})$.

The vector space axioms only describe algebraic properties of the elements of the space: vector addition, scalar multiplication, and other combinations of these. For the topological concepts such as openness, closure, convergence, and completeness, we need a measure of distance in a space.

Definition 1.2.3 *Let X be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $f : X \to \mathbb{R}^+$ a function. Then f is said to be a norm if the following properties hold:*

(N_1) $f(x) = 0$ *if and only if* $x = 0$; *(strict positivity)*

(N_2) $f(\lambda x) = |\lambda| f(x)$ *for all* $x \in X$ *and* $\lambda \in \mathbb{K}$; *(absolute homogeneity)*

(N_3) $f(x + y) \leq f(x) + f(y)$ *for all* $x, y \in X$. *(triangle inequality or subadditivity)*

The ordered pair (X, f) is called a normed space.

Observation

- $f(x) \geq 0$ for all $x \in X$.

- $|f(x) - f(y)| \leq f(x - y)$ and $|f(x) - f(y)| \leq f(x + y)$ for all $x, y \in X$.
- f is a continuous function, i.e., $x_n \to x \Rightarrow f(x_n) \to f(x)$.
- f is a convex function, i.e., $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$.
- Addition and scalar multiplication are jointly continuous, i.e., if $x_n \to x$ and $x_n \to y$, then $x_n + y_n \to x + y$ and if $x_n \to x$ and $\lambda_n \to \lambda$, then $\lambda_n x_n \to \lambda x$.

We use the notation $\|\cdot\|$ for norm. Then every normed space $(X, \|\cdot\|)$ is a metric space (X, d) with induced metric $d(x, y) = \|x-y\|$ and a topological space with the induced topology. It means that the induced metric $d(x, y) = \|x - y\|$ in turn, defines a topology on X, the norm topology.

Observation

- In every linear space X, we can easily define a function $\rho : X \times X \to \mathbb{R}^+$ by

$$\rho(x, y) = \begin{cases} 0 & if \quad x = y, \\ 1 & if \quad x \neq y, \end{cases} \tag{1.2}$$

  which is a metric on X. It shows that every linear space (not necessarily normed space) is always a metric space.

At this stage, there arises a natural question:

Under what conditions will any metric on a linear space be a normed space? Such sufficient conditions are given in following proposition:

Proposition 1.2.4 *Let d be a metric on a linear space X. Then function $\|\cdot\| : X \to \mathbb{R}^+$ defined by*

$$\|x\| = d(x, 0) \ for\ all\ x \in X$$

is a norm if d satisfies the following conditions:

(d_1) *d is homogeneous : $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in X$ and $\lambda \in \mathbb{K}$;*

(d_2) *d is translation invariant : $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.*

Remark 1.2.5 *The metric ρ defined by (1.2) is not homogeneous and the linear space X is a metric space under metric ρ, but not a normed space.*

The following example also demonstrates that a metric space is not necessarily a normed space.

Example 1.2.6 *Let X be a space of all complex sequences $\{x_i\}_{i=1}^{\infty}$ and $d(\cdot, \cdot)$ a metric on X defined by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|}, \quad x = \{x_i\},\ y = \{y_i\} \in X. \tag{1.3}$$

Then d is not a norm under the relation $d(x,y) = \|x-y\|$. In fact,

$$d(\lambda x, \lambda y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|\lambda||x_i - y_i|}{1 + |\lambda||x_i - y_i|} \neq |\lambda| \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|} = |\lambda| d(x,y),$$

i.e., d is not homogeneous.

Remark 1.2.7 *The metric d defined by (1.3) is bounded, because*

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

This metric is called a Fréchet metric for X.

We now consider some examples of normed spaces:

Example 1.2.8 *Let $X = \mathbb{R}^n, n > 1$ be a linear space. Then $\mathbb{R}^n$ is a normed space with the following norms:*

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n; \\ \|x\|_p &= \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \text{ and } p \in (1, \infty); \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i| \text{ for all } x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n. \end{aligned}$$

Remark 1.2.9 *(a) $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ is denoted by ℓ_p^n for all $1 \leq p < \infty$.*
(b) $\mathbb{R}^n$ equipped with the norm defined by $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$ is denoted by ℓ_∞^n.

Example 1.2.10 *Let $X = \ell_1$, the linear space whose elements consist of all absolutely convergent sequences $(x_1, x_2, \cdots, x_i, \cdots)$ of scalars (real or complex numbers), i.e.,*

$$\ell_1 = \left\{ x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty \right\}.$$

Then ℓ_1 is a normed space with the norm defined by $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 1.2.11 *Let $X = \ell_p$ $(1 < p < \infty)$, the linear space whose elements consist of all p-summable sequences $(x_1, x_2, \cdots, x_i, \cdots)$ of scalars, i.e.,*

$$\ell_p = \left\{ x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

Then ℓ_p is a normed space with the norm defined by $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 1.2.12 *Let* $X = \ell_\infty$, *the linear space whose elements consist of all bounded sequences* $(x_1, x_2, \cdots, x_i, \cdots)$ *of scalars, i.e.,*

$$\ell_\infty = \{x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \{x_i\}_{i=1}^\infty \text{ is bounded}\}.$$

Then ℓ_∞ *is a normed space with the norm defined by* $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

Example 1.2.13 *Let* $X = c$, *the sequence space of all convergent sequences of scalars, i.e.,*

$$c = \{x : x = (x_1, x_2, \cdots, x_i, \cdots) \text{ and } \{x_i\}_{i=1}^\infty \text{ is convergent}\}.$$

Then c *space is a normed space with the norm* $\| \cdot \|_\infty$.

Example 1.2.14 *Let* $X = c_0$, *the sequence space of all convergent sequences of scalars with limit zero, i.e.,*

$$c_0 = \{x = (x_1, x_2, \cdots, x_i, \cdots) : \{x_i\}_{i=1}^\infty \text{ is convergent to zero}\}.$$

The c_0 *space is a normed space with norm* $\| \cdot \|_\infty$.

Example 1.2.15 *Let* $X = c_{00}$, *the sequence space defined by*

$$c_{00} = \{x = \{x_i\}_{i=1}^\infty \in \ell_\infty : \{x_i\}_{i=1}^\infty \text{ has only a finite number of nonzero terms}\}.$$

Then c_{00} *space is a normed space with norm* $\| \cdot \|_\infty$.

Observation

- $c_{00} \subset \ell_p \subset c_0 \subset c \subset \ell_\infty$ for all $1 \leq p < \infty$.
- If $1 \leq p < q \leq \infty$, then $\ell_p \subset \ell_q$. In fact, let $x = (1, 1/2, \cdots, 1/n, \cdots)$, and we have

$$\sum_{i=1}^\infty |x_i| = \sum_{i=1}^\infty \frac{1}{i} = \infty, \text{ and } \sum_{i=1}^\infty |x_i|^2 = \sum_{i=1}^\infty \frac{1}{i^2} = \frac{\pi^2}{6} < \infty.$$

Note that $x \in \ell_2$, but $x \notin \ell_1$. Hence an element of ℓ_2 is not necessarily an element of ℓ_1. But each element of ℓ_1 is an element of ℓ_2.

Example 1.2.16 *Let* $X = L_p[a, b]$ $(1 \leq p < \infty)$, *the linear space of all equivalence classes of* p*-integrable functions on* $[a, b]$. *Then* $L_p[a, b]$ *space is a normed space with the norm defined by*

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p} < \infty.$$

Example 1.2.17 *Let* $X = L_\infty[a, b]$, *the linear space of all equivalence classes of essentially bounded functions on* $[a, b]$. *Then* $L_\infty[a, b]$ *space is a normed space with the norm defined by*

$$\|f\|_\infty = ess\ sup|f(t)| < \infty.$$

Example 1.2.18 *Let $X = C[a,b]$, the set of all continuous scalar-valued functions and let "$+$" and "$\cdot$" be operations defined by*

$$\begin{aligned}(f+g)(t) &= f(t)+g(t) \text{ for all } f,g \in C[a,b];\\ (\lambda f)(t) &= \lambda f(t) \text{ for all } f \in C[a,b] \text{ and scalar } \lambda \in \mathbb{K}.\end{aligned}$$

Then $C[a,b]$ is a linear space and is also a normed space with the norms:

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \qquad 1 \le p < \infty; \tag{1.4}$$

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|. \tag{1.5}$$

Observation

- The norm $\|\cdot\|_p$ defined by (1.4) on $C[a,b]$ is called a L_p-norm.
- The norm $\|\cdot\|_\infty$ defined by (1.5) on $C[a,b]$ is called a uniform convergence norm.

Equivalent norms – Let X be a linear space over $\mathbb{K}$ and let $\|\cdot\|'$ and $\|\cdot\|''$ be two norms on X. Then $\|\cdot\|'$ is said to be *equivalent* to $\|\cdot\|''$ (written as $\|\cdot\|' \sim \|\cdot\|''$) if there exist positive numbers a and b such that

$$a\|x\|' \le \|x\|'' \le b\|x\|' \text{ for all } x \in X,$$

or

$$a\|x\|'' \le \|x\|' \le b\|x\|'' \text{ for all } x \in X.$$

Observation

- The relation $\sim$ is an equivalence relation on the set of all norms on X.
- In a finite-dimensional normed space X, all norms on X are equivalent.
- If $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent norms on a linear space X, then a sequence $\{x_n\}$ that is convergent (Cauchy) with respect to $\|\cdot\|'$ is also convergent (Cauchy) with respect to $\|\cdot\|''$ and vice versa.
- If $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent norms on a linear space X, then the class of open sets with respect to $\|\cdot\|'$ is same as the class of open sets with respect to $\|\cdot\|''$ and vice versa.

Seminorm – Let X be a linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Then a function $p : X \to \mathbb{R}^+$ is said to be a *seminorm* on X if (N_2) and (N_3) (see Definition 1.2.3) are satisfied. The ordered pair (X, p) is called *seminormed space*. Note that a seminorm p is a norm if $p(x) = 0 \Rightarrow x = 0$.

Example 1.2.19 *Let $X = \mathbb{R}^2$ and define $p : X \to \mathbb{R}^+$ by*

$$p(x) = p((x_1, x_2)) = |x_1|, \quad x \in X.$$

Then p is a seminorm, but not a norm, because $p(x_1, x_2) = 0$ implies that only the first component of x is zero, i.e., $x_1 = 0$.

We now consider the notion of topological linear spaces.

Definition 1.2.20 *A linear space X over $\mathbb{K}$ is said to be a topological linear space if on X, there exists a topology τ such that $X \times X$ and $\mathbb{K} \times X$ with the product topology have the property that vector addition $+ : X \times X \to X$ and scalar multiplication $\cdot : \mathbb{K} \times X \to X$ are continuous functions.*

In this case, τ is called a *linear topology* on X.

Definition 1.2.21 *A linear topology on a topological linear space X is said to be a locally convex topology if every neighborhood of 0 (the zero of X) includes a convex neighborhood of 0. Then X is called a locally convex topological space.*

Then we have the following interesting result.

Proposition 1.2.22 *If X is a locally convex topological linear space over $\mathbb{K}$, then a topology of X is determined by a family of seminorms $\{p_i\}_{i \in I}$.*

Inner product – Let X be a linear space over field $\mathbb{C}$. An *inner product* on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ with the following three properties:

(I_1) $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(I_2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where the bar denotes complex conjugation;

(I_3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{C}$.

The ordered pair $(X, \langle \cdot, \cdot \rangle)$ is called an *inner product space*. Sometimes, it is called a *pre-Hilbert space*. $\langle x, y \rangle$ is called inner product of two elements $x, y \in X$.

Example 1.2.23 *Let $X = \mathbb{R}^n$, the set of n-tuples of real numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \ \textit{for all}\ x = (x_1, x_2, \cdots, x_n),\ y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$$

is an inner product on $\mathbb{R}^n$. $\mathbb{R}^n$ with this inner product is called real Euclidean n-space.

Example 1.2.24 *Let $X = \mathbb{C}^n$, the set of n-tuples of complex numbers. Then the function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by*

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y}_i \ \textit{for all}\ x = (x_1, x_2, \cdots, x_n),\ y = (y_1, y_2, \cdots, y_n) \in \mathbb{C}^n$$

is an inner product on $\mathbb{C}^n$. $\mathbb{C}^n$ with this inner product is called a complex Euclidean n-space.

Example 1.2.25 *Let* $X = \ell_2$, *the set of all sequences of complex numbers* $(a_1, a_2, \cdots, a_i, \cdots)$ *with* $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. *Then the function* $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{C}$ *defined by*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y}_i \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in X \tag{1.6}$$

is an inner product on ℓ_2.

We note that the series (1.6) converges by the Cauchy-Schwarz inequality (see Proposition 1.2.28).

Example 1.2.26 *Let* $X = C[a, b]$, *the linear space of all scalar-valued continuous functions on* $[a, b]$. *Then the function* $\langle \cdot, \cdot \rangle : C[a, b] \times C[a, b] \to \mathbb{C}$ *defined by*

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}dt \text{ for all } f, g \in C[a, b] \tag{1.7}$$

is an inner product on $C[a, b]$.

We now give some interesting characterizations of linear spaces having inner products.

Proposition 1.2.27 *Let* X *be an inner product space. Then the function* $\| \cdot \| : X \to \mathbb{R}^+$ *defined by*

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in X$$

is a norm on X.

Proposition 1.2.28 (The Cauchy-Schwarz inequality) – *Let* X *be an inner product space. Then the following holds:*

$$|\langle x, y \rangle|^2 \le \langle x, x \rangle \langle y, y \rangle \quad \text{for all } x, y \in X,$$

i.e.,

$$|\langle x, y \rangle| \le \|x\| \cdot \|y\| \quad \text{for all } x, y \in X.$$

Proposition 1.2.29 (The parallelogram law) – *Let* X *be an inner product space. Then* $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *for all* $x, y \in X$.

Proposition 1.2.30 *The norm on a normed linear space* X *is given by an inner product if and only if the norm satisfies the parallelogram law:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all} \quad x, y \in X.$$

Proposition 1.2.31 (The polarization identity) – *Let* X *be an inner product space. Then*

$$\langle x, y \rangle = \frac{1}{4}\left\{ \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right\} \text{ for all } x, y \in X.$$

Orthogonality of vectors – Let x and y be two vectors in an inner product space X. Then x and y are said to be *orthogonal* if $\langle x, y \rangle = 0$.

Remark 1.2.32 *If x and y are orthogonal, then we denote $x \perp y$ and we say "x is perpendicular to y."*

Proposition 1.2.33 *Let X be an inner product space and let $x, y \in X$ such that $x \perp y$. Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Observation

- $0 \perp x$ for all $x \in X$.
- $x \perp x$ if and only if $x = 0$.
- Every inner product space is a normed space.
- Every normed space is an inner product space if and only if its norm satisfies the parallelogram law.

Convergent sequence – A sequence $\{x_n\}$ in a normed space X is said to be *convergent to x* if $\lim_{n\to\infty} \|x_n - x\| = 0$. In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Observation

- $x_n \to x \Rightarrow \|x_n\| \to \|x\|$ (this fact can be easily shown by the continuity of norm). The converse of this fact is not true in general (see Theorem 2.2.13).
- The limit of convergent sequence is unique. To see it, suppose $x_n \to x$ and $x_n \to y$. Then $\|x - y\| \leq \|x_n - x\| + \|x_n - y\| \to 0$.

Cauchy sequence – A sequence $\{x_n\}$ in a normed space X is said to be *Cauchy* if $\lim_{m,\, n\to\infty} \|x_m - x_n\| = 0$, i.e., for $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq n_o$.

Observation

- A sequence in $(\mathbb{R}, |\cdot|)$ is convergent if and only if it is Cauchy sequence.
- Every convergent sequence is a Cauchy, but the converse need not be true in general. In fact, if $x_n \to x$, then

  $$\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\| \to 0 \; as \; m, n \to \infty.$$

  Conversely, suppose $X = c_{00}$ is the linear space of finitely nonzero sequences $(x_1, x_2, \cdots, x_i, 0, \cdots)$ with the norm $\|x\| = \sup_{i\in\mathbb{N}} |x_i|$. Let $\{x_n = (1, 1/2, 1/3, \cdots, 1/n, \cdots)\}$ be a sequence in X. Now

  $$\|x_n - x_m\| = \max\{1/n, 1/m\} \to 0 \quad as \quad m, n \to \infty,$$

  i.e., $\{x_n\}$ is a Cauchy sequence. Clearly, the limit x has infinitely nonzero elements. Thus, $x \notin X$. Therefore, a Cauchy sequence is not convergent in X.

- Every Cauchy sequence is bounded.
- Every Cauchy sequence is convergent if and only if it has a convergent subsequence.

Hilbert space and Banach space – A normed space $(X, \|\cdot\|)$ is said to be *complete if* it is complete as a metric space (X, d), i.e., every Cauchy sequence is convergent in X.

A complete normed space (inner product space) is called a *Banach space* (*Hilbert space*).

Example 1.2.34 ℓ_p^n $(1 \leq p \leq \infty)$ *are (finite-dimensional) Banach spaces.*

Example 1.2.35 ℓ_p *and* $L_p[0,1], 1 \leq p \leq \infty$ *are (infinite-dimensional) Banach spaces.*

Example 1.2.36 *The linear space* $C[a,b]$ *of continuous functions on closed and bounded interval* $[a,b]$ *is a Banach space with the uniform convergence norm* $\|f\|_\infty = \sup\limits_{t\in[a,b]} |f(t)|$, *but an incomplete normed space with the norm*

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}, \ 1 \leq p < \infty.$$

Example 1.2.37 c_{00} *is not complete.*

Theorem 1.2.38 *Every finite-dimensional normed space is a Banach space.*

The topological property closedness has an important role in the construction of Banach spaces from its subspaces. A point x in a normed space X is said to be a *limit point* of a subset $C \subseteq X$ if there exists a sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} x_n = x$. Also *a* subset C of a normed space is said to be *closed* if it contains all of its limit points, i.e., $C = \overline{C}$.

Theorem 1.2.39 *A closed subspace of a Banach space is a Banach space.*

Theorem 1.2.40 *Let* C *be a subset of a normed space* X *and let* $x \in X$. *Then* $x \in \overline{C}$ *if and only if there exists a sequence* $\{x_n\}$ *in* C *such that* $\lim\limits_{n\to\infty} x_n = x$.

Observation

- The subspaces c and c_0 are closed subspaces of ℓ_∞ (and hence are Banach spaces). The space c_{00} is only a subspace in c_0, but not closed in c_0 (and hence not in ℓ_∞). Therefore, c_{00} is not a Banach space.
- The subspace $C[a,b]$ is not closed in $L_p[a,b]$ for $1 \leq p < \infty$. Hence $C[a,b]$ is not a Banach space with the L_p-norm $\|\cdot\|_p$ $(1 \leq p < \infty)$ defined by (1.4).

We now give examples of Banach spaces that are not Hilbert spaces.

Example 1.2.41 *ℓ_p^n is a finite-dimensional Banach space that is not a Hilbert space for $p \neq 2$. Indeed, for $x = (1, 1, 0, 0, \cdots)$ and $y = (1, -1, 0, 0, \cdots)$, we have $x + y = (2, 0, 0, 0, \cdots)$ and $x - y = (0, 2, 0, 0, \cdots)$. Hence*

$$\begin{aligned} \|x\| &= \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = (1^p + 1^p)^{1/p} = 2^{1/p}, \\ \|y\| &= (1^p + 1^p) = 2^{1/p}, \\ \|x + y\| &= (2^p)^{1/p} = 2, \\ \|x - y\| &= (2^p)^{1/p} = 2. \end{aligned}$$

If $p = 2$, then the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is satisfied, which shows that ℓ_2^n is a Hilbert space. If $p \neq 2$, then the parallelogram law is not satisfied. Therefore, ℓ_p^n is not a Hilbert space for $p \neq 2$.

The following example shows that there exists an infinite-dimensional Banach space that is not a Hilbert space.

Example 1.2.42 *Let $X = C[0, 2\pi]$, the space of all real-valued continuous functions on $[0, 2\pi]$ with "sup" norm. Then $(C[0, 2\pi], \|\cdot\|_\infty)$ is a Banach space, but $\|\cdot\|_\infty$ does not satisfy the parallelogram law. In fact, for $x(t) = \max\{\sin t, 0\}$, $y(t) = \min\{\sin t, 0\}$, we have*

$$\|x\|_\infty = 1, \|y\|_\infty = 1, \|x + y\|_\infty = 1, \|x - y\|_\infty = 1,$$

i.e., the parallelogram law:

$$\|x + y\|_\infty^2 + \|x - y\|_\infty^2 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2$$

is not satisfied.

Remark 1.2.43 *$C[a, b]$ is an inner product space with the inner product defined by (1.7), but not a Hilbert space.*

Observation

- ℓ_2^n, ℓ_2, $L_2[a, b]$ are Hilbert spaces.
- ℓ_p^n, ℓ_p, $L_p[a, b]$ $(p \neq 2)$ are not Hilbert spaces.

We conclude this section with some important facts about the completeness property.

Definition 1.2.44 *A subset C of a normed space X is said to be complete if every Cauchy sequence in C converges to a point in C.*

Definition 1.2.45 *Let $\sum_{n=1}^{\infty} x_n$ be an infinite series of elements $x_1, x_2, \cdots$, $x_n, \cdots$ in a normed space X. Then the series $\sum_{n=1}^{\infty} x_n$ is said to converge to an element $x \in X$ if $\lim\limits_{n\to\infty} \|s_n - x\| = 0$, where $s_n = x_1 + x_2 + \cdots + x_n$ is n^{th} partial sum of series $\sum_{n=1}^{\infty} x_n$.*

Definition 1.2.46 *The series* $\sum_{n=1}^{\infty} x_n$ *in a normed space* X *is said to be absolutely convergent if* $\sum_{n=1}^{\infty} \|x_n\|$ *converges.*

The following result shows that completeness and closure are equivalent in a Banach space.

Theorem 1.2.47 *In a Banach space, a subset is complete if and only if it is closed.*

Remark 1.2.48 *Notice every normed space is closed, but not necessarily complete.*

Theorem 1.2.49 *A normed space* X *is a Banach space if and only if every absolutely convergent series of elements in* X *is convergent in* X.

Theorem 1.2.50 (Cantor's intersection theorem) – *A normed space* X *is a Banach space if and only if given any descending sequence* $\{F_n\}$ *of closed bounded subsets of* X,

$$\lim_{n\to\infty} diam(F_n) = 0 \Rightarrow \cap_{n=1}^{\infty} F_n \neq \emptyset. \tag{1.8}$$

Proof. Let X be a Banach space and $\{F_n\}$ a descending sequence of nonempty closed bounded subsets of X for which $\lim_{n\to\infty} diam(F_n) = 0$. For each n, select $x_n \in F_n$. Then given $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow diam(F_n) < \varepsilon$. If $m, n \geq n_0$, both x_n and x_m are in F_{n_0}, then $\|x_n - x_m\| \leq \varepsilon$. Hence $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. This shows that $x \in \overline{F}_n = F_n$ if $n \geq n_0$. Because the sequence $\{F_n\}$ is descending, $x \in \cap_{n=1}^{\infty} F_n$.

Conversely, suppose that the condition (1.8) holds. Suppose $\{x_n\} \subseteq X$ is a Cauchy sequence. For each $n \in \mathbb{N}$, let $F_n = \{x_n, x_{n+1}, \cdots\}$. Then $\{\overline{F}_n\}$ is a descending sequence of nonempty closed subsets of X for which $\lim_{n\to\infty} diam(\overline{F}_n) = 0$. By assumption, there exists a point $x \in \bigcap_{n=1}^{\infty} \overline{F}_n$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so large that

$$n \geq N \Rightarrow diam(\overline{F}_n) < \varepsilon.$$

Then clearly for such n we have that $\|x_n - x\| \leq \varepsilon$. Hence $\lim_{n\to\infty} x_n = x$. Therefore, X is complete. ∎

1.3 Dense set and separable space

A sequence $\{x_n\}$ in a normed space X is said to be a *(Schauder) basis* of X if each $x \in X$ has a unique expansion $x = \sum_{n=1}^{\infty} \alpha_n x_n$ for some scalars $\alpha_1, \alpha_2, \cdots,$ $\alpha_n, \cdots$.

Observation

- $\{x_n\}$ is a basis of a normed space X if for each $x \in X$, there exists a unique sequence $\{\alpha_n\}$ of scalars such that $\lim_{n\to\infty} \|x - \sum_{i=1}^{n} \alpha_i x_i\| = 0$.
- The elements

$$e_n = (0, 0, 0, \cdots, \underset{\substack{\uparrow \\ n^{th} \text{ position}}}{1}, 0, \cdots), \quad n \in \mathbb{N}$$

  from a basis for c_{00}, c_0 and ℓ_p $(1 \le p < \infty)$.

- $\{e_n\}_{n\in\mathbb{N}}$ is not a Schauder basis of ℓ_∞.
- The sequence $(\mathbf{1}, e_1, e_2, \cdots)$ is a basis for c, where $\mathbf{1} = (1, 1, 1, \cdots)$.

A subset C of a metric space (X, d) is said to be *dense* in X if $\overline{C} = X$. This means that C is dense in X if and only if $C \cap B_r(x) \ne \emptyset$ for all $x \in X$ and $r > 0$.

A metric space (X, d) is said to be *separable* if it contains a countable dense subset, i.e., there exists a countable set C in X such that $\overline{C} = X$.

Observation

- If X is a separable metric space, then $C \subset X$ is separable in the induced metric.
- A metric space X is separable if and only if there is a countable family $\{G_i\}$ of open sets such that for any open set $G \subset X$,

$$G = \cup_{G_i \subset G} G_i.$$

Next, we give some examples of separable and nonseparable spaces.

Example 1.3.1 *The space $\ell_p, 1 \le p < \infty$ is separable metric space.*

Example 1.3.2 *The ℓ_∞ space is not a separable space.*

Example 1.3.3 *The linear space X of all infinite sequences of real numbers with metric d defined by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|},$$

$$x = (x_1, x_2, \cdots, x_i, \cdots), y = (y_1, y_2, \cdots, y_i, \cdots) \in X$$

is a separable complete metric space.

Theorem 1.3.4 *Every normed space with basis is separable.*

Theorem 1.3.5 *Every subset of a separable normed space is separable.*

Theorem 1.3.6 *Every finite-dimensional normed space is separable.*

Observation

- $\mathbb{R}$, $\mathbb{R}^n$, c, $C[0, 1]$, ℓ_p, L_p $(1 \le p < \infty)$ are separable normed spaces.
- ℓ_∞, L_∞ are not separable.

1.4 Linear operators

Let X and Y be two linear spaces over the same field $\mathbb{K}$ and $T : X \to Y$ an operator with domain $Dom(T)$ and range $R(T)$. Then T is said to be a *linear operator* if

(i) T is additive: $T(x + y) = Tx + Ty$ for all $x, y \in X$;

(ii) T is homogeneous: $T(\alpha x) = \alpha Tx$ for all $x \in X, \alpha \in \mathbb{K}$.

One may easily check that T is linear if and only if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \text{ for all } x, y \in X \text{ and } \alpha, \beta \in \mathbb{K}.$$

Otherwise, the operator is called *nonlinear*. The linear operator is called a *linear functional* if $Y = \mathbb{R}$.

Example 1.4.1 *Let $X = \mathbb{R}^n$, $Y = \mathbb{R}$, and $T : X \to \mathbb{R}$ an operator defined by*

$$Tx = \sum_{i=1}^{n} x_i y_i \; for \; all \; x = (x_1, x_2, \cdots, x_n),$$

where $y = (y_1, y_2, \cdots, y_n)$ is the fixed element in $\mathbb{R}^n$. Then T is a linear functional on $\mathbb{R}^n$.

Example 1.4.2 *Let $X = Y = \ell_2$ and $T : \ell_2 \to \ell_2$ an operator defined by*

$$Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots\right) \text{ for all } x = (x_1, x_2, x_3, \cdots, x_n, \cdots) \in \ell_2.$$

Then T is a linear operator on ℓ_2.

Example 1.4.3 *Let $X = C[a, b]$, the linear space of all continuous real-valued functions on closed bounded interval $[a, b]$. Then the operator $T : C[a, b] \to C[a, b]$ defined by*

$$T(f)t = \int_a^t f(u)du, \quad t \in [a, b]$$

is a linear operator.

Example 1.4.4 *Let $X = L_2[0, 1]$, $Y = \mathbb{R}$ and $T : X \to \mathbb{R}$ an operator defined by*

$$Tx = \int_0^1 x(t)y(t)dt \; for \; all \; x \in L_2[0, 1],$$

where y is a fixed element in $L_2[0, 1]$. Then T is a linear functional on $L_2[0, 1]$.

The following result is very useful for linear operators:

Proposition 1.4.5 *Let X and Y be two linear spaces over the same field $\mathbb{K}$ and $T : X \to Y$ a linear operator. Then we have the following:*

(a) $T(0) = 0$.

(b) $R(T) = \{y \in Y : y = Tx$ *for some* $x \in X\}$, *the range of* T *is a linear subspace of* Y.

(c) T *is one-one if and only if* $Tx = 0 \Rightarrow x = 0$.

(d) *If* T *is one-one operator, then* T^{-1} *exists on* $R(T)$ *and* $T^{-1} : R(T) \to X$ *is also a linear operator.*

(e) *If* $dim(Dom(T)) = n < \infty$ *and* T^{-1} *exists, then* $dim(R(T)) = dim(Dom(T))$.

Recall an operator T from a normed space X into another normed space Y is continuous if for any sequence $\{x_n\}$ in X with $x_n \to x \in X \Rightarrow Tx_n \to Tx$. The following Theorem 1.4.6 is very interesting because the continuity of any linear operator can be verified by only verifying $Tx_n \to 0$ for any sequence $\{x_n\} \subseteq X$ with $x_n \to 0$.

Theorem 1.4.6 *Let* X *and* Y *be two normed spaces and* $T : X \to Y$ *a linear operator. If* T *is continuous at a single point in* X, *then* T *is continuous throughout space* X.

Proof. Suppose T is continuous at a point $x_0 \in X$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x \in X$. By the linearity of T, we have

$$\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\|.$$

Because T is continuous at x_0,

$$\lim_{n\to\infty}(x_n - x + x_0) = x_0 \Rightarrow \lim_{n\to\infty} T(x_n - x + x_0) = Tx_0,$$

it follows that $\|Tx_n - Tx\| = \|T(x_n - x + x_0) - Tx_0\| \to 0$ as $n \to \infty$. Thus, T is a continuous operator at an arbitrary point $x \in X$. ∎

Boundedness of linear operator – Let X and Y be two normed spaces and $T : X \to Y$ a linear operator. Then T is said be *bounded* if there exists a constant $M > 0$ such that

$$\|Tx\| \le M\|x\| \text{ for all } x \in X.$$

A linear functional $f : X \to \mathbb{R}$ is called *bounded* if there exists a constant $M > 0$ such that

$$|f(x)| \le M\|x\| \text{ for all } x \in X.$$

We now present an example of a linear operator that is unbounded.

Example 1.4.7 *Let* $X = c_{00}$, *the linear space of finitely nonzero real sequences with "sup" norm and* $T : X \to \mathbb{R}$ *a functional defined by*

$$Tx = \sum_{i=1}^{n} ix_i \text{ for all } x = (x_1, x_2, \cdots, x_n, 0, 0, \cdots) \in X.$$

Then T *is clearly a linear functional, but it is unbounded.*

With this example, we remark that linearity of the operator does not imply boundedness. Hence we require additional assumption for boundedness of any linear operator. The following important result shows that such an additional assumption is continuity of the linear operator.

Theorem 1.4.8 *A linear operator on a normed space is bounded if and only if it is continuous.*

Proof. Let T be a bounded linear operator from a normed space X into another normed space Y. Then there exists a constant $M > 0$ such that

$$\|Tx\| \leq M\|x\| \text{ for all } x \in X.$$

Then if $x_n \to 0$, we have that

$$\|Tx_n\| \leq M\|x_n\| \to 0 \text{ as } n \to \infty,$$

and it follows that T is continuous at zero. By Theorem 1.4.6, we conclude that T is continuous on X.

Conversely, suppose T is continuous. We show that T is bounded. Suppose, for contradiction, that T is unbounded. Hence there exists a sequence $\{x_n\}$ in X such that

$$\|Tx_n\| > n\|x_n\| \text{ for all } n \in \mathbb{N}.$$

Because $T0 = 0$, this implies that $x_n \neq 0$. Set $y_n := x_n/(n\|x_n\|), n \in \mathbb{N}$. Then $\|y_n\| = \|x_n/(n\|x_n\|)\| = 1/n \to 0$, which implies that $\lim_{n\to\infty} y_n = 0$. Observe that

$$\|Ty_n\| = \|T\left(\frac{x_n}{n\|x_n\|}\right)\| = \frac{1}{n\|x_n\|}\|Tx_n\| > 1 \text{ for all } n \in \mathbb{N}$$

and hence $\{Ty_n\}$ does not converge to zero. This means that T is not continuous at zero, a contradiction. ∎

If the dimension of X is finite, it also forces the boundedness of a linear operator.

Theorem 1.4.9 *Let X and Y be two normed spaces. If X is a finite-dimensional normed space, then all linear operators $T : X \to Y$ are continuous (hence bounded).*

Remark 1.4.10 *Example 1.4.7 shows that the conclusion of Theorem 1.4.9 is not true in general (in infinite-dimensional normed spaces). Thus, linear operators may be discontinuous in infinite-dimensional normed spaces.*

1.5 Space of bounded linear operators

Let X and Y be two normed spaces. Given two bounded linear operators $T_1, T_2 : X \to Y$, we define

$$\begin{aligned} (T_1 + T_2)x &= T_1x + T_2x, \\ (\alpha T_1)x &= \alpha T_1 x \text{ for all } x \in X \text{ and } \alpha \in \mathbb{K}. \end{aligned}$$

We denote by $B(X,Y)$, the family of all bounded linear operators from X into Y. Then $B(X,Y)$ is a linear space. The space $B(X,Y)$ becomes a normed space by assigning a norm as below:

$$\begin{aligned} \|T\|_B &= \inf\{M : \|Tx\| \le M\|x\|, x \in X\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0, x \in X\right\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| \le 1\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| = 1\}. \end{aligned}$$

Theorem 1.5.1 *The normed space $B(X,Y)$ is a Banach space if Y is a Banach space.*

We now state an important result:

Theorem 1.5.2 (Uniform boundedness principle) – *Let X be a Banach space, Y a normed space, and $\{T_i\}_{i\in\Lambda} \subseteq B(X,Y)$ a family of bounded linear operators of X into Y such that $\{T_ix\}$ is bounded set in Y for each $x \in X$, i.e., for each $x \in X$, there exists $M_x > 0$ such that*

$$\|T_ix\| \le M_x \textit{ for all } i \in \mathbb{N}.$$

Then $\{\|T_i\|_B\}$ is a bounded set in $\mathbb{R}^+$, i.e., T_i are uniformly bounded.

As an immediate consequence of Theorem 1.5.2 (uniform boundedness principle), we have

Theorem 1.5.3 *Let X and Y be two Banach spaces and $\{T_n\}$ a sequence in $B(X,Y)$. For each $x \in X$, let $\{T_nx\}$ converges to Tx. Then we have the following:*

(a) T is a bounded linear operator, i.e., $T \in B(X,Y)$;

(b) $\|T\|_B \le \liminf_{n\to\infty} \|T_n\|_B$.

Proof. (a) Because each T_n is linear, it follows that

$$\begin{aligned} T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) &= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} T_n(\beta y) \\ &= \alpha \lim_{n\to\infty} T_nx + \beta \lim_{n\to\infty} T_ny \\ &= \alpha Tx + \beta Ty \end{aligned}$$

for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$. Further, because the norm is continuous,

$$\lim_{n\to\infty} \|T_n x\| = \|Tx\| \text{ for all } x \in X,$$

it follows that $\{T_n x\}$ is a bounded set in Y. By the uniform boundedness principle, there exists a positive constant $M > 0$ such that $\sup_{n\in\mathbb{N}} \|T_n\|_B \le M$. Thus,

$$\|T_n x\| \le \|T_n\|_B \|x\| \le M\|x\|.$$

Taking the limit as $n \to \infty$, we have

$$\|Tx\| \le M\|x\|,$$

so T is bounded. Therefore, $T \in B(X, Y)$.

(b) Because

$$\|T_n x\| \quad \le \quad \|T_n\|_B \|x\|,$$

this implies that

$$\liminf_{n\to\infty} \|T_n x\| \quad \le \quad \liminf_{n\to\infty} \|T_n\|_B \|x\|.$$

Hence $\|Tx\| \le \liminf_{n\to\infty} \|T_n\|_B \|x\|$. Thus, $\|T\|_B \le \liminf_{n\to\infty} \|T_n\|_B$. ∎

Dual space – The space of all bounded linear functionals on a normed space X is called *the dual* of X and is denoted by X^*. Clearly, $X^* = B(X, \mathbb{R})$ and is a normed space with norm denoted and defined by

$$\|f\|_* = \sup\{|f(x)| : x \in S_X\}.$$

In view of Theorem 1.5.1, we have the following interesting result, which is very useful for the construction of Banach spaces from normed spaces.

Corollary 1.5.4 *The dual space $(X^*, \|\cdot\|_*)$ of a normed space X is always a Banach space.*

We now give basic dual spaces:

The dual of $\mathbb{R}^n$ – Let $\mathbb{R}^n$ be a normed space of vectors $x = (x_1, x_2, \cdots, x_n)$ with norm $\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$. Then for $y = (y_1, y_2, \cdots, y_i, \cdots, y_n) \in \mathbb{R}^n$, any functional $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = \sum_{i=1}^n x_i y_i, \qquad x = (x_1, x_2, \cdots, x_i, \cdots, x_n) \in \mathbb{R}^n$$

is linear. Further, from the Cauchy-Schwarz inequality,

$$|f(x)| = \left|\sum_{i=1}^n x_i y_i\right| \le \left(\sum_{i=1}^n x_i^2\right)^{1/2} \left(\sum_{i=1}^n y_i^2\right)^{1/2} = \left(\sum_{i=1}^n y_i^2\right)^{1/2} \|x\|_2,$$

which shows that f is bounded with $\|f\|_* \le (\sum_{i=1}^n y_i^2)^{1/2}$. However, because for $x = (y_1, y_2, \cdots, y_n)$ equality is achieved in the Cauchy-Schwarz inequality, we must in fact have $\|f\|_* = (\sum_{i=1}^n y_i^2)^{1/2}$.

Now, let j be any bounded linear functional on $X = \mathbb{R}^n$. Define the basis vectors e_i in $\mathbb{R}^n$ by

$$e_i = (0, 0, \cdots, \underset{\substack{\uparrow \\ i^{th} \text{ position}}}{1}, 0, \cdots, 0).$$

Suppose $j(e_i) = a_i$. Then for any $x = (x_1, x_2, \cdots, x_n)$, we have $x = \sum_{i=1}^n x_i e_i$. By the linearity of j, we have

$$j(x) = \sum_{i=1}^n j(e_i x_i) = \sum_{i=1}^n j(e_i) x_i = \sum_{i=1}^n a_i x_i.$$

Thus, the dual space X^* of $X = \mathbb{R}^n$ is itself $\mathbb{R}^n$ in the sense that the space X^* consists of all functionals of the form $f(x) = \sum_{i=1}^n a_i x_i$ and the norm on X^* is $\|f\|_* = (\sum_{i=1}^n |a_i|^2)^{1/2} = \|a\|$, where $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$.

The dual of $\boldsymbol{\ell_p, 1 \le p < \infty}$ – For $1 \le p < \infty$, the dual space of ℓ_p is ℓ_q $(1/p + 1/q = 1)$ in the sense that there is a one-one correspondence between elements $y \in \ell_q$ and bounded linear functionals f_y on ℓ_p such that

$$f_y(x) = \sum_{i=1}^\infty x_i y_i, \qquad x = \{x_i\}_{i=1}^\infty \in \ell_p,$$

where

$$y = \{y_i\}_{i=1}^\infty \in \ell_q$$

and

$$\|f_y\|_* = \|y\|_q = \begin{cases} (\sum_{i=1}^\infty |y_i|^q)^{1/q}, & \text{if } 1 < p < \infty, \\ \sup_{i \in \mathbb{N}} |y_i| & \text{if } p = 1. \end{cases}$$

Observation

- The dual of ℓ_1 is ℓ_∞.
- The dual of ℓ_p is ℓ_q, $1 < p < \infty$ and $1/p + 1/q = 1$.
- The dual of ℓ_∞ is not ℓ_1.

The dual of $\mathbf{c_0}$ – The Banach space c_0 of all real sequences $x = \{x_i\}$ such that $\lim_{i \to \infty} x_i = 0$ with norm $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$ is a subspace of ℓ_∞. The dual of c_0 is ℓ_1 in the usual sense that the bounded linear functionals on c_0 can be represented as

$$f_y(x) = \sum_{i=1}^\infty x_i y_i, \qquad x = \{x_i\}_{i=1}^\infty \in c_0,$$

where $y = \{y_i\}_{i=1}^\infty \in \ell_1$ and $\|f_y\|_* = \|y\|_1 = \sum_{i=1}^\infty |y_i|$.

The dual of $\mathbf{L_p[0,1], 1 \le p < \infty}$ – For $1 \le p < \infty$, the dual space of $L_p[0,1]$ is $L_q[0,1], (1/p + 1/q = 1)$ in the sense that there is one-one correspondence between elements $y \in L_q[0,1]$ and bounded linear functionals $f_y : L_p[0,1] \to \mathbb{R}$ such that

$$f_y(x) = \int_0^1 x(t)y(t)dt \text{ and } \|f_y\|_* = \|y\|_q.$$

We now state an important theorem in Hilbert space that is called the *Riesz representation theorem.* This theorem demonstrates that any bounded linear functional on a Hilbert space H can be represented as an inner product with a unique element in H.

Theorem 1.5.5 (Reisz representation theorem) – *Let H be a Hilbert space and $f \in H^*$. Then we have the following:*

(1) There exists a unique element $y_0 \in H$ such that $f(x) = \langle x, y_0 \rangle$ for each $x \in H$.

(2) Moreover, $\|f\|_* = \|y_0\|$.

Remark 1.5.6 *In a Hilbert space H, (distinct) bounded linear functionals f on H are generated by (distinct) elements y of the space H itself, i.e., there is one-one correspondence between $f \in H^*$ and $y \in H$. Therefore, $H^* = H$.*

1.6 Hahn-Banach theorem and applications

The Hahn-Banach theorem is one of the most important theorems in functional analysis. To state it, we need the following definitions:

Sublinear functional – Let X be a linear space and $p : X \to \mathbb{R}$ a functional. Then p is said to be a *sublinear functional* on X if

(i) p is subadditive: $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$,

(ii) p is positive homogeneous: $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \ge 0$.

It is evident that every norm is a sublinear functional.

The sublinear functional p on X is called *convex functional* on X if $p(x) \ge 0$ for all $x \in X$. Obviously, every norm is a convex functional also.

Example 1.6.1 *Let $p : \ell_\infty \to \mathbb{R}$ be a functional defined by*

$$p(x) = \limsup_{n\to\infty} x_n \textit{ for all } x = (x_1, x_2, \cdots, x_n, \cdots) \in \ell_\infty.$$

Then p is a sublinear functional on ℓ_∞.

Extension mapping – Let C be a proper subset of a linear space X and f a mapping from C into another linear space Y. If there exists a mapping $F : X \to Y$ such that

$$F(x) = f(x), \quad x \in C,$$

then F is called an *extension of f*.

Example 1.6.2 *Let* $X = [0,1], C = [0,1)$ *and* $f : C \to \mathbb{R}$ *defined by*

$$f(x) = x^2, \quad x \in [0,1).$$

Then

$$F_1(x) = \begin{cases} f(x) & if\ x \in C, \\ 0 & if\ x = 1 \end{cases}$$

and

$$F_2(x) = \begin{cases} f(x) & if\ x \in C, \\ 1 & if\ x = 1 \end{cases}$$

are two extensions of f*, where* F_2 *is continuous, but* F_1 *is not.*

Simply, the Hahn-Banach theorem states that a bounded linear functional f defined only on a subspace C of a normed space X can be extended to a bounded linear functional F defined on the entire space and with norm equal to that of f on C, i.e.,

$$\|F\|_X = \|f\|_C = \sup_{x \in C} \frac{|f(x)|}{\|x\|}.$$

We now state the theorem without proof.

Theorem 1.6.3 (Hahn-Banach theorem) – *Let* C *be a subspace of a real linear space* X*,* p *a sublinear functional on* X*, and* f *a linear functional defined on* C *satisfying the condition:*

$$f(x) \leq p(x)\ for\ all\ x \in C.$$

Then there exists a linear extension F *of* f *such that* $F(x) \leq p(x)$ *for all* $x \in X$.

Corollary 1.6.4 *Let* C *be a subspace of a real normed space* X *and* f *a bounded linear functional on* C*. Then there exists a bounded linear functional* F *defined on* X *that is an extension of* f *such that* $\|F\|_* = \|f\|_C$.

Proof. Take $p(x) = \|f\|_C \|x\|, x \in X$. ∎

The following corollary gives the existence of nontrivial bounded linear functionals on an arbitrary normed space.

Corollary 1.6.5 *Let* x *be an element of a normed space* X*. Then there exists (nonzero)* $j \in X^*$ *such that* $j(x) = \|j\|_* \|x\|$ *and* $\|j\|_* = \|x\|$.

Corollary 1.6.6 *Let* x *be a nonzero element of a normed space* X*. Then there exists* $j \in X^*$ *such that* $j(x) = \|x\|$ *and* $\|j\|_* = 1$.

Corollary 1.6.7 *Let* X *be a normed space. Then for any* $x \in X$*,*

$$\|x\| = \sup_{\|j\|_* \leq 1} |j(x)|.$$

Corollary 1.6.8 *If X is a normed space and $x_0 \in X$ such that $j(x_0) = 0$ for all $j \in X^*$, then $x_0 = 0$.*

Proof. Suppose $x_0 \neq 0$. By Corollary 1.6.6, there exists a functional $j \in X^*$ such that

$$j(x_0) = \|x_0\| \text{ and } \|j\|_* = 1.$$

This implies that $j(x_0) \neq 0$, which is a contradiction. Hence $j(x_0) = 0$ for all $j \in X^* \Rightarrow x_0 = 0$. ∎

The following theorems are very useful in many applications.

Theorem 1.6.9 *Let C be a subspace of a normed space X and x_0 an element in X such that $d(x_0, C) = d > 0$. Then there exists a bounded linear functional $j \in X^*$ with norm 1 such that $j(x_0) = d$ and $j(x) = 0$ for all $x \in C$.*

Theorem 1.6.10 (Separability) – *If X^* is the dual space of a normed space X and X^* is separable, then X is also separable.*

Next, we discuss geometric forms of the Hahn-Banach theorem. We need the following:

Hyperplane – A subset H of a linear space X is said to be a *hyperplane* if there exists a linear functional $f \neq 0$ on X such that

$$H = \{x \in X : f(x) = \alpha\}, \quad \alpha \in \mathbb{R}.$$

$f(x) = \alpha$ is called the *equation of the hyperplane.*

Example 1.6.11 *Let $X = \mathbb{R}$, $f(x) = 3x$, $\alpha = 2$. Then the set*

$$H = \{x \in X : f(x) = \alpha\} = \{x \in X : 3x = 2\} = \{2/3\}.$$

Hence H is a hyperplane.

We have the following interesting result.

Proposition 1.6.12 *Let X be a topological linear space. Then the hyperplane $\{x \in X : f(x) = \alpha\}$ is closed if and only if f is continuous.*

Let $f(x) = \alpha$, $\alpha \in \mathbb{R}$, be the equation of hyperplane in a linear space X. Then we have the following:

(i) $\{x \in X : f(x) < \alpha\}$ and $\{x \in X : f(x) > \alpha\}$ are open half-spaces.

(ii) $\{x \in X : f(x) \leq \alpha\}$ and $\{x \in X : f(x) \geq \alpha\}$ are closed half-spaces.

It is easy to see that the boundary of each of the four half-spaces is just a hyperplane.

Remark 1.6.13 *In a topological linear space X, we have*

(i) open half-spaces are open sets,

(ii) the closed half-spaces are closed sets if and only if f is continuous, i.e., the hyperplane $\{x \in X : f(x) = \alpha\}$ is closed.

Let X be a linear space. We say that the hyperplane $\{x \in X : f(x) = \alpha\}$ *separates* two sets $A \subset X$ and $B \subset X$ if $f(x) \leq \alpha$ for all $x \in A$ and $f(x) \geq \alpha$ for all $x \in B$. We say that the hyperplane $\{x \in X : f(x) = \alpha\}$ *strictly separates* two sets $A \subset X$ and $B \subset X$ if $f(x) < \alpha$ for all $x \in A$ and $f(x) > \alpha$ for all $x \in B$.

Theorem 1.6.14 (Hahn-Banach separation theorem) – *Let X be a normed space and let $A \subset X, B \subset X$ be two nonempty disjoint convex sets. Suppose that A is open. Then there exists a closed hyperplane that separates A and B, i.e., there exist $j \in X^*$ and a number $\alpha \in \mathbb{R}$ such that*

$$j(x) > \alpha \text{ if } x \in A \text{ and } j(x) \leq \alpha \text{ if } x \in B.$$

Proposition 1.6.15 *Let C be a nonempty open convex subset of a normed space X. Then for $x_0 \in X$, $x_0 \notin C$, there exists $f \in X^*$ such that*

$$f(x) < \alpha \quad \text{for all } x \in C,$$

where $f(x_0) = \alpha$.

An immediate consequence of the separation theorem shows that $\overline{co}(C)$ is the intersection of all closed half-spaces containing C. Indeed,

Theorem 1.6.16 *Let C be a nonempty subset of a normed space X. Then*

$$\overline{co}(C) = \{x \in X : f(x) \leq \sup_{y \in C} f(y) \text{ for all } f \in X^*\}.$$

Theorem 1.6.17 *Let C be a nonempty closed convex subset of a normed space X. If x is not an element in C, there exists a continuous linear functional $j \in X^*$ such that*

$$j(x) < \inf\{j(y) : y \in C\}.$$

Theorem 1.6.18 (Hahn-Banach strictly separation theorem) – *Let A and B be two nonempty disjoint convex subsets of a normed space X. Suppose A is closed and B is compact. Then there exists a closed hyperplane that strictly separates A and B.*

Supporting hyperplane – Let C be a convex subset of a normed space X with $int(C) \neq \emptyset$ and $x_0 \in \partial C$. Then a nonzero functional $f \in X^*$ is said to be a *support functional for C at x_0* if $f(x) \leq f(x_0)$ for all $x \in C$. The corresponding hyperplane $\{x \in X : f(x) = f(x_0)\}$ is called a *supporting hyperplane* for C at x_0.

A point of C through which a supporting hyperplane passes is called *a point of support of C*.

Observation

- Any supporting hyperplane of a set C with nonempty interior is closed.
- An interior point of C cannot be a point of support.

We give some conditions on C under which a boundary point is a point of support.

Theorem 1.6.19 *Let C be a convex subset of a normed space X with $int(C) \neq \emptyset$. Then every boundary point of C is a point of support, i.e., for every $x_0 \in \partial C$, there exists an $f \in X^*$ such that $f \neq 0$ and $f(x_0) = \sup_{x \in C} f(x)$.*

1.7 Compactness

Let (X, d) be a metric space. Recall that a subset C of X is called *compact* if every open cover of C has a finite subcover. Equivalently, a subset C of X is compact if every sequence in C contains a convergent subsequence with a limit in C.

A subset C of X is said to be *totally bounded* if for each $\varepsilon > 0$, there exists a finite number of elements $x_1, x_2, \cdots, x_n$ in X such that $C \subseteq \cup_{i=1}^n B_\varepsilon(x_i)$. The set $\{x_1, x_2, \cdots, x_n\}$ is called a finite ε-net.

Observation

- Every subset of a totally bounded set is totally bounded.
- Every totally bounded set is bounded, but a bounded set need not be totally bounded.

Proposition 1.7.1 *A subset of a compact metric space is compact if and only if it closed.*

Proposition 1.7.2 *Let X be a metric space. Then the following are equivalent:*

(a) X is compact.

(b) Every sequence in X has a convergent subsequence.

(c) X is complete and totally bounded.

Proposition 1.7.3 *Let C be a subset of a complete metric space X. Then we have the following:*

(a) C is compact if and only if C is closed and totally bounded.

(b) $\overline{C}$ is compact if and only if C is totally bounded.

Observation

- $X = (0, 1)$ with usual metric is totally bounded, but not compact.

- $X = \mathbb{R}$ with usual metric is complete. But it is not totally bounded and hence not compact.

A subset C of a topological space is said to be *relatively compact* if its closure is compact, i.e., $\overline{C}$ is compact. In particular, we have an interesting result:

Proposition 1.7.4 *Let C be a closed subset of a complete metric space. Then C is compact if and only if it is relatively compact.*

We now state the following fundamental theorems concerning compactness.

Theorem 1.7.5 (The Heine-Borel theorem) – *A subset C of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Corollary 1.7.6 *A set $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 1.7.7 (Weierstrass theorem) – *Let C be a nonempty compact subset of a metric space (X, d) and $f : C \to \mathbb{R}$ a continuous function. Then f attains its maximum and minimum, i.e., there exist $\underline{x}$, $\overline{x} \in C$ such that*

$$f(\underline{x}) = \inf_{x \in C} f(x) \text{ and } f(\overline{x}) = \sup_{x \in C} f(x).$$

Theorem 1.7.8 (Mazur's theorem) – *The closed convex hull $\overline{co}(C)$ of a compact set C of a Banach space is compact.*

Observation

- $\mathbb{R}^n$, $n \geq 1$ is not compact. However, every closed bonded subset of $\mathbb{R}^n$ is compact. For example, $C = [0, 1] \subset \mathbb{R}$ is compact, but $\mathbb{R}$ itself is not compact.
- $C[0, 1]$ and ℓ_2 are not compact.
- The subset $C = \{\{x_n\} \in \ell_2 : |x_n| \leq 1/n, n \in \mathbb{N}\}$ of ℓ_2 is compact.
- The closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ in infinite-dimensional normed space is not compact in the topology induced by norm (see Proposition 1.7.14).

Proposition 1.7.9 *A subset C of ℓ_p space is compact if C is bounded and for $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} |x_i|^p < \varepsilon^p$ for all $n \geq n_0$ and $x = \{x_i\}_{i=1}^{\infty} \in C$.*

Proposition 1.7.10 *Every compact subset of a normed space X is closed, but the converse may not be true.*

Observation

- $\mathbb{R}^n$ is closed.

Proposition 1.7.11 *Every compact subset of a normed space X is complete, but the converse may not be true.*

Proposition 1.7.12 *Every compact subset of a normed space is bounded, but the converse may not be true.*

Proposition 1.7.13 *Every compact subset of a normed space is separable.*

Proposition 1.7.14 *A closed and bounded subset of a normed space need not be compact.*

Proof. Let $X = \ell_2$. Then the unit ball $B_X = \{x \in \ell_2 : \|x\|_2 = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} \leq 1\}$ is closed and bounded. We now show that B_X is not compact. Let $\{x_n\}$ be a sequence in B_X defined by

$$x_n = (0, 0, \cdots, 1, 0, \cdots), \quad n \in \mathbb{N}.$$

$$\uparrow$$
$$n^{th} \text{ position}$$

Hence for $m \neq n$,

$$\|x_n - x_m\|_2 = \sqrt{2},$$

i.e., there is no convergent subsequence of $\{x_n\}$. Therefore, B_X is not totally bounded and hence it is not compact. ∎

Remark 1.7.15 *B_{ℓ_2} is compact in the weak topology (see Theorem 1.9.26).*

Proposition 1.7.16 *A normed space X is finite-dimensional if and only if every closed and bounded subset of X is compact.*

1.8 Reflexivity

Let $X_1, X_2, \cdots, X_m$ be m linear spaces over the same field $\mathbb{K}$. Then a functional $f : X_1 \times X_2 \times \cdots \times X_m \to \mathbb{R}$ is said to be an *m-linear (multilinear) functional* on $X = X_1 \times X_2 \times \cdots \times X_m$ if it is linear with respect to each of the variables separately. For $m = 2$, such a functional is called a *bilinear functional*.

Duality pairing - Given a normed space X and its dual X^*, we define the duality pairing as the functional $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{K}$ such that

$$\langle x, j \rangle = j(x) \quad \text{for all } x \in X \text{ and } j \in X^*.$$

The properties of duality pairing can be easily derived from the definition:

Proposition 1.8.1 *Let X^* be the dual of a normed space X. Then we have the following:*

(a) The duality pairing is a bilinear functional on $X \times X^*$:
(i) $\langle ax + by, j \rangle = a\langle x, j \rangle + b\langle y, j \rangle$ for all $x, y \in X; j \in X^*$ and $a, b \in \mathbb{K}$;
(ii) $\langle x, \alpha j_1 + \beta j_2 \rangle = \alpha \langle x, j_1 \rangle + \beta \langle y, j_2 \rangle$ for all $x \in X; j_1, j_2 \in X^*;\ \alpha, \beta \in \mathbb{K}$.

(b) $\langle x, j \rangle = 0$ for all $x \in X$ implies $j = 0$.

(c) $\langle x, j \rangle = 0$ for all $j \in X^*$ implies $x = 0$.

Natural embedding mapping - Let $(X, \|\cdot\|)$ be a normed space. Then $(X^*, \|\cdot\|_*)$ is a Banach space. Let $j \in X^*$. Hence for given $x \in X$, the equation

$$f_x(j) = \langle x, j \rangle$$

defines a functional f_x on the dual space X^*. The functional f_x is linear by Proposition 1.8.1. Moreover, for $j \in X^*$ we have

$$|f_x(j)| = |\langle x, j \rangle| \leq \|x\|\|j\|_*. \tag{1.9}$$

This shows that f_x is bounded and hence f_x is a bounded linear functional on X^*.

The space of all bounded linear functionals on X^* is denoted by X^{**} and is called the *second dual of* X. Then $f_x \in X^{**}$. Note that X^{**} is a Banach space. Let $\|\cdot\|_{**}$ denote a norm on X^{**}. From (1.9), we have

$$\|f_x\|_{**} \leq \|x\|.$$

By Corollary 1.6.5, there exists a nonzero functional $j \in X^*$ such that

$$\langle x, j \rangle = \|x\|\|j\|_* \text{ and } \|j\|_* = \|x\|.$$

This implies that $\|f_x\|_{**} = \|x\|$.

Define a mapping $\varphi : X \to X^{**}$ by $\varphi(x) = f_x$, $x \in X$. Then φ is called the *natural embedding mapping* from X into X^{**}. It has the following properties:

(i) φ is linear: $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ for all $x, y \in X, \alpha, \beta \in \mathbb{K}$;

(ii) $\varphi(x)$ is isometry: $\|\varphi(x)\| = \|x\|$ for all $x \in X$.

Generally, however, the natural embedding mapping φ from X into X^{**} is not onto. It means that there may be elements in X^{**} that cannot be represented by elements in X.

In the case when φ is onto, we have an important class of normed spaces.

Definition 1.8.2 *A normed space X is said to be reflexive if the natural embedding mapping $\varphi : X \to X^{**}$ is onto. In this case, we write $X \cong X^{**}$ or $X = X^{**}$.*

Observation

- $\mathbb{R}^n$ is reflexive. (In fact, every finite-dimensional Banach space is reflexive.)
- ℓ_p and L_p for $1 < p < \infty$ are reflexive Banach spaces.
- Every Hilbert space is a reflexive Banach space, i.e., $H^{**} = H$.
- ℓ_1, ℓ_∞, L_1 and L_∞ are not reflexive.
- c and c_0 are not reflexive Banach spaces.

We now state the following facts for the class of reflexive Banach spaces.

Proposition 1.8.3 *(a) Any reflexive normed space must be complete and, hence, is a Banach space.*

(b) A closed subspace of a reflexive Banach space is reflexive.

(c) The Cartesian product of two reflexive spaces is reflexive.

(d) The dual of a reflexive Banach space is reflexive.

Theorem 1.8.4 (James theorem) – *A Banach space X is reflexive if and only if for each $j \in S_{X^*}$, there exists $x \in S_X$ such that $j(x) = 1$.*

1.9 Weak topologies

Let X^* be the dual space of a Banach space X. The convergence of a sequence in a Banach space X is the usual *norm convergence* or *strong convergence*, i.e., $\{x_n\}$ in X converges to x if $\lim_{n\to\infty} \|x_n - x\| = 0$. This is related to the strong topology on X with neighborhood base $B_r(0) = \{x \in X : \|x\| < r\}, r > 0$ at the origin. There is also a weak topology on X generated by the bounded linear functionals on X. Indeed, $G \subset X$ is open in the weak topology (we say G is w-open) if and only if for every $x \in G$, there are bounded linear functionals $f_1, f_2, \cdots, f_n$ and positive real numbers $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ such that

$$\{y \in X : |f_i(x) - f_i(y)| < \varepsilon_i, i = 1, 2, \cdots, n\} \subset G.$$

Hence a subbase σ for the weak topology on X generated by a base of neighborhoods of $x_0 \in X$ is given by the following sets:

$$V(f_1, f_2 \cdots, f_n : \varepsilon) = \{x \in X : |\langle x - x_0, f_i\rangle| < \varepsilon, \quad \text{for every} \quad i = 1, 2, \cdots, n\}.$$

In particular, a sequence $\{x_n\}$ in X converges to $x \in X$ for weak topology $\sigma(X, X^*)$ if and only if $\langle x_n, f\rangle \to \langle x, f\rangle$ for all $f \in X^*$.

Observation

- The weak topology is not metrizable if X is infinite-dimensional.
- Under the weak topology, the normed space X is a locally convex topological space.
- The weak topology of a normed space is a Hausdorff topology.

We are now in a position to define convergence, closedness, completeness, and compactness with respect to the weak topology.

Weakly convergent – A sequence $\{x_n\}$ in a normed space X is said to converge weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, we write $x_n \rightharpoonup x$ or weak-$\lim_{n\to\infty} x_n = x$.

Weakly closed – A subset C of a Banach space X is said to be a *weakly closed* if it is closed in the weak topology.

Weak Cauchy sequence – A sequence $\{x_n\}$ in a normed space X is said to be a *weak Cauchy* if for each $f \in X^*, \{f(x_n)\}$ is a Cauchy sequence in $\mathbb{K}$.

Weakly complete – A normed space X is said to be *weakly complete* if every weak Cauchy sequence in X converges weakly to some element in X.

Weakly compact – A subset C of a normed space X is said to be *weakly compact* if C is compact in the weak topology.

Schur property – A Banach space is said to satisfy *Schur property* if there exist weakly convergent sequences that are norm convergent.

Theorem 1.9.1 (Schur's theorem) – *In ℓ_1, weak and norm convergences of sequences coincide.*

We have the following basic properties of weakly convergent sequences in normed spaces:

Proposition 1.9.2 (Uniqueness of weak limit) – *Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$. Then $x = y$.*

Proof. Because $\{f(x_n)\}$ is a sequence of scalars such that $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$, it follows that $f(x) = f(y)$. This implies that $f(x-y) = 0$. Therefore, $x = y$ by Corollary 1.6.8. ∎

Proposition 1.9.3 (Strong convergence implies weak convergence) – *Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \to x$. Then $x_n \rightharpoonup x$.*

Proof. Because $x_n \to x$, $\|x_n - x\| \to 0$. Hence

$$|f(x_n) - f(x)| \leq \|f\|_* \|x_n - x\| \to 0 \text{ for all } f \in X^*.$$

Therefore, $x_n \rightharpoonup x$. ∎

The converse of Proposition 1.9.3 is not true in general. It can be seen from the following example:

Example 1.9.4 *Let $X = \ell_2$ and $\{x_n\}$ be a sequence in ℓ_2 such that*

$$x_n = (0, 0, 0, \cdots, \underset{\substack{\uparrow \\ n^{th}\ position}}{1}, 0, \cdots), \qquad n \in \mathbb{N}.$$

For any $y = (y_1, y_2, \cdots, y_n, \cdots) \in X^* = \ell_2$, we have

$$(x_n, y) = y_n \to 0 \text{ as } n \to \infty.$$

Hence $x_n \rightharpoonup 0$ as $n \to \infty$. However, $\{x_n\}$ does not converge strongly because $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Therefore, a weakly convergent sequence need not be convergent in norm.

Theorem 1.9.5 (Weak convergence in $\boldsymbol{\ell_p}$ space, $\boldsymbol{1 < p < \infty}$) – *For $1 < p < \infty$, let*

$$x_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_i^n, \cdots) \in \ell_p, \quad n \in \mathbb{N}$$

and

$$x = (\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots) \in \ell_p.$$

Then $x_n \rightharpoonup x$ *if and only if*

(i) $\{x_n\}$ *is bounded, i.e.,* $\|x_n\| \leq M$ *for all* $n \in \mathbb{N}$ *and for some* $M \geq 0$*;*

(ii) *for each* i, $\alpha_i^{(n)} \to \alpha_i$ *as* $n \to \infty$.

Theorem 1.9.6 *Let* X *be a finite-dimensional normed space. Then strong convergence is equivalent to weak convergence.*

Theorem 1.9.7 *Every reflexive normed space is weakly complete.*

Convergence of sequences in B(X, Y) – Let X and Y be two normed spaces. A sequence $\{T_n\}$ in $B(X,Y)$ is said to be

(i) *uniformly convergent to* $T \in B(X,Y)$ in the norm of $B(X,Y)$ if $\|T_n - T\|_B \to 0$ as $n \to \infty$, i.e., for $\varepsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\sup\limits_{\|x\|\leq 1} \|T_n x - Tx\| < \varepsilon$ for all $n \geq n_0$,

[uniform convergence of $\{\mathbf{T_n}\}$]

(ii) *strongly convergent to* $T \in B(X,Y)$ if $\lim\limits_{n\to\infty} \|T_n x - Tx\| = 0$ for all $x \in X$,

[strong convergence of $\{\mathbf{T_n}\}$]

(iii) *weakly convergent to* $T \in B(X,Y)$ if $|f(T_n x) - f(Tx)| \to 0$ for all $x \in X$ and $f \in Y^*$.

[weak convergence of $\{\mathbf{T_n}\}$]

It follows immediately from the inequality

$$\|T_n x - Tx\| \leq \|T_n - T\|_B \|x\|, \quad x \in X$$

that the uniform convergence implies strong convergence. It can be easily observed for the sequence of operators in $B(X,Y)$ that

uniform convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

We note that the converse is not true in general.

Weak* topology - We have seen that if τ is the norm topology of a normed space X, then the weak topology $\sigma(X, X^*)$ is a subset of the original norm topology τ. Let τ^* be the norm topology of X^* generated by the norm $\|\cdot\|_*$ (of X^*). Then there exists a topology denoted by $\sigma(X^*, X)$ on X^* such that $\sigma(X^*, X) \subset \tau^*$. The topology $\sigma(X^*, X)$ is called the *weak* topology* on X^*. Thus, we can speak about strong neighborhood, strongly closed, strongly bounded, weak convergence in $(X^*, \|\cdot\|_*)$ and weak* neighborhood, weak*ly closed, weak*ly bounded, weak*ly convergence in $(X^*, \sigma(X^*, X))$, respectively.

We now study some basic properties of the weak topology and weak* topology. We begin with a simple characterization for the convergence of sequences in the weak topologies.

Proposition 1.9.8 *Let X be a normed space and $\{f_n\}$ a sequence in X^*. Then we have the following:*

(a) $\{f_n\}$ converges strongly to f in the norm topology on X^* (denoted by $f_n \to f$) if

$$\|f_n - f\|_* \to 0.$$

(b) $\{f_n\}$ converges to f in the weak topology on X^* (denoted by $f_n \rightharpoonup f$) if

$$\langle f_n - f, g\rangle \to 0 \textit{ for all } g \in X^{**}.$$

(c) $\{f_n\}$ converges to f in the weak topology on X^* (denoted by $f_n \to f$ weak*ly or $f_n \rightharpoonup^* f$) if*

$$\langle x, f_n - f\rangle \to 0 \textit{ for all } x \in X.$$

On the other hand, the following result is an immediate consequence of Theorem 1.5.3.

Corollary 1.9.9 *Let C be a nonempty subset of a Banach space X. For each $f \in X^*$, let $f(C) = \cup_{x\in C}\langle x, f\rangle$ be a bounded set in $\mathbb{R}$. Then C is bounded.*

Proof. Set $X := X^*$, $Y := \mathbb{R}$, and $T_x(f) := \langle x, f\rangle$, $x \in C$. Then $T_x \in B(X^*, \mathbb{R})$. Because $f(C)$ is bounded, it follows that

$$\sup_{x\in C} |T_x(f)| = \sup_{x\in C} |\langle x, f\rangle| \leq K,$$

for some $K > 0$. By the uniform boundedness principle, there exists a constant $M > 0$ such that

$$\|T_x\| \leq M \text{ for all } x \in C.$$

This implies that

$$|\langle x, f\rangle| \quad = \quad |T_x(f)| \leq \|T_x\|\|f\|_* \leq M\|f\|_*.$$

By Corollary 1.6.7, we have

$$\|x\| \leq M \text{ for all } x \in C.$$

Therefore, C is bounded. ∎

Applying Corollary 1.9.9, we have

Theorem 1.9.10 *Let $\{x_n\}$ be a sequence in a Banach space X. Then we have the following:*

(a) $x_n \rightharpoonup x$ (in X) implies $\{x_n\}$ is bounded and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

(b) $x_n \rightharpoonup x$ in X and $f_n \to f$ in X^* imply $f_n(x_n) \to f(x)$ in $\mathbb{R}$.

Proof. (a) Because $x_n \rightharpoonup x$, then $f(x_n) \to f(x)$ for all $f \in X^*$. Hence $\{f(x_n)\}$ is bounded for all $f \in X^*$. Thus, by Corollary 1.9.9, $\{x_n\}$ is bounded.

Moreover,

$$|\langle x_n, f\rangle| \leq \|x_n\|\|f\|_*.$$

Taking liminf in the above inequality, we have

$$|\langle x, f\rangle| \leq \liminf_{n\to\infty} \|x_n\|\|f\|_*.$$

By Corollary 1.6.7, we obtain

$$\|x\| = \sup_{\|f\|_*\leq 1} |\langle x, f\rangle| \leq \sup_{\|f\|_*\leq 1} (\liminf_{n\to\infty} \|x_n\|\|f\|_*) \leq \liminf_{n\to\infty} \|x_n\|.$$

(b) Because $x_n \rightharpoonup x$ in X, it follows that $\langle x_n - x, f\rangle = f(x_n) - f(x) \to 0$ and $\{x_n\}$ is bounded (by part (a)). Hence

$$\begin{aligned} |\langle x_n, f_n\rangle - \langle x, f\rangle| &\leq |\langle x_n, f_n\rangle - \langle x_n, f\rangle| + |\langle x_n, f\rangle - \langle x, f\rangle| \\ &= |\langle x_n, f_n - f\rangle| + |\langle x_n - x, f\rangle| \\ &\leq \|x_n\|\ \|f_n - f\|_* + |\langle x_n - x, f\rangle| \\ &\leq M\|f_n - f\|_* + |\langle x_n - x, f\rangle| \to 0 \end{aligned}$$

for some constant $M > 0$. Therefore, $f_n(x_n) \to f(x)$. ∎

Observation

- Let $\{x_n\}$ be a sequence in a Banach space X with $x_n \rightharpoonup x \in X$ and $\{\alpha_n\}$ a sequence of scalars such that $\alpha_n \to \alpha$. Then $\{\alpha_n x_n\}$ converges weakly to αx.

Theorem 1.9.11 *Let X be a Banach space and $\{x_n\}$ a sequence in X such that $x_n \rightharpoonup x \in X$. Then there exists a sequence of convex combinations of $\{x_n\}$ that converges strongly to x, i.e., there exists convex combination $\{y_n\}$ such that*

$$y_n = \sum_{i=n}^{m} \lambda_i x_i, \ \text{where} \sum_{i=n}^{m} \lambda_i = 1 \text{ and } \lambda_i \geq 0, \quad n \leq i \leq m,$$

which converges strongly to x.

Corollary 1.9.12 *Let C a nonempty subset of a Banach space X and $\{x_n\}$ a sequence in C such that $x_n \rightharpoonup x \in X$. Then $x \in \overline{co}(C)$.*

The weak topology is weaker than the norm topology, and every w-closed set is also norm closed. The following result shows that for convex sets, the converse is also true.

Proposition 1.9.13 *Let C be a convex subset of a normed space X. Then C is weakly closed if and only if C is closed.*

The following proposition is a generalization of Theorem 1.7.8.

Proposition 1.9.14 *Let C be a weakly compact subset of a Banach space X. Then $\overline{co}(C)$ is also weakly compact.*

The following result is a direct consequence of the uniform bounded principle:

Proposition 1.9.15 *Let C be a weakly compact subset of a Banach space X. Then C is bounded.*

Theorem 1.9.16 (Eberlein-Smulian theorem) – *Let C be a weakly closed subset of a Banach space. Then the following are equivalent:*

(a) C is weakly compact.

(b) C is weakly sequentially compact, i.e., each sequence $\{x_n\}$ in C has a subsequence that converges weakly to a point in C.

Corollary 1.9.17 *Let C be a closed convex subset of a Banach space. Then the following are equivalent:*

(a) C is weakly compact.

(b) Each sequence $\{x_n\}$ in C has a subsequence that converges weakly to a point in C.

Proposition 1.9.18 *Any closed convex subset of a weakly compact set is itself weakly compact.*

Theorem 1.9.19 (Kakutani's theorem) – *Let X be a Banach space. Then X is reflexive if and only if the unit closed ball $B_X := \{x \in X : \|x\| \leq 1\}$ is weakly compact (i.e., B_X is compact in the weak topology of X).*

Using Proposition 1.9.13 and Kakutani's theorem, we obtain

Theorem 1.9.20 *Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset of X is weakly compact (compact in weak topology).*

Theorem 1.9.21 *Let C be a subset of a reflexive Banach space. Then*

$$\begin{array}{ccc} C \text{ is weakly compact} & \Leftrightarrow & C \text{ is bounded} \\ \text{(compactness in weak topology)} & & \text{(boundedness in strong topology)} \end{array}$$

Theorem 1.9.22 *Let $\{x_n\}$ be a sequence in a weakly compact convex subset of a Banach space X and $\omega_w(\{x_n\})$ denote the set of all weak subsequential limits of $\{x_n\}$. Then $\overline{co}(\omega_w(\{x_n\})) = \cap_{n=1}^{\infty}\overline{co}(\{x_k\}_{k\geq n})$.*

Proof. Set $W := \omega_w(\{x_n\})$, $A_n := \overline{co}(\{x_k\}_{k\geq n})$, and $A := \cap_{n=1}^{\infty} A_n$. We now show that $\overline{co}(W) = A$. The inclusion $W \subset A$ (and hence $\overline{co}(W) \subset A$) is trivial.

Hence it suffices to prove that $A \subset \overline{co}(W)$. Suppose, for contradiction, that $x \in A \setminus \overline{co}(W)$. Then there exists $j \in X^*$ such that

$$\langle x, j\rangle > \sup\{\langle y, j\rangle : y \in \overline{co}(W)\} = \sup\{\langle y, j\rangle : y \in W\rangle. \tag{1.10}$$

Because $x \in A \subset A_n$,

$$\langle x, j\rangle \leq \sup\{\langle y, j\rangle : y \in A_n\} = \sup\{\langle x_k, j\rangle : k \geq n\}.$$

Therefore,

$$\langle x, j\rangle \leq \limsup_{n\to\infty}\langle x_n, j\rangle.$$

It follows from the Eberlein-Smulian theorem that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup x' \quad \text{and} \quad \langle x, j\rangle \leq \langle x', j\rangle.$$

Because $x' \in W$ by definition, this is a contradiction of (1.10). ∎

Corollary 1.9.23 *Let X be a Banach space and $\{x_n\}$ a sequence in X weakly convergent to z. Let $A_n = \overline{co}(\{x_k\}_{k\geq n})$. Then $\cap_{n=1}^{\infty} A_n = \{z\}$.*

Proposition 1.9.24 *Let $\{x_n\}$ be a bounded sequence in reflexive Banach space X and $A_n = \overline{co}(\{x_n\}_{k\geq n})$. If $\cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty}\overline{co(\{x_n, x_{n+1}, \cdots\})} = \{x\}$, then $x_n \rightharpoonup x$.*

Proposition 1.9.25 *Let $\{x_n\}$ be a weakly null sequence in a Banach space X and $\{j_n\}$ a bounded sequence in X^*. Then for each $\varepsilon > 0$, there exists an increasing sequence $\{n_k\}$ in $\mathbb{N}$ such that $|\langle x_{n_i}, j_{n_k}\rangle| < \varepsilon$ if $i \neq k$.*

Proof. Without loss of generality, we may assume that X is a separable space. We can assume that $\{j_n\}$ converges weak*ly to some $j \in B_{X^*}$. Given $\varepsilon > 0$, we find n_1 such that $|\langle x_n, j\rangle| < \varepsilon/2$ for all $n \geq n_1$. Next, having $n_1 < n_2 < \cdots < n_{k-1}$, we pick $n_k > n_{k-1}$ with $|\langle x_{n_k}, j_{n_i}\rangle| < \varepsilon$ and $|\langle x_{n_i}, j_{n_k} - j\rangle| < \varepsilon/2$ for all $i = 1, 2, \cdots, k-1$. Then $|\langle x_{n_i}, j_{n_k}\rangle| < \varepsilon$. ∎

We now list several properties that characterize reflexivity.

Theorem 1.9.26 *Let X be a Banach space. Then following statements are equivalent:*

(a) X is reflexive.

(b) B_X is weakly compact.

(c) Every bounded sequence in X in strong topology has a weakly convergent subsequence.

(d) For any $f \in X^*$, there exists $x \in B_X$ such that $f(x) = \|f\|_*$.

(e) *X^* is reflexive.*

(f) *$\sigma(X^*, X) = \sigma(X^*, X^{**})$, i.e., on X^* the weak topology and the weak* topology coincide.*

(g) *If $\{C_n\}$ is any descending sequence of nonempty closed convex bounded subsets of X, then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

(h) *For any closed convex bounded subset C of X and any $j \in X^*$, there exists $x \in C$ such that $\langle x, j \rangle = \sup\{\langle y, j \rangle : y \in C\}$.*

Finally, we give the fundamental result concerning the weak* topology.

Theorem 1.9.27 (Banach-Alaoglu's theorem) – *The unit ball B_{X^*} of the dual of a normed space X is compact in the weak* topology.*

1.10 Continuity of mappings

In this section, we discuss various forms of continuity of mappings with their properties.

Definition 1.10.1 *Let T be a mapping from a metric space (X, d) into another metric space (Y, ρ). Then T is said to be*

(i) *continuous at $x_0 \in X$ if $x_n \to x_0$ implies $Tx_n \to Tx_0$ in Y, i.e., for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, x_0) > 0$ such that $\rho(Tx_0, Ty) < \varepsilon$ whenever $d(x_0, y) < \delta$ for all $y \in X$,*

(ii) *uniformly continuous on X if for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\rho(Tx, Ty) < \varepsilon \text{ whenever } d(x, y) < \delta \text{ for all } x, y \in X.$$

Example 1.10.2 *Let $X = (0, 1]$ and $Y = \mathbb{R}$ and let X and Y have usual metric defined by absolute value. Then the mapping $T : X \to Y$ defined by $Tx = 1/x$ is continuous, but not uniformly continuous.*

Observation

- Every uniformly continuous mapping from X into Y is continuous at each point of X, but pointwise continuity does not necessary imply uniform continuity.
- Every uniformly continuous mapping T from a metric space X into another metric space Y maps a Cauchy sequence in X into a Cauchy sequence in Y.

Proposition 1.10.3 *Let T be a continuous mapping from a compact metric space (X, d) into another metric space (Y, ρ). Then T is uniformly continuous.*

A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy *Lipschitz condition* on X if there exists a constant $L > 0$ such that

$$\rho(Tx, Ty) \leq Ld(x, y) \text{ for all } x, y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called *Lipschitz constant.* In this case, we say that T is an *L-Lipschitz mapping* or simply a *Lipschitzian mapping* with Lipschitz constant L. Otherwise, it is called *non-Lipschitzian mapping.* An L-Lipschitz mapping T is said to be *contraction* if $L < 1$ and *nonexpansive* if $L = 1$. The mapping T is said to be *contractive* if

$$\rho(Tx, Ty) < d(x, y) \text{ for all } x, y \in X, \ x \neq y.$$

Remark 1.10.4 *Every Lipschitz continuous mapping T from a metric space X into another metric space Y is uniformly continuous on X. Indeed, choose $\delta < \varepsilon/L$ (independent of x), and we get*

$$\rho(Tx, Tx_0) \leq Ld(x, x_0) < \varepsilon.$$

The following example shows that the distance functional $f(x) = d(x, C)$ is nonexpansive.

Example 1.10.5 *Let C be a nonempty subset of a normed space X. Then for each pair x, y in X*

$$|d(x, C) - d(y, C)| \leq \|x - y\|.$$

In particular, the function $x \mapsto d(x, C)$ is nonexpansive and hence uniformly continuous.

The following proposition guarantees the existence of Lipschitzian mappings.

Proposition 1.10.6 *Let $T : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a, b). Suppose T' is continuous on $[a, b]$. Then T is a Lipschitz continuous function (and hence is uniformly continuous).*

Proof. By the Lagrange's theorem, we have

$$Ty - Tx = T'(c)(y - x) \text{ for all } a \leq x < y \leq b,$$

where $c \in (x, y) \subset [a, b]$. Because T' is continuous and interval $[a, b]$ is compact in $\mathbb{R}$, by Weierstrass theorem, there exists $x_0 \in [a, b]$ such that

$$L = |T'(x_0)| = \sup_{c \in [a,b]} |T'(c)|.$$

Thus, $|Tx - Ty| \leq L|x - y|$, which proves that T is Lipschitz continuous. ∎

The following example shows that there is a Lipschitzian mapping for which T' does not exist.

Example 1.10.7 *The function* $Tx = |x|,\ x \in [-1,1]$ *satisfies Lipschitz condition with* $L = 1$, *i.e.,* $|Tx - Ty| \leq |x - y|$ *for all* $x, y \in [-1,1]$. *Note* T *is not differentiable at zero.*

We now give an example of a non-Lipschitzian mapping that is continuous.

Example 1.10.8 *Let* $T : \left[-\frac{1}{\pi}, \frac{1}{\pi}\right] \to \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ *be a mapping defined by*

$$Tx = \begin{cases} 0 & \textit{if} \quad x = 0, \\ \frac{x}{2}\sin(1/x) & \textit{if} \quad x \neq 0. \end{cases}$$

Then T *is continuous, but not Lipschitz continuous.*

For linear mappings, the continuity condition can be restated in terms of uniform continuity.

Proposition 1.10.9 *Let* X *and* Y *be two normed spaces and* $T : X \to Y$ *a linear mapping. Then the following conditions are equivalent:*

(a) T *is continuous.*

(b) T *is Lipschitz function: there exists* $M > 0$ *such that* $\|Tx\| \leq M\|x\|$ *for all* $x \in X$.

(c) T *is uniformly continuous.*

Let X and Y be two Banach spaces and let T be a mapping from X into Y. Then the mapping T is said to be

1. *bounded* if C is bounded in X implies $T(C)$ is bounded;
2. *locally bounded* if each point in X has a bounded neighborhood U such that $T(U)$ is bounded;
3. *weakly continuous* if $x_n \rightharpoonup x$ in X implies $Tx_n \rightharpoonup Tx$ in Y;
4. *demicontinuous* if $x_n \to x$ in X implies $Tx_n \rightharpoonup Tx$ in Y;
5. *hemicontinuous* at $x_0 \in X$ if for any sequence $\{x_n\}$ converging to x_0 along a line implies $Tx_n \rightharpoonup Tx_0$, i.e., $Tx_n = T(x_0 + t_n x) \rightharpoonup Tx_0$ as $t_n \to 0$ for all $x \in X$;
6. *closed* if $x_n \to x$ in X and $Tx_n \to y$ in Y imply $Tx = y$;
7. *weakly closed* if $x_n \rightharpoonup x \in X$ and $Tx_n \rightharpoonup y$ in Y imply $Tx = y$;
8. *demiclosed* if $x_n \rightharpoonup x$ in X and $Tx_n \to y$ in Y imply $Tx = y$;
9. *compact* if C is bounded implies $T(C)$ is relatively compact ($\overline{T(C)}$ is compact), i.e., for every bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has convergent subsequence in Y;

10. *completely continuous* if it is continuous and compact;
11. *demicompact* if any bounded sequence $\{x_n\}$ in X such that $\{x_n - Tx_n\}$ converges strongly has a convergent subsequence.

In the case of linear mappings, the concepts of continuity and boundedness are equivalent, but it is not true in general.

Proposition 1.10.10 *Every continuous linear mapping $T : X \to Y$ is weakly continuous.*

Proposition 1.10.11 *Let X be a reflexive Banach space and Y a general Banach space. Then every weakly continuous mapping $T : X \to Y$ is bounded.*

Proposition 1.10.12 *A completely continuous mapping maps a weakly convergent sequence into a strongly convergent.*

Proposition 1.10.13 *Every linear mapping is hemicontinuous.*

Proof. Every linear and demicontinuous mapping is continuous. ∎

It is clear that every demicontinuous mapping is hemicontinuous, but the converse is not true.

Example 1.10.14 *Let $X = \mathbb{R}^2, Y = \mathbb{R}$, and $T : X \to Y$ a mapping defined by*

$$T(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Then T is hemicontinuous at $(0,0)$, but not demicontinuous at $(0,0)$.

Let X and Y be two sets. A *multivalued* T from X to Y, denoted by $T : X \to Y$, is a subset $T \subseteq X \times Y$. The inverse of $T : X \to Y$ is a multivalued function $T^{-1} : Y \to X$ defined by $(y,x) \in T^{-1}$ if and only if $(x,y) \in T$. The values of T are the sets $Tx = \{y \in Y : (x,y) \in T\}$; the *fibers* of T are the sets $T^{-1}(y) = \{x \in X : (x,y) \in T\}$ for $y \in Y$.

For $A \subset X$, the set

$$T(A) = \cup_{x \in A} Tx = \{y \in Y : T^{-1}(y) \cap A \neq \emptyset\}$$

is called the *image of* A under T; for $B \subset Y$, the set

$$T^{-1}(B) = \cup_{y \in B} T^{-1}(y) = \{x \in X : Tx \cap B \neq \emptyset\},$$

the image of B under T^{-1}, is called *inverse image* of B under T. A point of a set that is invariant under any transformation is called a *fixed point* of the transformation. A point $x_0 \in X$ is said to be a fixed point of T if $x_0 \in Tx_0$.

Let X and Y be two topological spaces. Then a multivalued function $T : X \to Y$ is said to be *upper semicontinuous* (*lower semicontinuous*) if the inverse image of a closed set (open set) is closed (open). A multivalued function is continuous if it is both upper and lower semicontinuous.

Finally, we conclude the chapter with the following important fixed point theorems.

Theorem 1.10.15 (Brouwer's fixed point theorem) – *Every continuous mapping from the unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Theorem 1.10.16 (Schauder's fixed point theorem) – *Let C be a nonempty closed convex bounded subset of a Banach space X. Then every continuous compact mapping $T : C \to C$ has a fixed point.*

Theorem 1.10.17 (Tychonoff's fixed point theorem) – *Let C be a nonempty compact convex subset of a locally convex topological linear space X and $T : C \to C$ a continuous mapping. Then T has a fixed point.*

Exercises

1.1 Let (X, d) be a metric space. Show that $\rho(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$ is also a metric space.

1.2 Give an example of a seminorm that is not a norm.

1.3 Let $\langle \cdot, \cdot \rangle$ be an inner product on a linear space X and $T : X \to X$ a one-one linear mapping. Let $\langle x, y\rangle_T = \langle Tx, Ty\rangle$ for all $x, y \in X$. Show that $\langle \cdot, \cdot\rangle_T$ is an inner product space.

1.4 Show that the space c_0 of all real sequences converging to 0 is a normed space with norm $\|x\| = \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty$.

1.5 Let c_{00} be a normed space with ℓ_p-norm $(1 \le p \le \infty)$ and $\{f_n\}$ a sequence of functional on c_{00} defined by $f_n(x) = nx_n$ for all $x = (x_1, x_2, \cdots, x_n, \cdots)$. Show that $f_n(x) \to 0$ for every $x \in c_{00}$, but $\|f_n\| = n$ for all n.

1.6 Show that the space ℓ_p $(1 < p < \infty)$ is reflexive, but ℓ_1 is not reflexive.

1.7 Let C be a nonempty closed convex subset of a normed space X and $\{x_n\}$ a sequence in C such that $x_n \rightharpoonup x$ in X. Show that $x \in C$.

1.8 Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \rightharpoonup x$. Show that $x \in \text{span}\,\{x_n\}$.

1.9 Let $\{x_n\}$ be a sequence in normed space X such that $x_n \rightharpoonup x$. Show that $\{x_n\}$ is bounded.

1.10 Let $X = c_{00}$ or c_0 with norm $\|\cdot\|_\infty$. Show that $x_n \rightharpoonup x$ in X if and only if $\{x_n\}$ is bounded in X and $x_{n,i} \to x_i$ as $n \to \infty$ for each $i = 1, 2, \cdots$.

Chapter 2

Convexity, Smoothness, and Duality Mappings

Geometric structures such as convexity and smoothness of Banach spaces play an important role in the existence and approximation of fixed points of nonlinear mappings. This chapter presents a substantial number of useful properties of duality mappings and Banach spaces having these geometric structures.

2.1 Strict convexity

Let X be a linear space. The *line segment* or *interval joining* the two points $x, y \in X$ is the set $[x, y] := \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$, i.e., $[x, y] = co(\{x, y\})$ is convex hull of x and y.

The basic property of a norm of a Banach space X is that it is always convex, i.e.,

$$\|(1-\lambda)x + \lambda y\| \leq (1-\lambda)\|x\| + \lambda\|y\| \text{ for all } x, y \in X \text{ and } \lambda \in [0, 1].$$

A number of Banach spaces do not have equality when $x \neq y$, i.e.,

$$\|(1-\lambda)x + \lambda y\| < (1-\lambda)\|x\| + \lambda\|y\| \quad \text{for all } x, y \in X \text{ with } x \neq y \text{ and } \lambda \in (0, 1). \tag{2.1}$$

We use S_X to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ on Banach space X. If $x, y \in S_X$ with $x \neq y$, then (2.1) reduces to

$$\|(1-\lambda)x + \lambda y\| < 1 \text{ for all } \lambda \in (0, 1),$$

which says that the unit sphere S_X contains no line segments. This suggests strict convexity of norm.

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Definition 2.1.1 *A Banach space X is said to be strictly convex if*

$$x, y \in S_X \text{ with } x \neq y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1 \text{ for all } \lambda \in (0,1).$$

This says that the midpoint $(x+y)/2$ of two distinct points x and y in the unit sphere S_X of X does not lie on S_X. In other words, if $x, y \in S_X$ with $\|x\| = \|y\| = \|(x+y)/2\|$, then $x = y$.

Example 2.1.2 *Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|x\|_2$ defined by*

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}, \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then X is strictly convex.

Example 2.1.3 *Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|\cdot\|_1$ defined by*

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|, \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then X is not strictly convex. To see it, let

$$x = (1, 0, 0, \cdots, 0) \text{ and } y = (0, 1, 0, \cdots, 0).$$

It is easy to see that $x \neq y, \|x\|_1 = 1 = \|y\|_1$, but $\|x+y\|_1 = 2$.

Example 2.1.4 *Consider $X = \mathbb{R}^n, n \geq 2$ with norm $\|\cdot\|_\infty$ defined by*

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n.$$

Then X is not strictly convex. Indeed, for $x = (1, 0, 0, \cdots, 0)$ and $y = (1, 1, 0, \cdots, 0)$, we have, $x \neq y$, $\|x\|_\infty = 1 = \|y\|_\infty$, but $\|x+y\|_\infty = 2$.

The other equivalent conditions of strict convexity are given in the following:

Proposition 2.1.5 *Let X be a Banach space. Then the following are equivalent:*

(a) X is strictly convex.

(b) For each nonzero $f \in X^*$, there exists at most one point x in X with $\|x\| = 1$ such that $\langle x, f \rangle = f(x) = \|f\|_*$.

Proof. $(a) \Rightarrow (b)$. Let X be a strictly convex Banach space and f an element in X^*. Suppose there exist two distinct points x, y in X with $\|x\| = \|y\| = 1$ such that $f(x) = f(y) = \|f\|_*$. If $t \in (0,1)$, then

$$\begin{aligned}
\|f\|_* &= tf(x) + (1-t)f(y) \qquad (as\ f(x) = f(y) = \|f\|_*) \\
&= f(tx + (1-f)y) \\
&\leq \|f\|_* \|tx + (1-t)y\| \\
&< \|f\|_*, \qquad (as\ \|tx + (1-t)y\| < 1)
\end{aligned}$$

which is a contradiction. Therefore, there exists at most one point x in X with $\|x\| = 1$ such that $f(x) = \|f\|_*$.

$(b) \Rightarrow (a)$. Suppose $x, y \in S_X$ with $x \neq y$ such that $\|(x+y)/2\| = 1$. By Corollary 1.6.6, there exists a functional $j \in S_{X^*}$ such that

$$\|j\|_* = 1 \text{ and } \langle (x+y)/2, j\rangle = \|(x+y)/2\|.$$

Because $\langle x, j\rangle \leq 1$ and $\langle y, j\rangle \leq 1$, we have $\langle x, j\rangle = \langle y, j\rangle$. This implies, by hypothesis, that $x = y$. Therefore, $(b) \Rightarrow (a)$ is proved. ∎

Proposition 2.1.6 *Let X be a Banach space. Then the following statements are equivalent:*

(a) X is strictly convex.

(b) For every $1 < p < \infty$,

$$\|tx+(1-t)y\|^p < t\|x\|^p+(1-t)\|y\|^p \text{ for all } x, y \in X,\ x \neq y \text{ and } t \in (0,1).$$

Proof. $(a) \Rightarrow (b)$. Let X be strictly convex. Suppose $x, y \in X$ with $x \neq y$. By strict convexity of X,

$$\|tx + (1-t)y\|^p < (t\|x\| + (1-t)\|y\|)^p \text{ for all } t \in (0,1). \tag{2.2}$$

If $\|x\| = \|y\|$, then

$$\|tx + (1-t)y\|^p < \|x\|^p = t\|x\|^p + (1-t)\|y\|^p.$$

We now assume that $\|x\| \neq \|y\|$. Consider the function $t \mapsto t^p$ for $1 < p < \infty$. Then it is a convex function and

$$\left(\frac{a+b}{2}\right)^p < \frac{a^p + b^p}{2} \text{ for all } a, b \geq 0 \text{ and } a \neq b.$$

Hence from (2.2) with $t = 1/2$, we have

$$\left\|\frac{x+y}{2}\right\|^p \leq \left(\frac{\|x\| + \|y\|}{2}\right)^p < \frac{1}{2}(\|x\|^p + \|y\|^p). \tag{2.3}$$

If $t \in (0, 1/2]$, then from (2.2), we have

$$\begin{aligned}
\|tx + (1-t)y\|^p &= \left\|2t\frac{x+y}{2} + (1-2t)y\right\|^p \\
&\leq \left(2t\left\|\frac{x+y}{2}\right\| + (1-2t)\|y\|\right)^p \\
&< 2t\left\|\frac{x+y}{2}\right\|^p + (1-2t)\|y\|^p \\
&\leq t\|x\|^p + (1-t)\|y\|^p. \qquad \text{(by (2.3))}
\end{aligned}$$

The proof is similar if $t \in (1/2, 1)$.

$(b) \Rightarrow (a)$. It is obvious. ∎

Proposition 2.1.7 *Let X be a strictly convex Banach space. If $\|x+y\| = \|x\|+\|y\|$ for $0 \neq x \in X$ and $y \in X$, then there exists $t \geq 0$ such that $y = tx$.*

Proof. Let $x, y \in X\backslash\{0\}$ be such that $\|x+y\| = \|x\|+\|y\|$. From Corollary 1.6.6, there exists $j \in X^*$ such that

$$\langle x+y, j\rangle = \|x+y\| \text{ and } \|j\|_* = 1.$$

Because $\langle x, j\rangle \leq \|x\|$ and $\langle y, j\rangle \leq \|y\|$, we must have $\langle x, j\rangle = \|x\|$ and $\langle y, j\rangle = \|y\|$. This means that $\langle x/\|x\|, j\rangle = \langle y/\|y\|, j\rangle = 1$. By strict convexity of X, it follows from Proposition 2.1.5 that $x/\|x\| = y/\|y\|$. Therefore, result holds. ∎

We now present the existence and uniqueness of elements of minimal norm in convex subsets of strictly convex Banach spaces.

Proposition 2.1.8 *Let X be a strictly convex Banach space and C a nonempty convex subset of X. Then there is at most one point x in C such that $\|x\| = inf\{\|z\| : z \in C\}$.*

Proof. Suppose, there exist two points $x, y \in C, x \neq y$ such that

$$\|x\| = \|y\| = \inf\{\|z\| : z \in C\} = d \ (say).$$

If $t \in (0, 1)$, then by strict convexity of X we have that

$$\|tx + (1-t)y\| < d,$$

which is a contradiction, as $tx + (1-y) \in C$ by the convexity of C. ∎

Proposition 2.1.9 *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X. Then there exists a unique point $x \in C$ such that $\|x\| = \inf\{\|z\| : z \in C\}$.*

Proof. *Existence*: Let $d := \inf\{\|z\| : z \in C\}$. Then there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n\| = d$. By the reflexivity of X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to an element x in C. The weak lower semicontinuity (w-lsc) of the norm (see Theorem 1.9.10) gives

$$\|x\| \leq \lim_{n\to\infty} \|x_n\| = d.$$

Therefore, $d = \|x\|$.

Uniqueness: It follows from Proposition 2.1.8 . ∎

The following result has important applications in the existence and uniqueness of best approximations.

Proposition 2.1.10 *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X. Then for $x \in X$, there exists a unique point $z_x \in C$ such that $\|x - z_x\| = d(x, C)$.*

Proof. Let $x \in C$. Because C is a nonempty closed convex subset Banach space X, then $D = \{y - x : y \in C\}$ is a nonempty closed convex subset of X. By Proposition 2.1.9, there exists a unique point $u_x \in D$ such that $\|u_x\| = \inf\{\|y - x\| : y \in C\}$. For $u_x \in D$, there exists a point $z_x \in C$ such that $u_x = z_x - x$. Thus, there exists a unique point $z_x \in C$ such that $\|z_x - x\| = d(x, C)$. ∎

2.2 Uniform convexity

The strict convexity of a normed space X says that the midpoint $(x + y)/2$ of the segment joining two distinct points $x, y \in S_X$ with $\|x - y\| \geq \varepsilon > 0$ does not lie on S_X, i.e.,

$$\left\| \frac{x + y}{2} \right\| < 1.$$

In such spaces, we have no information about $1 - \|(x + y)/2\|$, the distance of midpoints from the unit sphere S_X. A stronger property than strict convexity that provides information about the distance $1 - \|(x+y)/2\|$ is uniform convexity.

Definition 2.2.1 *A Banach space X is said to be uniformly convex*[1] *if for any ε, $0 < \varepsilon \leq 2$, the inequalities $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|(x + y)/2\| \leq 1 - \delta$.*

This says that if x and y are in the closed unit ball $B_X := \{x \in X : \|x\| \leq 1\}$ with $\|x - y\| \geq \varepsilon > 0$, the midpoint of x and y lies inside the unit ball B_X at a distance of at least δ from the unit sphere S_X.

Example 2.2.2 *Every Hilbert space H is a uniformly convex space. In fact, the parallelogram law gives us*

$$\|x + y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x - y\|^2 \text{ for all } x, y \in H.$$

Suppose $x, y \in B_H$ with $x \neq y$ and $\|x - y\| \geq \varepsilon$. Then

$$\|x + y\|^2 \leq 4 - \varepsilon^2,$$

so it follows that

$$\|(x + y)/2\| \leq 1 - \delta(\varepsilon),$$

where $\delta(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}$. Therefore, H is uniformly convex.

Example 2.2.3 *The spaces ℓ_1 and ℓ_∞ are not uniformly convex. To see it, take $x = (1, 0, 0, 0 \cdots), y = (0, -1, 0, 0, \cdots) \in \ell_1$ and $\varepsilon = 1$. Then*

$$\|x\|_1 = 1, \|y\|_1 = 1, \|x - y\|_1 = 2 > 1 = \varepsilon.$$

[1] The concept of uniform convexity was introduced by Clarkson in 1936.

However, $\|(x+y)/2\|_1 = 1$ *and there is no* $\delta > 0$ *such that* $\|(x+y)/2\|_1 \leq 1-\delta$. *Thus,* ℓ_1 *is not uniformly convex.*

Similarly, if we take $x = (1,1,1,0,0,\cdots), y = (1,1,-1,0,0\cdots) \in \ell_\infty$ *and* $\varepsilon = 1$, *then*

$$\|x\|_\infty = 1, \|y\|_\infty = 1, \|x-y\|_\infty = 2 > 1 = \varepsilon.$$

Because $\|(x+y)/2\|_\infty = 1$, ℓ_∞ *is not uniformly convex.*

Observation

- The Banach spaces ℓ_p, ℓ_p^n (whenever n is a nonnegative integer), and $L_p[a,b]$ with $1 < p < \infty$ are uniformly convex.
- The Banach spaces ℓ_1, c, c_0, ℓ_∞, $L_1[a,b]$, $C[a,b]$ and $L_\infty[a,b]$ are not strictly convex.

We derive some consequences from the definition of uniform convexity.

Theorem 2.2.4 *Every uniformly convex Banach space is strictly convex.*

Proof. Let X be a uniformly convex Banach space. It easily follows from Definition 2.2.1 that X is strictly convex. ∎

Remark 2.2.5 *The converse of Theorem 2.2.4 is not true in general. Let* $\beta > 0$ *and let* $X = c_o$ *with the norm* $\|\cdot\|_\beta$ *defined by*

$$\|x\|_\beta = \|x\|_{c_o} + \beta\bigg(\sum_{i=1}^{\infty}\Big(\frac{x_i}{i}\Big)^2\bigg)^{1/2}, \quad x = \{x_i\} \in c_o.$$

The spaces $(c_o, \|\cdot\|_\beta)$ *for* $\beta > 0$ *are strictly convex, but not uniformly convex, while* c_0 *with its usual norm is not strictly convex.*

Theorem 2.2.6 *Let* X *be a uniformly convex Banach space. Then we have the following:*

(a) For any r *and* ε *with* $r \geq \varepsilon > 0$ *and elements* $x, y \in X$ *with* $\|x\| \leq r$, $\|y\| \leq r$, $\|x-y\| \geq \varepsilon$, *there exists a* $\delta = \delta(\varepsilon/r) > 0$ *such that*

$$\|(x+y)/2\| \quad \leq \quad r[1-\delta(\varepsilon/r)].$$

(b) For any r *and* ε *with* $r \geq \varepsilon > 0$ *and elements* $x, y \in X$ *with* $\|x\| \leq r$, $\|y\| \leq r$, $\|x-y\| \geq \varepsilon$, *there exists a* $\delta = \delta(\varepsilon/r) > 0$ *such that*

$$\|tx + (1-t)y\| \leq r[1 - 2\min\{t, 1-t\}\delta(\varepsilon/r)] \text{ for all } t \in (0,1).$$

Proof. (a) Suppose that $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon > 0$. Then we have that

$$\left\|\frac{x}{r}\right\| \leq 1, \left\|\frac{y}{r}\right\| \leq 1 \text{ and } \left\|\frac{x}{r} - \frac{y}{r}\right\| \geq \frac{\varepsilon}{r} > 0.$$

By the definition of uniform convexity, there exists $\delta = \delta(\varepsilon/r) > 0$ such that

$$\left\|\frac{x+y}{2r}\right\| \leq 1-\delta,$$

which yields

$$\left\|\frac{x+y}{2}\right\| \leq r(1-\delta).$$

(b) When $t = 1/2$, we are done by Part (a). If $t \in (0, 1/2]$, we have

$$\|tx + (1-t)y\| = \|t(x+y) + (1-2t)y\| \leq 2t\|\frac{x+y}{2}\| + (1-2t)\|y\|. \quad (2.4)$$

From part (a), there exists a $\delta = \delta(\varepsilon/r) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq r\left[1-\delta\left(\frac{\varepsilon}{r}\right)\right].$$

From (2.4), we have

$$\begin{aligned} \|tx + (1-t)y\| &\leq 2t\left[1-\delta\left(\frac{\varepsilon}{r}\right)\right]r + (1-2t)r \quad (\text{as } \|y\| \leq r) \\ &\leq r\left[1-2t\delta\left(\frac{\varepsilon}{r}\right)\right]. \end{aligned}$$

Now by the choice of $t \in [1/2, 1)$, we have

$$\begin{aligned} \|tx + (1-t)y\| &= \|(2t-1)x + (1-t)(x+y)\| \\ &\leq (2t-1)\|x\| + 2(1-t)\left\|\frac{x+y}{2}\right\| \\ &\leq (2t-1)r + 2(1-t)r\left[1-\delta\left(\frac{\varepsilon}{r}\right)\right] \\ &= r\left[1-2(1-t)\delta\left(\frac{\varepsilon}{r}\right)\right]. \end{aligned}$$

Therefore,

$$\|tx + (1-t)y\| \leq r\left[1-2\min\{t, 1-t\}\delta\left(\frac{\varepsilon}{r}\right)\right]. \quad \blacksquare$$

Theorem 2.2.7 *Let X be a Banach space. Then the following are equivalent:*

(a) X is uniformly convex.

(b) For two sequences $\{x_n\}$ and $\{y_n\}$ in X,

$$\|x_n\| \leq 1, \|y_n\| \leq 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (2.5)$$

Proof. $(a) \Rightarrow (b)$. Suppose X is uniformly convex. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Suppose, for contradiction, that $\lim_{n\to\infty} \|x_n - y_n\| \neq 0$. Then for some $\varepsilon > 0$, there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - y_{n_i}\| \geq \varepsilon.$$

Because X is uniformly convex, there exists $\delta(\varepsilon) > 0$ such that

$$\|x_{n_i} + y_{n_i}\| \leq 2(1 - \delta(\varepsilon)). \tag{2.6}$$

Because $\lim_{n\to\infty} \|x_n + y_n\| = 2$, it follows from (2.6) that

$$2 \leq 2(1 - \delta(\varepsilon)),$$

a contradiction.

$(b) \Rightarrow (a)$. Suppose condition (2.5) is satisfied. If X is not uniformly convex, for $\varepsilon > 0$, there is no $\delta(\varepsilon)$ such that

$$\|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \Rightarrow \|x + y\| \leq 2(1 - \delta(\varepsilon)),$$

and then we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(i) $\|x_n\| \leq 1, \ \|y_n\| \leq 1$,

(ii) $\|x_n + y_n\| \geq 2(1 - 1/n)$,

(iii) $\|x_n - y_n\| \geq \varepsilon$.

Clearly $\|x_n - y_n\| \geq \varepsilon$, which contradicts the hypothesis, as (ii) gives $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Thus, X must be uniformly convex. ∎

For the class of uniform convex Banach spaces, we have the following important results:

Theorem 2.2.8 *Every uniformly convex Banach space is reflexive.*

Proof. Let X be a uniformly convex Banach space. Let $S_{X^*} := \{j \in X^* : \|j\|_* = 1\}$ be the unit sphere in X^* and $f \in S_{X^*}$. Suppose $\{x_n\}$ is a sequence in S_X such that $f(x_n) \to 1$. We show that $\{x_n\}$ is a Cauchy sequence. Suppose, for contradiction, that there exist $\varepsilon > 0$ and two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_i} - x_{n_j}\| \geq \varepsilon$. The uniform convexity of X guarantees that there exists $\delta(\varepsilon) > 0$ such that $\|(x_{n_i} + x_{n_j})/2\| < 1 - \delta$. Observe that

$$|f((x_{n_i} + x_{n_j})/2)| \leq \|f\|_* \|(x_{n_i} + x_{n_j})/2\| < \|f\|_*(1 - \delta) = 1 - \delta$$

and $f(x_n) \to 1$, yield a contradiction. Hence $\{x_n\}$ is a Cauchy sequence and there exists a point x in X such that $x_n \to x$. Clearly, $x \in S_X$. In fact,

$$\|x\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty} \|x_n\| = 1.$$

Using the James theorem (which states that a Banach space is reflexive if and only if for each $f \in S_{X^*}$, there exists $x \in S_X$ such that $f(x) = 1$), we conclude that X is reflexive. ∎

Remark 2.2.9 *Every finite-dimensional Banach space is reflexive, but it need not be uniformly convex, for example,* $X = \mathbb{R}^n, n \geq 2$ *with the norm* $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

Combining Proposition 2.1.9 and Theorems 2.2.4 and 2.2.8, we obtain the following interesting result:

Theorem 2.2.10 *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* X*. Then* C *has a unique element of minimum norm, i.e., there exists a unique element* $x \in C$ *such that* $\|x\| = \inf\{\|z\| : z \in C\}$.

We now introduce a useful property.

Definition 2.2.11 *A Banach space* X *is said to have the Kadec-Klee property if for every sequence* $\{x_n\}$ *in* X *that converges weakly to* x *where also* $\|x_n\| \to \|x\|$*, then* $\{x_n\}$ *converges strongly to* x.

Remark 2.2.12 *In Definition 2.2.11, the sequence* $\{x_n\}$ *can be replaced by the net* $\{x_\alpha\}$ *for the definition of the Kadec property.*

The following result has a very useful property:

Theorem 2.2.13 *Every uniformly convex Banach space has the Kadec-Klee property.*

Proof. Let X be a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$. If $x = 0$, then $\lim_{n\to\infty} \|x_n\| = 0$, which yields that $\lim_{n\to\infty} x_n = 0$.

Suppose $x \neq 0$. Then we show that $x_n \to x$. Suppose, for contradiction, that $\lim_{n\to\infty} x_n \neq x$, i.e., $x_n/\|x_n\| \nrightarrow x/\|x\|$. Then for $\varepsilon > 0$, there exists a subsequence $\{x_{n_i}/\|x_{n_i}\|\}$ of $\{x_n/\|x_n\|\}$ such that

$$\left\| \frac{x_{n_i}}{\|x_{n_i}\|} - \frac{x}{\|x\|} \right\| \geq \varepsilon > 0.$$

Because X is uniformly convex, there exists $\delta(\varepsilon) > 0$ such that

$$\frac{1}{2}\left\| \frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|} \right\| \leq 1 - \delta.$$

Because $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n/\|x_n\| \rightharpoonup x/\|x\|$, it follows that

$$\left\| \frac{x}{\|x\|} \right\| \quad \leq \quad \liminf_{n\to\infty} \frac{1}{2}\left\| \frac{x_{n_i}}{\|x_{n_i}\|} + \frac{x}{\|x\|} \right\| \leq 1 - \delta,$$

a contradiction. Therefore, $\{x_n\}$ converges strongly to $x \in X$. ∎

2.3 Modulus of convexity

Definition 2.3.1 *Let X be a Banach space. Then a function $\delta_X : [0,2] \to [0,1]$ is said to be the modulus of convexity of X if*

$$\delta_X(\varepsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

It is easy to see that $\delta_X(0) = 0$ and $\delta_X(t) \geq 0$ for all $t \geq 0$.

Example 2.3.2 *For the case of a Hilbert space H (see Example 2.2.2),*

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}},\ \varepsilon \in (0,2].$$

We now give the modulus of convexity for ℓ_p $(2 \leq p < \infty)$ spaces. The following result gives an analogue of the parallelogram law in ℓ_p $(2 \leq p < \infty)$ spaces.

Proposition 2.3.3 *In ℓ_p $(2 \leq p < \infty)$ spaces,*

$$\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) \text{ for all } x, y \in \ell_p. \tag{2.7}$$

Proof. We observe from Lemma A.1.1 of Appendix A that for $a, b \in \mathbb{R}$ and $p \in [2, \infty)$

$$\begin{aligned} |a+b|^p + |a-b|^p &\leq [|a+b|^2 + |a-b|^2]^{p/2} \\ &\leq [2|a|^2 + 2|b|^2]^{p/2} \\ &= 2^{p/2}(|a|^2 + |b|^2)^{p/2} \\ &\leq 2^{p/2}2^{(p-2)/2}(|a|^p + |b|^p) \\ &= 2^{p-1}(|a|^p + |b|^p). \end{aligned}$$

Hence for $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty} \in \ell_p$, we have

$$\sum_{i=1}^{\infty} |x_i + y_i|^p + \sum_{i=1}^{\infty} |x_i - y_i|^p \leq 2^{p-1}\left(\sum_{i=1}^{\infty} |x_i|^p + \sum_{i=1}^{\infty} |y_i|^p\right),$$

which implies that points $x, y \in \ell_p$ $(2 \leq p < \infty)$ satisfy the following analogue of the parallelogram law:

$$\|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p). \quad \blacksquare$$

Example 2.3.4 *For the ℓ_p $(2 \leq p < \infty)$ space,*

$$\delta_{\ell_p}(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p},\quad \varepsilon \in (0,2).$$

To see this, let $\varepsilon \in (0,2)$ *and* $x, y \in \ell_p$ *such that* $\|x\| \leq 1$, $\|y\| \leq 1$ *and* $\|x-y\| \geq \varepsilon$. *Then from (2.7), we have*

$$\|x+y\|^p \leq 2^p - \|x-y\|^p,$$

which implies that

$$\begin{aligned}\left\|\frac{x+y}{2}\right\| &\leq \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} = 1-\left[1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right] \\ &\leq 1-\delta_{\ell_p}(\varepsilon),\end{aligned}$$

where $\delta_{\ell_p}(\varepsilon) \geq 1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}$.

Observation

- $\delta_H(\varepsilon) = 1-\sqrt{1-(\varepsilon/2)^2}$.
- $\delta_{\ell_p}(\varepsilon) = 1-(1-(\varepsilon/2)^p)^{p/2}$.
- $\delta_{\ell_p}(\varepsilon)$, the modulus of convexity for ℓ_p $(1 < p \leq 2)$ satisfies the following implicit formula:
$$\left|1-\delta_{\ell_p}(\varepsilon)+\frac{\varepsilon}{2}\right|^p + \left|1-\delta_{\ell_p}(\varepsilon)-\frac{\varepsilon}{2}\right|^p = 2.$$
- $\delta_{\ell_p}(\varepsilon) > 0$ for all $\varepsilon > 0$ $(1 < p < \infty)$.
- $\delta_X(\varepsilon) \leq \delta_H(\varepsilon)$ for any Banach spaces X and any Hilbert space H, i.e., a Hilbert space is the most convex Banach space.

We now give some important properties of the modulus of convexity of Banach spaces.

Theorem 2.3.5 *A Banach space* X *is strictly convex if and only if* $\delta_X(2) = 1$.

Proof. Let X be a strictly convex Banach space with modulus of convexity δ_X. Suppose $\|x\| = \|y\| = 1$ and $\|x-y\| = 2$ with $x \neq -y$. By strict convexity of X, we have

$$1 = \left\|\frac{x-y}{2}\right\| = \left\|\frac{x+(-y)}{2}\right\| < 1,$$

a contradiction. Hence $x = -y$. Therefore, $\delta_X(2) = 1$.

Conversely, suppose $\delta_X(2) = 1$. Let $x, y \in X$ such that $\|x\| = \|y\| = \|(x+y)/2\| = 1$. Then

$$\left\|\frac{x-y}{2}\right\| = \left\|\frac{x+(-y)}{2}\right\| \leq 1-\delta_X(\|x-(-y)\|) = 1-\delta_X(2) = 0,$$

which implies that $x = y$. Thus, $\|x\| = \|y\|$ and $\|x+y\| = 2 = \|x\|+\|y\|$ imply that $x = y$. Therefore, X is strictly convex. ∎

Theorem 2.3.6 *A Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*

Proof. Let X be a uniformly convex Banach space. Then for $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$0 < \delta(\varepsilon) \leq 1 - \left\| \frac{x+y}{2} \right\|$$

for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. Therefore, from the definition of modulus of convexity, we have $\delta_X(\varepsilon) > 0$.

Conversely, suppose X is a Banach space with modulus of convexity δ_X such that $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let $x, y \in X$ such that $\|x\| = 1$, $\|y\| = 1$ with $\|x - y\| \geq \varepsilon$ for fixed $\varepsilon \in (0, 2]$. By the modulus of convexity $\delta_X(\varepsilon)$, we have

$$0 < \delta_X(\varepsilon) \leq 1 - \left\| \frac{x+y}{2} \right\|,$$

which implies that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon),$$

where $\delta(\varepsilon) = \delta_X(\varepsilon)$, which is independent of x and y. Therefore, X is uniformly convex. ∎

Theorem 2.3.7 *Let X be a Banach space with modulus of convexity δ_X. Then we have the following:*

(a) For all ε_1 and ε_2 with $0 \leq \varepsilon_1 < \varepsilon_2 \leq 2$,

$$\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}(1 - \delta_X(\varepsilon_1)) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

In particular, δ_X is a continuous function on $[0, 2)$.

(b) $\delta_X(s)/s$ is a nondecreasing function on $(0, 2]$.

(c) δ_X is a strictly increasing function if X is uniformly convex.

Proof. (a) We define the set

$$S_{u,v} = \{(x, y) : x, y \in B_X; x - y = au, x + y = bv \text{ for some } u, v \in X \text{ and } a, b \geq 0\}$$

and the function

$$\delta_{u,v}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_{u,v}, \|x - y\| \geq \varepsilon \right\}.$$

Note that $\delta_{u,v}(0) = 0$. For given ε_1 and ε_2 in $(0, 2]$ and $\eta > 0$, we can choose (x_i, y_i) in $S_{u,v}$ such that

$$\|x_i - y_i\| \geq \varepsilon_i \text{ and } \delta_{u,v}(\varepsilon_i) + \eta \geq 1 - \left\| \frac{x_i + y_i}{2} \right\| \text{ for } i = 1, 2.$$

Now for $t \in [0,1]$, let $x_3 = tx_1 + (1-t)x_2$ and $y_3 = ty_1 + (1-t)y_2$. Because $x_i, y_i \in B_X$ for $i = 1, 2$, it follows that

$$\|x_3\| \leq t\|x_1\| + (1-t)\|x_2\| \leq 1$$

and

$$\|y_3\| \leq t\|y_1\| + (1-t)\|y_2\| \leq 1.$$

Because $(x_i, y_i) \in S_{u,v}$, there exist positive constants $a_i, b_i \geq 0$ with $i = 1, 2$ such that $x_i - y_i = a_i u$ and $x_i + y_i = b_i v$. Set $\alpha := ta_1 + (1-t)a_2$ and $\beta := tb_1 + (1-t)b_2$. Then

$$\begin{aligned} x_3 - y_3 &= t(x_1 - y_1) + (1-t)(x_2 - y_2) \\ &= ta_1 u + (1-t)a_2 u \\ &= (ta_1 + (1-t)a_2)u \\ &= \alpha u. \end{aligned}$$

Similarly, $x_3 + y_3 = \beta v$. Thus, (x_3, y_3) is in $S_{u,v}$.

Observe that

$$\begin{aligned} \|x_3 - y_3\| &= (ta_1 + (1-t)a_2)\|u\| \\ &= t\|x_1 - y_1\| + (1-t)\|x_2 - y_2\| \\ &\geq t\varepsilon_1 + (1-t)\varepsilon_2 \text{ by the choice of } x_i, y_i, \end{aligned}$$

and $\|x_3 + y_3\| = t\|x_1 + y_1\| + (1-t)\|x_2 + y_2\|$.

By the definition of the function $\delta_{u,v}(\cdot)$, we have

$$\begin{aligned} \delta_{u,v}(t\varepsilon_1 + (1-t)\varepsilon_2) &\leq 1 - \left\|\frac{x_3 + y_3}{2}\right\| \\ &\leq 1 - t\left\|\frac{x_1 + y_1}{2}\right\| - (1-t)\left\|\frac{x_2 + y_2}{2}\right\| \\ &= t\left(1 - \left\|\frac{x_1 + y_1}{2}\right\|\right) + (1-t)\left(1 - \left\|\frac{x_2 + y_2}{2}\right\|\right) \\ &\leq t\left(\delta_{u,v}(\varepsilon_1) + \frac{\eta}{2}\right) + (1-t)\left(\delta_{u,v}(\varepsilon_2) + \frac{\eta}{2}\right) \\ &= t\delta_{u,v}(\varepsilon_1) + (1-t)\delta_{u,v}(\varepsilon_2) + \frac{\eta}{2}. \end{aligned}$$

Because η is arbitrary, it follows that $\delta_{u,v}(\varepsilon)$ is a convex function of ε.

Note that

$$\delta_X(\varepsilon) \leq \delta_{u,v}(\varepsilon) \text{ for all } u, v$$

and

$$(x, y) \in S_{u,v} \text{ with } \|x\| \leq 1 \text{ and } \|y\| \leq 1 \text{ for some } u, v \in X;$$

and hence

$$\delta_X(\varepsilon) = \inf\{\delta_{u,v}(\varepsilon) : u, v \in X \setminus \{0\}\}.$$

Now for any real number $\varepsilon > 0$, there exist $u, v \in X$ such that

$$\delta_{u,v}(\varepsilon_1) \leq \delta_X(\varepsilon_1) + \varepsilon.$$

Hence

$$\begin{aligned} \delta_{u,v}(\varepsilon_2) &= \delta_{u,v}\left(2\,\frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1} + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)\varepsilon_1\right) \\ &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\delta_{u,v}(2) + \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\right)\delta_{u,v}(\varepsilon_1), \end{aligned}$$

which implies that

$$\begin{aligned} \delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}(\delta_{u,v}(2) - \delta_{u,v}(\varepsilon_1)) \\ &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}(1 - \delta_X(\varepsilon_1)). \end{aligned}$$

Then we have

$$\begin{aligned} \delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) &\leq \delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1) + \varepsilon \\ &\leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}(1 - \delta_X(\varepsilon_1)) + \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we have

$$\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}\left(1 - \delta_X(\varepsilon_1)\right).$$

Because $\delta_X(\varepsilon_1) \geq 0$, we have

$$\delta_X(\varepsilon_2) - \delta_X(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1},$$

which implies that $\delta_X(\cdot)$ is continuous on $[0, 2)$.

(b) Fix $s \in (0, 2]$ with $s \leq \varepsilon$ and $x, y \in S_X$ and $\|x - y\| = \varepsilon$.

Set

$$t := \frac{s}{\varepsilon}, u := tx + (1-t)\frac{x+y}{\|x+y\|} \text{ and } v := ty + (1-t)\frac{x+y}{\|x+y\|}.$$

Then

$$u - v = t(x - y), \|u - v\| = s \text{ and } \frac{u+v}{2} = \frac{x+y}{\|x+y\|}\left(\frac{t}{2}\,\|x+y\| + 1 - t\right).$$

Thus,

$$\begin{aligned} \left\|\frac{x+y}{\|x+y\|} - \frac{u+v}{2}\right\| &= t - t\left\|\frac{x+y}{2}\right\| \\ &= 1 - \left(1 - t + t\left\|\frac{x+y}{2}\right\|\right) \\ &= 1 - \left\|\frac{u+v}{2}\right\|. \end{aligned}$$

Observe that

$$\left\| \frac{x+y}{\|x+y\|} - \frac{x+y}{2} \right\| = \left(\frac{1}{\|x+y\|} - \frac{1}{2} \right) \|x+y\| = 1 - \left\| \frac{x+y}{2} \right\|$$

and

$$\begin{aligned} \left\| \frac{x+y}{\|x+y\|} - \frac{u+v}{2} \right\| \Big/ \|u-v\| &= \left(1 - \left\| \frac{u+v}{2} \right\|\right) \Big/ s \\ &= \left(1 - (1-t) - t\left\| \frac{x+y}{2} \right\|\right) \Big/ s \\ &= \left(1 - \left\| \frac{x+y}{2} \right\|\right) \Big/ \|x-y\|. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\delta_X(s)}{s} &\le (1 - \|(u+v)/2\|)/\|u-v\| \\ &= (\|(x+y)/\|x+y\| - (u+v)/2\|)/\|u-v\| = (1 - \|(x+y)/2\|)/\varepsilon. \end{aligned}$$

By taking the infimum over all possible x and y with $\varepsilon = \|x-y\|$ and $x, y \in S_X$, we obtain

$$\frac{\delta_X(s)}{s} \le \frac{\delta_X(\varepsilon)}{\varepsilon}.$$

(c) Observe that

$$\frac{\delta_X(s)}{s} \le \frac{\delta_X(t)}{t} \text{ for } s < t \le 2 \text{ and } \delta_X(t) > 0.$$

Hence

$$t\delta_X(s) \le s\delta_X(t) < t\delta_X(t),$$

which implies that

$$\delta_X(s) < \delta_X(t).$$

Therefore, δ_X is a strictly increasing function. ∎

Remark 2.3.8 *The modulus of convexity δ_X need not be convex on [0,2] and need not be continuous at $t = 2$.*

Theorem 2.3.9 *Let X be a Banach space with modulus of convexity δ_X. Then*

$$\|tx + (1-t)y\| \le 1 - 2\min\{t, 1-t\}\delta_X(\|x-y\|)$$

for all $x, y \in X$ with $\|x\| \le 1, \|y\| \le 1$ and all $t \in [0,1]$.

Proof. The result follows from Theorem 2.2.6(b). ∎

Corollary 2.3.10 *Let X be a Banach space with modulus convexity δ_X. Then*

$$\|(1-t)x + ty\| \le 1 - 2t(1-t)\delta_X(\|x-y\|)$$

for all $x, y \in X$ with $\|x\| \le 1, \|y\| \le 1$ and all $t \in [0,1]$.

Proof. Because $t(1-t) \leq \min\{t,\ 1-t\}$ for all $t \in [0,1]$, the result follows Theorem 2.3.9. $\blacksquare$

Corollary 2.3.11 *Let X be a uniformly convex Banach space with modulus of convexity δ_X. If $r > 0$ and $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$, then*

$$\|tx + (1-t)y\| \leq r\left[1 - 2\min\{t, 1-t\}\delta_X\left(\frac{\|x-y\|}{r}\right)\right] \text{ for all } t \in (0,1).$$

Theorem 2.3.12 *Let X be a uniformly convex Banach space X. Then there exists a strictly increasing continuous convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$2t(1-t)g(\|x-y\|) \leq 1 - \|(1-t)x + ty\|$$

for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and all $t \in [0,1]$.

Proof. Let δ_X be the modulus of convexity of X. Define a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(\lambda) = \begin{cases} \frac{1}{2}\int_0^\lambda \delta_X(s)ds & if \quad 0 \leq \lambda \leq 2, \\ g(2) + \frac{1}{2}\delta_X(2)(\lambda - 2) & if \quad \lambda > 2. \end{cases}$$

For $t \in (0,2]$, we have

$$0 < g(t) = \frac{1}{2}\int_0^t \delta_X(s)ds \leq \frac{t}{2}\,\delta_X(t) \leq \delta_X(t). \qquad (\text{as } \delta_X(s) \leq \delta_X(t))$$

From the definition of g, we have

$$g'(t) = \frac{1}{2}\delta_X(t) \text{ for all } t \in [0,2].$$

Hence g' is increasing with $g'(2) = \delta_X(2)/2 = 1/2$, and it follows that g is convex.

Now, let $\|x\| \leq 1$, $\|y\| \leq 1$ and $t \in [0,1]$. Then, we have (see Corollary 2.3.10)

$$\|(1-t)x + ty\| \leq 1 - 2t(1-t)\delta_X(\|x-y\|). \tag{2.8}$$

Hence from (2.8) we have

$$\begin{aligned} 2t(1-t)g(\|x-y\|) &= t(1-t)\int_0^{\|x-y\|} \delta_X(s)ds \\ &\leq t(1-t)\delta_X(\|x-y\|)\|x-y\| \\ &\leq 2t(1-t)\delta_X(\|x-y\|) \\ &\leq 1 - \|(1-t)x + ty\|. \end{aligned}$$

Moreover, for $rs < 2$, the function $s \mapsto g(rs)/s$ is increasing (as $(g(rs)/s)' = [rs\delta_X(rs)/2 - g(rs)]/s^2 \geq 0$). Therefore, g is a strictly increasing continuous convex function. ∎

Using Corollary 2.3.11, we obtain the following, which has important applications in approximation of fixed points of nonlinear mappings in Banach spaces.

Theorem 2.3.13 *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence of real numbers in (0,1) bounded away from 0 and 1. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq a, \quad \limsup_{n\to\infty} \|y_n\| \leq a \text{ and } \limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$$

for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. The case $a = 0$ is trivial. So, let $a > 0$. Suppose, for contradiction, that $\{x_n - y_n\}$ does not converge to 0. Then there exists a subsequence $\{x_{n_i} - y_{n_i}\}$ of $\{x_n - y_n\}$ such that $\inf_i \|x_{n_i} - y_{n_i}\| > 0$. Note $\{t_n\}$ is bounded away from 0 and 1, and there exist two positive numbers α and β such that $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Because $\limsup_{n\to\infty} \|x_n\| \leq a$ and $\limsup_{n\to\infty} \|y_n\| \leq a$, we may assume an $r \in (a, a+1)$ for a subsequence $\{n_i\}$ such that $\|x_{n_i}\| \leq r$, $\|y_{n_i}\| \leq r$, $a < r$. Choose $r \geq \varepsilon > 0$ such that

$$2\alpha(1-\beta)\delta_X(\varepsilon/r) < 1 \text{ and } \|x_{n_i} - y_{n_i}\| \geq \varepsilon > 0 \text{ for all } i \in \mathbb{N}.$$

From Corollary 2.3.11, we have

$$\begin{aligned} \|t_{n_i}x_{n_i} + (1-t_{n_i})y_{n_i}\| &\leq r[1 - 2t_{n_i}(1-t_{n_i})\delta_X(\varepsilon/r)] \\ &\leq r[1 - 2\alpha(1-\beta)\delta_X(\varepsilon/r)] < a \text{ for all } i \in \mathbb{N}, \end{aligned}$$

which contradicts the hypothesis. ∎

We now present the following intersection theorem:

Theorem 2.3.14 (Intersection theorem) – *Let $\{C_n\}$ be a decreasing sequence of nonempty closed convex bounded subsets of a uniformly convex Banach space X. Then $\cap_{n\in\mathbb{N}} C_n$ is a nonempty closed convex subset of X.*

Proof. Let $z \in X$ be a point such that $z \notin C_1, r_n = d(z, C_n)$ and $r = \lim_{n\to\infty} r_n$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers that decreases to zero. Set

$$\begin{aligned} D_n := B_{r+\varepsilon_n}[z] &= \{x \in C_n : \|z - x\| \leq r + \varepsilon_n\}, \\ d_n : &= diam(D_n), \\ d : &= \lim_{n\to\infty} d_n. \end{aligned}$$

Suppose x and y are two elements in D_n such that $\|x-y\| \geq d_n - \varepsilon_n$. Then Corollary 2.3.11 gives

$$\left\| z - \frac{x+y}{2} \right\| \leq \left(1 - \delta_X\left(\frac{\|x-y\|}{r+\varepsilon_n}\right)\right)(r+\varepsilon_n)$$

and hence

$$r_n \leq \left(1 - \delta_X\left(\frac{d_n-\varepsilon_n}{r+\varepsilon_n}\right)\right)(r+\varepsilon_n).$$

This yields a contradiction unless $d = 0$. This in turn implies that $\cap_{n\in\mathbb{N}} D_n$ is nonempty, and so is $\cap_{n\in\mathbb{N}} C_n$. ∎

Remark 2.3.15 *Theorem 2.3.14 remains valid if the sequence $\{C_n\}$ is replaced by an arbitrary decreasing net of nonempty closed convex bounded subsets of X.*

We now study a weaker type convexity of Banach spaces that is called locally uniform convexity.

Definition 2.3.16 *A Banach space X is said to be locally uniformly convex if for any $\varepsilon > 0$ and $x \in S_X$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that*

$$\|x-y\| \geq \varepsilon \text{ implies that } \left\|\frac{x+y}{2}\right\| \leq 1-\delta \text{ for all } y \in S_X.$$

The *modulus of local convexity* of the Banach space X is

$$\delta_X(x,\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : y \in S_X, \|x-y\| \geq \varepsilon\right\} \text{ for each } x \in S_X \text{ and } 0 < \varepsilon \leq 2.$$

One may easily see that the Banach space X is locally uniformly convex if $\delta_X(x, \varepsilon) > 0$ for all $x \in S_X$ and $\varepsilon > 0$.

Observation

- Every uniformly convex Banach space is locally uniformly convex.
- By Definition 2.3.16, every locally uniformly convex Banach space is strictly convex.

We now give interesting properties of locally uniformly convex Banach spaces:

Proposition 2.3.17 *Let X be a Banach space. Then the following are equivalent:*

(a) X is locally uniformly convex.

(b) Every sequence $\{x_n\}$ in S_X and $x \in S_X$ with $\|x_n + x\| \to 2$ implies that $x_n \to x$.

Proof. $(a) \Rightarrow (b)$. By locally uniformly convexity of X, $\delta_X(x, \varepsilon) > 0$ for all $\varepsilon > 0$. Therefore,

$$1 - \frac{\|x_n + x\|}{2} \to 0 \text{ implies that } \|x_n - x\| \to 0.$$

$(b) \Rightarrow (a)$. Let $\{x_n\}$ be a sequence in S_X such that $\|x_n + x\| \to 2$ implies that $x_n \to x$. Then

$$\|x_n - x\| \geq \varepsilon > 0 \text{ implies that } \left\| \frac{x_n + x}{2} \right\| < 1.$$

Hence, by the definition of modulus of locally uniform convexity, $\delta_X(x, \varepsilon) > 0$. Therefore, X is locally uniformly convex. $\blacksquare$

The following theorem is a generalization of Theorem 2.2.13.

Theorem 2.3.18 *Every locally uniformly convex Banach space has the Kadec-Klee property.*

Proof. Let X be a locally uniformly convex Banach space. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$. For $x = 0$, $\|x_n\| \to 0$ implies that $x_n \to 0$. Suppose $x \neq 0$. Then

$$\frac{x_n}{\|x_n\|} \rightharpoonup \frac{x}{\|x\|} \Rightarrow \frac{x_n}{\|x_n\|} + \frac{x}{\|x\|} \rightharpoonup 2\frac{x}{\|x\|}.$$

By w-lsc of the norm, we have

$$\begin{aligned} 2 = 2\left\|\frac{x}{\|x\|}\right\| &\leq \liminf_{n\to\infty} \left\| \frac{x_n}{\|x_n\|} + \frac{x}{\|x\|} \right\| \\ &\leq \limsup_{n\to\infty} \left(\frac{\|x_n\|}{\|x_n\|} + \frac{\|x\|}{\|x\|} \right) = 2, \end{aligned}$$

which implies that $\|x_n/(\|x_n\|) + x/(\|x\|)\| \to 2$. By Proposition 2.3.17, we conclude that $x_n/\|x_n\| \to x/\|x\|$. Therefore, $x_n \to x$. $\blacksquare$

2.4 Duality mappings

Definition 2.4.1 *Let X^* be the dual of a Banach space X. Then a multivalued mapping $J : X \to 2^{X^*}$ is said to be a (normalized) duality mapping if*

$$Jx = \{j \in X^* : \langle x, j \rangle = \|x\|^2 = \|j\|_*^2\}.$$

Example 2.4.2 *In a Hilbert space H, the normalized duality mapping is the identity. To see this, let $x \in H$ with $x \neq 0$. Note that $H = H^*$ and*

$$\langle x, x \rangle = \|x\| \cdot \|x\| \text{ implies } x \in Jx.$$

Suppose $y \in Jx$. Then by the definition of J, we have $\langle x, y \rangle = \|x\|\|y\|$ and $\|x\| = \|y\|$. Because

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle,$$

it follows that $x = y$. Therefore, $Jx = \{x\}$.

For a complex number, we define the "sign" function by

$$sgn\ \alpha = \begin{cases} 0 & \text{if } \alpha = 0, \\ \alpha/|\alpha| & \text{if } \alpha \neq 0. \end{cases}$$

Observation

- $|sgn\ \alpha| = \begin{cases} 0 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha \neq 0. \end{cases}$
- $\alpha\ sgn\ \overline{\alpha} = \begin{cases} 0 & \text{if } \alpha = 0, \\ \alpha\overline{\alpha}/|\alpha| = |\alpha| & \text{if } \alpha \neq 0. \end{cases}$

Example 2.4.3 *In the ℓ_2 space,*

$$Jx = (|x_1|sgn(x_1), |x_2|sgn(x_2), \cdots, |x_i|sgn(x_i), \cdots), \quad x = \{x_i\} \in \ell_2.$$

Example 2.4.4 *In the $L_2[0,1]$ $(1 < p < \infty)$ space, the duality mapping is given by*

$$Jx = \begin{cases} |x|\ sgn(x)/\|x\|, & \textit{if } \ x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

Before giving fundamental properties of duality mappings, we need the following notations and definitions:

Let $T : X \to 2^{X^*}$ a multivalued mapping. The domain $Dom(T)$, range $R(T)$, inverse T^{-1}, and graph $G(T)$ of T are defined as

$$\begin{aligned} Dom(T) &= \{x \in X : Tx \neq \emptyset\}, \\ R(T) &= \cup_{x \in Dom(T)} Tx, \\ T^{-1}(y) &= \{x \in X : y \in Tx\}, \\ G(T) &= \{(x,y) \in X \times X^* : y \in Tx, x \in Dom(T)\}. \end{aligned}$$

The graph $G(T)$ of T is a subset of $X \times X^*$.

The mapping T is said to be

(i) *monotone* if $\langle x-y, j_x-j_y\rangle \geq 0$ for all $x, y \in Dom(T)$ and $j_x \in Tx, j_y \in Ty$.

(ii) *strictly monotone* if $\langle x-y, j_x-j_y\rangle > 0$ for all $x, y \in Dom(T)$ with $x \neq y$ and $j_x \in Tx, j_y \in Ty$.

(iii) *α-monotone* if there exists a continuous strictly increasing function $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$ and $\alpha(t) \to \infty$ as $t \to \infty$ such that

$$\langle x-y, j_x-j_y\rangle \geq \alpha(\|x-y\|)\|x-y\|$$

for all $x, y \in Dom(T),\ j_x \in Tx, j_y \in Ty$.

(iv) *strongly monotone* if T is α-monotone with $\alpha(t) = kt$ for some constant $k > 0$.

(v) *injective* if $Tx \cap Ty = \emptyset$ for $x \neq y$.

The monotone operator $T : Dom(T) \subset X \to 2^{X^*}$ is said to be *maximal monotone* if it has no proper monotone extensions, i.e., if for $(x, y) \in X \times X^*$

$$\langle x - z, y - j_z\rangle \geq 0 \text{ for all } z \in Dom(T) \text{ and } j_z \in Tz \text{ implies } y \in Tx.$$

The mapping $T : Dom(T) \subset X \to X^*$ is said to be *coercive* on a subset C of $Dom(T)$ if there exists a function $c : (0, \infty) \to [-\infty, \infty]$ with $c(t) \to \infty$ as $t \to \infty$ such that $\langle x, Tx\rangle \geq c(\|x\|)\|x\|$ for all $x \in C$.

In other words, T is coercive on C if $\frac{\langle x,Tx\rangle}{\|x\|} \to \infty$ as $\|x\| \to \infty,\ x \in C$.

Observation

- Every monotonically increasing mapping is monotone.
- If H is a Hilbert space and $T : H \to H$ is nonexpansive, then $I - T$ is monotone.

We are now in a position to establish fundamental properties of duality mappings in Banach spaces.

Proposition 2.4.5 *Let X be a Banach space and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then we have the following:*

(a) $J(0) = \{0\}$.

(b) For each $x \in X, Jx$ is nonempty closed convex and bounded subset of X^*.

(c) $J(\lambda x) = \lambda Jx$ for all $x \in X$ and real λ, i.e., J is homogeneous.

(d) J is multivalued monotone, i.e., $\langle x - y, j_x - j_y\rangle \geq 0$ for all $x, y \in X$, $j_x \in Jx$ and $j_y \in J(y)$.

(e) $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle$ for all $x, y \in X$ and $j \in Jy$.

(f) If X^* is strictly convex, J is single-valued.

(g) If X is strictly convex, then J is one-one, i.e., $x \neq y \Rightarrow Jx \cap Jy = \emptyset$.

(h) If X is reflexive with strictly convex dual X^*, then J is demicontinuous.

(i) If X is uniformly convex, then for $x, y \in B_r[0]$, $j_x \in Jx$, $j_y \in Jy$

$$\langle x - y, j_x - j_y\rangle \geq w_r(\|x - y\|)\|x - y\|,$$

where $w_r : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfies the conditions:

$$w_r(0) = 0, w_r(t) > 0 \textit{ for all } t > 0 \textit{ and } t \leq s \Rightarrow w_r(t) \leq w_r(s).$$

Proof. (a) It is obvious.

(b) If $x = 0$, we are done by Part(a). If x is a nonzero element in X, then by the Hahn-Banach theorem (see Corollary 1.6.6), there exists $f \in X^*$ such that $\langle x, f\rangle = \|x\|$ and $\|f\|_* = 1$. Set $j := \|x\|f$. Then $\langle x, j\rangle = \|x\|\langle x, f\rangle = \|x\|^2$ and $\|j\|_* = \|x\|$, and it follows that Jx is nonempty for each $x \neq 0$.

Now suppose $f_1, f_2 \in Jx$ and $t \in (0, 1)$. Because

$$\langle x, f_1\rangle = \|x\|\|f_1\|_*, \|x\| = \|f_1\|_*$$

and

$$\langle x, f_2 \rangle = \|x\|\|f_2\|_*, \|x\| = \|f_2\|_*,$$

we obtain

$$\langle x, tf_1 + (1-t)f_2 \rangle = \|x\|(t\|f_1\|_* + (1-t)\|f_2\|_*) = \|x\|^2.$$

Observe that

$$\begin{aligned}\langle x, tf_1 + (1-t)f_2 \rangle &\leq \|tf_1 + (1-t)f_2\|_*\|x\| \\ &\leq (t\|f_1\|_* + (1-t)\|f_2\|_*)\|x\| \\ &= \|x\|^2.\end{aligned}$$

Then

$$\|x\|^2 \leq \|x\|\|tf_1 + (1-t)f_2\|_* \leq \|x\|^2,$$

which gives us

$$\|x\|^2 = \|x\|\|tf_1 + (1-t)f_2\|_*,$$

i.e.,

$$\|tf_1 + (1-t)f_2\|_* = \|x\|.$$

Thus,

$$\langle x, tf_1 + (1-t)f_2 \rangle = \|x\|\|tf_1 + (1-t)f_2\|_* \text{ and } \|x\| = \|tf_1 + (1-t)f_2\|_*,$$

and this means that $tf_1 + (1-t)f_2 \in Jx$, i.e., Jx is a convex set.

Similarly, one can show that Jx is a closed and bounded set in X^*.

(c) For $\lambda = 0$, it is obvious that $J(0x) = 0Jx$. Assume that $j \in J(\lambda x)$ for $\lambda \neq 0$. First, we show that $J(\lambda x) \subseteq \lambda Jx$. Because $j \in J(\lambda x)$, we have

$$\langle \lambda x, j \rangle = \|\lambda x\|\|j\|_* \text{ and } \|\lambda x\| = \|j\|_*,$$

and it follows that $\langle \lambda x, j \rangle = \|j\|_*^2$. Hence

$$\langle x, \lambda^{-1} j \rangle = \lambda^{-1}\langle \lambda x, \lambda^{-1} j \rangle = \lambda^{-2}\langle \lambda x, j \rangle = \lambda^{-2}\|\lambda x\|\|j\|_* = \|\lambda^{-1} j\|_*^2 = \|x\|^2.$$

This shows that $\lambda^{-1} j \in Jx$, i.e., $j \in \lambda Jx$. Thus, we have $J(\lambda x) \subseteq \lambda Jx$. Similarly, one can show that $\lambda Jx \subseteq J(\lambda x)$. Therefore, $J(\lambda x) = \lambda Jx$.

(d) Let $j_x \in Jx$ and $j_y \in Jy$ for $x,y \in X$. Hence

$$\begin{aligned}\langle x - y, j_x - j_y \rangle &= \langle x, j_x \rangle - \langle x, j_y \rangle - \langle y, j_x \rangle + \langle y, j_y \rangle \\ &\geq \|x\|^2 + \|y\|^2 - \|x\|\|j_y\|_* - \|y\|\|j_x\|_* \\ &\geq \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \\ &= (\|x\| - \|y\|)^2 \geq 0. \qquad (2.9)\end{aligned}$$

(e) Let $j \in Jx$, $x \in X$. Then

$$\begin{aligned}\|y\|^2 &- \|x\|^2 - 2\langle y - x, j \rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle y, j \rangle \\ &\geq \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \\ &= (\|x\| - \|y\|)^2 \geq 0. \qquad (2.10)\end{aligned}$$

(f) Let $j_1, j_2 \in Jx$ for $x \in X$. Then

$$\langle x, j_1 \rangle = \|j_1\|_*^2 = \|x\|^2$$

and

$$\langle x, j_2 \rangle = \|j_2\|_*^2 = \|x\|^2.$$

Adding the above identities, we have

$$\langle x, j_1 + j_2 \rangle = 2\|x\|^2.$$

Because

$$2\|x\|^2 = \langle x, j_1 + j_2 \rangle \leq \|x\|\|j_1 + j_2\|_*,$$

this implies that

$$\|j_1\|_* + \|j_2\|_* = 2\|x\| \leq \|j_1 + j_2\|_*.$$

It now follows from the fact $\|j_1 + j_2\|_* \leq \|j_1\|_* + \|j_2\|_*$ that

$$\|j_1 + j_2\|_* = \|j_1\|_* + \|j_2\|_*.$$

Because X^* is strictly convex and $\|j_1 + j_2\|_* = \|j_1\|_* + \|j_2\|_*$, then there exists $\lambda \in \mathbb{R}$ such that $j_1 = \lambda j_2$. Because

$$\langle x, j_2 \rangle = \langle x, j_1 \rangle = \langle x, \lambda j_2 \rangle = \lambda \langle x, j_2 \rangle,$$

this implies that $\lambda = 1$ and hence $j_1 = j_2$. Therefore, J is single-valued.

(g) Suppose that $j \in Jx \cap Jy$ for $x, y \in X$. Because $j \in Jx$ and $j \in Jy$, it follows from $\|j\|_*^2 = \|x\|^2 = \|y\|^2 = \langle x, j \rangle = \langle y, j \rangle$ that

$$\|x\|^2 = \langle (x+y)/2, j \rangle \leq \|(x+y)/2\|\|x\|,$$

which gives that

$$\|x\| = \|y\| \leq \|(x+y)/2\| \leq \|x\|.$$

Hence $\|x\| = \|y\| = \|(x+y)/2\|$. Because X is strictly convex and $\|x\| = \|y\| = \|(x+y)/2\|$, we have $x = y$. Therefore, J is one-one.

(h) It suffices to prove demicontinuity of J on the unit sphere S_X. For this, let $\{x_n\}$ be a sequence in S_X such that $x_n \to z$ in X. Then $\|Jx_n\|_* = \|x_n\| = 1$ for all $n \in \mathbb{N}$, i.e., $\{Jx_n\}$ is bounded. Because X is reflexive and hence X^* is also reflexive. Then there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ in X^* such that $\{Jx_{n_k}\}$ converges weakly to some j in X^*. Because $x_{n_k} \to z$ and $Jx_{n_k} \rightharpoonup j$, then we have

$$\langle z, j \rangle = \lim_{k \to \infty} \langle x_{n_k}, Jx_{n_k} \rangle = \lim_{k \to \infty} \|x_{n_k}\|^2 = 1.$$

Moreover,

$$\begin{aligned} \|j\|_* &\leq \lim_{k \to \infty} \|Jx_{n_k}\|_* = \lim_{k \to \infty} (\|Jx_{n_k}\|_* \, \|x_{n_k}\|) \\ &= \lim_{k \to \infty} \langle x_{n_k}, Jx_{n_k} \rangle = \langle z, j \rangle = \|j\|_*. \end{aligned}$$

This shows that

$$\langle z, j\rangle = \|j\|_*\|z\| \text{ and } \|j\|_* = \|z\|.$$

This implies that $j = Jz$. Thus, every subsequence $\{Jx_{n_i}\}$ converging weakly to $j \in X^*$. This gives $Jx_n \rightharpoonup Jz$. Therefore, J is demicontinuous.

(i) Let $r > 0$ and $w_r : \mathbb{R}^+ \to \mathbb{R}^+$ a function defined by

$$\begin{cases} w_r(0) & = 0; \\ w_r(t) & = \inf\{\frac{\langle x-y, j_x - j_y\rangle}{\|x-y\|} : x, y \in B_r[0], \|x-y\| \geq t, j_x \in Jx, j_y \in Jy\} \\ & \quad \text{if } t \in (0, 2r]; \\ w_r(t) & = w_r(2r); \text{if } t \in (2r, \infty). \end{cases}$$

By (d), we have

$$\langle x - y, j_x - j_y\rangle \geq 0,$$

and it follows that $w_r(t) \geq 0$ for all $t \in \mathbb{R}^+$. It can be readily seen that w_r is nondecreasing. So it remains to prove that $w_r(t) > 0$ for all $t > 0$.

Suppose, for contradiction, that there exists $\lambda \in (0, 2r]$ such that $w_r(\lambda) = 0$. Then there exist sequences $\{x_n\}, \{y_n\}$ in $B_r[0]$ such that

$$\|x_n - y_n\| \geq \lambda > 0 \text{ and } \langle x_n - y_n, j_{x_n} - j_{y_n}\rangle \to 0,$$

where $j_{x_n} \in Jx_n, j_{y_n} \in Jy_n$. We know from (2.9) that

$$(\|x_n\| - \|y_n\|)^2 \leq \langle x_n - y_n, j_{x_n} - j_{y_n}\rangle.$$

We may assume that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = a > 0 \text{ (say)}.$$

Notice

$$\begin{aligned} \langle x_n + y_n, j_{x_n} + j_{y_n}\rangle &= 2\|x_n\|^2 + 2\|y_n\|^2 - \langle x_n - y_n, j_{x_n} - j_{y_n}\rangle \\ &\to 4a^2 \end{aligned} \tag{2.11}$$

and

$$\limsup_{n\to\infty} \|x_n + y_n\| \leq \limsup_{n\to\infty}(\|x_n\| + \|y_n\|) = 2a.$$

Moreover, from (2.11), we have

$$\begin{aligned} 4a^2 &= \lim_{n\to\infty} \langle x_n + y_n, j_{x_n} + j_{y_n}\rangle \\ &\leq \liminf_{n\to\infty} \|x_n + y_n\|(\|x_n\| + \|y_n\|) = 2a \liminf_{n\to\infty} \|x_n + y_n\|, \end{aligned}$$

which implies that

$$2a \leq \liminf_{n\to\infty} \|x_n + y_n\|.$$

Thus, we have that $\lim_{n\to\infty} \|x_n + y_n\| = 2a$. By the uniform convexity of X (see Theorem 2.3.13), we obtain that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, which contradicts our assumption that $\|x_n - y_n\| \geq \lambda > 0$. ∎

The inequalities given in the following results are very useful in many applications.

Proposition 2.4.6 *Let X be a Banach space and $J : X \to 2^{X^*}$ the duality mapping. Then we have the following:*

(a) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j_x\rangle$ *for all* $x, y \in X$, *where* $j_x \in Jx$.

(b) $\|x+y\|^2 \leq \|y\|^2 + 2\langle x, j_{x+y}\rangle$ *for all* $x, y \in X$, *where* $j_{x+y} \in J(x+y)$.

Proof. (a) Replacing y by $x+y$ in (2.10), we get the inequality.

(b) Replacing x by $x+y$ in (2.10), we get the result. ∎

Proposition 2.4.7 *Let X be a Banach and $J : X \to 2^{X^*}$ a normalized duality mapping. Then for $x, y \in X$, the following are equivalent:*

(a) $\|x\| \leq \|x+ty\|$ *for all* $t > 0$.

(b) There exists $j \in Jx$ *such that* $\langle y, j\rangle \geq 0$.

Proof. (a) $\Rightarrow$ (b). For $t > 0$, let $f_t \in J(x+ty)$ and define $g_t = f_t/\|f_t\|_*$. Hence $\|g_t\|_* = 1$. Because $g_t \in \|f_t\|_*^{-1} J(x+ty)$, it follows that

$$\begin{aligned}\|x\| &\leq \|x+ty\| = \|f_t\|_*^{-1}\langle x+ty, f_t\rangle \\ &= \langle x+ty, g_t\rangle = \langle x, g_t\rangle + t\langle y, g_t\rangle \\ &\leq \|x\| + t\langle y, g_t\rangle. \quad (\text{as } \|g_t\|_* = 1)\end{aligned}$$

By the Banach-Alaoglu theorem (which states that the unit ball in X^* is weak*ly-compact), the net $\{g_t\}$ has a limit point $g \in X^*$ such that

$$\|g\|_* \leq 1, \ \langle x, g\rangle \geq \|x\| \text{ and } \langle y, g\rangle \geq 0.$$

Observe that

$$\|x\| \leq \langle x, g\rangle \leq \|x\|\|g\|_* = \|x\|,$$

which gives that

$$\langle x, g\rangle = \|x\| \text{ and } \|g\|_* = 1.$$

Set $j = g\|x\|$, then $j \in Jx$ and $\langle y, j\rangle \geq 0$.

(b) $\Rightarrow$ (a). Suppose for $x, y \in X$ with $x \neq 0$ there exists $j \in Jx$ such that $\langle y, j\rangle \geq 0$. Hence for $t > 0$,

$$\|x\|^2 = \langle x, j\rangle \leq \langle x, j\rangle + \langle ty, j\rangle = \langle x+ty, j\rangle \leq \|x+ty\|\|x\|,$$

which implies that

$$\|x\| \leq \|x+ty\|. \quad ∎$$

Observation

- *$Dom(J) = X$.*
- J is odd, i.e., $J(-x) = -Jx$.
- J is homogeneous (hence J is positive homogeneous, i.e., $J(\lambda x) = \lambda Jx$ for all $\lambda > 0$).
- J is bounded.

We now consider the duality mappings that are more general than the normalized duality mappings.

Definition 2.4.8 *A continuous strictly increasing function* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be gauge function if* $\mu(0) = 0$ *and* $\lim\limits_{t\to\infty} \mu(t) = \infty$.

Definition 2.4.9 *Let* X *be a normed space and* μ *a gauge function. Then the mapping* $J_\mu : X \to 2^{X^*}$ *defined by*

$$J_\mu(x) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \mu(\|x\|)\}, \quad x \in X$$

is called the duality mapping with gauge function μ.

In the particular case $\mu(t) = t$, the duality mapping $J_\mu = J$ is called the normalized duality mapping .

In the case $\mu(t) = t^{p-1}$, $p > 1$, the duality mapping $J_\mu = J_p$ is called *the generalized duality mapping* and it is given by

$$J_p(x) := \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|^{p-1}\}, \quad x \in X.$$

Note that if $p = 2$, then $J_p = J_2 = J$ is the normalized duality mapping.

Remark 2.4.10 *For the gauge function* μ, *the function* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$\Phi(t) = \int_0^t \mu(s)ds$$

is a continuous convex strictly increasing function on $\mathbb{R}^+$. *Therefore,* Φ *has a continuous inverse function* Φ^{-1}.

Example 2.4.11 *Let* $x = (x_1, x_2, \cdots) \in \ell_p \ \ (1 < p < \infty)$, *set*

$$J_\mu(x) = (|x_1|^{p-1}sgn(x_1), |x_2|^{p-1}sgn(x_2), \cdots)$$

and let $\mu(t) = t^{p-1} = t^{p/q}$, *where* $1/p + 1/q = 1$. *Observe that*

$$\left(\sum_{i=1}^{\infty} |x_i|^{(p-1)q}\right)^{1/q} = \left(\sum_{i=1}^{\infty} |x_i|^{p}\right)^{1/q} \text{ and } J_\mu(x) \in \ell_q.$$

Moreover,

$$\mu(\|x\|) = \|x\|^{p/q} = \|J_\mu(x)\|_*$$

and

$$\begin{aligned}\langle x, J_\mu(x)\rangle &= \sum_{i=1}^{\infty} x_i|x_i|^{p-1}sgn(x_i) = \sum_{i=1}^{\infty} |x_i|^p = \|x\|^p \\ &= \|x\|\|x\|^{p-1} = \|x\|\mu(\|x\|) = \|x\|\|J_\mu(x)\|_*.\end{aligned}$$

Thus, J_μ *is a duality mapping with gauge function* μ. *Therefore, the generalized duality mapping* J_p *in* ℓ_p *space is given by*

$$J_p(x) = (|x_1|^{p-1}sgn(x_1), |x_2|^{p-1}sgn(x_2), \cdots), \quad x \in \ell_p.$$

One can easily see the following facts:

(i) $J_\mu(x)$ is a nonempty closed convex set in X^* for each $x \in X$,

(ii) J_μ is a function when X^* is strictly convex.

(iii) If $J_\mu(x)$ is single-valued, then

$$J_\mu(\lambda x) = \frac{sign(\lambda)\mu(\|\lambda x\|)}{\mu(\|x\|)} J_\mu(x) \text{ for all } x \in X \text{ and } \lambda \in \mathbb{R}$$

and

$$\langle x - y, J_\mu(x) - J_\mu(y)\rangle \geq (\mu(\|x\|) - \mu(\|y\|))(\|x\| - \|y\|) \text{ for all } x, y \in X.$$

We now give other interesting properties of the duality mappings J_μ in reflexive Banach spaces.

Theorem 2.4.12 *Let X be a Banach space and J_μ a duality mapping with gauge function μ. Then X is reflexive if and only if $\bigcup_{x\in X} J_\mu(x) = X^*$, i.e., J_μ is onto.*

Proof. Let X be reflexive and let $j \in X^*$. By the Hahn-Banach theorem, there is an $x \in S_X$ such that $\langle x, j\rangle = \|x\|$.

Because μ has the property of Darboux, there exists a constant $t \geq 0$ such that

$$\mu(||tx||) = \mu(t) = \|j\|_*.$$

Because $\langle tx, j\rangle = \|tx\|\|j\|_*$, it follows that $j \in J_\mu(tx)$.

Conversely, suppose that for each $j \in X^*$, there is $x \in X$ such that $j \in J_\mu(x)$. Set $y := x/\|x\|$. Then $\|y\| = 1$ and $\langle y, j\rangle = \|j\|_*$. Hence each continuous functional attains its supremum on the unit ball. By the James theorem, X is reflexive. ∎

Theorem 2.4.13 *Let X be a reflexive Banach space and J a duality mapping with gauge function μ. Then J^{-1} is the duality mapping with gauge μ^{-1}.*

Proof. From Theorem 2.4.12, we obtain

$$J^{-1}(j) = \{x \in X : j \in J_\mu(x)\} \neq \emptyset \text{ for all } j \in X^*.$$

Let J^* be the duality mapping on X^* with gauge μ^{-1}. Observe that $x \in J^{-1}(j)$ if and only if $\langle x, j\rangle = \|x\|\|j\|_*$ and $\|x\| = \mu^{-1}(\|j\|_*)$ or equivalently if and only if $x \in J^*(j)$. Thus,

$$J^*(j) = J^{-1}(j) = \{x \in X : \langle x, j\rangle = \|x\|\|j\|_*, \|x\| = \mu^{-1}(\|j\|_*)\}. \quad ∎$$

Corollary 2.4.14 *Let X be a reflexive Banach space and $J^* : X^* \to X$ the inverse of the normalized duality mapping $J : X \to X^*$. Then*

$$J^*J = I \text{ and } JJ^* = I^* \quad (\text{identity mappings on } X \text{ and } X^*, \text{ respectively}).$$

Theorem 2.4.15 *Let X be a Banach space and let J_μ be the duality mapping with gauge function μ. If X^* is uniformly convex, then J_μ is uniformly continuous on each bounded set in X, i.e., for $\varepsilon > 0$ and $K > 0$, there is a $\delta > 0$ such that*

$$\|x\| \le K, \|y\| \le K \text{ and } \|x-y\| < \delta \Rightarrow \|J_\mu(x) - J_\mu(y)\|_* < \varepsilon.$$

Proof. Because X^* is strictly convex, J_μ is single-valued. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\|x_n\| \le K, \|y_n\| \le K$ and $\|x_n - y_n\| \to 0$.

Assume that $x_n \to 0$, then $y_n \to 0$. Moreover,

$$\|J_\mu(x_n)\|_* = \mu(\|x_n\|) \to 0 \quad \text{and} \quad \|J_\mu(y_n)\|_* = \mu(\|y_n\|) \to 0.$$

Hence $\|J_\mu(x_n) - J_\mu(y_n)\|_* \to 0$ and we are done.

Suppose $\{x_n\}$ does not converge strongly to zero. There exist $\alpha > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k}\| \ge \alpha$. Because $\|x_n - y_n\| \to 0$, one can assume that $\|y_{n_k}\| \ge \alpha/2$. Without loss of generality, we may assume that

$$\|x_n\| \ge \beta \text{ and } \|y_n\| \ge \beta \text{ for some } \beta > 0.$$

Set $u_n := x_n/\|x_n\|$ and $v_n := y_n/\|y_n\|$ so that $\|u_n\| = \|v_n\| = 1$ and

$$\begin{aligned}
\|u_n - v_n\| &= \left\| \frac{x_n\|y_n\| - \|x_n\|y_n}{\|x_n\|\|y_n\|} \right\| \\
&\le \frac{1}{\beta^2} \Big\| x_n\|y_n\| - x_n\|x_n\| + x_n\|x_n\| - \|x_n\|y_n \Big\| \\
&\le \frac{1}{\beta^2} \left(\Big| \|y_n\| - \|x_n\| \Big| \|x_n\| + \|x_n\|\|x_n - y_n\| \right) \\
&\le \frac{1}{\beta^2} (\|y_n - x_n\|K + \|x_n - y_n\|K) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Because $\|J_\mu(u_n)\|_* = \mu(\|u_n\|) = \mu(1)$ and $\|J_\mu(v_n)\|_* = \mu(\|v_n\|) = \mu(1)$, we have

$$\begin{aligned}
\mu(1) + \mu(1) - \mu(1)\|u_n - v_n\| &\le \langle u_n, J_\mu(u_n)\rangle + \langle v_n, J_\mu(v_n)\rangle + \langle u_n - v_n, J_\mu(v_n)\rangle \\
&= \langle u_n, J_\mu(u_n)\rangle + \langle u_n, J_\mu(v_n)\rangle \\
&= \langle u_n, J_\mu(u_n) + J_\mu(v_n)\rangle \\
&\le \|J_\mu(u_n) + J_\mu(v_n)\|_* \le 2\mu(1).
\end{aligned}$$

This shows that $\lim_{n\to\infty} \|J_\mu(u_n) + J_\mu(v_n)\|_* = 2\mu(1)$. Because X^* is uniformly convex, we have $\|J_\mu(u_n) - J_\mu(v_n)\|_* \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}
&J_\mu(x_n) - J_\mu(y_n) \\
&\quad = [\mu(\|x_n\|)(J_\mu(u_n) - J_\mu(v_n)) + (\mu(\|x_n\|) - \mu(\|y_n\|))J_\mu(v_n)]/\mu(1),
\end{aligned}$$

and it follows that $\|J_\mu(x_n) - J_\mu(y_n)\|_* \to 0$ as $n \to \infty$. ∎

Observation

- If $J_\mu : X \to 2^{X^*}$ is a duality mapping with gauge function μ then
  (i) J_μ is norm to weak* upper semicontinuous.
  (ii) for each $x \in X$, the set $J_\mu(x)$ is convex and weakly closed in X^*;
  (iii) $J_\mu(-x) = -J_\mu(x)$ and $J_\mu(\lambda x) = \frac{\mu(\|\lambda x\|)}{\mu(\|x\|)} J_\mu(x)$ for all $x \in X, \lambda > 0$;
  (iv) each selection of J_μ is a homogeneous single-valued mapping $j : X \to X^*$ satisfying $j(x) \in J_\mu(x)$ for all $x \in X$,
  (v) J_μ is monotone, i.e., $\langle x - y, j_x - j_y\rangle \geq 0$ for all $x, y \in X$ and $j_x \in J_\mu(x)$, $j_y \in J_\mu(y)$;
  (vi) the strict convexity of X implies that J_μ is strictly monotone, i.e.,
  $$\langle x - y, j_x - j_y\rangle > 0 \text{ for all } x, y \in X \text{ and } j_x \in J_\mu(x), j_y \in J_\mu(y);$$
  (vii) the reflexivity of X and strict convexity of X^* imply that J_μ is single-valued monotone and demicontinuous.

One can easily see that the following are reflexive Kadec-Klee Banach spaces:
(a) a Banach space of finite-dimension,
(b) a reflexive Banach space that is locally uniformly convex,
(c) a uniformly convex Banach space.

We now conclude this section with an interesting result concerning a Banach space whose dual has the Kadec-Klee property.

Theorem 2.4.16 *Let X be a reflexive Banach space such that X^* has the Kadec-Klee property. Let $\{x_\alpha\}_{\alpha\in D}$ be a bounded net in X and $x, y \in \omega_w(\{x_\alpha\}_{\alpha\in D})$. Suppose $\lim\limits_{\alpha\in D} \|tx_\alpha + (1-t)x - y\|$ exists for all $t \in [0, 1]$. Then $x = y$.*

Proof. Because $\lim\limits_{\alpha\in D} \|tx_\alpha + (1-t)x - y\|$ exists (say, r), for each $\varepsilon > 0$, there exists $\alpha_0 \in D$ such that

$$\|tx_\alpha + (1-t)x - y\| \leq r + \varepsilon \text{ for all } \alpha \succeq \alpha_0.$$

It follows that for all $\alpha \succeq \alpha_0$ and $j(x - y) \in J(x - y)$,

$$\langle tx_\alpha + (1-t)x - y, j(x - y)\rangle \leq (r + \varepsilon)\|x - y\|.$$

Because $x \in \omega_w(\{x_\alpha\}_{\alpha\in D})$, we obtain

$$\begin{aligned} \|x - y\|^2 &= \langle tx + (1-t)x - y, j(x - y)\rangle \\ &\leq \|x - y\|(\lim_{\alpha\in D} \|tx_\alpha + (1-t)x - y\| + \varepsilon), \\ &= (r + \varepsilon)\|x - y\|. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$\|x - y\| \leq r. \tag{2.12}$$

By Proposition 2.4.6 (b), we have

$$\|tx_\alpha + (1-t)x - y\|^2 \le \|x-y\|^2 + 2t\langle x_\alpha - x, j(tx_\alpha + (1-t)x - y)\rangle$$

for all $t \in (0,1]$ and $j(tx_\alpha + (1-t)x - y) \in J(tx_\alpha + (1-t)x - y)$. By (2.12), we have

$$\liminf_{\alpha \in D} \langle x_\alpha - x, j(tx_\alpha + (1-t)x - y)\rangle \ge 0.$$

Hence there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_n \succeq \alpha_m$ for $n \ge m$ and

$$\left\langle x_\alpha - x, j\left(\frac{1}{n}x_\alpha + \left(1 - \frac{1}{n}\right)x - y\right)\right\rangle \ge -\frac{1}{n} \text{ for all } n \in \mathbb{N} \text{ and } \alpha \succeq \alpha_n. \quad (2.13)$$

Set $D_1 = \{\alpha : \alpha \succeq \alpha_1\}$. Without loss of generality, we may assume that $D = D_1$,

$$\omega_w(\{x_\alpha\}_{\alpha\in D}) = \omega_w\{x_\alpha\}_{\alpha\in D_1}$$

and

$$\lim_{\alpha\in D} \|tx_\alpha + (1-t)x - y\| = \lim_{\alpha \in D_1} \|tx_\alpha + (1-t)x - y\| \text{ for all } t \in [0,1].$$

Set $t_\alpha = \inf\{1/n : \alpha \succeq \alpha_n\}$ for all $\alpha \in D$.

We now consider two cases:

Case 1. $\alpha \in D$ *and* $t_\alpha > 0$.

Set $j_\alpha := j(t_\alpha x_\alpha + (1-t_\alpha)x - y)$. Then

$$\langle x - y, j_\alpha\rangle = \|t_\alpha x_\alpha + (1-t_\alpha)x - y\|^2 - t_\alpha\langle x_\alpha - x, j_\alpha\rangle \quad (2.14)$$

and

$$\|j_\alpha\| = \|t_\alpha x_\alpha + (1-t_\alpha)x - y\|. \quad (2.15)$$

By (2.13), we have

$$\langle x_\alpha - x, j_\alpha\rangle \ge -t_\alpha. \quad (2.16)$$

Case 2. $\alpha \in D$ *and* $t_\alpha = 0$.

In this case, we can choose a subsequence $\{j((1/n_k)x_\alpha + (1-1/n_k)x - y)\}_{k\in\mathbb{N}}$ which is weakly convergent to j, and set $j_\alpha := j$. It follows from (2.13) that

$$\langle x_\alpha - x, j_\alpha\rangle \ge 0. \quad (2.17)$$

Observe that

$$\begin{aligned} \|j_\alpha\| &\le \liminf_{k\to\infty} \left\| j\left(\frac{1}{n_k}x_\alpha + \left(1 - \frac{1}{n_k}\right)x - y\right)\right\| \\ &= \lim_{k\to\infty} \left\| \frac{1}{n_k}x_\alpha + \left(1 - \frac{1}{n_k}\right)x - y\right\| = \|x - y\|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle x-y, j_\alpha\rangle &= \lim_{k\to\infty}\left\langle x-y, j\Big(\frac{1}{n_k}x_\alpha+\Big(1-\frac{1}{n_k}\Big)x-y\Big)\right\rangle\\ &= \lim_{k\to\infty}\Big(\|\frac{1}{n_k}x_\alpha+\Big(1-\frac{1}{n_k}\Big)x-y\|^2\\ &\quad -\frac{1}{n_k}\left\langle x_\alpha-x, j\Big(\frac{1}{n_k}x_\alpha+\Big(1-\frac{1}{n_k}\Big)x-y\Big)\right\rangle\Big)\\ &= \|x-y\|^2. \end{aligned} \tag{2.18}$$

Therefore,

$$\|j_\alpha\| = \|x-y\| \tag{2.19}$$

and $j_\alpha \in J(x-y)$.

We note that by the Kadec-Klee property of X^*, the sequence $\{j((1/n_k)x_\alpha+(1-1/n_k)x-y)\}_{k\in\mathbb{N}}$ converges strongly to j_α.

Now from the net $\{x_\alpha\}_{\alpha\in D}$, we choose a subset $\{\alpha_\beta\}_{\beta\in\overline{D}}$ such that $\{x_{\alpha_\beta}\}_{\beta\in\overline{D}}$ converges weakly to $y\in w_w(\{x_\alpha\}_{\alpha\in D})$ and $\{j_{\alpha_\beta}\}_{\beta\in\overline{D}}$ converges weakly to $\overline{j}$. Then by (2.15) and (2.19) we get

$$\|\overline{j}\|_* \le \|x-y\|$$

and by (2.14) and (2.18), we get

$$\langle x-y, \overline{j}\rangle = \|x-y\|^2.$$

Hence $\overline{j}\in J(x-y)$. Because X is reflexive and X^* has the Kadec-Klee property, the space X^* has also the Kadec property and this implies that $\{j_{\alpha_\beta}\}_{\beta\in\overline{D}}$ converges strongly to $\overline{j}$. It follows from (2.16) and (2.17) that

$$\langle y-x, \overline{j}\rangle \ge 0,$$

i.e., $\|x-y\|^2\le 0$. Therefore, $x=y$. ∎

Corollary 2.4.17 *Let X be a reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $p,q\in\omega_w(\{x_n\})$. Suppose $\lim_{n\to\infty}\|tx_n+(1-t)p-q\|$ exists for all $t\in[0,1]$. Then $p=q$.*

2.5 Convex functions

Let X be a linear space and $f: X\to(-\infty,\infty]$ a function. Then

(i) f is said to be *convex* if $f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$ for all $x,y\in X$ and $\lambda\in[0,1]$;

(ii) f is said to be *strictly convex* if $f(\lambda x+(1-\lambda)y)<\lambda f(x)+(1-\lambda)f(y)$ for all $\lambda\in(0,1)$ and $x,y\in X$ with $x\ne y$, $f(x)<\infty$, $f(y)<\infty$;

(iii) f is said to be *proper if* there exists $x \in X$ such that $f(x) < \infty$;
(iv) $Dom(f) = \{x \in X : f(x) < \infty\}$ is called *domain or effective domain*;
(v) f is said to be *bounded below* if there exists a real number α such that $\alpha \leq f(x)$ for all $x \in X$;
(vi) the set $epi f = \{(x, \alpha) : x \in X, \alpha \in \mathbb{R}, f(x) \leq \alpha\}$ is called the *epigraph* of f.

Let C be a subset of X. Then the function i_C on X defined by

$$i_C(x) = \begin{cases} 0 & if \quad x \in C, \\ \infty & if \quad x \notin C \end{cases}$$

is called the *indicator function.*

Observation

- i_C is proper if and only if C is nonempty.
- $dom(i_C) = C$.
- The set C is convex if and only if its indicator function i_C is convex.
- The domain of each convex function is convex.

Let X be a topological space and $f : X \to (-\infty, \infty]$ a proper function. Then f is said to be *lower semicontinuous* (*l.s.c.*) at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x_0) = \sup_{V \in U_{x_0}} \inf_{x \in V} f(x),$$

where U_{x_0} is a base of neighborhoods of the point $x_0 \in X$. f is said to be *lower semicontinuous on* X if it is lower semicontinuous on each point of X, i.e., for each $x \in X$

$$x_n \to x \Rightarrow f(x) \leq \liminf_{n \to \infty} f(x_n).$$

We now discuss some elementary properties of convex functions:

Proposition 2.5.1 *Let X be a linear space and $f : X \to (-\infty, \infty]$ a function. Then f is convex if and only if its epigraph is a convex subset of $X \times \mathbb{R}$.*

Proof. Suppose f is convex. Then for $(x, \alpha), (y, \beta)$ in $epi f$, we have

$$f((1-t)x + ty) \quad \leq \quad (1-t)f(x) + tf(y) \leq (1-t)\alpha + t\beta \;\text{ for all } t \in [0, 1].$$

This implies that $((1-t)x + ty, (1-t)\alpha + t\beta) \in epi f$.

Conversely, suppose that $epi f$ is convex. Then $Dom(f)$ is also convex. Because for $x, y \in Dom(f)$ and $(x, f(x))$, $(y, f(y)) \in epi f$, we have

$$((1-t)x + ty, (1-t)f(x) + tf(y)) \in \; epi f \text{ for all } t \in [0, 1].$$

Thus, by the definition of $epi f$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y). \quad \blacksquare$$

Proposition 2.5.2 *Let X be a topological space and $f : X \to (-\infty, \infty]$ a function. Then the following statements are equivalent:*

(a) f is lower semicontinuous.

(b) For each $\alpha \in \mathbb{R}$, the level set $\{x \in X : f(x) \le \alpha\}$ is closed.

(c) The epigraph of the function f, $\{(x, \alpha) \in X \times \mathbb{R} : f(x) \le \alpha\}$ is closed.

Proof. We recall that

$$\liminf_{x \to x_0} f(x) = \sup_{V \in U_{x_0}} \inf_{x \in V} f(x).$$

$(a) \Rightarrow (b)$. Let $\alpha \in \mathbb{R}$ and let $x_0 \in X$ with $f(x_0) > \alpha$. Because f is lower semicontinuous, there exists $V_0 \in U_{x_0}$ such that $\inf\limits_{x \in V_0} f(x) > \alpha$. Hence $V_0 \subset \{x \in X : f(x) > \alpha\}$. Consequently, $\{x \in X : f(x) > \alpha\}$ is open and hence $\{x \in X : f(x) \le \alpha\}$ is closed.

$(b) \Rightarrow (a)$. Let $x_0 \in Dom(f)$, $\varepsilon > 0$ and $V_\varepsilon = \{x \in X : f(x) > f(x_0) - \varepsilon\}$. Because each level set of f is closed, it follows that $V_\varepsilon \in U(x_0)$. Because $\inf\limits_{x \in V_\varepsilon} f(x) \ge f(x_0) - \varepsilon$, it follows that $\liminf\limits_{x \to x_0} f(x) \ge f(x_0) - \varepsilon$. As ε is arbitrarily chosen, we conclude that (a) holds.

$(a) \Leftrightarrow (c)$. Define $\varphi : X \times \mathbb{R} \to (-\infty, \infty]$ by $\varphi(x, \alpha) = f(x) - \alpha$. Then, f is *l.s.c.* on $X \Leftrightarrow \varphi$ is *l.s.c.* on $X \times \mathbb{R}$. Because *epif* is a level set of φ, therefore, the conclusion holds. ∎

Proposition 2.5.3 *Let C be a nonempty closed convex subset of a Banach space X and $f : C \to (-\infty, \infty]$ a convex function. Then f is lower semicontinuous in the norm topology if and only if f is lower semicontinuous in the weak topology.*

Proof. Set $F_\alpha := \{x \in C : f(x) \le \alpha\}$, $\alpha \in \mathbb{R}$. Then F_α is convex. Indeed, for $x, y \in F_\alpha$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\le \lambda f(x) + (1-\lambda) f(y) \\ &\le \lambda\alpha + (1-\lambda)\alpha = a \text{ for all } \lambda \in [0,1]. \end{aligned}$$

It follows from Proposition 1.9.13 (which states that for a convex subset C in a normed space X, C is closed if and only if C is weakly closed) that F_α is closed if and only if F_α is weakly closed, i.e., F_α is closed in the weak topology. ∎

Before presenting an important result, we first establish a preliminary result:

Theorem 2.5.4 *Let X be a compact topological space and $f : X \to (-\infty, \infty]$ a lower semicontinuous function. Then there exists an element $x_0 \in X$ such that*

$$f(x_0) = \inf\{f(x) : x \in X\}.$$

Proof. Set $G_\alpha := \{x \in X : f(x) > \alpha\}$, $\alpha \in \mathbb{R}$. One may easily see that each G_α is open and $X = \bigcup_{\alpha\in\mathbb{R}} G_\alpha$. By compactness of X, there exists a finite family $\{G_{\alpha_i}\}_{i=1}^n$ of $\{G_\alpha\}_{\alpha\in\mathbb{R}}$ such that

$$X = \bigcup_{i=1}^n G_{\alpha_i}.$$

Suppose $\alpha_0 = \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. This gives $f(x) > \alpha_0$ for all $x \in X$. It follows that $\inf\{f(x) : x \in X\}$ exists. Let $m = \inf\{f(x) : x \in X\}$. Let β be a number such that $\beta > m$. Set $F_\beta := \{x \in X : f(x) \leq \beta\}$. Then F_β is a nonempty closed subset of X; and hence, by the intersection property, we have

$$\bigcap_{\beta>m} F_\beta \neq \emptyset.$$

Therefore, for any point x_0 of this intersection, we have $m = f(x_0)$. ∎

Theorem 2.5.5 *Let C be a weakly compact convex subset of a Banach space and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function. Then there exists $x_0 \in Dom(f)$ such that $f(x_0) = \inf\{f(x) : x \in C\}$.*

Proof. Because f is proper, there exists $u \in C$ such that $f(u) < \infty$. Then the set $C_0 = \{x \in C : f(x) \leq f(u)\}$ is nonempty. Because the set C_0 is closed and convex subset of C, it follows that C_0 is weakly compact. Applying Proposition 2.5.3, we have that f is lower semicontinuous in the weak topology. By Theorem 2.5.4, there exists $x_0 \in C_0 \subset C$ such that

$$f(x_0) = \inf\{f(x) : x \in C_0\} = \inf\{f(x) : x \in C\}. \qquad ∎$$

Remark 2.5.6 *If f is strictly convex function in Theorem 2.5.5, then $x_0 \in C$ is the unique point such that $f(x_0) = \inf_{x\in C} f(x)$.*

Recall that every closed convex bounded subset of a reflexive Banach space is weakly compact. Using this fact, we have

Theorem 2.5.7 *Let X be a reflexive Banach space and $f : X \to (-\infty, \infty]$ a proper lower semicontinuous convex function. Then for every nonempty closed convex bounded subset C of X, there exists a point $x_0 \in Dom(f)$ such that $f(x_0) = \inf_{x\in C} f(x)$.*

In Theorem 2.5.7, the boundedness of C may be replaced by the weaker assumption

$$\lim_{x\in C, \|x\|\to\infty} f(x) = \infty.$$

Theorem 2.5.8 *Let C be a nonempty closed convex subset of a reflexive Banach space X and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function such that $f(x_n) \to \infty$ as $\|x_n\| \to \infty$. Then there exists $x_0 \in Dom(f)$ such that*

$$f(x_0) = \inf\{f(x) : x \in C\}.$$

Proof. Let $m = \inf\{f(x) : x \in C\}$. Choose a minimizing sequence $\{x_n\}$ in C, i.e., $f(x_n) \to m$. If $\{x_n\}$ is not bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}\| \to \infty$. From the hypothesis, we have $f(x_{n_i}) \to \infty$, which contradicts $m \neq \infty$. Hence $\{x_n\}$ is bounded. By the reflexivity X, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x_0 \in C$. Because f is lower semicontinuous in the weak topology, we have

$$m \leq f(x_0) \leq \liminf_{j\to\infty} f(x_{n_j}) = \lim_{n\to\infty} f(x_n) = m.$$

Therefore, $f(x_0) = m$. ∎

Differentiation of convex functions – Let X be a normed space and $\varphi : X \to (-\infty, \infty]$ a function. Then the limit

$$\lim_{t\to 0} \frac{\varphi(x+ty) - \varphi(x)}{t} = \inf_{t>0} \frac{\varphi(x+ty) - \varphi(x)}{t}$$

is said to be the *directional derivative* of φ at the point $x \in X$ in the direction $y \in X$. If it exists, it is denoted by $\varphi'(x, y)$.

The function φ is said to be *Gâteaux differentiable at a point* $x \in X$ if there exists a continuous linear functional j on X such that $\langle y, j\rangle = \varphi'(x, y)$ for all $y \in X$. The element j, denoted by $\varphi'(x)$ or $\nabla\varphi(x)$ (*i.e.*, $grad\varphi(x)$) is called the *Gâteaux derivative* of φ at x.

One can easily see from the definition of Gâteaux derivative of φ that

(i) $\varphi'(x)(0) = 0$,

(ii) $\varphi'(x)(\lambda y) = \lambda \lim_{t\to 0} \dfrac{\varphi(x + t\lambda y) - \varphi(x)}{t} = \lambda\varphi'(x)(y)$ for all $\lambda \in \mathbb{R}$, i.e., $\varphi'(x)(\cdot)$ is homogeneous over $\mathbb{R}$.

Remark 2.5.9 *If the function φ is Gâteaux differentiable at $x \in X$, then there exists $j = \varphi'(x) \in X^*$ such that*

$$\frac{d}{dt}\varphi(x+ty)\Big|_{t=0} = \langle y, \varphi'(x)\rangle = \langle y, j\rangle \textit{ for all } y \in X.$$

Let X be a normed space and $\varphi : X \to (-\infty, \infty]$ a function. The function φ is said to be *Fréchet differentiable* at a point $x \in X$ if there exists a continuous linear functional j on X such that

$$\lim_{\|y\|\to 0} \frac{|\varphi(x+y) - \varphi(x) - \langle y, j\rangle|}{\|y\|} = 0.$$

In this case, the element j denoted by $d\varphi(x)$ is called the *Fréchet derivative* of φ at the point x.

Proposition 2.5.10 *Let X be a normed space and $\varphi : X \to (-\infty, \infty]$ a function. If φ is Fréchet differentiable at x, then φ is Gâteaux differentiable at x.*

Proof. Because φ is Fréchet differentiable at x,

$$\lim_{\|y\|\to 0} \frac{|\varphi(x+y)-\varphi(x)-d\varphi(x)y|}{\|y\|} = 0. \tag{2.20}$$

Set $y = ty_0$ for $t > 0$ and for any fixed $y_0 \neq 0$. From (2.20), we obtain

$$\lim_{t\to 0} \frac{|\varphi(x+ty_0)-\varphi(x)-td\varphi(x)y_0|}{t\|y_0\|} = 0,$$

which implies that

$$\lim_{t\to 0} \frac{\varphi(x+ty_0)-\varphi(x)}{t} = d\varphi(x)y_0.$$

Hence $d\varphi \in X^*$ and φ is Gâteaux differentiable at x. ∎

The following example shows that the converse of Proposition 2.5.10 is not true.

Example 2.5.11 *Let $X = \mathbb{R}^2$ be a normed space with norm $\|\cdot\|_2$ and $\varphi : X \to \mathbb{R}$ a function defined by*

$$\varphi(x,y) = \begin{cases} x^3y/(x^4+y^2) & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

One may easily see that φ is Gâteaux differentiable at 0 with Gâteaux derivative $\varphi'(0) = 0$. Because for $(h,k) \in X$, we have

$$\frac{|\varphi(h,k)|}{\|(h,k)\|_2} = \frac{|h^3k|}{(h^4+k^2)(h^2+k^2)^{1/2}} = \frac{1}{2(1+h^2)^{1/2}} \textit{ for } k = h^2.$$

Therefore, φ is not Fréchet differentiable.

Observation

- Every Fréchet differentiable function is Gâteaux differentiable.
- If φ is Fréchet differentiable at x, then φ is continuous at x.
- If φ is Gâteaux differentiable at x, then φ is not necessarily continuous at x (e.g., the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

  $$\varphi(x,y) = \frac{2y\ exp(-x^{-2})}{y^2 + exp(-2x^{-2})},\ x \neq 0 \textit{ and } \varphi(x,y) = 0,\ x = 0$$

  is Gâteaux differentiable at zero, but not continuous at zero).
- If φ is Gâteaux differentiable at x, then $\varphi(x+ty) \to \varphi(x)$ as $t \to 0$ (i.e., if $x_n \to x$ along a line, then $\varphi(x_n) \to \varphi(x)$).

Let X be a Banach space and $\varphi : X \to (-\infty, \infty]$ a proper convex function. Then an element $j \in X^*$ is said to be a *subgradient of* φ *at the point* $x \in X$ if

$$\varphi(x) - \varphi(y) \leq \langle x - y, j \rangle \text{ for all } y \in X.$$

The set (possibly nonempty)

$$\{j \in X^* : \varphi(x) - \varphi(y) \leq \langle x - y, j \rangle \text{ for all } y \in X\},$$

of subgradients of φ at $x \in X$ is called the *subdifferential of* φ *at* $x \in X$. Thus, the subdifferential of a proper convex function φ is a mapping $\partial\varphi : X \to 2^{X^*}$ (generally multivalued) defined by

$$\partial\varphi(x) = \{j \in X^* : \varphi(x) - \varphi(y) \leq \langle x - y, j \rangle \text{ for all } y \in X\}.$$

The domain of the subdifferential $\partial\varphi$ is denoted and defined by

$$\text{Dom}(\partial\varphi) = \{x \in X : \partial\varphi(x) \neq \emptyset\}.$$

Remark 2.5.12 *If φ is not the constant ∞, then Dom$(\partial\varphi)$ is a subset of Dom(φ).*

Observation

- $\partial\varphi(x)$ is always for every $x \in X$ nonempty if φ is continuous.
- $\partial\varphi(x)$ is always a closed convex set in X^*.
- $\partial(\lambda\varphi(x)) = \lambda\partial\varphi(x)$, i.e., $\partial\varphi(x)$ is homogeneous.
- φ has a minimum value at $x_0 \in Dom(\partial\varphi)$ if and only if $0 \in \partial\varphi(x_0)$.
- $Dom(\partial\varphi) = Dom(\varphi)$ if φ is lower semicontinuous on X.
- For a lower semicontinuous proper convex function φ on a reflexive Banach space X, $\partial\varphi$ is maximal monotone.

The following results are of fundamental importance in the study of convex functions. We begin with a basic result.

Proposition 2.5.13 *Let C be a nonempty closed convex subset of a Banach space X and i_C the indicator function of C, i.e.,*

$$i_C(x) = \begin{cases} 0 & if\ x \in C, \\ \infty & otherwise. \end{cases}$$

Then $\partial i_C(x) = \{j \in X^* : \langle x - y, j \rangle \geq 0$ *for all* $y \in C\}, \ x \in C$.

Proof. Because the indicator function is convex and lower semicontinuous function on X, by the subdifferentiability of i_C, we have

$$\partial i_C(x) = \{j \in X^* : i_C(x) - i_C(y) \leq \langle x - y, j \rangle \text{ for all } y \in C\}. \quad \blacksquare$$

Remark 2.5.14 *$Dom(i_C) = Dom(\partial i_C) = C$ and $\partial i_C(x) = \{0\}$ for each $x \in int(C)$.*

We now give a relation between Gâteaux differentiability and subdifferentiability.

Theorem 2.5.15 *Let X be a Banach space and $\varphi : X \to (-\infty, \infty]$ a proper convex function. If φ is Gâteaux differentiable at a point $x_0 \in X$, then $\partial\varphi(x_0) = \{\varphi'(x_0)\}$, i.e., the subdifferential of φ at $x_0 \in X$ is a singleton set $\{\varphi'(x_0)\}$ in X^*.*

Conversely, if φ is continuous at x_0 and $\partial\varphi(x_0)$ contains a singleton element, then φ is Gâteaux differentiable at x_0 and $\varphi'(x_0) = \partial\varphi(x_0)$.

Proof. Let φ be Gâteaux differentiable at $x_0 \in X$. Then

$$\langle y, \varphi'(x_0)\rangle = \lim_{t\to 0} \frac{\varphi(x_0 + ty) - \varphi(x_0)}{t} \quad \text{for all } y \in X.$$

Notice

$$\varphi(x_0 + \lambda(z - x_0)) = \varphi((1-\lambda)x_0 + \lambda z) \leq (1-\lambda)\varphi(x_0) + \lambda\varphi(z) \text{ for all } \lambda \in (0,1).$$

Set $y := z - x_0$. Then, we have

$$\varphi(x_0 + \lambda y) \leq \varphi(x_0) + \lambda[\varphi(x_0 + y) - \varphi(x_0)].$$

Thus,

$$\frac{\varphi(x_0 + \lambda y) - \varphi(x_0)}{\lambda} \leq \varphi(x_0 + y) - \varphi(x_0),$$

which implies that

$$\varphi(x_0) - \varphi(x_0 + y) \quad \leq \quad -\langle y, \varphi'(x_0)\rangle = \langle x_0 - (x_0 + y), \varphi'(x_0)\rangle \text{ for all } y \in X,$$

i.e., $\varphi'(x_0) \in \partial\varphi(x_0)$.

Now, let $j_{x_0} \in \partial\varphi(x_0)$. Then, we have

$$\varphi(x_0) - \varphi(u) \leq \langle x_0 - u, j_{x_0}\rangle \text{ for all } u \in X.$$

Therefore,

$$\frac{\varphi(x_0 + \lambda h) - \varphi(x_0)}{\lambda} \geq \langle h, j_{x_0}\rangle \text{ for all } \lambda > 0,$$

it follows that

$$\langle h, \varphi'(x_0) - j_{x_0}\rangle \geq 0 \text{ for all } h \in X,$$

i.e., $j_{x_0} = \varphi'(x_0)$. Thus, φ is Gâteaux differentiable at x_0 and $\varphi'(x_0) = \partial\varphi(x_0)$. ∎

Corollary 2.5.16 *Let X be a Banach space and $\varphi : X \to (-\infty, \infty]$ a proper convex function. Then φ is Gâteaux differentiable at $x \in int(dom(\varphi))$ if and only if it has a unique subgradient $\partial\varphi(x) = \{\varphi'(x)\}$, i.e., the subdifferential of φ at x is a singleton set in X^*. In this case*

$$\frac{d}{dt}\varphi(x+ty)\Big|_{t=0} = \langle y, \partial\varphi(x)\rangle = \langle y, \varphi'(x)\rangle \text{ for all } y \in X.$$

Theorem 2.5.17 *Let X be a Banach space, $J_\mu : X \to 2^{X^*}$ a duality mapping with gauge function μ, and $\Phi(\|x\|) = \int_0^{\|x\|} \mu(s)ds$, $0 \neq x \in X$. Then*

$$J_\mu(x) = \partial\Phi(\|x\|).$$

Proof. Because μ is a strictly increasing and continuous function, it follows that Φ is differentiable and hence $\Phi'(t) = \mu(t)$, $t \geq 0$. Then Φ is a convex function.

First, we show $J_\mu(x) \subseteq \partial\Phi(\|x\|)$. Let $x \neq 0$, and $j \in J_\mu(x)$. Then $\langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \mu(\|x\|)$. In order to prove $j \in \partial\Phi(\|x\|)$, i.e., $\Phi(\|x\|) - \Phi(\|y\|) \leq \langle x-y, j\rangle$ for all $y \in X$, we assume that $\|y\| > \|x\|$. Then

$$\|j\|_* = \mu(\|x\|) = \Phi'(\|x\|) \leq \frac{\Phi(\|y\|) - \Phi(\|x\|)}{\|y\| - \|x\|},$$

which yields

$$\begin{aligned} \Phi(\|x\|) - \Phi(\|y\|) &\leq \|j\|_*(\|x\| - \|y\|) \\ &\leq \langle x, j\rangle - \langle y, j\rangle \\ &= \langle x-y, j\rangle. \end{aligned}$$

In a similar way, if $\|x\| > \|y\|$, we have

$$\Phi(\|x\|) - \Phi(\|y\|) \leq \langle x-y, j\rangle.$$

In the case when $\|x\| = \|y\|$, we have

$$\begin{aligned} \langle y-x, j\rangle &= \langle y, j\rangle - \langle x, j\rangle \\ &\leq \|y\|\|j\|_* - \|x\|\|j\|_* \quad (\text{as } \langle x, j\rangle = \|x\|\|j\|_*) \\ &\leq \|j\|_*(\|y\| - \|x\|), \end{aligned}$$

and it follows that

$$\Phi(\|x\|) - \Phi(\|y\|) = 0 = \|j\|_*(\|x\| - \|y\|) \leq \langle x-y, j\rangle.$$

Hence $j \in \partial\Phi(\|x\|)$. Thus, $J_\mu(x) \subseteq \partial\Phi(\|x\|)$ for all $x \neq 0$.

We now prove $\partial\Phi(\|x\|) \subseteq J_\mu(x)$ for all $x \neq 0$. Suppose $j \in \partial\Phi(\|x\|)$ for $0 \neq x \in X$. Then

$$\begin{aligned} \|x\|\|j\|_* &= \sup\{\langle y, j\rangle\|x\| : \|y\| = 1\} \\ &= \sup\{\langle y, j\rangle : \|x\| = \|y\| = 1\} \\ &\leq \sup\{\langle y, j\rangle : \|x\| = \|y\|\} \\ &\leq \sup\{\langle x, j\rangle + \Phi(\|y\|) - \Phi(\|x\|) : \|x\| = \|y\|\} \\ &\leq \|x\|\|j\|_*. \quad (\text{as } \langle y, j\rangle \leq \langle x, j\rangle + \Phi(\|y\|) - \Phi(\|x\|)). \end{aligned}$$

Thus, $\langle x, j\rangle = \|x\|\|j\|_*$. To see $j \in J_\mu(x)$, we show that $\|j\|_* = \mu(\|x\|) = \Phi'(\|x\|)$. Because

$$\Phi(\|x\|) - \Phi(t\|x\|) \leq \langle x - tx, j\rangle = (1-t)\|x\|\|j\|_* \text{ for all } t > 0,$$

this implies that

$$\|j\|_* \leq \frac{\Phi(t\|x\|) - \Phi(\|x\|)}{t\|x\| - \|x\|}. \tag{2.21}$$

It follows from (2.21) that

$$\|j\|_* \leq \frac{\Phi(t\|x\|) - \Phi(\|x\|)}{t\|x\| - \|x\|} \qquad \text{if } t > 1$$

and

$$\frac{\Phi(\|x\|) - \Phi(t\|x\|)}{\|x\| - t\|x\|} \leq \|j\|_* \qquad \text{if } t < 1.$$

Taking the limit as $t \to 1$, we get

$$\|j\|_* = \Phi'(\|x\|) = \mu(\|x\|).$$

Thus, $\partial\Phi(\|x\|) \subseteq J_\mu(x)$. Therefore, $J_\mu(x) = \partial\Phi(\|x\|)$ for all $x \neq 0$. ∎

Remark 2.5.18 *Both the sets $J_\mu(x)$ and $\partial\Phi(\|x\|)$ are equal to $\{0\}$ if $x = 0$.*

Corollary 2.5.19 *For $p \in (1, \infty)$, the generalized duality mapping J_p is the subdifferential of the functional $\|\cdot\|^p/p$.*

Proof. Define $\mu(t) = t^{p-1},\ p > 1$. Hence

$$\Phi(t) = \int_0^t \mu(s)ds = \int_0^t s^{p-1}ds = \frac{t^p}{p}.$$

Therefore, $J_p(\cdot) = \partial(\|\cdot\|^p/p)$. ∎

Corollary 2.5.20 *Let X be a Banach space and $\varphi(x) = \|x\|^2/2$. Then the subdifferential $\partial\varphi$ coincides with the normalized duality mapping $J : X \to 2^{X^*}$ defined by*

$$Jx = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|\}, x \in X.$$

Theorem 2.5.21 *Let X be a Banach space. Then*

$$\partial\|x\| = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = 1\} \text{ for all } x \in X \setminus \{0\}.$$

Proof. Let $j \in \partial\|x\|$. Then

$$\langle y - x, j \rangle \leq \|y\| - \|x\| \leq \|y - x\| \text{ for all } y \in X. \tag{2.22}$$

It follows that $j \in X^*$ and $\|j\|_* \leq 1$. It is clear from (2.22) that $\|x\| \leq \langle x, j \rangle$, which gives

$$\langle x, j \rangle = \|x\| \text{ and } \|j\|_* = 1.$$

Thus,

$$\partial\|x\| \subseteq \{j \in X^* : \langle x, j \rangle = \|x\| \text{ and } \|j\|_* = 1\}.$$

Now suppose $j \in X^*$ such that $j \in \{f \in X^* : \langle x, f \rangle = \|x\| \text{ and } \|f\|_* = 1\}$. Then $\langle x, j \rangle = \|x\|$ and $\|j\|_* = 1$. Thus,

$$\langle y - x, j \rangle = \langle y, j \rangle - \|x\| \leq \|y\| - \|x\| \text{ for all } y \in X,$$

i.e., $j \in \partial\|x\|$. It follows that

$$\{j \in X^* : \langle x, j \rangle = \|x\| \text{ and } \|j\|_* = 1\} \subseteq \partial\|x\|.$$

Therefore, $\partial\|x\| = \{j \in X^* : \langle x, j \rangle \text{ and } \|j\|_* = 1\}$. ∎

Using Corollary 2.5.19, we establish an inequality in a general Banach space that is a generalization of the inequality given in Proposition 2.4.6(b).

Theorem 2.5.22 *Let X be a Banach space and let $J_p : X \to 2^{X^*}$, $1 < p < \infty$ be the generalized duality mapping. Then for any $x, y \in X$, there exists $j_p(x+y) \in J_p(x+y)$ such that $\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y) \rangle$.*

Proof. By Corollary 2.5.19, J_p is the subdifferential of the functional $\|\cdot\|^p/p$. By the subdifferentiability of $\|\cdot\|^p/p$, for $x, y \in X$, there exists $j_p(x+y) \in J_p(x+y)$ such that $\|x+y\|^p \leq \|x\|^p + p\langle y, j_p(x+y) \rangle$. ∎

The following result is very useful in the approximation of solution of nonlinear operator equations.

Theorem 2.5.23 *Let X be a Banach space and $J_\mu : X \to 2^{X^*}$ a duality mapping with gauge function μ. If J_μ is single-valued, then*

$$\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\mu(x+ty) \rangle dt \textit{ for all } x, y \in X.$$

Proof. Because J_μ is single-valued, it follows from Theorem 2.5.17 that $\partial\Phi(\|x\|) = \{J_\mu(x)\}$. Hence Corollary 2.5.16 implies that J_μ is the Gâteaux gradient of $\Phi(\|x\|)$, i.e.,

$$\left.\frac{d}{dt}\Phi(\|x+ty\|)\right|_{t=0} = \langle y, J_\mu(x) \rangle.$$

Hence

$$\frac{d}{dt}\Phi(\|x+ty\|)\Big|_{t=r} = \frac{d}{ds}\Phi(\|x+ry+sy\|)\Big|_{s=0} = \langle y,\ J_\mu(x+ry)\rangle, \quad r\in\mathbb{R}.$$

Because the function $t \mapsto \langle y,\ J_\mu(x+ty)\rangle$ is continuous, hence

$$\int_0^1 \langle y, J_\mu(x+ry)\rangle dr = \int_0^1 \frac{d}{dt}\Phi(\|x+ty\|)\Big|_{t=r} dr = \Phi(\|x+y\|) - \Phi(\|x\|). \quad \blacksquare$$

Corollary 2.5.24 *Let X be a Banach space. If X^* is strictly convex, then we have the following:*

(*a*) $\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\mu(x+ty)\rangle dt$ *for all* $x, y \in X$;

(*b*) $\|x+y\|^p = \|x\|^p + p\int_0^1 \langle y, J_p(x+ty)\rangle dt$ *for all* $x, y\in X$ *and* $p>1$;

(*c*) $\|x+y\|^2 = \|x\|^2 + 2\int_0^1 \langle y, J(x+ty)\rangle dt$ *for all* $x, y\in X$.

Proposition 2.5.25 *Let X be a Banach space with strictly convex dual and C a nonempty convex subset of X. Let x_0 be an element in C and $J_\mu : X\to X^*$ a duality mapping with gauge function μ. Then*

$$\|x_0\| = \inf_{x\in C}\|x\| \text{ if and only if } \langle x_0 - x, J_\mu(x_0)\rangle \le 0 \text{ for all } x\in C.$$

Proof. Let x_0 be a point in C such that $\langle x_0 - x, J_\mu(x_0)\rangle \le 0$ for all $x \in C$. Then

$$\|x_0\|\|J_\mu(x_0)\|_* = \langle x_0, J_\mu(x_0)\rangle \le \|x\|\|J_\mu(x_0)\|_* \text{ for all } x\in C.$$

Therefore, $\|x_0\| = \inf_{x\in C}\|x\|$.

Conversely, suppose that $x_0\in C$ such that $\|x_0\| = \inf_{x\in C}\|x\|$. Then

$$\|x_0\| \le \|x_0 + t(x-x_0)\| \text{ for all } x\in C \text{ and } t\in[0,1],$$

which implies that

$$\Phi(\|x_0\|) - \Phi(\|x_0 + t(x-x_0)\|) \le 0.$$

Because $J_\mu(z) = \partial\Phi(\|z\|)$, it follows that

$$\Phi(\|x_0+t(x-x_0)\|) - \Phi(\|x_0\|) \le \langle x_0 + t(x-x_0) - x_0, J_\mu(x_0+t(x-x_0))\rangle,$$

which implies that

$$t\langle x_0 - x, J_\mu(x_0+t(x-x_0))\rangle \le \Phi(\|x_0\|) - \Phi(\|x_0+t(x-x_0)\|) \le 0.$$

Thus,

$$\langle x_0 - x, J_\mu(x_0+t(x-x_0))\rangle \le 0.$$

Letting $t\to 0$, we obtain $\langle x_0 - x, J_\mu(x_0)\rangle \le 0$. $\blacksquare$

2.6 Smoothness

Let C be a nonempty closed convex subset of a normed space X such that the origin belongs to the interior of C. A linear functional $j \in X^*$ is said to be *tangent* to C at a point $x_0 \in \partial C$ if $j(x_0) = \sup\{j(x) : x \in C\}$. If $H = \{x \in X : j(x) = 0\}$ is the hyperplane, then the set $H + x_0$ is called a *tangent hyperplane* to C at x_0.

Definition 2.6.1 *A Banach space X is said to be smooth if for each $x \in S_X$, there exists a unique functional $j_x \in X^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$.*

Geometrically, the smoothness condition means that at each point x of the unit sphere, there is exactly one supporting hyperplane $\{j_x = 1\}$. This means that the hyperplane $\{j_x = 1\}$ is tangent at x to the unit ball, and this unit ball is contained in the half space $\{j_x \leq 1\}$.

Observation

- $\ell_p,\ L_p\ (1 < p < \infty)$ are smooth Banach spaces.
- $c_0,\ \ell_1,\ L_1,\ \ell_\infty, L_\infty$ are not smooth.

Differentiability of norms of Banach spaces – Let X be a normed space and $S_X = \{x \in X : \|x\| = 1\}$, the unit sphere of X. Then the norm of X is *Gâteaux differentiable* at point $x_0 \in S_X$ if for $y \in S_X$

$$\frac{d}{dt}(\|x_0 + ty\|)\bigg|_{t=0} = \lim_{t\to 0} \frac{\|x_0 + ty\| - \|x_0\|}{t}$$

exists (say, $\langle y, \bigtriangledown\|x_0\|\rangle$). $\bigtriangledown\|x_0\|$ is called the *gradient of the norm* $\varphi(x) = \|x\|$ at $x = x_0$. The norm of X is said to *Gâteaux differentiable* if it is Gâteaux differentiable at each point of S_X. The norm of X is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, the limit is approached uniformly for $x \in S_X$.

Example 2.6.2 *Let H be a Hilbert space. Then the norm of H is Gâteaux differentiable with $\bigtriangledown\|x\| = x/\|x\|$, $x \neq 0$. Indeed, for each $x \in X$ with $x \neq 0$, we have*

$$\begin{aligned}\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} &= \lim_{t\to 0} \frac{\|x + ty\|^2 - \|x\|^2}{t(\|x + ty\| + \|x\|)} \\ &= \lim_{t\to 0} \frac{2t\langle y, x\rangle + t^2\|y\|^2}{t(\|x + ty\| + \|x\|)} = \langle y, x/\|x\|\rangle.\end{aligned}$$

Therefore, the norm of H is Gâteaux differentiable with $\bigtriangledown\|x\| = x/\|x\|$.

Remark 2.6.3 *In view of Example 2.6.2, we have the following:*
(i) at $x \neq 0, \varphi(x) = \|x\|$ is Gâteaux differentiable with $\bigtriangledown\|x\| = x/\|x\|$,
(ii) at $x = 0, \varphi(x) = \|x\|$ is not differentiable, but it is subdifferentiable. Indeed,

$$\begin{aligned}\partial\varphi(0) = \partial\|0\| &= \{j \in H : \langle x, j\rangle \leq \|x\| \text{ for all } x \in H\} \\ &= \{j \in H : \|j\|_* \leq 1\}.\end{aligned}$$

Theorem 2.6.4 *Let X be a Banach space. Then we have the following:*

(a) If X^* is strictly convex, then X is smooth.

(b) If X^* is smooth, then X is strictly convex.

Proof. (a) Suppose X is not smooth. There exist $x_0 \in S_X$ and $j_1, j_2 \in S_{X^*}$ with $j_1 \neq j_2$ such that $\langle x_0, j_1\rangle = \langle x_0, j_2\rangle = 1$. This means that x_0 determines a continuous linear functional on X^* that takes its maximum value on B_{X^*} at two distinct points j_1 and j_1. Hence X^* is not strictly convex.

(b) Suppose X is not strictly convex. There exist $j \in S_{X^*}$ and $x, y \in S_X$ with $x \neq y$ such that $\langle x, j\rangle = \langle y, j\rangle = 1$. Thus, two supporting hyperplanes pass through $j \in S_{X^*}$ such that

$$\langle x, f\rangle = \langle y, f\rangle = 1, f \in X^*.$$

Therefore, X^* is not smooth. ∎

It is well-known that for a reflexive Banach space X, the dual spaces X and X^* can be equivalently renormed as strictly convex spaces such that the duality is preserved. Using the above fact, we have

Theorem 2.6.5 *Let X be a reflexive Banach space. Then we have the following:*
(a) X is smooth if and only if X^* is strictly convex.
(b) X is strictly convex if and only if X^* is smooth.

The following theorem establishes a relation between smoothness and Gâteaux differentiability of the norm.

Theorem 2.6.6 *A Banach space X is smooth if and only if the norm is Gâteaux differentiable on $X\backslash\{0\}$.*

Proof. Because the proper convex continuous function φ is Gâteaux differentiable if and only if it has a unique subgradient, we have

norm is Gâteaux differentiable at x
$\Leftrightarrow \partial\|x\| = \{j \in X^* : \langle x, j\rangle = \|x\|, \|j\|_* = 1\}$ is singleton
$\Leftrightarrow$ there exists a unique $j \in X^*$ such that $\langle x, j\rangle = \|x\|$ and $\|j\|_* = 1$
$\Leftrightarrow$ smooth. ∎

Next, we establish a relation between smoothness of a Banach space and a property of the duality mapping with gauge function μ.

Theorem 2.6.7 *Let X be a Banach space. Then X is smooth if and only if each duality mapping J_μ with gauge function μ is single-valued; in this case*

$$\frac{d}{dt}\Phi(\|x + ty\|)\bigg|_{t=0} = \langle y, J_\mu(x)\rangle \textit{ for all } x, y \in X. \tag{2.23}$$

Proof. The Banach space X is smooth if and only if there exists a unique $j \in X^*$ satisfying

$$\langle x\mu(||x||), j\rangle = \|x\|\mu(\|x\|) \text{ and} \|j\|_* = 1;$$

in this case $\mu(\|x\|)j = J_\mu(x) = \partial\Phi(\|x\|)$, and hence by Corollary 2.5.16, we obtain the formula (2.23). ∎

Corollary 2.6.8 *Let X be a Banach space and $J_\mu : X \to 2^{X^*}$ a duality mapping with gauge function μ. Then $j \in J_\mu(x)$, $x \in X$ if and only if $H = \{y \in X : \langle y, j\rangle = \|x\|\mu(\|x\|)\}$ is a supporting hyperplane for the closed ball $B_{\|x\|}[0]$ at x.*

Corollary 2.6.9 *Let X be a Banach space and $J : X \to 2^{X^*}$ a duality mapping. Then the following are equivalent:*

(a) X is smooth.

(b) J is single-valued.

(c) The norm of X is Gâteaux differentiable with $\bigtriangledown\|x\| = \|x\|^{-1}Jx$.

We now study the continuity property of duality mappings.

Theorem 2.6.10 *Let X be a Banach space and $J : X \to X^*$ a single-valued duality mapping. Then J is norm to weak* continuous.*

Proof. We show that $x_n \to x \Rightarrow Jx_n \to Jx$ in the weak* topology. Let $x_n \to x$ and set $f_n := Jx_n$. Then

$$\langle x_n, f_n\rangle = \|x_n\|\|f_n\|_*, \|x_n\| = \|f_n\|_*.$$

Because $\{x_n\}$ is bounded, $\{f_n\}$ is bounded in X^*. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f \in X^*$ in the weak* topology. Because the norm of X^* is lower semicontinuous in weak* topology, we have

$$\|f\|_* \le \liminf_{k\to\infty} \|f_{n_k}\|_* = \liminf_{k\to\infty} \|x_{n_k}\| = \|x\|.$$

Because $\langle x, f - f_{n_k}\rangle \to 0$ and $\langle x - x_{n_k}, f_{n_k}\rangle \to 0$, it follows from the fact

$$\begin{aligned} |\langle x, f\rangle - \|x_{n_k}\|^2| &= |\langle x, f\rangle - \langle x_{n_k}, f_{n_k}\rangle| \\ &\le |\langle x, f - f_{n_k}\rangle| + |\langle x - x_{n_k}, f_{n_k}\rangle| \to 0 \end{aligned}$$

that

$$\langle x, f\rangle = \|x\|^2.$$

As a result

$$\|x\|^2 = \langle x, f\rangle \le \|f\|_*\|x\|.$$

Thus, we have $\langle x, f\rangle = \|x\|^2, \|x\| = \|f\|_*$. Therefore, $f = Jx$. ∎

Theorem 2.6.11 *Let X be a Banach space with a uniformly Gâteaux differentiable norm. Then the duality mapping $J : X \to X^*$ is uniformly demicontinuous on bounded sets, i.e., J is uniformly continuous from X with its norm topology to X^* with the weak* topology.*

Proof. Suppose the result is not true. Then there exist sequences $\{x_n\}$ and $\{z_n\}$, a point y_0 and a positive ε such that

$$\|x_n\| = \|z_n\| = \|y_0\| = 1, z_n - x_n \to 0 \text{ and } \langle y_0, Jz_n - Jx_n\rangle \geq \varepsilon \text{ for all } n \in \mathbb{N}.$$

Set

$$a_n := t^{-1}(\|x_n + ty_0\| - \|x_n\| - t\langle y_0, Jx_n\rangle)$$

and

$$b_n := t^{-1}(\|z_n - ty_0\| - \|z_n\| + t\langle y_0, Jz_n\rangle).$$

If $t > 0$ is sufficiently small, then both a_n and b_n are less than $\varepsilon/2$. On the other hand, we have

$$\begin{aligned} a_n &\geq t^{-1}(\langle x_n + ty_0, Jz_n\rangle - \langle x_n + ty_0, Jx_n\rangle) \\ &= \langle y_0, Jz_n - Jx_n\rangle + t^{-1}\langle x_n, Jz_n - Jx_n\rangle \end{aligned}$$

and

$$\begin{aligned} b_n &\geq t^{-1}(\langle z_n - ty_0, Jx_n\rangle - \langle z_n - ty_0, Jz_n\rangle) \\ &= \langle y_0, Jz_n - Jx_n\rangle - t^{-1}\langle z_n, Jz_n - Jx_n\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} a_n + b_n &\geq 2\langle y_0, Jz_n - Jx_n\rangle + t^{-1}\langle x_n - z_n, Jz_n - Jx_n\rangle \\ &\geq 2\varepsilon - 2t^{-1}\|x_n - z_n\|, \end{aligned}$$

a contradiction by choosing $t = 2\|x_n - z_n\|/\varepsilon$ for sufficiently large n. ∎

2.7 Modulus of smoothness

Recall that the modulus of convexity of a Banach space X is a function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(t) = \inf\{1 - \|(x + y)/2\| : x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq t\}.$$

We now introduce the modulus of smoothness of a Banach space.

Definition 2.7.1 *Let X be a Banach space. Then a function $\rho_X : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the modulus of smoothness of X if*

$$\begin{aligned} \rho_X(t) &= \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\} \\ &= \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \|x\| = \|y\| = 1\right\}, \quad t \geq 0. \end{aligned}$$

It is easy to check that $\rho_X(0) = 0$ and $\rho_X(t) \geq 0$ for all $t \geq 0$.

The following result contains important properties of the modulus of smoothness.

Proposition 2.7.2 *Let ρ_X be the modulus of smoothness of a Banach space X. Then ρ_X is an increasing continuous convex function.*

Proof. Because for fixed $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$, the function

$$f(t) = \frac{\|x + ty\| + \|x - ty\|}{2} - 1, \ t \in \mathbb{R}$$

is convex and continuous on $\mathbb{R}$, it follows that the modulus of smoothness ρ_X is also continuous and a convex function.

Moreover, $f(-t) = f(t)$ for each $t \in \mathbb{R}$, f is nondecreasing on $\mathbb{R}^+$. Hence ρ_X is nondecreasing. ∎

The following theorem gives us an important relation between the modulus of convexity of X (respectively, X^*) and that of smoothness of X^* (respectively, X).

Theorem 2.7.3 *Let X be a Banach space. Then we have the following:*

(a) $\rho_{X^*}(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta_X(\varepsilon) : 0 \leq \varepsilon \leq 2\right\}$ *for all* $t > 0$.

(b) $\rho_X(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 \leq \varepsilon \leq 2\right\}$ *for all* $t > 0$.

Proof. (a) By the definition of modulus of smoothness of X^*, we have

$$\begin{aligned}
2\rho_{X^*}(t) &= \sup\{\|x^* + ty^*\|_* + \|x^* - ty^*\|_* - 2 : x^*, y^* \in S_{X^*}\} \\
&= \sup\{\langle x, x^*\rangle + t\langle x, y^*\rangle + \langle y, x^*\rangle - t\langle y, y^*\rangle - 2 : x, y \in S_X, x^*, y^* \in S_{X^*}\} \\
&= \sup\{\|x + y\| + t\|x - y\| - 2 : x, y \in S_X\} \\
&= \sup\{\|x + y\| + t\varepsilon - 2 : x, y \in S_X, \|x - y\| = \varepsilon, 0 \leq \varepsilon \leq 2\} \\
&= \sup\{t\varepsilon - 2\delta_X(\varepsilon) : 0 \leq \varepsilon \leq 2\}.
\end{aligned}$$

Part (b) can be obtained in the same manner. ∎

As an immediate consequence of Theorem 2.7.3 (b), we have

Corollary 2.7.4 *Let X be a Banach space. Then $\rho_X(t)/t$ is increasing function and $\rho_X(t) \leq t$ for all $t > 0$.*

Theorem 2.7.3 allows us to estimate ρ_X for Hilbert spaces. Indeed, we have

Proposition 2.7.5 *Let H be a Hilbert space. Then for $t > 0$*

$$\rho_H(t) = \sup\left\{t\varepsilon/2 - 1 + (1 - \varepsilon^2/4)^{1/2} : 0 < \varepsilon \leq 2\right\} = (1 + t^2)^{1/2} - 1.$$

Observation

- If X is a Banach space and H is a Hilbert space, then $\rho_X(t) \geq \rho_H(t) = \sqrt{1+t^2}-1$ for all $t \geq 0$.

Let X be a Banach space. Then the *characteristic of convexity* or the *coefficient of convexity* of the Banach space X is the number

$$\epsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}.$$

The Banach space X is said to be *uniformly convex* if $\epsilon_0(X) = 0$ and *uniformly nonsquare* if $\epsilon_0(X) < 2$. One may easily see that the modulus of convexity δ_X is strictly increasing on $[\epsilon_0, 2]$.

Example 2.7.6 *Let $X = \mathbb{R}^2$ with norm $\|\cdot\|_\infty$ defined by*

$$\|x\|_\infty = \|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}.$$

Then X has a square-shaped unit ball for which $\delta_X(\varepsilon) = 0$ for $\varepsilon \in [0,2]$. Hence $\epsilon_0(X) = 2$.

The following theorem gives an important relation between the modulus of smoothness of a Banach space and the characteristic of convexity of its dual space.

Theorem 2.7.7 *Let X be a Banach space. Then the following statements are equivalent:*

(a) $\lim\limits_{t\to 0} \dfrac{\rho_X(t)}{t} < \varepsilon/2$ *for all* $\varepsilon \leq 2$.

(b) $\epsilon_0(X^*) < \varepsilon$ *for all* $\varepsilon \leq 2$.

Proof. (a) $\Rightarrow$ (b). Let $\varepsilon \in [0,2]$. Suppose, for contradiction, that $\epsilon_0(X^*) \geq \varepsilon$. Then there exist $\{f_n\}$ and $\{g_n\}$ in S_{X^*} such that

$$\|f_n - g_n\|_* \geq \varepsilon \text{ and } \lim_{n\to\infty} \|f_n + g_n\|_* = 2. \tag{2.24}$$

From the definition of ρ_X, we get

$$\rho_X(t) \geq \left\|\frac{x+ty}{2}\right\| + \left\|\frac{x-ty}{2}\right\| - 1 \text{ for all } t > 0 \text{ and } x, y \in S_X.$$

Therefore,

$$\rho_X(t) \geq \left|\frac{f(x)+g(x)}{2}\right| + t\left|\frac{f(y)-g(y)}{2}\right| - 1 \text{ for all } f, g \in S_{X^*}.$$

Because x and y were arbitrary, we get

$$\rho_X(t) \geq \left\|\frac{f+g}{2}\right\|_* + t\left\|\frac{f-g}{2}\right\|_* - 1.$$

In particular, we have

$$\rho_X(t) \geq \left\| \frac{f_n + g_n}{2} \right\|_* + t \left\| \frac{f_n - g_n}{2} \right\|_* - 1 \text{ for all } n \in \mathbb{N}.$$

It follows from (2.24) that

$$\rho_X(t) \geq \frac{t\varepsilon}{2}.$$

(b) $\Rightarrow$ (a). Assume that $\epsilon_0(X^*) < \varepsilon$ and let $\varepsilon' \in (\epsilon_0(X^*), \varepsilon)$. Set $t' = \delta_{X^*}(\varepsilon')$ and consider $t \in [0, 2]$. There are two possibilities :

(i) Assume that $t < \varepsilon'$. Then $t\lambda/2 < \lambda\varepsilon'/2$ and so $t\lambda/2 - \delta_{X^*}(t) < \lambda\varepsilon'/2$.

(ii) Assume that $\varepsilon' \leq t$. Then $\delta_{X^*}(t) \geq \delta_{X^*}(\varepsilon') = t'$, because the modulus of convexity is an increasing function. Therefore,

$$\frac{\lambda t}{2} \leq \lambda < t' < \delta_{X^*}(t) \text{ for any } \lambda < t'.$$

This implies that

$$\frac{t\lambda}{2} - \delta_{X^*}(t) < 0.$$

Therefore, in any case we have for $\lambda < t'$

$$\sup\left\{ \frac{t\lambda}{2} - \delta_{X^*}(t) : t \in [0, 2] \right\} \leq \frac{\lambda\varepsilon'}{2}.$$

Using Theorem 2.7.3, we get $\rho_X(\lambda) \leq \lambda\epsilon'/2$, which gives that $\lim_{\lambda\to 0} \rho_X(\lambda)/\lambda \leq \varepsilon'/2$. Our choice of ε' implies that (b) is true. ∎

Let X be a Banach space. Then the *characteristic of smoothness* of X is the number

$$\rho_0(X) = \lim_{t\to 0} \frac{\rho_X(t)}{t}.$$

The following theorem allows us to estimate $\rho_0(X)$ for Banach spaces X.

Theorem 2.7.8 *Let X be a Banach space. Then*

$$\rho_0(X) = \rho_X'(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} = \frac{\epsilon_0(X^*)}{2}.$$

Proof. Assume first that $\epsilon_0(X^*) = 2$. Then $\delta_{X^*}(\varepsilon) = 0$ for every $\varepsilon \in [0, 2]$. Therefore, using Theorem 2.7.3, we get $\rho_X(t) = t$ for every $t > 0$. Hence

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 1 = \frac{\epsilon_0(X^*)}{2}.$$

Now if we assume that $\epsilon_0(X^*) < 2$, then from Theorem 2.7.7 we get the desired conclusion. ∎

Using Theorem 2.7.3 and 2.7.8, we have

Theorem 2.7.9 *Let X be a Banach space. Then we have the following:*
(a) $\rho_0(X) = \epsilon_0(X^*)/2$.
(b) $\rho_0(X^*) = \epsilon_0(X)/2$.

2.8 Uniform smoothness

Recall that the Banach space X is uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

We now define uniform smoothness of a Banach space.

Definition 2.8.1 *A Banach space X is said to be uniformly smooth if*

$$\rho'_X(0) = \lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

Example 2.8.2 *The ℓ_p spaces $(1 < p \leq 2)$ are uniformly smooth. In fact,*

$$\lim_{t \to 0} \frac{\rho_{\ell_p}(t)}{t} = \lim_{t \to 0} \frac{(1 + t^p)^{1/p} - 1}{t} = 0.$$

Uniform smoothness has a close relation with differentiability of norm.

Theorem 2.8.3 *Every uniformly smooth Banach space X is smooth.*

Proof. Suppose, for contradiction, that X is not smooth. Then there exist $x \in X \backslash \{0\}$, and $i, j \in X^*$ such that $i \neq j$, $\|i\| = \|j\| = 1$ and $\langle x, i \rangle = \langle x, j \rangle = \|x\|$. Let $y \in X$ such that $\|y\| = 1$ and $\langle y, i - j \rangle > 0$. For each $t > 0$, we have

$$\begin{aligned} 0 &< t\langle y, i - j \rangle \\ &= t\langle y, i \rangle - t\langle y, j \rangle \\ &= \frac{\langle x + ty, i \rangle + \langle x - ty, j \rangle}{2} - 1 \\ &\leq \frac{\|x + ty\| + \|x - ty\|}{2} - 1, \end{aligned}$$

and it follows that

$$0 < \langle y, i - j \rangle \leq \frac{\rho_X(t)}{t} \text{ for each } t > 0.$$

Hence X is not uniformly smooth. ∎

Next, we establish the duality between uniform convexity and uniform smoothness.

Theorem 2.8.4 *Let X be a Banach space. Then X is uniformly smooth if and only if X^* is uniformly convex.*

Proof. Recall that

$$\rho_X(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 < \varepsilon \le 2\right\} \text{ for all } t > 0. \tag{2.25}$$

Suppose, for contradiction, that X^* is not uniformly convex. Then there exists $\varepsilon_0 \in (0, 2]$ with $\delta_{X^*}(\varepsilon_0) = 0$. From (2.25), we have

$$\frac{t\varepsilon_0}{2} - \delta_{X^*}(\varepsilon_0) \le \rho_X(t)$$

which gives us that

$$0 < \frac{\varepsilon_0}{2} \le \frac{\rho_X(t)}{t} \text{ for all } t > 0,$$

and this means that X is not uniformly smooth.

Conversely, assume that X is not uniformly smooth. Then $\rho'_X(0) = \lim_{t\to 0} \frac{\rho_X(t)}{t} \neq 0$. Hence for $\varepsilon > 0$ with $\lim_{t\to 0} \frac{\rho_X(t)}{t} = \varepsilon$, there exists a sequence $\{t_n\}$ in $(0, 1)$ such that

$$t_n \to 0 \text{ and } \lim_{n\to\infty} \frac{\rho_X(t_n)}{t_n} = \varepsilon.$$

From (2.25), there exists a sequence $\{\varepsilon_n\}$ in $(0, 2]$ such that

$$\frac{\varepsilon}{2} t_n \le \frac{t_n \varepsilon_n}{2} - \delta_{X^*}(\varepsilon_n),$$

which implies that

$$0 < \delta_{X^*}(\varepsilon_n) \le \frac{t_n}{2}(\varepsilon_n - \varepsilon).$$

It follows from the condition $t_n < 1$ that $\varepsilon < \varepsilon_n$. Because δ_{X^*} is a nondecreasing function, we have $\delta_{X^*}(\varepsilon) \le \delta_{X^*}(\varepsilon_n) \to 0$, i.e., X^* is not uniformly convex. ∎

Theorem 2.8.5 *Let X be a Banach space. Then X is uniformly convex if and only if X^* is uniformly smooth.*

Proof. Notice

$$\rho_{X^*}(t) = \sup\left\{\frac{t\varepsilon}{2} - \delta_X(\varepsilon) : 0 < \varepsilon \le 2\right\} \text{ for all } t > 0.$$

By interchanging the roles of X and X^*, we obtain the result by Theorem 2.8.4. ∎

Theorem 2.8.6 *Every uniformly smooth Banach space is reflexive.*

Proof. Let X be a uniformly smooth Banach space. Then X^* is uniformly convex and hence X^* is reflexive. It follows from Theorem 1.9.26 (which states that the reflexivity of X^* implies the reflexivity of X) that X is reflexive. ∎

Fréchet differentiability of norm and uniform smoothness

Uniform smoothness can be characterized by uniform Fréchet differentiability of the norm.

The norm of a Banach space X is said to *Fréchet differentiable* if for each $x \in S_X$, $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists uniformly for $y \in S_X$.

In the other words, there exists a function $\varepsilon_x(s)$ with $\varepsilon_x(s) \to 0$ as $s \to 0$ such that

$$\Big| \|x+ty\| - \|x\| - t\langle y, Jx\rangle \Big| \leq |t|\varepsilon_x(|t|) \text{ for all } y \in S_X.$$

In this case, the norm is Gâteaux differentiable and

$$\lim_{t\to 0} \sup_{y\in S_X} \left| \frac{\frac{1}{2}\|x+ty\|^2 - \frac{1}{2}\|x\|^2}{t} \quad - \quad \langle y, Jx\rangle \right| = 0 \text{ for all } x \in X.$$

On the other hand,

$$\frac{1}{2}\|x\|^2 + \langle h, Jx\rangle \quad \leq \quad \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, Jx\rangle + b(\|h\|)$$

for all bounded $x, h \in X$, where b is a function defined on $\mathbb{R}^+$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0.$

We say that the norm of a Banach space X is *uniformly Fréchet differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \text{ exists uniformly for all } x, y \in S_X.$$

We now establish some results concerning Fréchet differentiability of the norm of Banach spaces.

Theorem 2.8.7 *Let X be a Banach space with a Fréchet differentiable norm. Then the duality mapping $J : X \to X^*$ is norm to norm continuous.*

Proof. It is sufficient to prove that $x_n \to x \in S_X \Rightarrow Jx_n \to Jx \in S_{X^*}$. Let $\{x_n\}$ be a sequence in S_X such that $x_n \to x$. Then $x \in S_X$. Because X has a Fréchet differentiable norm,

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} = \langle y, Jx\rangle \text{ uniformly in } y \in S_X,$$

i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{\|x+ty\| - \|x\|}{t} - \langle y, Jx \rangle \right| < \varepsilon \text{ for all } y \in S_X \text{ and all } t \text{ with } 0 < |t| \leq \delta.$$

Hence

$$\|x+ty\| - \|x\| < t(\langle y, Jx \rangle + \varepsilon) \text{ and } \|x-ty\| - \|x\| < -t(\langle y, Jx \rangle - \varepsilon),$$

so that

$$\|x+ty\| - 1 < t(\langle y, Jx \rangle + \varepsilon) \text{ and } \|x-ty\| - 1 < t(\varepsilon - \langle y, Jx \rangle).$$

Note

$$\begin{aligned} 0 \leq 1 - \langle x, Jx_n \rangle &= \langle x_n, Jx_n \rangle - \langle x, Jx_n \rangle \\ &\leq \langle x_n - x, Jx_n \rangle \\ &\leq \|x_n - x\| \|Jx_n\|_* = \|x_n - x\| \to 0, \end{aligned}$$

i.e., $\langle x, Jx_n \rangle \to 1$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that

$$1 \leq \langle x, Jx_n \rangle + t\varepsilon \text{ for all } n \geq n_0.$$

Because

$$\begin{aligned} 1 - t\varepsilon \leq \langle x, Jx_n \rangle &= \langle x, Jx + Jx_n \rangle - 1 \\ &= \langle x+ty, Jx \rangle + \langle x-ty, Jx_n \rangle - t\langle y, Jx - Jx_n \rangle - 1 \\ &\leq \|x+ty\| \|Jx\|_* + \|x-ty\| \|Jx_n\|_* - t\langle y, Jx - Jx_n \rangle - 1 \\ &\leq t\langle y, Jx \rangle + t\varepsilon + 1 + 1 + t\varepsilon - t\langle y, Jx \rangle - t\langle y, Jx - Jx_n \rangle - 1 \\ &= 2t\varepsilon - t\langle y, Jx - Jx_n \rangle + 1, \end{aligned}$$

this implies that

$$\langle y, Jx - Jx_n \rangle \leq 3\varepsilon \text{ for all } y \in S_X.$$

Similarly, we can show that

$$\langle y, Jx_n - Jx \rangle \leq 3\varepsilon \text{ for all } y \in S_X.$$

Thus,

$$\big|\langle y, Jx_n - Jx \rangle\big| \leq 3\varepsilon \text{ for all } n \geq n_0 \text{ and } y \in S_X$$

which gives us

$$\|Jx_n - Jx\|_* < 3\varepsilon \text{ for all } n \geq n_0.$$

Therefore, $x_n \to x$ in X implies $Jx_n \to Jx$ in X^*. ∎

Theorem 2.8.8 *Let X be a Banach space. Then the following are equivalent:*

(a) X has a uniformly Fréchet differentiable norm.

(b) X^* is uniformly convex.

Proof. $(a) \Rightarrow (b)$. Suppose the norm of X is uniformly Fréchet differentiable. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{\|x+ty\| - \|x\|}{t} - \langle y, Jx\rangle \right| < \frac{\varepsilon}{8} \text{ for all } x, y \in S_X \text{ and all } t \text{ with } 0 < |t| \leq \delta.$$

Then for fixed t with $0 < t < \delta$, we have

$$\|x+ty\| < \frac{t\varepsilon}{8} + t\langle y, Jx\rangle + 1$$

and

$$\|x-ty\| < \frac{t\varepsilon}{8} - t\langle y, Jx\rangle + 1.$$

As a result

$$\|x+ty\| + \|x-ty\| < \frac{t\varepsilon}{4} + 2 \text{ for all } x, y \in S_X.$$

Now, let $i, j \in S_{X^*}$ with $\|i-j\|_* \geq \varepsilon > 0$, then there exists $y_0 \in S_X$ such that

$$\langle y_0, i-j\rangle > \frac{\varepsilon}{2}.$$

Note

$$\begin{aligned} \|i+j\|_* &= \sup_{x \in S_X} \langle x, i+j\rangle \\ &= \sup_{x \in S_X} (\langle x+ty_0, i\rangle + \langle x-ty_0, j\rangle - \langle ty_0, i-j\rangle) \\ &< \sup_{x \in S_X} \left(\|x+ty_0\| + \|x-ty_0\| - \frac{t\varepsilon}{2} \right) \\ &\leq \frac{t\varepsilon}{4} + 2 - \frac{t\varepsilon}{2} \\ &\leq 2 - \frac{t\varepsilon}{2}. \end{aligned}$$

This implies $\|(i+j)/2\|_* < 1 - \delta(\varepsilon)$. Hence X^* is uniformly convex.

$(b) \Rightarrow (a)$. Let $x, y \in S_X$. Then for $t > 0$,

$$\begin{aligned} \frac{\langle y, Jx\rangle}{\|x\|} &= \frac{\langle x+ty, Jx\rangle - \|x\|^2}{t\|x\|} \\ &\leq \frac{\|x+ty\|\|x\| - \|x\|^2}{t\|x\|} \\ &= \frac{\|x+ty\| - \|x\|}{t} \\ &= \frac{\|x+ty\|^2 - \|x+ty\|\|x\|}{t\|x+ty\|} \\ &\leq \frac{\langle x+ty, J(x+ty)\rangle - \langle x, J(x+ty)\rangle}{t\|x+ty\|} \\ &= \frac{\langle y, J(x+ty)\rangle}{\|x+ty\|} \end{aligned}$$

and for $t < 0$,

$$\frac{\langle y, J(x+ty)\rangle}{\|x+ty\|} \leq \frac{\|x+ty\| - \|x\|}{t} \leq \frac{\langle y, Jx\rangle}{\|x\|}.$$

By Theorem 2.4.15, X has a uniformly Fréchet differentiable norm. ∎

Theorem 2.8.9 *Let X be a Banach space with uniformly Fréchet differentiable norm. Then the duality mapping $J : X \to X^*$ is uniformly continuous on each bounded set in X.*

Proof. Because X^* is uniformly convex, the result follows from Theorem 2.4.15 ∎

We now study the duality mapping from X^* to X. To do so, we define the *conjugate function* $f^* : X^* \to (-\infty, \infty]$ of any function $f : X \to (-\infty, \infty]$ by

$$f^*(j) = \sup\{\langle x, j\rangle - f(x) : x \in X\}, \quad j \in X^*. \tag{2.26}$$

The conjugate of f^*, i.e., the function on X defined by

$$f^{**}(x) = \sup\{\langle x, j\rangle - f^*(j) : j \in X^*\}, \quad x \in X$$

is called the *biconjugate* of f.

Observation

- f is lower semicontinuous proper convex on X if and only if $f^{**} = f$.

Example 2.8.10 *Let C be a nonempty subset of normed space X. Then the conjugate of the indicator function i_C of C is given by*

$$i_C^*(j) = \sup\{\langle x, j\rangle : x \in C\}, \quad j \in X^*.$$

The function i_C^* is called the *support function* of C.

We now give some basic properties of conjugate functions.

Proposition 2.8.11 *Let f^* be the conjugate function f. Then*

$$f(x) + f^*(j) \geq \langle x, j\rangle \text{ for all } x \in X, j \in X^*. \tag{2.27}$$

Proof. It easily follows from (2.26). ∎

The inequality (2.27) is known as the *Young inequality.* Observe also that if f is a proper function, then the relation (2.26) can be written as

$$f^*(j) = \sup\{\langle x, j\rangle - f(x) : x \in Dom(f)\},\ j \in X^*.$$

Proposition 2.8.12 *Let f^* be the conjugate function of f. Then*

$$(cf)^*(j) = cf^*(c^{-1}j) \text{ for all } c > 0 \text{ and } j \in X^*.$$

Proof. For $j \in X^*$, we have

$$\begin{aligned} (cf)^*(j) &= \sup\{\langle x, j\rangle - (cf)(x) : x \in X\} \\ &= c\sup\{c^{-1}\langle x, j\rangle - f(x) : x \in X\} \\ &= c\sup\{\langle x, c^{-1}j\rangle - f(x) : x \in X\} \\ &= cf^*(c^{-1}j). \quad \blacksquare \end{aligned}$$

Proposition 2.8.13 *Let X be a normed space and $f : X \to (-\infty, \infty]$ a proper convex function. Then the following statements are equivalent:*

(a) $j \in \partial f(x)$ for $x \in X$.

(b) $f(x) + f^*(j) \leq \langle x, j\rangle$.

(c) $f(x) + f^*(j) = \langle x, j\rangle$.

Proof. $(b) \Leftrightarrow (c)$. The Young inequality (2.27) shows that (b) and (c) are equivalent.

$(c) \Leftrightarrow (a)$. Suppose condition (c) holds. Then from the Young inequality (2.27), we find that

$$f(y) - f(x) \geq \langle y - x, j\rangle \text{ for all } y \in X,$$

i.e., $j \in \partial f(x)$.

Using a similar argument, it follows that (c) $\Rightarrow$ (a). $\blacksquare$

Proposition 2.8.14 *Let X be a normed space and $f : X \to (-\infty, \infty]$ a lower semicontinuous proper convex function. Then $j \in \partial f(x) \Leftrightarrow x \in \partial f^*(j)$.*

Proof. Because f is a lower semicontinuous convex function, $f^{**} = f$. Observe that

$$\begin{aligned} j \in \partial f(x) &\Leftrightarrow f(x) + f^*(j) = \langle x, j\rangle \\ &\Leftrightarrow f^{**}(x) + f^*(j) = \langle x, j\rangle \\ &\Leftrightarrow x \in \partial f^*(j). \quad \blacksquare \end{aligned}$$

Proposition 2.8.15 *Let X be a Banach space. If $f(x) = \|x\|^p/p$, $p > 1$, then*

$$f^*(j) = \|j\|_*^q/q, \ 1/p + 1/q = 1.$$

Proof. Because $J_p(x) = \partial(\|x\|^p/p) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|^{p-1}\}$, we have

$$f^*(j) = \sup_{x \in X}\{\langle x, j\rangle - f(x)\rangle = \sup_{x \in X}\{\|x\|^p - \|x\|^p/p\} = \sup_{x \in X}\{\|x\|^p/q\}.$$

Note $\|j\|_* = \|x\|^{p-1}$ so $\|j\|_*^q = \|x\|^{q(p-1)} = \|x\|^p$. Therefore, $f^*(j) = \|j\|_*^q/q$. $\blacksquare$

Theorem 2.8.16 *Let $p > 1$. Let X be a uniformly smooth Banach space and let $J_p : X \to X^*$ and $J_q^* : X^* \to X$ be the duality mappings with gauge functions $\mu_p(t) = t^{p-1}$ and $\mu_q(t) = t^{q-1}$, respectively. Then $J_p^{-1} = J_q^*$.*

Proof. The uniform smoothness of X implies that X is reflexive (see Theorem 2.8.6) and that X^* is uniformly convex and reflexive. Note also J_μ is surjective if and only if X is reflexive. Because J_p is single-valued, it follows that the inverse $J_p^{-1} : X^* = Dom\ (J_p^{-1}) \to X = X^{**}$ exists and is given by

$$J_p^{-1}(j) = \{x \in X : j = J_p(x)\} \text{ for all } j \in X^*.$$

Now, let $\Phi(t) = t^p/p, t > 0$. It is easy to see that $\Phi(\|\cdot\|) = \|\cdot\|^p/p$ is a continuous convex function and that its conjugate is given by $\Phi^*(\|j\|_*) = \|j\|_*^q/q$ for all $j \in X^*$. Note $J_p(x) = \partial\Phi(\|x\|)$ and $J_q^*(j) = \partial\Phi^*(\|j\|_*)$ for all $x \in X, j \in X^*$. Using Proposition 2.8.14, we have

$$j \in \partial\Phi(\|x\|) \text{ if and only if } x \in \partial\Phi^*(\|j\|_*).$$

Therefore, $J_p^{-1}(j) = J_q^*(j)$ for all $j \in X^*$. ∎

The following inequality is very useful in the existence and approximation of solutions of nonlinear operator equations.

Theorem 2.8.17 *Let X be a Banach space. Then the following are equivalent:*

(a) X is uniformly convex.

(b) For any p, $1 < p < \infty$ and $r > 0$, there exists a strictly increasing convex function $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g_r(0) = 0$ and

$$\|tx + (1-t)y\|^p \le t\|x\|^p + (1-t)\|y\|^p - t(1-t)g_r(\|x-y\|) \tag{2.28}$$

for all $x, y \in B_r[0]$ and $t \in [0,1]$.

Proof. (a) $\Rightarrow$ (b). Let X be a uniformly convex Banach space. Assume that $1 < p < \infty$. It suffices to prove that (2.28) is true for $r = 1$. Now we define a function γ by

$$\begin{aligned} \gamma(\varepsilon) &= \inf\{2^{p-1}(\|x\|^p + \|y\|^p) - \|x+y\|^p : x, y \in B_X \text{ and } \|x-y\| \ge \varepsilon\} \\ &\quad \text{for all } \varepsilon \in (0,2]. \end{aligned}$$

Because

$$\left(\frac{a+b}{2}\right)^p < \frac{a^p + b^p}{2} \text{ for all } a, b \ge 0 \text{ and } a \ne b, \tag{2.29}$$

we have

$$\gamma(\varepsilon) \ge 0 \text{ for all } 0 < \varepsilon \le 2.$$

Suppose that $\gamma(\varepsilon) = 0$ for some $\varepsilon > 0$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in B_X such that $\|x_n - y_n\| \ge \varepsilon$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} 2^{p-1}(\|x_n\|^p + \|y_n\|^p) - \|x_n + y_n\|^p = 0.$$

We may assume a subsequence of $\{x_n\}$ denoted by $\{x_n\}$ such that

$$a = \lim_{n\to\infty} \|x_n\|, \; b = \lim_{n\to\infty} \|y_n\| \text{ and } c = \lim_{n\to\infty} \|x_n + y_n\|$$

exist. Thus,

$$\left(\frac{a+b}{2}\right)^p = \frac{a^p + b^p}{2},$$

i.e., equality of inequality (2.29) holds with $c = a + b$. For $a = b > 0$, $c = 2a = \lim_{n\to\infty} \|x_n + y_n\|$, it follows from Theorem 2.2.7 that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, a contradiction. Therefore,

$$\gamma(\varepsilon) > 0 \text{ for all } \varepsilon, \; 0 < \varepsilon \leq 2.$$

Now set

$$\mu(\varepsilon) := \inf \left\{ \frac{\lambda\|x\|^p + (1-\lambda)\|y\|^p - \|\lambda x + (1-\lambda)y\|^p}{\lambda(1-\lambda)} \right\},$$

where the infimum is taken over all $x, y \in B_X$ with $\|x - y\| \geq \varepsilon$ and $\lambda \in (0,1)$. Note $\mu(\varepsilon) \geq \gamma(\varepsilon)/2^{p-1} > 0$ for all ε, $0 < \varepsilon \leq 2$. Thus, it suffices to take as g_1 the double dual Young's function μ^{**}.

$(b) \Rightarrow (a)$. Suppose (2.28) is satisfied. For $x, y \in B_X$ and $\|x - y\| = \varepsilon$, we have

$$\begin{aligned} \left\|\frac{x+y}{2}\right\| &\leq 1 - \frac{1}{4}g_1(\varepsilon) \\ &\leq 1 - \delta_X(\varepsilon), \end{aligned}$$

i.e., $\delta_X(\varepsilon) \geq g_1(\varepsilon)/4$, which shows that X is a uniformly convex Banach space. ∎

2.9 Banach limit

In this section, we generalize the concept of limit by introducing Banach limits and we discuss its properties.

Let $\ell : c \to \mathbb{K}$ be the "limit functional" defined by

$$\ell(x) = \lim_{i\to\infty} x_i \text{ for } x = \{x_i\} \in c.$$

Then ℓ is a linear functional on c. In order to extend limit ℓ on ℓ_∞, use the following notations and results.

Let S be a nonempty set and let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm.

Example 2.9.1 *Let* $S = \mathbb{N} = \{1, 2, 3, \cdots\}$*. Then* $B(S) = l_\infty$*.*

Let X be a subspace of $B(S)$ and let j be an element of X^*. Let e be a constant function on X defined by $e(s) = 1$ for all $s \in S$. We will denote $j(e)$ by $j(1)$. When X contains constants, a linear functional j on X is called a *mean* on X if $\|j\|_* = j(1) = 1$.

The following example shows that there is a subspace of ℓ_∞ for which the mean exists.

Example 2.9.2 *Let* $\ell_\infty = \{x = \{x_i\} : \sup_{i\in\mathbb{N}} |x_i| < \infty\}$ *and* X *a subset of* ℓ_∞ *such that*

$$X = \left\{ x = \{x_i\} \in \ell_\infty : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \textit{ exists} \right\}.$$

Then X *is a linear subspace of* ℓ_∞*. In fact, for* $x = \{x_i\}$ *and* $y = \{y_i\}$ *in* X*, we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \textit{ exists and } \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} y_i \textit{ exists.} \tag{2.30}$$

Hence for scalars α, β*, we have*

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \cdots, \alpha x_i + \beta y_i, \cdots).$$

Using (2.30), we obtain that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\alpha x_i + \beta y_i) = \alpha(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i) + \beta(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} y_i)$$

exists. It follows that X *is a linear subspace of* ℓ_∞*. We now define* $j : X \to \mathbb{R}$ *by*

$$j(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \textit{ for all } x \in X.$$

Note $j(1) = 1$ *and*

$$\begin{aligned} |j(x)| &= \left| \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \right| \\ &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |x_i| \\ &\le \|x\|_\infty, \end{aligned}$$

and it follows that $\|j\|_* = 1$*. Therefore,* j *is linear and* $\|j\|_* = j(1) = 1$*, i.e.,* j *is a mean on* X*.*

We now give an equivalent condition for mean.

Theorem 2.9.3 *Let X be a subspace of $B(S)$ containing constants and $j \in X^*$. Then the following are equivalent:*

(a) j is a mean on X, i.e., $\|j\|_* = j(1) = 1$.

(b) The inequalities

$$\inf_{s\in S} x(s) \leq j(x) \leq \sup_{s\in S} x(s)$$

hold for each $x \in X$.

Proof. $(a) \Rightarrow (b)$. First, we show that $j(x) \geq 0$ for all $x \geq 0$. Suppose, for contraction, that $j(x) < 0$. Choose a positive number K with $x \leq K$. Then

$$j(K - x) = Kj(1) - j(x) = K - j(x) > K.$$

Because

$$j(K - x) \leq \|j\|_* \|K - x\| = \|K - x\| = \sup_{s\in S} |K - x(s)| \leq K,$$

it follows that

$$K < j(K - x) \leq K,$$

a contradiction. Therefore, $j(x) \geq 0$.

Observe that

$$\inf_{s\in S} x(s) \leq x \leq \sup_{s\in S} x(s) \text{ for each } x \in X.$$

Because $j(x) \geq 0$ for $x \geq 0$, we have

$$\inf_{s\in S} x(s) = j(\inf_{s\in S} x(s)) \leq j(x) \leq j(\sup_{s\in S} x(s)) = \sup_{s\in S} x(s).$$

$(b) \Rightarrow (a)$. For $x = 1$, we have $1 \leq j(1) \leq 1$ and hence $j(1) = 1$. Note for each $x \in X$,

$$j(x) \leq \sup_{s\in S} x(s) \leq \sup_{s\in S} |x(s)| = \|x\|$$

and

$$-j(x) = j(-x) \leq \|-x\| = \|x\|,$$

so $|j(x)| \leq \|x\|$ for each $x \in X$. Thus, $\|j\|_* = 1$. Therefore, $\|j\|_* = j(1) = 1$, i.e., j is a mean on X. ∎

Let $f \in \ell_\infty$. We denote $f_n(x_{n+m})$ for $f(x_{m+1}, x_{m+2}, x_{m+3}, \cdots, x_{m+n}, \cdots)$, $m = 0, 1, 2, \cdots$. A continuous linear functional j on l_∞ is called a *Banach limit* if

(L_1) $\|j\|_* = j(1) = 1$,

(L_2) $j_n(x_n) = j_n(x_{n+1})$ for each $x = (x_1, x_2, \cdots) \in l_\infty$.

It is denoted by LIM.

Theorem 2.9.4 (The existence of Banach limits) – *There exists a linear continuous functional j on l_∞ such that $\|j\|_* = j(1) = 1$ and $j_n(x_n) = j_n(x_{n+1})$ for each $x = \{x_n\}_{n\in\mathbb{N}} \in \ell_\infty$.*

Proof. Let $p : \ell_\infty \to \mathbb{R}$ be the functional defined by

$$p(x) = \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Then

$$-p(-x) = \liminf_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

For $x \in c$, we have

$$\ell(x) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = p(x).$$

Moreover,

$$p(x + y) \le p(x) + p(y) \text{ for all } x, y \in c$$

and

$$p(\alpha x) = \alpha p(x) \text{ for all } x \in c \text{ and } \alpha \ge 0.$$

Thus, p is a sublinear functional with $\ell(x) = p(x)$. By the Hahn-Banach theorem, there is an extension $L : \ell_\infty \to \mathbb{R}$ of ℓ (from c to ℓ_∞) such that

$$L(x) \le \ell(x) \text{ for all } x \in \ell_\infty$$

and

$$-p(-x) \le L(x) \le p(x) \text{ for all } x \in \ell_\infty.$$

Thus, we have

$$p(1,\ 1,\ 1, \cdots) = 1$$

and

$$p((x_1, x_2, \cdots, x_n, \cdots) - (x_2, x_3, \cdots,\ x_{n+1}, \cdots)) = \limsup_{n\to\infty} \frac{x_1 - x_{n+1}}{n} = 0.$$

Hence

$$L((x_1, x_2, \cdots, x_n, \cdots) - (x_2, x_3, \cdots, x_{n+1}, \cdots)) = 0,$$

which implies that

$$L(x_1, x_2, \cdots, x_n, \cdots) = L(x_2, x_3, \cdots, x_{n+1} \cdots)$$
$$\text{for all } x = (x_1, x_2, \cdots x_n, \cdots) \in \ell_\infty.$$

Therefore, L is a Banach limit. ∎

Observation

- Every Banach limit is a positive functional on ℓ_∞, i.e., $LIM_n(x) \ge 0$ for all $x \in \ell_\infty$.
- $LIM(1, 1, \cdots 1, \cdots) = 1$.

We now give elementary properties of Banach limits.

Proposition 2.9.5 *Let LIM be a Banach limit. Then*

$$\liminf_{n\to\infty} x_n \leq LIM(x) \leq \limsup_{n\to\infty} x_n \text{ for each } x = (x_1, x_2, \cdots) \in l_\infty.$$

Moreover, if $x_n \to a$, then $LIM(x) = a$.

Proof. For each $m \in \mathbb{N}$, we have

$$LIM_n(x_n) = LIM_n(x_{n+1}) = \cdots = LIM_n(x_{n+(m-1)}) \geq \inf_{n\geq m} x_n$$

and hence $LIM_n(x_n) \geq \sup_{m\in\mathbb{N}} \inf_{n\geq m} x_n = \liminf_{n\to\infty} x_n$.

Similarly, since $LIM_n(x_n) \leq \sup_{n\geq m} x_n$, we have $LIM_n(x_n) \leq \limsup_{n\to\infty} x_n$. Therefore,

$$\liminf_{n\to\infty} x_n \leq LIM(x) \leq \limsup_{n\to\infty} x_n \text{ for each } x = (x_1, x_2, \cdots) \in l_\infty.$$

Letting $x_n \to a$, we have $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a$ and hence $LIM(x) = a$. ∎

Proposition 2.9.6 *Let a be a real number and let $(x_1, x_2, \cdots) \in \ell_\infty$. Then the following are equivalent:*

(a) $LIM_n(x_n) \leq a$ for all Banach limits LIM.

(b) For each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\frac{x_n + x_{n+1} + \cdots + x_{n+m-1}}{m} < a + \varepsilon \quad \textit{for all} \quad m \geq m_0 \textit{ and } n \in \mathbb{N}. \tag{2.31}$$

Proof. $(a) \Rightarrow (b)$. Suppose that for $\{x_n\} \in \ell_\infty$, we have $LIM_n(x_n) \leq a$ for all Banach limits LIM. Define a sublinear functional $q : \ell_\infty \to \mathbb{R}$ by

$$q(y_1, y_2, \cdots) = \limsup_{m\to\infty} \left(\sup_{n\in\mathbb{N}} \frac{1}{m} \sum_{i=n}^{n+m-1} y_i \right), \ \{y_n\} \in \ell_\infty.$$

By the Hahn-Banach theorem, there exists a linear functional $j : \ell_\infty \to \mathbb{R}$ such that

$$j \leq q \text{ and } j_n(x_n) = q_n(x_n).$$

It is easy to see that j is a Banach limit. From the assumption, we have

$$q_n(x_n) = \limsup_{m\to\infty} \left(\sup_{n\in\mathbb{N}} \frac{1}{m} \sum_{i=n}^{n+m-1} x_i \right) \leq a.$$

Thus, for $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\frac{x_n + x_{n+1} + \cdots + x_{n+m-1}}{m} < a + \varepsilon \text{ for all } m \geq m_0 \text{ and } n \in \mathbb{N}.$$

$(b) \Rightarrow (a)$. Suppose for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that (2.31) holds.

Let LIM be a Banach limit. Then

$$LIM_n(x_n) = LIM_n\left(\frac{x_n + x_{n+1} + \cdots + x_{n+m_0-1}}{m_0}\right) \leq a + \varepsilon.$$

Because ε is an arbitrary positive real number, we have $LIM_n(x_n) \leq a$. ∎

Proposition 2.9.7 *Let a be a real number and let $(x_1, x_2, \cdots) \in \ell_\infty$ such that $LIM_n(x_n) \leq a$ for all Banach limits LIM and $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$. Then $\limsup_{n\to\infty} x_n \leq a$.*

Proof. Let $\varepsilon > 0$. By Proposition 2.9.6, there exists $m \geq 2$ such that

$$\frac{x_n + x_{n+1} + \cdots + x_{n+m-1}}{m} < a + \frac{\varepsilon}{2} \text{ for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that

$$x_{n+1} - x_n < \frac{\varepsilon}{m-1} \text{ for all } n \geq n_0.$$

Let $n \geq n_0 + m$. Observe that

$$\begin{aligned} x_n &= x_{n-i} + (x_{n-i+1} - x_{n-i}) + \cdots + (x_n - x_{n-1}) \\ &\leq x_{n-i} + \frac{i\varepsilon}{m-1} \text{ for each } i = 0, 1, \cdots, m-1. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} x_n \leq a + \varepsilon.$$

Because ε is arbitrary positive number, we get the conclusion. ∎

We note that if a linear functional j on l_∞ satisfying:

$$\liminf_{n\to\infty} x_n \leq j(x) \leq \limsup_{n\to\infty} x_n \text{ for each } x = (x_1, x_2, \cdots) \in l_\infty,$$

then j is a mean on ℓ_∞. Thus, every Banach limit on ℓ_∞ is a mean on ℓ_∞.

Let X be a Banach space, $\{x_n\}$ a bounded sequence in X, and LIM a Banach limit. Then a point $x_0 \in X$ is said to be a *mean point* of $\{x_n\}$ concerning a Banach limit LIM if

$$LIM_n\langle x_n, j\rangle = \langle x_0, j\rangle \text{ for all } j \in X^*.$$

We establish two preliminary results related to mean points.

Proposition 2.9.8 ***(Existence of mean points)*** – *Let X be a reflexive Banach space and $\{x_n\}$ a bounded sequence in X. Then, for a Banach limit LIM, there exists a point x_0 in X such that*

$$LIM_n\langle x_n, j\rangle = \langle x_0, j\rangle \textit{ for all } j \in X^*.$$

Proof. Note the function $LIM_n\langle x_n, j\rangle$ is linear in j. Further, as

$$|LIM_n\langle x_n, j\rangle| \leq (\sup_{n\in\mathbb{N}} \|x_n\|) \cdot \|j\|_*,$$

the function $LIM_n\langle x_n, j\rangle$ is also bounded in j. So, we have $j_0^* \in X^{**}$ such that

$$LIM_n\langle x_n, j\rangle = \langle j_0^*, j\rangle \text{ for every } j \in X^*.$$

Because X is reflexive, there exists $x_0 \in X$ such that $LIM_n\langle x_n, j\rangle = \langle x_0, j\rangle$ for all $j \in X^*$. ∎

Proposition 2.9.9 *Let $\{x_n\}$ be a bounded sequence in a Banach space X and $x_0 \in X$ a mean point of $\{x_n\}$ concerning a Banach limit LIM. Then $x_0 \in \bigcap_{n=1}^{\infty} \overline{co}(\{x_k\}_{k\geq n})$.*

Proof. If not, there exists $n_0 \in \mathbb{N}$ such that $x_0 \notin \overline{co}\{x_n : n \geq n_0\}$. By the separation theorem, we obtain a point $j \in X^*$ such that

$$\langle x_0, j\rangle < \inf\{\langle z, j\rangle : z \in \overline{co}\{x_n : n \geq n_0\}\}.$$

Thus, we have

$$\begin{aligned} LIM_n\langle x_n, j\rangle &= \langle x_0, j\rangle < \inf\{(x_n, j) : n \geq n_0\} \\ &\leq LIM_n\{\langle x_n, j\rangle : n \geq n_0\} = LIM_n\langle x_n, j\rangle, \end{aligned}$$

a contradiction. ∎

We now characterize the sequences in ℓ_∞ for which all Banach limits coincide. It is obvious that for any element $x \in c$,

$$LIM(x) = \ell(x) = \lim_{n\to\infty} x_n \text{ for all Banach limit } LIM.$$

However, there exist nonconvergent sequences for which all Banach limits coincide.

Example 2.9.10 *Let $x = (1, 0, 1, 0, \cdots) \in \ell_\infty$. Then*

$$(x_1, x_2, \cdots x_n, \cdots) + (x_2, x_3, \cdots, x_{n+1}, \cdots) = (1, 1, 1, \cdots),$$

and it follows that

$$LIM_n(x_n) + LIM_n(x_{n+1}) = LIM_n(1) = 1 \textit{ for all } LIM.$$

Using (L_2), we have

$$LIM_n(x_n) = \frac{1}{2} \textit{ for all Banach limit } LIM.$$

A bounded sequence $x = \{x_i\}$ is said to be *almost convergent* if all its Banach limits have the same value at x. Equivalently, $x = \{x_i\} \in \ell_\infty$ is almost convergent if

$$\lim_{i\to\infty} \frac{x_n + x_{n+1} \cdots + x_{n+i-1}}{i} \text{ exists uniformly in } n.$$

We have seen in Example 2.9.10 that the sequence $(1, 0, 1, 0, \cdots)$ is not convergent, but it is almost convergent.

In optimization theory, the structure of M defined in our next result is of much interest.

Theorem 2.9.11 *Let C be a nonempty closed convex subset of a reflexive Banach space X, $\{x_n\}$ a bounded sequence in C, LIM a Banach limit, and φ a real-valued function on C defined by $\varphi(z) = LIM_n\|x_n - z\|^2$, $z \in C$. Then the set M defined by*

$$M = \{u \in C : LIM_n\|x_n - u\|^2 = \inf_{z\in C} LIM_n\|x_n - z\|^2\} \tag{2.32}$$

is a nonempty closed convex bounded set. Moreover, if X is uniformly convex, then M has exactly one point.

Proof. First, we show that φ is continuous and convex. Let $\{y_m\}$ be a sequence in C such that $y_m \to y \in C$. Set $L := \sup\{\|x_n - y_m\| + \|x_n - y\| : m, n \in \mathbb{N}\}$. Observe that

$$\begin{aligned} \|x_n - y_m\|^2 - \|x_n - y\|^2 &\le (\|x_n - y_m\| + \|x_n - y\|)(\|x_n - y_m\| - \|x_n - y\|) \\ &\le L|\ \|x_n - y_m\| - \|x_n - y\|\ | \\ &\le L\|y_m - y\| \text{ for all } n, m \in \mathbb{N}. \end{aligned}$$

Then

$$LIM_n\|x_n - y_m\|^2 \le LIM_n\|x_n - y\|^2 + L\|y_m - y\|.$$

Similarly we have

$$LIM_n\|x_n - y\|^2 \le LIM_n\|x_n - y_m\|^2 + L\|y_m - y\|.$$

Thus, we have

$$|\varphi(y_m) - \varphi(x)| \le L\|y_m - x\|.$$

Hence φ is continuous on C. Now, let $x, y \in C$ and $\lambda \in [0, 1]$. It is easy to see that

$$\varphi((1 - \lambda)x + \lambda y) \le (1 - \lambda)\varphi(x) + \lambda\varphi(y).$$

Hence φ is convex.

Using the fact $((a + b)/2)^2 \le (a^2 + b^2)/2$ for all $a, b \ge 0$, we have

$$\|y_m\|^2 \le 2\|y_m - x_n\|^2 + 2\|x_n\|^2,$$

and hence

$$\|y_m\|^2 \le 2\varphi(y_m) + 2\sup_{n\in\mathbb{N}} \|x_n\|^2,$$

i.e., $\varphi(y_m) \to \infty$ as $\|y_m\| \to \infty$. Thus, φ is a continuous convex functional and $\varphi(z) \to \infty$ as $\|z\| \to \infty$. Because X is reflexive, φ attains its infimum over C by Theorem 2.5.8. Then M is a nonempty closed convex set. Moreover, M is bounded. Indeed, let $u \in M$. Because

$$\|u\|^2 \le 2\|u - x_n\|^2 + 2\|x_n\|^2 \text{ for all } n \in \mathbb{N},$$

this implies that

$$\|u\|^2 \quad \le \quad 2\varphi(u) + 2K = 2\inf_{z\in C} \varphi(z) + 2K$$

for some $K \ge 0$.

Now, suppose X is uniformly convex. Let $z_1, z_2 \in M$. Then $(z_1+z_2)/2 \in M$. Choose $r > 0$ large enough so that $\{x_n\} \cup M \subset B_r[0]$. Then $x_n - z_1, x_n - z_2 \in B_{2r}[0]$ for all $n \in \mathbb{N}$. By Theorem 2.8.17, we have

$$\left\|x_n - \frac{z_1+z_2}{2}\right\|^2 \le \frac{1}{2}\|x_n - z_1\|^2 + \frac{1}{2}\|x_n - z_2\|^2 - \frac{1}{4} g_{2r}(\|z_1 - z_2\|).$$

If $z_1 \ne z_2$, we have

$$\begin{aligned}\inf_{z\in C} \varphi(z) \le \varphi\left(\frac{z_1+z_2}{2}\right) \quad &\le \quad \frac{1}{2}\varphi(z_1) + \frac{1}{2}\varphi(z_2) - \frac{1}{4} g_{2r}(\|z_1 - z_2\|) \\ &= \quad \inf_{z\in C} \varphi(z) - \frac{1}{4} g_{2r}(\|z_1 - z_2\|) \\ &< \quad \inf_{z\in C} \varphi(z),\end{aligned}$$

a contradiction. Therefore, M has exactly one element. ▮

Let LIM be a Banach limit and let $\{x_n\}$ be a bounded sequence in a Banach space X. We observe that if $\psi : X \to \mathbb{R}$ is bounded, Gâteaux differentiable uniformly on bounded sets, then a function $f : X \to \mathbb{R}$ defined by $f(z) = LIM_n \psi(x_n + z)$ is Gâteaux differentiable with Gâteaux derivative given by $\langle y, f'(z)\rangle = LIM_n \langle y, \psi'(x_n + z)\rangle$ for each $y \in X$.

Using the above facts, we give the following result, which will be used in convergence of sequences $\{x_n\}$ in Banach spaces with Gâteaux differentiable norm.

Theorem 2.9.12 *Let X be a Banach space with a uniformly Gâteaux differentiable norm and $\{x_n\}$ a bounded sequence in X. Let LIM be a Banach limit and $u \in X$. Then*

$$LIM_n \|x_n - u\|^2 = \inf_{z\in X} LIM_n \|x_n - z\|^2$$

if and only if

$$LIM_n \langle z, J(x_n - u)\rangle = 0 \text{ for all } z \in X.$$

Proof. Let $u \in X$ be such that $LIM_n\|x_n - u\|^2 = \inf_{z \in X} LIM_n\|x_n - z\|^2$. Then u minimizes the continuous convex function $\phi : X \to \mathbb{R}^+$ defined by $\phi(z) = LIM_n\|x_n - z\|^2$, so we have $\phi'(u) = 0$.

Note that the norm of X is Gâteaux differentiable, and Jx is the subdifferential of the convex function $\varphi(x) = \|x\|^2/2$ at x as the Gâteaux differential of φ. Hence

$$LIM_n\langle z, J(x_n - u)\rangle = \langle z, \phi'(u)\rangle = 0 \text{ for all } z \in X.$$

Conversely, suppose that $LIM_n\langle u - z, J(x_n - u)\rangle = 0$ for all $z \in X$. If $x \in X$,

$$\|x_n - x\|^2 - \|x_n - u\|^2 \geq 2\langle u - x, J(x_n - u)\rangle \text{ for all } n \in \mathbb{N}.$$

Because $LIM_n\langle u - x, J(x_n - u)\rangle = 0$ for all $x \in X$, we obtain

$$LIM_n\|x_n - u\|^2 = \inf_{x \in X} LIM_n\|x_n - x\|^2. \qquad \blacksquare$$

Corollary 2.9.13 *Let X be a Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in C. Let LIM be a Banach limit and $u \in C$. Then*

$$u \in M \textit{ if and only if } LIM_n\langle z, J(x_n - u)\rangle \leq 0 \textit{ for all } z \in C.$$

2.10 Metric projection and retraction mappings

Let C be a nonempty subset of a normed space X and let $x \in X$. An element $y_0 \in C$ is said to be a *best approximation* to x if

$$\|x - y_0\| = d(x, C),$$

where $d(x, C) = \inf_{y \in C} \|x - y\|$. The number $d(x, C)$ is called *the distance from x to C* or *the error in approximating x by C.*

The (possibly empty) set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

This defines a mapping P_C from X into 2^C and is called the *metric projection* onto C. The metric projection mapping is also known as the *nearest point projection mapping, proximity mapping*, and *best approximation operator.*

The set C is said to be a *proximinal* [2] (respectively, *Chebyshev*) *set* if each $x \in X$ has at least (respectively, exactly) one best approximation in C.

[2]The term "proximinal" is a combination of the words "proximity" and "minimal" and was coined by Killgrove.

Observation

- C is proximinal if $P_C(x) \neq \emptyset$ for all $x \in X$.
- C is Chebyshev if $P_C(x)$ is singleton for each $x \in X$.
- The set of best approximations is convex if C is convex.

Some fundamental results on proximinal sets are the following:

First, we observe that every proximinal set must be closed.

Proposition 2.10.1 *Let C be a proximinal subset of a Banach space X. Then C is closed.*

Proof. Suppose, for contradiction, that C is not closed. Then there exists a sequence $\{x_n\}$ in C such that $x_n \to x$ and $x \notin C$, but $x \in X$. It follows that

$$d(x, C) \leq \|x_n - x\| \to 0,$$

so that $d(x, C) = 0$. Because $x \notin C$, it means that

$$\|x - y\| > 0 \quad \text{for all} \quad y \in C.$$

This implies $P_C(x) = \emptyset$. This contradicts $P_C(x) \neq \emptyset$. ∎

Theorem 2.10.2 (The existence of best approximations) – *Let C be a nonempty weakly compact convex subset of a Banach space X and $x \in X$. Then x has a best approximation in C, i.e., $P_C(x) \neq \emptyset$.*

Proof. The function $f : C \to \mathbb{R}^+$ defined by

$$f(y) = \|x - y\|, \quad y \in C$$

is obviously lower semicontinuous. Because C is weakly compact, we can apply Theorem 2.5.5, and then there exists $y_0 \in C$ such that $\|x - y_0\| = \inf_{y \in C} \|x - y\|$. ∎

Corollary 2.10.3 *Let C be a nonempty closed convex subset of a reflexive Banach space X. Then each element $x \in X$ has a best approximation in C.*

Theorem 2.10.4 (The uniqueness of best approximations) – *Let C be a nonempty convex subset of a strictly convex Banach space X. Then for each element $x \in X$, C has at most one best approximation.*

Proof. Suppose, for contradiction, that $y_1,\ y_2 \in C$ are best approximations to $x \in X$. Because the set of best approximations is convex, it follows that $(y_1 + y_2)/2$ is also a best approximation to x. Set $r := d(x, C)$. Then

$$0 \leq r = \|x - y_1\| = \|x - y_2\| = \|x - (y_1 + y_2)/2\|,$$

and it follows that

$$\|(x - y_1) + (x - y_2)\| = 2r = \|x - y_1\| + \|x - y_2\|.$$

By the strict convexity of X we have

$$x - y_1 = t(x - y_2), \quad t \geq 0.$$

Taking the norm in this relation, we obtain $r = tr$, i.e., $t = 1$, which gives us $y_1 = y_2$. ∎

The following example shows that the strict convexity cannot be dropped in Theorem 2.10.4.

Example 2.10.5 *Let $X = \mathbb{R}^2$ with norm $\|\cdot\|_1$. It is easy to check that X is not strictly convex. Now, let*

$$C = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \leq 1\} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}.$$

Then C is a closed convex set. The distance from $z = (-1, -1)$ to the set C is one, and this distance is realized by more than one point of C.

In Theorem 2.10.4, uniqueness of best approximations need not be true for nonconvex sets.

Example 2.10.6 *Let $X = \mathbb{R}^2$ with the norm $\|\cdot\|_2$ and $C = S_X = \{(x, y)\} \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then X is strictly convex and C is a nonconvex set. However, all points of C are best approximations to $(0, 0) \in X$.*

Theorem 2.10.7 *If in a Banach space X, every element possesses at most a best approximation with respect to every convex set, then X is strictly convex.*

Proof. Suppose, for contradiction, that X is not strictly convex. Then there exist $x, y \in X,\ x \neq y$ with

$$\|x\| = \|y\| = \|(x + y)/2\| = 1.$$

Furthermore,

$$\|tx + (1 - t)y\| = 1 \text{ for all } t \in [0, 1].$$

Set $C := co(\{x, y\})$. Then $\|0 - z\| = d(0, C)$ for all $z \in C$. This means that every element of C is the best approximation to zero and this clearly contradicts the uniqueness. ∎

From Corollary 2.10.3 and Theorem 2.10.4 (see also Proposition 2.1.10), we obtain some important results:

Theorem 2.10.8 *Let C be a nonempty weakly compact convex subset of a strictly convex Banach space X. Then for each $x \in X$, C has the unique best approximation, i.e., $P_C(\cdot)$ is a single-valued metric projection mapping from X onto C.*

Corollary 2.10.9 *Let C be a nonempty closed convex subset of a strictly convex reflexive (e.g., uniformly convex) Banach space X and let $x \in X$. Then there exists a unique element $x_0 \in C$ such that $\|x - x_0\| = d(x, C)$.*

Observation

- Every closed convex subset C of a reflexive Banach space is proximinal.
- Every closed convex subset C of a reflexive strictly convex Banach is a Chebyshev set.
- For every Chebyshev set C, we have
  (i) $P_C(x)$ is singleton set, i.e., P_C is a function from X onto C.
  (ii) $\|x - P_C(x)\| = d(x, C)$ for all $x \in X$.

We now study useful properties of metric projection mappings.

Theorem 2.10.10 *Let C be a subset of a normed space X and $\overline{x} \in X$. Then $P_C(\overline{x}) \subseteq \partial C$.*

Proof. Let $y \in P_C(\overline{x})$. Suppose $y \in int(C)$. Then there exists an $\varepsilon > 0$ such that $B_\varepsilon(y) \subset C$. For each $n \in \mathbb{N}$, let $z_n = (1/n)\overline{x} + (1 - 1/n)y$. Then

$$\|z_n - y\| = (1/n)\|\overline{x} - y\|.$$

For sufficiently large $N \in \mathbb{N}$, $\|z_N - y\| < \varepsilon$. Thus, $z_N \in B_\varepsilon(y) \subset C$. On the other hand,

$$\|\overline{x} - z_N\| = (1 - 1/N)\|\overline{x} - y\| < \|\overline{x} - y\| = d(\overline{x}, C),$$

which contradicts the fact that $y \in P_C(\overline{x})$. Therefore, $y \in \partial C$. ∎

Corollary 2.10.11 *Let C be a nonempty closed convex subset of a strictly convex reflexive Banach space X and let $x \in X$. Then we have the following:*

(a) If $x \in C$, then $P_C(x) = x$.

(b) If $x \notin C$, then $P_C(x) \in \partial C$.

Theorem 2.10.12 *Let C be a nonempty closed convex subset of a reflexive strictly convex Banach space X. If X has the Kadec-Klee property, then the projection mapping P_C of X onto C is continuous.*

Proof. Suppose, for contradiction, that P_C is not continuous. Then for the sequence $\{x_n\}$ in X with $\lim_{n\to\infty} x_n = x \in X$, there exists $\varepsilon > 0$ such that

$$\|P_C(x_n) - P_C(x)\| \geq \varepsilon \text{ for all } n \in \mathbb{N}.$$

Because

$$|d(x_n, C) - d(x, C)| \leq \|x_n - x\|,$$

it follows that

$$\big|\|x_n - P_C(x_n)\| - \|x - P_C(x)\|\big| \leq \|x_n - x\|.$$

This implies that

$$\lim_{n\to\infty} \|x_n - P_C(x_n)\| = \|x - P_C(x)\|. \tag{2.33}$$

Because $\{P_C(x_n)\}$ is bounded in C by (2.33), there exists a subsequence $\{P_C(x_{n_i})\}$ of $\{P_C(x_n)\}$ such that $w - \lim_{i\to\infty} P_C(x_{n_i}) = z \in C$. Note

$$w - \lim_{i\to\infty} (x_{n_i} - P_C(x_{n_i})) = x - z. \tag{2.34}$$

By w-lsc of the functional $\|\cdot\|$, we have

$$\|x - z\| \leq \liminf_{i\to\infty} \|x_{n_i} - P_C(x_{n_i})\| = \|x - P_C(x)\|.$$

This implies $z = P_C(x)$ by definition of the function P_C. From (2.33) and (2.34)

$$w - \lim_{i\to\infty} (x_{n_i} - P_C(x_{n_i})) = x - P_C(x) \text{ and } \lim_{i\to\infty} \|x_{n_i} - P_C(x_{n_i})\| = \|x - P_C(x)\|.$$

Because X has the Kadec-Klee property, we obtain

$$\lim_{i\to\infty} (x_{n_i} - P_C(x_{n_i})) = x - P_C(x),$$

which implies that $\lim_{i\to\infty} P_C(x_{n_i}) = P_C(x)$, which is a contradiction to the assumption that $\|P_C(x_n) - P_C(x)\| \geq \varepsilon$. ∎

Then following Proposition 2.5.25, we have

Theorem 2.10.13 *Let C be a nonempty convex subset of a smooth Banach space X and let $x \in X$ and $y \in C$. Then the following are equivalent:*

(a) y is a best approximation to x: $\|x - y\| = d(x, C)$.

(b) y is a solution of the variational inequality:

$$\langle y - z, J_\mu(x - y)\rangle \geq 0 \text{ for all } z \in C,$$

where J_μ is a duality mapping with gauge function μ.

As an immediate consequence of Theorem 2.10.13, we have

Corollary 2.10.14 *Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let x be an element in H. Then the following are equivalent:*

(a) $\|x - P_C(x)\| = d(x, C)$.

(b) $\langle x - P_C(x), P_C(x) - z\rangle \geq 0$ for all $z \in C$.

Proposition 2.10.15 *Let C be a nonempty closed convex subset of a Hilbert space X and P_C the metric projection from X onto C. Then the following hold:*

(a) P_C is "idempotent": $P_C(P_C(x)) = P_C(x)$ for all $x \in X$.

(b) P_C is "firmly nonexpansive":

$$\langle x-y, P_C(x)-P_C(y)\rangle \geq \|P_C(x)-P_C(y))\|^2 \quad \text{for all } x,y\in X.$$

(c) P_C is "nonexpansive": $\|P_C(x)-P_C(y)\| \leq \|x-y\|$ for all $x,y\in X$.

(d) P_C is "monotone": $\langle P_C(x)-P_C(y), x-y\rangle \geq 0$ for all $x,y\in X$.

(e) P_C is "demiclosed": $x_n \rightharpoonup x_0$ and $P_C(x_n)\to y_0 \Rightarrow P_C(x_0)=y_0$.

Proof. (a) Observe that $P_C(x)\in C$ for all $x\in X$ and $P_C(z)=z$ for all $z\in C$. Then $P_C(P_C(x))=P_C(x)$ for all $x\in X$, i.e., $P_C^2=P_C$.

(b) Set $j:=P_C(x)-P_C(y)$ for $x,y\in X$. We have

$$\langle x-y,j\rangle = \langle x-P_C(x),j\rangle + \langle j,j\rangle + \langle P_C(y)-y,j\rangle.$$

Because from Corollary 2.10.14, we get

$$\langle x-P_C(x),j\rangle \geq 0 \text{ and } \langle y-P_C(y),j\rangle \geq 0,$$

it follows that

$$\langle x-y,j\rangle \geq \|j\|^2.$$

(c) This is an immediate consequence of (b).

(d) It follows from (b).

(e) From Corollary 2.10.14, we have

$$\langle x_n-P_C(x_n), P_C(x_n)-z\rangle \geq 0 \quad \text{for all} \quad z\in C.$$

Because $x_n \rightharpoonup x_0$ and $P_C(x_n)\to y_0$, we have

$$\langle x_0-y_0, y_0-z\rangle \geq 0 \text{ for all } z\in C.$$

Using Theorem 2.10.13, we obtain $\|x_0-y_0\| = d(x_0,C)$. Therefore, $P_C(x_0)=y_0$. ∎

Remark 2.10.16 *Proposition 2.10.15(c) shows that in a Hilbert space, a metric projection operator is not only continuous, but also it is Lipschitz continuous and hence it is uniformly continuous.*

The following result is of fundamental importance. It shows that every point on line segment joining $x\in X$ to its best approximation $P_C(x)\in C$ has $P_C(x)$ as its best approximation.

Proposition 2.10.17 *Let C be a Chebyshev set in a Hilbert space H and $x\in H$. Then $P_C(x)=P_C(y)$ for all $y\in co(\{x,P_C(x)\})$.*

Proof. Suppose, for contradiction, that there exist $y \in co(\{x, P_C(x)\})$ and $z \in C$ such that

$$\|y - z\| < \|y - P_C(x)\|.$$

Set $y := \lambda x + (1-\lambda)P_C(x)$ for some $\lambda \in (0,1)$. Then

$$\begin{aligned} \|x - z\| &\le \|x - y\| + \|y - z\| \\ &< \|x - y\| + \|y - P_C(x)\| \\ &= (1-\lambda)\|x - P_C(x)\| + \lambda\|x - P_C(x)\| = d(x, C), \end{aligned}$$

a contradiction. ∎

If C is a Chebyshev set in a Hilbert space H, then

$$P_C[\lambda x + (1-\lambda)P_C(x)] = P_C(x),\ x \in H,\ 0 \le \lambda \le 1.$$

Motivated by this fact, we introduce the following:

A Chebyshev subset C of a normed space X is said to be *sun* if

$$P_C[\lambda x + (1-\lambda)P_C(x)] = P_C(x) \text{ for all } x \in X \text{ and } \lambda \ge 0.$$

In other words, C is a sun if and only if each point on the ray from $P_C(x)$ through x also has $P_C(x)$ as its best approximation in C.

Let C be a nonempty subset of a topological space X and D a nonempty subset of C. Then a continuous mapping $P : C \to D$ is said to be a *retraction* if $Px = x$ for all $x \in D$, i.e., $P^2 = P$. In such case, D is said to be a *retract* of C.

Example 2.10.18 *Every closed convex subset C of $\mathbb{R}^n$ is a retract of $\mathbb{R}^n$.*

We have seen in Theorem 2.10.8 that for every weakly compact convex subset C of a strictly convex Banach space, there exists a metric projection mapping $P_C : X \to C$ that may not be continuous. However, every single-valued metric projection mapping is a retraction if it is continuous.

Theorem 2.10.19 *Every closed convex subset C of a uniformly convex Banach space X is a retract of X.*

Proof. By Theorem 2.10.8, there exists a metric projection mapping $P_C : X \to C$ such that $P_C(x) = x$ for all $x \in C$. By Theorem 2.10.12, P_C is continuous. Therefore, P_C is retraction. ∎

We now show that every retraction P with condition (2.35) is sunny nonexpansive (and hence continuous).

Proposition 2.10.20 *Let C be a nonempty convex subset of a smooth Banach space X and D a nonempty subset of C. If P is a retraction of C onto D such that*

$$\langle x - Px, J(y - Px)\rangle \le 0 \text{ for all } x \in C \text{ and } y \in D, \tag{2.35}$$

then P is sunny nonexpansive.

Proof. <u>*P is sunny:*</u> For $x \in C$, set $x_t := Px + t(x - Px)$ for all $t > 0$. Because C is convex, it follows that $x_t \in C$ for all $t \in (0, 1]$. Hence

$$\langle x - Px, J(Px - Px_t)\rangle \geq 0 \text{ and } \langle x_t - Px_t, J(Px_t - Px)\rangle \geq 0. \quad (2.36)$$

Because $x_t - Px = t(x - Px)$ and $\langle t(x - Px), J(Px - Px_t)\rangle \geq 0$, we have

$$\langle x_t - Px, J(Px - Px_t)\rangle \geq 0. \quad (2.37)$$

Combining (2.36) and (2.37), we get

$$\begin{aligned} \|Px - Px_t\|^2 &= \langle Px - x_t + x_t - Px_t, J(Px - Px_t)\rangle \\ &\leq -\langle x_t - Px, J(Px - Px_t)\rangle + \langle x_t - Px_t, J(Px - Px_t)\rangle \\ &\leq 0. \end{aligned}$$

Thus, $Px = Px_t$. Therefore, P is sunny.

<u>*P is nonexpansive :*</u> For $x, z \in C$, we have from (2.35) that

$$\langle x - Px, J(Px - Pz)\rangle \geq 0 \text{ and } \langle z - Pz, J(Pz - Px)\rangle \geq 0.$$

Hence

$$\langle x - z - (Px - Pz), J(Px - Pz)\rangle \geq 0.$$

This implies that

$$\langle x - z, J(Px - Pz)\rangle \geq \|Px - Pz\|^2$$

and hence P is nonexpansive. ▮

We now give equivalent formulations of sunny nonexpansive retraction mappings.

Proposition 2.10.21 *Let C be a nonempty convex subset of a smooth Banach space X, D a nonempty subset of C, and $P : C \to D$ a retraction. Then the following are equivalent:*

(a) P is the sunny nonexpansive.

(b) $\langle x - Px, J(y - Px)\rangle \leq 0$ for all $x \in C$ and $y \in D$.

(c) $\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.

Proof. (a)⇒ (b). Let P be the sunny nonexpansive retraction and $x \in C$. Then $Px \in D$ and there exists a point $z \in D$ such that $Px = z$. Set $M := \{z + t(x - z) : t \geq 0\}$. Then M is nonempty convex set. Hence for $v \in M$

$$\begin{aligned} \|y - z\| &= \|Py - Pv\| \quad (\text{as } P \text{ is sunny, i.e., } Pv = z) \\ &\leq \|y - v\| = \|y - z + t(z - x)\| \text{ for all } y \in D. \end{aligned}$$

Hence from Proposition 2.4.7, we have

$$\langle x - Px, J(y - Px)\rangle \leq 0.$$

(b) $\Rightarrow$ (a). It follows from Proposition 2.10.20.

(b) $\Rightarrow$ (c). Let $x, y \in C$. Then $Px, Py \in D$ and hence from (b), we have

$$\langle x - Px, J(Py - Px)\rangle \le 0 \text{ and } \langle y - Py, J(Px - Py)\rangle \le 0.$$

Combining the above inequalities, we get

$$\langle Px - Py - (x - y), J(Px - Py)\rangle \le 0.$$

Hence

$$\begin{aligned} \|Px - Py\|^2 &= \langle Px - Py, J(Px - Py)\rangle \\ &= \langle Px - Py - (x - y), J(Px - Py)\rangle + \langle x - y, J(Px - Py)\rangle \\ &\le \langle x - y, J(Px - Py)\rangle. \end{aligned}$$

(c) $\Rightarrow$ (b). Suppose (c) holds. Let $x \in C$ and $y \in D$. Replacing y by $y = Py$ in (c), we have

$$\langle x - Py, J(Px - P^2y) \ge \|Px - P^2y\|^2,$$

which implies that

$$\langle x - y, J(Px - y)\rangle \ge \|Px - y\|^2.$$

Therefore,

$$\begin{aligned} \langle x - Px, J(Px - y)\rangle &= \langle x - y, J(Px - y)\rangle + \langle y - Px, J(Px - y) \\ &\ge \|Px - y\|^2 - \|Px - y\|^2 = 0. \quad \blacksquare \end{aligned}$$

Finally, we give uniqueness of sunny nonexpansive retraction mappings.

Proposition 2.10.22 *Let C be a nonempty convex subset of a smooth Banach space X and D a nonempty subset of C. If P is a sunny nonexpansive retraction from C onto D, then P is unique.*

Proof. Let Q be another sunny nonexpansive retraction from C onto D. Then, we have, for each $x \in C$

$$\langle x - Px, J(y - Px)\rangle \le 0 \text{ and } \langle x - Qx, J(y - Qx)\rangle \le 0 \quad \text{for all } y \in D.$$

In particular, because Px and Qx are in D, we have

$$\langle x - Px, J(Qx - Px)\rangle \le 0 \text{ and } \langle x - Qx, J(Px - Qx)\rangle \le 0,$$

which imply that $\|Px - Qx\|^2 \le 0$. Therefore, $Px = Qx$ for all $x \in C$. $\blacksquare$

Bibliographic Notes and Remarks

The results of Sections 2.1∼2.3 are well-known. These results are adapted from Cioranescu [41], Goebel and Kirk [59], Goebel and Reich [60], Istratescu [73], Martin [106], and Prus [121].

The results in Sections 2.4∼2.8 are based on Barbu [11], Goebel and Kirk [59], Prus [121], and Showalter [145]. Theorem 2.4.16 follows from Kaczor [82]. Theorem 2.6.11 is proved in Reich [127]. The interested reader should also consult the details on duality mappings in Cioranescu [41] and its review by Reich [131].

Some results in Section 2.9 are based on some results of Ha and Jung [65], Jung and Park [80], and Shioji and Takahashi [144]. Theorem 2.9.11 is a cornerstone of the "Optimization Method" expounded in Reich [126, 128]. The uniqueness of the minimizer is shown in the paper by Reich [130].

The results in Section 2.10 are adapted from several sources (Goebel and Kirk [59], Goebel and Reich [60], and Takahashi [155]). Theorem 2.10.12 is an important improvement of Proposition 3.2 of Martin [106]. Proposition 2.10.21 was first proved in Reich [123], where the term "sunny nonexpansive retraction" was coined. More updated information on (sunny) nonexpansive retractions can be found in Reich and Kopecká [132].

Exercises

2.1 Let X be a strictly convex Banach space and let $x, y \in X$ with $x \neq y$. If $\|x-z\| = \|x-w\|, \|z-y\| = \|w-y\|$ and $\|x-y\| = \|x-z\| + \|z-y\|$, show that $z = w$.

2.2 Let X be a uniformly convex Banach space and let δ_X be the modulus of convexity of X. Let $0 < \varepsilon < r \leq 2R$. Show that $\delta_X(\varepsilon/R) > 0$ and

$$\|\lambda x + (1-\lambda)y\| \leq r\left\{1 - 2\lambda(1-\lambda)\delta_X\left(\frac{\varepsilon}{R}\right)\right\}$$

for all $x, y \in X$ with $\|x\| \leq r, \|y\| \leq r$ and $\|x-y\| \geq \varepsilon$ and $\lambda \in [0,1]$.

2.3 Let X be a Banach space. Show that X is uniformly convex if and only if $\gamma(t) > 0$ for all $t \in (0,2]$, where

$$\gamma(t) = \inf\{\langle x-y, x^* - y^*\rangle : x, y \in S_X, \|x-y\| \geq t, x^* \in J(x), y^* \in J(y)\}.$$

2.4 If $1 < p < \infty$, and if the $X_n's$ are all strictly convex Banach spaces, show that

$$(\prod_{n\in\mathbb{N}} X_n)_p = \{x = \{x_n\} : x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n\in\mathbb{N}} \|x_n\|_{x_n}^p < \infty\}$$

endowed with norm

$$\|x\| = (\sum_{n\in\mathbb{N}} \|x_n\|_{x_n}^p)^{1/p}$$

is strictly convex.

2.5 On $L^2([0,1], dt)$, we consider the norm

$$\|f\| = \left[\frac{1}{2}(\|f\|_2^2 + \|f\|_1^2)\right]^{1/2}.$$

Show that this norm is equivalent to $\|\cdot\|_2$, but is not smooth.

2.6 On ℓ_1, we consider the norm $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{1/2}, \ x = \{x_n\}_{n\in\mathbb{N}}$ (where $\|x\|_1 = \sum_{n\in\mathbb{N}} |x_n|, \|x\|_2 = (\sum_{n\in\mathbb{N}} |x_n|^2)^{1/2}$).
Show that this norm is equivalent to the ℓ_1-norm and that it is strictly convex.

2.7 Let C be a nonempty closed convex subset of a strictly convex Banach space X and D a nonempty subset of C. Let $x \in C$ and P be a sunny nonexpansive retraction of C onto D such that $\|Px - y\| = \|x - y\|$ for some $y \in D$. Then $Px = x$.

Chapter 3

Geometric Coefficients of Banach Spaces

Geometric coefficients play a key role in the existence of fixed points of Lipschitzian as well as non-Lipschitzian mappings. In this chapter, we discuss normal structure coefficient, weak normal structure coefficient, Maluta constants, and other related coefficients.

3.1 Asymptotic centers and asymptotic radius

The concept of asymptotic center is introduced, and several useful results are discussed here.

Let C be a nonempty subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Consider the functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X.$$

The infimum of $r_a(\cdot, \{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C, \{x_n\})$. A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $\mathcal{Z}_a(C, \{x_n\})$. This set may be empty, a singleton, or certain infinitely many points. In fact, if $\{x_n\}$ converges strongly to $x \in C$, then

$$\mathcal{Z}_a(C, \{x_n\}) = \{x\}$$

and if $\{x_n\}$ converges strongly to x and $x \notin C$, then

$$r_a(C, \{x_n\}) = d(x, C) \text{ and } \mathcal{Z}_a(C, \{x_n\}) = \{y \in C : \|x - y\| = d(x, C)\} = P_C(x),$$

where P_C is the metric projection from X into 2^C.

R.P. Agarwal et al., *Fixed Point Theory for Lipschitzian-type Mappings with Applications,* Topological Fixed Point Theory and Its Applications 6, DOI 10.1007/978-0-387-75818-3_3,

For any $\lambda \geq 0$, the level set is

$$A_\lambda(C, \{x_n\}) = \{x \in C : r_a(x, \{x_n\}) \leq r_a(C, \{x_n\}) + \lambda\}.$$

It can be easily observed that

(*i*) $A_0(C, \{x_n\}) = \mathcal{Z}_a(C, \{x_n\})$, the asymptotic center of $\{x_n\}$ with respect to C,

(*ii*) $A_\lambda(C, \{x_n\}) \neq \emptyset$ for all $\lambda > 0$,

(*iii*) $A_{\lambda'}(C, \{x_n\}) \subset A_\lambda(C, \{x_n\})$ if $\lambda' < \lambda$,

(*iv*) $\mathcal{Z}_a(C, \{x_n\}) = A_0(C, \{x_n\}) = \cap_{\lambda>0} A_\lambda(C, \{x_n\})$ may be empty,

(*v*) $$\mathcal{Z}_a(C, \{x_n\}) = \cap_{\lambda>0} \overline{\{\cup_{n=1}^{\infty} \cap_{i=n}^{\infty} B_{r_a(C,\{x_n\})+\lambda}[x_i]\}} \cap C. \tag{3.1}$$

A bounded sequence $\{x_n\}$ in a Banach space X is said to be *regular* with respect to a bounded subset C of X if the asymptotic radii (with respect to C) of all subsequences of $\{x_n\}$ are the same, i.e.,

$$r_a(C, \{x_{n_i}\}) = r_a(C, \{x_n\}) \text{ for each subsequence } \{x_{n_i}\} \text{ of } \{x_n\}.$$

A regular sequence $\{x_n\}$ in X is said to be *asymptotically uniform with respect to C* if $\mathcal{Z}_a(C, \{x_{n_i}\}) = \mathcal{Z}_a(C, \{x_n\})$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Observation

- For an arbitrary subsequence $\{x_{n_i}\}$ of $\{x_n\}$, the following fact always holds:

$$\mathcal{Z}_a(C, \{x_{n_i}\}) \supseteq \mathcal{Z}_a(C, \{x_n\}).$$

- Asymptotic radius: $r_a(C, \{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in C\}$.
- Asymptotic center: $\mathcal{Z}_a(C, \{x_n\}) = \{z \in C : r_a(z, \{x_n\}) = r_a(C, \{x_n\})\}$.
- For $x \in X, r_a(x, \{x_n\}) = 0 \Leftrightarrow \lim\limits_{n\to\infty} x_n = x$.
- $r_a(\alpha x + \beta y, \{x_n\}) \leq \alpha r_a(x, \{x_n\}) + \beta r_a(y, \{x_n\})$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
- $|r_a(x, \{x_n\}) - r_a(y, \{x_n\})| \leq \|x - y\| \leq r_a(x, \{x_n\}) + r_a(y, \{x_n\})$ for all $x, y \in X$.
- $r_a(\cdot, \{x_n\})$ is convex and nonexpansive (and hence continuous).
- $\varphi(\cdot) = r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ is w-lsc. Indeed, by the continuity of $\varphi(\cdot)$, $\varphi^{-1}((-\infty, \alpha])$ is closed for every $\alpha \in \mathbb{R}$. Also convexity of $\varphi(\cdot)$ implies that $\varphi^{-1}((-\infty, \alpha])$ is convex. Thus, $\varphi^{-1}((-\infty, \alpha])$ is weakly closed.

We first establish two preliminary results:

Proposition 3.1.1 *Let C be a nonempty bounded subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then $\{x_n\}$ has a subsequence that is regular with respect to C.*

Proof. Set

$$r_0 := \inf\{r_a(C,\{x_{n_i}\}) : \{x_{n_i}\} \text{ is a subsequence of } \{x_n\}\}.$$

Then there is a subsequence $\{x_{n_{k(1)}}\}$ of $\{x_n\}$ such that

$$r_a(C,\{x_{n_{k(1)}}\}) \le r_0 + 1.$$

Set

$$r_1 := \inf\{r_a(C,\{x_{n_{k(1)_j}}\})\},$$

where the infimum is taken over all subsequences $\{x_{n_{k(1)_j}}\}$ of $\{x_{n_{k(1)}}\}$. Let $\{x_{n_{k(i)}}\}$ be a subsequence of $\{x_{n_{k(i-1)}}\}$ and set

$$r_i := \inf\{r_a(C,\{x_{n_{k(i)_j}}\})\},$$

where the infimum is taken over all subsequences $\{x_{n_{k(i)_j}}\}$ of $\{x_{n_{k(i)}}\}$. Select a subsequence $\{x_{n_{k(i+1)}}\}$ of $\{x_{n_{k(i)}}\}$ such that

$$r_a(C,\{x_{n_{k(i+1)}}\}) < r_i + \frac{1}{i+1}. \tag{3.2}$$

Because $r_1 \le r_2 \le r_3 \le \cdots$ and $\{r_i\}$ is bounded above, it follows that $\lim\limits_{i\to\infty} r_i$ exists (say r). Then from (3.2),

$$\lim_{i\to\infty} r_a(C,\{x_{n_{k(i+1)}}\}) = r.$$

Now consider the diagonal sequence $\{x_{n_{k(k)}}\}$ and $\overline{r} = r_a(C,\{x_{n_{k(k)}}\})$. Because $\{x_{n_{k(k)}}\}$ is a subsequence of $\{x_{n_{k(i)}}\}$, it follows that $r_i \le \overline{r}$.

Moreover, from (3.2) we have

$$\overline{r} \quad\le\quad r_a(C,\{x_{n_{k(i+1)}}\}) < r_i + \frac{1}{i+1},$$

which implies that

$$\overline{r} \le r.$$

Hence $\overline{r} = r$. Note any subsequence $\{y_n\}$ of $\{x_{n_{k(k)}}\}$ satisfies the following:

$$\{y_n\} \subseteq \{x_{n_{k(i)}}\} \text{ and } \{y_n\} \subseteq \{x_{n_{k(i+1)}}\}.$$

Hence $r_a(C,\{y_n\}) = r$, and we conclude that $\{x_{n_{k(k)}}\}$ is regular with respect to C. ∎

Proposition 3.1.2 *Let C be a separable bounded subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then $\{x_n\}$ has an asymptotically uniform subsequence.*

Proof. By Proposition 3.1.1, $\{x_n\}$ has a subsequence that is regular with respect to C. Because C is separable, a routine diagonalization argument can be used to obtain a subsequence of $\{x_n\}$, which we again denote by $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - y\|$ exists for all $y \in C$. Clearly, such a sequence must be asymptotically uniform. ∎

We now discuss the existence and uniqueness of asymptotic centers of bounded sequences.

Theorem 3.1.3 (The existence of of asymptotic centers) – *Let $\{x_n\}$ be a bounded sequence in a Banach space X and C a nonempty subset of X. Then we have the following:*

(a) If C is weakly compact, then $\mathcal{Z}_a(C, \{x_n\})$ is nonempty.

(b) If C is weakly compact and convex, then $\mathcal{Z}_a(C, \{x_n\})$ is a nonempty convex set.

Proof. (a) From (3.1), we obtain that $\mathcal{Z}_a(C, \{x_n\})$ can be characterized as the intersection of a decreasing family of weakly closed sets. Thus, $\mathcal{Z}_a(C, \{x_n\})$ is nonempty.

(*b*) Because C is weakly compact convex set and the function $r_a(\cdot, \{x_n\})$ is continuous, it follows from Theorem 2.5.5 that $\mathcal{Z}_a(C, \{x_n\}) = \{x \in C : r_a(x, \{x_n\}) = \inf_{z\in C} r_a(z, \{x_n\})\}$ is nonempty. Also $\mathcal{Z}_a(C, \{x_n\})$ is convex. Indeed, for $x, y \in \mathcal{Z}_a(C, \{x_n\})$ and $t \in [0, 1]$, we have

$$\begin{aligned} r_a((1-t)x + ty, \{x_n\}) &= \limsup_{n\to\infty} \|(1-t)x + ty - x_n\| \\ &\leq (1-t)\limsup_{n\to\infty} \|x_n - x\| + t\limsup_{n\to\infty} \|x_n - y\| \\ &= (1-t)r_a(C, \{x_n\}) + tr_a(C, \{x_n\}) = r_a(C, \{x_n\}), \end{aligned}$$

i.e., $(1-t)x + ty \in \mathcal{Z}_a(C, \{x_n\})$. ∎

Theorem 3.1.4 *Let C be a nonempty closed convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. Then*

$$diam(\mathcal{Z}_a(C, \{x_n\})) \leq \epsilon_0(X)\ r_a(C, \{x_n\}).$$

Proof. Set $d = diam(\mathcal{Z}_a(C, \{x_n\}))$. If $\mathcal{Z}_a(C, \{x_n\})$ is empty or a singleton, then we are done. So, we may assume that $d > 0$. Let $0 < r < d$ and $x, y \in \mathcal{Z}_a(C, \{x_n\})$ with $\|x - y\| \geq d - r$. By the convexity of $\mathcal{Z}_a(C, \{x_n\})$, $(x + y)/2 = z \in \mathcal{Z}_a(C, \{x_n\})$. Then from the property of modulus of convexity (see Corollary 2.3.11),

$$\begin{aligned} r_a(C, \{x_n\}) &= r_a(z, \{x_n\}) = \limsup_{n\to\infty} \|x_n - (x + y)/2\| \\ &\leq \left(1 - \delta_X\left(\frac{d - r}{r_a(C, \{x_n\})}\right)\right) r_a(C, \{x_n\}), \end{aligned}$$

so it follows that

$$\delta_X\left(\frac{d-r}{r_a(C,\{x_n\})}\right) \leq 0.$$

By the definition of $\epsilon_0(X)$,

$$d - r \leq \epsilon_0(X) r_a(C,\{x_n\}).$$

Because $r > 0$ is arbitrary, it follows that $d \leq \epsilon_0(X)\ r_a(C,\{x_n\})$. ∎

Using Theorem 3.1.4, we obtain

Theorem 3.1.5 (The uniqueness of asymptotic centers) – *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in X. Then $\mathcal{Z}_a(C,\{x_n\})$ is a singleton set.*

Proof. Because $r_a(\cdot,\{x_n\})$ is a continuous and convex functional and $r_a(z,\{x_n\}) \to \infty$ as $\|z\| \to \infty$, by Theorem 2.5.8, we obtain that $\mathcal{Z}_a(C,\{x_n\}) \neq \emptyset$. By the uniform convexity of X, $\epsilon_0(X) = 0$, it follows from Theorem 3.1.4 that $diam(\mathcal{Z}_a(C,\{x_n\}) = 0$, i.e., $\mathcal{Z}_a(C,\{x_n\})$ is a singleton set. ∎

The following theorem shows that the asymptotic center enjoys an interesting inequality.

Theorem 3.1.6 *Let C be a nonempty closed convex subset of a uniformly convex Banach space. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C, i.e., $\mathcal{Z}_a(C,\{x_n\}) = \{z\}$ and*

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - x\| \ for\ x \neq z.$$

Proof. The result follows from Theorem 3.1.5. ∎

Observation

- If C is weakly compact, then $\mathcal{Z}_a(C,\{x_n\})$ is nonempty.
- If C is closed, then $\mathcal{Z}_a(C,\{x_n\})$ is closed.
- If C is convex, then $\mathcal{Z}_a(C,\{x_n\})$ is convex.
- $\mathcal{Z}_a(C,\{x_n\}) \subset \partial C \cup \mathcal{Z}_a(X,\{x_n\})$.

We now give the following result, which is very useful in the study of multivalued mappings in Banach spaces.

Proposition 3.1.7 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $\{x_n\}$ a sequence in C with asymptotic center z and asymptotic radius r. For $t \in (0,1)$, let $z_n = (1-t)z + tx_n, n \in \mathbb{N}$. Then $Z_a(C,\{z_n\}) = z$ and $r_a(C,\{z_n\}) = tr$.*

Proof. Suppose, for contradiction, that $\mathcal{Z}_a(C,\{z_n\}) = v \neq z$. Because

$$\|z_n - z\| = t\|x_n - z\| \text{ for all } n \in \mathbb{N}, \tag{3.3}$$

it follows that

$$r_a(C,\{z_n\}) = \inf\{\limsup_{n\to\infty} \|z_n - w\| : w \in C\} \le tr.$$

Let $r_a(C,\{z_n\}) = r'$. Because the asymptotic center v of $\{z_n\}$ is unique, hence from Theorem 3.1.6, we have

$$r' = \limsup_{n\to\infty} \|z_n - v\| < \limsup_{n\to\infty} \|z_n - z\| = tr.$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n - v\| &= \|v - (1-t)z - tx_n + (1-t)z - (1-t)x_n\| \\ &\le \|v - [(1-t)z + tx_n]\| + (1-t)\|x_n - z\| \\ &= \|z_n - v\| + (1-t)\|x_n - z\|,\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|x_n - v\| \le r' + (1-t)r < r$$

contradicting $r_a(C,\{x_n\}) = r$. Thus, $\mathcal{Z}_a(C,\{z_n\}) = z$ and from (3.3), we have $r_a(C,\{z_n\}) = tr$. ∎

We present the following result, which has important applications in the study of fixed point theory of nonlinear mappings:

Theorem 3.1.8 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in C such that $\mathcal{Z}_a(C,\{x_n\}) = \{z\}$. If $\{y_m\}$ is a sequence in C such that $\lim_{m\to\infty} r_a(y_m,\{x_n\}) = r_a(C,\{x_n\})$, then $\lim_{m\to\infty} y_m = z$.*

Proof. Suppose, for contradiction, that $\{y_m\}$ does not converge strongly to z. Then there exists a subsequence $\{y_{m_i}\}$ of $\{y_m\}$ such that

$$\|y_{m_i} - z\| \ge d > 0 \text{ for all } i \in \mathbb{N}.$$

By the uniform convexity of X, there exists $\varepsilon > 0$ such that

$$(r_a(C,\{x_n\}) + \varepsilon)\left[1 - \delta_X\left(\frac{d}{r_a(C,\{x_n\}) + \varepsilon}\right)\right] < r_a(C,\{x_n\}).$$

Because $r_a(z,\{x_n\}) = r_a(C,\{x_n\})$, there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - z\| \le r_a(C,\{x_n\}) + \varepsilon \text{ for all } n \ge n_0.$$

Because $r_a(y_m,\{x_n\}) \to r_a(C,\{x_n\})$ as $m \to \infty$ and hence $r_a(y_{m_i},\{x_n\}) \to r_a(C,\{x_n\})$ as $i \to \infty$, then there exists an integer $n_0' \in \mathbb{N}$ such that

$$\|x_n - y_{m_i}\| \le r_a(C,\{x_n\}) + \varepsilon \text{ for all } n \ge n_0'.$$

Because X is uniformly convex,

$$\left\|x_n - \frac{z+y_{m_i}}{2}\right\| \leq \left[1-\delta_X\left(\frac{d}{(r_a(C,\{x_n\})+\varepsilon)}\right)\right](r_a(C,\{x_n\})+\varepsilon) < r_a(C,\{x_n\})$$

for all $n \geq \max\{n_0, n_0'\}$. This implies that

$$r_a\left(\frac{z+y_{m_i}}{2}, \{x_n\}\right) < r_a(C,\{x_n\}),$$

which contradicts the uniqueness of the asymptotic center z. ∎

Let C be a nonempty subset of a Banach space X. For $x \in C$, the *inward set of x relative to C* is the set

$$I_C(x) = \{(1-t)x + ty : y \in C \text{ and } t \geq 0\}.$$

Geometrically, it is the union of all rays beginning at x and passing through other points of C. Let $\overline{I_C(x)}$ denote the closure of $I_C(x)$. Then we have

Proposition 3.1.9 *Let C be a nonempty closed convex subset of a Banach space X. If $w \in \overline{I_C(x)}$, then $(1-t)x + tw \in \overline{I_C(x)}$ for all $t > 0$.*

Proof. Because $w \in \overline{I_C(x)}$, then there exists a sequence $\{w_n\}$ in $I_C(x)$ such that

$$w = \lim_{n\to\infty} w_n \text{ and } w_n = (1-c_n)x + c_n y_n, \; y_n \in C, \; c_n \geq 0.$$

For $t > 0$, we have

$$\begin{aligned}(1-t)x + tw_n &= (1-t)x + t[(1-c_n)x + c_n y_n] \\ &= (1-tc_n)x + tc_n y_n \in I_C(x),\end{aligned}$$

which implies that

$$(1-t)x + tw \in \overline{I_C(x)}. \quad ∎$$

Proposition 3.1.10 *Let C be a nonempty closed convex subset of a Banach space X. Then the following are equivalent:*

(a) $w \in \overline{I_C(x)}$.

(b) There exists $b > 0$ such that $(1-b)x + bw \in \overline{I_C(x)}$.

(c) $(1-b)x + bw \in \overline{I_C(x)}$ for all $b > 0$.

Proof. $(a) \Rightarrow (c)$. It follows from Proposition 3.1.9.

$(c) \Rightarrow (b)$. It is obvious.

$(b) \Rightarrow (a)$. Suppose that there exists $b > 0$ such that $(1-b)x + bw \in \overline{I_C(x)}$. By Proposition 3.1.9,

$$(1-a)x + a((1-b)x + bw) \in \overline{I_C(x)} \text{ for all } a > 0.$$

Taking $a = 1/b$, we have $w \in \overline{I_C(x)}$. ∎

Proposition 3.1.11 *Let C be a convex subset of a normed linear space X. Then $x - y \in \overline{I_C(x)}$ if and only if*

$$\lim_{h\to 0^+} d(x-hy, C)/h = 0. \tag{3.4}$$

Proof. Suppose that (3.4) holds. Let $x \in C$. Let $\varepsilon > 0$ be given. Then there exists $b \in (0,1)$ such that

$$b^{-1}d(x-by, C) < \frac{\varepsilon}{2}.$$

By the definition of distance, there exists $u \in C$ such that

$$\|x - by - u\| < d(x-by, C) + \frac{b\varepsilon}{2}.$$

Observe that $x + b^{-1}(u-x) \in I_C(x)$. Because

$$\begin{aligned} \|[x + b^{-1}(u-x)] - (x-y)\| &= b^{-1}\|u - (x-by)\| \\ &< b^{-1}\left[d(x-by, C) + \frac{b\varepsilon}{2}\right] \\ &< \varepsilon, \end{aligned}$$

it follows that $x - y$ is in the closure of $I_C(x)$.

Conversely, suppose that $x - y \in \overline{I_C(x)}$. Then there exists a sequence $\{x_n\}$ in $I_C(x)$ such that $x_n \to x - y$. Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - (x-y)\| < \varepsilon \text{ for all } n \geq n_0.$$

Observe that

$$h^{-1}d(x-hy, C) \leq h^{-1}\|x - hy - [(1-h)x + hx_{n_0}]\| + h^{-1}d((1-h)x + hx_{n_0}, C).$$

Because $x_{n_0} \in I_C(x)$ and C is convex, there exists $h_0 > 0$ such that $(1-h_0)x + h_0x_{n_0} \in C$. Thus, if $0 < h \leq h_0$, $h^{-1}d((1-h)x + hx_{n_0}, C) = 0$ and

$$\begin{aligned} h^{-1}d(x-hy, C) &\leq h^{-1}\|x - hy - [(1-h)x + hx_{n_0}]\| \\ &= \|x - y - x_{n_0}\| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{h\to 0^+} d(x-hy, C)/h = 0$. ∎

Proposition 3.1.12 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in C. If z is the asymptotic center of $\{x_n\}$ with respect to C, then it is also the asymptotic center with respect to $\overline{I_C(z)}$.*

Proof. Let v be the asymptotic center of $\{x_n\}$ with respect to $\overline{I_C(z)}$. Suppose that $v \neq z$. Because $v \neq z$ and $C \subseteq \overline{I_C(z)}$, we have $v \in \overline{I_C(z)} \setminus C$ and $r_a(v, \{x_n\}) < r_a(z, \{x_n\})$ by the uniqueness of the asymptotic center (see Theorem 3.1.6). By the continuity of $r_a(\cdot, \{x_n\})$, there exists $w \in \overline{I_C(z)} \setminus C$ such that $r_a(w, \{x_n\}) < r_a(z, \{x_n\})$. Hence $w = (1-t)z + ty$ for some $y \in C$ and $t > 1$. Because $r_a(\cdot, \{x_n\}))$ is a convex functional,

$$\begin{aligned} r_a(y, \{x_n\}) &\leq r_a(t^{-1}w + (1-t^{-1})z, \{x_n\}) \\ &\leq t^{-1}r_a(w, \{x_n\}) + (1-t^{-1})r_a(z, \{x_n\}) \\ &< r_a(z, \{x_n\}), \end{aligned}$$

a contradiction. Hence $v = z$. ∎

Proposition 3.1.13 *Let C be a nonempty closed convex subset of a reflexive Banach space X and $\{x_n\}$ a bounded sequence in X. Then there exists a point $v \in C$ such that*

(a) $r_a(v, \{x_n\}) = \inf\{r_a(z, \{x_n\}) : z \in \overline{I_C(v)}\}$.

(b) $\liminf_{n\to\infty} \langle x - v, J(x_n - v)\rangle \leq 0$ *for all* $x \in \overline{I_C(v)}$ *if the norm of X is uniformly Gâteaux differentiable.*

Proof. (a) Set $\varphi(\cdot) = r_a(\cdot, \{x_n\})$. Observe that $\varphi(\cdot)$ is a continuous and convex functional and $r_a(z, \{x_n\}) \to \infty$ as $\|z\| \to \infty$. By Theorem 2.5.8, we obtain that $\mathcal{Z}_a(C, \{x_n\}) \neq \emptyset$. Let v be an asymptotic center of $\{x_n\}$ with respect to C. We now show that v is also an asymptotic center of $\{x_n\}$ with respect to $\overline{I_C(v)}$, i.e.,

$$\varphi(v) = \inf\{\varphi(z) : z \in \overline{I_C(v)}\}.$$

Set $r := \inf\{\varphi(z) : z \in \overline{I_C(v)}\}$. Suppose that $r < \varphi(v)$. Now for $\varepsilon > 0$, $r+\varepsilon < \varphi(v)$. By the continuity of φ, there exists $z \in I_C(v)$ such that $\varphi(z) \leq r+\varepsilon$. Thus,

$$z = v + t(w - v) \text{ for some } w \in C \text{ and } t \geq 1.$$

By the convexity of φ, we obtain

$$\varphi(v) \leq \varphi(w) \leq t^{-1}\varphi(z) + (1-t^{-1})\varphi(v),$$

which implies that

$$\varphi(v) \leq \varphi(z) \leq r + \varepsilon,$$

a contradiction. This shows that $r = \varphi(v)$.

(b) For arbitrary $y \in \overline{I_C(v)}$ and $t > 0$, let $z_t = (1-t)v + ty$. Then $z_t \in \overline{I_C(v)}$. By Proposition 2.4.5, we have

$$2\langle z_t - v, J(x_n - z_t)\rangle \leq \|x_n - v\|^2 - \|x_n - z_t\|^2,$$

which implies that

$$2t \liminf_{n\to\infty} \langle y - v, J(x_n - z_t)\rangle \leq \varphi^2(v) - \varphi^2(z_t) \leq 0$$

and hence

$$\liminf_{n\to\infty}\langle y - v, J(x_n - x_t)\rangle \le 0.$$

Because X is reflexive with uniformly Gâteaux differential norm, J is uniformly demicontinuous on bounded subsets of X. Using the above fact and letting $t \to 0$, we obtain

$$\liminf_{n\to\infty}\langle y - v, J(x_n - v)\rangle \le 0 \text{ for all } y \in \overline{I_C(v)}. \qquad \blacksquare$$

Note that when X is separable, by a diagonalization argument, given a bounded sequence $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - z\|$ exists for all $z \in X$.

In that case, we have the following.

Proposition 3.1.14 *Let X be a separable reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X and $\{x_n\}$ a bounded sequence in X. Then there exist a point $v \in C$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\limsup_{k\to\infty}\langle x - v, J(x_{n_k} - v)\rangle \le 0 \textit{ for all } x \in \overline{I_C(v)}.$$

Proof. The result follows from Proposition 3.1.13. $\blacksquare$

Proposition 3.1.15 *Let X be a separable reflexive Banach space with a uniformly Gâteaux differentiable norm and $\{x_n\}$ a bounded sequence in X. Then there exist a point $v \in X$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{J(x_{n_k} - v)\}$ converges weakly to zero.*

Proof. Now for $C = X$, $\overline{I_C(v)} = X$. Hence $J(x_{n_k} - v) \rightharpoonup 0$ by Proposition 3.1.14. $\blacksquare$

3.2 The Opial and uniform Opial conditions

The Opial condition plays an important role in convergence of sequences and in the study of the demiclosedness principle of nonlinear mappings and the geometry of Banach spaces.

A Banach space X is said to satisfy the *Opial condition* if whenever a sequence $\{x_n\}$ in X converges weakly to $x_0 \in X$, then

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\| \text{ for all } x \in X,\ x \ne x_0. \tag{3.5}$$

We observe that (3.5) is equivalent to the analogous condition obtained by replacing lim inf by lim sup. Replacing the strict inequality " $<$ " in (3.5) with " $\le$," we obtain the definition of the so-called *non-strict (or weak) Opial condition.*

Example 3.2.1 *Every Hilbert space satisfies the Opial condition, i.e., if the sequence* $\{x_n\}$ *in a Hilbert space* H, *converges weakly to* $x \in H$, *then*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \text{ for all } y \in H \text{ and } y \neq x.$$

In fact, because every weakly convergent sequence is necessarily bounded, so we have that $\limsup_{n\to\infty} \|x_n - x\|$ *and* $\limsup_{n\to\infty} \|x_n - y\|$ *are finite. Note*

$$\|x_n - y\|^2 = \|x_n - x + x - y\|^2 = \|x_n - x\|^2 + \|x - y\|^2 + 2\langle x_n - x, x - y\rangle,$$

so that

$$\limsup_{n\to\infty} \|x_n - y\|^2 > \limsup_{n\to\infty} \|x_n - x\|^2. \quad \blacksquare$$

The following example shows that $L_p[0, 2\pi], 1 < p < \infty$ does not satisfy even the nonstrict Opial condition for any $p \neq 2$.

Example 3.2.2 *Let* f *be a periodic function with period* 2π *defined by*

$$f(t) = \begin{cases} 1 & if \quad 0 \leq t \leq 4\pi/3; \\ -2 & if \quad 4\pi/3 < t \leq 2\pi. \end{cases}$$

Set $x_n(t) = f(nt),\ n \in \mathbb{N}$. *Then* $\{x_n\}$ *is a weakly null sequence of Rademacher-like functions in* $L_p[0, 2\pi]$ *for every* $1 < p < \infty$. *Define a function*

$$\lambda_p(s) = \lim_{n\to\infty} \|x_n - s\|^p = \int_0^{2\pi} |f(t) - s|^p dt,$$

where $s \in \mathbb{R}$ *is treated as the constant function. Note*

$$\lambda'_p(0) = -p\int_0^{2\pi} |f(t)|^{p-1} sgn(f(t))dt = \frac{4\pi}{3} p(2^{p-2} - 1),$$

$\lambda'_p(0) \neq 0$ *whenever* $p \neq 2$. *It follows that* $\lambda_p(0)$ *is not the minimal value of* λ_p, *except for the case* $p = 2$. *Therefore,* $L_p[0, 2\pi]$ *does not satisfy even the nonstrict Opial condition for any* $p \neq 2$.

We now give necessary and sufficient condition for a space satisfying the Opial condition.

Proposition 3.2.3 *A Banach space* X *satisfies the Opial condition if and only if*

$$x_n \rightharpoonup 0 \text{ and } \liminf_{n\to\infty} \|x_n\| = 1 \Rightarrow \liminf_{n\to\infty} \|x_n - x\| > 1 \text{ for all } x \neq 0. \tag{3.6}$$

Proof. Suppose that the condition (3.6) is satisfied. Let $u_n \rightharpoonup u$ and $r = \liminf_{n\to\infty} \|u_n - u\|$. If $r = 0$, then the Opial condition (3.5) follows from the uniqueness of a weak limit.

If $r = \liminf_{n\to\infty} \|u_n - u\| > 0$, then $x_n = r^{-1}(u_n - u) \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| = 1$. Hence from (3.6) we have

$$\liminf_{n\to\infty} \|x_n - x\| > 1 \text{ for } x \neq 0,$$

which implies that

$$\liminf_{n\to\infty} \|r^{-1}(u_n - u) - x\| > \liminf_{n\to\infty} \|r^{-1}(u_n - u)\|,$$

i.e.,

$$\liminf_{n\to\infty} \|u_n - (u + rx)\| > \liminf_{n\to\infty} \|u_n - u\| \text{ for } u \neq u + rx.$$

Hence X satisfies the Opial condition. The inverse implication is obvious. ∎

Proposition 3.2.4 *Let $X_1, X_2, \cdots, X_k$ be Banach spaces with norm $\|\cdot\|_1$, $\|\cdot\|_2$, $\cdots$, $\|\cdot\|_k$, respectively. Let p be a constant in $[1, \infty)$ and put $X = X_1 \times X_2 \times \cdots \times X_k$, where the norm of X is given by*

$$\|(x_1, x_2, \cdots, x_k)\| = (\|x_1\|_2^p + \|x_2\|_2^p + \cdots + \|x_k\|_k^p)^{1/p} \textit{ for all } (x_1, x_2, \cdots, x_k) \in X.$$

Then the following are equivalent:

(a) X has the Opial condition.

(b) Each X_j has the Opial condition.

Proof. $(a) \Rightarrow (b)$. Let $\{x_n\}$ be a sequence in X_j such that $x_n \rightharpoonup z$. Then a sequence

$$\Big\{(0, 0, \cdots, 0, \underset{\substack{\uparrow \\ j^{th} \text{ position}}}{x_n}, 0, \cdots, 0)\Big\}$$

in X converges weakly to

$$(0, 0, \cdots, 0, \underset{\substack{\uparrow \\ j^{th} \text{ position}}}{z}, 0, \cdots, 0).$$

Using this fact, one can easily see that (a) implies (b).

Conversely, let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup z$ and let w belong to $X \setminus \{z\}$. Set $x_n := (x_n^{(1)}, x_n^{(2)}, \cdots, x_n^{(k)})$ for $n \in \mathbb{N}$, $z := (z^{(1)}, z^{(2)}, \cdots, z^{(k)})$ and $w := (w^{(1)}, w^{(2)}, \cdots, w^{(k)})$. Because $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\liminf_{n\to\infty} \|x_n - w\|^p = \lim_{i\to\infty} \|x_{n_i} - w\|^p,$$

and that the limit of $\{\|x_{n_i}^{(j)} - z^{(j)}\|_j\}$ exist for all $j = 1, 2, \cdots, k$. Because X_j for $j, 1 \leq j \leq k$ satisfies the Opial condition,

$$\lim_{i\to\infty} \|x_{n_i}^{(j)} - z^{(j)}\|_j^p \leq \liminf_{n\to\infty} \|x_{n_i}^{(j)} - w^{(j)}\|_j^p$$

and

$$\lim_{i\to\infty} \|x_{n_i}^{(\ell)} - z^{(\ell)}\|_\ell^p < \liminf_{n\to\infty} \|x_{n_i}^{(\ell)} - w^{(\ell)}\|_\ell^p$$

holds for some $\ell, 1 \le \ell \le k$ because $z \neq w$. Therefore, we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - z\|^p &\le \lim_{i\to\infty} \|x_{n_i} - z\|^p \\
&= \sum_{j=1}^{k} \lim_{i\to\infty} \|x_{n_i}^{(j)} - z^{(j)}\|_j^p \\
&< \sum_{j=1}^{k} \liminf_{n\to\infty} \|x_{n_i}^{(j)} - w^{(j)}\|_j^p \\
&\le \lim_{i\to\infty} \|x_{n_i} - w\|^p \\
&= \liminf_{n\to\infty} \|x_n - w\|^p.
\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - w\|. \quad \blacksquare$$

The following example shows that Proposition 3.2.4 is not true if the norm of X is $\|(x_1, x_2, \cdots, x_k)\| = \max\limits_{1\le i\le k} \|x\|_i$.

Example 3.2.5 *Let $X = \mathbb{R} \times \ell_2$ with the norm*

$$\|(a, y)\| = \max\{|a|, \|y\|_2\}.$$

Let e_n be the n^{th} element of the basis of ℓ_2 and $\{x_n = (0, e_n)\}$ be a sequence in X. Then we have

$$\|x_n\| = \max\{0, \|e_n\|_2\} = 1 \textit{ for all } n \in \mathbb{N}$$

and $x_n \rightharpoonup 0$ as $n \to \infty$, but for $x = (1, 0)$, we have

$$\|x_n - x\| = \|(1, e_n)\| = \max\{1, \|e_n\|_2\} = 1 \textit{ for all } n \in \mathbb{N}.$$

Therefore, X does not satisfy the Opial condition, even though $\mathbb{R}$ and ℓ_2 satisfy it.

We now consider some classes of Banach spaces that always imply the Opial condition.

Definition 3.2.6 *A Banach space X is said to have a* weakly continuous duality mapping *if there exists a gauge function μ such that the duality mapping J_μ (with gauge function μ) is single-valued and (sequentially) continuous from the weak topology of X to the weak topology of X^*.*

Example 3.2.7 *The spaces ℓ_p $(1 < p < \infty)$ possess duality mappings that are weakly continuous. To see this, for $(x_1, x_2, \cdots, x_i, \cdots) \in \ell_p$, let*

$$J_\mu(x) = (|x_1|^{p-1} sgn(x_1), |x_2|^{p-1} sgn(x_2), \cdots, |x_i|^{p-1} sgn(x_i), \cdots)$$

and $\mu(t) = t^{p/q}$, where $1/p + 1/q = 1$. Note

$$\ell_p \ni (\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_i^{(n)}, \cdots) = x_n \rightharpoonup x = (\alpha_1, \alpha_2, \cdots, \alpha_i, \cdots) \in \ell_p$$

if and only if

$$\|x_n\|_p \leq M \text{ for all } n \in \mathbb{N} \text{ and } \alpha_i^{(n)} \to \alpha_i \text{ as } n \to \infty.$$

Observe that

$$J_\mu(x_n) = (|\alpha_1^{(n)}|^{p-1} sgn(\alpha_1^{(n)}), |\alpha_2^{(n)}|^{p-1} sgn(\alpha_2^{(n)}), \cdots, |\alpha_i^{(n)}|^{p-1} sgn(\alpha_i^{(n)}), \cdots) \in \ell_q,$$

so it follows that

$$\|J_\mu(x_n)\|_q = \mu(\|x_n\|_p) \leq \mu(M) \text{ for all } n \in \mathbb{N}$$

and

$$|\alpha_i^{(n)}|^{p-1} sgn(\alpha_i^{(n)}) \to |\alpha_i|^{p-1} sgn(\alpha_i) \text{ as } n \to \infty.$$

Thus, we conclude that

$$\ell_q \ni J_\mu(x_n) \rightharpoonup J_\mu(x) \in \ell_q \text{ as } n \to \infty.$$

Therefore, J_μ is a weakly continuous mapping from $X = \ell_p$ into $X^* = \ell_q$.

The following theorem gives an important characterization of Banach spaces that possess weakly continuous mappings.

Theorem 3.2.8 *Let X be a Banach space with a weakly continuous duality mapping J_μ with function gauge μ. Then we have the following:*

(a) If $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x$, then

$$\widetilde{\lim_{n\to\infty}} \Phi(\|x_n - y\|) = \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x\|) + \Phi(\|x - y\|) \text{ for all } y \in X, \tag{3.7}$$

where $\widetilde{\lim_{n\to\infty}}$ is either $\liminf_{n\to\infty}$ or $\limsup_{n\to\infty}$.

(b) X has the Opial condition.

Proof. (a) Because $J_\mu(x)$ is the Gâteaux derivative of the convex functional $\Phi(\|x\|) = \int_0^{\|x\|} \mu(t)dt$, it follows (see Theorem 2.5.23) that

$$\Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\mu(x + ty)\rangle dt \text{ for all } x, y \in X.$$

Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x$ and let y be an element in C. Then $J_\mu(x_n + ty) \rightharpoonup J_\mu(x + ty)$, so

$$\begin{aligned}
\widetilde{\lim_{n\to\infty}} \Phi(\|x_n - y\|) &= \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x + x - y\|) \\
&= \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x\|) \\
&\quad + \widetilde{\lim_{n\to\infty}} \int_0^1 \langle x - y, J_\mu(x_n - x + t(x - y))\rangle dt \\
&= \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x\|) + \int_0^1 \langle x - y, J_\mu(t(x - y))\rangle dt \\
&= \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x\|) + \|x - y\| \int_0^1 \mu(t\|x - y\|) dt \\
&= \widetilde{\lim_{n\to\infty}} \Phi(\|x_n - x\|) + \Phi(\|x - y\|).
\end{aligned}$$

(b) Because Φ is strictly increasing, it follows from (3.7) that X has the Opial condition. ∎

Observation

- The duality mapping of each Hilbert space (e.g., ℓ_2 and $\mathbb{R}^n$) is the identity mapping and hence it is weakly continuous. Therefore, every Hilbert space satisfies the Opial condition (see also Example 3.2.1).

- ℓ^p $(1 < p < \infty)$ spaces have weakly sequentially continuous duality mappings (and hence the Opial condition), but the $L_p[0, 2\pi]$ space $(1 < p < \infty, p \neq 2)$ fails to satisfy the Opial condition. It means that the Opial condition is independent of uniform convexity.

The following Theorem 3.2.9 shows that weak limit of a bounded sequence is the asymptotic center under some geometric conditions.

Theorem 3.2.9 *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex subset of X. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$, then z is the asymptotic center of $\{x_n\}$ in C.*

Proof. From Theorem 3.1.6, $\mathcal{Z}_a(C, \{x_n\})$ is singleton. Let $\mathcal{Z}_a(C, \{x_n\}) = \{u\}$, $u \neq z$. Because $x_n \rightharpoonup z$, by the Opial condition,

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - u\|.$$

Using again Theorem 3.1.6, we obtain

$$\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \|x_n - z\|.$$

Therefore, $z = u$. ∎

Corollary 3.2.10 *Let X be a uniformly convex Banach space with a weakly continuous duality mapping and C a nonempty closed convex subset of X. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$, then z is the asymptotic center of $\{x_n\}$ in C.*

Corollary 3.2.11 *Let C be a nonempty closed convex subset of a Hilbert space H. Then the weak limit of a weakly convergent sequence in C coincides with its asymptotic center with respect to C.*

Remark 3.2.12 *Corollary 3.2.11 is valid in all sequence spaces ℓ^p $(1 < p < \infty)$, but it does not hold in the Lebesgue spaces $L_p[0, 2\pi]$ $(1 < p < \infty, p \neq 2)$.*

We now introduce notions of uniform Opial condition and locally uniform Opial condition:

Definition 3.2.13 *A Banach space X is said to satisfy the uniform Opial condition if for each $t > 0$, there exists an $r > 0$ such that*

$$1 + r \leq \liminf_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \geq t$ and each sequence $\{x_n\}$ in X such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$.

It is obvious from Proposition 3.2.3 that the uniform Opial condition implies the Opial condition.

Definition 3.2.14 *A Banach space X is said to satisfy the locally uniform Opial condition if for any weakly null sequence $\{x_n\}$ in X with $\liminf_{n\to\infty} \|x_n\| \geq 1$ and any $t > 0$, there is an $r > 0$ such that*

$$1 + r \leq \liminf_{n\to\infty} \|x_n + x\|$$

for every $x \in X$ with $\|x\| \geq t$.

Note that

uniform Opial condition ⇒ locally uniform Opial condition ⇒ Opial condition.

We now define the *Opial modulus of X*, denoted by r_X, as follows:

$$r_X(t) = \inf \left\{ \liminf_{n\to\infty} \|x_n + x\| - 1 \right\},$$

where $t \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq t$ and sequences $\{x_n\}$ in X such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$.

It is obvious that r_X is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(t) > 0$ for all $t > 0$.

Note that in the definition of locally uniform Opial condition, "$\liminf_{n\to\infty}$" can be replaced with "$\limsup_{n\to\infty}$."

Let $\{x_n\}$ be a weakly null sequence in a Banach space X with $\liminf_{n\to\infty} \|x_n\| \geq 1$. We define the local Opial modulus of X as follows:

$$r_{X,x_n}(t) = \inf\bigg\{ \liminf_{n\to\infty} \|x_n + x\| - 1 : x \in X \text{ with } \|x\| \geq t \bigg\}.$$

One may easily see that X has the local uniform Opial condition if

$$r_{X,x_n}(t) > 0 \quad \text{for all} \quad t > 0$$

and the uniform Opial condition if

$$r_X(t) = \inf\bigg\{ r_{X,x_n}(t) : x_n \rightharpoonup 0 \text{ with } \liminf_{n\to\infty} \|x_n\| \geq 1 \bigg\} > 0 \quad \text{for all} \quad t > 0.$$

We now establish fundamental properties of the Opial modulus.

Proposition 3.2.15 *Let X be a Banach space with Opial modulus r_X. Then*

$$r_X(t_2) - r_X(t_1) \leq (t_2 - t_1)\frac{1 + r_X(t_1)}{t_1} \textit{ for all } 0 < t_1 \leq t_2,$$

i.e., the Opial modulus r_X is continuous.

Proof. Let

$$G_t := \bigg\{ \{x_n\} \text{ in } X : x_n \rightharpoonup 0 \text{ and} \liminf_{n\to\infty} \|x_n\| \geq t \bigg\}.$$

Then for $0 < t_1 \leq t_2$, we have

$$\begin{aligned} 1 + r_X(t_2) &= \inf\bigg\{ \liminf_{n\to\infty} \|x_n + x\| : \{x_n\} \subset X, x_n \rightharpoonup 0, \liminf_{n\to\infty} \|x_n\| \geq 1, \ \|x\| \geq t_2 \bigg\} \\ &= \inf\bigg\{ \liminf_{n\to\infty} \|x_n + \frac{t_2}{t_1} x\| : \{x_n\} \in G_1, \|x\| \geq t_1 \bigg\}. \end{aligned}$$

Because $t_1/t_2 \leq 1$, it follows that

$$\begin{aligned} 1 + r_X(t_2) &= \frac{1}{t_1} \inf\{\liminf_{n\to\infty} \|t_1 x_n + t_2 x\| : \{x_n\} \in G_1, \|x\| \geq t_1\} \\ &= \frac{t_2}{t_1} \inf\{\liminf_{n\to\infty} \|z_n + z\| : \{z_n\} \in G_{\frac{t_1}{t_2}}, \|z\| \geq t_1\} \\ &\leq \frac{t_2}{t_1} \inf\{\liminf_{n\to\infty} \|z_n + z\| : \{z_n\} \in G_1, \|z\| \geq t_1\} \\ &\leq \frac{t_2}{t_1}\Big(1 + r_X(t_1)\Big). \qquad \blacksquare \end{aligned}$$

The following theorem allows us to estimate the Opial modulus of Banach spaces.

Theorem 3.2.16 *Let X be a Banach space with a weakly continuous duality mapping J_μ with gauge function μ. Then $r_X(t) = \Phi^{-1}(\Phi(1) + \Phi(t)) - 1$ for all $t \geq 0$.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup 0$. Then we obtain from (3.7) that

$$\liminf_{n\to\infty} \Phi(\|x_n + y\|) = \liminf_{n\to\infty} \Phi(\|x_n\|) + \Phi(\|y\|) \text{ for all } y \in X. \tag{3.8}$$

If $\liminf_{n\to\infty} \|x_n\| \geq 1$, then

$$\Phi(1) + \Phi(\|y\|) \leq \liminf_{n\to\infty} \Phi(\|x_n\|) + \Phi(\|y\|) = \liminf_{n\to\infty} \Phi(\|x_n + y\|).$$

Thus,

$$\Phi^{-1}(\Phi(1) + \Phi(\|y\|)) \leq \liminf_{n\to\infty} \|x_n + y\| \text{ for all } y \in X,$$

and it follows from the definition of r_X that

$$\Phi^{-1}(\Phi(1) + \Phi(t)) \leq r_X(t) + 1. \tag{3.9}$$

Now, let $x_n \in S_X$ with $x_n \rightharpoonup 0$ and $\|x\| = t$. Then from (3.8), we have

$$\begin{aligned} 1 + r_X(t) &\leq \liminf_{n\to\infty} \|x_n + x\| \\ &\leq \Phi^{-1}(\Phi(1) + \Phi(t)). \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$1 + r_X(t) = \Phi^{-1}(\Phi(1) + \Phi(t)) \text{ for all } t \geq 0. \quad \blacksquare$$

Because ℓ_p $(1 < p < \infty)$ space admits a weakly continuous duality mapping J_μ with the gauge function $\mu(t) = t^{p-1}$, we have

Corollary 3.2.17 *Let $1 < p < \infty$. Then $r_{\ell_p}(t) = (1 + t^p)^{1/p} - 1$, $t \geq 0$.*

Theorem 3.2.18 *Let X be a Banach space. Then the following are equivalent:*

(a) X has a nonstrict Opial condition.

(b) $r_X(t) \geq 0$ for all $t > 0$.

Proof. $(a) \Rightarrow (b)$. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup 0$ and $\liminf_{n\to\infty} \|x_n\| \geq 1$. Let $x \in X$ such that $\|x\| \geq t$ for $t > 0$. Then

$$1 \leq \liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x_n + x\|.$$

Hence $r_X(t) \geq 0$.

$(b) \Rightarrow (a)$. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup 0$ and $\alpha := \liminf_{n\to\infty} \|x_n\| > 0$. Let $x \in X$ such that $x \neq 0$. If $\liminf_{n\to\infty} \|x_n\| \leq \|x\|$, then by w-lsc of the norm, we have

$$\liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x_n + x\|$$

and hence we are done.

If $\|x\| \le \liminf_{n\to\infty} \|x_n\|$, then

$$1 + r_X\left(\frac{\|x\|}{\alpha}\right) \le \liminf_{n\to\infty} \left\|\frac{x_n}{\alpha} + \frac{x}{\alpha}\right\|,$$

so

$$\alpha\left(1 + r_X\left(\frac{\|x\|}{\alpha}\right)\right) \le \liminf_{n\to\infty} \|x_n + x\|. \tag{3.11}$$

Because $\|x\|/\alpha > 0$, by assumption, we have $r_X(\|x\|/\alpha) \ge 0$. It follows from (3.11) that

$$\alpha = \liminf_{n\to\infty} \|x_n\| \le \liminf_{n\to\infty} \|x_n + x\|.$$

Hence X satisfies a nonstrict Opial condition. ∎

Proposition 3.2.19 *A Banach space X satisfies the locally uniform Opial condition if and only if for any sequence $\{x_n\}$ in X that converges weakly to $x \in X$ and for any sequence $\{y_m\}$ in X,*

$$\limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_n - y_m\|) \le \limsup_{n\to\infty} \|x_n - x\| \text{ implies } y_m \to x.$$

Proof. Assume that X satisfies the locally uniform Opial condition. Let $\{x_n\}$ be a sequence in X with $x_n \rightharpoonup x \in X$ and $\{y_m\}$ a sequence in X such that

$$\limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_n - y_m\|) \le \limsup_{n\to\infty} \|x_n - x\|. \tag{3.12}$$

Set $d := \limsup_{n\to\infty} \|x_n - x\|$. If $d = 0$, then $y_m \to x$. If $d > 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $d = \lim_{i\to\infty} \|x_{n_i} - x\|$. Suppose, for contradiction, that $\{y_m\}$ does not converge to x in norm. Then there exist an $\varepsilon > 0$ and a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ such that

$$\|y_{m_j} - x\| \ge \varepsilon \text{ for all } j \in \mathbb{N}.$$

Set $z_i := (x_{n_i} - x)/d$. By the local uniform Opial condition of X, we have an $r > 0$ such that

$$1 + r \le \liminf_{i\to\infty} \|z_i + z\| \text{ for all } z \in X \text{ with } \|z\| \ge \frac{\varepsilon}{d}.$$

In particular, we have

$$\liminf_{i\to\infty} \|x_{n_i} - y_{m_j}\| \ge d(1 + r) \text{ for all } j \in \mathbb{N},$$

which gives that

$$\limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_n - y_m\|) \ge d(1 + r) > d = \limsup_{n\to\infty} \|x_n - x\|,$$

which contradicts the inequality (3.12).

Conversely, suppose that X does not satisfy the locally uniform Opial condition. Then there exist a sequence $\{x_n\}$ in X with $x_n \rightharpoonup 0$ and $\limsup_{n\to\infty} \|x_n\| \geq 1$, a constant $c > 0$, and a sequence $\{y_m\}$ in X with $\|y_m\| \geq c$ for all $m \in \mathbb{N}$ such that

$$1 + \frac{1}{m} > \limsup_{n\to\infty} \|x_n - y_m\| \text{ for } m \in \mathbb{N}.$$

Hence

$$\limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_n - y_m\|) \leq 1 \leq \limsup_{n\to\infty} \|x_n\|.$$

By assumption, we have $y_m \to 0$. This contradicts the fact that $\|y_m\| \geq c$ for all $m \in \mathbb{N}$. ∎

3.3 Normal structure

Let C be a nonempty bounded subset of a Banach space X. Then a point $x_0 \in C$ is said to be

(i) *a diametral point of* C if

$$\sup\{\|x_0 - x\| : x \in C\} = diam(C),$$

(ii) *a nondiametral point of* C if

$$\sup\{\|x_0 - x\| : x \in C\} < diam(C).$$

A nonempty convex subset C of a Banach space X is said to have *normal structure* if each convex bounded subset D of C with at least two points contains a nondiametral point, i.e., there exists $x_0 \in D$ such that

$$\sup\{\|x_0 - x\| : x \in D\} < diam(D).$$

Geometrically, C is said to have *normal structure* if for each convex bounded subset D of C with $diam(D) > 0$, there exist a point $x_0 \in D$ and $r < diam(D)$ such that

$$D \subseteq B_r[x_0].$$

The Banach space X is said to have *normal structure* if every closed convex bounded subset C of X with $diam(C) > 0$ has normal structure.

The following theorems state that compact convex subsets of any Banach space and closed convex bounded subsets of a uniformly convex Banach space have this geometric property.

Theorem 3.3.1 *Every compact convex subset C of a Banach space X has normal structure.*

Proof. Suppose, for contradiction, that C does not have normal structure. Let D be a convex subset of C that has at least two points. Because C does not have normal structure, all points of D are diametral. Now we construct a sequence $\{x_i\}_{i=1}^{\infty}$ in D such that

$$\|x_i - x_j\| = \ diam(D) \text{ for all } i, j \in \mathbb{N}, i \neq j.$$

For this, let x_1 be an arbitrary point in D. Then there exists a point $x_2 \in D$ such that $diam(D) = \|x_1 - x_2\|$. Because D is convex, there exists a point $(x_1 + x_2)/2 \in D$. Next we choose a point $x_3 \in D$ such that

$$diam(D) = \left\| x_3 - \frac{x_1 + x_2}{2} \right\|.$$

Proceeding in the same manner, we obtain a sequence $\{x_n\}$ in D such that

$$diam(D) = \left\| x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\|, \quad n \geq 2.$$

Because

$$\begin{aligned} diam(D) &= \left\| x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n} \right\| \\ &= \left\| \frac{(x_{n+1} - x_1) + (x_{n+1} - x_2) + \cdots + (x_{n+1} - x_n)}{n} \right\| \\ &\leq \frac{1}{n}(\|x_{n+1} - x_1\| + \|x_{n+1} - x_2\| + \cdots + \|x_{n+1} - x_n\|) \\ &\leq diam(D), \end{aligned}$$

it follows that $diam(D) = \|x_{n+1} - x_i\|, 1 \leq i \leq n$. This implies that the sequence $\{x_n\}$ has no convergent subsequences. This contradicts the compactness of C. ▮

Corollary 3.3.2 *Every finite-dimensional Banach space has normal structure.*

Proof. The result easily follows from Theorem 3.3.1 ▮

Theorem 3.3.3 *Every closed convex bounded subset C of a uniformly convex Banach space X has normal structure.*

Proof. Let D be a closed convex subset of C with $diam(D) = d > 0$. Let x_1 be an arbitrary point in D. Choose a point $x_2 \in D$ such that $\|x_1 - x_2\| \geq d/2$. Because D is convex, $(x_1 + x_2)/2 \in D$. Set $x_0 = (x_1 + x_2)/2$. By the uniform convexity,

$$\|u\| \leq r, \|v\| \leq r \text{ and } \|u - v\| \geq \varepsilon > 0 \Rightarrow \left\| \frac{u+v}{2} \right\| \leq \left(1 - \delta_X\left(\frac{\varepsilon}{r}\right)\right) r.$$

Hence for $x \in D$ we have

$$\begin{aligned} \|x - x_0\| &= \left\| x - \frac{x_1 + x_2}{2} \right\| = \left\| \frac{(x - x_1) + (x - x_2)}{2} \right\| \\ &\leq d\left(1 - \delta_X\left(\frac{d}{2d}\right)\right) \\ &= d\left(1 - \delta_X\left(\frac{1}{2}\right)\right) \qquad (3.13) \\ &< d. \qquad \text{(as } \delta_X(1/2) > 0) \end{aligned}$$

Consequently,

$$\sup\{\|x - x_0\| : x \in D\} < \ diam(D). \quad \blacksquare$$

Theorem 3.3.4 *Every uniformly convex Banach space has normal structure.*

Proof. It follows from Theorem 3.3.3. ∎

The following class of Banach spaces is more general than the class of uniformly convex Banach spaces:

Let X be a Banach space. Given an element $z \in S_X$ and a constant $\varepsilon \in [0, 2]$, we define

$$\delta_X(z, \varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon, x-y = tz \text{ for some } t \geq 0\right\}.$$

The number $\delta_X(z, \varepsilon)$ is called the *modulus of convexity in the direction* $z \in S_X$. Then Banach space X is said to be *uniformly convex in every direction* if $\delta_X(z, \varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ and $z \in S_X$. It is obvious that $\delta_X(\varepsilon) = \inf\{\delta_X(z, \varepsilon) : z \in S_X\}$.

Observation

- A Banach space X may be uniformly convex in every direction while failing to be uniformly convex.
- Uniformly convex Banach spaces in every direction are always strictly convex.
- In case of a Hilbert $H, \delta_H(z, \varepsilon) = 1 - \sqrt{1 - (\varepsilon/2)^2}$ for all $z \in S_H$ and $\varepsilon \in [0, 2]$.

The following Theorem 3.3.6 shows that a uniformly convex Banach space in every direction has normal structure.

Proposition 3.3.5 *Let X be a Banach space and C a convex bounded subset of X with $d =$ diam$(C) > 0$. Then, for $\varepsilon > 0$, there exists a point $x_0 \in C$ such that*

$$\sup\{\|x_0 - x\| : x \in C\} \leq d\left(1 - \delta_X\left(z, \frac{d - \varepsilon}{d}\right)\right) \text{ for some } z \in S_X.$$

Proof. Given $d > \varepsilon > 0$, we choose $x_1, x_2 \in C$ such that $\|x_1 - x_2\| \geq d - \varepsilon$. Set $x_0 := (x_1 + x_2)/2$ and $z := (x_1 - x_2)/\|x_1 - x_2\|$. Note $\|x - x_1\| \leq d$, $\|x - x_2\| \leq d$ and $(x - x_1) - (x - x_2) = x_1 - x_2 = z\|x_1 - x_2\|$. Hence

$$\|x - x_0\| = \left\| x - \frac{x_1 + x_2}{2} \right\| \leq d\left(1 - \delta_X\left(z, \frac{d - \varepsilon}{d}\right)\right). \quad \blacksquare$$

Theorem 3.3.6 *Let X be a uniformly convex Banach space in every direction. Then X has normal structure.*

Proof. Because X is uniformly convex in every direction, $\delta_X(z, \varepsilon) > 0$ for all $z \in X$ and $\varepsilon > 0$. It follows from Proposition 3.3.5 and the continuity of $\delta_X(z, \varepsilon)$ that X has normal structure. $\blacksquare$

Example 3.3.7 *The space $C[0,1]$ of continuous real-valued functions with "sup" norm does not have normal structure. To see it, consider the subset C of $X = C[0,1]$ defined by*

$$C = \{f \in C[0,1] : 0 = f(0) \leq f(t) \leq f(1) = 1, t \in [0,1]\}.$$

Let $f_1, f_2 \in C$ and $\lambda \in [0,1]$ and $f = \lambda f_1 + (1 - \lambda) f_2$. Then $f(0) = 0, f(1) = 1$ and $0 \leq f(t) \leq 1$ for all $t \in [0,1]$. Hence C is convex. Thus, C is a closed convex bounded subset of X with diam $(C) = \sup\{\|f - g\| : f, g \in C\} = 1$. Then each point of C is a diametral point. In fact, for $f_0 \in C$

$$\sup\{\|f_0 - f\| : f \in C\} = 1 = diam(C).$$

Therefore, C does not have normal structure.

The following notion plays an important role in the study of normal structure.

A bounded sequence $\{x_n\}$ in a Banach space is said to be a *diametral sequence* if

$$\lim_{n \to \infty} d(x_{n+1}, co(\{x_1, x_2, \cdots, x_n\})) = diam(\{x_n\}),$$

where $d(x, A) = \inf_{y \in A} d(x, y)$.

Remark 3.3.8 *Any subsequence of a diametral sequence is also diametral.*

The following result gives an important fact relating normal structure and nondiametral sequence.

Proposition 3.3.9 *A convex bounded subset C of a Banach space X has normal structure if and only if it does not contain a diametral sequence.*

Proof. Suppose C contains a diametral sequence $\{x_n\}$. Then the set $C_0 = co(\{x_n\})$ is a convex subset of C and each point of C_0 is a diametral point. Thus, C fails to have normal structure.

Conversely, suppose that C contains a convex subset D with $d = diam(D) > 0$ and each point of D is a diametral point. By induction, we construct a sequence $\{x_n\}$ in D such that

$$\begin{aligned} y_0 &= x_1, \\ y_{n-1} &= \sum_{i=1}^{n} \frac{x_i}{n}. \end{aligned}$$

Because y_{n-1} is a diametral point in D, then for $0 < \varepsilon < d$, there exists an $x_{n+1} \in D$ such that

$$\|x_{n+1} - y_{n-1}\| > d - \frac{\varepsilon}{n^2}.$$

Suppose $x \in co(\{x_1, x_2, \cdots, x_n\})$, *say* $x = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Set $0 < \lambda := max\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then

$$\frac{1}{n}\left(1 - \frac{\lambda_i}{\lambda}\right) \geq 0 \text{ and } \frac{1}{n\lambda} + \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda}\right) = 1.$$

Hence

$$\frac{1}{n\lambda}x + \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda}\right)x_i = \frac{1}{n\lambda}x + \frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{\lambda n}\sum_{i=1}^{n}\lambda_i x_i = y_{n-1}.$$

Observe that

$$\begin{aligned} d - \frac{\varepsilon}{n^2} &< \|x_{n+1} - y_{n-1}\| \\ &= \left\|\frac{1}{n\lambda}(x_{n+1} - x) + \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda}\right)(x_{n+1} - x_i)\right\| \\ &\leq \frac{1}{n\lambda}\|x_{n+1} - x\| + \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{\lambda_i}{\lambda}\right)\|x_{n+1} - x_i\| \\ &\leq \frac{1}{n\lambda}\|x_{n+1} - x\| + \left(1 - \frac{1}{n\lambda}\right)d. \end{aligned}$$

Hence

$$\begin{aligned} \|x_{n+1} - x\| &\geq n\lambda\left(d - \frac{\varepsilon}{n^2} - \left(1 - \frac{1}{n\lambda}\right)d\right) \\ &= n\lambda\left(\frac{d}{n\lambda} - \frac{\varepsilon}{n^2}\right) \\ &= d - \frac{\varepsilon\lambda}{n} \\ &\geq d - \frac{\varepsilon}{n}, \end{aligned}$$

and it follows that

$$dist\ (x_{n+1}, co(\{x_1, x_2, \cdots, x_n\})) \geq d - \frac{\varepsilon}{n}.$$

Now ε is an arbitrary constant, so therefore, $\{x_n\}$ is a diametral sequence in D. ∎

Example 3.3.10 *In the space ℓ_1, the basis vectors $\{e_n\}$ form a diametral sequence. Hence ℓ_1 does not have normal structure.*

Theorem 3.3.11 *Let X be a reflexive Banach space with the Opial condition. Then X has normal structure.*

Proof. Suppose, for contradiction, that X fails to have normal structure. Then X contains a diametral sequence $\{x_n\}$ that may converge weakly to 0. Because $\{x_n\}$ is diametral sequence, by the definition

$$\lim_{n\to\infty} d(x_{n+1}, co(\{x_1, x_2, \cdots, x_n\})) = diam(\{x_n\}).$$

In particular, $\lim_{n\to\infty} \|x_n - y\| = diam(\{x_n\})$ if $y \in co(\{x_1, x_2, \cdots\})$ and $\lim_{n\to\infty} \|x_n - y\| = diam(\{x_n\})$ if $y \in \overline{co}(\{x_1, x_2, \cdots\})$. Taking $y = 0$, we obtain

$$\lim_{n\to\infty} \|x_n\| = diam(\{x_n\}).$$

Because also

$$\lim_{n\to\infty} \|x_n - x_1\| = diam(\{x_n\}),$$

this contradicts the Opial condition. ∎

Many spaces satisfy a stronger property than normal structure.

Definition 3.3.12 *A nonempty convex subset C of a Banach space is said to have uniformly normal structure if there exists a constant $\alpha \in (0, 1)$, independent of C, such that each closed convex bounded subset D of C with $diam(D) > 0$ contains a point $x_0 \in C$ such that*

$$\sup\{\|x_0 - x\| : x \in D\} \leq \alpha\ diam(D).$$

Theorem 3.3.13 *Every uniformly convex Banach space X has uniformly normal structure.*

Proof. For a closed convex bounded subset C of X with $d = diam(C) > 0$ from (3.13), there exists a point $x_0 \in C$ such that

$$\|x - x_0\| \leq \left(1 - \delta_X\left(\frac{1}{2}\right)\right)d.$$

This implies that

$$\sup\{\|x - x_0\| : x \in C\} \leq \alpha\ diam(C),$$

where $\alpha = 1 - \delta_X(1/2) < 1$. Therefore, X has uniformly normal structure. ∎

Let C be a nonempty bounded subset of a Banach space X. We adopt the following notations:

$$\begin{aligned} r_x(C) &= \sup\{\|x-y\| : y \in C\}, \quad x \in C; \\ r(C) &= \inf\{r_x(C) : x \in C\} = \inf\{\sup_{y\in C} \|x-y\| : x \in C\}; \\ \mathcal{Z}(C) &= \{x \in C : r_x(C) = r(C)\}; \\ r_X(C) &= \inf\{r_x(C) : x \in X\}. \end{aligned}$$

The number $r(C)$ is called the *Chebyshev radius of* C and the set $\mathcal{Z}(C)$ is called the *Chebyshev center of* C. Note that for any $x \in C$

$$r(C) \leq r_x(C) \leq \ diam(C).$$

Clearly, a point $x_0 \in C$ is a *diametral (nondiametral)* point if $r_{x_0}(C) = diam(C)$ ($r_{x_0}(C) < \ diam(C)$). Thus, set C has normal structure if

$$r(C) < \ diam(C)$$

and uniformly normal structure if there exists a constant $\alpha \in (0,1)$, independent of C, such that

$$r(C) \leq \alpha \ diam(C).$$

The set C is called diametral if it consists only of diametral points, i.e.,

$$r_x(C) = \ diam(C) \text{ for all } x \in C,$$

equivalently

$$\mathcal{Z}(C) = \{x \in C : r_x(C) = \ diam(C) = r(C)\}.$$

Observation

- The set $\mathcal{Z}(C)$ may be empty.
- The set C is diametral if $r(C) = \ diam(C)$.
- $\frac{1}{2} diam(C) \leq r_X(C) \leq r(C) \leq \ diam(C)$.
- $co(C) = \cap\{B_{r_x(C)}(x) : x \in C\}$.
- $\sup\{\|x_0 - x\| : x \in C\} = \sup\{\|x_0 - y\| : y \in co(C)\}, x_0 \in C$.
- $r(co(C)) \leq r(C)$.
- $diam(C) = \ diam(co(C))$.

The following result gives an essential condition for the existence of Chebyshev centers.

Proposition 3.3.14 *Let C be a weakly compact convex subset of a Banach space X. Then $\mathcal{Z}(C)$ is a nonempty closed convex subset of C.*

Proof. For $x \in C$, set

$$C_n(x) := B_{r(C)+\frac{1}{n}}[x] = \left\{ y \in C : \|x - y\| \le r(C) + \frac{1}{n} \right\}, \quad n \in \mathbb{N}.$$

Then $C_n(x)$ is a nonempty closed convex subset of C and hence $C_n = \bigcap_{x \in C} C_n(x)$ is a nonempty closed convex subset of C and $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Because C is weakly compact, it follows that $\mathcal{Z}(C) = \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. ∎

Proposition 3.3.15 *Let X be a Banach space and C a weakly compact convex subset of X with $diam(C) > 0$. Suppose C has normal structure. Then*

$$diam(\mathcal{Z}(C)) < diam(C).$$

Proof. Because C has normal structure, there exists at least one nondiametral point $x_0 \in C$, i.e.,

$$r_{x_0}(C) = \sup\{\|x_0 - x\| : x \in C\} < \ diam(C).$$

Let u and v be any two points of $\mathcal{Z}(C)$. Then $r_u(C) = r_v(C) = r(C)$. Because

$$\|u - v\| \le \sup\{\|u - x\| : x \in \mathcal{Z}(C)\} \le r(C) \le r_{x_0}(C) < \ diam(C),$$

it follows that

$$diam(\mathcal{Z}(C)) < diam(C). \quad ∎$$

3.4 Normal structure coefficient

Let X be a Banach space. Then the number $N(X)$ is said to be the *normal structure coefficient* if

$$N(X) = \inf \left\{ \frac{diam(C)}{r(C)} \right\},$$

where the infimum is taken over all closed convex bounded subsets C of X with $diam(C) > 0$.

It is clear that $N(X) \ge 1$ and $N(X) > 1$ if and only if X has uniformly normal structure.

Example 3.4.1 *For a Hilbert space H, $N(H) = \sqrt{2}$. Indeed, let C be the positive part of the unit ball B_H in a Hilbert space $H = \ell_2$, i.e.,*

$$C = \{x = \{x_i\} : \|x\| \le 1 \text{ and } x_i \ge 0, i = 1, 2, \cdots\}.$$

Then $r(C) = \inf\{\sup_{y \in C} \|x - y\| : x \in C\} = 1$ and $diam(C) = \sqrt{2}$. Hence $diam(C) \ge \sqrt{2}\, r(C)$.

We now give some important properties of the normal structure coefficient.

Theorem 3.4.2 *Let X be a Banach space. Then*

$$N(X) \geq \frac{1}{1-\delta_X(1)}. \tag{3.14}$$

Proof. Let C be a closed convex bounded subset of X with $d = diam\ (C) > 0$ and let $d > \varepsilon > 0$. Choose x and y in C such that $\|x-y\| \geq d-\varepsilon$. Let u be an element in C and $v = (x+y)/2 \in C$ such that

$$\|u-v\| \geq r_v(C) - \varepsilon.$$

By the definition of δ_X,

$$\|u-v\| = \left\|\frac{u-x+u-y}{2}\right\| \leq d\left(1-\delta_X\left(\frac{d-\varepsilon}{d}\right)\right)$$

and by the definition $r_v(C)$,

$$r(C) \leq r_v(C) \leq \|u-v\| + \varepsilon.$$

Thus,

$$r(C) \leq \left(1-\delta_X\left(\frac{d-\varepsilon}{d}\right)\right) diam\ (C) + \varepsilon.$$

Hence by continuity of δ_X, we have

$$r(C) \leq (1-\delta_X(1)) diam(C).$$

Therefore, we get the desired result. ∎

Remark 3.4.3 *For the Hilbert space H, $\delta_H(\varepsilon) = 1-(1-(\varepsilon^2/4))^{1/2}$, which gives $N(H) \geq 4/\sqrt{3}$, i.e., the estimate (3.14) is not sharp.*

Before giving an important example, we observe that the property "uniformly normal structure" is stable under small norm perturbations.

Theorem 3.4.4 *Let X be a Banach space and let $X_1 = (X, \|\cdot\|')$ and $X_2 = (X, \|\cdot\|'')$, where $\|\cdot\|'$ and $\|\cdot\|''$ are two equivalent norms on X satisfying for $\alpha, \beta > 0$,*

$$\alpha\|x\|' \leq \|x\|'' \leq \beta\|x\|',\ x \in X.$$

If $k = \beta/\alpha$, then

$$k^{-1}N(X_1) \leq N(X_2) \leq kN(X_1).$$

Proof. Note for a nonempty bounded convex subset C of X,

$$\alpha\ diam_{\|\cdot\|'}(C) \leq\ diam_{\|\cdot\|''}(C) \leq \beta\ diam_{\|\cdot\|'}(C).$$

Hence the result follows from the definition of $N(X)$. ∎

The following example shows that a uniformly convex Banach space (Hilbert space) has an equivalent norm that fails to have normal structure.

Example 3.4.5 *Let $X_1 = \ell_2$ and let $X_\lambda, \lambda > 1$ denote the space obtained by renorming the Hilbert space $(\ell_2, \|\cdot\|)$ as follows:*

For $x = (x_1, x_2, \cdots) \in \ell_2$, set

$$\|x\|_\lambda : \;=\; \max\left\{\|x\|_\infty, \lambda^{-1}\|x\|\right\} = \max\left\{\max_{i\in\mathbb{N}}|x_i|,\ \lambda^{-1}\left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}\right\}.$$

Because

$$\lambda^{-1}\|x\| \le \|x\|_\lambda \le \|x\|,$$

it means that all the norms $\|\cdot\|_\lambda$ are equivalent to norm $\|\cdot\|$. However the spaces $(X_\lambda, \|\cdot\|_\lambda)$, $\lambda > 1$ are not uniformly convex. A simple calculation shows that

$$\epsilon_0(X_\lambda) = \begin{cases} 2(\lambda^2 - 1)^{1/2} & for \quad \lambda \le \sqrt{2}, \\ 2 & for \quad \lambda \ge \sqrt{2}. \end{cases}$$

Then we have the following:

(i) $\epsilon_0(X_{\sqrt{5}/2}) = 1$.

(ii) For $\lambda = \sqrt{2}$, $\epsilon_0(X_\lambda) = 2$ and $N(X_\lambda) = 1$, i.e., X_λ fails to have normal structure, because the set $C \subset X_{\sqrt{2}}$ defined by

$$C = \left\{x = \{x_i\} : \sum_{i=1}^{\infty} x_i^2 \le 1 \text{ and } x_i \ge 0, i = 1, 2, \cdots\right\}$$

satisfies $r(C) = diam(C) = 1$ with respect to the $X_{\sqrt{2}}$ norm.

Let us now check the validity of Theorem 3.3.13 for uniformly smooth Banach spaces. To do so, we introduce the notion of super-reflexivity.

Let X and Y be two Banach spaces. We say that Y is *finitely representable* in X if for every $\lambda > 0$, every finite-dimensional subspace Y_0 of Y, there exist a finite-dimensional subspace X_0 of X with $dim(X_0) = dim(Y_0)$ and an isomorphism $T : Y_0 \to X_0$ such that

$$\|T\|\ \|T^{-1}\| \le 1 + \lambda.$$

This property can be expressed in terms of the Banach-Mazur distance, which is defined as follows: The *Banach-Mazur distance* between two Banach spaces X and Y is denoted by $d(X, Y)$ and is defined by

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism from X onto Y}\}.$$

Thus, Y is finitely representable in X if for any $\varepsilon \in (0, 1)$ and any finite-dimensional subspace Y_0 of Y, there exists a subspace X_0 of X such that $dim(X_0) = dim(Y_0)$ and $d(X_0, Y_0) \le 1 + \varepsilon$.

Observation

- The Banach space X is finitely representable in itself.
- The relation finite representability is transitive: If a Banach space X is finitely representable in a Banach space Y and if the Banach space Y is finitely representable in another Banach space Z, then X is finitely representable in Z.

We now define the "super-$\mathcal{P}$" property of Banach spaces.

Definition 3.4.6 *Let $\mathcal{P}$ be a property defined for Banach spaces. Then a Banach space X is said to have super-$\mathcal{P}$ if every Banach space finitely representable in X has the property $\mathcal{P}$.*

Remark 3.4.7 *Every Banach space is finitely representable in itself, so it follows that if X has super-$\mathcal{P}$ for any $\mathcal{P}$, then X has $\mathcal{P}$.*

Now we are in a position to define super-reflexivity of Banach spaces.

Definition 3.4.8 *A Banach space X is said to be super-reflexive if every Banach space Y that is finitely representable in X is itself reflexive.*

Thus, any super-reflexive Banach space is necessarily reflexive (by Remark 3.4.7).

For uniform convexity of a Banach space, we have the following:

Theorem 3.4.9 *Let X and Y be two Banach spaces with respective moduli of convexity δ_X and δ_Y, and suppose Y is finitely representable in X. Then for each $\varepsilon \in [0,2)$, $\delta_X(\varepsilon) \leq \delta_Y(\varepsilon)$.*

Proof. Let $\varepsilon > 0$ and let $x, y \in Y$ with $x, y \in S_Y$ and $\|x - y\|_Y \geq \varepsilon$. Suppose $\lambda > 0$ and let T be an isomorphism of $\text{span}\{x, y\}$ onto some two-dimensional subspace X_0 of X that satisfies

$$\|T\| \leq 1 + \lambda \text{ and } \|T^{-1}\| = 1.$$

Set $x' := Tx$ and $y' := Ty$. Then

$$\|x'\|_X, \|y'\|_X \leq 1 + \lambda \text{ and } \|x' - y'\|_X = \|Tx - Ty\|_X \geq \|x - y\|_Y \geq \varepsilon.$$

By the definition of δ_X, we have

$$\left\|\frac{1}{2}\left(\frac{x'}{1+\lambda} + \frac{y'}{1+\lambda}\right)\right\|_X \leq 1 - \delta_X\left(\frac{\varepsilon}{1+\lambda}\right).$$

By the continuity of δ_X, we obtain

$$\left\|\frac{x'+y'}{2}\right\| \leq 1 - \delta_X(\varepsilon).$$

Because $\|T^{-1}\| = 1$, it follows that

$$\left\|\frac{x+y}{2}\right\|_Y = \left\|T^{-1}\left(\frac{x'+y'}{2}\right)\right\|_Y \leq \left\|\frac{x'+y'}{2}\right\|_X.$$

Therefore, by the definition of δ_Y, we conclude that $\delta_Y(\varepsilon) \leq \delta_X(\varepsilon)$. ∎

Corollary 3.4.10 *Let X be a Banach space. If Y is finitely representable in X, then $\epsilon_0(Y) \le \epsilon_0(X)$.*

Corollary 3.4.11 *Let X be a uniformly convex Banach space. If a Banach space Y is finitely representable in X, then Y is uniformly convex.*

Corollary 3.4.12 *Every uniformly convex Banach space is super-reflexive.*

Proof. Because every uniformly convex Banach space is reflexive, it follows from Corollary 3.4.11 that every uniformly convex space is super-reflexive. ∎

We now discuss the super-property by ultrapower of Banach spaces:

Because the ultrapower $\{X\}_{\mathcal{U}}$ of a Banach space X is finitely representable in X (see Proposition A.3.10 of Appendix A), we have:

If $\mathcal{P}$ is a Banach space property that is inherited by subspaces, then a Banach space X has super-$\mathcal{P}$ if and only if every ultrapower of X has $\mathcal{P}$.

Thus, we have a link between moduli of convexity and smoothness concepts and ultrapowers:

Let $\{X\}_{\mathcal{U}}$ be an ultrapower of a Banach space X. Then

$$\delta_X(\cdot) = \delta_{\{X\}_{\mathcal{U}}}(\cdot) \text{ and } \rho_X(\cdot) = \rho_{\{X\}_{\mathcal{U}}}(\cdot).$$

Consequently,

$$\epsilon_0(X) = \epsilon_0(\{X\}_{\mathcal{U}}) \text{ and } \rho'_X(0) = \rho'_{\{X\}_{\mathcal{U}}}(0).$$

Theorem 3.4.13 *Let X be a Banach space with modulus of smoothness ρ_X. If $\rho'_X(0) < 1/2$, then X is super-reflexive and has normal structure.*

Proof. Suppose, for contradiction, that X is not super-reflexive. Then for any $\theta < 1$, there exist $x, y \in B_X$ and $j_1, j_2 \in B_{X^*}$ such that

$$\langle y, j_1\rangle = \langle x, j_1\rangle = \langle x, j_2\rangle = \theta \text{ and } \langle y, j_2\rangle = 0.$$

Hence for all $t > 0$:

$$\begin{aligned} \rho_X(t) &\ge \frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 \\ &\ge \frac{1}{2}(\langle x+ty, j_1\rangle + \langle x-ty, j_2\rangle) - 1 \\ &= \theta(1+\frac{t}{2}) - 1. \end{aligned}$$

Because $\theta < 1$ is arbitrary, $\rho_X(t) \ge t/2$, a contradiction.

Again suppose, for contradiction, that X does not have normal structure. Then there exists a sequence $\{x_n\}$ in B_X such that

$$x_n \rightharpoonup 0, \quad \lim_{n\to\infty} \|x_n\| = 1 \text{ and } diam(\{x_n\}) \le 1.$$

Consider a sequence $\{j_n\}$ in S_{X^*} such that $\langle x_n, j_n\rangle = \|x_n\|, n \in \mathbb{N}$. Because X^* is reflexive, we may assume that $\{j_n\}$ converges weakly to $j \in X^*$. Select $i \in \mathbb{N}$ such that $|\langle x_i, j\rangle| < \frac{\varepsilon}{2}$ while $\|x_n\| > 1 - \varepsilon$ for all $n \ge i$. Then for $k > i$ sufficiently large,

$$\langle x_i, j_k - j\rangle < \frac{\varepsilon}{2} \text{ and } |\langle x_k, j_i\rangle| < \varepsilon.$$

Consequently, $|\langle x_i, j_k\rangle| < \varepsilon$. For all $t \in (0, 1)$, we have

$$\begin{aligned}
\rho_X(t) &\ge \frac{1}{2}(\|x_i - x_k + tx_i\| + \|x_i - x_k - tx_i\|) - 1 \\
&\ge \frac{1}{2}(|\langle (1+t)x_i - x_k, j_i\rangle| + |\langle x_k - (1-t)x_i, j_k\rangle|) - 1 \\
&\ge \frac{1}{2}((1+t)(1-\varepsilon) - \varepsilon + 1 - \varepsilon - (1-t)\varepsilon) - 1 \\
&= \frac{t}{2} - 2\varepsilon.
\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, $\rho_X(t) \ge t/2$, and it follows that $\rho'_X(0) \ge 1/2$, a contradiction to the hypothesis $\rho'_X(0) < 1/2$. ∎

Theorem 3.4.14 *Let X be a Banach space with $\rho'_X(0) < 1/2$. Then X has uniformly normal structure.*

Proof. Note X has normal structure by Theorem 3.4.13. Suppose, for contradiction, that it is not uniform. Then there exists a sequence $\{C_n\}$ of closed convex bounded sets of X, each containing 0 and having diameter 1, for which $\lim_{n\to\infty} r(C_n) = 1$, where $r(C_n) = \inf\{r_x(C_n) : x \in C_n\}$. Define the set $C \subset \{X\}_\mathcal{U}$ by

$$C = \{C_n\}_\mathcal{U} = \{x \in \{X\}_\mathcal{U} : x = \{x_n\}_\mathcal{U}, x_n \in C_n\}.$$

Then C is a closed convex set in $\{X\}_\mathcal{U}$ with $diam(C) = 1$. Moreover, for any $x = \{x_n\}_\mathcal{U} \in C$, there exists $y = \{y_n\}_\mathcal{U} \in C$ such that $\lim_{n\to\infty} \|x_n - y_n\| = 1$, and it means that $\|x - y\|_\mathcal{U} = 1$ and C is diametral. Because $\rho'_X(0) = \rho'_{\{X\}_\mathcal{U}}(0) < \frac{1}{2}$, it follows that $\{X\}_\mathcal{U}$ has normal structure. This is a contradiction. ∎

Corollary 3.4.15 *Let X be a uniformly smooth Banach space. Then X has uniformly normal structure.*

We call a subsequence $\{y_n\}$ a c-subsequence of the sequence $\{x_n\}$ of a Banach space X if there exists a sequence of integers $1 = p_1 \le q_1 < p_2 \le q_2 < \cdots$ and scalars $\alpha_i \ge 0$ such that, for each $n \in \mathbb{N}$,

$$y_n = \sum_{i=p_n}^{q_n} \alpha_i x_i, \quad \sum_{i=p_n}^{q_n} \alpha_i = 1.$$

Theorem 3.4.16 *Every Banach space with a uniformly normal structure is reflexive.*

Proof. Let X be a Banach space with a uniformly normal structure. Let $\{C_n\}$ be a decreasing sequence of nonempty closed convex bounded subsets of X. We need to show that $\cap_{n\in\mathbb{N}}C_n \neq \emptyset$. For each $n \in \mathbb{N}$, choose $x_n \in C_n$. Then for each $\varepsilon > 0$, there exists a c-subsequence $\{y_n\}$ of $\{x_n\}$ with $\|y_m - y_n\| < \varepsilon$ for each m, n. Suppose this is not true for some $\varepsilon > 0$. Let $B_m = \{x_n\}_{n=m}^{\infty}$. Then there exist h, $0 < h < 1$ and $y_1' \in \overline{co}(B_1)$ such that

$$\sup\{\|y_1' - y\| : y \in \overline{co}(B_1)\} \leq h \; diam(B_1).$$

Suppose that $0 < h < h_1 < 1$. Then there exists $y_1 \in co(B_1)$ such that

$$\sup\{\|y_1 - y\| : y \in \overline{co}(B_1)\} \leq h_1 \; diam(B_1).$$

Because y_1 is a finite linear combination of members of B_1, there exists a c-subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$\sup\{\|y_n - y\| : y \in \overline{co}(B_{p_n})\} \leq h_1 \; diam(B_{p_n}) \leq h_1 \; diam(B_1),$$

and it follows that $diam(\{y_n\}) \leq h_1 diam(B_1)$.

By repeating the argument, there exists a successive c-subsequence with diameter less than or equal to $h_1^2 \; diam(B_1)$. We need only repeat the argument a sufficient number k of times with $h_1^k \; diam(B_1) < \varepsilon$ to obtain a contradiction.

Next by the diagonal method, there exists a c-subsequence of $\{x_n\}$ that is norm Cauchy, and hence convergent to some y. Therefore, $y \in \cap_{n\in\mathbb{N}}C_n$. ∎

We now give the following constants for bounded sequences in Banach spaces that are very useful to define various geometric coefficients:

Definition 3.4.17 *Let $\{x_n\}$ be a bounded sequence in a Banach space X. Then*

(1) the real number

$$diam(\{x_n\}) := \sup\{\|x_m - x_n\| : m, n \in \mathbb{N}\}$$

is called the diameter of the sequence $\{x_n\}$,

(2) the real number

$$diam_a(\{x_n\}) := \lim_{n\to\infty} (\sup\{\|x_i - x_j\| : i, j \geq n\})$$

is called the asymptotic diameter of the sequence $\{x_n\}$,

(3) the real number

$$r_a(\{x_n\}) := \inf\{r_a(y, \{x_n\}) : y \in \overline{co}(\{x_n\}_{n\in\mathbb{N}})\}$$

is called the asymptotic radius of the sequence $\{x_n\}$ with respect to $\overline{co}(\{x_n\}_{n\in\mathbb{N}})$, *(4) the real number*

$$sep(\{x_n\}) := \inf\{\|x_m - x_n\| : m \neq n\}$$

is called the separation of the sequence $\{x_n\}$.

For a bounded sequence $\{x_n\}$ in a Banach space X, we set

$$D[\{x_n\}] := \limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_m - x_n\|).$$

It can be easily seen that $D[\{x_n\}] \leq diam_a(\{x_n\})$.

The following example shows that $D[\{x_n\}] \neq diam_a(\{x_n\})$ in general.

Example 3.4.18 *Let J be the James quasi-reflexive space, consisting of all real sequences $x := \{x_n\} = \sum_{n=1}^{\infty} x_n e_n$ for which*

$$\lim_{n\to\infty} x_n = 0,\ e_n = (0, 0, \cdots, \underset{\substack{\uparrow \\ n^{th}\ position}}{1}, 0, \cdots) \text{ and } \|x\|_J < \infty,$$

where

$$\|x\|_J = \sup\{[(x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \cdots + (x_{p_{m-1}} - x_{p_m})^2 + (x_{p_m} - x_{p_1})^2]^{1/2}\}$$

and the supremum is taken over all choices of m and $p_1 < p_2 < \cdots < p_m$. Then J is a Banach space with the norm $\|\cdot\|_J$. Consider the sequence $\{x_n\}$ defined by

$$x_n = e_n - e_{n+1}, \quad n \in \mathbb{N}.$$

Then $x_n \in J$ and $\|x_n\|_J = \sqrt{6}$ for all $n \in \mathbb{N}$. We now show that

$$D[\{x_n\}] < diam_a(\{x_n\}).$$

Note $\|x_m - x_n\|_J = 2\sqrt{3}$ for each fixed $n \in \mathbb{N}$ and for all $m \geq n+3$, and hence

$$D[\{x_n\}] = \limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_m - x_n\|_J) = 2\sqrt{3}.$$

On the other hand, for each $k \in \mathbb{N}$, if we take $n = k,\quad m = n+1 \geq k$, then

$$\|x_m - x_n\|_J = 2\sqrt{5}$$

and hence

$$\sup\{\|x_m - x_n\|_J : m, n \geq k\} = 2\sqrt{5} \text{ for each } k \in \mathbb{N},$$

which implies that $diam_a(\{x_n\}) = 2\sqrt{5}$. Therefore,

$$D[\{x_n\}] = 2\sqrt{3} < 2\sqrt{5} = diam_a(\{x_n\}).$$

Remark 3.4.19 *For each bounded sequence $\{x_n\}$ in a Banach space X, we have*

$$r_a(\{x_n\}) \leq D[\{x_n\}] \leq diam_a(\{x_n\}) \leq diam(\{x_n\}).$$

A different form of uniformly normal structure coefficient $N(X)$ is defined by

$$N(X) = \inf\left\{\frac{diam_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a bounded sequence which is not norm convergent}\right\}.$$

We now prove an interesting result concerning the uniformly normal structure coefficient $N(X)$.

Theorem 3.4.20 *Let X be a Banach with $\tilde{N}(X) = N(X)^{-1} < 1$. Then, for every bounded sequence $\{x_n\}$ in X, there exists a point $z \in \overline{co}(\{x_n\})$ such that*

(a) for every $y \in X, \|z - y\| \leq r_a(y, \{x_n\})$,

(b) $r_a(z, \{x_n\}) \leq \tilde{N}(X)\ diam_a(\{x_n\})$.

Proof. (a) For each $k \in \mathbb{N}$, set $A_k := \overline{co}(\{x_n\}_{n=k}^{\infty})$ and $A := \cap_{k\in\mathbb{N}} A_k$. Observe that any point $z \in A$ satisfies (a). In fact, $z \in A_k$ for each $k \in \mathbb{N}$, and hence

$$\|z - y\| \leq \sup\{\|x - y\| : x \in A_k\} \text{ for all } y \in X,$$

which implies that

$$\|z - y\| \leq \lim_{k\to\infty} r_y(A_k) = r_a(y, \{x_n\}).$$

(b) The reflexivity X implies that each A_k is weakly compact. Hence the sets A, $\mathcal{Z}_a(A_k, \{x_n\})$ and $\mathcal{Z}_a(A, \{x_n\})$ are all nonempty. For each k, choose z_k in $\mathcal{Z}_a(A_k, \{x_n\})$ and consider a weakly convergent subsequence $\{z_{k_i}\}$ of $\{z_k\}$ such that $z_{k_i} \rightharpoonup z$. Because $z \in \overline{co}(\{z_{k_i}\}_{i=j}^{\infty}) \subset A_{k_j}$ for any j, by the monotonicity of the sequence $\{A_k\}$, we obtain that $z \in \cap_{j=1}^{\infty} A_{k_j} = A$. Note $r_a(z_k, \{x_n\})$ is a monotonic nondecreasing sequence that has $r_a(A, \{x_n\})$ as an upper bound. Moreover, because $r_a(\cdot, \{x_n\})$ is weakly lower semicontinuous, we have

$$\begin{aligned}\lim_{k\to\infty} r_a(z_k, \{x_n\}) &= \lim_{j\to\infty} r_a(z_{k_j}, \{x_n\}) \\ &\geq r_a(z, \{x_n\}) \\ &\geq r_a(A, \{x_n\}).\end{aligned}$$

Hence

$$\lim_{k\to\infty} r_a(z_k, \{x_n\}) = r_a(z, \{x_n\}) = r_a(A, \{x_n\}).$$

Note, for any k,

$$r_a(z_k, \{x_n\}) = r_a(z_k, \{x_n\}_{n=k}^{\infty}) \leq \tilde{N}(X)\ diam_a(\{x_n\}_{n=k}^{\infty}) = \tilde{N}(X)\ diam_a(\{x_n\}).$$

Therefore,

$$r_a(z, \{x_n\}) \leq \tilde{N}(X)\ diam_a(\{x_n\}). \quad \blacksquare$$

3.5 Weak normal structure coefficient

A Banach space X is said to have *weak normal structure* if every weakly compact convex subset C of X with more than one point contains a nondiametral point, that is, an $x_0 \in C$ for which

$$\sup\{\|x_0 - y\| : y \in C\} < diam(C).$$

Every Banach space that has normal structure also has weak normal structure, but the converse is not true. For reflexive Banach spaces, these properties are equivalent.

Example 3.5.1 *The space ℓ_1 does not have normal structure. Because weak compactness coincides with compactness in ℓ_1, it follows that ℓ_1 does have weak normal structure.*

Let X be a Banach space. Then the number $WCS(X)$ is said to be *weak normal structure coefficient* or *weakly convergent sequence coefficient* if

$$\begin{aligned} WCS(X) &= \inf\left\{\frac{diam(C)}{r(C)} : C \text{ is weakly compact convex subset of } X \right.\\ &\qquad \left. \text{with } diam(C) > 0\right\} \\ &= \inf\left\{\frac{diam_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence}\right.\\ &\qquad \left.\text{which is not norm convergent}\right\}, \end{aligned}$$

where $r_a(\{x_n\}) = \inf\{r_a(x, \{x_n\}) : x \in \overline{co}(\{x_n\})$ is the asymptotic radius of $\{x_n\}$ relative to $\overline{co}(\{x_n\})\}$.

Observation

- $WCS(X) \geq 1$, as $r(C) \leq diam(C)$.
- A Banach space X has weak uniformly normal structure if $WCS(X) > 1$.
- If $WCS(X) > 1$, then X has weak normal structure.

We now give a sharp expression for $WCS(X)$ in terms of $D[\{x_n\})]$. We begin with the following lemmas:

Proposition 3.5.2 *Let $\{y_n\}$ and $\{z_n\}$ be two sequences in a Banach space X such that $\alpha = \lim_{n\to\infty} \|y_n\| \neq 0$ and $z_n = y_n/\|y_n\|$. Then $D[\{z_n\}] = D[\{y_n\}]/\alpha$.*

Proof. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\|z_m - z_n\| &= \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\| \\ &= \left\| \frac{y_m}{\|y_m\|} - \frac{y_m}{\alpha} + \frac{y_m}{\alpha} - \frac{y_n}{\alpha} + \frac{y_n}{\alpha} - \frac{y_n}{\|y_n\|} \right\| \\ &\leq \|y_m\| \left| \frac{1}{\|y_m\|} - \frac{1}{\alpha} \right| + \frac{1}{\alpha}\|y_m - y_n\| + \|y_n\| \left| \frac{1}{\alpha} - \frac{1}{\|y_n\|} \right|.\end{aligned}$$

Hence

$$\begin{aligned}D[\{z_n\}] &= \limsup_{m\to\infty}(\limsup_{n\to\infty} \|z_m - z_n\|) \\ &\leq \limsup_{m\to\infty}(\|y_m\| \left| \frac{1}{\|y_m\|} - \frac{1}{\alpha} \right| + \frac{1}{\alpha} \limsup_{n\to\infty} \|y_m - y_n\|) \\ &= \frac{1}{\alpha} \limsup_{m\to\infty}(\limsup_{n\to\infty} \|y_m - y_n\|) = \frac{1}{\alpha} D[\{y_n\})].\end{aligned}$$

Similarly, we can obtain

$$\frac{1}{\alpha} D[\{y_n\}] \leq D[\{z_n\}]$$

Therefore, $D[\{z_n\}] = D[\{y_n\}]/\alpha$. ∎

Proposition 3.5.3 *Let X be a Banach space and $M > 0$. Then the following are equivalent:*

(a) $M \limsup_{n\to\infty} \|x_n - x\| \leq diam_a(\{x_n\})$ for any $x_n \rightharpoonup x$ (not strongly convergent).

(b) $M \limsup_{n\to\infty} \|y_n - y\| \leq D[\{y_n\}]$ for any $y_n \rightharpoonup y$ (not strongly convergent).

Proof. $(b) \Rightarrow (a)$. Because $D[\{x_n\}] \leq diam_a(\{x_n\})$, it is obvious that $(b) \Rightarrow (a)$.

$(a) \Rightarrow (b)$. Let $x_n \rightharpoonup x$ (not strongly convergent) and $\alpha := \limsup_{n\to\infty} \|x_n - x\| \neq 0$. Then we can choose a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\alpha := \lim_{m\to\infty} \|x_m - x\|$. Set $z_m := (x_m - x)/\alpha$. Then $z_m \rightharpoonup 0$ and $\|z_m\| = 1$, by using the diagonal method, we can choose a subsequence $\{z_{m_i}\}$ of $\{z_m\}$ such that $\lim_{i,j\to\infty, i\neq j} \|z_{m_i} - z_{m_j}\|$ exists. From (a), we have

$$\begin{aligned}M &= M \lim_{i\to\infty} \|z_{m_i}\| \\ &\leq diam_a(\{z_{m_i}\}) = D[\{z_{m_i}\}] \\ &\leq D[\{z_m\}] = \frac{1}{\alpha} D[\{x_m\}] \quad \text{(by Proposition 3.5.2)} \\ &\leq \frac{1}{\alpha} D[\{x_n\}],\end{aligned}$$

and it follows that

$$M \limsup_{n\to\infty} \|x_n - x\| = M\alpha \leq D[\{x_n\}]. \quad \blacksquare$$

We are now able to give an expression for $WCS(X)$ in terms of $D[\{x_n\}]$.

Theorem 3.5.4 *Let X be a Banach space. Then*

$$WCS(X) = \sup\{M > 0 : x_n \rightharpoonup u \Rightarrow M \limsup_{n\to\infty} \|x_n - u\| \leq D[\{x_n\}]\}.$$

Proof. The result follows from Proposition 3.5.3. $\blacksquare$

We now establish a relation between the Opial modulus and weak normal structure coefficient of a Banach space.

Theorem 3.5.5 *Let X be a Banach space with the Opial modulus r_X. Then*

$$WCS(X) \geq 1 + r_X(1).$$

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x \in X$. Set $b := r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|$. Without loss of generality, we may assume that $b > 0$ and $\lim_{n\to\infty} \|x_n - x\|$ exists. Otherwise, we can consider a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \|x_n - x\| = \lim_{i\to\infty} \|x_{n_i} - x\|.$$

Set $z_n := (x_n - x)/b$. From the definition of r_X, we get

$$1 + r_X(t) \leq \liminf_{n\to\infty} \|z_n + y\| \text{ for all } y \in X \text{ with } \|y\| \geq t.$$

In particular for $y = (x - x_m)/b$ and $t = \|y\|$, we have

$$1 + r_X\left(\frac{\|x - x_m\|}{b}\right) \leq \liminf_{n\to\infty} \frac{\|x_n - x_m\|}{b} \leq \limsup_{n\to\infty} \frac{\|x_n - x_m\|}{b}.$$

It follows that

$$b(1 + r_X(1)) \leq D[\{x_n\}].$$

Therefore, $WCS(X) \geq 1 + r_X(1)$. $\blacksquare$

Theorem 3.5.6 *Let X be Banach space with the Opial modulus r_X. If $r_X(1) > 0$, then $WCS(X) > 1$, i.e., X has weak uniformly normal structure.*

Proof. The result follows from the fact that $WCS(X) \geq 1 + r_X(1)$. $\blacksquare$

3.6 Maluta constant

Let X be a Banach space. Then the number $D(X)$ is called the *Maluta constant* if

$$D(X) = \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam(\{x_n\})}\right\},$$

where the supremum is taken over are all nonconstant bounded sequences in X.

The following result shows that $D(X)$ can be defined in several ways.

Proposition 3.6.1 *Let X be a Banach space. Then we have the following:*

(a) $D(X) = \sup\left\{\frac{\liminf_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam(\{x_n\})} : \{x_n\} \subset X\right\}.$

(b) If X is reflexive, then

$$D(X) = \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam(\{x_n\})} : \{x_n\} \text{ a weakly convergent sequence in } X\right\}.$$

(c) $D(X) = \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam_a(\{x_n\})} : \{x_n\} \text{ a nonconvergent sequence in } X\right\}.$

(d) If X is an infinite-dimensional reflexive space, then

$$D(X) = \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam_a(\{x_n\})} : \{x_n\} \text{ a weakly, but not strongly convergent sequence in } X\right\}.$$

Proof. (a) Let $\{x_n\}$ be a sequence in X and set $\alpha := \limsup_{n\to\infty} d(x_n, co(\{x_i\}_{i=1}^{n-1}))$. Then for a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha = \lim_{k\to\infty} d(x_{n_{k'}}, co(\{x_i\}_{i=1}^{n_k-1}))$, we have

$$\alpha = \lim_{k\to\infty} d(x_{n_k}, co(\{x_{n_i}\}_{i=1}^{n_k-1})) \leq \liminf_{k\to\infty} d(x_{n_k}, co(\{x_i\}_{i=1}^{k-1})).$$

Hence for every $\{x_n\}$ we can find a sequence $\{y_n\}(=\{x_{n_k}\})$ with $diam(\{y_n\}) \leq diam(\{x_n\})$ such that

$$\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam(\{x_n\})} \leq \frac{\liminf_{n\to\infty} d(y_{n+1}, co(\{y_i\}_{i=1}^n))}{diam(\{y_n\})}.$$

Thus, our assertion is proved.

(b) Let $\{x_n\}$ be a sequence in X with a weakly convergent subsequence $\{x_{n_k}\}$. Then

$$d(x_{n_k}, co(\{x_i\}_{i=1}^{n_k-1})) \leq d(x_{n_k}, co(\{x_i\}_{i=1}^{k-1})),$$

and it follows that

$$\frac{\liminf_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^{n}))}{diam(\{x_n\})} \leq \frac{\limsup_{n\to\infty} d(x_{n_k}, co(\{x_i\}_{i=1}^{k-1}))}{diam(\{x_{n_k}\})}.$$

From part (a) we can conclude (b).

(c) Let $\{x_n\}$ be a nonconvergent sequence in X. Then for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^{n})) &\leq \limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=k}^{n})) \\ &\leq D(X)\ diam(\{x_i\}_{i=k}^{\infty}).\end{aligned}$$

Thus

$$\begin{aligned}\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^{n})) &\leq \lim_{k\to\infty} (\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=k}^{n}))) \\ &\leq D(X)\ diam_a(\{x_n\}).\end{aligned}$$

Using the fact that $diam_a(\{x_n\}) \leq diam(\{x_n\})$, we obtain

$$D(X) \leq \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^{n}))}{diam_a(\{x_n\})}\right\} \leq D(X).$$

(d) It follows easily from (b). ∎

Theorem 3.6.2 *Let X be a Banach space. Then we have the following:*

(a) If X is a finite-dimensional space, then $D(X) = 0$.

(b) If X is an infinite-dimensional space with modulus of convexity δ_X, then

$$D(X) \geq \frac{1}{2(1-\delta_X(1))} \geq \frac{1}{2}.$$

Proof. (a) Let X be a finite-dimensional Banach space. Then for every convergent sequence $\{x_n\}$, we have $\lim_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^{n})) = 0$. It follows from Proposition 3.6.1(b) that $D(X) = 0$.

(b) Let X be an infinite-dimensional Banach space and $0 < r < 1$. We now construct a sequence in the following way:

$$\begin{cases} x_1 \in S_X, \\ x_{n+1} \in S_X \ \text{ such that } d(x_{n+1},\ span\{x_i\}_{i=1}^{n}) > r \text{ for all } n \in \mathbb{N}. \end{cases}$$

Then for any $i, j,\ i \neq j$ we have $\|x_i + x_j\| > r$ so

$$\frac{1}{2}\|x_i - x_j\| \leq 1 - \delta_X(r).$$

Hence

$$\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam(\{x_n\})} \geq \frac{r}{2(1-\delta_X(r))}.$$

Because δ_X is continuous at 1, we obtain

$$D(X) \geq \frac{1}{2(1-\delta_X(1))}. \qquad \blacksquare$$

Theorem 3.6.3 *Let X be a Banach space. Then we have the following:*

(a) $D(X) \leq \tilde{N}(X)$.

(b) If X is an infinite-dimensional reflexive Banach space, then

$$D(X) \leq \frac{1}{WCS(X)}.$$

Proof. (a) For every sequence $\{x_n\}$ in X, we denote $co(\{x_n\})$ by C. Then $diam(C) = diam(\{x_n\})$ and $r_x(C) = \sup_{y\in C} \|x-y\| = r_x(\{x_n\})$ for all $x \in X$. If

$$z \in C,\ z = \sum_{i=1}^N \lambda_i x_i \qquad (\lambda_i \geq 0 \text{ and } \sum_{i=1}^N \lambda_i = 1),$$

then

$$\begin{aligned} r_z(\{x_n\}) &= \sup_{n\in\mathbb{N}} \|x_n - \sum_{i=1}^N \lambda_i x_i\| \\ &\geq \limsup_{n\to\infty} d(x_n, co(\{x_i\}_{i=1}^N)) \\ &\geq \limsup_{n\to\infty} d(x_n, co(\{x_i\}_{i=1}^{n-1})). \end{aligned}$$

Hence we obtain

$$r(C) = \inf_{x\in C} r_x(C) = \inf_{x\in C} r_x(\{x_n\}) \geq \limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n)$$

and

$$\tilde{N}(X) \geq \sup\left\{\frac{r(C)}{diam(C)} : C = co(\{x_n\})\right\} \geq D(X).$$

(b) Let $x \in co(\{x_n\}), x = \sum_{i=1}^N \lambda_i x_i$ $(\lambda_i \geq 0,\ \sum_{i=1}^N \lambda_i = 1)$. Then

$$\begin{aligned} r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\| &\geq \limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^N) \\ &\geq \limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n)). \end{aligned}$$

Hence

$$\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n) \leq \inf\{r_a(z, \{x_n\}) : z \in co(\{x_n\})\} \text{ for all } \{x_n\} \text{ in } X.$$

By Proposition 3.6.1(d), we obtain

$$\begin{aligned} D(X) &= \sup\left\{\frac{\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n))}{diam_a(\{x_n\})} : \{x_n\} \text{ a weakly, but not strongly convergent sequence in } X\right\} \\ &\le \sup\left\{\frac{r_a(\{x_n\})}{diam_a(\{x_n\})} : \{x_n\} \text{ a weakly, but not strongly convergent sequence in } X\right\} \\ &= \frac{1}{WCS(X)}. \quad \blacksquare \end{aligned}$$

Theorem 3.6.4 *Let X be an infinite-dimensional reflexive Banach space X. Then*

$$WCS(X) = D'(X) = \frac{1}{D(X)},$$

where $D'(X) = \inf\{diam(\{x_n\}) : \{x_n\}$ *in* X *with* $x_n \rightharpoonup 0$ *and* $\limsup_{n\to\infty} \|x_n\| = 1\}$.

Proof. By Theorem 3.6.3(b), it suffices to show that

$$\frac{1}{D(X)} \le D'(X) \le WCS(X).$$

First, we show that $1/D(X) \le D'(X)$. We take a positive number $\varepsilon < 1$ and choose a sequence $\{x_n\}$ in S_X with $x_n \rightharpoonup 0$ and $diam(\{x_n\}) < D'(X) + \varepsilon$. Let $\{j_n\}$ be a sequence in X^* such that $\langle x_n, j_n\rangle = \|j_n\|_* = 1$ for all $n \in \mathbb{N}$. By Proposition 1.9.25, we can assume that

$$|\langle x_m, j_n\rangle| < \varepsilon \text{ whenever } m \neq n.$$

Suppose that $\lambda_i, i = 1, 2, \cdots n$ are nonnegative constants such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Then we have

$$\|x_{n+1} - \sum_{i=1}^n \lambda_i x_i\| \ge |\langle x_{n+1} - \sum_{i=1}^n \lambda_i x_i, j_{n+1}\rangle| \ge 1 - \sum_{i=1}^n \lambda_i |\langle x_i, j_{n+1}\rangle| > 1 - \varepsilon.$$

Thus, $\limsup_{n\to\infty} d(x_{n+1}, co(\{x_i\}_{i=1}^n)) \ge 1 - \epsilon$ and hence

$$\frac{1}{D(X)} \le \frac{diam(\{x_n\})}{1-\varepsilon} < \frac{D'(X)+\varepsilon}{1-\varepsilon}.$$

Because $\varepsilon \in (0,1)$ is arbitrary, we have $D(X)^{-1} \le D'(X)$.

We now show that $D'(X) \leq WCS(X)$. Let $\varepsilon > 0$ and choose a weakly convergent non-norm convergent sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}(\sup\{\|x_i - x_j\| : i, j \geq n\}) < (WCS(X) + \varepsilon)r_a(\overline{co}(\{x_n\}), \{x_n\}),$$

where $r_a(C, \{x_n\}) = \inf\{r_a(z, \{x_n\}) : z \in C\}$.

Let x be the weak limit of $\{x_n\}$ and $\limsup_{n\to\infty}\|x_n - x\| = d$. Hence for k a sufficiently large number, we have

$$diam(\{x_n\}_{n=k}^{\infty}) - \varepsilon < d\ (WCS(X) + \varepsilon).$$

Set $y_n := d^{-1}(x_n - x)$. Then $\limsup_{n\to\infty}\|y_n\| = 1$ and $y_n \rightharpoonup 0$. Therefore,

$$D'(X) \leq diam(\{y_n\}_{n=k}^{\infty}) = d^{-1}diam(\{x_n\}_{n=k}^{\infty}) < WCS(X) + (1 + d^{-1})\varepsilon,$$

and it follows that $D'(X) \leq WCS(X)$. ∎

We now establish equivalent expressions for $WCS(X)$.

Theorem 3.6.5 *Let X be an infinite-dimensional reflexive Banach space. Then we have the following:*

(a) $WCS(X) = \inf\{diam(\{x_n\}) : x_n \rightharpoonup 0$ *and* $\limsup_{n\to\infty}\|x_n\| = 1\} = D'(X)$.

(b) $WCS(X) = \inf\{diam_a(\{x_n\}) : x_n \rightharpoonup 0$ *and* $\limsup_{n\to\infty}\|x_n\| = 1\}$.

(c) $WCS(X) = \inf\left\{\dfrac{\lim_{n,m;n\neq m}\|x_n - x_m\|}{\limsup_{n\to\infty}\|x_n\|} : x_n \rightharpoonup 0 \text{ and } \limsup_{n,m;\ n\neq m}\|x_n - x_m\| \text{ exists}\right\}$.

Proof. (a) This part follows easily Theorem 3.6.4.

(b) It follows from part (a) that

$$\begin{aligned} WCS(X) \quad = \quad & \inf\{diam_a(\{x_n\}) : \{x_n\} \text{ converges weakly to zero and} \\ & \limsup_{n\to\infty}\|x_n\| = 1\}. \end{aligned} \tag{3.15}$$

(c) The equality (3.15) allows us to conclude (c) because for every sequence, we can obtain a subsequence $\{x_n\}$ such that $\lim_{n,m;n\neq m}\|x_n - x_m\|$ exists by a diagonal argument. ∎

Theorem 3.6.6 *Let X be a reflexive Banach space. Let*

$$r = \inf\left\{\frac{\lim_{n,m;\ n\neq m}\|x_n - x_m\|}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all weakly convergent sequences that are not convergent and such that $\lim_{n,m;\ n\neq m}\|x_n - x_m\|$ exists and $\lim_{n\to\infty}\|x_n - z\|$ exists for every $z \in \overline{co}(\{x_n\})$. Then $WCS(X) = r$.

Proof. Because $\lim_{n,m;\ n\neq m} \|x_n - x_m\| = diam_a(\{x_n\})$ if this limit exists, it follows that

$$r \geq WCS(X).$$

We now show that $WCS(X) \leq r$. Let $\{x_n\}$ be a sequence in X with $x_n \rightharpoonup 0$ such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists and $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in \overline{co}(\{x_n\})$. Set $A_k := \overline{co}(\{x_n\}_{n\geq k})$. The weak convergence of $\{x_n\}$ to zero implies that $\bigcap_{k=1}^{\infty} A_k = \{0\}$. Because the function φ on X defined by

$$\varphi(x) = \lim_{n\to\infty} \|x_n - x\|,\ x \in X$$

is weakly lower semicontinuous and A_k is weakly compact, then φ attains a minimum at a point z_k in A_k. Because 0 is the unique point that can be weakly adherent to $\{z_k\}$ we infer that $\{z_k\}$ is weakly null. By the weak lower semicontinuity of φ, we have

$$\varphi(0) \leq \lim_{k\to\infty} \varphi(z_k).$$

Because $\{\varphi(z_k)\}$ is a nondecreasing sequence that is bounded by $\varphi(0)$, it follows that

$$\lim_{k\to\infty} \varphi(z_k) \leq \varphi(0).$$

Thus,

$$\lim_{k\to\infty} \varphi(z_k) = \varphi(0).$$

Observe that

$$\begin{aligned} r_a(\{x_n\}_{n\geq k}) &= r_a(co(\{x_n\}_{n\geq k}),\ \{x_n\}_{n\geq k}) \\ &= \inf\{\varphi(y) : y \in co(\{x_n\}_{n\geq k})\} \\ &= \varphi(z_k). \end{aligned}$$

By the definition of r, we have

$$r \lim_{n\to\infty} \|x_n - z_k\| \leq \lim_{n,m;\ n\neq m} \|x_n - x_m\|.$$

Taking the limit as $k \to \infty$, we obtain

$$r \lim_{n\to\infty} \|x_n\| \leq \lim_{n,m;\ n\neq m} \|x_n - x_m\|.$$

Hence

$$r \leq \inf\left\{ \frac{\lim_{n,m;\ n\neq m} \|x_n - x_m\|}{\limsup_{n\to\infty} \|x_n\|} \right\},$$

where the infimum is taken over all weakly null sequence such that $\lim_{n,m;\ n\neq m} \|x_n - x_m\|$ exists and $\lim_{n\to\infty} \|x_n - z\|$ exists for every $z \in \overline{co}(\{x_n\})$. Using Theorem 3.6.5 (c), it is clear that this infimum is $WCS(X)$. Hence $r \leq WCS(X)$. Therefore, $WCS(X) = r$. ∎

We now apply Theorem 3.6.6 to estimate $WCS(X)$ for reflexive Banach spaces with weakly continuous duality mappings.

Theorem 3.6.7 *Let X be a reflexive Banach space with a weakly continuous duality mapping J_μ with gauge function μ. Then $1 + r_X(1) = WCS(X) = \Phi^{-1}(2\Phi(1))$, where $\Phi(t) = \int_0^t \mu(s)ds$, $t \geq 0$.*

Proof. Using Theorem 3.2.16 and Theorem 3.5.5, we have

$$WCS(X) \geq 1 + r_X(1) = \Phi^{-1}(2\Phi(1)).$$

To prove equality, we take a weakly null sequence $\{x_n\}$ in S_X. We may assume (through a subsequence if necessary) that $\lim\limits_{n,m;n\neq m} \|x_n - x_m\|$ exists. By Theorem 3.6.6, we have

$$\begin{aligned}
WCS(X) &\leq \lim_{n,m;n\neq m} \|x_n - x_m\| \\
&= \lim_{m\to\infty} (\lim_{n\to\infty} \|x_n - x_m\|) \\
&= \lim_{m\to\infty} (\lim_{n\to\infty} \Phi^{-1}(\Phi(\|x_n\|) + \Phi(\| - x_m\|))) \\
&= \lim_{m\to\infty} \Phi^{-1}(\Phi(1) + \Phi(\|x_m\|)) \\
&= \Phi^{-1}(2\Phi(1)).
\end{aligned}$$

This proves the desired result. ∎

Because ℓ_p $(1 < p < \infty)$, admits a weakly continuous duality mapping J_μ with the gauge $\mu(t) = t^{p-1}$, we have

Corollary 3.6.8 *Let $1 < p < \infty$. Then $WCS(\ell_p) = 2^{1/p}$.*

We now establish a relation between $WCS(X)$ and $N(X)$.

Theorem 3.6.9 *Let X be a Banach space. Then $WCS(X) \geq N(X)$.*

Proof. Let $\{x_n\}$ be a weakly null sequence in X such that $\lim\limits_{n\to\infty} \|x_n\| = c > 0$ and $\lim\limits_{m,n;m\neq n} \|x_n - x_m\| = d$ exist. Given $\varepsilon > 0$ with $c/2 > \varepsilon$, we may assume that

$$\|x_n\| \geq c - \varepsilon \text{ and } \|x_n - x_m\| \leq d + \varepsilon \text{ for all } n, m \in \mathbb{N}.$$

Let $\{j_n\}$ be a sequence in S_{X^*} such that $\langle x_n, j_n\rangle = \|x_n\|$ for all $n \in \mathbb{N}$. Using Proposition 1.9.25, we may assume that $|\langle x_m, j_n\rangle| < \varepsilon$ whenever $m \neq n$. Note $x \in co(\{x_n\})$ implies that $|\langle x, j_n\rangle| < \varepsilon$ for n sufficiently large. Hence

$$c - 2\varepsilon \leq \|x_n\| - \varepsilon = \langle x_n - x, j_n\rangle \leq \|x_n - x\|,$$

and it gives us $r(\overline{co}(\{x_n\})) \geq c - 2\varepsilon$. Thus,

$$N(X) \leq \frac{diam(\overline{co}(\{x_n\}))}{r(\overline{co}(\{x_n\}))} \leq \frac{d + \varepsilon}{c - 2\varepsilon}.$$

This shows that

$$N(X) \leq \frac{d}{c}.$$

Taking the infimum, we obtain that $N(X) \leq WCS(X)$. ▮

Corollary 3.6.10 *Let X be a Banach space. If X has uniformly normal structure, then X has weak uniformly normal structure.*

3.7 GGLD property

Definition 3.7.1 *A Banach space X is said to be have generalized Gossez-Lami Dozo property (GGLD) whenever $D[\{x_n\}] > 1$ for every weakly null sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n\| = 1$.*

Example 3.7.2 *Consider the space c_o equivalently renormed by*

$$\|\{x_n\}\| = \|\{x_n\}\|_\infty + \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n|.$$

It enjoys the Opial condition and it has a weak normal structure but lacks the GGLD property.

The following coefficient is very useful for fixed point theory of nonlinear mappings:

$$\beta(X) := \inf\{D[\{x_n\}] : x_n \rightharpoonup 0, \|x_n\| \to 1\}.$$

Observation

- A Banach space X has the GGLD property if $\beta(X) > 1$.

Proposition 3.7.3 *Let X be a Banach space with the GGLD property. If $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x \in X$ with $\lim_{n\to\infty} \|x_n - x\| \neq 0$, then*

$$\lim_{n\to\infty} \|x_n - x\| < D[\{x_n\}].$$

Proof. Let $\alpha := \lim_{n\to\infty} \|x_n - x\|$ and $y_n := \alpha^{-1}(x_n - x)$. Then $y_n \rightharpoonup 0$ and $\lim_{n\to\infty} \|y_n\| = 1$. By the GGLD property, $1 < D[\{y_n\}]$. Hence $\alpha < D[\{x_n\}]$. ▮

A Banach space X is said to have the *semi-Opial condition* (SO in short) if for any nonconstant bounded sequence $\{x_n\}$ in X with $x_n - x_{n+1} \to 0$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x \in X$ and $\lim_{k\to\infty} \|x_{n_k} - x\| < diam(\{x_n\})$.

The following theorem shows that for reflexive Banach spaces, the SO condition is more general than the GGLD property.

Theorem 3.7.4 *Every reflexive Banach space with the GGLD property satisfies the SO condition.*

Proof. Let $\{x_n\}$ be a bounded sequence in X such that $\|x_{n+1} - x_n\| \to 0$. Because X is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in X$. We may assume that $r := \lim_{k\to\infty} \|x_{n_k} - x\| > 0$, otherwise the result follows immediately.

Now set $y_k := r^{-1}(x_{n_k} - x)$. From the GGLD property of space X, we obtain $1 < D[\{y_k\}]$. Therefore,

$$r < \limsup_{k\to\infty}(\limsup_{j\to\infty} \|x_{n_j} - x_{n_k}\|) \leq diam(\{x_{n_k}\}) \leq diam(\{x_n\}),$$

which completes the proof. ∎

Let us give some examples concerning the GGLD property.

Example 3.7.5 *The Banach space* $X_\beta := (\ell_2, \|\cdot\|_\beta)$*, where* $\|x\|_\beta = \max\{\|x\|_2, \beta\|x\|_\infty\}$ *as has the SO condition for* $1 < \beta < 2$*, but if* $\sqrt{2} < \beta$*,* X_β *does not have normal structure and hence* X_β *cannot have the GGLD property.*

Example 3.7.6 *Consider the James space* J *that consists of sequences* $x = \{x_n\} \in c_o$ *such that*

$$\|x\|_J = \sup\{(x_{p_1} - x_{p_2})^2 + (x_{p_2} - x_{p_3})^2 + \cdots + (x_{p_{n-1}} - x_{p_n})^2\} < \infty,$$

where the supremum is taken over all increasing sequences of positive integers $\{p_i\}$*. The James space* J *fails to be uniformly convex in every direction (in fact, does not have normal structure), but* J *satisfies the GGLD property.*

Example 3.7.7 *Consider the classic space* c_o *of sequences with norm* $\|\cdot\|$ *defined by* $\|x\| = \left(\|x\|_\infty^2 + \sum_{i=1}^\infty x_i^2/2^i\right)^{1/2}$*. Then* $(c_o, \|\cdot\|)$ *is uniformly convex in every direction, but* c_o *fails to have the GGLD property.*

Bibliographic Notes and Remarks

Many of the topics and techniques in Section 3.1 follow from Bose and Laskar [21], Downing and Kirk [49], Goebel and Kirk [59], and Lan and Webb [95].

The results of Section 3.2 are due to Dalby [43], Lami Dozo [50], Lin, Tan, and Xu [101], and Suzuki [152].

The notion of normal structure was introduced by Brodskii and Milman in [23]. The uniformly convex and uniformly smooth Banach spaces enjoy uniformly normal structure. The results of Sections 3.3∼3.4 are well described in Goebel and Kirk [59] and Prus [121]. Theorem 3.4.20 is proved in Casini and Maluta [36].

The normal structure always implies weak normal structure. The results of Section 3.5 may be found in Kim and Kim [86] and Lin, Tan, and Xu [101].

The Maluta constant given in Section 3.6 is due to Maluta [103]. The results described in Section 3.6 can be found in Benavides and Acedo [15], Lin, Tan, and Xu [101], and Maluta [103]. Theorem 3.6.6 is proved in Benavides [14].

Exercises

3.1 Let H be a Hilbert space and $\varphi : H \to (-\infty, \infty]$ be a convex, lower semicontinuous, and proper function. For $\lambda > 0$ and $x \in H$, set

$$\varphi_\lambda(x) := \inf_{y \in H} \left[\varphi(y) + \frac{\lambda}{2}\|x - y\|^2 \right].$$

Let $\partial\varphi$ be the subdifferential of φ and $J_\lambda = (I + \lambda\partial\varphi)^{-1}$. Show that

(a) φ_λ is convex and $\varphi_\lambda(x) = \varphi(J_\lambda x) + \frac{\lambda}{2}\|(\partial\varphi)_\lambda(x)\|^2$.

(b) $\partial(\varphi_\lambda) = (\partial\varphi)_\lambda$; in particular, φ_λ is continuously differentiable and has Lipschitz continuous derivative.

(c) $\varphi_\lambda(x)$ increases to $\varphi(x)$ as $\lambda \downarrow 0$.

3.2 Let X be a Banach space, and $x_1, \cdots, x_n$ a finite number of points in X. Define

$$\varphi(z) = \frac{1}{n}\sum_{i=1}^{n} \|x_i - z\|^2 \text{ for all } z \in X.$$

Show that φ is a convex function and that, if X is reflexive, φ attains its minimum.

3.3 Let H be a Hilbert space. Let $C = \{x \in H : \|x - a\| \leq r\}$ and $D = \{x \in H : \|x\| \leq \|a\| + r\}$ be two sets, where $a \neq 0$ and $r > 0$. Let P_C and P_D be metric projection mappings, respectively. Show that $P_C P_D$ is a nonexpansive retraction of H onto C that is different from P_C.

3.4 Let C be a nonempty closed convex subset of a Hilbert space H, $P_C : H \to C$ the metric projection mapping onto C, and $\{x_n\}$ a sequence in H such that $x_n \rightharpoonup x$. Show that the asymptotic center of $\{x_n\}$ with respect to C is $P_C x$.

3.5 Let X be a Banach space. Show that X^* has a Fréchet differentiable norm iff X is reflexive and strictly convex, and has the following property:

if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\{x_n\}$ converges strongly to x.

Chapter 4

Existence Theorems in Metric Spaces

In this chapter, we study asymptotic fixed point theorems for contraction mappings and for mappings that are more general than contraction mappings in metric spaces.

4.1 Contraction mappings and their generalizations

In this section, we establish a fundamental asymptotic fixed point theorem that is known as the "Banach contraction principle" and further we give its generalizations in metric spaces.

By an asymptotic fixed point theorem for the mapping T, we mean a theorem that guarantees the existence of a fixed point of T, if the iterative T^n possess certain properties. Before to establish the Banach contraction principle, we discuss some basic definitions and results:

Let (X, d) be a metric space and let $Lip(X)$ denote the class of mappings $T : X \to X$ such that

$$\sigma(T^n) = \sup\left\{\frac{d(T^n x, T^n y)}{d(x,y)} : x, y \in X, x \neq y\right\} < \infty$$

for all $n \in \mathbb{N}$.

Members of $Lip(X)$ are called *Lipschitzian mappings* and $\sigma(T^n)$ is the *Lipschitz constant* of T^n. Note that $\sigma(T) = 0$ if and only if T is constant on X. For two Lipschitzian mappings $T : X \to X$ and $S : X \to X$ such that $S(X) \subseteq Dom(T)$, we have

$$\sigma(T \circ S) \leq \sigma(T)\sigma(S).$$

R.P. Agarwal et al., *Fixed Point Theory for Lipschitzian-type Mappings with Applications,* Topological Fixed Point Theory and Its Applications 6, DOI 10.1007/978-0-387-75818-3_4,

It is clear that the mapping $T \in Lip(X)$ if there exists a constant $L_n \geq 0$ such that

$$d(T^n x, T^n y) \leq L_n d(x,y) \text{ for all } x, y \in X \text{ and } \ n \in \mathbb{N}. \tag{4.1}$$

Moreover, the smallest constant L_n for which (4.1) holds is the *Lipschitz constant of* T^n. A Lipschitzian mapping $T : X \to X$ is said to be *uniformly L-Lipschitzian* if $L_n = L$ for all $n \in \mathbb{N}$. A Lipschitzian mapping is said to be *contraction* (*nonexpansive*) if $\sigma(T) < 1$ ($\sigma(T) = 1$).

The following result plays an important role in proving several existence theorems in metric spaces.

Proposition 4.1.1 *Let (X,d) be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a bounded below lower semicontinuous function. Suppose that $\{x_n\}$ is a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}) \textit{ for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Then $\{x_n\}$ converges to a point $v \in X$ and $d(x_n, v) \leq \varphi(x_n) - \varphi(v)$ for all $n \in \mathbb{N}_0$.

Proof. Because

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \quad n \in \mathbb{N}_0,$$

it follows that $\{\varphi(x_n)\}$ is a decreasing sequence. Moreover, for $m \in \mathbb{N}_0$

$$\begin{aligned} \sum_{n=0}^{m} d(x_n, x_{n+1}) &\leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_m, x_{m+1}) \\ &\leq \varphi(x_0) - \varphi(x_{m+1}) \\ &\leq \varphi(x_0) - \inf_{n \in \mathbb{N}_0} \varphi(x_n). \end{aligned}$$

Letting $m \to \infty$, we have

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Because X is complete, there exists $v \in X$ such that $\lim_{n \to \infty} x_n = v$. Let $m, n \in \mathbb{N}_0$ with $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \varphi(x_n) - \varphi(x_m). \end{aligned}$$

Letting $m \to \infty$, we obtain

$$d(x_n, v) \leq \varphi(x_n) - \lim_{m \to \infty} \varphi(x_m) \leq \varphi(x_n) - \varphi(v) \text{ for all } n \in \mathbb{N}_0. \quad \blacksquare$$

We now begin with Caristi's fixed point theorem. To prove it, we need the following important result.

Theorem 4.1.2 *Let X be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a proper, bounded below and lower semicontinuous function. Suppose that, for each $u \in X$ with $\inf_{x\in X} \varphi(x) < \varphi(u)$, there exists a $v \in X$ such that*

$$u \neq v \quad \text{and} \quad d(u, v) \leq \varphi(u) - \varphi(v).$$

Then there exists an $x_0 \in X$ such that $\varphi(x_0) = \inf_{x\in X} \varphi(x)$.

Proof. Suppose that $\inf_{x\in X} \varphi(x) < \varphi(y)$ for every $y \in X$. Let $u_0 \in X$ with $\varphi(u_0) < \infty$. If $\inf_{x\in X} \varphi(x) = \varphi(u_0)$, then we are done. Otherwise $\inf_{x\in X} \varphi(x) < \varphi(u_0)$, and there exists a $u_1 \in X$ such that $u_0 \neq u_1$ and $d(u_0, u_1) \leq \varphi(u_0) - \varphi(u_1)$.

Define inductively a sequence $\{u_n\}$ in X, starting with u_0. Suppose $u_{n-1} \in X$ is known. Then choose $u_n \in S_n$, where

$$S_n := \{w \in X : d(u_{n-1}, w) \leq \varphi(u_{n-1}) - \varphi(w)\}$$

such that

$$\varphi(u_n) \leq \inf_{w\in S_n} \varphi(w) + \frac{1}{2}\{\varphi(u_{n-1}) - \inf_{w\in S_n} \varphi(w)\}. \tag{4.2}$$

Because $u_n \in S_n$, we get

$$d(u_{n-1}, u_n) \leq \varphi(u_{n-1}) - \varphi(u_n), \quad n \in \mathbb{N}.$$

Proposition 4.1.1 implies that $u_n \to v \in X$ and $d(u_{n-1}, v) \leq \varphi(u_{n-1}) - \varphi(v)$. By hypothesis, there exists a $z \in X$ such that $z \neq v$ and $d(v, z) \leq \varphi(v) - \varphi(z)$. Observe that

$$\begin{aligned}
\varphi(z) &\leq \varphi(v) - d(v, z) \\
&\leq \varphi(v) - d(v, z) + \varphi(u_{n-1}) - \varphi(v) - d(u_{n-1}, v) \\
&= \varphi(u_{n-1}) - [d(v, z) + d(u_{n-1}, v)] \\
&\leq \varphi(u_{n-1}) - d(u_{n-1}, z).
\end{aligned}$$

This implies that $z \in S_n$. It follows from (4.2) that

$$2\varphi(u_n) - \varphi(u_{n-1}) \leq \inf_{w\in S_n} \varphi(w) \leq \varphi(z).$$

Thus,

$$\varphi(z) < \varphi(v) \leq \lim_{n\to\infty} \varphi(u_n) \leq \varphi(z),$$

a contradiction. Therefore, there exists a point $x_0 \in X$ such that $\varphi(x_0) = \inf_{x\in X} \varphi(x)$. ∎

Theorem 4.1.3 (Caristi's fixed point theorem) – *Let X be a complete metric space and $\varphi : X \to (-\infty, \infty]$ a proper, bounded below and lower semicontinuous function. Let $T : X \to X$ be a mapping such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \quad \textit{for all } x \in X. \tag{4.3}$$

Then there exists a point $v \in X$ such that $v = Tv$ and $\varphi(v) < \infty$.

Proof. Because φ is proper, there exists $u \in X$ such that $\varphi(u) < \infty$. Let

$$C = \{x \in X : d(u, x) \leq \varphi(u) - \varphi(x)\}.$$

Then C is a nonempty closed subset of X. We show that C is invariant under T. For each $x \in C$, we have

$$d(u, x) \leq \varphi(u) - \varphi(x)$$

and hence from (4.3), we have

$$\begin{aligned} \varphi(Tx) &\leq \varphi(x) - d(x, Tx) \\ &\leq \varphi(x) - d(x, Tx) + \varphi(u) - \varphi(x) - d(u, x) \\ &= \varphi(u) - [d(x, Tx) + d(u, x)] \\ &\leq \varphi(u) - d(u, Tx), \end{aligned}$$

and it follows that $Tx \in C$.

Suppose, for contradiction, that $x \neq Tx$ for all $x \in C$. Then, for each $x \in C$, there exists $w \in C$ such that

$$x \neq w \text{ and } d(x, w) \leq \varphi(x) - \varphi(w).$$

By Theorem 4.1.2, there exists an $x_0 \in C$ with $\varphi(x_0) = \inf\limits_{x \in C} \varphi(x)$. Hence for such an $x_0 \in C$, we have

$$\begin{aligned} 0 < d(x_0, Tx_0) &\leq \varphi(x_0) - \varphi(Tx_0) \qquad (\inf_{x \in C} \varphi(x) = \varphi(x_0) \leq \varphi(Tx_0)) \\ &\leq \varphi(Tx_0) - \varphi(Tx_0) \\ &= 0, \end{aligned}$$

a contradiction. ∎

Remark 4.1.4 *The fixed point of the mapping T in Theorem 4.1.3 need not be unique.*

We now state and prove the Banach contraction principle, which gives a unique fixed point of the mapping.

Theorem 4.1.5 (Banach's contraction principle) – *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with Lipschitz constant $k \in (0, 1)$. Then we have the following:*

(a) There exists a unique fixed point $v \in X$.

(b) For arbitrary $x_0 \in X$, the Picard iteration process defined by

$$x_{n+1} = Tx_n, \; n \in \mathbb{N}_0$$

converges to v.

(c) $d(x_n, v) \leq (1-k)^{-1} k^n d(x_0, x_1)$ for all $n \in \mathbb{N}_0$.

Proof. (a) Define the function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = (1-k)^{-1} d(x, Tx)$, $x \in X$. Hence φ is a continuous function. Because T is a contraction mapping,

$$d(Tx, T^2x) \leq kd(x, Tx), \quad x \in X, \tag{4.4}$$

which implies that

$$d(x, Tx) - kd(x, Tx) \leq d(x, Tx) - d(Tx, T^2x).$$

Hence

$$\begin{aligned} d(x, Tx) &\leq \frac{1}{1-k}[d(x, Tx) - d(Tx, T^2x)] \\ &= \varphi(x) - \varphi(Tx). \end{aligned} \tag{4.5}$$

Let x be an arbitrary element in X and define the sequence $\{x_n\}$ in X by

$$x_n = T^n x, \quad n \in \mathbb{N}_0.$$

From (4.5), we have

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \quad n \in \mathbb{N}_0,$$

and it follows from Proposition 4.1.1 that

$$\lim_{n \to \infty} x_n = v \in X$$

and

$$d(x_n, v) \leq \varphi(x_n), \quad n \in \mathbb{N}_0. \tag{4.6}$$

Because T is continuous and $x_{n+1} = Tx_n$, it follows that $v = Tv$. Suppose z is another fixed point of T. Then

$$0 < d(v, z) = d(Tv, Tz) \leq kd(v, z) < d(v, z),$$

a contradiction. Hence T has unique fixed point $v \in X$.

(b) It follows from part (a).

(c) From (4.4) we have that $\varphi(x_n) \leq k^n \varphi(x_0)$. This implies from (4.6) that $d(x_n, v) \leq k^n \varphi(x_0)$. ∎

Let us give some examples of contraction mappings.

Example 4.1.6 *Let $X = [a, b]$ and $T : X \to X$ a mapping such that T is differentiable at every $x \in (a, b)$ such that $|T^{'}(x)| \le k < 1$. Then, by the mean value theorem, if $x, y \in X$, there is a point ξ between x and y such that*

$$Tx - Ty = T^{'}(\xi)(x - y).$$

Thus,

$$|Tx - Ty| = |T^{'}(\xi)|\ |x - y| \le k|x - y|.$$

Therefore, T is contraction and it has a unique fixed point.

Example 4.1.7 *Let $X = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ a mapping defined by*

$$Tx = \frac{1}{2}x + 1, \quad x \in \mathbb{R}.$$

Then T is contraction and $F(T) = \{2\}$.

The following example shows that there exists a mapping that is not a contraction, but it has a unique fixed point.

Example 4.1.8 *Let $X = [0, 1]$ and $T : [0, 1] \to [0, 1]$ a mapping defined by*

$$Tx = 1 - x,\ x \in [0, 1].$$

Then T has a unique fixed point $1/2$, but T is not a contraction.

Let (X, d) be a metric space. Then a mapping $T : X \to X$ is said to be *contractive* if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X,\ x \ne y.$$

It is clear that the class of contractive mappings falls between the class of contraction mappings and that of nonexpansive mappings.

Observation

- A contractive mapping can have at most one fixed point.

The contractive mapping may not have a fixed point. It can be seen from the following example.

Example 4.1.9 *Let X be the space c_0 consisting of all real sequences $x = \{x_i\}$ with $\lim\limits_{i\to\infty} x_i = 0$ and $d(x, y) = \|x - y\| = \sup\limits_{i\in\mathbb{N}} |x_i - y_i|, x = \{x_i\}, y = \{y_i\} \in c_0$. Let $B_X = \{x \in c_0 : \|x\| \le 1\}$. For each $x \in B_X$, define*

$$T(x_1, x_2, \cdots, x_i, \cdots) = (y_1, y_2, \cdots, y_i, \cdots),$$

where $y_1 = (1 + \|x\|)/2$ and $y_i = (1 - 1/2^{i+1})x_{i-1}$ for $i = 2, 3, \cdots$. Note that $|y_1| \le 1$ and $|y_i| \le |x_{i-1}| \le 1$ for all $i = 2, 3, \cdots$. Hence $T : B_X \to B_X$.

Suppose x and y are two distinct points in B_X. Then

$$\begin{aligned}\|Tx - Ty\| &= \sup\left\{\frac{\|x\| - \|y\|}{2}, \left(1 - \frac{1}{2^{i+1}}\right)|x_{i-1} - y_{i-1}| : i = 2, 3, \cdots\right\} \\ &\le \sup\left\{\frac{\|x - y\|}{2}, \left(1 - \frac{1}{2^{i+1}}\right)|x_{i-1} - y_{i-1}| : i = 2, 3, \cdots\right\} \\ &< \|x - y\|.\end{aligned}$$

Suppose that there is a point $v \in B_X$ such that $Tv = v$. Then $v_1 = (1 + \|v\|)/2 > 0$ and for $i \ge 2$

$$|v_i| = \left(1 - \frac{1}{2^{i+1}}\right)|v_{i-1}|.$$

This implies for all $i \ge 2$

$$\begin{aligned}|v_i| &= \left(1 - \frac{1}{2^{i+1}}\right)|v_{i-1}| \\ &= \left(1 - \frac{1}{2^{i+1}}\right)\left(1 - \frac{1}{2^{i}}\right)|v_{i-2}| \\ &\cdots \\ &= \prod_{k=0}^{i-2}\left(1 - \frac{1}{2^{i+1-k}}\right)|v_1| \\ &\ge \left(1 - \sum_{k=0}^{i-2}\frac{1}{2^{i+1-k}}\right)|v_1| \\ &= \left(1 - \sum_{j=3}^{i+1}\frac{1}{2^j}\right)|v_1| > \frac{3}{4}|v_1|.\end{aligned}$$

This is not possible, because $v_i \to 0$ as $i \to \infty$. Thus, T has no fixed point in B_X. ∎

We note that completeness and boundedness of a metric space do not ensure the existence of fixed points of contractive mappings. However, contractive mappings always have fixed points in compact metric spaces.

Theorem 4.1.10 *Let X be a compact metric space and $T : X \to X$ a contractive mapping. Then T has a unique fixed point v in X. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ of iterates converges to v.*

Proof. For each $x \in X$, define a function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = d(x, Tx)$. Then φ is continuous on X and by compactness of X, φ attains its minimum, say $\varphi(v)$, at $v \in X$. Then $\varphi(v) = \min_{x \in X} \varphi(x)$. If $v \ne Tv$, then

$$\varphi(Tv) = d(Tv, T^2 v) < d(v, Tv) = \varphi(v),$$

a contradiction. Hence $v = Tv$. Uniqueness of v follows from the contractive condition of T.

Now, let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Set $c_n := d(T^n x_0, v), n \in \mathbb{N}_0$. Because

$$c_{n+1} = d(T^{n+1}x_0, v) < d(T^n x_0, v) = c_n,$$

$\{c_n\}$ is a nonincreasing sequence in $\mathbb{R}^+$. Hence $\lim_{n\to\infty} c_n$ exists. Suppose $\lim_{n\to\infty} c_n = c \geq 0$. Assume that $c > 0$. Because X is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in X$. Observe that

$$0 < c = \lim_{i\to\infty} c_{n_i} = \lim_{i\to\infty} d(T^{n_i} x_0, v) = d(z, v),$$

i.e., $z \neq v$. Because T is contractive and continuous,

$$c = \lim_{i\to\infty} d(T^{n_i+1} x_0, v) = d(Tz, v) < d(z, v) = c,$$

a contradiction. Thus, $c = 0$, i.e., $z = v$. This means that every convergent subsequence of $\{T^n x_0\}$ must converge to v. Therefore, $\{T^n x_0\}$ converges to v. ∎

The following example shows that in general, even in a Hilbert space for contractive mappings we cannot have that $T^n x \to x_0$ for every $x \in B_X$ and $x_0 = Tx_0$.

Example 4.1.11 *Let $X = \ell_2 = \{(x_1, x_2, \cdots, x_i, \cdots) : x_i$ real for each $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $B_X = \{x \in X : \|x\|_2 = (\sum_{i=1}^{\infty} |x_i\|^2)^{1/2} \leq 1\}$. Define a mapping $T : B_X \to B_X$ by*

$$Tx = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_i x_i \cdots), \quad x = (x_1, x_2, \cdots, x_i, \cdots) \in B_X,$$

where $\alpha_1 = 1$; $\alpha_i = (1 - 1/i^2)$, $i = 2, 3, \cdots$. It is easy to see that T is contractive with fixed point $(0, 0, \cdots, 0, \cdots)$.

Now, let $x = (1, 0, \cdots, 0, \cdots) \in B_X$, then

$$T^n x = (0, 0, \cdots, \prod_{i=1}^{n} \alpha_i, 0, \cdots) \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\|T^n x\| = \frac{n+2}{2(n+1)} \to \frac{1}{2} \text{ as } n \to \infty,$$

and hence $T^n x \nrightarrow 0$.

We now consider some important generalizations of the Banach contraction principle in which the Lipschitz constant k is replaced by some real-valued control function.

Theorem 4.1.12 (Boyd and Wong's fixed point theorem) – *Let X be a complete metric space and $T : X \to X$ a mapping that satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X, \tag{4.7}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is upper semicontinuous function from the right (i.e.,* $\lambda_i \downarrow \lambda \geq 0 \Rightarrow \limsup_{i\to\infty} \psi(\lambda_i) \leq \psi(\lambda)$*) such that* $\psi(t) < t$ *for each* $t > 0$*. Then* T *has a unique fixed point* $v \in X$*. Moreover, for each* $x \in X$*,* $\lim_{n\to\infty} T^n x = v$*.*

Proof. Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x$, $n \in \mathbb{N}_0$. Set $d_n := d(x_n, x_{n+1})$. We divide the proof into three steps:

Step 1. $\lim_{n\to\infty} d_n = 0$.

Note

$$d_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \psi(d_n), \quad n \in \mathbb{N}_0.$$

Hence $\{d_n\}$ is monotonic decreasing and bounded below. Hence $\lim_{n\to\infty} d_n$ exists. Let $\lim_{n\to\infty} d_n = \delta \geq 0$. Assume that $\delta > 0$. By the right continuity of ψ,

$$\delta = \lim_{n\to\infty} d_{n+1} \leq \lim_{n\to\infty} \psi(d_n) \leq \psi(\delta) < \delta,$$

so $\delta = 0$.

Step 2. $\{x_n\}$ *is Cauchy sequence.*

Assume that $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}_0$ such that $m_k > n_k \geq k$ and

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{for} \quad k = 0, 1, 2, \cdots .$$

Also, choosing m_k as small as possible, it may be assumed that

$$d(x_{m_k-1}, x_{n_k}) < \varepsilon.$$

Hence for each $k \in \mathbb{N}_0$, we have

$$\begin{aligned} \varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k-1}, x_{m_k}) + \varepsilon \\ &= d_{m_k-1} + \varepsilon, \end{aligned}$$

and it follows from the fact $d_{m_k} \to 0$ that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. Observe that

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq d_{m_k} + \psi(d(x_{m_k}, x_{n_k})) + d_{n_k}. \end{aligned}$$

Letting $k \to \infty$ and using the upper semicontinuity of ψ from the right, we obtain

$$\varepsilon = \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \lim_{k\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \psi(\varepsilon),$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X.

Step 3. Existence and uniqueness of fixed points.

Because $\{x_n\}$ is Cauchy and X is complete, $\lim_{n\to\infty} x_n = v \in X$. By continuity of T, we have $v = Tv$. Uniqueness of v easily follows from condition (4.7). ∎

Let Φ denote the class of all mappings $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:
(i) φ is continuous,
(ii) $\varphi(t) < t$ for all $t > 0$.

As an immediate consequence of the Boyd-Wong's fixed point theorem, we have the following important result, which will be useful in establishing existence theorems concerning asymptotic contraction mappings.

Corollary 4.1.13 *Let X be a complete metric space and $T : X \to X$ a mapping that satisfies*

$$d(Tx, Ty) \le \varphi(d(x, y)) \text{ for all } x, y \in X,$$

where $\varphi \in \Phi$. Then T has a unique fixed point $v \in X$. Moreover, for each $x \in X$, $\lim_{n\to\infty} T^n x = v$.

We now introduce a wider class of mappings that we call "*asymptotic contractions.*"

Definition 4.1.14 *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an asymptotic contraction if for each $n \in \mathbb{N}$*

$$d(T^n x, T^n y) \le \varphi_n(d(x, y)) \text{ for all } x, y \in X, \tag{4.8}$$

where $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d.

The following theorem shows that asymptotic contractions have unique fixed points.

Theorem 4.1.15 *Let X be a complete metric space and $T : X \to X$ a continuous asymptotic contraction for which the mappings φ_n in (4.8) are also continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $v \in X$ and for each $x \in X$, $\{T^n x\}$ converges to v.*

Proof. Because the sequence $\{\varphi_i\}$ is uniformly convergent, it follows that φ is continuous. For any $x, y \in X, x \ne y$, we have

$$\limsup_{n\to\infty} d(T^n x, T^n y) \le \limsup_{n\to\infty} \varphi_n(d(x, y)) = \varphi(d(x, y)) < d(x, y).$$

If there exist $x, y \in X$ and $\varepsilon > 0$ such that $\limsup_{n\to\infty} d(T^n x, T^n y) = \varepsilon$, then there exists $k \in \mathbb{N}$ such that $\varphi(d(T^k x, T^k y)) < \varepsilon$ because φ is continuous, and $\varphi(\varepsilon) < \varepsilon$. It follows that

$$\begin{aligned} \limsup_{n\to\infty} d(T^n x, T^n y) &= \limsup_{n\to\infty} d(T^n(T^k x), T^n(T^k y)) \\ &\le \limsup_{n\to\infty} \varphi_n(d(T^k x, T^k y)) \\ &= \varphi(d(T^k x, T^k y)) < \varepsilon, \end{aligned}$$

a contraction. Hence

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0 \text{ for } \text{ any } x, y \in X. \tag{4.9}$$

Thus, all sequences of the Picard iterates defined by T, are equi-convergent and bounded.

Now let $z_0 \in X$ be arbitrary, $\{z_n\}$ be a sequence of Picard iterates of T at the point z_0, $C = \overline{\{z_n\}}$ and $F_n = \{x \in C : d(x, T^k x) \leq 1/n, k = 1, \cdots, n\}$. Because $\{z_n\}$ is bounded, C is bounded. It follows from (4.9) that F_n is nonempty. Because T is continuous, we have F_n is closed, for any n. Also, we have $F_{n+1} \subseteq F_n$. Let $\{x_n\}$ and $\{y_n\}$ be two arbitrary sequences such that $x_n, y_n \in F_n$. Let $\{n_j\}$ be a sequence of integers such that $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = \limsup\limits_{n\to\infty} d(x_n, y_n)$.

Observe that

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_{n_j}) &\leq \lim_{j\to\infty} (d(x_{n_j}, T^{n_j} x_{n_j}) + d(T^{n_j} x_{n_j}, T^{n_j} y_{n_j}) + d(y_{n_j}, T^{n_j} y_{n_j})) \\
&= \lim_{j\to\infty} \varphi_{n_j}(d(x_{n_j}, y_{n_j})) \\
&= \varphi(\lim_{j\to\infty} d(x_{n_j}, y_{n_j})),
\end{aligned}$$

and hence $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = \varphi(\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}))$, which implies that $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = 0$, because C is bounded. Thus, $\limsup\limits_{n\to\infty} d(x_n, y_n) = 0$ and hence $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. This implies that $\lim\limits_{n\to\infty} diam(F_n) = 0$. By the completeness of C, it follows that there exists $v \in X$ such that $\cap_{n=1}^{\infty} F_n = \{v\}$. Because $d(v, Tv) \leq 1/n$ for any n, we have $Tv = v$. From (4.9), we have $\lim\limits_{n\to\infty} d(T^n x, v) = 0$ for any $x \in X$. ∎

We now study an important generalization of the Boyd and Wong's fixed point theorem in which the control function φ is extended in a different direction. Interestingly, in the following result the continuity condition on φ is replaced by $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

Theorem 4.1.16 (Matkowski's fixed point theorem) – *Let X be a complete metric space and $T : X \to X$ a mapping that satisfies*

$$d(Tx, Ty) \leq \psi(d(x, y)) \textit{ for all } x, y \in X,$$

where $\psi : (0, \infty) \to (0, \infty)$ is nondecreasing and satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point $v \in X$ and for each $x \in X$, $\lim\limits_{n\to\infty} T^n x = v$.

Proof. Fix $x_0 \in X$ and let $x_n = T^n x_0,\ n \in \mathbb{N}$. Observe that

$$0 \leq \limsup_{n\to\infty} d(x_n, x_{n+1}) \leq \limsup_{n\to\infty} \psi^n(d(x_0, x_1)) = 0.$$

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Because $\psi^n(t) \to 0$ for $t > 0$, $\psi(s) < s$ for any $s > 0$. Because $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, given any $\varepsilon > 0$, it is possible to choose n such that

$$d(x_{n+1}, x_n) \leq \varepsilon - \psi(\varepsilon).$$

Now for $z \in B_\varepsilon[x_n] = \{x \in X : d(x, x_n) \leq \varepsilon\}$, we have

$$\begin{aligned} d(Tz, x_n) &\leq d(Tz, Tx_n) + d(Tx_n, x_n) \\ &\leq \psi(d(z, x_n)) + d(x_{n+1}, x_n) \\ &\leq \psi(\varepsilon) + (\varepsilon - \psi(\varepsilon)) = \varepsilon. \end{aligned}$$

Therefore, $T : B_\varepsilon[x_n] \to B_\varepsilon[x_n]$ and it follows that $d(x_m, x_n) \leq \varepsilon$ for all $m \geq n$. Hence $\{x_n\}$ is a Cauchy sequence. The conclusion of the proof follows as in Theorem 4.1.12. ▮

We now introduce the concept of nearly Lipschitzian mappings:

Let (X, d) be a metric space and fix a sequence $\{a_n\}$ in $\mathbb{R}^+$ with $a_n \to 0$. A mapping $T : X \to X$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n) \text{ for all } x, y \in C. \tag{4.10}$$

The infimum of constants k_n for which (4.10) holds is denoted by $\eta(T^n)$ and called the *nearly Lipschitz constant.*

Notice that

$$\eta(T^n) = \sup\left\{\frac{d(T^n x, T^n y)}{d(x, y) + a_n} : x, y \in C, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with sequence $\{(\eta(T^n), a_n)\}$ is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$,

(ii) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) \leq 1$,

(iv) *nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(v) *nearly uniformly k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Example 4.1.17 *Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$ and $T : X \to X$ a mapping defined by*

$$Tx = \begin{cases} 1/2 \text{ if } x \in [0, 1/2], \\ 0 \text{ if } x \in (1/2, 1]. \end{cases}$$

Thus, T is discontinuous and non-Lipschitzian. However, it is nearly nonexpansive mapping. Indeed, for a sequence $\{a_n\}$ with $a_1 = 1/2$ and $a_n \to 0$, we have

$$d(Tx, Ty) \leq d(x, y) + a_1 \text{ for all } x, y \in X$$

and

$$d(T^n x, T^n y) \le d(x,y) + a_n \text{ for all } x, y \in X \text{ and } n \ge 2,$$

because

$$T^n x = \frac{1}{2} \text{ for all } x \in [0,1] \text{ and } n \ge 2.$$

We now develop a technique for studying the existence and uniqueness of fixed points of nearly Lipschitzian mappings.

Theorem 4.1.18 *Let X be a complete metric space and $T : X \to X$ a continuous nearly Lipschitzian mapping with sequence $\{(\eta(T^n), a_n)\}$, i.e., for a fixed sequence $\{a_n\}$ in $\mathbb{R}^+$ with $a_n \to 0$ and for each $n \in \mathbb{N}$, there exists a constant $\eta(T^n) > 0$ such that*

$$d(T^n x, T^n y) \le \eta(T^n)(d(x,y) + a_n) \quad \text{for all} \quad x, y \in X.$$

Suppose $\eta_\infty(T) = \limsup_{n\to\infty} [\eta(T^n)]^{1/n} < 1$. Then we have the following:

(a) T has a unique fixed point $v \in X$.

(b) For each $x \in X$, the sequence $\{T^n x\}$ converges to v.

(c) $d(T^n x, v) \le \sum_{i=n}^{\infty} \eta(T^i)(d(x,Tx) + M)$ for all $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$.

Proof. (a) Fix $x \in X$ and let $x_n = T^n x$, $n \in \mathbb{N}$. Set $d_n := d(x_n, x_{n+1})$. Hence

$$d_n = d(T^n x, T^{n+1} x) \le \eta(T^n)(d(x,Tx) + a_n),$$

which implies that

$$\sum_{n=1}^{\infty} d_n \le (d(x,Tx) + M) \sum_{n=1}^{\infty} \eta(T^n)$$

for some $M > 0$, because $\lim_{n\to\infty} a_n = 0$. By the Root Test for convergence of series, if $\eta_\infty(T) = \limsup_{n\to\infty} [\eta(T^n)]^{1/n} < 1$, then $\sum_{n=1}^{\infty} \eta(T^n) < \infty$. It follows that $\sum_{n=1}^{\infty} d_n < \infty$ and hence $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n\to\infty} x_n$ exists (say $v \in X$). By the continuity of T, v is fixed point of T. Let w be another fixed point T. Then

$$\begin{aligned} \infty = \sum_{n=1}^{\infty} d(v,w) &= \sum_{n=1}^{\infty} d(T^n v, T^n w) \le \sum_{n=1}^{\infty} \eta(T^n)(d(v,w) + a_n) \\ &\le (d(u,w) + M) \sum_{n=1}^{\infty} \eta(T^n) < \infty, \end{aligned}$$

a contradiction, hence T has a unique fixed point $v \in X$.

(b) It follows from part (a).

(c) If $m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+m}) &= d(T^n x, T^{n+m} x) \\ &\leq \sum_{i=n}^{n+m-1} d(T^i x, T^{i+1} x) \\ &\leq \sum_{i=n}^{n+m-1} \eta(T^i)(d(x, Tx) + a_i) \\ &\leq \sum_{i=n}^{n+m-1} \eta(T^i)(d(x, Tx) + M). \end{aligned}$$

Letting $m \to \infty$, we obtain

$$d(x_n, v) \leq \textstyle\sum_{i=n}^{\infty} \eta(T^i)(d(x, Tx) + M). \quad \blacksquare$$

Remark 4.1.19 *In the case of a nearly uniformly k-Lipschitzian mapping, we have*

$$\limsup_{n\to\infty} [\eta(T^n)]^{1/n} = \limsup_{n\to\infty} (k)^{1/n} = 1.$$

Therefore, the assumptions of Theorem 4.1.18 do not hold for nearly uniformly k-Lipschitzian mappings.

4.2 Multivalued mappings

Let A be a nonempty subset of a metric space X. For $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $CB(X)$ denote the set of nonempty closed bounded subsets of X and $\mathcal{K}(X)$ denote the set of nonempty compact subsets of X. It is clear that $\mathcal{K}(X)$ is included in $CB(X)$.

For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\},$$

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\} = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

Example 4.2.1 *Let* $X = \mathbb{R}$, $A = [1, 2]$ *and* $B = [2, 3]$. *Then*

$$\delta(A, B) = \sup_{a\in A} d(a, B) = 1 \text{ and } \delta(B, A) = \sup_{b\in B} d(b, A) = 1.$$

Hence $H(A, B) = \max\{\delta(A, B), \delta(B, A)\} = 1.$

Note that set distance δ is not symmetric. However, δ and H have the following properties:

Proposition 4.2.2 *Let (X, d) be a metric space. Let $A, B, C \in CB(X)$. Then we have the following:*

(a) $\delta(A, B) = 0 \Leftrightarrow A \subset B$.

(b) $B \subset C \Rightarrow \delta(A, C) \leq \delta(A, B)$.

(c) $\delta(A \cup B, C) = \max\{\delta(A, C), \delta(B, C)\}$.

(d) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.

Proof. (a) By the definition δ, we have

$$\begin{aligned} \delta(A, B) = 0 &\Leftrightarrow \sup_{x \in A} d(x, B) = 0 \\ &\Leftrightarrow d(x, B) = 0 \text{ for all } x \in A. \end{aligned}$$

Because B is closed in X,

$$d(x, B) = 0 \Leftrightarrow x \in B.$$

Thus,

$$\delta(A, B) = 0 \Leftrightarrow A \subset B.$$

(b) Observe that

$$B \subset C \Rightarrow d(x, C) \leq d(x, B) \text{ for all } x \in X.$$

(c) Observe that

$$\delta(A \cup B, C) = \sup_{x \in A \cup B} d(x, C) = \max\{\sup_{x \in A} d(x, C), \sup_{x \in B} d(x, C)\}.$$

(d) Let $a \in A$, $b \in B$ and $c \in C$. Then

$$d(a, b) \leq d(a, c) + d(c, b),$$

which implies that

$$d(a, B) \leq d(a, c) + d(c, B)$$

and hence

$$d(a, B) \leq d(a, c) + \delta(C, B).$$

Because $c \in C$ is arbitrary, we have

$$d(a, B) \leq d(a, C) + \delta(C, B).$$

Similarly, because $a \in A$ is arbitrary, we have

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B). \quad \blacksquare$$

Proposition 4.2.3 *Let (X, d) be a metric space. Then H is a metric on $CB(X)$.*

Proof. By the definition of H, we have

$$H(A,B) \geq 0 \text{ and } H(A,B) = H(B,A).$$

Observe that

$$\begin{aligned} H(A,B) = 0 \quad &\Leftrightarrow \quad \max\{\delta(A,B), \delta(B,A)\} = 0 \\ &\Leftrightarrow \quad \delta(A,B) = 0 \quad \text{and} \quad \delta(B,A) = 0 \\ &\Leftrightarrow \quad A \subset B \quad \text{and} \quad B \subset A \\ &\Leftrightarrow \quad A = B. \end{aligned}$$

Using Proposition 4.2.2, we obtain

$$\begin{aligned} H(A,B) &= \max\{\delta(A,B), \delta(B,A)\} \\ &\leq \max\{\delta(A,C) + \delta(C,B), \delta(B,C) + \delta(C,A)) \\ &\leq \max\{\delta(A,C), \delta(C,A)\} + \max\{\delta(B,C), \delta(C,B)\} \\ &= H(A,C) + H(C,B). \quad \blacksquare \end{aligned}$$

The metric H on $CB(X)$ is called the *Hausdorff metric*. The metric H depends on the metric d. It is easy to see that the completeness of (X,d) implies the completeness of $(CB(X), H)$ and $(\mathcal{K}(X), H)$.

Remark 4.2.4 *Let $A, B \in CB(X)$ and $a \in A$. Then for $\varepsilon > 0$, there must exist a point $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$.*

The following proposition gives a characteristic property of the Hausdorff metric that will be used in Section 8.1.

Proposition 4.2.5 *Let X be a metric space. Then*

$$H(A \cup B, C \cup D) \leq \max\{H(A,C), H(B,D)\} \text{ for all } A, B, C, D \in CB(X).$$

Proof. Observe that

$$\begin{aligned} \delta(A \cup B, C \cup D) &= \max\{\delta(A, C \cup D), \delta(B, C \cup D)\} \\ &\leq \max\{\delta(A,C), \delta(B,D)\} \\ &\leq \max\{H(A,C), H(B,D)\}. \end{aligned}$$

Similarly, we have

$$\delta(C \cup D, A \cup B) \leq \max\{H(A,C), H(B,D)\}$$

By definition of H, we have

$$\begin{aligned} H(A \cup B, C \cup D) &= \max\{\delta(A \cup B, C \cup D), \delta(C \cup D, A \cup B)\} \\ &\leq \max\{H(A,C), H(B,D)\} \text{ for all } A, B, C, D \in CB(X). \quad \blacksquare \end{aligned}$$

Let $F(X)$ denote the family of nonempty closed subsets of a metric space (X,d). Then we have

Proposition 4.2.6 *Let C be a nonempty subset of a metric space (X, d). Suppose the mapping $T : C \to F(X)$ is an upper semicontinuous at $x_0 \in C$. Then the mapping $\varphi : C \to \mathbb{R}^+$ defined by $\varphi(x) = d(x, Tx)$, $x \in C$ is lower semicontinuous at x_0.*

Proof. Let $\varepsilon > 0$. By the upper semicontinuity of T at x_0, there exists $\delta > 0$ such that $y \in B_\delta[x_0] \cap C$ implies Ty lies in an $\varepsilon/4$-neighborhood of Tx_0, and moreover we may suppose $\delta \leq \varepsilon/4$. Select $u \in Ty$ such that

$$d(y, u) \leq d(y, Ty) + \frac{\varepsilon}{2}$$

and select $v \in Tx_0$ so that $d(u, v) \leq \varepsilon/4$. Then

$$\begin{aligned} d(x_0, Tx_0) - \left[d(y, Ty) + \frac{\varepsilon}{2}\right] &\leq d(x_0, Tx_0) - d(y, u) \\ &\leq d(x_0, v) - d(y, u) \\ &\leq d(x_0, y) + d(y, u) + d(u, v) - d(y, u) \\ &\leq d(x_0, y) + d(u, v) \\ &\leq \delta + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

and hence

$$d(x_0, Tx_0) \leq d(y, Ty) + \varepsilon.$$

Therefore, φ is lower semicontinuous at x_0. ∎

We now introduce the class of multivalued contraction mappings and obtain a fixed point theorem for this class of mappings:

Let T be a mapping from a metric space (X, d) into $CB(X)$. Then T is said to be *Lipschitzian* if there exists a constant $k > 0$ such that

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

A multivalued Lipschitzian mapping T is said to be *contraction* (*nonexpansive*) if $k < 1$ ($k = 1$). Let $F(T)$ denote the set of fixed points of T, i.e., $F(T) = \{x \in X : x \in Tx\}$.

Theorem 4.2.7 (Nadler's fixed point theorem) – *Let X be a complete metric space and $T : X \to CB(X)$ a contraction mapping. Then T has a fixed point in X.*

Proof. Let k, $0 < k < 1$ be the Lipschitz constant of T. Let $x_0 \in X$ and $x_1 \in Tx_0$. By Remark 4.2.4, there must exist $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + k.$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2.$$

Thus, there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + k^n \quad \text{for all } n \in \mathbb{N}.$$

Notice for each $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$ and so

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + k^n \\ &\leq kd(x_{n-1}, x_n) + k^n \\ &\leq k[kd(x_{n-2}, x_{n-1}) + k^{n-1}] + k^n \\ &\leq k^2 d(x_{n-2}, x_{n-1}) + 2k^n \\ &\cdots \\ &\leq k^n d(x_0, x_1) + nk^n. \end{aligned}$$

Because $\sum_{n=0}^{\infty} k^n < \infty$ and $\sum_{n=0}^{\infty} nk^n < \infty$, we have

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq d(x_0, x_1) \sum_{n=0}^{\infty} k^n + \sum_{n=0}^{\infty} nk^n < \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Again, by the continuity of T,

$$\lim_{n\to\infty} H(Tx_n, Tv) = 0.$$

Because $x_{n+1} \in Tx_n$,

$$\lim_{n\to\infty} d(x_{n+1}, Tv) = 0,$$

which implies that $d(v, Tv) = 0$. Because Tv is closed, it follows that $v \in Tv$. ∎

Example 4.2.8 *Let $X = [0, 1]$ and $f : [0, 1] \to [0, 1]$ a mapping such that*

$$f(x) = \begin{cases} x/2 + 1/2, & 0 \leq x \leq 1/2, \\ -x/2 + 1, & 1/2 \leq x \leq 1. \end{cases}$$

Define $T : X \to 2^X$ by $Tx = \{f(x)\} \cup \{0\}$, $x \in X$. Then T is a multivalued contraction mapping with $F(T) = \{0, 2/3\}$.

Remark 4.2.9 *Example 4.2.8 shows that the fixed point of a multivalued contraction mapping is not necessarily unique.*

We now discuss a stability result (Theorem 4.2.11) for multivalued contraction mappings.

Proposition 4.2.10 *Let X be a complete metric space and let $S, T : X \to CB(X)$ be two contraction mappings each having Lipschitz constant $k < 1$, i.e.,*

$$H(Sx, Sy) \leq kd(x, y) \text{ and } H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

Then $H(F(S), F(T)) \leq (1 - k)^{-1} \sup_{x\in X} H(Sx, Tx)$.

Proof. Let $\varepsilon > 0$ and $c > 0$ be such that $c\sum_{n=1}^{\infty} nk^n < 1$. For $x_0 \in F(S)$, select $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \le H(Sx_0, Tx_0) + \varepsilon.$$

Because $H(Tx_0, Tx_1) \le kd(x_0, x_1)$, it is possible to select $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\le H(Tx_0, Tx_1) + \frac{c\varepsilon k}{1-k} \\ &\le kd(x_0, x_1) + \frac{c\varepsilon k}{1-k}. \end{aligned}$$

Define $\{x_n\}$ inductively by

$$x_{n+1} \in Tx_n \text{ and } d(x_{n+1}, x_n) \le kd(x_n, x_{n-1}) + \frac{c\varepsilon k^n}{1-k}.$$

Set $\eta := c\varepsilon/(1-k)$. Observe that

$$\begin{aligned} d(x_{n+1}, x_n) &\le kd(x_n, x_{n-1}) + \eta k^n \\ &\le k(kd(x_{n-1}, x_{n-2}) + \eta k^{n-1}) + \eta k^n \\ &\le k^2 d(x_{n-1}, x_{n-2}) + 2\eta k^n \\ &\cdots \\ &\le k^n d(x_0, x_1) + n\eta k^n. \end{aligned}$$

Because $\sum_{n=1}^{\infty} k^n < \infty$ and $\sum_{n=1}^{\infty} nk^n < \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X and it converges to some point $v \in X$. Because $\lim_{n\to\infty} H(Tx_n, Tv) = 0$ by continuity of T, it follows from $x_{n+1} \in Tx_n$ that $v \in F(T)$. Observe that

$$\begin{aligned} d(x_0, v) &\le \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le \sum_{n=0}^{\infty} k^n d(x_0, x_1) + \eta \sum_{n=0}^{\infty} nk^n \\ &\le (1-k)^{-1} d(x_0, x_1) + \eta \sum_{n=0}^{\infty} nk^n \\ &\le (1-k)^{-1}(d(x_0, x_1) + \varepsilon) \\ &\le (1-k)^{-1}(H(Sx_0, Tx_0) + 2\varepsilon). \end{aligned}$$

Interchanging the roles of S and T, we conclude:

For each $y_0 \in F(T)$, there exist $y_1 \in Sy_0$ and $u \in F(S)$ such that

$$d(y_0, u) \le (1-k)^{-1}(H(Sy_0, Ty_0) + 2\varepsilon).$$

Because $\varepsilon > 0$ is arbitrary, the conclusion follows. ∎

Theorem 4.2.11 *Let X be a complete metric space and let $T_n : X \to CB(X)$ $(n = 1, 2, \cdots)$ be contraction mappings each having Lipschitz constant $k < 1$, i.e.,*

$$H(T_n x, T_n y) \le kd(x, y) \quad \textit{for all } x, y \in X \textit{ and } \; n \in \mathbb{N}.$$

If $\lim_{n\to\infty} H(T_n x, T_0 x) = 0$ uniformly for $x \in X$, then $\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0$.

Proof. Let $\varepsilon > 0$. Because $\lim_{n\to\infty} H(T_nx, T_0x) = 0$ uniformly for $x \in X$, it is possible to select $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in X} H(T_nx, T_0x) \leq (1-k)\varepsilon \text{ for all } n \geq n_0.$$

By Proposition 4.2.10, we have $H(F(T_n), F(T_0)) < \varepsilon$ for all $n \geq n_0$. ∎

Next, we extend Nadler's fixed point theorem for non-self multivalued mappings in a metric space. First, we define a metrically convex metric space.

Definition 4.2.12 *A metric space (X,d) is said to be metrically convex*[1] *if for any $x, y \in X$ with $x \neq y$, there exists $z \in X$, $x \neq y \neq z$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

We note that in such a space, each two points are the end points of at least one metric segment. This fact immediately yields a very useful lemma.

Lemma 4.2.13 *If C is a nonempty closed subset of a complete and metrically convex metric space (X,d), then for any $x \in C$, $y \notin C$, there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Now we are in a position to establish a fundamental result on the existence of fixed points for non-self multivalued contraction mappings.

Theorem 4.2.14 (Assad and Kirk's fixed point theorem) – *Let (X,d) be a complete and metrically convex metric space, C a nonempty closed subset of X, and $T : C \to CB(X)$ a contraction mapping, i.e.,*

$$H(Tx, Ty) \leq kd(x,y) \text{ for all } x, y \in X,$$

where $k \in (0,1)$. If $Tx \subset C$ for each $x \in \partial C$, then T has a fixed point in C.

Proof. We construct a sequence $\{p_n\}$ in C in the following way:

Let $p_0 \in C$ and $p_1' \in Tp_0$. If $p_1' \in C$, let $p_1 = p_1'$. Otherwise, select a point $p_1 \in \partial C$ such that

$$d(p_0, p_1) + d(p_1, p_1') = d(p_0, p_1').$$

Thus, $p_1 \in C$. By Remark 4.2.4, we may choose $p_2' \in Tp_1$ such that

$$d(p_1', p_2') \leq H(Tp_0, Tp_1) + k.$$

Now, if $p_2' \in C$, let $p_2' = p_2$, otherwise, let $p_2 \in \partial C$ such that

$$d(p_1, p_2) + d(p_2, p_2') = d(p_1, p_2').$$

Continuing in this manner, we obtain sequences $\{p_n\}$ and $\{p_n'\}$ such that for $n \in \mathbb{N}$,

[1]The concept of metric convexity was introduced by K. Menger in 1953.

(i) $p'_{n+1} \in Tp_n$;

(ii) $d(p'_{n+1}, p'_n) \leq H(Tp_n, Tp_{n-1}) + k^n$,

where $p'_{n+1} = p_{n+1}$, if $p'_{n+1} \in C$ or

$$d(p_n, p_{n+1}) + d(p_{n+1}, p'_{n+1}) = d(p_n, p'_{n+1}) \text{ if } p'_{n+1} \notin C \text{ and } p_{n+1} \in \partial C. \quad (4.11)$$

Now, set

$$\begin{aligned} P: &= \{p_i \in \{p_n\} : p_i = p'_i, i \in \mathbb{N}\}; \\ Q: &= \{p_i \in \{p_n\} : p_i \neq p'_i, i \in \mathbb{N}\}. \end{aligned}$$

Observe that if $p_i \in Q$ for some i, then $p_{i+1} \in P$ be the boundary condition.

We wish to estimate the distance $d(p_n, p_{n+1})$ for $n \geq 2$. For this, we consider three cases:

Case I. $p_n \in P$ and $p_{n+1} \in P$.

In this case, we have

$$\begin{aligned} d(p_n, p_{n+1}) = d(p'_n, p'_{n+1}) &\leq H(Tp_n, Tp_{n-1}) + k^n \\ &\leq kd(p_n, p_{n-1}) + k^n. \end{aligned}$$

Case II. $p_n \in P$ and $p_{n+1} \in Q$.

By (4.11), we have

$$\begin{aligned} d(p_n, p_{n+1}) &\leq d(p_n, p'_{n+1}) = d(p'_n, p'_{n+1}) \\ &\leq H(Tp_{n-1}, Tp_n) + k^n \\ &\leq kd(p_{n-1}, p_n) + k^n. \end{aligned}$$

Case III. $p_n \in Q$ and $p_{n+1} \in P$.

By the above observation, two consecutive terms of $\{p_n\}$ cannot be in Q, hence $p_{n-1} \in P$ and $p'_{n-1} = p_{n-1}$. Using this fact, we obtain

$$\begin{aligned} d(p_n, p_{n+1}) &\leq d(p_n, p'_n) + d(p'_n, p_{n+1}) \\ &= d(p_n, p'{}_n) + d(p'_n, p'_{n+1}) \\ &\leq d(p_n, p'_n) + H(Tp_{n-1}, Tp_n) + k^n \\ &\leq d(p_n, p'_n) + \alpha d(p_{n-1}, p_n) + k^n \\ &< d(p_{n-1}, p'_n) + k^n \\ &= d(p'_{n-1}, p'_n) + k^n \\ &\leq H(Tp_{n-2}, Tp_{n-1}) + k^{n-1} + k^n \\ &\leq kd(p_{n-2}, p_{n-1}) + k^{n-1} + k^n. \end{aligned}$$

The only other possibility, $p_n \in Q,\ p_{n+1} \in Q$ cannot occur. Thus, for $n \geq 2$, we have

$$d(p_n, p_{n+1}) = \begin{cases} kd(p_n, p_{n-1}) + k^n, & \text{or} \\ kd(p_{n-2}, p_{n-1}) + k^n + k^{n-1}. \end{cases} \quad (4.12)$$

Set $\delta := k^{-1/2} \max\{d(p_0, p_1), d(p_1, p_2)\}$. We now prove that

$$d(p_n, p_{n+1}) \leq k^{n/2}(\delta + n), \quad n \in \mathbb{N}. \tag{4.13}$$

For $n = 1$

$$d(p_1, p_2) \leq k^{1/2}(\delta + 1).$$

For $n = 2$, we use (4.12) and taking each case separately, we obtain

$$\begin{aligned} d(p_2, p_3) &\leq kd(p_1, p_2) + k^2 \\ &\leq kk^{1/2}(\delta + 1) + k^2 \\ &\leq k(\delta + 2); \\ d(p_2, p_3) &\leq kd(p_0, p_1) + k^2 + k \\ &\leq k(k^{1/2}\delta + k + 1) \\ &\leq k(\delta + 2). \end{aligned}$$

Now assume that (4.13) holds for $1 \leq n \leq m$. Observe that for $m \geq 2$

$$\begin{aligned} d(p_{m+1}, p_{m+2}) &\leq kd(p_m, p_{m+1}) + k^{m+1} \\ &\leq k[k^{m/2}(\delta + m)] + k^{m+1} \\ &\leq k^{(m+1)/2}(\delta + m) + k^{(m+1)/2}k^{(m+1)/2} \\ &\leq k^{(m+1)/2}[\delta + (m + 1)] \end{aligned}$$

or

$$\begin{aligned} d(p_{m+1}, p_{m+2}) &\leq kd(p_{m-1}, p_m) + k^{m+1} + k^m \\ &\leq k[k^{(m-1)/2}(\delta + m - 1)] + k^{m+1} + k^m \\ &\leq k^{(m+1)/2}(\delta + m - 1) + k^{(m+1)/2}k^{(m+1)/2} + k^{(m+1)/2}k^{(m-1)/2} \\ &\leq k^{(m+1)/2}(\delta + m - 1) + k^{(m+1)/2} + k^{(m+1)/2} \\ &= k^{(m+1)/2}(\delta + m + 1), \end{aligned}$$

and it follows that (4.13) is true for all $n \in \mathbb{N}$. Using (4.13) we obtain

$$d(p_n, p_m) \leq \delta \sum_{i=m}^{\infty} (k^{1/2})^i + \sum_{i=m}^{\infty} i(k^{1/2})^i, \quad n > m \geq 1.$$

This means that $\{p_n\}$ is a Cauchy sequence. Because C is closed, $\{p_n\}$ converges to a point $z \in C$. By our choice of $\{p_n\}$, there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \in P$, i.e., $p_{n_i} = p'_{n_i}$, $\quad i = 1, 2, \cdots$. Note $p'_{n_i} \in Tp_{n_i - 1}$ for $i \in \mathbb{N}$ by (i) and $p_{n_i - 1} \to z$ imply that $Tp_{n_i - 1} \to Tz$ as $i \to \infty$ in the Hausdorff metric H. Because

$$d(p_{n_i}, Tz) \leq H(Tp_{n_i - 1}, Tz) \to 0 \text{ as } i \to \infty,$$

it follows that $d(z, Tz) = 0$. As Tz is closed, $z \in Tz$. ∎

4.3 Convexity structure and fixed points

Let C be a nonempty subset of a metric space X and $T : C \to C$ a mapping. Then a sequence $\{x_n\}$ in C is said to be an *approximating fixed point sequence* (in short AFPS) of T if $\lim_{n\to\infty} d(x_n,\ Tx_n) = 0$.

We have seen in the Banach contraction principle that every contraction mapping has an approximating fixed point sequence in a metric space. In fact, the Picard iterative sequence $(x_{n+1} = Tx_n, n \in \mathbb{N})$ is an approximating fixed point sequence of the contraction mapping T.

The following example shows that the Picard iterative sequence is not necessarily an approximating fixed point sequence of nonexpansive mappings.

Example 4.3.1 *Let $X = \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ a mapping defined by*

$$Tx = -x \text{ for all } x \in \mathbb{R}.$$

Note that T is nonexpansive with $F(T) = \{0\}$. However for $x_0 > 0$, the iterative sequence of the Picard iteration process is

$$x_{n+1} = Tx_n = (-1)^n x_0, \quad n \in \mathbb{N}_0.$$

Hence $d(x_n,\ Tx_n) = |(-1)^{n-1} - (-1)^n| x_0 = 2x_0 \nrightarrow 0$ as $n \to \infty$.

The following Proposition 4.3.9 shows that the convexity structure has an important role in the existence of AFPS for nonexpansive mappings. We define convexity structure in a metric space.

Definition 4.3.2 *Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure*[2] *on X, if for all $x, y \in X$ and $\lambda \in [0, 1]$ the following condition is satisfied:*

$$d(u, W(x, y; \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y) \text{ for all } u \in X. \tag{4.14}$$

A metric space X with convex structure is called a convex metric space.

A subset C of a convex metric space X is said to be *convex* if $W(x, y; \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. A convex metric space X is said to have *property (B)* if

$$d(W(u, x; \lambda), W(u, y; \lambda)) = (1 - \lambda)d(x, y) \text{ for all } u, x, y \in X \text{ and } \lambda \in (0, 1).$$

Example 4.3.3 *A normed space and each of its convex subsets are convex metric spaces with convexity structure $W(x, y; \lambda) = \lambda x + (1 - \lambda)y$.*

[2]The convexity structure in a metric space was introduced by W. Takahashi in 1970.

Example 4.3.4 *Let X be a linear space that is also a metric space with the following properties:*

(i) $d(x,y) = d(x-y,0)$ for all $x,y \in X$;

(ii) $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0)$ for all $x,y \in X$ and $\lambda \in [0,1]$.

Then X is a convex metric space.

Example 4.3.5 *A Fréchet space is not necessarily a convex metric space.*

The following propositions are very useful in various applications.

Proposition 4.3.6 *Let $\{C_\alpha : \alpha \in \Lambda\}$ be a family of convex subsets of a convex metric space X. Then $\cap_{\alpha\in\Lambda} C_\alpha$ is also a convex subset of X.*

Proposition 4.3.7 *The open balls $B_r(x)$ and the closed balls $B_r[x]$ in a convex metric space X are convex subsets of X.*

Proof. For $y,z \in B_r(x)$ and $\lambda \in [0,1]$, there exists $W(y,z;\lambda) \in X$. Because X is a convex metric space,

$$\begin{aligned} d(x, W(y,z;\lambda)) &\leq \lambda d(x,y) + (1-\lambda)d(x,z) \\ &< \lambda r + (1-\lambda)r = r. \end{aligned}$$

Therefore, $W(y,z;\lambda) \in B_r(x)$. Similarly, $B_r[x]$ is a convex subset of X. ∎

Proposition 4.3.8 *Let X be a convex metric space. Then*

$$d(x,y) = d(x, W(x,y;\lambda)) + d(W(x,y;\lambda), y) \text{ for all } x,y \in X \text{ and } \lambda \in [0,1].$$

Proof. Because X is a convex metric space, we obtain

$$\begin{aligned} d(x,y) &\leq d(x, W(x,y;\lambda)) + d(W(x,y;\lambda), y) \\ &\leq \lambda d(x,x) + (1-\lambda)d(x,y) + \lambda d(x,y) + (1-\lambda)d(y,y) \\ &= \lambda d(x,y) + (1-\lambda)d(x,y) \\ &= d(x,y) \end{aligned}$$

for all $x,y \in X$ and $\lambda \in [0,1]$. Therefore,

$$d(x,y) = d(x, W(x,y;\lambda) + d(W(x,y;\lambda), y) \text{ for all } x,y \in X \text{ and } \lambda \in [0,1].$$ ∎

We now apply the convexity structure defined in Definition 4.3.2 to obtain AFPS for nonexpansive mappings in a metric space. Note, similar results are also discussed in Chapter 5.

Proposition 4.3.9 *Let X be a complete convex metric space with property (B), C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping. Then we have the following:*

(a) For $u \in C$ and $t \in (0,1)$, there exists exactly one point $x_t \in C$ such that

$$x_t = W(u, Tx_t; 1-t)$$

(b) If C is bounded, then $d(x_t, Tx_t) \to 0$ as $t \to 1$, i.e., T has an AFPS.

Proof. (a) For $t \in (0,1)$, consider the mapping $T_t : C \to C$ defined by

$$T_t x = W(u, Tx; 1-t).$$

By property (B), we have

$$d(T_t x, T_t y) = td(Tx, Ty) \leq td(x, y) \text{ for all } x, y \in C.$$

By the Banach contraction principle, T_t has exactly one fixed point x_t in C. Therefore,

$$x_t = W(u, Tx_t; 1-t).$$

(b) By boundedness of C, we get

$$\begin{aligned} d(x_t, Tx_t) &= d(Tx_t, W(u, Tx_t; 1-t)) \\ &\leq (1-t)d(Tx_t, u) \leq (1-t)\ diam(C) \to 0 \text{ as } t \to 1. \end{aligned}$$ ∎

Applying Proposition 4.3.9, we have

Theorem 4.3.10 *Let X be a complete convex metric space X with property (B), C a nonempty compact convex subset of X, and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof. By Proposition 4.3.9, there exists a sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{4.15}$$

Because C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to v \in C$. Hence from (4.15), we have $v = Tv$. ∎

In Theorem 4.3.14, we will see that compactness can be dropped if C has normal structure. To see this, we extend the notion of normal structure in metric space X.

For $C \subset X$, we denote the following, which will be used throughout the remainder of this chapter:

$$\begin{aligned} r_x(C) &= \sup\{d(x,y) : y \in C\},\ x \in C, \\ r(C) &= \inf\{r_x(C) : x \in C\}, \\ \mathcal{Z}_C &= \{x \in C : r_x(C) = r(C)\}. \end{aligned}$$

A point $x_0 \in C$ is said to be a *diametral point* of C if

$$\sup\{d(x_0, y) : y \in C\} = diam(C).$$

A convex metric space X is said to have *normal structure* if for each closed convex bounded subset C of X that contains at least two points, there exists $x_0 \in C$ that is not a diametral point of C.

Example 4.3.11 *Every compact convex metric space has normal structure.*

A convex metric space X is said to have *property (C)* if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection. By Smulian's theorem, every weakly compact convex subset of a Banach space has property (C).

Using property (C), we have

Proposition 4.3.12 *If a convex metric space X has property (C), then $\mathcal{Z}_C$ is nonempty, closed, and convex.*

Proof. Let $C_n(x) = \{y \in C : d(x,y) \leq r(C) + 1/n\}$ for $n \in \mathbb{N}$ and $x \in X$. It is easily seen that the sets $C_n = \cap_{x\in X} C_n(x)$ form a decreasing sequence of nonempty closed convex sets, and hence $\cap_{n=1}^{\infty} C_n$ is nonempty closed convex by property (C). Because $\mathcal{Z}_C = \cap_{n=1}^{\infty} C_n$, the proof is complete. ∎

Proposition 4.3.13 *Let C be a nonempty compact subset of a convex metric space X and let D be the least closed convex set containing C. If $diam(C) > 0$, then there exists an element $x_0 \in D$ such that* $\sup\{d(x, x_0) : x \in C\} < diam(C)$.

Proof. Because C is compact, we may find $x_1, x_2 \in C$ such that $d(x_1, x_2) = diam(C)$. Let $C_0 \subset C$ be maximal so that $C_0 \supset \{x_1, x_2\}$ and $d(x,y) = 0$ or $diam(C)$ for all $x, y \in C_0$. It is easy to see that C_0 is finite. Let $C_0 = \{x_1, x_2, \cdots, x_n\}$. Because X is a convex metric space, we can define

$$\begin{aligned} y_1 &= W(x_1, x_2; \frac{1}{2}), \\ y_2 &= W(x_3, y_1; \frac{1}{3}), \\ &\cdots \\ y_{n-2} &= W(x_{n-1}, y_{n-3}; \frac{1}{n-1}), \\ y_{n-1} &= W(x_n, y_{n-2}; \frac{1}{n}) = u. \end{aligned}$$

Because C is compact, we can find $y_0 \in C$ such that

$$d(y_0, u) = \sup\{d(x,u) : x \in C\}.$$

From (4.14), we obtain

$$\begin{aligned} d(y_0, u) &\leq \frac{1}{n}\, d(y_0, x_n) + \frac{n-1}{n}\, d(y_0, y_{n-2}) \\ &\leq \frac{1}{n}\, d(y_0, x_n) + \frac{n-1}{n}\left(\frac{1}{n-1}\, d(y_0, x_{n-1}) + \frac{n-2}{n-1}\, d(y_0, y_{n-3})\right) \\ &= \frac{1}{n}\, d(y_0, x_n) + \frac{1}{n}\, d(y_0, x_{n-1}) + \frac{n-2}{n}\, d(y_0, y_{n-3}) \\ &\cdots \\ &\leq \frac{1}{n}\sum_{k=1}^{n} d(y_0, x_k) \leq diam(C). \end{aligned}$$

Therefore, if $d(y_0, u) = diam(C)$, then we must have $d(y_0, x_k) = diam(C) > 0$ for all $k = 1, 2, \cdots, n$. Hence $y_0 \in C_0$ by definition of C_0. But, then we must have $y_0 = x_k$ for some $k = 1, 2, \cdots, n$. This is a contradiction. Therefore,

$$\sup\{d(x, u) : x \in C\} = d(y_0, u) < diam(C). \qquad \blacksquare$$

A closed convex subset C of a convex metric space X is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive $T : C \to C$ has a fixed point.

We now prove that every closed convex subset of a convex metric space has fixed point property for nonexpansive mappings under normal structure.

Theorem 4.3.14 *Let X be a convex metric space with property (C). Let C be a nonempty closed convex bounded subset of X with normal structure and T a nonexpansive mapping from C into itself. Then T has a fixed point in C.*

Proof. Let $\mathcal{F}$ be the family of all nonempty closed convex subsets of C, each of which is mapped into itself by T. By property (C) and Zorn's lemma, $\mathcal{F}$ has a minimal element C_0. We show that C_0 consists of a single point. Let $x \in \mathcal{Z}_{C_0}$. Then

$$d(Tx, Ty) \leq d(x, y) \leq r_x(C_0) \text{ for all } y \in C_0.$$

Hence $T(C_0)$ is contained in the ball $B = B_{r_{(C_0)}}[Tx]$. Because $T(C_0 \cap B) \subset C_0 \cap B$, the minimality of C_0 implies that $C_0 \subset B$. Hence $r_{Tx}(C_0) \leq r(C_0)$. Because $r(C_0) \leq r_x(C_0)$ for all $x \in C_0$, we have $r_{Tx}(C_0) = r(C_0)$. Hence $Tx \in \mathcal{Z}_{C_0}$ and $T(\mathcal{Z}_{C_0}) \subset \mathcal{Z}_{C_0}$. By Proposition 4.3.12, $\mathcal{Z}_{C_0} \in \mathcal{F}$. If $z, w \in \mathcal{Z}_{C_0}$, then $d(z, w) \leq r_z(C_0) = r(C_0)$. Hence, by normal structure,

$$\delta(\mathcal{Z}_{C_0}) \leq r(C_0) < \delta(C_0).$$

Because this contradicts the minimality of C_0, $diam(C_0) = 0$ and C_0 consists of a single point. $\blacksquare$

4.4 Normal structure coefficient and fixed points

In this section, we discuss another convexity structure on metric space and the existence of fixed points of uniformly L-Lipschitzian mappings in a metric space with uniformly normal structure.

Let $\mathcal{F}(X)$ denote a nonempty family of subsets of a metric space (X, d). We say that $\mathcal{F}(X)$ defines a *convexity structure* on X if $\mathcal{F}(X)$ is stable by intersection and that $\mathcal{F}(X)$ has *property* (R) if any decreasing sequence $\{C_n\}$ of nonempty closed bounded subsets of X with $C_n \in \mathcal{F}(X)$ has nonvoid intersection.

A subset of X is said to be *admissible* if it is an intersection of closed balls. We denote by $\mathcal{A}(X)$ the family of all admissible subsets of X. It is obvious that

$\mathcal{A}(X)$ defines a convexity structure on X. In this section, any other convexity structure $\mathcal{F}(X)$ on X is always assumed to contain $\mathcal{A}(X)$.

For a bounded subset C of X, we define the admissible hull of C, denoted by $ad(C)$, as the intersection of all those admissible subsets of X that contain C, i.e.,

$$ad(C) = \cap\{B : C \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.$$

A basic property of admissible hull is given in the following proposition.

Proposition 4.4.1 *Let C be a bounded subset of a metric space X and $x \in X$. Then*

$$r_x(ad(C)) = r_x(C).$$

Proof. Suppose $r = r_x(ad(C)) > r_x(C)$. Then $C \subseteq B_{\overline{r}}[x]$ for any $\overline{r}$ with $r_x(C) < \overline{r} < r$. It follows that $ad(C) \subseteq B_{\overline{r}}[x]$. Hence

$$r_x(ad(C)) = \sup\{d(x, y) : y \in ad(C)\} \leq \overline{r} < r,$$

a contradiction. ∎

We introduce normal structure and uniformly normal structure with respect to convexity structure $\mathcal{F}(X)$ in a metric space X, respectively.

Definition 4.4.2 *A metric space (X, d) is said to have normal structure if there exists a convexity structure $\mathcal{F}(X)$ such that $r(C) < diam(C)$ for all $C \in \mathcal{F}(X)$ that is bounded and consists of more than one point. We say that $\mathcal{F}(X)$ is normal.*

Definition 4.4.3 *A metric space (X, d) is said to have uniformly normal structure if there exists a convexity structure $\mathcal{F}(X)$ such that $r(C) \leq \alpha \cdot diam(C)$ for some constant $\alpha \in (0, 1)$ and for all $C \in \mathcal{F}(X)$ that is bounded and consists of more than one point. We also say that $\mathcal{F}(X)$ is uniformly normal.*

We now define the normal structure coefficient of X (with respect to a given convexity structure $\mathcal{F}(X)$):

The number $N(X)$ is said to be the *normal structure coefficient* if

$$N(X) = \inf\left\{\frac{diam(C)}{R(C)}\right\},$$

where the infimum is taken over all bounded $C \in \mathcal{F}(X)$ with $diam(C) > 0$. It is easy to see that X has uniformly normal structure if and only if $N(X) > 1$.

The following theorem shows that every convexity structure with uniformly normal structure has property (R).

Theorem 4.4.4 *Let X be a complete metric space with a convexity structure $\mathcal{F}(X)$ that is uniformly normal. Let $\{C_n\}$ be a decreasing sequence of nonempty bounded subsets of X with $C_n \in \mathcal{F}(X)$. Then $\cap_{n=1}^{\infty}\overline{C}_n \neq \emptyset$.*

Proof. Without loss of generality, we may assume that $diam(C_n) > 0$ for all $n \in \mathbb{N}$. Let η be a real number with $\tilde{N}(X) < \eta < 1$, where $\tilde{N}(X) = N(X)^{-1}$. Define a sequence $\{x_{n,k}\}$ in X as follows:

For arbitrary $x_{n,1} \in C_n, n \in \mathbb{N}$, select $x_{n,k} \in ad(\{x_{m,k-1}\}_{m\geq n})$ such that

$$\sup\{d(x_{n,k}, x) : x \in ad(\{x_{m,k-1}\}_{m\geq n}) \leq \eta \; diam(ad\{x_{m,k-1}\}_{m\geq n}).$$

Set $A_{n,k} := ad(\{x_{m,k}\}_{m\geq n})$. Observe that

$$A_{n,k} \subseteq C_n \text{ for all } n, k \in \mathbb{N}$$

and for $m \geq n$,

$$\begin{aligned} d(x_{n,k}, x_{m,k}) &\leq \sup\{d(x_{n,k}, x) : x \in A_{n,k-1}\} \\ &\leq \eta \; diam(A_{n,k-1}) \\ &\leq \eta \; diam(\{x_{i,k-1}\}_{i\geq 1}). \end{aligned}$$

For $k \geq 2$, we have

$$\begin{aligned} diam(\{x_{n,k}\}) &\leq \eta \; diam(\{x_{n,k-1}\}) \\ &\leq \eta^2 \; diam(\{x_{n,k-2}\}) \\ &\cdots \\ &\leq \eta^{k-1} \; diam(\{x_{n,1}\}) \\ &\leq \eta^{k-1} \; diam(C_1). \end{aligned}$$

Now we consider a subsequence $\{x_{n,n}\}$ of $\{x_{n,k}\}$. Then $\{x_{n,n}\}$ is Cauchy, because

$$d(x_{n,n}, x_{m,m}) \leq \eta^{n-1} \; diam(C_1) \text{ for } m \geq n.$$

Therefore, there exists an $x \in \cap_{n=1}^{\infty} \overline{C}_n$ such that $\{x_{n,n}\}$ converges to x, i.e., $\cap_{n=1}^{\infty} \overline{C}_n \neq \emptyset$. ∎

Corollary 4.4.5 *Let X be a complete bounded metric space and $\mathcal{F}(X)$ a convexity structure of X with uniformly normal structure. Then $\mathcal{F}(X)$ has property (R).*

We now introduce the property $(\mathcal{P})$ for metric spaces.

Definition 4.4.6 *A metric space (X, d) is said to have property $(\mathcal{P})$ if given any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \cap_{n=1}^{\infty} ad(\{z_j : j \geq n\})$ such that*

$$\limsup_{n\to\infty} d(z, x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, x_n).$$

Remark 4.4.7 *If X has property (R), then $\cap_{n=1}^{\infty} ad(\{z_j : j \geq n\}) \neq \emptyset$. Also, if X is a weakly compact convex subset of a normed space, then admissible hulls are closed convex and $\cap_{n=1}^{\infty} ad\{z_j : j \geq n\} \neq \emptyset$ by weak compactness of X, and that X possesses property $(\mathcal{P})$ follows directly from the w-lsc of the function $\limsup_{n\to\infty} \|x_n - x\|$.*

We establish the following key result that can be viewed as the metric space formulation of Theorem 3.4.20.

Theorem 4.4.8 *Let (X, d) be a complete bounded metric space with both property $(\mathcal{P})$ and uniformly normal structure. Let $N(X)$ be the normal structure coefficient with respect to the given convexity structure $\mathcal{F}(X)$. Then for any sequence $\{x_n\}$ in X and any constant $\alpha > \tilde{N}(X)$, there exists a point $z \in X$ satisfying the properties:*

(a) $d(z, y) \leq \limsup_{n\to\infty} d(x_n, y)$ *for all* $y \in X$,

(b) $\limsup_{n\to\infty} d(z, x_n) \leq \alpha\ diam(\{x_n\})$.

Proof. (a) For each $n \in \mathbb{N}$, let $A_n = ad(\{x_j : j \geq n\})$. Then $\{A_n\}$ is a decreasing sequence of admissible subsets of X and hence $A := \cap A_n \neq \emptyset$ by Corollary 4.4.5. We observe by Proposition 4.4.1 that

$$\begin{aligned}
diam(A_n) &= \sup\{r_x(A_n) : x \in A_n\} \\
&= \sup_{x\in A_n}\sup_{j\geq n} d(x, x_j) \\
&= \sup_{j\geq n}\sup_{x\in A_n} d(x, x_j) = \sup_{j\geq n} r_{x_j}(A_n) \\
&= \sup_{j\geq n}\sup_{i\geq n} d(x_j, x_i) \\
&\leq \sup\{d(x_i, x_j) : i, j \in \mathbb{N}\} = diam(\{x_n\}).
\end{aligned}$$

For any $z \in A$ and any $y \in X$, we have

$$\sup_{j\geq n} d(y, x_j) = r_y(A_n) \geq r_y(A) \geq d(y, z).$$

It follows that

$$d(y, z) \leq \limsup_{n\to\infty} d(y, x_n).$$

(b) Without loss of generality, we may assume that $diam(\{x_n\}) > 0$. Then for $\alpha > \tilde{N}(X)$, we choose $\varepsilon > 0$ so small that

$$\tilde{N}(X) diam(\{x_n\}) + \varepsilon \leq \alpha\ diam(\{x_n\}).$$

By definition, one can find $a\ z_n \in A_n$ such that

$$\begin{aligned}
r_{z_n}(A_n) &< r(A_n) + \varepsilon \\
&\leq \tilde{N}(X)\ diam(A_n) + \varepsilon \\
&\leq \tilde{N}(X) diam(\{x_n\}) + \varepsilon \\
&\leq \alpha\ diam(\{x_n\}).
\end{aligned}$$

Hence for each $i \geq 1$,

$$\limsup_{m\to\infty} d(z_i, x_m) \leq \alpha\ diam(\{x_i\}).$$

Now property $(\mathcal{P})$ yields a point $z \in \cap_{i=1}^{\infty} ad(\{z_n : n \geq i\})$ such that

$$\limsup_{m\to\infty} d(z, x_m) \leq \limsup_{i\to\infty} \limsup_{m\to\infty} d(z_i, x_m).$$

Thus, $z \in A$ and satisfies

$$\limsup_{m\to\infty} d(z, x_m) \leq \alpha \ diam(\{x_i\}). \qquad \blacksquare$$

We now present the existence theorem for uniformly L-Lipschitzian mappings in a metric space.

Theorem 4.4.9 *Let (X, d) be a complete bounded metric space with both property $(\mathcal{P})$ and uniformly normal structure and $T : X \to X$ a uniformly L-Lipschitzian mapping with $L < \tilde{N}(X)^{-1/2}$. Then T has a fixed point in X.*

Proof. Choose a constant $\alpha, 1 > \alpha > \tilde{N}(X)$, such that $L < \alpha^{-1/2}$. Let $x_0 \in X$. By Theorem 4.4.8, we can inductively construct a sequence $\{x_m\}_{m=1}^{\infty}$ in X:

for each integer $m \geq 0$,

(a) $\limsup_{i\to\infty} d(x_{m+1}, T^i x_m) \leq \alpha \ diam(\{T^i x_m\})$;

(b) $d(x_{m+1}, y) \leq \limsup_{i\to\infty} d(T^i x_m, y)$ for all $y \in X$.

Set $r_m := \limsup_{i\to\infty} d(x_{m+1}, T^i x_m)$ and $h := \alpha L^2 < 1$. Note for each $i > j \geq 0$,

$$\begin{aligned} d(T^j x_m, T^i x_m) &\leq Ld(x_m, T^{i-j} x_m) \\ &\leq L \limsup_{n\to\infty} d(T^n x_{m-1}, T^{i-j} x_m) \qquad (by\ (b)) \\ &\leq L^2 \limsup_{n\to\infty} d(T^n x_{m-1}, x_m) \\ &\leq L^2 r_{m-1}. \end{aligned}$$

Observe that

$$\begin{aligned} r_m &= \limsup_{i\to\infty} d(x_{m+1}, T^i x_m) \\ &\leq \alpha \ diam(\{T^i x_m\}) \\ &\leq \alpha L^2 r_{m-1} = h r_{m-1} \\ &\cdots \\ &\leq h^m r_0. \end{aligned}$$

Hence for each integer $i \geq 0$,

$$\begin{aligned} d(x_{m+1}, x_m) &\leq d(x_{m+1}, T^i x_m) + d(x_m, T^i x_m) \\ &\leq d(x_{m+1}, T^i x_m) + \limsup_{j\to\infty} d(T^j x_{m-1}, T^i x_m) \\ &\leq d(x_{m+1}, T^i x_m) + L \limsup_{j\to\infty} d(T^j x_{m-1}, x_m) \\ &\leq d(x_{m+1}, T^i x_m) + L r_{m-1}, \end{aligned}$$

which implies that

$$d(x_{m+1}, x_m) \le r_m + Lr_{m-1} \le (h^m + Lh^{m-1})r_0.$$

This implies that $\{x_m\}$ is Cauchy. Let $\lim_{m\to\infty} x_m = v \in X$. Observe that

$$\begin{aligned} d(v, Tv) &\le d(v, x_{m+1}) + d(x_{m+1}, T^i x_m) + d(T^i x_m, Tv) \\ &\le d(z, x_{m+1}) + d(x_{m+1}, T^i x_m) + Ld(T^{i-1}x_m, v) \\ &\le d(v, x_{m+1}) + d(x_{m+1}, T^i x_m) + Ld(T^{i-1}x_m, x_{m+1}) + Ld(x_{m+1}, v), \end{aligned}$$

which implies that

$$d(v, Tv) \le (1+L)d(v, x_{m+1}) + (1+L)r_m \to 0 \text{ as } m \to \infty.$$

Therefore, v is a fixed point of T. ▮

4.5 Lifschitz's coefficient and fixed points

In this section, we give an existence theorem concerning uniformly L-Lipschitzian mappings in a metric space.

First, we define the Lifschitz's coefficient of a metric space:

Let (X, d) be a metric space. Then the Lifschitz's coefficient $\kappa(X)$ is a number defined by

$$\begin{aligned} \kappa(X) &= \sup\{\beta > 0 : \exists\ \alpha > 1 \text{ such that for all } x, y \in X, \text{ for all } r > 0, \\ &\quad [d(x,y) > r \Rightarrow \exists\ z \in X \text{ such that} B_{\alpha r}[x] \cap B_{\beta r}[y] \subseteq B_r[z]]\}. \end{aligned}$$

It is clear that $\kappa(X) \ge 1$ for any metric space X. For a strictly convex Banach space X, $\kappa(X) > 1$ and for a Hilbert space H, $\kappa(H) = \sqrt{2}$.

We are now in a position to prove a fundamental existence theorem for uniformly L-Lipschitzian mappings in a metric space with Lifschitz's coefficient $\kappa(X)$.

Theorem 4.5.1 *Let (X, d) be a bounded complete metric space and $T : X \to X$ a uniformly L-Lipschitzian mapping with $L < \kappa(X)$. Then T has a fixed point in X.*

Proof. If $\kappa(X) = 1$, then T is contraction and hence T has a unique fixed point. So, suppose $\kappa(X) > 1$. For $b \in (L, \kappa(X))$, there exists $a > 1$ such that

$$\forall\ u, v \in X,\ r > 0 \text{ with } d(x,y) > r \Rightarrow \exists z \in X :\ B_{br}[u] \cap B_{ar}[v] \subset B_r[z]. \quad (4.16)$$

For any $x \in X$, let

$$r(x) = \inf\{R > 0 :\ \text{there exists } y \in X \text{ such that } \limsup_{n\to\infty} d(T^n x, y) \le R\}.$$

Observe that r is a lower semicontinuous and $r(x) = 0$ implies $x = Tx$.

Take $\lambda \in (0, 1)$ such that $\gamma = \min\{\lambda a, \lambda b/L\} > 1$. We now show that there exists a sequence $\{y_m\}$ in X that satisfies the following:

$$r(y_{m+1}) \leq \lambda r(y_m) \text{ and } d(y_m, y_{m+1}) \leq (\lambda + \gamma) r(y_m) \text{ for all } m \in \mathbb{N}_0. \quad (4.17)$$

Indeed, consider an arbitrary point $y_0 \in X$ and assume that $y_0, y_1, \cdots, y_m$ are given. We now construct y_{m+1}. If $r(y_m) = 0$, then $y_{m+1} = y_m$. If $r(y_m) > 0$, then for a number $\lambda r(y_m)$, there exists $n \in \mathbb{N}$ such that

$$d(y_m, T^n y_m) > \lambda r(y_m).$$

From the definition of $r(y_m)$, there exists $x \in X$ such that

$$\limsup_{n\to\infty} d(y_m, T^n x) \leq r(y_m) < \gamma r(y_m).$$

Hence for $i > j$

$$d(T^i x, T^j y_m) \leq L\ d(T^{i-j} x, y_m),$$

which implies that

$$\limsup_{i\to\infty} d(T^i x, T^j y_m) \quad \leq \quad L \ \limsup_{i\to\infty} d(T^{i-j} x, y_m) \leq L\gamma\ r(m).$$

Because

$$B_{\gamma r(y_m)}[y_m] \cap B_{L\gamma r(y_m)}[T^n y_m] \subset B_{a\lambda r(y_m)}[y_m] \cap B_{b\lambda r(y_m)}[T^n y_m] = C,$$

the set C is contained in a closed ball centered at w with radius $\lambda r(y_m)$ (Condition (4.16)). Thus, $\limsup_{n\to\infty} d(T^n x, w) \leq \lambda r(y_m)$. Take $w = y_{m+1}$, and it follows from above that $\{y_m\}$ satisfies (4.17).

Note

$$r(y_{m+1}) \leq \lambda r(y_m) \leq \cdots \leq \lambda^{m+1} r(y_0) \to 0 \text{ as } m \to \infty$$

and

$$d(y_m, y_{m+1}) \leq (\lambda + \gamma) r(y_m) \to 0 \text{ as } m \to \infty.$$

Hence $\{y_m\}$ converges to $v \in M$. But because $r(v) = 0$, v is a fixed point of T. ∎

Bibliographic Notes and Remarks

Most of the results presented in Section 4.1 may be found in Goebel and Kirk [59], Khamsi and Kirk [85], Kirk and Sims [91], and Martin [106]. Theorem 4.1.16 first appeared in Matkowski [108] as a generalization of the result of Boyd and Wong [22], and it was recently extended for non-self mappings in Reich and

Zaslavski [133](see also recent results of Agarwal, O'Regan and their coworkers [1, 117]).

The notion of nearly non-Lipschitzian mappings was introduced by Sahu [137]. Some existence theorems for demicontinuous nearly contraction and nearly asymptotically nonexpansive mappings were also established in [137].

The fixed point theory of multivalued contraction self-mappings was first proved in Nadler [115]. It was extended for multivalued non-self contraction mappings by Assad and Kirk [4].

The results describes in Sections 4.3∼4.5 can be found in Lifschitz [97], Lim and Xu [99], and Takahashi [154].

Exercises

4.1 Let X be a complete metric space and $T : X \to X$ a mapping such that T^m is contraction for some $m \in \mathbb{N}$. Show that T has a unique fixed point.

4.2 Let (X, d) be a metric space and $T : X \to X$ a mapping. T is said to be a Zamfirescu mapping if there exist the real number a, b, and c satisfying $0 < a < 1, 0 < b, c < 1/2$ such that for each pair x, y in X, at least one of the following is true:

(Z_1) $d(Tx, Ty) \le ad(x, y)$,

(Z_2) $d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)]$,

(Z_3) $d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)]$.

If (X, d) is a complete metric space and $T : X \to X$ a Zamfirescu mapping, show that T has a unique fixed point $z \in X$ and for each $x \in X, \{T^n x\}$ converges to z.

4.3 Let T be a mapping from a complete metric space X into itself satisfying the condition:

$$d(Tx, Ty) \le ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(y, Tx) + d(x, Ty)]$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that $a + 2b + 2c < 1$. Show that T has a unique fixed point $z \in X$ and for each $x \in X, \{T^n x\}$ converges to z.

4.4 Let T be a mapping from a complete metric space into itself. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, Tx) < \delta \Rightarrow T(B_\varepsilon[x]) \subset B_\varepsilon[x].$$

If $d(T^n x, T^{n+1} x) \to 0$ for some $x \in X$, show that the sequence $\{T^n x\}$ converges to z, which is a fixed point of T.

4.5 Let X be a complete metric space and $T : X \to X$ an expansion mapping, i.e., there exists constant $k > 1$ such that

$$d(Tx, Ty) \geq kd(x, y) \text{ for all } x, y \in X.$$

Assume that $T(X) = X$. Show that

(a) T is one to one,

(b) T has a unique fixed point $z \in X$ with $T^n x \to z$ as $n \to \infty$ for some $x \in X$.

4.6 Let (X, d) be a complete metric space and $T : X \to CB(X)$ a mapping. If α is a function from $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r \to t^+} \alpha(r) < t$ for every $t \in [0, \infty)$ and if

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$$

for each $x, y \in X$, show that T has a fixed point in X.

Chapter 5

Existence Theorems in Banach Spaces

This chapter is devoted to a demiclosed principle and existence of fixed points of Lipschitzian and non-Lipschitzian mappings in Banach spaces.

5.1 Non-self contraction mappings

In Chapter 4, we studied fixed point theorems for single-valued and multivalued contraction mappings in metric spaces. In this section, we discuss fixed point theorems concerning non-self contraction mappings in Banach spaces.

Let C be a nonempty subset of a Banach space X. For $x \in C$, the *inward set of x relative to C* is the set

$$I_C(x) = \{x + t(y - x) : y \in C \text{ and } t \geq 0\}$$

and the *outward set of x relative to C* is the set

$$O_C(x) = \{x - t(y - x) : y \in C \text{ and } t \geq 0\}.$$

Let $\overline{I_C(x)}$ and $\overline{O_C(x)}$ denote closures of $I_C(x)$ and $O_C(x)$, respectively.

Set

$$\overline{I}_C(x) := x + \{y \in X : \liminf_{h \to 0^+} \frac{d(x + hy, C)}{h} = 0\}, \quad x \in C.$$

Note that for a convex set C, we have
(i) $\overline{I}_C(x) = \overline{I_C(x)}$,
(ii) $C \subseteq I_C(x)$.

We now define weakly inward and weakly outward mappings:

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Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. Then T is said to be a

(i) *inward mapping* if $Tx \in I_C(x)$ for all $x \in C$,

(ii) *weakly inward mapping* if $Tx \in \overline{I_C(x)}$ for all $x \in C$,

(iii) *weakly outward mapping* if $Tx \in \overline{O_C(x)}$ for all $x \in C$.

Let us compare inwardness and weak inwardness conditions with other conditions. Set

(C1) *Rothe's condition*: $T(\partial C) \subseteq C$;

(C2) inwardness condition: $Tx \in I_C(x)$ for all $x \in C$;

(C3) weak inwardness condition: $Tx \in \overline{I_C(x)}$ for all $x \in C$;

(C4) the *Leray-Schauder* condition (if the interior $int(C)$ of C is nonempty): there exists a $z \in int(C)$ such that

$$Tx - z \neq \mu(x - z) \quad \text{for all} \quad x \in \partial C \text{ and } \mu > 1.$$

These boundary conditions hold the implications:

$$(C1) \Rightarrow (C2) \Rightarrow (C3) \Rightarrow (C4).$$

The following proposition gives an equivalent formulation of the weakly inwardness condition.

Proposition 5.1.1 *Let C be a nonempty closed convex subset of a Banach space X. Then $T : C \to X$ is weakly inward if and only if*

$$\lim_{h \to 0^+} \frac{d((1-h)x + hTx, C)}{h} = 0 \quad \text{for all } x \in C. \tag{5.1}$$

Proof. Suppose that condition (5.1) holds. Fix $x \in C$. For $\varepsilon > 0$, we may assume $t \in (0, 1)$ and $y \in C$ such that

$$\|(1-t)x + tTx - y\| \leq d((1-t)x + tTx, C) + t\varepsilon.$$

It follows that

$$\|Tx - [(1-t^{-1})x + t^{-1}y]\| \leq t^{-1}d((1-t)x + tTx, C) + \varepsilon.$$

It is easy to see that $Tx \in \overline{I_C(x)}$. Hence T is weakly inward.

Conversely, suppose that T is weakly inward, i.e., $Tx \in \overline{I_C(x)}$ for all $x \in C$. Hence for $\varepsilon > 0$, there exists $y \in I_C(x)$ such that

$$\|y - Tx\| \leq \varepsilon.$$

Because C is convex, there exists $h_0 > 0$ such that

$$(1-h)x + hy \in C \text{ for } 0 < h \leq h_0.$$

Hence for these h, we have

$$\begin{aligned}\frac{d((1-h)x+hTx,C)}{h} &\leq \frac{\|(1-h)x+hTx-\{(1-h)x+hy\}]\|}{h}\\ &\leq \varepsilon.\end{aligned}$$

Therefore, the condition (5.1) holds. ∎

The following result is an extension of the Banach contraction principle for non-self contraction mappings.

Theorem 5.1.2 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a weakly inward contraction mapping. Then T has a unique fixed point in C.*

Prof. Let k, $0 < k < 1$ denote Lipschitz constant of T. Choose $\varepsilon > 0$ so small that $k < (1-\varepsilon)/(1+\varepsilon)$. By Proposition 5.1.1, T satisfies the condition (5.1). Hence for $x \in C$ with $x \neq Tx$, there exists $h \in (0,1)$ such that

$$d((1-h)x+hTx,C) < h\varepsilon\|x-Tx\|.$$

By the definition of distance, there exists $y \in C$ such that

$$\|(1-h)x+hTx-y\| < h\varepsilon\|x-Tx\|. \tag{5.2}$$

By (5.2), we have

$$\begin{aligned}h\varepsilon\|x-Tx\| &> \|x-y-h(x-Tx)\|\\ &\geq \|x-y\|-h\|x-Tx\|,\end{aligned}$$

which implies that

$$\|x-y\| < (1+\varepsilon)h\|x-Tx\|. \tag{5.3}$$

Using (5.2), we have

$$\begin{aligned}\|y-Ty\| &\leq \|y-[(1-h)x+hTx]\|+\|(1-h)x+hTx-Tx\|+\|Tx-Ty\|\\ &\leq h\varepsilon\|x-Tx\|+(1-h)\|x-Tx\|+k\|x-y\|\\ &= \|x-Tx\|+(\varepsilon-1)h\|x-Tx\|+k\|x-y\|\\ &= \|x-Tx\|+(\varepsilon-1)h\|x-Tx\|+\frac{1-\varepsilon}{1+\varepsilon}\|x-y\|\\ &\quad -\left(\frac{1-\varepsilon}{1+\varepsilon}-k\right)\|x-y\|\\ &< \|x-Tx\|-\left(\frac{1-\varepsilon}{1+\varepsilon}-k\right)\|x-y\|. \qquad \text{(by (5.3))}\end{aligned}$$

If $x \neq Tx,\ x \in C$, denote $y \in C$ as above by $f(x)$, where f is a self-mapping on C. Then by putting

$$\varphi(x) = \left(\frac{1-\varepsilon}{1+\varepsilon}-k\right)^{-1}\|x-Tx\|,$$

$\varphi : C \to \mathbb{R}^+$ is a continuous function and

$$\|x - fx\| < \varphi(x) - \varphi(fx). \tag{5.4}$$

By Caristi's theorem, f has a fixed point, which contradicts the strict inequality (5.4). ∎

We now turn our attention to study fixed points of multivalued mappings in Banach spaces.

Let C be a nonempty subset of a Banach space X. We say that a mapping T of C into the family of nonempty subsets of X is weakly inward if $Tx \subset \overline{I_C(x)}$ for each $x \in C$. Let $F(X)$ denote the family of nonempty closed subsets of X and $T : C \to F(X)$ a multivalued mapping. Given $x \in C$ and $\alpha \geq 1$, let $T_\alpha(x)$ denote the set $\{z \in Tx : \|x - z\| \leq \alpha\, d(x, Tx)\}$.

Theorem 5.1.3 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to F(X)$ an upper semicontinuous mapping satisfying the conditions:*

(a) For each $x \in C$, there exists $\delta = \delta(x) > 0$ such that

$$y \in B_\delta[x] \cap C \Rightarrow \ d(y, Ty) \leq d(y, Tx) + k\|x - y\|,$$

where $k \in (0, 1)$.

(b) $T_1(x) \cap \overline{I_C(x)} \neq \emptyset$ for each $x \in C$.

Then T has a fixed point in C.

Proof. Suppose, for contradiction, that T has no fixed point. We may assume that $d(x, Tx) > 0$ for each $x \in C$. Select $\varepsilon > 0$ such that $k < (1 - \varepsilon)/(1 + \varepsilon)$. Given $x \in C$, condition (b) implies the existence of an element $z \in T_1(x) \cap \overline{I_C(x)}$, and by Proposition 3.1.11, there exists $h \in (0, 1)$ such that

$$h^{-1} d((1 - h)x + hz, C) < \varepsilon\, d(x, Tx). \tag{5.5}$$

Set $\overline{z} := (1 - h)x + hz$. Observe that $\|\overline{z} - x\| = h\|z - x\|$, and moreover we may suppose h has been chosen so small that $\overline{z} \in B_{\delta/2}[x]$ (where $\delta = \delta(x)$ is from condition (a)). By (5.5), there exists $y \in C$, $y \neq x$ such that

$$\|\overline{z} - y\| < h\varepsilon\, d(x, Tx). \tag{5.6}$$

Hence

$$\begin{aligned}
\|x - y\|/\|\overline{z} - x\| &\leq [\|x - \overline{z}\| + \|\overline{z} - y\|]/\|\overline{z} - x\| \\
&= 1 + \|\overline{z} - y\|/\|\overline{z} - x\| \\
&< 1 + \|\overline{z} - y\|/(hd(x, Tx)) \\
&< 1 + \varepsilon,
\end{aligned}$$

which implies that

$$(1 + \varepsilon)^{-1}\|x - y\| < \|\overline{z} - x\|. \tag{5.7}$$

Because $\overline{z} \in B_{\delta/2}[x]$, it follows from (5.7) that $y \in B_\delta[x]$ and thus

$$\begin{aligned} d(y,Ty) &\leq d(y,Tx) + k\|x-y\| \\ &\leq \|y-\overline{z}\| + d(\overline{z},Tx) + k\|x-y\|. \end{aligned}$$

Combining this with (5.6), (5.7), and using the definition of $\overline{z}$ along with the fact that $z \in T_1(x)$, we obtain

$$\begin{aligned} d(y,Ty) &\leq \|y-\overline{z}\| + d(x,Tx) - \|x-\overline{z}\| + k\|x-y\| \\ &< \varepsilon\|x-\overline{z}\| + d(x,Tx) - \|x-\overline{z}\| + k\|x-y\|. \\ &= d(x,Tx) + k\|x-y\| - (1-\varepsilon)\|x-\overline{z}\| \\ &< d(x,Tx) + k\|x-y\| - (1-\varepsilon)(1+\varepsilon)^{-1}\|x-y\| \\ &= d(x,Tx) + [k-(1-\varepsilon)(1+\varepsilon)^{-1}]\|x-y\|. \end{aligned}$$

Set $\eta = -[k-(1-\varepsilon)(1+\varepsilon)^{-1}]$. Then $\eta > 0$ and

$$\eta\|x-y\| \leq d(x,Tx) - d(y,Ty).$$

We now define $g : C \to C$ by $g(x) = y$ with y determined as above, and let $\varphi(x) = \eta^{-1}d(x,Tx)$. Proposition 4.2.6 implies that φ is lower semicontinuous, so Caristi's theorem implies the existence of an $x_0 \in C$ such that $x_0 = g(x_0)$. But $g(x) = y \neq x$ for all $x \in C$ by definition, and our assumption that T has no fixed points is contradicted. ∎

The following example shows that condition (b) in Theorem 5.1.3 cannot be altered to $Tx \cap I_C(x) \neq \emptyset$.

Example 5.1.4 *Let $X = \mathbb{R}$ and $C = [0,1]$. Define $T : C \to \mathcal{K}(X)$ by $Tx = \{-1, 2\}$ for all $x \in C$. Then T is a constant and*

$$I_C(x) = \begin{cases} [0,\infty) & \text{if } x = 0, \\ \mathbb{R} & \text{if } x \in (0,1), \\ (-\infty,1] & \text{if } x = 1. \end{cases}$$

Therefore, $Tx \cap I_C(x) \neq \emptyset$ for all $x \in C$, but T has no fixed point in C.

Theorem 5.1.5 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to F(X)$ an upper semicontinuous mapping that satisfies the conditions:*

(a) For each $x \in C$, there exists $\delta = \delta(x) > 0$ such that

$$y \in B_\delta[x] \cap C \Rightarrow \ d(y,Ty) \leq d(y,Tx) + k\|x-y\|,$$

where $k \in (0,1)$.

(b′) Corresponding with each $x \in C$, there exist constants $\alpha = \alpha(x) > 1$, $\mu = \mu(x) \in (0,1)$ such that

$$(1-\mu)x + \mu T_\alpha(x) \subset C.$$

Then T has a fixed point in C.

Proof. Suppose, for contradiction, that T has no fixed point. Select $k' \in (k, 1)$. Fix $x \in C$, let $\alpha = \alpha(x)$, and choose μ so that $\mu \leq \min\{\mu(x), \delta(x)\}$ and $\alpha\mu\, d(x, Tx) \leq \delta(x)$. Then

$$\|x - y\| \geq \mu\, d(x, Tx) \text{ for all } y \in (1-\mu)x + \mu T_\alpha(x).$$

Thus, if $\xi \in (1, \alpha)$ is chosen so that $\xi - 1 < \mu(k' - k)$, we obtain

$$(\xi - 1)\, d(x, Tx) \leq (k' - k)\|x - y\| \text{ for all } y \in (1-\mu)x + \mu T_\alpha(x) \tag{5.8}$$

Now fix $z \in T_\xi(x)$ and let $y = (1-\mu)x + \mu z$. By (5.8), we have

$$k\|x - y\| \leq k'\|x - y\| - (\xi - 1)\, d(x, Tx)$$

and therefore (using (a) because $\mu\|x - z\| \leq \mu\xi\, d(x, Tx) \leq \delta(x)$):

$$\begin{aligned} d(y, Ty) &\leq d(y, Tx) + k\|x - y\| \\ &\leq \|y - z\| + k\|x - y\| \\ &= \|x - z\| - \|x - y\| + k\|x - y\| \\ &\leq \xi d(x, Tx) - \|x - y\| + k'\|x - y\| - (\xi - 1)\, d(x, Tx) \\ &= d(x, Tx) + (k' - 1)\|x - y\|. \end{aligned}$$

Hence

$$\|x - y\| \leq (1 - k')^{-1}[d(x, Tx) - d(y, Ty)]$$

and the proof is completed as in Theorem 5.1.3 by taking $g(x') = y$ and $\varphi(x) = (1 - k')^{-1} d(x, Tx)$. ∎

We now derive some existence theorems from Theorems 5.1.3 and 5.1.5.

Theorem 5.1.6 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to CB(X)$ a multivalued contraction mapping that satisfies either condition (b) or condition (b′). Then T has a fixed point in C.*

Proof. It is easy to see that T is automatically upper semicontinuous. Because

$$d(y, Ty) \leq d(y, Tx) + H(Tx, Ty) \text{ for all } x, y \in C,$$

it follows that T satisfies condition (a). ∎

Theorem 5.1.7 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to \mathcal{K}(X)$ a multivalued contraction mapping for which $Tx \subset \overline{I_C(x)}$, $x \in C$. Then T has a fixed point in C.*

Proof. Under the stated assumptions, condition (b) is automatically satisfied. Hence the result follows from Theorem 5.1.6. ∎

As a slightly different result, we have

Theorem 5.1.8 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to CB(X)$ a contraction mapping. If for each $x \in C$,*

$$\lim_{h\to 0^+} \frac{d((1-h)x+hz, C)}{h} = 0 \text{ uniformly for } z \in Tx,$$

then T has a fixed point.

Proof. Let k be the Lipschitz constant of T. Choose real numbers k' and q such that $k < k' < 1, 0 < q < 1$ and $k' < (1-q)/(1+q)$. Suppose, for contradiction, that T has no fixed point, i.e., $d(x, Tx) > 0$ for all $x \in C$. For each $x \in C$, take $\varepsilon > 0$ such that

$$qd(x, Tx) - \varepsilon d(x, Tx) > 0.$$

By assumption, there exists $h \in (0, 1)$ such that

$$d((1-h)x + hz, C) < h(q-\varepsilon)d(x, Tx) \quad \text{for all } z \in Tx.$$

Choose $z \in Tx$ such that

$$\|x - z\| < d(x, Tx) + h\varepsilon d(x, Tx). \tag{5.9}$$

For such a z, take $y \in C$ such that

$$\|(1-h)x + hz - y\| < h(q-\varepsilon)d(x, Tx). \tag{5.10}$$

Set $w := (1-h)x + hz$. From (5.10), we have

$$\begin{aligned} \|w - y\| &< h(q-\varepsilon)d(x, Tx) \\ &\leq hq\|x - z\| - h\varepsilon d(x, Tx) \\ &= q\|w - x\| - h\varepsilon d(x, Tx) \quad (\text{since } h(z-x) = w - x) \\ &< q\|w - x\| \end{aligned}$$

and

$$\begin{aligned} \|x - y\| &\leq \|x - w\| + \|w - y\| \\ &< \|x - w\| + q\|w - x\| \\ &< (1+q)\|x - w\|. \end{aligned}$$

Let $\varepsilon' = [k'\|x - y\| - H(Tx, Ty)]/2$. Then, because T is contraction with Lipschitz constant k and $x \neq y$ by (5.10), we have $\varepsilon' > 0$. Choose $u \in Tx$ and $v \in Ty$ satisfying

$$\|w - u\| < d(w, Tx) + \varepsilon' \text{ and } \|u - v\| \leq H(Tx, Ty) + \varepsilon'.$$

Then we have

$$\begin{aligned}
d(y,Ty) &\le \|y-v\| \\
&\le \|y-w\|+\|w-u\|+\|u-v\| \\
&< \|y-w\|+d(w,Tx)+\varepsilon'+H(Tx,Ty)+\varepsilon' \\
&\le \|y-w\|+\|w-z\|+k'\|x-y\| \\
&= \|y-w\|+\|x-z\|-h\|x-z\|+k'\|x-y\| \\
&= \|y-w\|+\|x-z\|-\|w-x\|+k'\|x-y\| \\
&< q\|w-x\|-h\varepsilon\, d(x,Tx)+\|x-z\|-\|w-x\|+k'\|x-y\| \\
&= (q-1)\|w-x\|+\|x-z\|-h\varepsilon\, d(x,Tx)+k'\|x-y\| \\
&< \frac{q-1}{q+1}\,\|x-y\|+d(x,Tx)+k'\|x-y\| \quad \text{(from (5.9))} \\
&= d(x,Tx)-\left(\frac{1-q}{q+1}-k'\right)\|x-y\| \\
&= d(x,Tx)-r\|x-y\|,
\end{aligned}$$

where $r=[(1-q)/(1+q)]-k'$. Now for each $x\in C$, denote $y\in C$ as above by $f(x)$. Then

$$\|x-f(x)\|<\varphi(x)-\varphi(f(x)) \quad \text{for all } x\in C, \tag{5.11}$$

where $\varphi : C\to[0,\infty)$ is the continuous function defined by $\varphi(x)=r^{-1}d(x,Tx)$ for all $x\in C$. By Caristi's fixed point theorem, there exists an $x_0\in C$ such that $x_0=fx_0$. This contradicts the inequality (5.11). Therefore, T has a fixed point. ∎

We now consider the existence of fixed points of multivalued contraction mappings when the domain is not necessarily convex.

Let C be a nonempty subset of a Banach space X. For given $x\in C$ and $\alpha\in\mathbb{R}^+$, let $I_C(x,\alpha)$ denote the set $\{z\in X : z=x+\lambda(y-x)$ for some $y\in C$ and $\lambda\ge\alpha\}$. Obviously, $I_C(x)=I_C(x,0)$.

Theorem 5.1.9 *Let C be a nonempty closed subset of a Banach space X and $T : C\to\mathcal{K}(X)$ a contraction mapping satisfying the condition:*

$$Tx\subset\overline{I_C(x,1)},\quad x\in C.$$

Then T has a fixed point.

Proof. Suppose, for contradiction, that T has no fixed point. Let k be the Lipschitz constant of T. Choose q such that $0<q<1$ and $k<(1-q)/(1+q)$. Let $x\in C$. Then there exists a point $z\in Tx$ such that

$$0<d(x,Tx)=\|x-z\|.$$

Because $z \in Tx \subset \overline{I_C(x,1)}$, there exists a $y_n \in C$ and $\lambda_n \geq 1$ such that

$$\|z - (x + \lambda_n(y_n - x))\| < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

If $\lambda_n = 1$ for all sufficiently large n, then we define $y = z$. Otherwise, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_i} > 1$. We choose N sufficiently large number such that

$$\|z - (x + \lambda_N(y_N - x))\| < \frac{1}{N} < q\|x - z\|$$

and we define $y_N = y$. Set $h := 1/\lambda_N$ and $w := (1-h)x + hz$. Observe that

$$\|w - x\| = h\|x - z\|$$

and

$$\left\|\frac{1}{\lambda_N}z - \frac{1}{\lambda_N}x - (y - x)\right\| < \frac{q}{\lambda_N}\left\|x - z\right\|,$$

i.e.,

$$\|w - y\| < qh\ \|x - z\|.$$

It follows that

$$\begin{aligned}
\|x - y\| &\leq \|x - w\| + \|w - y\| \\
&\leq \|x - w\| + qh\ \|x - z\| \\
&= (1+q)\|w - x\|.
\end{aligned}$$

Choosing $u \in Tx$ and $v \in Ty$ such that

$$\|w - u\| = d(w, Tx) \text{ and } \|u - v\| \leq H(Tx, Ty),$$

we have that

$$\begin{aligned}
d(y, Ty) &\leq \|y - v\| \\
&\leq \|y - w\| + \|w - u\| + \|u - v\| \\
&\leq \|y - w\| + \|w - z\| + H(Tx, Ty) \\
&= \|y - w\| + \|z - x\| - h\|x - z\| + k\|x - y\| \\
&< qh\|x - z\| + \|x - z\| - h\|x - z\| + k\|x - y\| \\
&= (q-1)\|w - x\| + \|x - z\| + k\|x - y\| \\
&< \frac{q-1}{q+1}\|x - y\| + \|x - z\| + k\|x - y\| \\
&= d(x, Tx) - r\|x - y\|,
\end{aligned}$$

where $r = [(1-q)/(1+q)] - k > 0$.

We define the mapping $f : C \to C$ by $f(x) = y$ for $x \in C$. Then

$$\|x - fx\| < \varphi(x) - \varphi(fx) \quad \text{for all } x \in C, \tag{5.12}$$

where $\varphi : C \to [0, \infty)$ is the continuous function defined by $\varphi(x) = r^{-1}d(x, Tx)$ for all $x \in C$. By Caristi's fixed point theorem, there exists an $x_0 \in C$ such that $x_0 = fx_0$. This contradicts the inequality (5.12). Therefore, T has a fixed point. $\blacksquare$

In the following theorem, we assume that each point $x \in C$ has a nearest point in the set Tx.

Theorem 5.1.10 *Let C be a nonempty closed subset of a Banach space X and $T : C \to 2^X \backslash \{\emptyset\}$ a contraction with closed-values and satisfying the condition:*

$$Tx \subset x + \overline{\{\lambda(y - x) : \lambda \geq 1, y \in C\}} \quad \textit{for all } x \in C.$$

Assume that each $x \in C$ has a nearest point in Tx. Then T has a fixed point.

Proof. Without loss of generality, we may assume that

$$Tx \subseteq x + \overline{\{\lambda(y - x) : \lambda > 1, y \in C\}} \quad \text{for all } x \in C. \tag{5.13}$$

Choose $q \in (0, 1)$ and $\varepsilon \in (0, 1)$ such that $k < q < (1-\varepsilon)/(1+\varepsilon)$. By assumption, each $x \in C$ has a nearest point in Tx, then there exists $z \in Tx$ such that

$$\|x - z\| = d(x, Tx).$$

Set $z = f(x)$. Then $f(x) \in Tx$ and $\|x - f(x)\| = d(x, Tx)$. By (5.13), there exists $y_n \in C$ and $\lambda_n > 1$ such that

$$\|f(x) - (x + \lambda_n(y_n - x))\| \to 0 \text{ as } n \to \infty.$$

Suppose, for contradiction, that T has no fixed point. Then we have a sufficient large natural number N such that

$$\|f(x) - (x + \lambda_N(y_N - x))\| < \varepsilon d(x, Tx). \tag{5.14}$$

Set

$$h := \frac{1}{\lambda_N}, w := \left(1 - \frac{1}{\lambda_N}\right)x + \frac{1}{\lambda_N}f(x) = (1 - h)x + hf(x) \text{ and } g(x) := y_N.$$

By (5.14), we have

$$\|y_N - w\| = \|y_N - ((1 - h)x + hf(x))\| < \varepsilon hd(x, Tx) = \varepsilon h\|x - f(x)\|.$$

Also

$$\|w - f(x)\| = (1 - h)\|x - f(x)\| = (1 - h)d(x, Tx).$$

Observe that

$$\begin{aligned}
d(y_N, Ty_N) &\le \|y_N - w\| + d(w, Tx) + H(Tx, Ty_N) \\
&\le \|y_N - w\| + \|w - f(x)\| + k\|x - y_N\| \\
&\le \|y_N - w\| + (1-h)\|x - f(x)\| + k\|x - y_N\| \\
&< \varepsilon h\|x - f(x)\| + (1-h)\|x - f(x)\| + k\|x - y_N\| \\
&= (\varepsilon - 1)h\|x - f(x)\| + \|x - f(x)\| + k\|x - y_N\| \\
&= (\varepsilon - 1)\|w - x\| + \|x - f(x)\| + k\|x - y_N\| \\
&= (\varepsilon - 1)\|w - x\| + q\|x - y_N\| \\
&\quad + \|x - f(x)\| - (q-k)\|x - y_N\|. \qquad (5.15)
\end{aligned}$$

By the choice of the integer N, we see that

$$\begin{aligned}
\varepsilon d(x, Tx) &> \|f(x) - x - \lambda_N(y_N - x)\| \\
&\ge \lambda_N\|y_N - x\| - \|f(x) - x\| \\
&= \lambda_N\|y_N - x\| - d(x, Tx),
\end{aligned}$$

which implies that

$$\|y_N - x\| < \frac{1+\varepsilon}{\lambda_N} d(x, Tx).$$

By the choice of q, we have

$$\begin{aligned}
\|y_N - x\| &< \frac{(1+\varepsilon)}{\lambda_N} d(x, Tx) \\
&< \frac{1-\varepsilon}{q} h d(x, Tx) \quad (\text{since } \ \|w - x\| = hd(x, Tx)) \\
&= \frac{1-\varepsilon}{q}\|w - x\|.
\end{aligned}$$

It follows from (5.15) that

$$d(y_N, Ty_N) < \|x - f(x)\| - (q-k)\|x - y_N\|.$$

Thus, we have

$$\|x - g(x)\| < \varphi(x) - \varphi(g(x)) \quad \text{for all } x \in C, \qquad (5.16)$$

where $\varphi : C \to \mathbb{R}^+$ is the continuous function defined by $\varphi(x) = (q-k)^{-1}d(x, Tx)$, $x \in C$. By Caristi's theorem, g has a fixed point that contradicts the strict inequality (5.16). Therefore, T has a fixed point in C. ∎

Theorem 5.1.11 *Let C be a nonempty closed subset of a Banach space X and $T : C \to F(X)$ a contraction. Assume that T is weakly inward on C and that each $x \in C$ has a nearest point in Tx. Then T has a fixed point.*

Proof. The proof is similar to the proof of Theorem 5.1.10. ∎

5.2 Nonexpansive mappings

Let C be a nonempty subset of a normed space X and $T : C \to X$ a mapping. Then T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

Recall that a sequence $\{x_n\} \subset C$ is an *approximating fixed point sequence of* T if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

The approximating fixed point sequence has a fundamental role in the study of fixed point theory of nonexpansive mappings. We begin with the existence and basic properties of approximating fixed point sequences of nonexpansive mappings.

Proposition 5.2.1 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a nonexpansive mapping that is weakly inward. Then for $u \in C$ and $t \in (0,1)$, there exists exactly one point $x_t \in C$ such that*

$$x_t = (1-t)u + Tx_t.$$

If C is bounded, then $x_t - Tx_t \to 0$ as $t \to 1$.

Proof. For $t \in (0,1)$, the mapping $T_t : C \to X$ defined by

$$T_t x = (1-t)u + tTx, \quad x \in C \tag{5.17}$$

is a contraction with Lipschitz constant t. By Theorem 5.1.2, there exists exactly one point $x_t \in C$ such that

$$x_t = T_t x_t = (1-t)u + tTx_t.$$

If C is bounded, then

$$\|x_t - Tx_t\| = (1-t)\|u - Tx_t\| \leq (1-t)\ diam(C) \to 0 \ \ as \ \ t \to 1. \quad \blacksquare$$

As an immediate consequence of Proposition 5.2.1, we have

Corollary 5.2.2 *Let C be a nonempty closed convex bounded subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. Then there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. For $t \in (0,1)$, the mapping $T_t : C \to C$ defined by (5.17) is a contraction and it has exactly one fixed point x_t in C. Now the result follows from Proposition 5.2.1 $\blacksquare$

It is clear from the proof of Corollary 5.2.2 that one does not need the convexity of C. Indeed, this assumption can be replaced by the assumption that C is star-shaped, i.e., there exists $u \in C$ such that $(1-t)u + tx \in C$ for all $x \in C$ and $t \in [0,1]$. The point u is called star-center of C.

There is another way to obtain an approximating fixed point sequence of nonexpansive mappings defined in a nonconvex domain.

Before proving our next theorem, we need the following lemma:

Lemma 5.2.3 *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of a normed space* X*. If there is a sequence* $\{t_n\}$ *of real numbers satisfying the conditions:*

(i) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $a_{n+1} = (1-t_n)a_n + t_n b_n$ *for all* $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \|a_n\| = d$,

(iv) $\limsup_{n\to\infty} \|b_n\| \le d$ *and* $\{\sum_{i=1}^{n} t_i b_i\}$ *is bounded,*

then $d = 0$.

Proof. Suppose, for contradiction, that $d > 0$. It follows from (iv) that $\{\sum_{i=n}^{n+m-1} t_i b_i\}$ is bounded for all n and m. Set $M = \sup\{\|\sum_{i=n}^{n+m-1} t_i b_i\| : n, m \in \mathbb{N}\}$. Choose a number N such that $N > \max\{2M/d, 1\}$. We can choose a positive ε such that $1 - 2\varepsilon\ exp((N+1)/(1-t)) > 1/2$. It follows from (i) that there exists a natural k such that $N < \sum_{i=1}^{k} t_i \le N+1$. Because $\lim_{n\to\infty} \|a_n\| = d$, $\limsup_{n\to\infty} \|b_n\| \le d$ and ε is independent of n, without loss of generality we may assume that for all $n \in \mathbb{N}$,

$$d(1-\varepsilon) < \|a_n\| < d(1+\varepsilon) \ \text{ and } \ \|b_n\| < d(1+\varepsilon).$$

Set $T = \sum_{i=1}^{k} t_i, S = \prod_{i=1}^{k} s_i$ and $s_n = 1 - t_n$ for all $n \in \mathbb{N}$. From (ii), we obtain

$$\begin{aligned} a_{k+1} &= s_1 s_2 \cdots s_k a_1 + t_1 s_2 s_3 \cdots s_k b_1 + \cdots + t_{k-1} s_k b_{k-1} \\ &\quad + t_k b_k, a_{k+1} \in B := co\{a_1, b_1, b_2, \cdots, b_k\}. \end{aligned}$$

Let $x = T^{-1} \sum_{i=1}^{k} t_i b_i$ and

$$\begin{aligned} y &= S(1-S)^{-1}\{a_1 + t_1(s_1^{-1} - T^{-1})b_1 \\ &\quad + t_2(s_1^{-1}s_2^{-1} - T^{-1})b_2 + \cdots + t_k(S^{-1} - T^{-1})b_k\}. \end{aligned}$$

Then $x, y \in B$ and $a_{k+1} = Sx + (1-S)y$. Hence

$$\begin{aligned} d(1-\varepsilon) &< \|a_{k+1}\| \le S\|x\| + (1-S)\|y\| \\ &\le S\|x\| + (1-S)d(1+\varepsilon). \end{aligned}$$

It follows that

$$\begin{aligned}
\|x\| &> d(1-S^{-1}(2-S)\varepsilon)\\
&> d(1-2\varepsilon S^{-1})\\
&= d\left(1-2\varepsilon\prod_{i=1}^{k}(1-t_i)^{-1}\right)\\
&= d\left[1-2\varepsilon\ exp\left(\sum_{i=1}^{k}\log\left(1+\frac{t_i}{1-t_i}\right)\right)\right]\\
&\geq d\left(1-2\varepsilon\ exp\left(\sum_{i=1}^{k}\frac{t_i}{1-t_i}\right)\right)\\
&\geq d(1-2\varepsilon\ exp(T/(1-t)))\\
&\geq d(1-2\varepsilon\ exp((N+1)/(1-t)))\\
&> d/2,
\end{aligned}$$

because $\log(1+u)\leq u$ for $-1<u<\infty$. Thus, we have

$$\|x\| = T^{-1}\left\|\sum_{i=1}^{k}t_ib_i\right\| \leq T^{-1}M \leq \frac{d}{2M}M = \frac{d}{2},$$

a contradiction. ∎

Theorem 5.2.4 *Let C be a nonempty subset of a normed space and $T : C \to C$ a nonexpansive mapping. For $x_0 \in C$, suppose we have a sequence $\{t_n\}$ of real numbers and a sequence $\{T_{t_n}\}$ of mappings from C into itself satisfying the conditions:*

(i) $0 \leq t_n \leq t < 1$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,

(ii) $x_{n+1} = T_{t_n}x_n$ *and* $T_{t_n} = (1-t_n)I + t_nT,\ n\in\mathbb{N}_0$.

If $\{x_n\}$ is bounded sequence in C, then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Moreover,

$$\lim_{n\to\infty}\|T_{t_n}T_{t_{n-1}}\cdots T_{t_0}x_0 - T_{t_{n-1}}T_{t_{n-2}}\cdots T_{t_0}x_0\| = 0.$$

Proof. Because

$$\begin{aligned}
\|x_{n+1}-Tx_{n+1}\| &\leq (1-t_n)\|x_n - Tx_{n+1}\| + t_n\|Tx_n - Tx_{n+1}\|\\
&\leq (1-t_n)(\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\|) + t_n\|x_{n+1}-x_n\|\\
&\leq (1-t_n)\|x_{n+1} - Tx_{n+1}\| + \|x_{n+1} - x_n\|\\
&\leq (1-t_n)\|x_{n+1} - Tx_{n+1}\| + t_n\|x_n - Tx_n\|,
\end{aligned}$$

this yields

$$\|x_{n+1}-Tx_{n+1}\| \leq \|x_n - Tx_n\| \text{ for all } n\in\mathbb{N}_0.$$

The sequence $\{\|x_n - Tx_n\|\}$ is nonincreasing; it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| \text{ exists.} \tag{5.18}$$

Suppose $\lim_{n\to\infty} \|x_n - Tx_n\| = r$. Without loss of generality, we may assume that $t_n > 0$ for all $n \in \mathbb{N}_0$. Set $a_n := x_n - Tx_n$ and $b_n := t_n^{-1}(Tx_n - Tx_{n+1})$. Then

$$a_{n+1} = (1 - t_n)a_n + t_n b_n.$$

Because

$$\|b_n\| = t_n^{-1}\|Tx_n - Tx_{n+1}\| \leq t_n^{-1}\|x_n - x_{n+1}\| = \|x_n - Tx_n\|,$$

this implies that

$$\limsup_{n\to\infty} \|b_n\| \leq r.$$

Observe that

$$\begin{aligned}\|\sum_{i=0}^{n} t_i b_i\| &= \|\sum_{i=0}^{n}(Tx_i - Tx_{i+1})\| \\ &= \|Tx_0 - Tx_{n+1}\| \\ &\leq \|x_0 - x_{n+1}\|,\end{aligned}$$

Hence $\{\sum_{i=0}^{n} t_i b_i\}$ is bounded, because $\{x_n\}$ is bounded. By Lemma 5.2.3, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Now, for all $n \in \mathbb{N}$

$$\|x_{n+1} - x_n\| = t_n\|x_n - Tx_n\| \leq t\|x_n - Tx_n\|,$$

which implies that

$$\lim_{n\to\infty} \|T_{t_n}T_{t_{n-1}} \cdots T_{t_0}x_0 - T_{t_{n-1}}T_{t_{n-2}} \cdots T_{t_0}x_0\| = 0. \quad \blacksquare$$

The notion of asymptotic regularity is of fundamental importance in the study of fixed point theory of nonlinear mappings.

Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. Then T is said to be

(i) *asymptotically regular at* $x_0 \in C$ if $\lim_{n\to\infty} \|T^n x_0 - T^{n+1}x_0\| = 0$;

(ii) *weakly asymptotically regular at* $x_0 \in C$ if $T^n x_0 - T^{n+1}x_0 \rightharpoonup 0$;

(iii) *asymptotically regular on* C if for any $x \in C$, $\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0$;

(iv) *uniformly asymptotically regular* if $\lim_{n\to\infty} (\sup_{x\in C} \|T^n x - T^{n+1}x\|) = 0$;

(v) *reasonable wanderer in C* if starting at any $x \in C$,

$$\sum_{n=0}^{\infty} \|T^n x - T^{n+1}x\| < \infty.$$

Note that every uniformly asymptotically regular mapping is asymptotically regular.

Remark 5.2.5 *The asymptotic regularity of a mapping T at a point x_0 implies the existence of an approximating fixed point sequence of that mapping, but the converse is not true.*

It can be easily seen that a contraction mapping enjoys all this properties. The following example shows that there exists a nonexpansive mapping that is not necessarily asymptotically regular.

Example 5.2.6 *Let $X = \mathbb{R}$ and $T : X \to X$ defined by $Tx = -x$. Note that T is nonexpansive, but T is not asymptotically regular.*

However, a convex combination of nonexpansive mappings turns out to be asymptotically regular in a general Banach space. Indeed, we have

Theorem 5.2.7 *Let C be a nonempty convex subset of a normed space X and $T : C \to C$ a nonexpansive mapping. For a $t \in (0, 1)$, define a mapping $T_t : C \to C$ by*

$$T_t = (1 - t)I + tT.$$

If for $x_0 \in C, \{T_t^n x_0\}$ is bounded, then T_t is asymptotically regular at x_0, i.e., $\lim\limits_{n\to\infty} \|T_t^n x_0 - T_t^{n+1} x_0\| = 0$.

We now turn to study a demiclosedness principle for nonexpansive mappings in Banach spaces.

Definition 5.2.8 *Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. Then T is said to be demiclosed at $v \in X$ if for any sequence $\{x_n\}$ in C the following implication holds:*

$$x_n \rightharpoonup u \in C \text{ and } Tx_n \to v \quad \text{imply} \quad Tu = v.$$

Our first result concerning the demiclosedness principle of nonexpansive mappings is in an Opial space.

Theorem 5.2.9 *Let X be a Banach space that satisfies the Opial condition, C a nonempty weakly compact subset of X, and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and $\lim\limits_{n\to\infty} \|(I - T)x_n - y\| = 0$ for some $y \in X$. We show that $(I - T)x = y$.

Observe that

$$\|x_n - Tx - y\| \leq \|x_n - Tx_n - y\| + \|Tx_n - Tx\|,$$

which implies that

$$\liminf_{n\to\infty} \|x_n - Tx - y\| \leq \liminf_{n\to\infty} \|x_n - x\|.$$

By the Opial condition, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - (Tx + y)\|,$$

a contradiction. Therefore, $(I - T)x = y$. ∎

Corollary 5.2.10 *Let X be a reflexive Banach space that satisfies the Opial condition, C a nonempty closed convex subset of X, and $T : C \to X$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

We now extend the demiclosedness principle of nonexpansive mappings in a uniformly convex Banach space without Opial's condition. To do so, we need the following:

Proposition 5.2.11 *Let C be a nonempty convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then, for any $\varepsilon > 0$, there exists positive number $\zeta(\varepsilon) > 0$ such that $\|x - Tx\| < \varepsilon$ for all $x \in co(\{x_0, x_1\})$, whenever for $x_0, x_1 \in C$ with $\|x_0 - Tx_0\| \leq \zeta(\varepsilon)$ and $\|x_1 - Tx_1\| \leq \zeta(\varepsilon)$.*

Proof. Let $x = (1 - \lambda)x_0 + \lambda x_1$ for some $\lambda \in [0, 1]$. Suppose $\|x_0 - x_1\| < \varepsilon/3$, then

$$\|x - x_0\| = \lambda\|x_1 - x_0\| < \varepsilon/3.$$

If $\zeta(\varepsilon) < \varepsilon/3$, then we have

$$\begin{aligned} \|Tx - x\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| + \|x_0 - x\| \\ &\leq 2\|x - x_0\| + \zeta(\varepsilon) \\ &< \varepsilon. \end{aligned} \tag{5.19}$$

Hence we need only consider pairs of points x_0 and x_1 with $\|x_1 - x_0\| \geq \varepsilon/3$. Set $d := diam(C)$. Then for any nonnegative number λ with $\lambda < \varepsilon/(3d)$,

$$\|x - x_0\| = \lambda\|x_1 - x_0\| < \frac{\varepsilon}{3}.$$

Thus, if $\zeta(\varepsilon) < \varepsilon/3$ and $\lambda < \varepsilon/(3d)$, from (5.19), we have $\|Tx - x\| < \varepsilon$.

Now let $\lambda \geq \varepsilon/(3d)$. If $(1 - \lambda) < \varepsilon/(3d)$, then because $\|x - x_1\| = (1 - \lambda)\|x_1 - x_0\| < \varepsilon/3$, we have $\|x - Tx\| < \varepsilon$.

So, without loss of generality we may assume that $\lambda \in [\varepsilon/(3d), 1-\varepsilon/(3d)]$ and $\|x_0 - x_1\| \geq \varepsilon/3$. Note

$$\begin{aligned} \|Tx - x_0\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| \\ &\leq \lambda\|x_1 - x_0\| + \zeta(\varepsilon) \end{aligned}$$

and

$$\begin{aligned} \|Tx - x_1\| &\leq \|Tx - Tx_1\| + \|Tx_1 - x_1\| \\ &\leq (1-\lambda)\|x_1 - x_0\| + \zeta(\varepsilon). \end{aligned}$$

Set

$$u := \frac{Tx - x_0}{\lambda\|x_1 - x_0\|} \text{ and } v := \frac{x_1 - Tx}{(1-\lambda)\|x_1 - x_0\|}.$$

Then

$$\|u\| = \frac{\|Tx - x_0\|}{\lambda\|x_1 - x_0\|} \leq 1 + \frac{\zeta(\varepsilon)}{\lambda\|x_1 - x_0\|} \leq 1 + \frac{9d\zeta(\varepsilon)}{\varepsilon^2}$$

and

$$\|v\| = \frac{\|Tx - x_1\|}{(1-\lambda)\|x_1 - x_0\|} \leq 1 + \frac{\zeta(\varepsilon)}{(1-\lambda)\|x_1 - x_0\|} \leq 1 + \frac{9d\zeta(\varepsilon)}{\varepsilon^2}.$$

Observe that

$$\|\lambda u + (1-\lambda)v\| = \frac{\|x_1 - x_0\|}{\|x_1 - x_0\|} = 1 \quad \text{for all} \quad \lambda \in \left[\frac{\varepsilon}{3d}, 1 - \frac{\varepsilon}{3d}\right].$$

By the uniform convexity of X, if $\zeta(\varepsilon)$ is sufficiently small and positive, it follows that $\|u - v\| \leq \varepsilon/d$. Because $x = \lambda x_1 + (1-\lambda)x_0$, it follows that

$$\begin{aligned} \|Tx - x\| &\leq \|\lambda(Tx - x_1) + (1-\lambda)(Tx - x_0)\| \\ &\leq \lambda(1-\lambda)\|u - v\|\|x_1 - x_0\| < \lambda(1-\lambda)\left(\frac{\varepsilon}{d}\right)\|x_1 - x_0\| \leq \varepsilon. \end{aligned}$$ ∎

Theorem 5.2.12 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $I - T$ is demiclosed on X.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n - y\| = 0$ for some $y \in X$. Set $T_y x := Tx + y, x \in C$. Then $(I - T_y)x_n = (I - T)x_n - y \to 0$. If $(I - T_y)x = 0$, then $(I - T)x = y$. Hence, we may assume without loss of generality that $y = 0$. Set $\varepsilon_n := \|x_n - Tx_n\|$. Because $\varepsilon_n \to 0$ as $n \to \infty$, we may thin out the sequence to make the convergence faster, and we do this in such a way that for each n

$$\varepsilon_n \leq \zeta(\varepsilon_{n-1}) < \varepsilon_{n-1},$$

where $\zeta(\varepsilon)$ for any $\varepsilon > 0$ is constant as described in the conclusion of Proposition 5.2.11. Hence for each point $z \in \overline{co}(\{x_k : k \geq n\})$, we have $\|z - Tz\| \leq \varepsilon_{n-1}$. Because $\overline{co}(\{x_k : k \geq n\})$ is weakly compact (and hence weakly closed) and contains the weak limit x of the sequence $\{x_n\}$, it follows that $\|x - Tx\| \leq \varepsilon_{n-1} \to 0$ as $n \to \infty$. Therefore, $x = Tx$. ∎

Theorem 5.2.13 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $(I - T)(C)$ is closed.*

Proof. Suppose $u \in \overline{(I - T)(C)}$. Then there is a sequence $\{x_n\}$ in C such that $x_n - Tx_n \to u$ as $n \to \infty$. Because C is a weakly closed and bounded set in a reflexive Banach space X, it is weakly compact. Hence we may assume that $x_n \rightharpoonup x \in C$. By the conclusion of Theorem 5.2.12, we have $(I - T)x = u$, i.e., $(I - T)(C)$ is closed. ∎

We now prove some fundamental existence theorems for nonexpansive mappings.

Theorem 5.2.14 *Let C be a nonempty closed convex bounded subset of a Banach space X and $T : C \to X$ a nonexpansive mapping that is weakly inward. If $I - T$ is closed, then T has a fixed point in C.*

Proof. By Proposition 5.2.1, $x_t - Tx_t \to 0$ as $t \to 1$. Hence 0 lies in the closure of $(I - T)(C)$. Because $I - T$ is closed, there exists a point $v \in C$ such that $(I - T)v = 0$. ∎

Theorem 5.2.15 *Let X be a reflexive Banach space with the Opial condition. Let C be a nonempty closed convex bounded subset of X and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof. By Corollary 5.2.2, there exists a sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By the reflexivity of X, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in C$. By Corollary 5.2.10, $I - T$ is demiclosed at zero, i.e., $x_{n_k} \rightharpoonup x \in C$ and $x_{n_k} - Tx_{n_k} \to 0$ imply $x - Tx = 0$. Therefore, x is a fixed point of T. ∎

Using Theorem 5.2.14, we prove some fundamental existence theorems for nonexpansive mappings.

Theorem 5.2.16 (Browder's theorem and Göhde's theorem) – *Let X be a uniformly convex Banach space and C a nonempty closed convex bounded subset of X. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

Proof. Because $(I - T)(C)$ is closed by Theorem 5.2.13, it follows from Theorem 5.2.14 that T has fixed point in C. ∎

Corollary 5.2.17 (Browder's theorem) – *Let H be a Hilbert space and C a nonempty closed convex bounded subset of H. Then every nonexpansive mapping $T : C \to C$ has a fixed point in C.*

The following result is slightly more general than Theorem 5.2.16.

Theorem 5.2.18 (Kirk's fixed point theorem) – *Let X be a Banach space and C a nonempty weakly compact convex subset of X with normal structure. Then every nonexpansive mapping $T : C \to C$ has a fixed point.*

Proof. Let

$$\mathcal{F} = \{D_\alpha \subset C : D_\alpha \text{ is nonempty closed convex set such that } T(D_\alpha) \subset D_\alpha\}.$$

Because $C \in \mathcal{F}$, it follows that $\mathcal{F}$ is nonempty and it can be partially ordered by set inclusion. Then, using Zorn's Lemma (see Appendix A), $\mathcal{F}$ has a minimal element, say C_0. We show that C_0 has only one element. Suppose, for contradiction, that C_0 contains two elements. Hence $diam(C_0) > 0$. Because C_0 is weakly compact convex, $\mathcal{Z}(C_0)$ is nonempty. Let $x_0 \in \mathcal{Z}(C_0)$. Then for $x \in C_0$

$$\|Tx_0 - Tx\| \leq \|x_0 - x\| \leq r_{x_0}(C_0) = r(C_0),$$

i.e., Tx is contained in $B = B_{r(C_0)}[Tx_0]$. Thus, $T(C_0) \subset B$ and hence $T(B \cap C_0) \subset B \cap C_0$. The minimality of C_0 implies that $C_0 \subset B$. Hence $r_{Tx_0}(C_0) \leq r(C_0)$. Because $r(C_0) \leq r_{Tx_0}(C_0)$, it follows that $r(C_0) = r_{T_{x_0}}(C_0)$. Thus, $Tx_0 \in \mathcal{Z}(C_0)$, i.e., $\mathcal{Z}(C_0)$ is mapped into itself by T. By Proposition 3.3.14, $\mathcal{Z}(C_0)$ is a nonempty closed convex subset of C_0 such that $T(\mathcal{Z}(C_0)) \subseteq \mathcal{Z}(C_0)$. It means that $\mathcal{Z}(C_0) \in \mathcal{F}$ and also $\mathcal{Z}(C_0)$ is properly contained in C_0 by Proposition 3.3.15. This contradicts the minimality of C_0. Therefore, C_0 consists of a single point and hence T has a fixed point in C. ∎

The following examples show that nonexpansive mappings may fail to have fixed points in general Banach spaces.

Example 5.2.19 *Let X be a Banach space and $T : X \to X$ a translation mapping defined by*

$$Tx = x + a, \;\; a \neq 0.$$

Then T is nonexpansive and a fixed point free mapping.

Example 5.2.20 (Sadovski) – *Let c_0 be the Banach space of null sequences and $C = \{x \in c_0 : \|x\| \leq 1\}$, the unit closed ball in c_0. Define a mapping $T : C \to C$ by*

$$T(x_1, x_2, \cdots, x_i, \cdots) = (1, x_1, x_2, x_3, \cdots).$$

It is obvious that T is nonexpansive on the closed convex bounded set C and $x = (1, 1, 1, \cdots)$ is a fixed point of T. But $(1, 1, 1, \cdots) \notin c_0$. In this case, the Banach space $X = c_0$ is not reflexive and C does not have normal structure.

The following example shows that there exists a non-self nonexpansive mapping that has a fixed point.

Example 5.2.21 *Let* $X = \mathbb{R}$, $C = [-1, 1]$ *and* $T : C \to X$ *defined by* $Tx = 1 - x$, $x \in C$. *Then* $T(-1) = 2 \notin C$, *i.e.,* T *is not a self-mapping. But* T *does have a unique fixed point in* C.

We now give another example that shows that the boundedness of C in Browder's existence theorem may not be essential (even for non-self nonexpansive mappings).

Example 5.2.22 *Let* $X = \mathbb{R}$, $C = \mathbb{R}^+$ *and* $T : C \to X$ *defined by* $Tx = 1/(1 + x)$, $x \in C$. *Then* T *is nonexpansive,* $T(C) = (0, 1]$, *which is bounded, and* $Tv = v = (\sqrt{5} - 1)/2$. *However,* C *is unbounded.*

Examples 5.2.21 and 5.2.22 indicate that we may be able to extend Browder-Göhde-Kirk theorem for non-self nonexpansive mappings. We begin with the following fundamental theorem.

Theorem 5.2.23 *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $T : C \to H$ *a nonexpansive mapping with* $T(C)$ *bounded. Then there exists a* $z \in C$ *such that* $\|z - Tz\| = d(Tz, C)$.

Proof. Let P be the metric projection mapping from H onto C. Then $PT : C \to C$ is a nonexpansive mapping. Set $D := \overline{co(PT(C))}$. Then $PT(C)$ is bounded, because $T(C)$ is bounded. Thus, D is a closed convex bounded set and PT is nonexpansive self-mapping on D. By Browder-Göhde-Kirk's theorem, PT has a fixed point z in D. Therefore,

$$\|Tz - z\| = \|Tz - PTz\| = d(Tz, C). \qquad \blacksquare$$

In the following corollary, we replace boundedness of $T(C)$ by "boundedness of C":

Corollary 5.2.24 *Let* C *be a nonempty closed convex bounded subset of a Hilbert space* H *and* $T : C \to H$ *a nonexpansive mapping. Then there exists a* $z \in C$ *such that* $\|z - Tz\| = d(Tz, C)$.

We now apply Theorem 5.2.23 to derive an existence theorem for fixed points of non-self nonexpansive mappings.

Theorem 5.2.25 *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $T : C \to H$ *a nonexpansive mapping. Let* $T(C)$ *be bounded and* $T(\partial C) \subseteq C$. *Then* T *has a fixed point.*

Proof. By Theorem 5.2.23, there exists a point $z \in C$ such that

$$\|z - Tz\| = d(Tz, C).$$

If $Tz \in C$, then z is a fixed point of T. Otherwise, $z \in \partial C$, and hence $Tz \in C$, since $T(\partial C) \subseteq C$. It follows that $z \in F(T)$. $\blacksquare$

Theorem 5.2.26 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a weakly inward nonexpansive mapping. Then T has a fixed point.*

Proof. By Proposition 5.2.1, there exists a path $\{x_t\}_{t\in(0,1)} \subset C$ such that $x_t - Tx_t \to 0$ as $t \to 1$. Set $x_n := x_{t_n}$, where $t_n \to 1$ as $n \to \infty$. Let z be the asymptotic center of $\{x_n\}$ with respect to C. Then

$$\begin{aligned} r_a(Tz, \{x_n\}) &= \limsup_{n\to\infty} \|x_n - Tz\| \\ &\le \limsup_{n\to\infty} \|Tx_n - Tz\| \\ &\le \limsup_{n\to\infty} \|x_n - z\| = r_a(z, \{x_n\}). \end{aligned}$$

Because $Tz \in \overline{I_C(z)}$ and by Proposition 3.1.12, z is the asymptotic center of $\{x_n\}$ with respect to $\overline{I_C(z)}$, we conclude that $Tz = z$ by the uniqueness of the asymptotic center. ∎

We now discuss the structure of the set of fixed points of nonexpansive mappings.

Theorem 5.2.27 *Let C be a convex subset of a strictly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $F(T)$, the set of fixed points of T is either empty or convex.*

Proof. The example $Tx = x+a$ $(a \neq 0)$ shows that $F(T) = \emptyset$. Next, we assume that $F(T) \neq \emptyset$. Set $x, y \in F(T)$ and $\alpha \in [0, 1]$. Then for $z = \alpha x + (1 - \alpha)y$, we have

$$\|x - Tz\| = \|Tx - Tz\| \le \|x - z\| = (1 - \alpha)\|x - y\|$$

and

$$\|y - Tz\| = \|Ty - Tz\| \le \|y - z\| = \alpha\|x - y\|.$$

Hence

$$\|x - y\| \le \|x - Tz\| + \|Tz - y\| \le \|x - z\| + \|y - z\| = \|x - y\|.$$

This implies that

$$\|x - y\| = \|x - Tz\| + \|Tz - y\|.$$

Let $a = x - Tz$ and $b = Tz - y$. Then $\|a + b\| = \|a\| + \|b\|$. Because X is strictly convex, $a = \lambda b$ for some positive constant λ (see Proposition 2.1.7). This means that Tz is a linear combination of x and y, i.e., $Tz = \beta x + (1 - \beta)y$ for some real β. Hence

$$\begin{aligned} \|Tz - x\| &= \|z - x\| = (1 - \alpha)\|x - y\| = (1 - \beta)\|x - y\|, \\ \|Tz - y\| &= \|z - y\| = \alpha\|x - y\| = \beta\|x - y\|. \end{aligned}$$

Consequently, $\alpha = \beta$, i.e., $z = Tz$. ∎

Remark 5.2.28 *By the continuity of T, $F(T)$ is always closed.*

Corollary 5.2.29 *Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then $F(T)$ is closed and convex.*

Theorem 5.2.27 is not true in a general Banach space. This fact is shown in the following example:

Example 5.2.30 *Let $X = \mathbb{R}^2$ be a Banach space with maximum norm defined by*

$$\|(a,b)\| = \max\{|a|,|b|\} \text{ for all } x = (a,b) \in \mathbb{R}^2.$$

Let $T : X \to X$ be a mapping defined by

$$T(a,b) = (|b|,b) \quad \text{for all } (a,b) \in \mathbb{R}^2.$$

Then T is nonexpansive and $(1,1)$ and $(1,-1)$ are fixed points of T. However, no other point in the segment joining these two points is a fixed point of T.

We now introduce the class of mappings that is properly included in the class of nonexpansive mappings.

Let Γ denote the set of strictly increasing continuous convex functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a nonempty convex subset of a Banach space X. Then a mapping $T : C \to X$ is said to be of type (γ) if there exists $\gamma \in \Gamma$ such that

$$\gamma(\|(1-t)Tx + tTy - T((1-t)x + ty)\|) \leq \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ and $t \in [0,1]$.

It is clear that every mapping of type (γ) is nonexpansive, but the converse is not true in general. We derive the following interesting result, which shows that every nonexpansive mapping defined on a convex bounded subset of a uniformly convex Banach space is a mapping of type (γ).

Theorem 5.2.31 *Let C be a nonempty convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then there exists a strictly increasing continuous convex function (depending on diam(C)) $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ such that*

$$\gamma(\|(1-t)Tx + tTy - T((1-t)x + ty)\|) \leq \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ and $t \in [0,1]$.

Proof. Because X is uniformly convex, there exists a strictly increasing continuous convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$2t(1-t)g(\|u - v\|) \leq 1 - \|(1-t)u + tv\| \tag{5.20}$$

for all $u, v \in X$ with $\|u\| \le 1$, $\|v\| \le 1$ and all $t \in [0,1]$ (see Theorem 2.3.12). It suffices to show Theorem 5.2.31 when $t \in (0,1)$.

Let $x, y \in C$ and $z = (1-t)x + ty$ for $t \in (0,1)$. Set

$$d := diam(C),\ u := \frac{Ty - Tz}{(1-t)\|x-y\|} \text{ and } v := \frac{Tz - Tx}{t\|x-y\|}.$$

Then we have $\|u\| \le 1, \|v\| \le 1$,

$$(1-t)u + tv = \frac{Ty - Tx}{\|x-y\|} \text{ and } u - v = \frac{(1-t)Tx + tTy - Tz}{t(1-t)\|x-y\|}.$$

Take $r = \|(1-t)Tx + tTy - Tz\|$ and $s^{-1} = t(1-t)\|x-y\|$. It follows from (5.20) that

$$2t(1-t)\|x-y\|g(rs) \le \|x-y\| - \|Tx - Ty\|,$$

which implies that

$$2\frac{g(rs)}{s} \le \|x-y\| - \|Tx-Ty\|. \tag{5.21}$$

Observe that $rs \le 2$. Because $t(1-t) \le 1/4$, it follows that $t(1-t)\|x-y\| \le d/4$ and hence $4/d \le s$. Note that for $rs \le 2$, the function $s \mapsto g(rs)/s$ is nondecreasing, and then from (5.21) we have

$$\frac{d}{2}\, g\left(\frac{4r}{d}\right) = 2\,\frac{g(4r/d)}{4/d} \le 2\,\frac{g(rs)}{s} \le \|x-y\| - \|Tx-Ty\|.$$

Therefore,

$$\gamma(\|(1-t)Tx + tTy - T((1-t)x + ty)\|) \le \|x-y\| - \|Tx-Ty\|,$$

where $\gamma(t) = dg(4t/d)/2,\ t \ge 0$. It can be easily verified that $\gamma \in \Gamma$. ∎

We denote

$$\triangle^{n-1} = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) : \lambda_i \ge 0\ (i = 1, 2, \cdots, n) \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

Proposition 5.2.32 *Let C be a nonempty convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then for each positive integer p, there exists $\gamma_p \in \Gamma$ such that for any $\lambda \in \triangle^{p-1}$ and $x_1, x_2, \cdots, x_p \in C$,*

$$\gamma_p\left(\left\|T\left(\sum_{i=1}^{p}\lambda_i x_i\right) - \sum_{i=1}^{p}\lambda_i Tx_i\right\|\right) \le \max_{1 \le i,j \le p}(\|x_i - x_j\| - \|Tx_i - Tx_j\|). \tag{5.22}$$

Proof. Once γ_p has been defined, we define γ_{p+1} to be any function in Γ satisfying the condition:

$$\gamma_{p+1}^{-1}(t) \geq \gamma_2^{-1}(t) + \gamma_p^{-1}(t + \gamma_2^{-1}(t)).$$

We must verify (5.22) for $p+1$. Fix $\lambda \in \triangle^p$ and $x_1, x_2, \cdots, x_{p+1} \in C$. The case $\lambda_{p+1} = 1$ is trivial. We assume that $\lambda_{p+1} \neq 1$. Set

$$u_j := (1-\lambda_{p+1})x_j + \lambda_{p+1}x_{p+1}, \ \mu_j := \frac{\lambda_j}{1-\lambda_{p+1}}$$

and

$$u_j' := (1-\lambda_{p+1})Tx_j + \lambda_{p+1}Tx_{p+1} \text{ for } j = 1, 2, \cdots, p.$$

Observe that

$$\sum_{j=1}^{p} \mu_j = \sum_{j=1}^{p} \frac{\lambda_j}{1-\lambda_{p+1}} = 1;$$

$$\sum_{i=1}^{p+1} \lambda_i x_i = \sum_{i=1}^{p} \frac{\lambda_i}{1-\lambda_{p+1}} \{(1-\lambda_{p+1})x_i + \lambda_{p+1}x_{p+1}\} = \sum_{j=1}^{p} \mu_j u_j;$$

$$\sum_{i=1}^{p+1} \lambda_i Tx_i = \sum_{j=1}^{p} \mu_j u_j';$$

$$\begin{aligned} &\left\| T\left(\sum_{i=1}^{p+1} \lambda_i x_i\right) - \sum_{i=1}^{p+1} \lambda_i Tx_i \right\| \\ &= \left\| T\left(\sum_{j=1}^{p} \mu_j u_j\right) - \sum_{j=1}^{p} \mu_j u_j' \right\| \\ &\leq \left\| T\left(\sum_{j=1}^{p} \mu_j u_j\right) - \sum_{j=1}^{p} \mu_j Tu_j \right\| + \sum_{j=1}^{p} \mu_j \|Tu_j - u_j'\|; \end{aligned} \tag{5.23}$$

$$\begin{aligned} &\gamma_p\left(\left\| T\left(\sum_{j=1}^{p} \mu_j u_j\right) - \sum_{j=1}^{p} \mu_j Tu_j \right\|\right) \\ &\leq \max_{1 \leq j,k \leq p} (\|u_j - u_k\| - \|Tu_j - Tu_k\|); \end{aligned} \tag{5.24}$$

$$\begin{aligned} \|u_j - u_k\| - \|Tu_j - Tu_k\| &\leq \|u_j - u_k\| - \|u_j' - u_k'\| \\ &+ \|u_k' - Tu_k\| + \|u_j' - Tu_j\|; \end{aligned} \tag{5.25}$$

$$\gamma_2(\|Tu_j - u_j'\|) \leq \|x_j - x_{p+1}\| - \|x_j' - x_{p+1}'\|; \tag{5.26}$$

$$\begin{aligned} \|u_j - u_k\| - \|u_j' - u_k'\| &= (1-\lambda_{p+1})(\|x_j - x_k\| - \|x_j' - x_k'\|) \\ &\leq \|x_j - x_k\| - \|x_j' - x_k'\|. \end{aligned} \tag{5.27}$$

Put $t := \max_{1\le i,k\le p+1}\{\|x_i - x_k\| - \|x'_i - x'_k\|\}$. By (5.26), we have

$$\|Tu_j - u'_j\| \le \gamma_2^{-1}(t).$$

By (5.25) and (5.27)

$$\begin{aligned}\|u_j - u_k\| - \|Tu_j - Tu_k\| &\le \|x_j - x_k\| - \|x'_j - x'_k\| + 2\gamma_2^{-1}(t)\\ &\le t + 2\gamma_2^{-1}(t). \qquad (5.28)\end{aligned}$$

It follows from (5.24) that

$$\left\|T\left(\sum_{j=1}^{p}\mu_j u_j\right) - \sum_{j=1}^{p}\mu_j Tu_j\right\| \le \gamma_p^{-1}(t + \gamma_2^{-1}(t)). \qquad (5.29)$$

From (5.23), (5.28), and (5.29), we have

$$\begin{aligned}\left\|T\left(\sum_{i=1}^{p+1}\lambda_i x_i\right)\right. &- \left.\sum_{i=1}^{p+1}\lambda_i Tx_i\right\|\\ &\le \gamma_p^{-1}(t + \gamma_2^{-1}(t)) + \gamma_2^{-1}(t)\\ &\le \gamma_{p+1}^{-1}(t). \qquad \text{(by definition of } \gamma_{p+1})\end{aligned}$$

Therefore,

$$\gamma_{p+1}\left(\left\|T\left(\sum_{i=1}^{p+1}\lambda_i x_i\right) - \sum_{i=1}^{p+1}\lambda_i Tx_i\right\|\right) \le t = \max_{1\le i,k\le p+1}(\|x_i - x_k\| - \|Tx_i - Tx_k\|). \quad \blacksquare$$

A Banach space X is said to have the *convex combination property* (CAP) if for each $\varepsilon > 0$, there exists an integer $p(= p(\varepsilon)) \ge 1$ such that for all subsets D in X whose diameters are uniformly bounded,

$$co(D) \subset co_p(D) + B_r[0],$$

where

$$co_p(D) = \left\{\sum_{i=1}^{p}\lambda_i x_i : \lambda \in \triangle^{p-1}, x_1, x_2, \cdots, x_p \in D\right\}.$$

We note that every uniformly convex Banach space X has the CAP. The product of uniformly convex Banach spaces being uniformly convex implies $X \times X$ has the CAP.

Theorem 5.2.33 *Let C be a nonempty convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a nonexpansive mapping. Then there exists $\gamma \in \Gamma$ such that for any finite many elements $\{x_i\}_{i=1}^n$ in C and any finite many nonnegative numbers $\{\lambda_i\}_{i=1}^n$ with $\sum_{i=1}^n \lambda_i = 1$, the following inequality holds:*

$$\gamma\left(\left\|T\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i Tx_i\right\|\right) \le \max_{1\le i,j\le n}(\|x_i - x_j\| - \|Tx_i - Tx_j\|).$$

Proof. First, determine $\gamma_p \in \Gamma$ for $p = 2, 3, \cdots$ from Proposition 5.2.32. Because $X \times X$ has the CAP and hence, given $\varepsilon > 0$, we can determine p so that

$$co(D) \subset co_p(D) + B_{\varepsilon/3}[0] \times B_{\varepsilon/3}[0]$$

for every $D \subset C \times C$. Set $\delta = \gamma_p(\varepsilon/3)$. Suppose $x_1, x_2, \cdots, x_n \in C$ satisfy

$$\|x_i - x_j\| - \|Tx_i - Tx_j\| \leq \delta \text{ for all } i, j.$$

Consider $D = \{(x_i, Tx_i) \in X \times X : i = 1, 2, \cdots, n\}$. Thus, for each $\lambda \in \triangle^{n-1}$, there exist $\mu \in \triangle^{p-1}$ and $i_1, i_2, \cdots, i_p \in \{1, 2, \cdots, n\}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{j=1}^{p} \mu_j x_{i_j} \right\| < \frac{\varepsilon}{3}$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i T x_i - \sum_{j=1}^{p} \mu_j T x_{i_j} \right\| < \frac{\varepsilon}{3}.$$

In other words, the CAP on $X \times X$ guarantees simultaneous approximability in X. Observe that

$$\begin{aligned}
\left\| T\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) - \sum_{i=1}^{n} \lambda_i T x_i \right\| &\leq \left\| T\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) - T\Big(\sum_{j=1}^{p} \mu_j x_i\Big) \right\| \\
&\quad + \left\| T\Big(\sum_{j=1}^{p} \mu_j x_i\Big) - \sum_{j=1}^{p} \mu_j T x_{i_j} \right\| \\
&\quad + \left\| \sum_{j=1}^{p} \mu_j T x_{i_j} - \sum_{i=1}^{n} \lambda_i T x_i \right\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, whenever $\|x_i - x_j\| - \|Tx_i - Tx_j\| \leq \delta$ for all i, j, we have

$$\left\| T\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) - \sum_{i=1}^{n} \lambda_i T x_i \right\| \leq \varepsilon.$$

Therefore, the construction of $\gamma \in \Gamma$ such that $\gamma(\varepsilon) \leq \delta$ for this $\varepsilon - \delta$ prescription is a simple calculation. ∎

5.3 Multivalued nonexpansive mappings

In this section, we consider the problem of solving the operator equation

$$x \in Tx, \tag{5.30}$$

where T is a multivalued nonexpansive mapping in a Banach space.

Definition 5.3.1 *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to CB(X)$ is said to be nonexpansive if*

$$H(Tx, Ty) \leq \|x - y\| \text{ for all } x, y \in C,$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

Recall that the graph $G(A)$ of a multivalued mapping $A : C \to 2^Y$ is

$$G(A) = \{(x, y) \in X \times Y : x \in C, y \in Ax\},$$

where Y is another Banach space.

The mapping A is said to be *demiclosed* at $y \in Y$ if

$$x_n \text{ (in } C) \rightharpoonup x \text{ and } y_n \in Ax_n \to y \Rightarrow y \in Ax.$$

First, we show that for every compact-valued nonexpansive mapping T, $I - T$ is demiclosed in a Banach space with the Opial condition.

Theorem 5.3.2 *Let C be a nonempty weakly compact subset of a Banach space X with the Opial condition and $T : C \to \mathcal{K}(X)$ a nonexpansive mapping. Then $I - T$ is demiclosed.*

Proof. Because the domain of $I - T$ is weakly compact, we must prove that the graph of $I - T$ is only sequentially closed. Let $(x_n, y_n) \in G(I - T)$ be such that $x_n \rightharpoonup x$ and $y_n \to y$. Hence $x \in C$.

We now show that $y \in (I - T)x$. Because $y_n \in x_n - Tx_n$, $y_n = x_n - z_n$ for some $z_n \in Tx_n$. By the nonexpansiveness of T, there exists $z_n' \in Tx$ such that

$$\|z_n - z_n'\| \leq H(Tx_n, Tx) \leq \|x_n - x\|. \tag{5.31}$$

It follows from (5.31) that

$$\begin{aligned} \liminf_{n\to\infty} \|x_n - x\| &\geq \liminf_{n\to\infty} \|z_n - z_n'\| \\ &= \liminf_{n\to\infty} \|x_n - y_n - z_n'\|. \end{aligned} \tag{5.32}$$

Because Tx is compact, $z_n' \in Tx$ and $y_n \to y$, and then there exists a subsequence $\{z_{n_i}'\}$ of $\{z_n'\}$ such that

$$z_{n_i}' \to z \in Tx \text{ and } y_{n_i} \to y.$$

Hence from (5.32) we have

$$\liminf_{i\to\infty} \|x_{n_i} - x\| \geq \liminf_{i\to\infty} \|x_{n_i} - y - z\|.$$

By the Opial condition

$$\liminf_{i\to\infty} \|x_{n_i} - x\| < \liminf_{i\to\infty} \|x_{n_i} - y - z\|,$$

which implies that $y + z = x$. Therefore, $y = x - z \in x - Tx$. ∎

The fixed point theory of multivalued nonexpansive mappings is however much more complicated than the corresponding theory of single-valued nonexpansive mappings. We will concentrate on some important existence theorems.

We begin with the existence of fixed points of compact-valued nonexpansive mappings in Banach spaces.

Theorem 5.3.3 *Let X be a Banach space with the Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to \mathcal{K}(C)$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof. Let u be an element in C and let $\{a_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} a_n = 1$. For each $n \in \mathbb{N}$, define $T_n : C \to \mathcal{K}(C)$ by

$$T_n x = (1 - a_n)u + a_n T x, \quad x \in C. \tag{5.33}$$

Then T_n is a contraction mapping. By Theorem 4.2.7 (Nadler's fixed point theorem), there exists $x_n \in C$ such that $x_n \in T_n x_n$. Because C is weakly compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. From (5.33) we have

$$x_n = (1 - a_n)u + a_n z_n,$$

where $z_n \in T x_n$. Observe that

$$\|x_n - z_n\| = (1 - a_n)\|z_n - u\|.$$

Hence $y_n = x_n - z_n \in (I - T)x_n$ and $y_n \to 0$. Thus, $(x_{n_i}, y_{n_i}) \in G(I - T)$ and $x_{n_i} \rightharpoonup x$ and $y_{n_i} \to 0$, it follows from the demiclosedness of $I - T$ at zero that $0 \in (I - T)v$, Therefore, $v \in T v$. ∎

Proposition 5.3.4 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(C)$ a nonexpansive mapping. Suppose there exists a bounded sequence $\{x_n\}$ in C such that $d(x_n, T x_n) \to 0$ as $n \to \infty$. Then T has a fixed point in C.*

Proof. We may assume that $\{x_n\}$ is regular and thus asymptotically uniform. Let $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$ and $r_a(C, \{x_n\}) = r$. Choose $y_n \in T x_n$ such that $\|x_n - y_n\| \to 0$ as $n \to \infty$. By the compactness of Tz, select $z_n \in Tz$ such that

$$\|y_n - z_n\| \le H(T x_n, Tz) \le \|x_n - z\|. \tag{5.34}$$

Because Tz is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to v \in Tz$.

By the regularity of $\{x_n\}$, we have

$$r_a(C, \{x_{n_k}\}) = r_a(C, \{x_n\}) = r \text{ and } \mathcal{Z}_a(C, \{x_n\}) = \mathcal{Z}_a(C, \{x_{n_k}\}) = \{z\}.$$

Because

$$\|x_{n_k} - v\| \le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - v\|,$$

it follows from (5.34) that

$$\limsup_{k\to\infty} \|x_{n_k} - v\| \leq r$$

and hence $\mathcal{Z}_a(C, \{x_n\}) = z = v$. ∎

Theorem 5.3.5 *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X (with $0 \in C$), and $T : C \to \mathcal{K}(C)$ a nonexpansive mapping. Suppose the set*

$$E = \{x \in C : \lambda x \in Tx \ for\ some\ \lambda > 1\}$$

is bounded. Then T has a fixed point in C.

Proof. Let $\{t_n\}$ be a sequence in (0,1) with $t_n \to 1$. Then for each $n \in \mathbb{N}$, $t_nT : C \to \mathcal{K}(C)$ is a contraction mapping. Then Nadler's theorem implies that $x_n \in t_nTx_n$ for some $x_n \in C$. Now select $y_n \in Tx_n$ such that $x_n = t_ny_n$, which yields

$$d(x_n, Tx_n) \leq \|x_n - y_n\| = (t_n^{-1} - 1)\|x_n\|.$$

Because $\{x_n\}$ is in E and E is bounded, we obtain that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Therefore, there exists a fixed point of T in C by Proposition 5.3.4. ∎

Theorem 5.3.6 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(X)$ a nonexpansive mapping. If $Tx \subset I_C(x)$ for all $x \in C$, then T has a fixed point in C.*

Proof. Let $u \in C$ be fixed and let $\{a_n\}$ be a sequence in $(0, 1)$ such that $\lim\limits_{n\to\infty} a_n = 1$. For each $n \in \mathbb{N}$, define $T_n : C \to 2^X$ by

$$T_nx = (1 - a_n)u + a_nTx, \quad x \in C.$$

Then T_n is a multivalued contraction with Lipschitz constant a_n. Because $I_C(x)$ is convex for each $x \in C$, it follows that

$$T_nx \subset I_C(x), \quad x \in C. \tag{5.35}$$

Observe that $T(C) = \cup_{x\in C}Tx$ is a bounded set. Now let $x \in C, z \in Tx$ and let $K = \|u - z\| + diam(\cup_{x\in C}Tx)$. If $y \in T_nx$, then $y = (1 - a_n)u + a_nw$ for some $w \in Tx$ and

$$d(y, Tx) \leq (1 - a_n)\|u - z\| \leq (1 - a_n)K. \tag{5.36}$$

Also for any $\overline{x} \in Tx$, we have $(1 - a_n)u + a_n\overline{x} \in T_nx$ and so

$$d(\overline{x}, T_nx) \leq (1 - a_n)K. \tag{5.37}$$

Together (5.36) and (5.37) imply that $H(Tx, T_nx) \leq (1 - a_n)K \to 0$ uniformly for $x \in C$ as $n \to \infty$.

Because $T_n x \subset I_C(x), x \in C$ by (5.35) and the values of T are compact, T_n has a fixed point $x_n \in C$ by Theorem 5.1.7. This means that

$$x_n \in T_n x_n = (1-a_n)u + a_n T x_n.$$

Thus

$$d(x_n, Tx_n) \le H(T_n x_n, Tx_n).$$

By the uniform convergence of $\{T_n\}$ we have

$$d(x_n, Tx_n) \to 0 \text{ as } n \to \infty.$$

For each $n \in \mathbb{N}$, choose $y_n \in Tx_n$ such that

$$\|x_n - y_n\| = d(x_n, Tx_n).$$

By Propositions 3.1.1 and 3.1.2, passing to a subsequence if necessary, we may assume that $\{x_n\}$ is regular and asymptotically uniform. Let $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$ and $r_a(C, \{x_n\}) = r$.

Because T is compact-valued mapping, we can select $z_n \in Tz$ such that

$$\|y_n - z_n\| \le H(Tx_n, Tz) \le \|x_n - z\|. \tag{5.38}$$

Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \to v$ as $i \to \infty$. Hence $v \in Tz$. Because $v \in Tz \subset I_C(z)$, there exists $t \in (0,1)$ such that

$$(1-t)z + tv \in C.$$

Note $\mathcal{Z}_a(C, \{x_{n_i}\}) = z$, $r_a(C, \{x_{n_i}\}) = r$ and $\{x_n\}$ is regular and asymptotically uniform. Set $w_i := (1-t)z + tx_{n_i},\ i \in \mathbb{N},\ t \in (0,1)$. Proposition 3.1.7 implies that $Z_a(C, \{w_i\}) = z$ and $r_a(C, \{w_i\}) = tr$. Set

$$\begin{aligned} p_i : &= (1-t)z + ty_{n_i} \\ q_i : &= (1-t)z + tz_{n_i}, \\ w : &= (1-t)z + tv. \end{aligned}$$

Then for each $i \in \mathbb{N}$,

$$\begin{aligned} \|w_i - w\| &\le \|w_i - p_i\| + \|p_i - q_i\| + \|q_i - w\| \\ &\le t\|x_{n_i} - y_{n_i}\| + t\|y_{n_i} - z_{n_i}\| + t\|z_{n_i} - v\| \\ &\le t(\|x_{n_i} - y_{n_i}\| + \|x_{n_i} - z\| + \|z_{n_i} - v\|) \qquad \text{(from (5.38))} \end{aligned}$$

From $z_{n_i} \to v$ and $x_n - y_n \to 0$, we have

$$\limsup_{i\to\infty} \|w_i - w\| \quad \le \quad t \ \limsup_{i\to\infty} \|x_{n_i} - z\| = tr.$$

Because $\mathcal{Z}_a(C, \{w_i\}) = \{z\}$ and $r_a(C, \{w_i\}) = tr$, by the uniqueness of asymptotic center we have $w = z$. Thus, $z = v \in Tz$. ∎

Finally, we obtain some existence theorems for multivalued nonexpansive non-self mappings in which the convexity of domain is not necessary.

Theorem 5.3.7 *Let C be a nonempty weakly compact star-shaped subset of a uniformly convex Banach space X and $T : C \to \mathcal{K}(X)$ a nonexpansive mapping. If for each $x \in \partial C$, $Tx \subset C$ and $\lambda x + (1-\lambda)Tx \subset C$ for some $\lambda \in (0,1)$ or $Tx \subset int(C)$, then T has a fixed point in C.*

Proof. Let p be the star-center of C and $\{a_n\}$ a sequence in $(0,1)$ with $a_n \to 1$. Define

$$T_n x = (1-a_n)p + a_n Tx, \ \ x \in C, n \in \mathbb{N}.$$

Then T_n is a multivalued contraction. By Theorem 4.2.14, each T_n has a fixed point x_n such that

$$x_n = (1-a_n)p + a_n Tx_n. \tag{5.39}$$

Because $\{x_n\}$ is bounded, we have $r_a(C, \{x_{n_i}\}) = r_a(C, \{x_n\})$ and $\mathcal{Z}_a(C, \{x_n\}) \subset \mathcal{Z}_a(C, \{x_{n_i}\})$. Let $z \in Z_a(C, \{x_n\})$. Because Tz is compact, there exists $z_n \in Tz$ such that for $y_n \in Tx_n$

$$\|y_n - z_n\| \leq H(Tx_n, Tz) \leq \|x_n - z\|. \tag{5.40}$$

Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \to v \in Tz$. Because $\mathcal{Z}_a(C, \{x_n\}) \subset \mathcal{Z}_a(C, \{x_{n_i}\})$, it follows that $z \in \mathcal{Z}_a(C, \{x_{n_i}\})$. From (5.39), we get

$$\|x_{n_i} - y_{n_i}\| = (1-a_{n_i})\|x_{n_i} - p\| \to 0 \ as \ i \to \infty.$$

Hence

$$\begin{aligned}
\limsup_{i\to\infty} \|x_{n_i} - v\| &\leq \limsup_{i\to\infty} (\|x_{n_i} - y_{n_i}\| + \|y_{n_i} - z_{n_i}\| + \|z_{n_i} - v\|) \\
&= \limsup_{i\to\infty} \|y_{n_i} - z_{n_i}\| \\
&\leq \limsup_{i\to\infty} \|x_{n_i} - z\| \qquad \text{(from (5.40))} \\
&= \inf\{\limsup_{i\to\infty} \|x_{n_i} - x\| : x \in C\},
\end{aligned}$$

and this means that $v \in \mathcal{Z}_a(C, \{x_{n_i}\})$.

If $z \in \partial C$, then by hypothesis for $v \in Tz$, there exists $w \in C$ such that

$$w = (1-\lambda)v + \lambda z, \quad \lambda \in (0,1).$$

Suppose that $v \neq z$. By uniform convexity of X, we have some $\delta \in (0,1)$

$$\limsup_{i\to\infty} \|x_{n_i} - w\| \leq (1-\delta) \inf\{\limsup_{i\to\infty} \|x_{n_i} - y\| : y \in C\},$$

which is a contradiction of the choice of w. If $z \in \mathcal{Z}_a(X, \{x_n\})$, we have

$$\begin{aligned}
r_a(X, \{x_{n_i}\}) &\leq \limsup_{i\to\infty} \|v - x_{n_i}\| \\
&\leq \limsup_{i\to\infty} (\|v - z_{n_i}\| + \|z_{n_i} - y_{n_i}\| + \|y_{n_i} - x_{n_i}\|)
\end{aligned}$$

$$\leq \limsup_{i\to\infty} \|z_{n_i} - y_{n_i}\|$$
$$\leq \limsup_{i\to\infty} \|z - x_{n_i}\| = r_a(X, \{x_{n_i}\}).$$

Hence $v \in \mathcal{Z}_a(X, \{x_{n_i}\})$. By uniform convexity of X, we obtain $v = z \in Tz$. ∎

Theorem 5.3.8 *Let C be a nonempty closed bounded subset of a Banach space X and $T : C \to \mathcal{K}(X)$ a mapping that satisfies the following conditions:*

(i) $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in C$,

(ii) $Tx \subset \overline{I_C(x, 1)}$ for all $x \in C$,

(iii) $\bigcup_{x\in C}(x - Tx) = \bigcup_{x\in C}\bigcup_{y\in Tx}(x - y)$ is a closed subset of X.

Then T has a fixed point in C.

Proof. Without loss of generality, we may assume that $0 \in C$. Let $\{a_n\}$ be a sequence in $(0, 1)$ such that $a_n \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define

$$T_n x = a_n Tx, \quad x \in C.$$

It is easy to see that $T_n x \in \mathcal{K}(X)$ and $T_n x \subset \overline{I_C(x, 1)}$ for each $n \in \mathbb{N}$ and all $x \in C$. Furthermore,

$$H(T_n x, T_n y) \leq a_n \|x - y\| \text{ for all } x, y \in C, \ n \in \mathbb{N}.$$

By Theorem 5.1.9, there exists $x_n \in C$ such that

$$x_n = T_n x_n = a_n T x_n.$$

We choose $y_n \in Tx_n$ satisfying $x_n = a_n y_n$. By the boundedness of C,

$$\|x_n - y_n\| = \left(\frac{1}{a_n} - 1\right)\|x_n\| \to 0.$$

Because $\cup_{x\in C}(x - Tx)$ is closed, we have $0 \in \cup_{x\in C}(x - Tx)$. Therefore, there exists some $z \in C$ such that $z \in Tz$. ∎

5.4 Asymptotically nonexpansive mappings

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* [1] if for each $n \in \mathbb{N}$, there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

The following example shows that the class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings.

[1]The notion of asymptotically nonexpansive mappings was introduced by Goebel and Kirk in 1972.

Example 5.4.1 *Let B_H be the closed unit ball in the Hilbert space $H = \ell_2$ and $T : B_H \to B_H$ a mapping defined by*

$$T(x_1, x_2, x_3, \cdots) = (0, x_1^2, a_2x_2, a_3x_3, \cdots),$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = 1/2$. Then

$$\|Tx - Ty\| \leq 2\|x - y\| \text{ for all } x, y \in B_H,$$

i.e., T is Lipschitzian, but not nonexpansive. Observe that

$$\|T^n x - T^n y\| \leq 2\prod_{i=2}^{n} a_i\|x - y\| \text{ for all } x, y \in B_H \text{ and } n \geq 2.$$

Here $k_n = 2\prod_{i=2}^{n} a_i \to 1$ as $n \to \infty$. Therefore, T is asymptotically nonexpansive, but not nonexpansive.

There is also a connection between the demiclosedness principle and the fixed point theory of asymptotically nonexpansive mappings. Some simple results concerning the demiclosedness principle of asymptotically nonexpansive mappings are given in the following theorems:

We first establish Proposition 5.4.2, which shows that the asymptotic center of every bounded $AFPS$ of an asymptotically nonexpansive mapping is a fixed point of the mapping in uniformly convex Banach spaces.

Proposition 5.4.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. If $\{y_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ and $\mathcal{Z}_a(C, \{y_n\}) = \{v\}$, then v is a fixed point in C.*

Proof. We define a sequence $\{z_n\}$ in C by $z_m = T^m v, m \in \mathbb{N}$. For integers $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \|z_m - y_n\| &\leq \|T^m v - T^m y_n\| + \|T^m y_n - T^{m-1} y_n\| + \cdots + \|Ty_n - y_n\| \\ &\leq k_m\|v - y_n\| + (\|Ty_n - y_n\| + \sum_{i=1}^{m-1} k_i\|y_n - Ty_n\|). \end{aligned} \tag{5.41}$$

Then by (5.41) we have

$$r_a(z_m, \{y_n\}) = \limsup_{n\to\infty} \|y_n - z_m\| \leq k_m r_a(v, \{y_n\}) = k_m r_a(C, \{y_n\}).$$

Hence

$$|r_a(z_m, \{y_n\}) - r_a(C, \{y_n\})| \leq (k_m - 1)r_a(C, \{y_n\}) \to 0 \text{ as } m \to \infty.$$

It follows from Theorem 3.1.8 that $T^m v \to v$. By the continuity of T, we have

$$Tz = T(\lim_{m\to\infty} T^m z) = \lim_{m\to\infty} T^{m+1} z = z. \quad \blacksquare$$

Theorem 5.4.3 *Let X be a uniformly convex Banach space with the Opial condition, C a nonempty closed convex (not necessarily bounded) subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$ as $n \to \infty$. By Theorem 3.2.9 (which states that if a sequence $\{u_n\}$ of a nonempty closed convex subset of a uniformly convex Banach space having the Opial condition converges weakly to $u \in C$, then u is asymptotic center of $\{u_n\}$ with respect to C), the asymptotic center of $\{x_n\}$ is x. It follows from Proposition 5.4.2 that x is a fixed point of T. ∎

To prove the next theorem concerning a demiclosedness principle, we need the following:

Proposition 5.4.4 *Let X be a Banach space with the Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $\{T^n x\}$ converges weakly to x.*

Proof. Set $A_m := \overline{co}(\{T^i x\}_{i \geq m}), m \in \mathbb{N}$ and $A := \bigcap_{m=1}^{\infty} A_m$. Because C is weakly compact, then A is nonempty and $A = \overline{co}(\omega_w(\{T^n x\}))$ by Theorem 1.9.22. We show that $T^n x \rightharpoonup x$, and this means that $A = \{x\}$. Because $\{x_n\}$ is bounded, we define a functional $f : C \to \mathbb{R}$ by $f(y) = \limsup_{n\to\infty} \|x_n - y\|, \quad y \in C$. Suppose, for contradiction, that $y_0 \in A$ such that $y_0 \neq x$. Then by the Opial condition, we have

$$f(x) < f(y_0).$$

Because $k_m - 1 \to 0$, then for $\varepsilon := (f(y_0) - f(x))/(1 + f(x)) > 0$, there exists an integer $m_0 \in \mathbb{N}$ such that $k_m - 1 < \varepsilon$ for all $m \geq m_0$. Because $y_0 \in A_{m_0+1}$, there exist an integer $p \in \mathbb{N}$ and nonnegative numbers $t_1, t_2, \cdots, t_p$ with $\sum_{i=1}^{p} t_i = 1$ such that

$$\left\| y_0 - \sum_{j=1}^{p} t_j T^{m_0+j} x \right\| < \varepsilon. \tag{5.42}$$

Note

$$\begin{aligned} f(y_0) &= \limsup_{n\to\infty} \|x_n - y_0\| \\ &\leq \limsup_{n\to\infty} \left(\|x_n - \sum_{j=1}^{p} t_j T^{m_0+j} x\| + \|\sum_{j=1}^{p} t_j T^{m_0+j} x - y_0\| \right) \\ &< \sum_{j=1}^{p} t_j \limsup_{n\to\infty} \|x_n - T^{m_0+j} x\| + \varepsilon \quad \text{(by (5.42))} \end{aligned}$$

$$\begin{aligned}
&\le \sum_{j=1}^{p} t_j \limsup_{n\to\infty} \|T^{m_0+j}x_n - T^{m_0+j}x\| + \varepsilon \qquad (\text{as } x_n - Tx_n \to 0) \\
&\le \sum_{j=1}^{p} t_j k_{m_0+j} \limsup_{n\to\infty} \|x_n - x\| + \varepsilon \\
&\le \sum_{j=1}^{p} t_j(\varepsilon+1)f(x) + \varepsilon \qquad (\text{as } k_m < \varepsilon + 1 \text{ for } m \ge m_0) \\
&= (\varepsilon+1)f(x) + \varepsilon = f(y_0),
\end{aligned}$$

a contradiction. Therefore, $x = y_0$, i.e., $A = \{x\}$. ∎

Theorem 5.4.5 *Let X be a Banach space with the locally uniform Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Proof. Suppose $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$. By Proposition 5.4.4, we have $T^n x \rightharpoonup x$. Because T is asymptotically nonexpansive, we have

$$\begin{aligned}
\limsup_{m\to\infty}(\limsup_{n\to\infty} \|T^n x - T^m x\|) &\le \limsup_{m\to\infty}(\limsup_{n\to\infty} k_m \|T^{n-m}x - x\|) \\
&= \limsup_{n\to\infty} \|T^n x - x\|.
\end{aligned}$$

Proposition 3.2.19 implies that $T^m x \to x$. By the continuity of T, we have $x = Tx$. ∎

We have already shown in Section 5.2 that every nonexpansive mapping is demiclosed in a uniformly convex Banach space. The following Theorem 5.4.6 shows that Theorem 5.2.12(a) is valid also for asymptotically nonexpansive mappings.

Theorem 5.4.6 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$ as $n \to \infty$. We show that $T^n x \to x$. Indeed, because $\{x_n\}$ is weakly convergent to x, there exists for each integer $n \in \mathbb{N}$ a convex combination

$$y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n} \quad (t_i^{(n)} \ge 0 \text{ and } \sum_{i=1}^{m(n)} t_i^{(n)} = 1)$$

such that $\|y_n - x\| < 1/n$. For an arbitrary but fixed $j \in \mathbb{N}$, because $(I-T)x_n \to 0$, there is an $n_0 = n_0(\varepsilon, j)$ so large that $1/n_0 < \varepsilon$ and $\|(I - T^j)x_n\| < \varepsilon$ for $n \ge n_0$.

Because X is uniformly convex, by Theorem 5.2.33, there exists a strictly increasing convex and continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+, g(0) = 0$ such that for any nonexpansive mapping $S : C \to X$, for any finite many elements $\{u_i\}_{i=1}^n$ in C, and for any finite many nonnegative numbers $\{t_i\}_{i=1}^n$ with $\sum_{i=1}^n t_i = 1$, the following inequality holds:

$$g(\|S(\sum_{i=1}^n t_i u_i) - \sum_{i=1}^n t_i S u_i\|) \leq \max_{1\leq i,\ j\leq n} (\|u_i - u_j\| - \|Su_i - Su_j\|). \quad (5.43)$$

Suppose L_j is the Lipschitz constant of T^j. Then ${L_j}^{-1}T^j$ is nonexpansive in C, and it follows from (5.43) that

$$\begin{aligned}
\|T^j y_n - y_n\| &\leq \|T^j y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n}\| + \|\sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n} - y_n\| \\
&\leq \|T^j \left(\sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}\right) - \sum_{i=1}^{m(n)} t_i^{(n)} T^j x_{i+n}\| \\
&+ \sum_{i=1}^{m(n)} t_i^{(n)} \|T^j x_{i+n} - x_{i+n}\| \\
&\leq L_j g^{-1}\Bigg(\max_{1\leq i,\ k\leq m(n)} (\|x_{i+n} - x_{k+n}\| \\
&\quad - L_j^{-1}\|T^j x_{i+n} - T^j x_{k+n}\|)\Bigg) + \varepsilon \\
&\leq L_j g^{-1}(2\varepsilon + (1 - L_j^{-1})\ diam(C)) + \varepsilon, \quad (5.44)
\end{aligned}$$

because

$$\begin{aligned}
\|x_{i+n} - x_{k+n}\| &- L_j^{-1}\|T^j x_{i+n} - T^j x_{k+n}\| \\
&\leq \|x_{i+n} - T^j x_{i+n}\| + \|x_{k+n} - T^j x_{k+n}\| \\
&+ (1 - L_j^{-1})\|T^j x_{i+n} - T^j x_{k+n}\| \\
&\leq 2\varepsilon + (1 - L_j^{-1})\ diam(C).
\end{aligned}$$

Taking the limit superior as $n \to \infty$ in (5.44), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \|T^j y_n - y_n\| &\leq L_j g^{-1}(2\varepsilon + (1 - L_j^{-1})\ diam(C)) \\
&+ \varepsilon \text{ for all } j \in \mathbb{N}. \quad (5.45)
\end{aligned}$$

For each $j \in \mathbb{N}$, we have

$$\begin{aligned}
\|T^j x - x\| &\leq \|T^j x - T^j y_n\| + \|T^j y_n - y_n\| + \|y_n - x\| \\
&\leq (L_j + 1)\|y_n - x\| + \|T^j y_n - y_n\|. \quad (5.46)
\end{aligned}$$

Because $y_n \to x$, it follows from (5.45) and (5.46) that

$$\begin{aligned} \|T^j x - x\| &\leq \limsup_{n\to\infty}((L_j+1)\|y_n - x\| + \|T^j y_n - y_n\|) \\ &\leq L_j\ g^{-1}(2\varepsilon + (1-L_j^{-1})\ diam(C)) + \varepsilon \\ &= g^{-1}(0) = 0 \text{ as } j \to \infty \text{ and } \varepsilon \to 0. \end{aligned}$$

This shows that $T^n x \to x$ and by the continuity of T, we obtain that $x = Tx$. ▮

We now give a fundamental existence theorem for asymptotically nonexpansive mappings in a uniformly convex Banach space.

Theorem 5.4.7 (Goebel and Kirk's fixed point theorem) – *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then T has a fixed point in C.*

Proof. For fixed $y \in C$ and $r > 0$, set

$R_y := \{r :$ there exists $k \in \mathbb{N}$ with $C \cap (\cap_{i=k}^{\infty} B_r[T^i y]) \neq \emptyset\}$ and $d := diam(C)$.

Then $d \in R_y$. Hence $R_y \neq \emptyset$. Let $r_0 = \inf\{r : r \in R_y\}$. For each $\varepsilon > 0$, we define

$$C_\varepsilon = \cup_{k=1}^{\infty}(\cap_{i=k}^{\infty} B_{r_0+\varepsilon}[T^i y]).$$

Thus, for each $\varepsilon > 0$, the set $C_\varepsilon \cap C$ is nonempty and convex. The reflexivity of X implies that

$$\cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C) \neq \emptyset.$$

Note that for $x \in \cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C)$ and $\eta > 0$, there exists an integer n_0 such that

$$\|x - T^n y\| \leq r_0 + \eta \ \text{ for all } \ n \geq n_0.$$

Now, let $x \in \cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C)$ and suppose, for contradiction, that the sequence $\{T^n x\}$ does not converge strongly to x. Then there exist $\varepsilon > 0$ and a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that

$$\|T^{n_i} x - x\| \geq \varepsilon \ \text{ for all } \ i = 1, 2, \cdots .$$

Suppose k_n is the Lipschitz constant of T^n. Then for $m > n$, we have

$$\|T^n x - T^m x\| \leq k_n \|x - T^{m-n} x\|.$$

Assume that $r_0 > 0$ and choose $\alpha > 0$ such that

$$\left(1 - \delta_X\left(\frac{\varepsilon}{r_0 + \alpha}\right)\right)(r_0 + \alpha) < r_0.$$

Select n such that

$$\|x - T^n x\| \geq \varepsilon \ \text{ and } \ k_n\left(r_0 + \frac{\alpha}{2}\right) \leq r_0 + \alpha.$$

If $n_0 \geq n$ is sufficiently large, then $m > n_0$ implies

$$\|x - T^{m-n}y\| \leq r_0 + \frac{\alpha}{2}.$$

Because

$$\|T^n x - T^m y\| \leq k_n \|x - T^{m-n}y\| \leq k_n\left(r_0 + \frac{\alpha}{2}\right) \leq r_0 + \alpha$$

and

$$\|x - T^m y\| \leq r_0 + \alpha,$$

it follows from the uniform convexity of X that for $m > n_0$,

$$\|\frac{1}{2}(x + T^n x) - T^m y\| \leq \left(1 - \delta_X\left(\frac{\varepsilon}{r_0 + \alpha}\right)\right)(r_0 + \alpha) < r_0.$$

This contradicts the definition of r_0. Hence we conclude that $r_0 = 0$ or $x = Tx$. But $r_0 = 0$ implies that $\{T^n y\}$ is a Cauchy sequence and hence $\lim_{n\to\infty} T^n y = x = Tx$. Therefore, the set $\cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C)$ is a singleton that is a fixed point of T. ∎

In our next existence theorem, the boundedness of C is not necessary. In particular, we will show that the existence of a fixed point of an asymptotically nonexpansive mapping in a uniformly convex Banach space is not only equivalent to the existence of a bounded orbit at a point, but it is also equivalent to the existence of a bounded $AFPS$.

Theorem 5.4.8 *Let C be a nonempty closed convex (but not necessarily bounded) subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then the following statements are equivalent:*

(a) T has a fixed point.

(b) There exists a point $x_0 \in C$ such that the sequence $\{T^n x_0\}$ is bounded.

(c) There exists a bounded sequence $\{y_n\}$ in C such that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$.

Proof. $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ follows easily.

$(b) \Rightarrow (a)$. Let $x_0 \in C$ be a point such that the sequence $\{x_n = T^n x_0\}$ is bounded. By Theorem 3.1.5, there exists a unique point $z \in C$ such that $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$. Define a sequence $\{y_m\}$ in C by $y_m = T^m z$ for all $m \in \mathbb{N}$. Let k_m be the Lipschitz constant of the iterates T^m. For $n > m \geq 1$, we have

$$\|x_n - y_m\| \ = \ \|T^m x_{n-m} - T^m z\| \leq k_m \|x_{n-m} - z\|.$$

This implies that

$$r_a(y_m, \{x_n\}) \leq k_m r_a(z, \{x_n\}) = k_m r_a(C, \{x_n\}).$$

This shows that $\lim_{m\to\infty} r_a(y_m, \{x_n\}) = r_a(C, \{x_n\})$. By Theorem 3.1.8, $\lim_{m\to\infty} y_m = z$. By the continuity of T, v is a fixed point of T.

$(c) \Rightarrow (a)$. Let $\{y_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. Let $\mathcal{Z}_a(C, \{y_n\}) = \{v\}$. Therefore, Proposition 5.4.2 implies that v is a fixed point of T. ∎

We have seen in a Corollary 5.2.29 that $F(T)$ is closed and convex in strictly convex Banach space for nonexpansive mappings. However, Corollary 5.2.29 is not true for asymptotically nonexpansive mappings. In fact, we have

Theorem 5.4.9 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping. Then $F(T)$ is closed and convex.*

Proof. The closedness of $F(T)$ is obvious. To show convexity, it is sufficient to prove that $z = (x + y)/2 \in F(T)$ for $x, y \in F(T)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x - T^n z\| &= \|T^n x - T^n z\| \leq k_n \|x - z\| = \frac{1}{2} k_n \|x - y\|, \\ \|y - T^n z\| &= \|T^n y - T^n z\| \leq k_n \|y - z\| = \frac{1}{2} k_n \|x - y\|. \end{aligned}$$

By the uniform convexity of X, we have

$$\|z - T^n z\| \leq \frac{1}{2}\left[1 - \delta_X\left(\frac{2}{k_n}\right)\right] k_n \|x - y\| \leq \frac{1}{2}\left[1 - \delta_X\left(\frac{2}{k_n}\right)\right] k_n diam(C)$$

and hence $T^n z \to z$ as $n \to \infty$. It follows from the continuity of T that z is a fixed point of T. ∎

5.5 Uniformly L-Lipschitzian mappings

Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. Then T is said to be *uniformly L-Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$.

Note that every nonexpansive mapping is uniformly L-Lipschitzian with $L = 1$ and every asymptotically nonexpansive mapping with sequence $\{k_n\}$ is also uniformly L-Lipschitzian with $L = \sup_{n\in\mathbb{N}} k_n$.

The following proposition shows that the class of uniformly L-Lipschitzian mappings on C can be characterized as the class of mappings on C that are nonexpansive relative to some metric on C that is equivalent to the norm.

Proposition 5.5.1 *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a uniformly L-Lipschitzian mapping. Then there exists a metric d on C that is equivalent to the norm metric such that T is nonexpansive with respect to d.*

Proof. Define the metric d on C by

$$d(x,y) = \sup\{\|T^n x - T^n y\| : n = 0, 1, 2, \cdots\}, x, y \in C.$$

Because

$$\|x - y\| \leq d(x,y) \leq L\|x - y\|,$$

this means that the metric d on C is equivalent to the norm metric. Furthermore, T is nonexpansive with respect to d. ∎

Proposition 5.5.2 *Let C be a nonempty subset of a Banach space and ρ a metric on C satisfying the condition:*

$$\alpha\|x - y\| \leq \rho(x,y) \leq \beta\|x - y\| \text{ for all } x, y \in C. \tag{5.47}$$

If $T : C \to C$ is a nonexpansive mapping with respect to ρ, then T is uniformly β/α-Lipschitzian with respect to $\|\cdot\|$.

Proof. Because

$$\alpha\|x - y\| \leq \rho(x,y) \leq \beta\|x - y\| \text{ for all } x, y \in C$$

and $T : C \to C$ is nonexpansive with respect to $\rho(x,y)$, then for $n = 1, 2, \cdots$,

$$\|T^n x - T^n y\| \leq \frac{1}{\alpha}\rho(T^n x, T^n y) \leq \frac{1}{\alpha}\rho(x,y) \leq \frac{\beta}{\alpha}\|x - y\|.$$

Therefore, T is uniformly β/α-Lipschitzian. ∎

The following Theorem 5.5.3 shows that the Goebel and Kirk's fixed point theorem (see Theorem 5.4.7) for asymptotically nonexpansive mappings remains valid for a broader class of uniformly L-Lipschitzian mappings with $L < \gamma$, where γ is sufficiently near one.

Theorem 5.5.3 (Goebel and Kirk's fixed point theorem) – *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X. Then every uniformly L-Lipschitzian mapping $T : C \to C$ with $L < \gamma$ has a fixed point in C, where $\gamma > 1$ is the unique solution of the equation*

$$t\left(1 - \delta_X\left(\frac{1}{t}\right)\right) = 1.$$

Proof. We take γ to be the solution of the equation $t(1 - \delta_X(1/t)) = 1$ and assume that $1 \leq L < \gamma$, i.e., L satisfies the inequality:

$$L\left(1 - \delta_X\left(\frac{1}{L}\right)\right) < 1. \tag{5.48}$$

For $x \in C$, set $d(x) := \limsup_{n\to\infty} \|x - T^n x\|$. Let

$$R = \{r > 0 : \text{there exists } n \in \mathbb{N} \text{ with } C \cap (\cap_{i=n}^{\infty} B_r[T^i x]) \neq \emptyset\}.$$

Then R is nonempty (because R contains the diameter of C), so we can define

$$r_0 = r_0(x) = \inf\{r > 0 : r \in R\}.$$

For each $\varepsilon > 0$, we define

$$C_\varepsilon = \cup_{n=1}^{\infty} (\cap_{i=n}^{\infty} B_{r_0+\varepsilon}[T^i x]).$$

Hence for each $\varepsilon > 0$, the sets C_ε are nonempty and convex. The reflexivity of X implies that

$$\cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C) \neq \emptyset.$$

Let $z = z(x) \in \cap_{\varepsilon>0}(\overline{C}_\varepsilon \cap C)$. Notice that z and r_0 have the following properties:

(i) for each $\varepsilon > 0$, $B_{r_0+\varepsilon}[z]$ contains almost all terms of the sequence $\{T^i x\}$,

(ii) given $u \in C$ and $r < r_0$, the set $\{i : \|u - T^i x\| > r\}$ is infinite.

Now if $r_0 = 0$ or if $d(z) = 0$, then $\lim_{i\to\infty} T^i x = z$ yielding $z = Tz$. So we may assume that $r_0 > 0$ and $d(z) = \limsup_{n\to\infty} \|z - T^i z\| > 0$.

Let $\varepsilon > 0$ with $0 < \varepsilon \leq d(z)$ and select $j \in \mathbb{N}$ such that $\|z - T^i z\| \geq d(z) - \varepsilon$. By (i), there exists an integer n_0 such that if $i \geq n_0$, then

$$\|z - T^i x\| \leq r_0 + \varepsilon \leq L(r_0 + \varepsilon), \qquad (\text{as } 1 \leq L)$$

and it follows for $i - j \geq n_0$ that

$$\begin{aligned} \|T^j z - T^i x\| &= \|T^j z - T^j (T^{i-j}) x\| \\ &\leq L\|z - T^{i-j} x\| \\ &\leq L(r_0 + \varepsilon). \end{aligned}$$

Set $w := (z + T^j z)/2$. By a property of δ_X, we have

$$\begin{aligned} \|w - T^i x\| &= \left\| \frac{z - T^i x + T^j z - T^i x}{2} \right\| \\ &\leq \left(1 - \delta_X\left(\frac{d(z) - \varepsilon}{L(r_0 + \varepsilon)}\right)\right) L(r_0 + \varepsilon) \text{ for all } i \geq n_0 + j. \end{aligned}$$

This implies (by (ii)) that

$$r_0 \leq \left(1 - \delta_X\left(\frac{d(z) - \varepsilon}{L(r_0 + \varepsilon)}\right)\right) L(r_0 + \varepsilon).$$

By the continuity of δ_X, we have

$$L\left(1-\delta_X\left(\frac{d(z)}{Lr_0}\right)\right) \geq 1.$$

This implies that

$$d(z) \leq L\delta_X^{-1}\left(1-\frac{1}{L}\right)r_0.$$

From (ii), we have $r_0 \leq d(x)$, and hence

$$d(z) \quad \leq \quad L\delta_X^{-1}\left(1-\frac{1}{L}\right)d(x) = \alpha d(x),$$

where $\alpha = L\delta_X^{-1}(1-1/L) < 1$ because L satisfies (5.48).

To complete the proof, fix $x_0 \in C$ and define the sequence $\{x_n\}$ by

$$x_{m+1} = z(x_m), \quad m = 0, 1, 2, \cdots,$$

where $z(x_m)$ is selected in the same manner as $z(x)$. Now if for any m we have $r_0(x_m) = 0$, then $Tx_{m+1} = x_{m+1}$. Otherwise,

$$\|x_m - x_{m+1}\| \quad \leq \quad 2d(x_m) \leq 2\alpha^m d(x_0),$$

which implies that $\{x_m\}$ is a Cauchy sequence. Hence $x_m \to v \in C$. Note

$$\begin{aligned}
\|v - T^i v\| &\leq \|v - x_m\| + \|x_m - T^i x_m\| + \|T^i x_m - T^i v\| \\
&\leq (1+L)\|v - x_m\| + \|x_m - T^i x_m\| \\
&\leq (1+L)\|v - x_m\| + d(x_m) \to 0 \text{ as } m \to \infty.
\end{aligned}$$

By the continuity of T, we have $v = Tv$. ∎

Corollary 5.5.4 *Let C be a nonempty closed convex bounded subset of Hilbert space H and $T : C \to C$ uniformly L-Lipschitzian with $L < \gamma = \sqrt{5}/2$. Then T has a fixed point.*

Let us give an example of a uniformly L-Lipschitzian mapping that is fixed point free.

Example 5.5.5 *Let $\|\cdot\|_2$ be the usual Euclidean norm on the Hilbert space $H = \ell_2$ and let B_H be the closed unit ($\|\cdot\|_2$-unit) ball and let $S : \ell_2 \to \ell_2$ be the right shift operator defined by*

$$S(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots).$$

Then the mapping $T : B_H \to B_H$ defined by

$$Tx = \frac{(1-\|x\|_2)e + Sx}{\|(1-\|x\|_2)e + Sx\|_2}, \ x \in B_H, \ e = (1, 0, 0, \cdots)$$

is uniformly L-Lipschitzian with $L = 2$, but it has no fixed point in B_H.

Applying Theorem 5.5.3, we have

Theorem 5.5.6 *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X. If $T : C \to C$ is nonexpansive with respect to a metric $\rho(x, y)$ on C satisfying (5.47), where $\beta/\alpha < \gamma$ for γ as in Theorem 5.5.3, then T has a fixed point in C.*

The following theorem has a sharper estimate for L than Goebel and Kirk's fixed point theorem even in a more general Banach space.

Theorem 5.5.7 (Casini and Maluta's fixed point theorem) - *Every Banach space X with uniformly normal structure has the fixed point property for uniformly L-Lipschitzian mappings with $L < \sqrt{N(X)}$.*

Proof. Let C be a nonempty closed convex bounded subset of X and $T : C \to C$ a uniformly L-Lipschitzian mapping. For any $x_0 \in C$, consider the sequence $\{T^n x_0\}$ in C. By Theorem 3.4.20(b) for $\{T^n x_0\}$, there exists $x_1 \in \overline{co}(\{T^n x_0\})$ such that

$$r_a(x_1, \{T^n x_0\}) \leq \tilde{N}(X)\ diam_a(\{T^n x_0\}), \tag{5.49}$$

where $\tilde{N}(X) = 1/N(X)$. Observe that

$$\begin{aligned} diam_a(\{T^n x_0\}) &= \lim_{k\to\infty} (\sup\{\|T^i x_0 - T^j x_0\| : i,\ j \geq k\}) \\ &\leq \sup_{i\geq j\geq 0} \|T^i x_0 - T^j x_0\| \\ &\leq L \sup_{i\geq j\geq 0} \|x_0 - T^{i-j} x_0\| \\ &\leq L \sup_{n\in\mathbb{N}} \|x_0 - T^n x_0\|. \end{aligned}$$

From (5.49), we have

$$r_a(x_1, \{T^n x_0\}) \leq L\tilde{N}(X) \sup_{n\in\mathbb{N}} \|x_0 - T^n x_0\|. \tag{5.50}$$

Moreover, for $\ell \in \mathbb{N}$, we have

$$\begin{aligned} r_a(T^\ell x_1, \{T^n x_0\}) &= \limsup_{n\to\infty} \|T^\ell x_1 - T^n x_0\| \\ &\leq L \limsup_{n\to\infty} \|x_1 - T^{n-\ell} x_0\| \\ &= L r_a(x_1, \{T^n x_0\}). \end{aligned} \tag{5.51}$$

From (5.50) and (5.51), we have

$$r_a(T^\ell x_1, \{T^n x_0\}) \leq L^2 \tilde{N}(X)\ \sup_{n\in\mathbb{N}} \|x_0 - T^n x_0\|.$$

By Theorem 3.4.20(a), we have

$$\|x_1 - T^\ell x_1\| \leq r_a(T^\ell x_1, \{T^n x_0\}) \leq L^2 \tilde{N}(X) \sup_{n\in\mathbb{N}} \|x_0 - T^n x_0\|,$$

which implies that

$$\sup_{\ell\in\mathbb{N}} \|x_1 - T^\ell x_1\| \le L^2\tilde{N}(X) \sup_{n\in\mathbb{N}} \|x_0 - T^n x_0\|.$$

Thus, for any $x_0 \in C$, we can inductively define a sequence $\{x_m\}_{m\ge 0}$ in the following manner:

$$\sup_{n\in\mathbb{N}} \|x_{m+1} - T^n x_{m+1}\| \le L^2\tilde{N}(X) \sup_{n\in\mathbb{N}} \|x_m - T^n x_m\| \text{ for all } m \in \mathbb{N}_0$$

and

$$r_a(x_{m+1}, \{T^n x_m\}) \le L\tilde{N}(X) \sup_{n\in\mathbb{N}} \|x_m - T^n x_m\| \text{ for all } m \in \mathbb{N}_0.$$

Set $D_m := \sup_{n\in\mathbb{N}} \|x_m - T^n x_m\|$ for $m \ge 0$ and $\eta := L^2\tilde{N}(X) < 1$. Then

$$D_{m+1} \le \eta D_m \le \eta^2 D_{m-1} \le \cdots \le \eta^{m+1} D_0 \to 0 \text{ as } m \to \infty.$$

Observe that

$$\|x_m - x_{m+1}\| \le \|x_m - T^n x_m\| + \|x_{m+1} - T^n x_m\| \le D_m + \|x_{m+1} - T^n x_m\|.$$

Taking the limit superior as $n \to \infty$, we have

$$\|x_m - x_{m+1}\| \le (1 + L\tilde{N}(X))D_m,$$

and it follows that $\{x_m\}$ is a Cauchy sequence in C. Let $\lim_{m\to\infty} x_m = v \in C$. Hence

$$\begin{aligned} \|v - Tv\| &\le \|v - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tv\| \\ &\le (1 + L)\|x_m - v\| + D_m \to 0 \text{ as } m \to \infty. \end{aligned}$$ ∎

The following result is a slight generalization of Theorem 5.5.7.

Theorem 5.5.8 *Let X be a Banach space with uniformly normal structure, C a nonempty bounded subset of X, and $T : C \to C$ a uniformly L-Lipschitzian mapping with $L < \sqrt{N(X)}$. Suppose that there exists a nonempty closed convex bounded subset M of C with the following property (P):*

$$x \in M \text{ implies } \omega_w(\{T^n x\}) \subset M. \tag{5.52}$$

Then T has a fixed point in M.

Proof. For any $x_0 \in M$ and each $n \in \mathbb{N}$, consider a sequence $\{T^j x_0\}_{j\ge n}$ in C. By Theorem 3.4.20(b), we have $y_n \in \overline{co}(\{T^j x_0\}_{j\ge n})$ such that

$$\limsup_{j\to\infty} \|T^j x_0 - y_n\| \le \tilde{N}(X)\ diam_a(\{T^j x_0\}_{j\ge n}). \tag{5.53}$$

Theorem 3.4.16 implies that X is reflexive, and there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup x_1 \in X$. From (5.53) and w-lsc of the functional $r_a(\cdot, \{T^n x_0\})$, we have

$$\begin{aligned} r_a(x_1, \{T^j x_0\}) &\leq \liminf_{i\to\infty} r_a(y_{n_i}, \{T^j x_0\}) \\ &\leq \limsup_{n\to\infty} r_a(y_n, \{T^j x_0\}) \\ &\leq \tilde{N}(X)\ diam_a(\{T^n x_0\}). \end{aligned}$$

It can be easily seen that $x_1 \in \cap_{n=1}^{\infty} \overline{co}(\{T^j x_0\}_{j\geq n})$ and that

$$\|x_1 - y\| \leq \limsup_{n\to\infty} \|T^n x_0 - y\| \text{ for all } y \in X.$$

Using Theorem 1.9.22, we obtain that $\overline{co}(\omega_w(\{T^n x_0\})) = \cap_{n=1}^{\infty} \overline{co}(\{T^j x_0\}_{j\geq n})$. It follows that $x_1 \in \overline{co}(\omega_w(\{T^n x_0\}))$. Using property (P) we obtain that $x_1 \in M$. By repeating the above process, we can obtain a sequence $\{x_m\}$ in M with the properties:

for all integers $m \geq 0$,

$$\limsup_{n\to\infty} \|x_{m+1} - T^n x_m\| \leq \tilde{N}(X)\ diam_a(\{T^n x_m\}) \tag{5.54}$$

and

$$\|x_{m+1} - y\| \leq \limsup_{n\to\infty} \|T^n x_m - y\| \text{ for all } y \in X. \tag{5.55}$$

Set $D_m := \sup_{n\in\mathbb{N}} \|x_m - T^n x_m\|$ for all $m = 0, 1, 2, \cdots$. Note

$$\begin{aligned} diam_a(\{T^n x_m\}) &= \lim_{k\to\infty} (\sup\{\|T^i x_m - T^j x_m\| : i, j \geq k\}) \\ &\leq \sup_{i\geq j\geq 0} \|T^i x_m - T^j x_m\| \\ &\leq L \sup_{i\geq j\geq 0} \|x_m - T^{i-j} x_m\| \\ &\leq LD_m. \end{aligned}$$

Moreover, from (5.54) we have, for $\ell \in \mathbb{N}$

$$\begin{aligned} \limsup_{n\to\infty} \|T^\ell x_{m+1} - T^n x_m\| &\leq L \limsup_{n\to\infty} \|x_{m+1} - T^{n-\ell} x_m\| \\ &\leq L \limsup_{n\to\infty} \|x_{m+1} - T^n x_m\| \\ &\leq L\tilde{N}(X)\ diam_a(\{T^n x_m\}) \\ &\leq L^2 \tilde{N}(X) D_m. \end{aligned}$$

From (5.55), we have

$$\begin{aligned} \|x_{m+1} - T^\ell x_{m+1}\| &\leq \limsup_{n\to\infty} \|T^n x_m - T^\ell x_{m+1}\| \\ &\leq L^2 \tilde{N}(X) D_m, \end{aligned}$$

which implies that

$$D_{m+1} \leq \eta D_m \text{ for all } m = 0, 1, 2, \cdots,$$

where $\eta = L^2 \tilde{N}(X) < 1$. Now proceeding with the same argument as in the proof of Theorem 5.5.7, we conclude that $\{x_m\}$ converges strongly to a fixed point of T in M. ▮

Using weak uniformly normal structure coefficient $WCS(X)$, we now establish an existence theorem for asymptotically regular Lipschitzian mappings. Before proving Theorem 5.5.10, we first establish a preliminary result.

Proposition 5.5.9 *Let C be a nonempty closed subset of a Banach space X and $T : C \to C$ an asymptotically regular mapping such that for some $m \in \mathbb{N}$, T^m is continuous. If $\limsup_{i\to\infty} \|T^{n_i}x - z\| = 0$ for some $x \in C$ and $z \in C$, then $z \in F(T)$.*

Proof. Note $T^{n_i}x \to z$ as $i \to \infty$. So

$$\begin{aligned} \|z - T^{n_i+m}x\| &\leq \|z - T^{n_i}x\| + \|T^{n_i}x - T^{n_i+m}x\| \\ &\leq \|z - T^{n_i}x\| + \sum_{\nu=0}^{m-1} \|T^{n_i+\nu}x - T^{n_i+\nu+1}x\|. \end{aligned}$$

By the asymptotic regularity of T, $T^{n_i+m}x \to z$ as $i \to \infty$. Because T^m is continuous, it follows that

$$T^m z = T^m(\lim_{i\to\infty} T^{n_i}x) = \lim_{i\to\infty} (T^{n_i+m}x) = z.$$

Because $z = T^m z = T^{2m} z = \cdots = T^{ms} z$ for all $s \in \mathbb{N}$,

$$\|z - Tz\| = \|T^{ms}z - T^{ms+1}\| \to 0 \text{ as } s \to \infty.$$

Therefore, $z \in F(T)$. ▮

Theorem 5.5.10 *Let X be a Banach space with $WCS(X) > 1$, C a nonempty weakly compact convex subset of X, $T : C \to C$ a Lipschitzian mapping such that $\liminf_{n\to\infty} \sigma(T^n) < \sqrt{WCS(X)}$. If T is asymptotically regular on C, then T has a fixed point.*

Proof. Because one can easily construct a nonempty closed convex separable subset C_0 of C such that C_0 is invariant under T, i.e., $T(C_0) \subset C_0$, we may assume that C is itself separable. The separability of C makes it possible to select a subsequence $\{n_i\}$ of natural numbers such that

$$\liminf_{n\to\infty} \sigma(T^n) = \lim_{i\to\infty} \sigma(T^{n_i}) < \sqrt{WCS(X)}$$

and

$$\{T^{n_i}x\} \text{ converges weakly for every } x \in C.$$

Now we can construct a sequence $\{x_m\}$ in C in the following way:

$$\begin{cases} x_0 \in C \quad \text{arbitrary}, \\ x_{m+1} = w - \lim_{i\to\infty} T^{n_i} x_m, m \geq 0. \end{cases}$$

By the asymptotic regularity of T on C, we have

$$x_{m+1} = w - \lim_{i\to\infty} T^{n_i+k} x_m \text{ for all } k \geq 0.$$

Set

$$r_m := \limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\| \text{ and } L := \limsup_{i\to\infty} \sigma(T^{n_i}).$$

By the definition of $WCS(X)$, we obtain

$$WCS(X) = \sup\{M > 0 : M \cdot \limsup_{n\to\infty} \|x_n - u\| \leq D[\{x_n\}]\},$$

where the supremum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in X with $x_n \rightharpoonup u$. Then we have

$$r_m = \limsup_{i\to\infty} \|x_{m+1} - T^{n_i} x_m\| \leq \frac{1}{WCS(X)} D[\{T^{n_i} x_m\}].$$

However, from the w-lsc of the norm of X, we obtain that

$$\begin{aligned}
D[\{T^{n_i} x_m\}] &= \limsup_{i\to\infty}(\limsup_{j\to\infty} \|T^{n_i} x_m - T^{n_j} x_m\|) \\
&\leq \limsup_{i\to\infty}(\limsup_{j\to\infty}(\|T^{n_i} x_m - T^{n_i+n_j} x_m\| \\
&\quad +\|T^{n_i+n_j} x_m - T^{n_j} x_m\|)) \\
&\leq \limsup_{i\to\infty}(\limsup_{j\to\infty}(\sigma(T^{n_i})\|x_m - T^{n_j} x_m\| \\
&\quad + \sum_{\nu=0}^{n_i-1} \|T^{n_j+\nu} x_m - T^{n_j+\nu+1} x_m\|)) \\
&= \limsup_{i\to\infty} \sigma(T^{n_i}) \times \limsup_{j\to\infty} \|x_m - T^{n_j} x_m\| \\
&\leq L \limsup_{j\to\infty}(\limsup_{k\to\infty} \|T^{n_k} x_{m-1} - T^{n_j} x_m\|) \\
&= L \limsup_{j\to\infty}(\limsup_{k\to\infty}(\|T^{n_j} x_m - T^{n_j+n_k} x_{m-1}\| \\
&\quad +\|T^{n_j+n_k} x_{m-1} - T^{n_k} x_{m-1}\|) \\
&\leq L \limsup_{j\to\infty}(\limsup_{k\to\infty}(\sigma(T^{n_j})\|x_m - T^{n_k} x_{m-1}\| \\
&\quad + \sum_{\nu=0}^{n_j-1} \|T^{n_k+\nu} x_{m-1} - T^{n_k+\nu+1} x_{m-1}\|)) \\
&= L^2 r_{m-1}.
\end{aligned}$$

Hence

$$r_m \le \frac{L^2}{WCS(X)} r_{m-1} = \eta r_{m-1} \text{ for all } m \in \mathbb{N},$$

where $\eta = L^2/WCS(X) < 1$.

Again by w-lsc of the norm of X, we have

$$\begin{aligned}
\|x_m - x_{m+1}\| &\le \limsup_{i\to\infty}(\|x_m - T^{n_i}x_m\| + \|T^{n_i}x_m - x_{m+1}\|) \\
&\le \limsup_{i\to\infty}\|x_m - T^{n_i}x_m\| + \limsup_{i\to\infty}\|x_{m+1} - T^{n_i}x_m\| \\
&\le \limsup_{i\to\infty}(\limsup_{j\to\infty}\|T^{n_j}x_{m-1} - T^{n_i}x_m\|) + r_m \\
&\le \limsup_{i\to\infty}(\limsup_{j\to\infty}(\|T^{n_i}x_m - T^{n_i+n_j}x_{m-1}\| \\
&\quad +\|T^{n_i+n_j}x_{m-1} - T^{n_j}x_{m-1}\|)) + r_m \\
&\le \limsup_{i\to\infty}(\limsup_{j\to\infty}(\sigma(T^{n_i})\|x_m - T^{n_j}x_{m-1}\| \\
&\quad + \sum_{\nu=0}^{n_i-1}\|T^{n_j+\nu}x_{m-1} - T^{n_j+\nu+1}x_{m-1}\|)) + r_m \\
&= Lr_{m-1} + r_m \\
&\le (L+\eta)r_{m-1},
\end{aligned}$$

and it follows that $\{x_m\}$ is a Cauchy sequence in C. Let $\lim_{m\to\infty} x_m = p \in C$. Note

$$\begin{aligned}
\|p - T^{n_i}p\| &\le \|p - x_{m+1}\| + \|x_{m+1} - T^{n_i}x_m\| + \|T^{n_i}x_m - T^{n_i}p\| \\
&\le \|p - x_{m+1}\| + \|x_{m+1} - T^{n_i}x_m\| + \sigma(T^{n_i})\|x_m - p\|,
\end{aligned}$$

which implies that

$$\limsup_{i\to\infty}\|p - T^{n_i}p\| \le \|x_{m+1} - p\| + r_m + L\|x_m - p\| \to 0 \text{ as } m \to \infty.$$

Hence $T^{n_i}p \to p$ as $i \to \infty$. Therefore, $Tp = p$ by Proposition 5.5.9. ∎

5.6 Non-Lipschitzian mappings

Let C be a nonempty subset of a Banach space X, $T : C \to C$ a mapping, and fix a sequence $\{a_n\}$ in $\mathbb{R}^+$ with $a_n \to 0$. Recall that

$$\eta(T^n) = \sup\left\{\frac{\|T^nx - T^ny\|}{\|x-y\| + a_n} : x, y \in C, x \ne y\right\}$$

is nearly Lipschitz constant of T^n. Then T is *nearly Lipschitzian with sequence* $\{(\eta(T^n), a_n)\}$ if

$$\|T^nx - T^ny\| \le \eta(T^n)(\|x-y\| + a_n) \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

T is *nearly asymptotically nonexpansive* with sequence $\{(\eta(T^n), a_n)\}$ if for each $n \in \mathbb{N}$, $\eta(T^n) \geq 1$ with $\lim_{n\to\infty} \eta(T^n) = 1$ and

$$\|T^n x - T^n y\| \leq \eta(T^n)(\|x - y\| + a_n) \text{ for all } x, y \in C.$$

In this section, we study fixed point theorems for non-Lipschitzian mappings in Banach spaces. We begin with the following preliminary result.

Proposition 5.6.1 *Let C be a nonempty closed subset of a Banach space and $T : C \to C$ a demicontinuous mapping. Suppose that $T^n u \to x^*$ as $n \to \infty$ for some $u, x^* \in C$. Then x^* is a fixed point of T.*

Proof. Let $u_n = T^n u$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ and $\{Tu_n\}$ converge strongly to x^*. By the demicontinuity of T, $\{Tu_n\}$ converges weakly to Tx^*. By uniqueness of weak limits of $\{Tu_n\}$, we have $x^* = Tx^*$. ∎

In view of Proposition 5.6.1, we remark that Theorem 4.1.18 is valid for demicontinuous nearly Lipschitzian mappings. In fact,

Theorem 5.6.2 *Let C be a nonempty closed subset of a Banach space and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(\eta(T^n), a_n)\}$. Suppose $\eta_\infty(T) = \limsup_{n\to\infty}[\eta(T^n)]^{1/n} < 1$. Then we have the following:*

(a) T has a unique fixed point $v \in C$.

(b) For each $x \in C$, the sequence $\{T^n x\}$ converges to v.

(c) $\|T^n x - v\| \leq \sum_{i=n}^{\infty} \eta(T^i)(\|x - Tx\| + M)$ for all $x \in C$ and $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}} a_n$.

Proof. Let $x \in C$. By Theorem 4.1.18, $\{T^n x\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} T^n x = v \in C$. It follows from Proposition 5.6.1 that v is a fixed point of T. ∎

We now give demiclosedness principle for nearly Lipschitzian mappings in a Banach space.

Theorem 5.6.3 *Let X be a Banach space with a weakly continuous duality mapping $J_\mu : X \to X^*$ with gauge function μ. Let C be a nonempty closed convex subset of X and $T : C \to C$ a uniformly continuous nearly Lipschitzian mapping with sequence $\{(\eta(T^n), a_n)\}$ such that $\lim_{n\to\infty} \eta(T^n) = 1$. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $x \in C$ because C is weakly closed. The uniform continuity of T implies that

$$\lim_{n\to\infty} \|x_n - T^m x_n\| = 0 \text{ for all } m \in \mathbb{N}.$$

It follows that $T^m x_n \rightharpoonup x$ for all $m \in \mathbb{N}$. Set $r_m := \limsup_{n\to\infty} \|T^m x_n - x\|$, $m \in \mathbb{N}$. Let $m, s \in \mathbb{N}$. Because $T^{m+s}x_n \rightharpoonup x$ as $n \to \infty$, by the Opial condition, we have

$$\begin{aligned} r_{m+s} = \limsup_{n\to\infty} \|T^{m+s}x_n - x\| &< \limsup_{n\to\infty} \|T^{m+s}x_n - T^s x\| \\ &\le \limsup_{n\to\infty} \eta(T^s)(\|T^m x_n - x\| + a_s) \\ &= \eta(T^s)(r_m + a_s). \end{aligned}$$

It follows that

$$\limsup_{s\to\infty} r_s \le r_m,$$

which implies that

$$\limsup_{s\to\infty} r_s \le \liminf_{m\to\infty} r_m.$$

Thus, $\lim_{m\to\infty} r_m$ exists. Suppose $\lim_{m\to\infty} r_m = r$ for some $r > 0$. Noting by Theorem 2.5.23 that

$$\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\mu(x+ty)\rangle dt \text{ for all } x, y \in X.$$

For $m, s \in \mathbb{N}$, we have

$$\begin{aligned} \Phi(\|T^{m+s}x_n - x\|) &= \Phi(\|T^{m+s}x_n - T^m x + T^m x - x\|) \\ &= \Phi(\|T^{m+s}x_n - T^m x\|) \\ &\quad + \int_0^1 \langle T^m x - x, J_\mu(T^{m+s}x_n - T^m x + t(T^m x - x))\rangle dt \\ &\le \Phi(\eta(T^m)(\|T^s x_n - x\| + a_m)) \\ &\quad + \int_0^1 \langle T^m x - x, J_\mu(T^{m+s}x_n - T^m x + t(T^m x - x))\rangle dt. \end{aligned}$$

Because $T^{m+s}x_n \rightharpoonup x$ as $n \to \infty$, we obtain

$$\begin{aligned} \Phi(r_{m+s}) &= \Phi(\limsup_{n\to\infty} \|T^{m+s}x_n - x\|) \\ &\le \Phi(\eta(T^m)(r_s + a_m)) \\ &\quad - \int_0^1 \langle T^m x - x, J_\mu((1-t)(T^m x - x))\rangle dt \\ &= \Phi(\eta(T^m)(r_s + a_m)) \\ &\quad - \int_0^1 \|T^m x - x\| \mu(t\|T^m x - x\|) dt \\ &= \Phi(\eta(T^m)(r_s + a_m)) - \Phi(\|T^m x - x\|), \end{aligned}$$

which implies that

$$\Phi(\|T^m x - x\|) \le \Phi(\eta(T^m)(r_s + a_m)) - \Phi(r_{m+s}).$$

Because $\lim_{s\to\infty} r_s$ exists, we have

$$\Phi(\|T^m x - x\|) \leq \Phi(\eta(T^m)(r + a_m)) - \Phi(r) \to 0 \text{ as } m \to \infty.$$

Thus, $T^m x \to x$. Therefore, by the continuity of T, we have $x = Tx$. ∎

Corollary 5.6.4 *Let X be a Banach space with a weakly continuous duality mapping $J_\mu : X \to X^*$ with gauge function μ. Let C be a nonempty closed convex subset of X and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

The following theorem is an extension of Theorem 5.4.8 for demicontinuous nearly asymptotically nonexpansive mappings.

Theorem 5.6.5 *Let C be a nonempty closed convex (but not necessarily bounded) subset of a uniformly convex Banach space X and $T : C \to C$ a demicontinuous nearly asymptotically nonexpansive mapping. Then the following statements are equivalent:*

(a) T has a fixed point in C.

(b) There is a point $x_0 \in C$ such that the sequence $\{T^n x_0\}$ is bounded.

Proof. $(a) \Rightarrow (b)$ follows easily.

$(b) \Rightarrow (a)$. Assume that $x_0 \in C$ is such that the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ is bounded. By Theorem 3.1.5, let $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$, and let $\{y_m\}$ be a sequence in C defined by $y_m = T^m z$ for $m = 1, 2, \cdots$.

For two integers $n > m \geq 1$, we have

$$\|x_n - y_m\| = \|T^n x_0 - T^m z\| = \|T^m(T^{n-m}x_0) - T^m z\|$$

and hence

$$r_a(y_m, \{x_n\}) \leq \eta(T^m)(r_a(z, \{x_n\}) + a_m).$$

This shows that $r_a(y_m, \{x_n\}) \to r_a(C, \{x_n\})$ as $m \to \infty$. By Theorem 3.1.8, this would imply that $T^m z \to z$ as $m \to \infty$. Because T is demicontinuous, hence by Proposition 5.6.1, $z \in F(T)$. ∎

In the following results, $WCS(X)$ plays an important role in the existence of fixed points of nearly Lipschitzian mappings.

Theorem 5.6.6 *Let X be a Banach space with $WCS(X) > 1$, C a nonempty weakly compact convex subset of X, and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{(a_n, \eta(T^n))\}$ such that $\eta(T^n) \to 1$ as $n \to \infty$. If T is weakly asymptotically regular on C, i.e., $T^n x - T^{n+1}x \rightharpoonup 0$ for all $x \in C$, then T has a fixed point in C.*

Proof. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We then define a mapping S on C by

$$Sx = w - \lim_{\mathcal{U}} T^n x, \quad x \in C.$$

Because C is weakly compact, Sx is well defined for all $x \in C$. The asymptotic nonexpansiveness of T clearly implies that S is a nonexpansive on C. Hence, S has a fixed point $v \in C$, i.e.,

$$w - \lim_{\mathcal{U}} T^n v = v.$$

This yields a subsequence $\{T^{n_i} v\}$ of $\{T^n v\}$ converging weakly to v. By the property of $WCS(X)$

$$\limsup_{i\to\infty} \|T^{n_i} v - v\| \leq \frac{1}{WCS(X)} D[\{T^{n_i} v\}]. \tag{5.56}$$

By the weak asymptotic regularity of T, we have

$$T^{n_t+p} v \rightharpoonup v \ as \ t \to \infty \text{ for any } p \geq 0.$$

On the other hand, for each $i, j \in \mathbb{N}$ with $i > j$, by the w-lsc of the norm $\|\cdot\|$ we have

$$\begin{aligned} \|T^{n_i} v - T^{n_j} v\| &\leq \eta(T^{n_j})(\|v - T^{n_i - n_j} v\| + a_{n_j}) \\ &\leq \eta(T^{n_j})(\liminf_{t\to\infty} \|T^{n_t+p} v - T^{n_i - n_j} v\| + a_{n_j}) \\ &\qquad\qquad (with\ p = n_i - n_j) \\ &\leq \eta(T^{n_j})[\eta(T^{n_i - n_j})(\liminf_{t\to\infty} \|v - T^{n_t} v\| + a_{n_i - n_j}) + a_{n_j}]. \end{aligned}$$

Taking the limit superior as $i \to \infty$, we get

$$\limsup_{i\to\infty} \|T^{n_i} v - T^{n_j} v\| \leq \eta(T^{n_j}) \limsup_{t\to\infty} \|v - T^{n_t} v\| + a_{n_j}$$

and hence from (5.56), we have

$$\begin{aligned} \limsup_{i\to\infty} \|T^{n_i} v - v\| &\leq \frac{1}{WCS(X)} D[\{T^{n_i} v\}] \\ &\leq \frac{1}{WCS(X)} \limsup_{j\to\infty} (\lim_{i\to\infty} \|T^{n_i} v - T^{n_j} v\|) \\ &\leq \frac{1}{WCS(X)} \limsup_{t\to\infty} \|v - T^{n_t} v\|, \end{aligned}$$

which implies that

$$(WCS(X) - 1) \limsup_{i\to\infty} \|T^{n_i} v - v\| \leq 0,$$

i.e., $\lim_{i\to\infty} T^{n_i} v = v$. By the demicontinuity of T, $\lim_{i\to\infty} T^{n_i} v = v$ implies $w - \lim_{i\to\infty} T^{n_i+1} v = Tv$. By weak asymptotic regularity of T, we have $T^{n_i} v - T^{n_i+1} v \to 0$ implies $w - \lim_{i\to\infty} T^{n_i+1} v = v$. Hence by the uniqueness of weak limit of $\{T^{n_i+1} v\}$, we conclude that $v = Tv$. ∎

Corollary 5.6.7 *Let X be a Banach space with $WCS(X) > 1$, C a nonempty weakly compact convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. If T is weakly asymptotically regular on C, then T has a fixed point in C.*

5.7 Pseudocontractive mappings

In this section, our aim is to study a class of continuous pseudocontractive mappings in Banach spaces. Strongly pseudocontractive mappings will play an important role in many of the existence theorems for pseudocontractive mappings.

Let X be a Banach space with dual X^*. Then a mapping T with domain $Dom(T)$ and range $R(T)$ in X is said to be *strongly pseudocontractive* if there exists a positive constant k and such that

$$\|x-y\| \leq \|(1+t)(x-y) - kt(Tx-Ty)\|$$

for all $x, y \in Dom(T)$ and all $t > 0$.

For $k = 1$, such mappings are called *pseudocontractive.*

Following Proposition 2.4.7, we are able to formulate an equivalent definition of strongly pseudocontractive mapping as follows:

The mapping $T : Dom(T) \subseteq X \to X$ is strongly pseudocontractive if for each $x, y \in Dom(T)$, there exist a positive constant k and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2.$$

It is easy to see that the mapping T is pseudocontractive if for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2. \tag{5.57}$$

We note that every nonexpansive (contraction) mapping is pseudocontractive (strongly pseudocontractive), but the converse is not true. In fact, if T is nonexpansive with domain $Dom(T)$, for each $x, y \in Dom(T)$ and $j(x-y) \in J(x-y)$, we have

$$\langle Tx - Ty, j(x-y)\rangle \quad \leq \quad \|Tx - Ty\|\|j(x-y)\|_* \leq \|x-y\|^2.$$

We now give examples of pseudocontractive mappings that are not nonexpansive.

Example 5.7.1 *Let $H = \mathbb{R}^2$ be the Hilbert space under the usual Euclidean inner product. If $x = (a, b) \in H$, we define $x^\perp = (b, -a) \in H$. Trivially, we have*

$$\begin{aligned} \langle x, x^\perp\rangle &= 0, \|x^\perp\| = \|x\|; \\ \langle x^\perp, y^\perp\rangle &= \langle x, y\rangle, \ \|x^\perp - y^\perp\| = \|x-y\|; \end{aligned}$$

and

$$\langle x, y^\perp\rangle + \langle x^\perp, y\rangle = 0 \text{ for all } x, y \in H.$$

Let C be the closed unit ball in H, $C_1 = \{x \in H : \|x\| \leq 1/2\}$ and $C_2 = \{x \in H : 1/2 \leq \|x\| \leq 1\}$.

We define the mapping $T : C \to C$ by

$$Tx = \begin{cases} x + x^{\perp} & \text{if} \quad x \in C_1, \\ x/\|x\| - x + x^{\perp} & \text{if} \quad x \in C_2. \end{cases}$$

We now show that T is Lipschitz continuous. One easily shows that

$$\|Tx - Ty\| = \sqrt{2}\|x - y\| \text{ for all } x, y \in C_1.$$

For $x, y \in C_2$, we have

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 &= \frac{2}{\|x\| \cdot \|y\|}(\|x\| \cdot \|y\| - \langle x, y \rangle) \\ &= \frac{1}{\|x\| \cdot \|y\|}(\|x - y\|^2 - (\|x\| - \|y\|)^2) \\ &\leq \frac{2}{\|x\| \cdot \|y\|}\|x - y\|^2 \\ &\leq 8\|x - y\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|Tx - Ty\| &\leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \|x - y\| + \|x^{\perp} - y^{\perp}\| \\ &\leq 5\|x - y\|. \end{aligned}$$

Now let x and y be in the interiors of C_1 and C_2, respectively. Then there exist $\lambda \in (0, 1)$ and $z \in C_1 \cap C_2$ for which $z = \lambda x + (1 - \lambda)y$. Hence

$$\begin{aligned} \|Tx - Ty\| &\leq \|Tx - Tz\| + \|Tz - Ty\| \\ &\leq \sqrt{2}\|x - z\| + 5\|z - y\| \\ &\leq 5\|x - z\| + 5\|z - y\| \\ &= 5\|x - y\|. \end{aligned}$$

Thus, $\|Tx - Ty\| \leq 5\|x - y\|$ for all $x, y \in C$, i.e., T is Lipschitzian on C.

We now show that T is pseudocontractive. For $x, y \in C$, set $\Gamma(x, y) := \|x - y\|^2 - \langle Tx - Ty, x - y \rangle$. Hence to show T is a pseudocontractive, we need to prove that $\Gamma(x, y) \geq 0$ for all $x, y \in C$. We consider the following three cases:

Case 1. $x, y \in C_1$:

Obviously, $\Gamma(x, y) \geq 0$ for all $x, y \in C_1$.

Case 2. $x \in C_1$ and $y \in C_2$:

We have

$$\begin{aligned}\Gamma(x,y) &= \|x-y\|^2 - \left\{\|x\|^2 - \|y\|^2 + \|y\| - \frac{\langle x,y\rangle}{\|y\|}\right\} \\ &= 2\|y\|^2 - \|y\| + (1-2\|y\|)\frac{\langle x,y\rangle}{\|y\|}.\end{aligned}$$

Because $1-2\|y\| \le 0$ *for* $y \in C_2$, $\langle x,y\rangle/(\|x\|\|y\|)$ *has its minimum, for fixed* $\|x\|$ *and* $\|y\|$ *when* $\langle x,y\rangle/(\|x\|\|y\|) = 1$. *We conclude that*

$$\begin{aligned}\Gamma(x,y) &\ge 2\|y\|^2 - \|y\| + \|x\| - 2\|x\|\|y\| \\ &= (\|y\| - \|x\|)(2\|y\|-1) \\ &\ge 0 \textit{ for all } x \in C_1,\ y \in C_2.\end{aligned}$$

Case 3. $x, y \in C_2$*:*

Observe that

$$\begin{aligned}\langle Tx - Ty, x-y\rangle &= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 + \left(2 - \frac{1}{\|x\|} - \frac{1}{\|y\|}\right)\langle x,y\rangle \\ &= \|x\| - \|x\|^2 + \|y\| - \|y\|^2 + (2\|x\|\|y\| - \|x\| - \|y\|)\frac{\langle x,y\rangle}{\|x\|\|y\|}.\end{aligned}$$

Hence $\Gamma(x,y) = 2\|x\|^2 + 2\|y\|^2 - \|x\| - \|y\| - (4\|x\|\|y\| - \|x\| - \|y\|)\langle x,y\rangle/(\|x\|\|y\|)$. *It is easy to see that* $4\|x\|\|y\| - \|x\| - \|y\| \ge 0$ *for all* $x, y \in C_2$. *Hence for fixed* $\|x\|$ *and* $\|y\|, \Gamma(x,y)$ *has a minimum when* $\langle x,y\rangle/(\|x\|\|y\|) = 1$. *This minimum is* $2\|x\|^2 + 2\|y\|^2 - 4\|x\|\|y\| = 2(\|x\| - \|y\|)^2$. *Thus,* $\Gamma(x,y) \ge 0$ *for all* $x, y \in C_2$. *Therefore, T is Lipschitz continuous pseudocontractive, but it is not nonexpansive.*

Example 5.7.2 *Let* $X = \mathbb{R}$ *and* $T : Dom(T) = [0,1] \to \mathbb{R}$ *be defined by*

$$Tx = (1 - x^{2/3})^{3/2}, \quad x \in [0,1].$$

Because T is monotonically decreasing, T is pseudocontractive. Observe that

$$\begin{aligned}\left|T\left(\frac{1}{4^3}\right) - T\left(\frac{1}{2^3}\right)\right| &= \left|\left(\frac{15}{16}\right)^{3/2} - \left(\frac{3}{4}\right)^{3/2}\right| \\ &= \frac{|(15)^{3/2} - (12)^{3/2}|}{64} > \frac{7}{64} = \left|\frac{1}{4^3} - \frac{1}{2^3}\right|.\end{aligned}$$

Hence T is not nonexpansive. Thus, continuous pseudocontractive is not necessarily nonexpansive.

We now establish an equivalent definition of pseudocontractive mapping in a Hilbert space.

Proposition 5.7.3 *Let H be a Hilbert space. Then following are equivalent:*

(a) T is a pseudocontractive mapping with domain $Dom(T)$.

(b) $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$ for all $x, y \in Dom(T)$.

Proof. $(a) \Rightarrow (b)$. Let T be a pseudocontractive with $Dom(T)$. Then from (5.57), we have

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \text{ for all } x, y \in Dom(T).$$

Observe that

$$\begin{aligned} \|Tx - Ty\|^2 &= \|(I - T)x - (I - T)y - (x - y)\|^2 \\ &= \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \\ &\quad -2\langle (I - T)x - (I - T)y, x - y\rangle \\ &= \|x - y\|^2 + \|(I - T)x - (I - T)\|^2 \\ &\quad -2\{\|x - y\|^2 - \langle Tx - Ty, x - y\rangle\} \\ &\leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2. \end{aligned}$$

$(b) \Rightarrow (a)$. Suppose for all $x, y \in Dom(T)$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$$

holds. Then we have

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x - y\|^2 + \|x - y - (Tx - Ty)\|^2 \\ &\leq \|x - y\|^2 + \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle, \end{aligned}$$

and it follows that $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$. ∎

Proposition 5.7.4 *Let H be a Hilbert space and T a nonlinear mapping on H with domain $Dom(T)$. Then T is strongly pseudocontractive if the following inequality is satisfied:*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \tag{5.58}$$

for all $x, y \in Dom(T)$, where $k \in (0, 1)$.

Proof. From (5.58), we have for all $x, y \in Dom(T)$

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\{\|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle\},$$

which implies that

$$\langle Tx - Ty, x - y\rangle \leq t(k)\|x - y\|^2 - \frac{1 - k}{2k}\|Tx - Ty\|^2,$$

where $t(k) = (1 + k)/(2k)$. ∎

The pseudocontractive mappings are easily seen to be more general than the nonexpansive mappings. They derive their importance in nonlinear functional analysis via their connection with an important class of nonlinear operators defined as follows:

Let X be a Banach space. An operator $A : Dom(A) \subseteq X \to X$ is said to be *accretive* if for each $x, y \in Dom(T)$ and $t > 0$, the following inequality holds:

$$\|x - y\| \leq \|x - y + t(Ax - Ay)\|. \tag{5.59}$$

An operator A is said to be *dissipative* if $-A$ is accretive and A is *expansive* if

$$\|Ax - Ay\| \geq \|x - y\| \text{ for all } x, y \in Dom(A).$$

Example 5.7.5 *Let $X = \mathbb{R}$ and $A : Dom(A) \subseteq X \to \mathbb{R}$ a real-valued increasing (nonincreasing) function. Then A is accretive (dissipative).*

By Proposition 2.4.7, we obtain

Proposition 5.7.6 *Let X be a Banach space and T a nonlinear operator on X with domain $Dom(T)$. Then the following are equivalent:*

(a) T is an accretive operator.

(b) For each $x, y \in Dom(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0.$$

Proposition 5.7.7 *Let X be a Banach space and $T : Dom(T) \subseteq X \to X$ a mapping. Then T is pseudocontractive if and only if $I - T$ is accretive.*

Proposition 5.7.8 *Let X be a Banach space and $A : Dom(T) \subseteq X \to X$ an operator. If $(I + tA)$ is expansive for all $t > 0$, then the following hold:*

(i) A is accretive.

(ii) $(I+tA)^{-1}$ exists and $(I+tA)^{-1}$ is a nonexpansive mapping from $R(I+tA)$ into $Dom(A)$.

Proof. Proposition 5.7.8 follows from (5.59). ∎

We now introduce a semigroup of type ω:

Let C be a nonempty closed subset of a Banach space X and ω a real number. A *semigroup of type* ω *on* C is a function $S : \mathbb{R}^+ \times C \to C$ satisfying the following conditions:

(i) $S(t_1 + t_2)x = S(t_1)S(t_2)x$ for $t_1, t_2 \geq 0$ and $x \in C$;

(ii) $\|S(t)x - S(t)y\| \leq e^{\omega t}\|x - y\|$ for $t \geq 0$ and $x, y \in C$;

(iii) $S(0)x = x$ for $x \in C$;

(iv) $S(t)x$ is continuous in $t \geq 0$ for each $x \in C$.

For each $t > 0$ and $x \in C$, let $A^t x = (S(t)x - x)/t$, $Dom(A) = \{x \in C : \lim_{t\to 0^+} A^t x \text{ exists}\}$ and $Ax = \lim_{t\to 0^+} A^t x$ for all $x \in Dom(A)$. Then A is called the (strong) *generator of a semigroup* S.

If $\omega = 0$, S is said to be *semigroup of nonexpansive mappings.*

Remark 5.7.9 *If C is a nonempty closed convex subset of a Banach space X and $A : C \to X$ a continuous mapping, the following are equivalent:*

(i) A is the generator of a semigroup S of type ω on C.

(ii) For $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \leq \omega\|x-y\|^2$$

and $\lim_{h\to 0^+} \dfrac{d(x+hAx, C)}{h} = 0$ *for all* $x \in C$.

The following proposition plays a key role in the existence of solutions of nonlinear operator equations.

Proposition 5.7.10 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a continuous strongly pseudocontractive mapping with constant $k \in (0,1)$ such that*

$$\lim_{h\to 0^+} \frac{d(x+hTx, C)}{h} = 0.$$

Then for each $\varepsilon > 0$ with $\varepsilon k < 1$, the range of $(I - \varepsilon T)$ contains C.

Proof. Let u be an element in C and ε a positive number such that $\varepsilon k < 1$. For each $x \in C$, define $Bx = \varepsilon Tx + u - x$. Then B is continuous on C and for $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned} \langle Bx - By, j(x-y)\rangle &= \langle \varepsilon(Tx - Ty) - (x-y), j(x-y)\rangle \\ &\leq (\varepsilon k - 1)\|x-y\|^2. \end{aligned}$$

Let $x \in C$ and for each $h > 0$, let x_h be an element of C such that

$$d(x+hTx, C) \geq \|x + hTx - x_h\| - h^2.$$

Note that

$$\lim_{h\to 0^+} \frac{\|x+hTx-x_h\|}{h} = 0.$$

Define

$$y_h = \frac{\varepsilon x_h + hu + (1-h)x}{1+\varepsilon}.$$

Because C is convex and x_h, u and x are in C, it follows that $y_h \in C$, whenever $h \in (0,1)$. Hence if $h \in (0,1)$ and $\lambda = (1+\varepsilon)^{-1}h$, we have

$$\begin{aligned} d(x+\lambda Bx, C) &\leq \|x+\lambda Bx - y_h\| \\ &= \|[(1+\varepsilon)x + h\varepsilon Tx + hu - hx] \\ &\quad -[\varepsilon x_h + hu + (1-h)x]\|(1+\varepsilon)^{-1} \\ &= \varepsilon(1+\varepsilon)^{-1}\|x + hTx - x_h\|. \end{aligned}$$

Hence $\lim\limits_{\lambda\to 0^+} d(x+\lambda Bx, C)/\lambda = 0$. So B is the generator of a semigroup V of type $(k\varepsilon - 1)$ on C. Because

$$\|V(t)x - V(t)y\| \leq e^{(k\varepsilon-1)t}\|x-y\| \text{ for } t>0,$$

$V(t)$ is a contraction from C into itself. Hence $V(t)$ has a unique fixed point $x_t \in C$. Because

$$V(s)x_t = V(s)V(t)x_t = V(t)V(s)x_t,$$

there is a unique point $z \in C$ such that $V(t)z = z$ for all $t \geq 0$. Thus $Bz = 0$ and $z - \varepsilon Tz = u$. Therefore, the range of $(I - \varepsilon T)$ contains C. ∎

The following example shows that convexity of C cannot be removed from Proposition 5.7.10.

Example 5.7.11 *Let $X = \mathbb{R}^2$ be the Euclidean space, $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $A(x,y) = (y,-x)$ for all $(x,y) \in C$. Then A is generator of a semigroup U of type 0 on C (in particular, $U(t)(x,y) = (x\ \cos t + y\ \sin t, -x\ \sin t + y\ \cos\ t)$), but the image of C under $I - \varepsilon A$ does not intersect C for any $\varepsilon > 0$.*

We now introduce more general classes of nonlinear operators:

Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\alpha(0) = 0$ and $\liminf\limits_{r\to r_0} \alpha(r) > 0$ for all $r_0 > 0$.

A mapping $A : C \subseteq X \to X$ is said to be *α-strongly accretive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha(\|x-y\|)\|x-y\|. \tag{5.60}$$

A mapping T is said to be *α-strongly pseudocontractive* if $I - T$ is α-strongly accretive.

If $\alpha(r) = kr$ for some $k > 0$, then A is strongly accretive (with strongly accretive constant k) as in this case (5.60) reduces to

$$\langle Ax - Ay, j(x-y)\rangle \geq k\|x-y\|^2$$

and $T = I - A$ $(k \in (0,1))$ is strongly pseudocontractive (with strongly pseudocontractive constant $1-k$) as in this case (5.60) takes the form of

$$\langle Tx - Ty, j(x-y)\rangle \leq (1-k)\|x-y\|^2.$$

One may easily see that the pseudocontractivity of T implies the strong pseudocontractivity of tT for $t \in (0,1)$. Also, the accretiveness of A implies the strong accretiveness of $tI + A$ for $t > 0$.

Note that every continuous strongly accretive self-mapping on X is surjective.

The following proposition shows that the sum of accretive and strongly accretive mappings is strongly accretive.

Proposition 5.7.12 *Let X be a smooth Banach space, and $T : X \to X$ a strongly accretive mapping with the strongly accretive constant $k \in (0,1)$. Let $S : X \to X$ be an accretive mapping. Then $T + S : X \to X$ is also a strongly accretive mapping with the strongly accretive constant k.*

Proof. Because S is accretive and T is strongly accretive with the strongly accretive constant k, then for any $x, y \in X$, we have

$$\langle Sx - Sy, J(x-y)\rangle \geq 0 \text{ and } \langle Tx - Ty, J(x-y)\rangle \geq k\|x-y\|^2.$$

Hence

$$\begin{aligned}\langle (T+S)x - (T+S)y, J(x-y)\rangle &= \langle Tx - Ty, J(x-y)\rangle + \langle Sx - Sy, J(x-y)\rangle \\ &\geq k\|x-y\|^2. \quad \blacksquare\end{aligned}$$

Applying Proposition 5.7.10, we obtain the existence of zeros for continuous and α-strongly accretive mappings.

Theorem 5.7.13 *Let C be a nonempty closed convex subset of a Banach space X and $A : C \to X$ a continuous and α-strongly accretive mapping such that*

(i) $\liminf_{r\to\infty} \alpha(r) > \|Ax_0\|$ *for some* $x_0 \in C$,

(ii) $\lim_{h\to 0^+} \dfrac{d(x - hAx, C)}{h} = 0$ *for all* $x \in C$.

Then A has a unique zero in C.

Proof. We may assume that $x_0 = 0$. Proposition 5.7.10 implies that $C \subset (I + A)(C)$. Because $(I + A)$ is invertible, the mapping $g = (I + A)^{-1}$ is a nonexpansive self-mapping on C and the fixed points of g are zeros of A. It suffices to show that g has a fixed point.

Let $D = \{x \in C : Ax = tx \text{ for some } t < 0\}$. Then D is bounded. Indeed, for $x \in D$, we have

$$Ax = tx \text{ for some } t < 0 \text{ and } \langle Ax - A0, j\rangle \geq \alpha(\|x\|)\|x\| \text{ for some } j \in J(x),$$

which imply that

$$\begin{aligned}\alpha(\|x\|)\|x\| &\leq \|Ax\|\|x\| + \|A0\|\|x\| \\ &= t\|x\|^2 + \|A0\|\|x\|.\end{aligned}$$

Because $t < 0, \alpha(\|x\|) < \|A0\|$. It follows that D is bounded.

Similarly, one can show that $E = \{y \in C : g(y) = \lambda y \text{ for some } \lambda > 1\}$ is also bounded.

Next, we show that $(I - g)(C)$ is a closed set of X. For this, let $\{y_n\}$ be a sequence in C such that $y_n - g(y_n) \to u$ for some $u \in X$. Set $x_n := g(y_n)$. Then

$$y_n - g(y_n) = Ax_n \to u \text{ as } n \to \infty.$$

Because for $m, n \in \mathbb{N}$, there exists $j(x_n - x_m) \in J(x_n - x_m)$ such that

$$\langle Ax_n - Ax_m, j(x_n - x_m)\rangle \geq \alpha(\|x_n - x_m)\|x_n - x_m\|,$$

this yields

$$\alpha(\|x_n - x_m\|) \leq \|Ax_n - Ax_m\|.$$

Hence $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$ for some $x \in C$. By the continuity of $I + A$, $y_n \to y$ for some $y \in C$. Thus, $(I - g)(y) = u$, i.e., $(I - g)(C)$ is closed.

Now, let $\{t_n\}$ be a sequence in $(0, 1)$ with $t_n \to 1$. Then $t_n g(y_n) = y_n$ for some $y_n \in C$, and it follows that

$$y_n - g(y_n) = (1 - t_n^{-1})y_n.$$

Because $\{y_n\}$ is in E and E is bounded, $y_n - g(y_n) \to 0 \in (I - g)(C)$. Therefore, g has a fixed point in C. ∎

Note that the mapping A satisfies condition (ii) of Theorem 5.7.13 if and only if $I - A$ is weakly inward on C. Therefore, Theorem 5.7.13 yields the following useful existence results for strongly pseudocontractive mappings in Banach spaces.

Corollary 5.7.14 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a weakly inward continuous α-strongly pseudocontractive mapping. Then T has a unique fixed point in C.*

Corollary 5.7.15 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a continuous α-strongly pseudocontractive mapping. Then T has a unique fixed point in C.*

We now give fundamental properties and existence results for pseudocontractive mappings in Banach spaces.

Proposition 5.7.16 *Let C be a nonempty subset of a Banach space X and $T : C \to X$ a continuous pseudocontractive mapping. Let $A_T : C \to X$ be a mapping defined by $A_T := I + r(I - T)$ for any $r > 0$. Then we have the following:*

(a) A_T is one-one and A_T^{-1} is nonexpansive.

(b) $F(T) = F(A_T^{-1})$.

(c) If C is closed, then $A_T(C)$ is closed.

(d) If C is closed and convex and T is weakly inward, then the range of A_T contains C, i.e., $C \subset A_T(C)$.

Proof. (a) By pseudocontractivity of T,

$$\|x-y\| \le \|[I+r(I-T)]x-[I+r(I-T)]y\| = \|A_Tx-A_Ty\| \text{ for all } x,y\in C,$$

and it follows that A_T is one-one. Therefore, A_T^{-1} is nonexpansive.

(b) and (c) are obvious.

(d) Let z be a point in C. Then it suffices to show that there exists $x \in C$ such that $z = A_Tx$. Define $g : C \to X$ by $g(x) = (1+r)^{-1}(rTx+z)$. Then g is weakly inward and continuous. Let $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \le \|x-y\|^2,$$

which implies that

$$\begin{aligned} \langle g(x)-g(y), j(x-y)\rangle &= \frac{r}{1+r}\langle Tx-Ty, j(x-y)\rangle \\ &\le \frac{r}{1+r}\|x-y\|^2. \end{aligned}$$

Then g is continuous and a $r/(1+r)$-strongly pseudocontractive mapping. By Corollary 5.7.14, there exists $x \in C$ with $g(x) = x$, i.e., $z = A_T(x)$. ∎

Theorem 5.7.17 *Let X be a uniformly convex Banach space satisfying the Opial condition. Let C be a nonempty closed convex subset of X and $T : C \to X$ a weakly inward continuous pseudocontractive mapping. Then $I - T$ is demi-closed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup z$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By Theorem 3.2.9, we have $\mathcal{Z}_a(C, \{x_n\}) = \{z\}$.

Let $A_T : C \to X$ be a mapping defined by $A_T := I + r(I-T)$ for any $r > 0$. Then Proposition 5.7.16 (d) implies that $C \subset A_T(C)$ and because A_T is one-one, we conclude that $g : C \to C$ defined by $g = A_T^{-1}$ is nonexpansive. Because $A_T(x_n) = x_n + r(x_n - Tx_n)$, it follows that $x_n = g(x_n + r(x_n - Tx_n))$. Now

$$\begin{aligned} \|x_n - g(x_n)\| &= \|g(x_n + r(x_n - Tx_n)) - g(x_n)\| \\ &\le r\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Because

$$\begin{aligned} r_a(g(z), \{x_n\}) &= \limsup_{n\to\infty} \|x_n - g(z)\| \\ &\le \limsup_{n\to\infty} \|g(x_n) - g(z)\| \le \limsup_{n\to\infty} \|x_n - z\|, \end{aligned}$$

it follows that $g(z) = z$. ∎

We now give existence results for pseudocontractive mappings.

Proposition 5.7.18 *Let $\{x_n\}$ be a bounded sequence in a Hilbert space H and $\{r_n\}$ a strictly decreasing sequence in $\mathbb{R}^+$ such that*

$$\langle r_n x_n - r_m x_n, x_n - x_m\rangle \leq 0 \text{ for all } m, n \in \mathbb{N}.$$

Then there exists $x \in H$ such that $x_n \to x$.

Proof. Observe that

$$2\langle r_n x_n - r_m x_n, x_n - x_m\rangle = (r_n + r_m)\|x_n - x_m\|^2 + (r_n - r_m)(\|x_n\|^2 - \|x_m\|^2)$$

for all $m, n \in \mathbb{N}$. This implies that $\{x_n\}$ is a Cauchy sequence and hence there exists $x \in H$ such that $x_n \to x$. ∎

Theorem 5.7.19 *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to H$ a weakly inward continuous pseudocontractive mapping. Then T has a fixed point.*

Proof. Let u be an element in C and let $\{t_n\}$ be a strictly increasing sequence in $(0, 1)$ with $t_n \to 1$. Define a mapping $T_n : C \to X$ by

$$T_n x = (1 - t_n)u + t_n Tx, \ x \in C, \ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, T_n is a continuous strongly pseudocontractive mapping and also T_n is weakly inward because C is convex. By Corollary 5.7.14, there exists exactly one point $x_n \in C$ such that $x_n = (1 - t_n)u + t_n Tx_n$.

Set $r_n = t_n^{-1} - 1$. Then

$$\begin{aligned}\langle r_n x_n - r_m x_m, x_n - x_m\rangle &= (r_n - r_m)\langle u, x_n - x_m\rangle + \langle Tx_n - Tx_m \\ &\quad -(x_n - x_m), x_n - x_m\rangle \\ &\leq (r_n - x_m)\langle u, x_n - x_m\rangle.\end{aligned}$$

Without loss of generality, we may assume that $u = 0$. By Proposition 5.7.18, $x_n \to x$. It follows from the fact $x_n - Tx_n \to 0$ and continuity of T that $x = Tx$. ∎

Theorem 5.7.20 *Let X be a uniformly convex Banach space and C a nonempty closed convex subset of X (with $0 \in C$). Let $T : C \to X$ be a weakly inward continuous pseudocontractive mapping. Then T has fixed point in C if and only if the set $E = \{x \in C : Tx = \lambda x \text{ for some } \lambda > 1\}$ is bounded.*

Proof. Suppose that E is bounded. Define a mapping $A_T : C \to X$ by $A_T := I + r(I - T)$ for any $r > 0$. Proposition 5.7.16 implies that $C \subset A_T(C)$ and because A_T is one-one, we conclude that $g : C \to C$ defined by $g = A_T^{-1}$ is nonexpansive.

Now, by Theorem 5.3.5, it suffices to show that the set

$$D = \{y \in C : g(y) = \mu y \text{ for some } \mu > 1\}$$

is bounded. Suppose that $g(y) = \mu y$ for $\mu > 1$. Select $x \in C$ such that $y = A_T x$. Then $Tx = (1 + (\mu - 1)/(\mu r))x$, i.e., $x \in E$. Because $x = \mu y$, it follows that D is bounded.

Conversely, suppose that v is a fixed point of T and $x \in E$. Then $Tx = \lambda x$ for some $\lambda > 1$. By the pseudocontractivity of T,

$$\|x - v\| \leq \|(1 + r)(x - v) - r(\lambda x - v)\| = \|(1 + r - r\lambda)x - v\|.$$

By choosing $r = (\lambda - 1)^{-1}$, we have $\|x - v\| \leq \|v\|$. Therefore, E is bounded. ∎

Corollary 5.7.21 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a Lipschitzian pseudocontractive mapping. Suppose C has the fixed point property for nonexpansive mappings. Then T has a fixed point in C.*

Theorem 5.7.22 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a weakly inward continuous pseudocontractive mapping. Suppose C has the fixed point property for nonexpansive self-mappings. Then T has a fixed point in C.*

Proof. Note the mapping $g : C \to C$ defined by $g := A_T^{-1}$ is nonexpansive by Proposition 5.7.16. Therefore, there exists $v \in C$ such that $v = A_T v = v + r(v - Tv)$ from which $v \in F(T)$. ∎

Corollary 5.7.23 *Let C be a nonempty closed convex subset of a Banach space and $T : C \to X$ a weakly inward nonexpansive mapping. Suppose C has the fixed point property for nonexpansive self-mappings. Then T has a fixed point in C.*

Finally, we discuss the structure of the set of fixed points of pseudocontractive mappings.

Theorem 5.7.24 *Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to X$ a weakly inward continuous pseudocontractive mapping. Then $F(T)$ is closed and convex.*

Proof. Proposition 5.7.16 implies that the mapping $g := A_T^{-1}$ is nonexpansive and $F(T) = F(g)$. Hence $F(T)$ is closed and convex by Corollary 5.2.29. ∎

Bibliographic Notes and Remarks

The existence theorem for single-valued non-self contraction mappings in the form presented here is due to Caristi [35], but the techniques for existence results for multivalued non-self contraction mappings are based on those of Downing and Kirk [49], Mizoguchi and Takahashi [110], Xu [166], and Yi and Zhao [170].

The properties of nonexpansive mappings presented in Section 5.2 are based on Bruck [32] and Goebel and Kirk [59]. An extension of Theorem 5.2.7 can be found in Baillon, Reich, and Bruck [8]. The first fixed point theorem in the noncompact setting for nonexpansive mappings was proved by Browder [26] in Hilbert space in 1965. Theorem 5.2.16 was proved independently by Browder [27] and Göhde [61] in 1965. Theorem 5.2.18 was proved in Kirk [88] in the same year in a slightly more general situation. Theorem 5.2.23 and 5.2.25 are due to Singh and Watson [149]. Theorem 5.2.26 is based on the interesting result of Caristi [35].

Different techniques for solving operator equation (5.30) presented in Section 5.3 are based on the results of Downing and Kirk [49], Lami Dozo [50], Morales [112], and Yi and Zhao [170]. Theorem 5.3.6 can also be found in Reich [124]. Theorems 5.4.7 and 5.5.3 were proved by Goebel and Kirk in [57] and [58], respectively. Other important results presented in Sections 5.4∼5.5 can be found in Benavides and Xu [17], Casini and Maluta [36], Gornicki [62], Lim and Xu [98], and Xu [164].

Section 5.6 contains some new results for non-Lipschitzian mappings. Corollary 5.6.7 is the main result of Benavides, Acedo, and Xu [16].

Example 5.7.1 was constructed in Chidume and Mutangadura [39]. An important result Proposition 5.7.10 is due to Martin [105]. Other results presented in Section 5.7 are discussed in Goebel and Reich [60], Khamsi and Kirk [85], Kirk and Schoneberg [90], Molales [113], and Reinermann and Schoneberg [134].

The approximate fixed point property plays an important role in the fixed point theory of nonexpansive, asymptotically nonexpansive, and pseudocontractive mappings. This property is also described in Matoušková and Reich [109], Reich [129], and Shafrir [143].

Exercises

5.1 Let C be a nonempty closed subset of a Banach space X and $T : C \to X$ a continuous mapping with the property that for each $x \in C$, there is an α_x, $0 < \alpha_x \leq 1$ such that $(1-\alpha_x)x + \alpha_x Tx \in C$. Let $x_1 \in C$ and inductively for $n \in \mathbb{N}$ define $x_{n+1} \in C$ by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n$, where $\alpha_n \in (0,1]$ is chosen so that $x_{n+1} \in C$. Show that

(a) if $z = \lim_{n\to\infty} x_n$ exists and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then z is a fixed point of T,

(b) if T is a contraction mapping, then $z = \lim_{n\to\infty} x_n$ exists.

5.2 Let X be a Banach space and $T : X \to X$ a mapping satisfies the condition:

$$\|Tx - Ty\| \leq \varphi(\|x-y\|) \text{ for all } x, y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function and $\varphi(t) < t$ for $t > 0$. Show that $I - T$ is bijective and $(I-T)^{-1}$ is continuous on X.

5.3 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a weakly inward contraction mapping. Show that T has a unique fixed point in C.

5.4 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. If $A = I - T$, show that $R(I + \lambda A) \supset C$ for every $\lambda > 0$.

5.5 Let C be a nonempty subset of a normed space X. A function α of $C \times C \to [0, \infty)$ is symmetric if $\alpha(x, y) = \alpha(y, x)$ for all $x, y \in C$. Let $T : C \to C$ be a mapping. T is said to be generalized nonexpansive if there exist symmetric functions $\alpha_i, i = 1, 2, 3, \cdots, 5$ of $C \times C$ into $[0, \infty)$ such that $\sup\left\{\sum_{i=1}^{5} \alpha_i(x, y) : x, y \in C\right\} \leq 1$ and for all x, y in X,

$$\|Tx-Ty\| \leq a_1\|x-y\|+a_2\|x-Ty\|+a_3\|y-Tx\|+a_4\|x-Tx\|+a_5\|y-Ty\|,$$

where $a_i = \alpha_i(x, y)$.

If C is a nonempty convex subset of a uniformly convex Banach space X, and $T : C \to C$ is a generalized nonexpansive mapping with $F(T) \neq \emptyset$, show that for each $t \in (0, 1)$, the mapping defined by

$$T_t x = (1 - t)x + tTx, x \in C$$

is asymptotically regular.

5.6 Let X be a strictly convex Banach space and C a weakly compact convex subset of X that has normal structure. Let $\mathcal{S} = \{T_1, T_2, \cdots, T_n\}$ be a finite commuting family of nonexpansive mappings of C into itself. Show that $\cap_{i=1}^{n} F(T_i) \neq \emptyset$.

5.7 Let X be a Banach space, C a nonempty closed bounded subset of X that is star-shaped with respect to 0, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ and uniformly asymptotically regular (i.e., for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|T^n x - T^{n+1}x\| \leq \varepsilon$ for all $n \geq n_0$ and all $x \in C$). Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. Show that

(a) for each $n \in \mathbb{N}$, there exists exactly one $x_n \in C$ such that $x_n = (\lambda_n/k_n)T^n x_n$,

(b) $x_n - Tx_n \to 0$ as $n \to \infty$.

5.8 Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. T is said to be weakly asymptotically semicontractive if there exist a mapping $S : C \times C \to C$ and a sequence $\{k_n\} \subset [1, \infty)$ such that $Tx = S(x, x)$ for all $x \in C$ while for each fixed $x \in C, S(\cdot, x)$ is asymptotically nonexpansive with sequence $\{k_n\}$ and for fixed $x \in C$ and fixed $n \in \mathbb{N}$, the mapping $y \to S(\cdot, y)^n x$ is compact on C.

If X is a reflexive Banach space possessing a weakly continuous duality mapping, C is a nonempty closed convex bounded subset of X, and $T : C \to C$ is weakly asymptotically semicontractive with data $(S, \{k_n\})$ and satisfies the condition:

for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $z \in C$,

$$\|S(\cdot, z)^{n+1}(z) - S(\cdot, z)^n(z)\| < \varepsilon,$$

show that

(a) $\inf_{x \in C} \{\|x - Tx\|\} = 0$,

(b) if $(I - T)(C)$ is closed, it follows that $F(T) \neq \emptyset$.

5.9 Let X be a Banach space with GGLD and the Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Show that $I - T$ is demiclosed at 0.

5.10 Let $X = \mathbb{R}^2$ with the ℓ_1 norm, i.e., $\|(x, y)\| = |x| + |y|$. Let A and B be operators in X defined by

$$D(A) = D(B) = \{(0, 0), (0, 1)\}$$

and

$$A(0, 0) = (0, 0) = B(0, 1), A(0, 1) = (1, 1/2), B(0, 0) = (1, 1).$$

Show that both A and B are accretive, but $A + B$ is not.

Chapter 6

Approximation of Fixed Points

The purpose of this chapter is to develop iterative techniques for approximation of fixed points of nonlinear mappings by using the Picard, Mann, and Ishikawa iteration processes.

6.1 Basic properties and lemmas

In this section, we develop preliminary results for approximation of fixed points of nonlinear mappings.

Proposition 6.1.1 *Let X be a Banach space satisfying the Opial condition, C a nonempty weakly compact subset of X, and $T : C \to C$ a mapping such that*

(i) $F(T) \neq \emptyset$,

(ii) $I - T$ is demiclosed at zero.

Let $\{x_n\}$ be a sequence in C satisfying the following properties:

(D_1) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$;

(D_2) $\{x_n\}$ is an AFPS, i.e., $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Because C is weakly compact, it follows that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to p. Because $\{x_{n_j}\} \subset C$ and C is weakly closed, then $p \in C$. From (D_2), $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and because $I - T$ is demiclosed at zero, we have $(I - T)p = 0$, so that $p \in F(T)$. To complete the proof, we show that $\{x_n\}$ converges weakly to a fixed point of T; it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, p. Suppose there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $q \neq p$. As in the case of p, we must have $q \in C$ and $q \in F(T)$. It follows

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from (D_1) that $\lim_{n\to\infty} ||x_n - p||$ and $\lim_{n\to\infty} ||x_n - q||$ exist. Because X satisfies the Opial condition, we have

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{j\to\infty} ||x_{n_j} - p|| < \lim_{j\to\infty} ||x_{n_j} - q|| = \lim_{n\to\infty} ||x_n - q||,$$

$$\lim_{n\to\infty} ||x_n - q|| = \lim_{k\to\infty} ||x_{n_k} - q|| < \lim_{k\to\infty} ||x_{n_k} - p|| = \lim_{n\to\infty} ||x_n - p||,$$

a contradiction. Hence $p = q$ and $\{x_n\}$ converges weakly to p. ∎

Proposition 6.1.2 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a mapping such that*

(i) $F(T) \neq \emptyset$,

(ii) $I - T$ is demiclosed at zero.

Let $\{x_n\}$ be a sequence in C that satisfies properties (D_1) and (D_2). Suppose $\{x_n\}$ holds one of the following conditions:

(a) X is uniformly convex with Fŕechet differentiable norm and

$$\lim_{n\to\infty} \langle x_n, J(p-q)\rangle \text{ exists for all } p, q \in F(T). \tag{6.1}$$

(b) X is reflexive, X^* has the Kadec-Klee property, and $\lim_{n\to\infty} \|tx_n+(1-t)p-q\|$ exists for all $t \in [0, 1]$ and for some $p, q \in \omega_w(\{x_n\})$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. We show that $\omega_w(\{x_n\})$ has exactly one point. Let $u, v \in \omega_w(\{x_n\})$ with $u \neq v$. Then for some subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, we have $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. By (D_2), $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, which implies by the demiclosedness of $I - T$ at zero that $u, v \in \omega_w(\{x_n\}) \subset F(T)$.

(a) From (6.1), we have

$$\langle u, J(p-q)\rangle = d \text{ (say)}, \quad \text{and } \langle v, J(p-q)\rangle = d;$$

so

$$\langle u - v, J(p-q)\rangle = 0 \text{ for all } p, q \in F(T). \tag{6.2}$$

From (6.2) we obtain that

$$\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0,$$

a contradiction. Hence $\omega_w(\{x_n\})$ is singleton. Therefore, $\{x_n\}$ converges weakly to a fixed point of T.

(b) By assumption, $\lim_{n\to\infty} \|tx_n+(1-t)u-v\|$ exists. Corollary 2.4.17 guarantees that $u = v$. Hence $\omega_w(\{x_n\})$ is singleton. Therefore, $\{x_n\}$ converges weakly to a fixed point of T. ∎

Proposition 6.1.3 *Let X be a reflexive Banach space, C a nonempty closed convex subset of X, $\{x_n\}$ a bounded sequence in C, and $T : C \to C$ a nonexpansive mapping. Suppose $\{x_n\}$ satisfies one of the following conditions:*

(D_2) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$

(D_3) $\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0.$

Then the set M defined by (2.32) is a nonempty closed convex bounded and T-invariant subset of C.

Proof. Define a real-valued function φ on C by $\varphi(z) = LIM_n\|x_n - z\|^2$ for each $z \in C$. Theorem 2.9.11 implies that M is a nonempty closed convex bounded set. If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then for $y \in M$

$$\begin{aligned}\varphi(Ty) &= LIM_n\|x_n - Ty\|^2 \le LIM_n\|Tx_n - Ty\|^2 \\ &\le LIM_n\|x_n - y\|^2 = \varphi(y).\end{aligned}$$

Hence $Ty \in M$, i.e., M is T-invariant.

Suppose now that $\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0$. Observe that for $y \in M$

$$\begin{aligned}\varphi(Ty) &= LIM_n\|x_n - Ty\|^2 \\ &= LIM_n\|x_{n+1} - Ty\|^2 \ \ (\text{as } LIM_n(a_n) = LIM_n(a_{n+1})) \\ &\le LIM_n\|Tx_n - Ty\|^2 \\ &\le LIM_n\|x_n - y\|^2 = \varphi(y).\end{aligned}$$

Hence M is T-invariant. ∎

We now give some useful lemmas:

Lemma 6.1.4 *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Suppose that $\beta_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$. Then $\liminf_{n\to\infty} \beta_n = 0$.*

Proof. Suppose, for contradiction, that $\liminf_{n\to\infty} \beta_n = \delta$ for some $\delta > 0$. Then there exists $n_0 \in$ such that $\beta_n > \delta/2$ for all $n \ge n_0$. Then

$$\sum_{n\ge n_0} \alpha_n\beta_n > \frac{\delta}{2}\sum_{n\ge n_0} \alpha_n = \infty,$$

a contradiction. Therefore, $\delta = 0$. ∎

Lemma 6.1.5 *Let $\{\delta_n\}$ be a sequence of nonnegative numbers satisfying:*

$$\delta_{n+1} \le \beta_n\delta_n + \gamma_n \text{ for all } n \in \mathbb{N},$$

where $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences of nonnegative numbers such that*

$$\{\beta_n\} \subseteq [1,\infty), \quad \sum_{n=1}^{\infty}(\beta_n - 1) < \infty, \tag{6.3}$$

$$\sum_{n=1}^{\infty}\gamma_n < \infty. \tag{6.4}$$

Then $\lim_{n\to\infty}\delta_n$ *exists. If* $\liminf_{n\to\infty}\delta_n = 0$*, then* $\lim_{n\to\infty}\delta_n = 0$.

Proof. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
\delta_{n+m+1} &\le \beta_{n+m}\delta_{n+m} + \gamma_{n+m} \\
&\le \beta_{n+m}(\delta_{n+m} + \gamma_{n+m}) \\
&\le \beta_{n+m}(\beta_{n+m-1}(\delta_{n+m-1} + \gamma_{n+m-1}) + \gamma_{n+m}) \\
&\cdots \\
&\le \left(\prod_{i=n}^{n+m}\beta_i\right)\left(\delta_n + \sum_{i=n}^{n+m}\gamma_i\right).
\end{aligned}$$

Hence

$$\limsup_{m\to\infty}\delta_m \le \left(\prod_{i=n}^{\infty}\beta_i\right)\left(\delta_n + \sum_{i=n}^{\infty}\gamma_i\right). \tag{6.5}$$

By the conditions (6.3) and (6.4), we have $\lim_{n\to\infty}\left(\prod_{i=n}^{\infty}\beta_i\right) = 1$ and $\lim_{n\to\infty}\sum_{i=n}^{\infty}\gamma_i = 0$. It follows from (6.5) that $\limsup_{n\to\infty}\delta_n \le \liminf_{n\to\infty}\delta_n$. Therefore, $\lim_{n\to\infty}\delta_n$ exists. Suppose $\liminf_{n\to\infty}\delta_n = 0$. Then $\lim_{n\to\infty}\delta_n = \liminf_{n\to\infty}\delta_n = 0$. ∎

Lemma 6.1.6 *Let* $\{a_n\}$, $\{b_n\}$, *and* $\{t_n\}$ *be three sequences of nonnegative numbers such that*

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n, \quad n \in \mathbb{N}, \tag{6.6}$$

where $t_n \in [0,1], \sum_{n=1}^{\infty} t_n = \infty$ *and* $\lim_{n\to\infty} b_n = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Proof. By (6.6),

$$0 \le a_{n+1} \le \prod_{i=k}^{n}(1 - t_i)a_k + \sum_{i=k}^{n}\left[t_i \prod_{j=i+1}^{n}(1 - t_j)\right]b_i. \tag{6.7}$$

Observe that

$$\sum_{i=k}^{n} t_i \prod_{j=i+1}^{n}(1 - t_j) \le 1 \text{ for all } n, k \in \mathbb{N} \text{ and } \prod_{i=k}^{n}(1 - t_i) \le exp\left(-\sum_{i=k}^{n} t_i\right) \to 0.$$

Given $\varepsilon > 0$, pick k such that $b_i \le \varepsilon$ for all $i \ge k$, from (6.7), we have

$$0 \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} a_n = 0$. ∎

Lemma 6.1.7 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a uniformly convex Banach space X such that*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n \text{ and } \|y_n\| \le \|x_n\|, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence of nonnegative numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \min\{\alpha_n, 1-\alpha_n\} = \infty$. Then $0 \in \overline{\{x_n - y_n\}}$.

Proof. Suppose, for contradiction, that $\|x_n - y_n\| \ge \varepsilon > 0$ for all $n \in \mathbb{N}$. Observe that

$$\|x_{n+1}\| \le \|x_n\| \le \cdots \le \|x_1\| \text{ for all } n \in \mathbb{N}.$$

Then

$$\begin{aligned} \|x_{n+1}\| &= \|(1-\alpha_n)x_n + \alpha_n y_n\| \\ &\le \|x_n\|\left[1 - 2\min\{\alpha_n, 1-\alpha_n\}\delta_X\left(\frac{\varepsilon}{\|x_1\|}\right)\right]. \end{aligned}$$

Inductively, we have

$$\|x_n\| \le \prod_{i=1}^{n-1}\left[1 - 2\min\{\alpha_i, 1-\alpha_i\}\delta_X\left(\frac{\varepsilon}{\|x_1\|}\right)\right]\|x_1\| \text{ for all } n > 1.$$

Because $\sum_{i=1}^{\infty} \min\{\alpha_i, 1-\alpha_i\} = \infty$, it follows that

$$\|x_n\| \le \|x_1\| exp\left(-2\delta_X\left(\frac{\varepsilon}{\|x_1\|}\right)\sum_{i=1}^{n-1}\min\left\{\alpha_i, 1-\alpha_i\right\}\right) \to 0 \text{ as } n \to \infty$$

and hence $\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\| = 0$. This is a contradiction. ∎

Lemma 6.1.8 *Let C be a nonempty closed convex bounded subset of a Banach space X and $\{T_n\}$ a sequence of Lipschitzian self-mappings of C such that*

(i) $L_n(\ge 1)$ is the Lipschitz constant of T_n with $\sum_{n=1}^{\infty}(L_n - 1) < \infty$,

(ii) $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n)$, the set of common fixed points of $\{T_n\}$ is nonempty.

For a given $x_1 \in C$, define the sequence $\{x_n\}$ by

$$x_{n+1} = T_n x_n, \quad n \in \mathbb{N}. \tag{6.8}$$

Then the following hold:

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in \mathcal{F}$.

(b) If X *is uniformly convex, then* $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ *exists for all* $f_1, f_2 \in \mathcal{F}$, *and* $t \in [0,1]$.

(c) If X *is uniformly convex with Fréchet differentiable norm, then* $\lim_{n\to\infty} \langle x_n - p, J(p-q)\rangle$ *exists for all* $p, q \in \mathcal{F}$.

Proof. (a) Let $p \in \mathcal{F}$. Then from (6.8), we have

$$\|x_{n+1} - p\| = \|T_n x_n - p\| \le L_n \|x_n - p\|, \quad n \in \mathbb{N},$$

and it follows from Lemma 6.1.5 that $\lim_{n\to\infty} \|x_n - p\|$ exists, because $\sum_{n=1}^{\infty} (L_n - 1) < \infty$.

(b) Let $p, q \in \mathcal{F}$. Set

$$\begin{aligned} a_n(t) :&= \|tx_n + (1-t)p - q\|, \\ S_{n,m} :&= T_{n+m-1}T_{n+m-2}\cdots T_n, \\ b_{n,m} &= \|S_{n,m}(tx_n + (1-t)p) - (tx_{n+m} + (1-t)p)\|. \end{aligned}$$

We show $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$. Note that $\lim_{n\to\infty} a_n(0)$ and $\lim_{n\to\infty} a_n(1)$ exist, so it remains to show that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$.

Observe that for $x, y \in C$

$$\begin{aligned} x_{n+m} &= S_{n,m}x_n; \\ \|S_{n,m}x - S_{n,m}y\| &\le \left(\prod_{i=n}^{n+m-1} L_i\right)\|x-y\| \le \left(\prod_{i=n}^{\infty} L_i\right)\|x-y\|; \end{aligned}$$

and

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\le \|tx_{n+m} + (1-t)p - S_{n,m}(tx_n + (1-t)p)\| \\ &\quad + \|S_{n,m}(tx_n + (1-t)p) - q\| \\ &\le b_{n,m} + \left(\prod_{i=n}^{n+m-1} L_i\right)\|tx_n + (1-t)p - q\| \\ &\le b_{n,m} + \left(\prod_{i=n}^{\infty} L_i\right)a_n(t) = b_{n,m} + H_n a_n(t), \end{aligned} \tag{6.9}$$

where $H_n = \prod_{i=n}^{\infty} L_i$.

By Theorem 5.2.31, there exists a strictly increasing continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ such that

$$\gamma(\|S(tx + (1-t)y) - (tSx + (1-t)Sy)\|) \le \|x-y\| - \|Sx - Sy\|$$

for all nonexpansive $S : C \to X$ and $t \in [0,1]$. It then follows that

$$\begin{aligned}\gamma(H_n^{-1} b_{n,m}) &= \gamma(H_n^{-1}\|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)p)\|) \\ &\leq \|x_n - p\| - H_n^{-1}\|x_{n+m} - p\|,\end{aligned}$$

which implies that

$$b_{n,m} \leq H_n \gamma^{-1}(\|x_n - p\| - H_n^{-1}\|x_{n+m} - p\|). \tag{6.10}$$

Because $\lim_{n\to\infty} \|x_n - p\|$ exists by part (a) and $\lim_{n\to\infty} H_n = 1$, then from (6.10) we obtain that $\lim_{n,m\to\infty} b_{n,m} = 0$. Hence from (6.9), we have

$$\limsup_{m\to\infty} a_m(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} H_n a_n(t) = \liminf_{n\to\infty} a_n(t).$$

Therefore, $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$.

(c) Because the norm of X is Fréchet differentiable,

$$\frac{1}{2}\|x\|^2 + \langle h, Jx\rangle \leq \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, Jx\rangle + b(\|h\|) \tag{6.11}$$

for all bounded $x, h \in X$, where J is normalized duality mapping and b is the function defined on $\mathbb{R}^+$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0$.

Taking $x = p - q$ and $h = t(x_n - p)$ in (6.11), we get

$$\begin{aligned}\frac{1}{2}\|p-q\|^2 &+ t\langle x_n - p, J(p-q)\rangle \leq \frac{1}{2}a_n^2(t) \\ &\leq \frac{1}{2}\|p-q\|^2 + t\langle x_n - p, J(p-q)\rangle + b(t\|x_n - p\|).\end{aligned}$$

Because for each $t \in (0,1)$, $\lim_{n\to\infty} a_n(t)$ exists, it follows that

$$\begin{aligned}\frac{1}{2}\|p-q\|^2 &+ t\limsup_{n\to\infty}\langle x_n - p, J(p-q)\rangle \\ &\leq \frac{1}{2}\lim_{n\to\infty} a_n^2(t) \\ &\leq \frac{1}{2}\|p-q\|^2 + t\liminf_{n\to\infty}\langle x_n - p, J(p-q)\rangle + o(t).\end{aligned}$$

This yields

$$\limsup_{n\to\infty}\langle x_n - p, J(p-q)\rangle \leq \liminf_{n\to\infty}\langle x_n - p, J(p-q)\rangle + \frac{o(t)}{t}.$$

On letting $t \to 0^+$, we obtain that $\lim_{n\to\infty} \langle x_n - p, J(p-q)\rangle$ exists. ∎

6.2 Convergence of successive iterates

In this section, we develop techniques for convergence of $\{T^n x\}$ to fixed points of nonlinear operators in Banach spaces. Recall for contraction mapping T, the sequence of iterates $\{T^n x\}$ converges strongly in Banach spaces. The following example shows that even if a nonexpansive mapping T has a unique fixed point, then $\{T^n x\}$ need not converge to it.

Example 6.2.1 *Let $C = B_H$ be the unit ball of the Hilbert space $H = \ell_2$ and $\{a_n\}$ a sequence of real numbers in $[0,1]$ such that $\prod_{n=1}^{\infty} a_n > 0$. Consider the linear mapping $T : C \to C$ defined by*

$$T(x_1, x_2, \cdots) = (0, a_1 x_1, a_2 x_2, \cdots).$$

The origin is the only fixed point of T. It is easy to see that the sequence of iterates $\{T^n e\}$ with $e = (1, 0, 0, \cdots)$ converges weakly to 0, but it does not converge strongly to 0.

We first study strong convergence of $\{T^n x\}$ for nonexpansive mappings in a Banach space.

Theorem 6.2.2 (Browder and Petryshyn's theorem) – *Let X be a Banach space and T an asymptotically regular nonexpansive self-mapping of X. Suppose that T has a fixed point and that $I - T$ maps closed bounded subsets of X into closed subsets of X. Then for each $x \in X, \{T^n x\}$ converges strongly to an element of $F(T)$.*

Proof. Let $p \in F(T)$. Then $\{\|T^n x - p\|\}$ is a nonincreasing sequence. It suffices therefore to show that there exists a subsequence of $\{T^n x\}$ that converges strongly to a fixed point of T. Let S be the strong closure of sequence $\{T^n x\}$. By the asymptotic regularity of T,

$$(I - T)T^n x \to 0 \text{ as } n \to \infty.$$

Hence 0 lies in the strong closure of $(I-T)(S)$ and because the latter is closed by hypothesis (as S is closed and bounded), 0 lies in $(I-T)(S)$. Hence there exists a subsequence $\{T^{n_i} x\}$ such that $T^{n_i} x \to v \in S$ such that $(I - T)v = 0$. Hence $T^n x \to v$. ∎

We now turn our attention to the study of weak convergence of the iterates of nonlinear mappings.

Theorem 6.2.3 *Let X be a Banach space satisfying the Opial condition, C a nonempty weakly compact convex subset of X, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If for $x \in C$, $T^n x - T^{n+1} x \to 0$ as $n \to \infty$, then $\{T^n x\}$ converges weakly to an element of $F(T)$.*

Proof. For $x \in C$, define a sequence $\{x_n\}$ in C by $x_n = T^n x,\ n \in \mathbb{N}_0$. Then for $v \in F(T)$,

$$\|x_{n+1} - v\| = \|T^{n+1}x - v\| \leq \|T^n x - v\| \leq \|x_n - v\| \text{ for all } n \in \mathbb{N}_0,$$

and it follows that $\lim_{n\to\infty} \|x_n - v\|$ exists. By assumption

$$T^n x - T^{n+1}x = x_n - Tx_n \to 0 \text{ as } n \to \infty.$$

Thus, $\{T^n x\}$ is an AFPS of T. By Theorem 5.2.9, $I - T$ is demiclosed at zero. Applying Proposition 6.1.1, we conclude that $\{T^n x\}$ converges weakly to some $z \in F(T)$. ∎

We now study weak convergence of iterates of mappings that are more general than nonexpansive mappings. We begin with the following proposition:

Proposition 6.2.4 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial condition and $T : C \to C$ a nearly asymptotically nonexpansive mapping. Suppose that x_0 is the asymptotic center of the bounded sequence $\{T^n x\}$ for some $x \in C$. If the weak limit z of a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ is a fixed point of T, then x_0 coincides with z.*

Proof. It is obvious that $r_a(C, \{T^n x\}) \geq r_a(C, \{T^{n_i}x\})$. Because $T^{n_i}x \rightharpoonup z$, it follows from Theorem 3.2.9 that $\mathcal{Z}_a(C, \{T^{n_i}x\}) = \{z\}$ and so, for any $\varepsilon > 0$, we can choose an integer i_o such that

$$\|z - T^{n_{i_0}}x\| \leq r_a(C, \{T^{n_i}x\}) + \frac{\varepsilon}{2}.$$

Because z is a fixed point of T and T is nearly asymptotically nonexpansive, we can choose an integer j_0 such that for all $j \geq j_0$

$$\begin{aligned} \|z - T^{n_{i_0}+j}x\| &\leq \eta(T^j)(\|z - T^{n_{i_0}}x\| + a_j) \\ &\leq \eta(T^j)(r_a(C, \{T^{n_i}x\}) + \frac{\varepsilon}{2} + a_j) \\ &\leq \eta(T^j)(r_a(C, \{T^n x\}) + \varepsilon + a_j), \end{aligned}$$

and it follows that

$$\limsup_{n\to\infty} \|z - T^n x\| = r_a(C, \{T^n x\}).$$

By the uniqueness of asymptotic center, we have $z = x_0$. ∎

Theorem 6.2.5 *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex (but not necessarily bounded) subset of X. Let $T : C \to C$ be a demicontinuous nearly asymptotically nonexpansive mapping and $x \in C$. Then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Proof. Assume that $T^n x \rightharpoonup z$ as $n \to \infty$. We show that $z \in F(T)$. By Theorem 3.2.9, $\mathcal{Z}_a(C, \{T^n x\}) = \{z\}$. As in proof of Theorem 5.6.5, we have $z \in F(T)$. Because $T^n x \rightharpoonup z$ as $n \to \infty$, it follows that $T^{n+1}x - T^n x \rightharpoonup 0$ as $n \to \infty$.

Conversely, suppose that $T^{n+1}x - T^n x \rightharpoonup 0$ as $n \to \infty$. First, we show that $\omega_w(\{T^n x\}) \subseteq F(T)$. Let $y \in \omega_w(\{T^n x\})$. Then we have a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ such that $T^{n_i}x \rightharpoonup y$. By the weak asymptotic regularity of T, we have

$$T^{n_i+m}x \rightharpoonup y \text{ as } i \to \infty \text{ for } m = 0, 1, \cdots .$$

By Theorem 3.2.9, we have

$$\mathcal{Z}_a(C, \{T^{n_i+m}x\}) = \{y\} \text{ for } m = 0, 1, 2, \cdots .$$

Let $\{y_s\}$ be a sequence in C defined by $y_s = T^s y$ for $s \in \mathbb{N}$. For $m, s \in \mathbb{N}$ with $m > s$, we have

$$\begin{aligned} \|y_s - T^{n_i+m}x\| &= \|T^s y - T^s(T^{n_i+m-s}x)\| \\ &\le \eta(T^s)(\|y - T^{n_i+m-s}x\| + a_s), \end{aligned}$$

which implies that

$$r_a(y_s, \{T^m x\}) \le \eta(T^s)(r_a(y, \{T^m x\}) + a_s).$$

By Theorem 3.1.8, $T^s y \to y$ as $s \to \infty$. By the demicontinuity of T, we obtain from Proposition 5.6.1 that $Ty = y$. Thus, $\omega_w(\{T^n x\}) \subseteq F(T)$ is verified. To complete the proof, we show that $\omega_w(\{T^n x\})$ is singleton. Let $u, v \in \omega_w(\{T^n x\})$. Then we have two subsequences $\{T^{n_j}x\}$ and $\{T^{n_k}x\}$ of $\{T^n x\}$ such that $T^{n_j}x \rightharpoonup u$ and $T^{n_k}x \rightharpoonup v$. Then $u, v \in F(T)$. Let $\mathcal{Z}_a(C, \{T^n x\}) = \{z\}$. It follows from Proposition 6.2.4 that $u = v = z$. This proves that $\omega_w(\{T^n x\}) = \{z\}$. ∎

Corollary 6.2.6 *Let X be a uniformly convex Banach space satisfying the Opial condition and C a nonempty closed convex (but not necessarily bounded) subset of X. If $T : C \to C$ is an asymptotically nonexpansive mapping and $x \in C$, then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Proof. Because every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive mapping, the result follows from Theorem 6.2.5.

6.3 Mann iteration process

We have already seen in Section 6.2 that asymptotic regularity of nonlinear mappings T is required even for weak convergence of $\{T^n x\}$. We drop asymptotic regularity of nonlinear mappings when using the Mann iteration process.

Definition 6.3.1 *Let C be a nonempty convex subset of a linear space X and $T : C \to C$ a mapping. Let $A = [a_{i,j}]$ be an infinite real matrix satisfying:*

(A_1) A is a lower matrix with nonnegative entries ($a_{n,i} \geq 0$ for all n, $i \in \mathbb{N}$ and $a_{n,i} = 0$ for all $i > n$), i.e.,

$$A = \begin{pmatrix} a_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

(A_2) the sum of each row is 1, i.e., $\sum_{i=1}^{n} a_{n,i} = 1$ for all $n \in \mathbb{N}$,

(A_3) $\lim_{n\to\infty} a_{n,i} = 0$ for all $i \in \mathbb{N}$.

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = T(\sum_{i=1}^{n} a_{n,i}x_i), \quad n \in \mathbb{N}. \tag{6.12}$$

Then the sequence $\{x_n\}$ defined by (6.12) is called the Mann iteration.

Such an iteration process is called the *Mann iteration process.*[1]

Example 6.3.2 *Let Λ define the Cesaro matrix, i.e.,*

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is easy to see that Λ satisfies all the hypothesis related to the matrix A. Then the sequence $\{x_n\}$ in C defined by (6.12) reduces to

$$x_{n+1} = T(\frac{1}{n}\sum_{i=1}^{n} x_i), \ n \in \mathbb{N}.$$

In Definition 6.3.1, the matrix A is very general. The most useful Mann iteration process can be obtained by choosing the matrix A as follows:

$$a_{n,i} = (1 - a_{n,n})a_{n-1,i}, i = 1, 2, \cdots, n \text{ and } n = 2, 3, \cdots, \tag{6.13}$$

and

$$\text{either } a_{n,n} = 1 \text{ or } a_{n,n} < 1 \text{ for all } n \in \mathbb{N}. \tag{6.14}$$

[1]It was introduced by W.R. Mann in 1953.

The entries of matrix A satisfying conditions (6.13) and (6.14) can be constructed by choosing a sequence of nonnegative numbers in $[0, 1]$ as below:

Choose a sequence $\{\alpha_n\}$ of nonnegative numbers satisfying the conditions:

$$0 \leq \alpha_n < 1 \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty \tag{6.15}$$

and then we can define the entries of A by

$$\begin{cases} a_{1,1} = 1, \ a_{1,i} = 0 \text{ for } i > 1; \\ a_{n+1,n+1} = \alpha_n \text{ for } n \in \mathbb{N}; \\ a_{n+1,i} = a_{i,i} \ \prod_{k=i}^{n} (1 - \alpha_k) \text{ for } i = 1, 2, \cdots, n; \\ a_{n+1,i} = 0 \text{ for } i > n+1, n \in \mathbb{N}. \end{cases}$$

More precisely, we now define the Mann iteration process, which will be used to approximate fixed points of nonlinear mappings.

Definition 6.3.3 *Let C be a nonempty convex subset of a linear space X and $T : C \to C$ a mapping. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying (6.15). Define a sequence $\{x_n\}$ in C by*

$$\begin{cases} x_1 \in C; \\ x_{n+1} = M(x_n, \alpha_n, T), \ n \in \mathbb{N}; \end{cases} \tag{6.16}$$

where $M(x_n, \alpha_n, T) = (1 - \alpha_n)x_n + \alpha_n T x_n$. Then sequence $\{x_n\}$ is called the (normal) Mann iteration.

Using convexity structure defined in Section 4.3, we now define the Mann iteration in a metric space.

Definition 6.3.4 *Let C be a nonempty convex subset of a convex metric space X and $T : C \to C$ a mapping. Let $\{\alpha_n\}$ be a sequence satisfying (6.15). Define a sequence $\{x_n\}$ in C by*

$$\begin{cases} x_1 \in C; \\ x_{n+1} = W(Tx_n, x_n; \alpha_n), n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ is called the Mann iteration.

First, we study a convergent Mann iteration for arbitrary continuous mappings.

Theorem 6.3.5 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a continuous mapping. If the Mann iteration $\{x_n\}$ defined by (6.16) converges strongly to a point $p \in C$, then p is a fixed point of T.*

Proof. Let $\lim_{n\to\infty} x_n = p$. Suppose, for contradiction, that $p \neq Tp$. Set $\varepsilon_n := x_n - Tx_n - (p - Tp)$. Because $\lim_{n\to\infty} x_n = p$ and T is continuous, it follows that

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} [(x_n - Tx_n) - (p - Tp)] = 0.$$

Because $\|p - Tp\| > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\varepsilon_n\| < \|p - Tp\|/3$ and $\|x_n - x_m\| < \|p - Tp\|/3$ for all $n, m \geq n_0$. Let N be any positive integer such that $\sum_{i=n_0}^{n_0+N} \alpha_i \geq 1$. Because $x_{i+1} - x_i = \alpha_i(Tx_i - x_i)$, it follows that

$$\begin{aligned} \|x_{n_0} - x_{n_0+N+1}\| &= \left\| \sum_{i=n_0}^{n_0+N} (x_i - x_{i+1}) \right\| = \left\| \sum_{i=n_0}^{n_0+N} \alpha_i(p - Tp + \varepsilon_i) \right\| \\ &\geq \left\| \sum_{i=n_0}^{n_0+N} \alpha_i(p - Tp) \right\| - \left\| \sum_{i=n_0}^{n_0+N} \alpha_i \varepsilon_i \right\| \\ &\geq \sum_{i=n_0}^{n_0+N} \alpha_i \left(\|p - Tp\| - \frac{\|p - Tp\|}{3} \right) \\ &\geq \frac{2\|p - Tp\|}{3}. \end{aligned}$$

The contradiction proves the result. ∎

We have shown that if the Mann iteration is convergent to v for a continuous mapping T, then v is a fixed point of T. But if T is not continuous, then there is no guarantee that, even if the Mann iteration converges strongly to z, then z will be a fixed point of T. Let us give an example of a discontinuous mapping.

Example 6.3.6 *Let $X = C = [0, 1]$ and $T : C \to C$ a mapping defined by*

$$Tx = \begin{cases} 0, & \text{if} \quad x = 0, \\ 1, & \text{if} \quad 0 < x < 1, \\ 0, & \text{if} \quad x = 1. \end{cases}$$

Then $T0 = 0$ and the Mann iteration $\{x_n\}$ defined by (6.16) with $x_1 \in (0, 1)$ and $\alpha_n = 1/n,\ n \in \mathbb{N}$ converges to 1, which is not a fixed point of T.

The following results are very useful for approximation of fixed points of nonexpansive type mappings.

Proposition 6.3.7 *Let C be a nonempty convex subset of a normed space X and $T : C \to C$ a mapping with a fixed point p in C such that*

$$\|Tx - p\| \leq \|x - p\| \text{ for all } x \in C.$$

Then for the Mann iteration $\{x_n\}$ defined by (6.16) with $\{\alpha_n\}$ in $[0, 1]$, $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Because

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \leq \|x_n - p\| \text{ for all } n \in \mathbb{N},$$

it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. ∎

Proposition 6.3.8 *Let C be a nonempty convex subset of a uniformly convex Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$ satisfying the condition:*

$$\|Tx - p\| \leq \|x - p\| \text{ for all } x \in C \text{ and } p \in F(T).$$

Define a sequence $\{x_n\}$ in C by (6.16) with the restriction that $\sum_{n=1}^{\infty} \min\{\alpha_n, 1-\alpha_n\} = \infty$. Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Proposition 6.3.7 implies that $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in F(T)$. Observe that

$$\|Tx_n - p\| \leq \|x_n - p\| \text{ for all } n \in \mathbb{N}$$

and

$$x_{n+1} - p = (1-\alpha_n)x_n + \alpha_n(Tx_n - p) \text{ for all } n \in \mathbb{N}.$$

Applying Lemma 6.1.7, we obtain that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. ∎

6.4 Nonexpansive and quasi-nonexpansive mappings

We begin with a basic result on approximation of fixed points of nonexpansive mappings in a uniformly convex Banach space with compact setting.

Theorem 6.4.1 (Krasnoselski) – *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and T a nonexpansive mapping of C into a compact subset of C. Let $x_1 \in C$ be an arbitrary point in C. Then the sequence $\{x_n\}$ defined by*

$$(K) \qquad x_{n+1} = \frac{1}{2}(x_n + Tx_n) = M(x_n, \frac{1}{2}, T), \quad n \in \mathbb{N}$$

converges strongly to a fixed point of T in C.

Proof. Note $F(T) \neq \emptyset$ by Schauder's theorem. Let $p \in F(T)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists by Proposition 6.3.7. Proposition 6.3.8 implies that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{6.17}$$

Note

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \frac{1}{2}(\|x_n - Tx_{n+1}\| + \|Tx_n - Tx_{n+1}\|) \\ &\leq \frac{1}{2}(\|x_n - x_{n+1}\| + \|x_{n+1} - Tx_{n+1}\| + \|x_n - x_{n+1}\|), \end{aligned}$$

which gives

$$\|x_{n+1} - Tx_{n+1}\| \leq \|x_n - Tx_n\| \text{ for all } n \in \mathbb{N}.$$

This means that $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists. Using (6.17), we obtain

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty. \tag{6.18}$$

Because $\{Tx_n\}$ is in a compact set, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to v \in C$. Hence from (6.18), we have $x_{n_k} \to v$. Because T is continuous, v is a fixed point of T. Note $\lim_{n\to\infty} \|x_n - v\| = \lim_{k\to\infty} \|x_{n_k} - v\|$ exists. Therefore, $\{x_n\}$ converges strongly to a fixed point of T in C. ∎

The following example shows that without the asymptotic regularity condition, the Picard iteration cannot be used to approximate fixed points of nonexpansive mapping, but the iteration procedure given by (K) can be used to locate fixed point of the same mapping.

Example 6.4.2 *Let $X = \mathbb{C}$, which has the usual absolute value metric for complex numbers, $C = \{z \in \mathbb{C} : |z| \le 1\}$, and $T : C \to C$ a mapping defined by*

$$Tz = iz, \ \ z \in C, \ \ \textit{where } i = \sqrt{-1}.$$

It is easy to see that T is nonexpansive mapping with a fixed point $0 \in C$.

Now, let $z_0 \neq 0$ be an arbitrary point in C. Then the Picard iteration of T is given by

$$z_{n+1} = Tz_n = i^{n+1}z_0, \quad n = 0, 1, 2, \cdots .$$

Hence

$$\begin{aligned} |T^n z_0 - T^{n+1} z_0| &= |i^n z_0 - i^{n+1} z_0| \\ &= |i^n| \cdot |1 - i| \cdot |z_0| = \sqrt{2}|z_0| \nrightarrow 0 \ \textit{as } n \to \infty. \end{aligned}$$

Note T is not asymptotically regular at z_0, and $\{T^n z_0 = i^n z_0\}$ does not converge to zero.

However, from (K) we have

$$z_{n+1} = \frac{1}{2}(z_n + Tz_n) = \frac{1+i}{2} z_n = \left(\frac{1+i}{2}\right)^{n+1} z_0.$$

Because

$$\begin{aligned} \sum_{n=0}^{\infty} |z_n - z_{n+1}| &= \sum_{n=0}^{\infty} \left|\frac{1+i}{2}\right|^n \left|1 - \frac{1+i}{2}\right| |z_0| \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{n+1} |z_0| \\ &= (\sqrt{2}+1)|z_0| < \infty, \end{aligned}$$

it follows that $\{z_n\}$ is Cauchy sequence. Let $\lim_{n\to\infty} z_n = p$. Because $z_n - Tz_n \to 0$ as $n \to \infty$, it follows that $p = 0$. Therefore, $\{z_n\}$ converges to the fixed point 0.

The following result shows that the sequence $\{x_n\}$ defined by

$$x_{n+1} = M(x_n, \alpha_n, T), \quad n \in \mathbb{N}$$

converges strongly to a fixed point of nonexpansive mapping T without the assumption of convexity of domain.

Theorem 6.4.3 (Ishikawa) – *Let C be a nonempty closed subset of a Banach space X and let T be a nonexpansive mapping from C into a compact subset of X. Suppose there exist $x_1 \in C$ and a sequence $\{\alpha_n\}$ of real numbers satisfying the conditions:*

(i) $0 \le \alpha_n \le \alpha < 1$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $x_n \in C$ *for all* $n \in \mathbb{N}$, *where* $x_{n+1} = M(x_n, \alpha_n, T)$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof. Let D denote $\overline{co}(T(C) \cup \{x_1\})$. Then D is compact by Mazur's theorem. The sequence $\{x_n\}$ is clearly in D. It follows from assumptions (i) $\sim$ (ii) that $\{x_n\}$ is a compact sequence in C. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to v \in C$, as C is closed. As in proof of Theorem 5.2.4, $\{x_n\}$ is an AFPS for T, i.e., $x_n - Tx_n \to 0$. Thus, $v \in F(T)$. Note $\lim_{n\to\infty} \|x_n - v\|$ exists by Proposition 6.3.7. Therefore, $\lim_{n\to\infty} \|x_n - v\| = \lim_{k\to\infty} \|x_{n_k} - v\| = 0$. ∎

The next result is similar to Theorem 6.2.2 (Browder and Petryshyn's theorem), but the asymptotic regularity condition of T is not necessary.

Theorem 6.4.4 (Groetsch) – *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ a nonexpansive mapping that has at least one fixed point, and $\{\alpha_n\}$ a sequence of nonnegative numbers such that $0 \le \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Suppose that $I - T$ maps closed bounded subsets of C into closed subsets of C. Then the Mann iteration $\{x_n\}$ defined by (6.16) converges strongly to a fixed point of T.*

Proof. By Proposition 6.3.8, we have $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. It is easy to show that $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists (see Theorem 5.2.4). It follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{6.19}$$

Let S be the strong closure of $\{x_n\}$. By (6.19) and the fact that $(I - T)(S)$ is closed, $0 \in (I - T)(S)$. Hence there exists a subsequence $\{x_{n_i}\}$ converging to v, where $(I - T)v = 0$. Therefore, $\{x_n\}$ converges strongly to v, as $\lim_{n\to\infty} \|x_n - v\|$ exists. ∎

We now consider a class of mappings that properly includes the class of nonexpansive mappings with fixed points.

Definition 6.4.5 *Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping that has at least one fixed point p in C. Then T is said to be quasi-nonexpansive if*

$$\|Tx - p\| \leq \|x - p\| \text{ for all } x \in C.$$

Observation

- A nonexpansive mapping with at least one fixed point is quasi-nonexpansive.
- A linear quasi-nonexpansive is nonexpansive.

The following example shows that there exists a nonlinear continuous quasi-nonexpansive mapping that is not nonexpansive.

Example 6.4.6 *Let $X = l_\infty$, $C = B_X = \{x \in l_\infty : \|x\|_\infty \leq 1\}$ and $T : C \to C$ a mapping defined by*

$$Tx = (0, x_1^2, x_2^2, x_3^2, \cdots) \quad \text{for } x = (x_1, x_2, x_3, \cdots) \in C.$$

It is clear that T is a nonlinear continuous self-mapping on C with unique fixed point 0. Moreover,

$$\|Tx - 0\|_\infty = \|(0, x_1^2, x_2^2, \cdots)\|_\infty \leq \|(0, x_1, x_2, \cdots)\|_\infty = \|x - p\| \text{ for all } x \in C,$$

i.e., T is quasi-nonexpansive mapping. However, T is not nonexpansive. Indeed, for $x = (1/2, 1/2, \cdots)$ and $y = (3/4, 3/4, \cdots)$, we have

$$\|Tx - Ty\|_\infty = \left\| \left(0, \frac{5}{16}, \frac{5}{16}, \cdots \right) \right\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty.$$

We now show that Theorem 5.2.4 is also true for quasi-nonexpansive mappings in a uniformly convex Banach space.

Theorem 6.4.7 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a quasi-nonexpansive mapping that has at least one fixed point p. Let $\{x_n\}$ be the Mann iteration defined by*

$$x_{n+1} = T_{\alpha_n} x_n = M(x_n, \alpha_n, T), \; n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence of nonnegative numbers that is bounded away from 0 and 1. Then $\{x_n\}$ has the following properties:

(D_1) $\lim_{n \to \infty} \|x_n - p\|$ *exists.*

(D_2) $\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$

Moreover, $\lim_{n \to \infty} \|T_{\alpha_n} T_{\alpha_{n-1}} \cdots T_{\alpha_1} x_1 - T_{\alpha_{n-1}} T_{\alpha_{n-2}} \cdots T_{\alpha_1} x_1\| = 0.$

Proof. (a) It follows that from Proposition 6.3.7.

(b) Suppose $\lim_{n \to \infty} \|x_n - p\| = r$. Because

$$x_{n+1} - p = (1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p) \text{ and } \|Tx_n - p\| \leq \|x_n - p\|,$$

it follows from Theorem 2.3.13 that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$ ∎

Corollary 6.4.8 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a quasi-nonexpansive mapping that has at least one fixed point p. Let $\{x_n\}$ be the Mann iterative sequence defined by*

$$x_{n+1} = T_\alpha x_n = M(x_n, \alpha, T), \ n \in \mathbb{N}.$$

Then T_α is asymptotically regular for each $x_1 \in C$, i.e., $\lim_{n\to\infty} \|T_\alpha^n x_1 - T_\alpha^{n+1} x_1\| = 0$.

Recall that a self-mapping T on a nonempty subset C of a Banach space X is *demicompact* if every bounded $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges strongly contains a convergent subsequence.

The following example demonstrates that there is no connection between continuity and demicompactness of mappings.

Example 6.4.9 *Let $X = C = [0, 1]$ with the usual metric and $T : C \to C$ a mapping defined by*

$$Tx = \begin{cases} x/2, & if \ x \neq 0, \\ 1, & if \ x = 0. \end{cases}$$

Then T is not continuous. However, T is demicompact. In fact, if $\{x_n\}$ is a bounded sequence in C such that $x_n - Tx_n \to 0$ as $n \to \infty$, then by the Bolzano-Weierstrass theorem, it follows that $\{x_n\}$ has a convergent subsequence.

The following example shows that there is a demicompact mapping that is not compact.

Example 6.4.10 *Let $X = \ell_2$ and $C = \{e_1, e_2, \cdots, e_n, \cdots\}$ be the usual orthonormal basis for ℓ_2. Define $T : C \to C$ by*

$$T(e_i) = e_{i+1}, \quad i \in \mathbb{N}.$$

Then T is continuous (in fact, an isometry), but not compact. However, T is demicompact. Indeed, if $\{e_i\}_{i\in\mathbb{N}}$ is a bounded sequence in C such that $e_i - Te_i$ converges, $\{e_i\}_{i\in\mathbb{N}}$ must be finite.

We now introduce a condition that ensures strong convergence of iterative sequences to fixed points of nonexpansive type mappings.

Condition I. Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$. Then T is said to satisfy *Condition I* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for $t \in (0, \infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)) \text{ for all } x \in C.$$

T is said to satisfy *Condition II* if there exists a constant $c > 0$ such that

$$\|x - Tx\| \geq c\ d(x,\ F(T)) \text{ for all } x \in C.$$

It is easy to see that mappings that satisfy Condition II also satisfy Condition I.

The following example shows that there exists a mapping that is quasi-nonexpansive mapping and satisfies Condition II.

Example 6.4.11 *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping such that*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\| \text{ for all } x, y \in C,$$

where $a, b, c \geq 0$ with $a + b + c \leq 1$. If $F(T) \neq \emptyset$, then T satisfies condition II. Indeed, if $p \in F(T)$, then for $x \in C$

$$\|Tx - p\| \leq a\|x - p\| + b\|x - Tx\| \leq a\|x - p\| + b(\|x - p\| + \|p - Tx\|),$$

which implies that

$$\|Tx - p\| \leq \frac{a + b}{1 - b}\|x - p\|.$$

Hence T is quasi-nonexpansive. Observe that

$$\|Tx - p\| \geq \big|\|Tx - x\| - \|x - p\|\big| \geq \|x - p\| - \|x - Tx\|$$

and

$$\|Tx - p\| \leq a\|x - p\| + b\|x - Tx\|.$$

Hence

$$a\|x - p\| + b\|x - Tx\| \geq \|x - p\| - \|x - Tx\|,$$

which gives

$$\|x - Tx\| \geq \frac{1 - a}{a + b}\|x - p\|.$$

The constant $(1 - a)/(1 + b)$ is positive because $0 < a, b < 1$. Thus, Condition II holds.

We now establish a relationship between mappings that satisfy Condition I and those that are demicompact.

Proposition 6.4.12 *Let C be a nonempty closed bounded subset of a Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$. If $I - T$ maps closed bounded subsets of C onto closed subsets of X, then T satisfies Condition I on C.*

Proof. Let $M = \sup\{d(x, F(T)) : x \in C\}$. If $M = 0$, then $F(T) = C$ and Condition I follows trivially. If $M > 0$, then for $0 < r < M$, define

$$C_r = \{x \in C : d(x, F(T)) \geq r\}$$

and

$$f(r) = \inf\{\|x - Tx\| : x \in C_r\}.$$

Note that C_r is nonempty closed bounded. We show that $f(r) > 0$ for arbitrary r, $0 < r < M$.

By hypothesis, $(I-T)(C_r) = \{x - Tx : x \in C_r\}$ is closed. If $0 \in (I-T)(C_r)$ then $0 = z - Tz$ for some $z \in C_r$ and hence $z \in F(T)$, but $d(z, F(T)) \geq r > 0$ a contradiction. Therefore, $0 \neq (I-T)(C_r)$.

Suppose now that $f(r) = \inf\{\|x - Tx\| : x \in C_r\} = 0$. Then there exists a sequence $\{x_n\}$ in C_r such that $\|x_n - Tx_n\| \to 0$. Note $\{x_n - Tx_n\} \subseteq (I-T)(C_r)$ is a closed set. Thus, we obtain $0 \in (I-T)(C_r)$, contradicting our statement above that $0 \notin (I-T)(C_r)$. Therefore, $f(r) > 0$ for $r < M$.

We extend the domain of f to $\mathbb{R}^+$ by defining $f(0) = 0$ and $f(r) = \sup\{f(s) : s < M\}$ for $r \geq M$. It is easy to verify that f so defined fulfills the hypotheses of Condition I; in particular, $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$. ∎

We now give strong convergence of the Mann iteration for quasi-nonexpansive mappings satisfying Condition I.

Theorem 6.4.13 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a continuous quasi-nonexpansive mapping. If T satisfies Condition I, then for arbitrary $x_1 \in C$, the Mann iteration $\{x_n\}$ defined by (6.16), where $\{\alpha_n\}$ is a sequence of nonnegative numbers in $[0,1]$ that is bounded away from 0 and 1, converges strongly to a fixed point of T.*

Proof. Because for $p \in F(T)$, $\|x_{n+1} - p\| \leq \|x_n - p\|$, it follows that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Thus, $\lim\limits_{n\to\infty} d(x_n, F(T))$ exists. By Theorem 6.4.7, $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$. Note

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))),\ n \in \mathbb{N},$$

which gives that $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$. Then for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{2} \text{ for all } n \geq n_0.$$

Note for all $n, m \geq n_0$ and $p \in F(T)$

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|p - x_m\| \\ &\leq 2\|x_{n_0} - p\|, \end{aligned}$$

which implies that

$$\|x_n - x_m\| \leq \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence and $\lim\limits_{n\to\infty} x_n = z \in C$. Therefore, $x_n - Tx_n \to 0$ implies by the continuity of T that $z \in F(T)$. ∎

A consequence of Proposition 6.4.12 and Theorem 6.4.13 is the following:

Corollary 6.4.14 (Browder and Petryshyn) – *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a nonexpansive mapping. For $t \in (0,1)$, let T_t be given by $T_t = tI + (1-t)T$. If $I - T$ maps closed bounded sets of C onto closed subsets of X and $F(T) \neq \emptyset$, then for each $x \in C, \{T_t^n x\}$ converges strongly to a fixed point of T.*

In many applications, compactness is a strong condition. We now study the problem of approximation of fixed points of nonexpansive and quasi-nonexpansive mappings in the noncompact setting.

Theorem 6.4.15 *Let X be a Banach space satisfying the Opial condition, C a weakly compact subset of X, and $T : C \to X$ a nonexpansive mapping. Given a sequence $\{x_n\}$ in C defined by (6.16), where $\{\alpha_n\}$ is a sequence of nonnegative numbers such that $0 \leq \alpha_n \leq \alpha < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Theorem 5.2.4 implies that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Because C is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to $p \in C$. By Theorem 5.2.9, $I - T$ is demiclosed at zero, $p = Tp$. Thus, all the assumptions of Proposition 6.1.1 are satisfied. Therefore, $\{x_n\}$ converges weakly to a fixed point of T by Proposition 6.1.1. ∎

Theorem 6.4.16 *Let X be a uniformly convex Banach space satisfying the Opial condition, C a nonempty closed convex subset of X, and $T : C \to C$ a quasi-nonexpansive mapping that has at least one fixed point. If $I - T$ is demiclosed at zero, then the Mann iteration $\{x_n\}$ defined in Theorem 6.4.7 converges weakly to a fixed point of T.*

Proof. Theorem 6.4.7 implies that $\{x_n\}$ has properties $(D_1) \sim (D_2)$. Therefore, the conclusion follows from Proposition 6.1.1. ∎

We have seen in Section 3.2 that there exists a class of uniformly convex Banach spaces without the Opial condition (e.g., L_p spaces, $p \neq 2$). Therefore, Theorem 6.4.15 is not true for such Banach spaces. The following theorem deals with the problem of approximation of fixed points of nonexpansive mappings in a uniformly convex Banach space without the Opial condition.

Theorem 6.4.17 *Let X be a uniformly convex Banach space with a Fŕechet differentiable norm, C a nonempty closed convex bounded subset of X, and $T : C \to C$ a nonexpansive mapping. Then for each $x_1 \in C$, the Mann iteration $\{x_n\}$ defined by (6.16) with the restriction that $\sum_{n=1}^{\infty} \min\{\alpha_n, 1 - \alpha_n\} = \infty$ converges weakly to a fixed point of T.*

Proof. Set

$$T_n := (1 - \alpha_n)I + \alpha_n T, \quad n \in \mathbb{N}. \tag{6.20}$$

It is easy to see that $F(T) \subseteq F(T_n)$ and T_n is nonexpansive. It follows from Lemma 6.1.8 that $\lim_{n\to\infty} \langle x_n, J(p - q)\rangle$ exists for all $p, q \in F(T)$.

Observe that

(i) $I - T$ is demiclosed at zero by Theorem 5.2.12;

(ii) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ by Proposition 6.3.7,

(iii) $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ by Proposition 6.3.8 and $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists by (5.18) imply that $\|x_n - Tx_n\| \to 0$.

Thus, all the assumptions of Proposition 6.1.2 are satisfied. Therefore, $\{x_n\}$ converges weakly to a fixed point of T. ∎

The following example shows that there exists a Banach space that does not satisfy the Opial condition and its norm is not Fŕechet differentiable. However, its dual does have the Kadec-Klee property.

Example 6.4.18 *Let $X = \mathbb{R}^2$ with the norm given by $\|x\| = \sqrt{\|x\|_1^2 + \|x\|_2^2}$ and $Y = L_p[0,1]$ with $1 < p < \infty$ and $p \neq 2$. Then the Cartesian product $X \times Y$ equipped with the ℓ_2-norm is uniformly convex, it does not satisfy the Opial condition, and its norm is not Fŕechet differentiable. However, its dual does have the Kadec-Klee property.*

The following theorem is more general than Theorem 6.4.17.

Theorem 6.4.19 *Let X be a uniformly convex Banach space such that its dual has the Kadec-Klee property, C a nonempty closed convex bounded subset of X, and $T : C \to C$ a nonexpansive mapping. Then the Mann iteration defined by (6.16) with the restriction that $\sum_{n=1}^{\infty} \min\{\alpha_n, 1-\alpha_n\} = \infty$ converges weakly to a fixed point of T.*

Proof. It follows from the proof of Theorem 6.4.17 that T_n defined by (6.20) is a nonexpansive mapping. Observe that

(i) $I - T$ is demiclosed at zero;

(ii) $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $p, q \in F(T)$ and $t \in [0,1]$ by Lemma 6.1.8(b);

(iii) $x_n - Tx_n \to 0$.

Therefore, $\{x_n\}$ converges weakly to a fixed point of T by Proposition 6.1.2. ∎

6.5 The modified Mann iteration process

In this section, we study weak convergence of the modified Mann iteration process to fixed points of mappings that are more general than nonexpansive mappings in Banach spaces.

First, we modify the Mann iteration process and prove a useful lemma:

Let C be a nonempty convex subset of a linear space X, $T : C \to C$ a mapping, and $\{\alpha_n\}$ a real sequence such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ in C defined by

$$\begin{cases} x_1 \in C; \\ x_{n+1} = M(x_n, \alpha_n, T^n), n \in \mathbb{N} \end{cases} \tag{6.21}$$

is called *the modified Mann iteration.*

Lemma 6.5.1 *Let C be a nonempty convex subset of a Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define the modified Mann iteration $\{x_n\}$ by (6.21). Then we have the following:*

(a) If p is a fixed point of T, it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(b) If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. (a) Let p be the fixed point of T. From (6.21), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n x_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n k_n\|x_n - p\| \\ &\leq k_n\|x_n - p\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Because $\sum_{n=1}^{\infty}(k_n - 1)$ is convergent, it follows from Lemma 6.1.5 that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(b) For each $n \in \mathbb{N}$, set $d_n := \|x_n - T^n x_n\|$ and $L = \sup_{n\in\mathbb{N}} k_n$. Then we have

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\leq d_{n+1} + L\|x_{n+1} - T^n x_{n+1}\| \\ &\leq d_{n+1} + L(\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\quad +\|T^n x_n - T^n x_{n+1}\| \\ &\leq d_{n+1} + L(\alpha_n d_n + d_n + L\|x_n - x_{n+1}\|) \\ &\leq d_{n+1} + L(2 + L)d_n \to 0 \text{ as } n \to \infty. \quad \blacksquare \end{aligned}$$

Using Lemma 6.5.1, we prove weak convergence of the modified Mann iteration $\{x_n\}$ defined by (6.21) in uniformly convex Banach spaces.

Theorem 6.5.2 *Let X be a uniformly convex Banach space satisfying the Opial condition, C a nonempty closed convex bounded subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers in $(0, 1)$ bounded away from 0 and 1. Then the modified Mann iteration $\{x_n\}$ defined by (6.21) converges weakly to a fixed point of T.*

Proof. Let p be a fixed point of T. By Lemma 6.5.1(a), $\lim_{n\to\infty} \|x_n - p\|$ exists. Because

$$\limsup_{n\to\infty} \|T^n x_n - p\| \leq \limsup_{n\to\infty} (k_n \|x_n - p\|) \leq \lim_{n\to\infty} \|x_n - p\|$$

and

$$\lim_{n\to\infty} \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\|,$$

it follows from Theorem 2.3.13 that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$, which in turn implies by Lemma 6.5.1(b) that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By Theorem 5.4.3, $I - T$ is demiclosed at zero. Therefore, $\{x_n\}$ converges weakly to a fixed point of T by Proposition 6.1.1. ∎

The following convergence theorems extend Theorems 6.4.17 and 6.4.19 for asymptotically nonexpansive mappings, respectively.

Theorem 6.5.3 *Let X be a uniformly convex Banach space with Fréchet differentiable norm, C a nonempty closed convex bounded subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Then for each $x_1 \in C$, the Mann iteration $\{x_n\}$ defined by (6.21), where $\{\alpha_n\}$ is a sequence of nonnegative numbers bounded away from 0 and 1, converges weakly to a fixed point of T.*

Proof. Set $T_n := (1-\alpha_n)I + \alpha_n T^n, n \in \mathbb{N}$. It is easily seen that $F(T) \subseteq F(T_n)$ and T_n is Lipschitzian with Lipschitz constant $L_n = (1-\alpha_n) + \alpha_n k_n \geq 1$. Because

$$\sum_{n=1}^{\infty}(L_n - 1) = \sum_{n=1}^{\infty} \alpha_n(k_n - 1) \leq \sum_{n=1}^{\infty}(k_n - 1) < \infty,$$

it follows from Lemma 6.1.8 that $\lim_{n\to\infty} \langle x_n, J(p-q)\rangle$ exists for all $p, q \in F(T)$. Observe that

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$,

(ii) $x_n - Tx_n \to 0$,

(iii) $I - T$ is demiclosed at zero,

(iv) $\lim_{n\to\infty} \langle x_n, J(p-q)\rangle$ exists for all $p, q \in F(T)$.

Hence result follows from Proposition 6.1.2. ∎

Theorem 6.5.4 *Let X be a uniformly convex Banach space such that X^* has the Kadec-Klee property, C a nonempty closed convex bounded subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Then for each $x_1 \in C$, the Mann iteration $\{x_n\}$ defined by (6.21), where $\{\alpha_n\}$ is a sequence of nonnegative numbers bounded away from 0 and 1, converges weakly to a fixed point of T.*

Proof. As in proof of Theorem 6.5.3, we have the following:

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$,

(ii) $x_n - Tx_n \to 0$,

(iii) $I - T$ is demiclosed at zero,

(iv) $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $p, q \in F(T)$ and $t \in [0,1]$ by Lemma 6.1.8.

Hence Theorem 6.5.4 follows from Proposition 6.1.2. ∎

6.6 The Ishikawa iteration process

In this section, we discuss the problem of approximation of fixed points of pseudocontractive mappings and develop iterative methods to deal with such problems in Hilbert spaces. We have seen in Section 6.4 that the Mann iteration converges (strongly) to fixed points of nonexpansive mapping in finite-dimensional Banach spaces. The following example shows that there exists a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iteration fails to converge.

Example 6.6.1 *Let H, C_1, C_2, C, and T be as in Example 5.7.1. Observe that*

$$\|Tx\|^2 = 2\|x\|^2 \text{ for all } x \in C_1$$

and

$$\|Tx\|^2 = 1 + 2\|x\|^2 - 2\|x\| \text{ for all } x \in C_2.$$

It is clear that the origin is the only fixed point of T.

We now show that no Mann iteration for T is convergent to the origin for any nonzero starting point. Let $x \in C$ be such that $x \neq 0$. If $x \in C_1$, then

$$\|(1-\lambda)x + \lambda Tx\|^2 = (1+\lambda^2)\|x\|^2 > \|x\|^2 \text{ for } \lambda \in (0,1),$$

i.e., the Mann iterate of x is actually further away form zero than x is. If $x \in C_2$, then

$$\begin{aligned}\|(1-\lambda)x + \lambda Tx\|^2 &= \left\|\left(\frac{\lambda}{\|x\|} + 1 - 2\lambda\right)x + \lambda x^{\perp}\right\|^2 \\ &= \left[\left(1 + \frac{\lambda}{\|x\|} - 2\lambda\right)^2 + \lambda^2\right]\|x\|^2 \\ &> 0 \text{ for } \lambda \in (0,1).\end{aligned}$$

Thus, the Mann iteration $\{x_n\}$ defined by (6.16) has the following properties:

(i) if $x_1 \in C_1$, then $\|x_{n+1}\| > \|x_n\|$ for all $n \in \mathbb{N}$,

(ii) if $x_1 \in C_2$, then $\|x_{n+1}\| \geq \|x_n\|/\sqrt{2}$ for all $n \in \mathbb{N}$.

For convergence of such a sequence to origin, x_n would have to lie in neighborhood C_1 of the origin for all $n > n_0$ for some $n_0 \in \mathbb{N}$. But this is not possible because we already established for C_1 that $\|x_{n+1}\| > \|x_n\|$ for all $n > n_0$.

We now introduce an iteration process for approximation of fixed points of pseudocontractive mappings:

Definition 6.6.2 *Let C be a nonempty convex subset of a linear space X and $T : C \to C$ a mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers in [0,1] satisfying the following conditions:*

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

For arbitrary $x_1 \in C$, define a sequence $\{x_n\}$ in C by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n \quad\;\; = (1-\beta_n)x_n + \beta_n T x_n, \;\; n \in \mathbb{N}. \end{cases} \tag{6.22}$$

Then $\{x_n\}$ is called the Ishikawa iteration.[2]

Before proving a theorem, we first establish two preliminary results:

Proposition 6.6.3 *Let C be a nonempty convex subset of a Hilbert space H and $T : C \to C$ a pseudocontractive mapping. Then*

$$\|(1-\alpha)(x-y) + \alpha(Tx-Ty)\|^2 \leq \|x-y\|^2 + \alpha^2\|x-y-(Tx-Ty)\|^2$$

for all $x, y \in C$ and $\alpha \in [0,1]$.

Proof. Let $x, y \in C$. Then from the identity

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \lambda \in [0,1];$$

we have

$$\begin{aligned} \|(1 \;\; - \;\; & \alpha)(x-y) + \alpha(Tx-Ty)\|^2 \\ = \;\; & (1-\alpha)\|x-y\|^2 + \alpha\|Tx-Ty\|^2 - \alpha(1-\alpha)\|x-y-(Tx-Ty)\|^2 \\ \leq \;\; & (1-\alpha)\|x-y\|^2 + \alpha(\|x-y\|^2 + \|x-y-(Tx-Ty)\|^2) \\ & -\alpha(1-\alpha)\|x-y-(Tx-Ty)\|^2 \\ \leq \;\; & \|x-y\| + \alpha^2\|x-y-(Tx-Ty)\|^2. \quad \blacksquare \end{aligned}$$

[2] This iteration process was introduced by Ishikawa in 1974.

Proposition 6.6.4 *Let C be a nonempty convex subset of a Hilbert space H and $T : C \to C$ a pseudocontractive mapping. For $0 \leq \alpha \leq \beta \leq 1$, define a mapping $T_{\alpha,\beta} : C \to C$ by $T_{\alpha,\beta}x = (1-\alpha)x + \alpha T[(1-\beta)x + \beta Tx]$, $x \in C$. Then*

$$\begin{aligned}\|T_{\alpha,\beta}x - T_{\alpha,\beta}y\| &\leq \|x-y\|^2 - \alpha\beta(1-2\beta)\|x-y-(Tx-Ty)\|^2 \\ &\quad -\alpha(\beta-\alpha)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\quad +\alpha\beta\|Tx-Ty-(Tu_x-Tu_y)\|^2.\end{aligned}$$

for all $x, y \in C$, where $u_x = (1-\beta)x + \beta Tx$ and $u_y = (1-\beta)y + \beta Ty$.

Proof. Let $x, y \in C$. By Proposition 6.6.3, we have

$$\begin{aligned}\|u_x - u_y\|^2 &= \|(1-\beta)(x-y) + \beta(Tx-Ty)\|^2 \\ &\leq \|x-y\|^2 + \beta^2\|x-y-(Tx-Ty)\|^2, \\ \|u_x - Tu_x\|^2 &= \|(1-\beta)(x-Tu_x) + \beta(Tx-Tu_x)\|^2 \\ &= (1-\beta)\|x-Tu_x\|^2 + \beta\|Tx-Tu_x\|^2 - \beta(1-\beta)\|x-Tx\|^2.\end{aligned}$$

Because T is pseudocontractive,

$$\begin{aligned}\|Tu_x - Tu_y\|^2 &\leq \|u_x-u_y\|^2 + \|u_x-u_y-(Tu_x-Tu_y)\|^2 \\ &\leq \|u_x-u_y\|^2 \\ &\quad +\|(1-\beta)(x-y) + \beta(Tx-Ty) - (Tu_x-Tu_y)\|^2 \\ &\leq \|x-y\|^2 + \beta^2\|x-y-(Tx-Ty)\|^2 \\ &\quad +(1-\beta)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\quad +\beta\|Tx-Ty-(Tu_x-Tu_y)\|^2 \\ &\quad -\beta(1-\beta)\|x-y-(Tx-Ty)\|^2 \\ &\leq \|x-y\|^2 - \beta(1-2\beta)\|x-y-(Tx-Ty)\|^2 \\ &\quad +(1-\beta)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\quad +\beta\|Tx-Ty-(Tu_x-Tu_y)\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|T_{\alpha,\beta}x &- T_{\alpha,\beta}y\|^2 \\ &= \|(1-\alpha)(x-y) + \alpha(Tu_x-Tu_y)\|^2 \\ &= (1-\alpha)\|x-y\|^2 + \alpha\|Tu_x-Tu_y\|^2 \\ &\quad -\alpha(1-\alpha)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\leq (1-\alpha)\|x-y\|^2 + \alpha\{\|x-y\|^2 - \beta(1-2\beta)\|x-y-(Tx-Ty)\|^2 \\ &\quad +(1-\beta)\|x-y-(Tu_x-Tu_y)\|^2 + \beta\|Tx-Ty-(Tu_x-Tu_y)\|^2\} \\ &\quad -\alpha(1-\alpha)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\leq \|x-y\|^2 - \alpha\beta(1-2\beta)\|x-y-(Tx-Ty)\|^2 \\ &\quad -\alpha(\beta-\alpha)\|x-y-(Tu_x-Tu_y)\|^2 \\ &\quad +\alpha\beta\|Tx-Ty-(Tu_x-Tu_y)\|^2. \qquad \blacksquare\end{aligned}$$

Theorem 6.6.5 *Let C be a nonempty compact convex subset of a Hilbert space H and $T : C \to C$ a Lipschitzian pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iteration defined by (6.22). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. (a) Let $p \in F(T)$. Set $T_n x_n := T_{\alpha_n,\beta_n} x_n$. Then $x_{n+1} = T_n x_n,\ n \in \mathbb{N}$. From Proposition 6.6.4, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \quad &\leq \quad \|x_n - p\|^2 - \alpha_n\beta_n(1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad -\alpha_n(\beta_n - \alpha_n)\|x_n - Ty_n\|^2 + \alpha_n\beta_n\|Tx_n - Ty_n\|^2.\end{aligned}$$

Because $\alpha_n \leq \beta_n$, it follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 \quad &\leq \quad \|x_n - p\|^2 - \alpha_n\beta_n(1 - 2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + \alpha_n\beta_n\|Tx_n - Ty_n\|^2. \qquad (6.23)\end{aligned}$$

Suppose T is L-Lipschitzian mapping. Then

$$\|Tx_n - Ty_n\| \quad \leq \quad L\|x_n - y_n\| \leq L\beta_n\|x_n - Tx_n\|$$

Hence from (6.23)

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n\beta_n(1 - 2\beta_n - L^2\beta_n^2)\|x_n - Tx_n\|^2.$$

Because $\lim\limits_{n\to\infty} \beta_n = 0$, there exists a number $n_0 \in \mathbb{N}$ such that $2\beta_n + L^2\beta_n^2 \leq 1/2$ for all $n \geq n_0$. Hence

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \frac{1}{2}\alpha_n\beta_n\|x_n - Tx_n\|^2 \text{ for all } n \geq n_0, \qquad (6.24)$$

which gives that

$$\frac{1}{2}\sum_{i=n_0}^{n} \alpha_i\beta_i\|x_i - Tx_i\|^2 \leq \|x_{n_0} - p\|^2 - \|x_{n+1} - p\|^2.$$

Because C is bounded, $\{\|x_{n+1} - p\|\}$ is bounded. Therefore, the series on the left-hand side is bounded. From condition (ii), this implies that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0,$$

which in turn implies from the compactness of C that there exists subsequence $\{x_{n_j}\}$ such that $\lim\limits_{j\to\infty} x_{n_j} = v$, where $v \in F(T)$.

Because $v \in F(T)$, it follows from (6.24) that

$$\|x_{n+1} - v\| \leq \|x_n - v\| \text{ for all } n \geq n_0. \qquad (6.25)$$

Let $\varepsilon > 0$. Then there exists an $N_{i,0}$ such that

$$\|x_{N_{i,0}} - v\| \leq \varepsilon \text{ for all } N_{i,0} \geq n_0.$$

Hence from (6.25), we get

$$\|x_n - v\| \leq \varepsilon \text{ for all } n \geq N_{i,0}.$$

This completes the proof of Theorem 6.6.5. ∎

6.7 The S-iteration process

For C a convex subset of a linear space X and T a mapping of C into itself, the iterative sequence $\{x_n\}$ of *the S-iteration process* is generated from $x_1 \in C$ and is defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N}, \end{cases} \tag{6.26}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty. \tag{6.27}$$

Let us compare the rate of convergence of the Picard, Mann, and S-iteration processes for contraction mappings.

Proposition 6.7.1 *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction mapping with Lipschitz constant k and a unique fixed point p. For $u_1 = v_1 = w_1 \in C$, define sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ in C as follows:*

Picard iteration: $u_{n+1} = Tu_n, n \in \mathbb{N}$

Mann iteration: $v_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n, n \in \mathbb{N}$

S-iteration: $w_{n+1} = (1-\alpha_n)Tw_n + \alpha_n Ty_n,$
$y_n = (1-\beta_n)w_n + \beta_n Tw_n, n \in \mathbb{N},$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. Then we have the following:

(a) $\|u_{n+1} - p\| \le k\|u_n - p\|$ for all $n \in \mathbb{N}$.

(b) $\|v_{n+1} - p\| \le \|v_n - p\|$ for all $n \in \mathbb{N}$.

(c) $\|w_{n+1} - p\| \le k[1-(1-k)\alpha_n\beta_n]\|w_n - p\|$ for all $n \in \mathbb{N}$.

Proof. Part (a) is obvious.

(b) Now part (b) follows from

$$\begin{aligned} \|v_{n+1} - p\| &= \|(1-\alpha_n)(v_n - p) + \alpha_n(Tv_n - p)\| \\ &\le (1-\alpha_n)\|v_n - p\| + k\alpha_n\|v_n - p\| \\ &\le [1-(1-k)\alpha_n]\|v_n - p\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

(c) For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|w_{n+1} - p\| &\le (1-\alpha_n)k\|w_n - p\| + \alpha_n k\|y_n - p\| \\ &\le k[(1-\alpha_n)\|w_n - p\| + \alpha_n((1-\beta_n)\|w_n - p\| + k\beta_n\|w_n - p\|)] \\ &= k[1-(1-k)\alpha_n\beta_n]\|w_n - p\|. \quad \blacksquare \end{aligned}$$

It is obvious that the rate convergence of the S-iteration process is faster than the Picard iteration process and the Picard iteration process is faster than the Mann iteration process for contraction mappings.

We now discuss the S-iteration process for nonexpansive mappings.

Lemma 6.7.2 *Let X be a normed space, C a nonempty convex subset of X, and $T : C \to C$ a nonexpansive mapping. If $\{x_n\}$ is the iterative process defined by (6.26), then $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists.*

Proof. Set $a_n := x_n - Tx_n$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|a_{n+1}\| &\leq (1-\alpha_n)\|Tx_n - Tx_{n+1}\| + \alpha_n\|Ty_n - Tx_{n+1}\| \\ &\leq (1-\alpha_n)\|x_n - x_{n+1}\| + \alpha_n\|y_n - x_{n+1}\|. \end{aligned} \tag{6.28}$$

Because

$$\begin{aligned} \|y_n - Ty_n\| &\leq (1-\beta_n)\|x_n - Ty_n\| + \beta_n\|Tx_n - Ty_n\| \\ &\leq (1-\beta_n)\|x_n - Ty_n\| + \beta_n^2\|a_n\|, \\ \|x_{n+1} - y_n\| &\leq (1-\alpha_n)\|y_n - Tx_n\| + \alpha_n\|y_n - Ty_n\| \\ &= (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n((1-\beta_n)\|x_n - Ty_n\| + \beta_n^2\|a_n\|) \\ &\leq (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n((1-\beta_n)(\|x_n - Tx_n\| \qquad (6.29) \\ &\quad +\|Tx_n - Ty_n\|) + \beta_n^2\|a_n\|) \\ &\leq [(1-\alpha_n)(1-\beta_n) + \alpha_n(1-\beta_n)(1+\beta_n) + \alpha_n\beta_n^2]\|a_n\| \\ &= (1-\beta_n + \alpha_n\beta_n)\|a_n\|, \\ \|x_{n+1} - x_n\| &\leq (1-\alpha_n)\|a_n\| + \alpha_n\|x_n - Ty_n\| \\ &\leq (1-\alpha_n)\|a_n\| + \alpha_n(\|x_n - Tx_n\| + \|Tx_n - Ty_n\|) \qquad (6.30) \\ &\leq (1+\alpha_n\beta_n)\|a_n\|. \end{aligned}$$

From (6.28), (6.29), and (6.30), we have

$$\begin{aligned} \|a_{n+1}\| &\leq [(1-\alpha_n)(1+\alpha_n\beta_n) + \alpha_n(1-\beta_n+\alpha_n\beta_n)]\|a_n\| \\ &= \|a_n\|, \end{aligned}$$

so that $\{\|a_n\|\}$ is nonincreasing and hence $\lim_{n\to\infty} \|a_n\|$ exists. ∎

Theorem 6.7.3 *Let C be a nonempty closed convex (not necessary bounded) subset of a uniformly convex Banach space X and $T : C \to C$ a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (6.26) with the restriction:*

$$\lim_{n\to\infty} \alpha_n\beta_n(1-\alpha_n) \text{ exists and } \lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0. \tag{6.31}$$

Then, for arbitrary initial value $x_1 \in C, \{\|x_n - Tx_n\|\}$ converges to some constant $r_C(T)$, which is independent of the choice of the initial value $x_1 \in C$.

Proof. Lemma 6.7.2 implies that $\lim_{n\to\infty} \|x_n - Tx_n\|$ exists. We denote this limit by $r(x_1)$. Let $\{x_n^*\}$ be another iterative sequence generated by (6.26) with the same restriction on iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ as the sequence $\{x_n\}$, but with the initial value $x_1^* \in C$. It follows from Lemma 6.7.2 that $\lim_{n\to\infty} \|x_n^* - Tx_n^*\| = r(x_1^*)$.

Observe that

$$\begin{aligned}\|Ty_n - Ty_n^*\| &\le \|y_n - y_n^*\|\\ &\le (1-\beta_n)\|x_n - x_n^*\| + \beta_n\|Tx_n - Tx_n^*\|\\ &\le \|x_n - x_n^*\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x_{n+1}^*\| &= \|(1-\alpha_n)(Tx_n - Tx_n^*) + \alpha_n(Ty_n - Ty_n^*)\|\\ &\le (1-\alpha_n)\|x_n - x_n^*\| + \alpha_n\|Ty_n - Ty_n^*\|\\ &\le (1-\alpha_n)\|x_n - x_n^*\| + \alpha_n\|y_n - y_n^*\| \qquad (6.32)\\ &\le \|x_n - x_n^*\|.\end{aligned}$$

This shows that $\lim_{n\to\infty}\|x_n - x_n^*\|$ exists. Let $\lim_{n\to\infty}\|x_n - x_n^*\| = d$ for some $d > 0$.

By using Corollary 2.3.10, we obtain that

$$\begin{aligned}\|y_n - y_n^*\| &= \|(1-\beta_n)(x_n - x_n^*) + \beta_n(Tx_n - Tx_n^*)\|\\ &\le \left[1 - 2\beta_n(1-\beta_n)\delta_X\left(\frac{\|x_n - x_n^* - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right)\right]\|x_n - x_n^*\|,\end{aligned}$$

it follows from (6.32) that

$$\begin{aligned}\|x_{n+1} - x_{n+1}^*\| \le \|x_n - x_n^*\|\\ -2\alpha_n\beta_n(1-\beta_n)\|x_n - x_n^*\|\delta_X\left(\frac{\|x_n - x_n^* - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right).\end{aligned}$$

This gives us

$$\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\|x_n - x_n^*\|\delta_X\left(\frac{\|x_n - x_n^* - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right) \le \|x_1 - x_1^*\| < \infty.$$

Because $\lim_{n\to\infty}\alpha_n\beta_n(1-\beta_n) \neq 0$ and $\lim_{n\to\infty}\|x_n - x_n^*\| = d$, we

$$\lim_{n\to\infty}\delta_X\left(\frac{\|x_n - x_n^* - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right) = 0.$$

Because δ_X is strictly increasing and continuous and $\lim_{n\to\infty}\|x_n - x_n^*\| = d > 0$, we have

$$\lim_{n\to\infty}\|x_n - x_n^* - (Tx_n - Tx_n^*)\| = 0.$$

Observe that

$$\big|\|x_n - Tx_n\| - \|x_n^* - Tx_n^*\|\big| \le \|x_n - Tx_n - (x_n^* - Tx_n^*)\|,$$

which implies that

$$\lim_{n\to\infty}\big|\|x_n - Tx_n\| - \|x_n^* - Tx_n^*\|\big| = 0.$$

Thus, $r(x_1) = r(x_1^*)$. Because

$$\|x_{n+1} - Tx_{n+1}\| \leq \|x_n - Tx_n\| \leq \|x_1 - Tx_1\| \text{ for all } n \in \mathbb{N}, \text{ and } x_1 \in C,$$

it follows that

$$r_C(T) = \inf\{\|x - Tx\| : x \in C\}. \quad \blacksquare$$

Theorem 6.7.4 *Let X be a real uniformly convex Banach space with a Fréchet differentiable norm or that satisfies Opials condition. Let C be a nonempty closed convex (not necessary bounded) subset of and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (6.26) with the restriction (6.31). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Set $T_n x := (1 - \alpha_n)Tx + \alpha_n T((1 - \beta_n)x + \beta_n Tx)$ for all $x \in C$ and $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, the mapping $T_n : C \to C$ is also nonexpansive and the S-iterative sequence $\{x_n\}$ defined by (6.26) can be written as

$$x_{n+1} = T_n x_n \text{ for all } n \in \mathbb{N}.$$

Furthermore, we have $F(T) \subseteq F(T_n)$ for all $n \in \mathbb{N}$. Because $F(T) \neq \emptyset$, by Theorem 6.7.3, we see that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The remainder of the proof is followed by Theorems 6.4.16 and 6.4.17. $\blacksquare$

Bibliographic Notes and Remarks

The important tools presented in Section 6.1 were presented in Deng [46], Groetsch [64], Osilike and Aniagbosor [118], Reich [125], and Tan and Xu [159, 160].

Theorem 6.2.2 is established by Browder and Petryshyn [30]. Other results of Section 6.2 belong to Gornicki [62]. Theorem 6.2.5 is an improvement of the main result of Bose [20]. Theorem 6.3.5 is proved in Chidume and Mutangadura [39].

Theorems 6.4.3 and 6.4.4 are established by Ishikawa [71] and Groetsch [63], respectively. Theorem 6.4.17 is similar to the result established by Reich in [125]. Other results presented in Section 6.4 are discussed in Krasnoselskii [93] Senter and Dotson [142], and Singh and Whitfield [147]. A survey of iteration processes can be found in an article [135] by Rhoades. The convergence of the modified Mann iteration process for asymptotically nonexpansive mappings presented here is discussed in Chidume, Shahzad, and Zegeye [40] and Tan and Xu [160]. Theorem 6.6.5 is proved in Ishikawa [70]. Notice that S-iteration process is faster than the Picard iteration process. The S-iteration process was introduced in Agarwal, O'Regan, and Sahu [2].

Exercises

6.1 Let C be a compact convex subset of a Hilbert space H and let $\mathcal{P}(C)$ denote family of all bounded proximinal subsets of C. Let $T : C \to \mathcal{P}(C)$ be a nonexpansive mapping with fixed point p. Define the sequence $\{x_n\}$ of Mann iterates by $x_1 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad n \in \mathbb{N},$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$ and $\{\alpha_n\}$ is a real sequence such that $0 \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a fixed point of T.

6.2 Let X be a uniformly convex Banach space that satisfies the Opial condition or has a Fréchet differentiable norm, C a closed convex bounded subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose that x_1 is a given point in C and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1. Show that the sequence $\{x_n\}$ defined by the modified Ishikawa iteration process:

$$\text{(I)} \qquad x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n((1 - \beta_n) x_n + \beta_n T^n x), \quad n \in \mathbb{N}$$

converges weakly to a fixed point of T.

6.3 Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\| \text{ for all } x \in C, p \in F(T) \text{ and } n \in \mathbb{N}.$$

If C is a nonempty compact convex subset of a uniformly convex Banach space X and $T : C \to C$ is a uniformly continuous asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, show that sequence $\{x_n\}$ defined by (I) converges strongly to a fixed point of T.

6.4 Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. T is said to be asymptotically nonexpansive in the intermediate sense provided T is uniformly continuous and

$$\limsup_{n \to \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

If C is a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ is completely continuous and asymptotically nonexpansive in the intermediate sense with

$$c_n = \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0 \text{ such that } \sum_{n=1}^{\infty} c_n < \infty,$$

show that sequence $\{x_n\}$ defined by (I) converges strongly to a fixed point of T.

6.5 Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction mapping with Lipschitz constant k and a unique fixed point p. For $u_1 = v_1 = w_1 \in C$, define sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ in C as follows:

Picard iteration: $u_{n+1} = Tu_n, n \in \mathbb{N};$

Mann iteration: $v_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n, n \in \mathbb{N};$

S-iteration: $w_{n+1} = (1-\alpha_n)Tw_n + \alpha_n Ty_n,$
$y_n = (1-\beta_n)w_n + \beta_n Tw_n, n \in \mathbb{N};$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. Show that

(a) $\|u_{n+1} - p\| \leq k\|u_n - p\|$ for all $n \in \mathbb{N}$.

(b) $\|v_{n+1} - p\| \leq \|v_n - p\|$ for all $n \in \mathbb{N}$.

(c) $\|w_{n+1} - p\| \leq k\|w_n - p\|$ for all $n \in \mathbb{N}$.

6.6 Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a nonexpansive mapping. Show that the following are equivalent:

(a) $F(T) \neq \emptyset$.

(b) For any $x \in C$, $\{\frac{1}{n}\sum_{i=0}^{n-1} T^i x\}$ converges weakly to a fixed point of T.

6.7 Let C be a nonempty subset of a Hilbert space and $T : C \to C$ a mapping. The mapping T is said to satisfy condition (A) if $F(T) \neq \emptyset$ and there exists a real positive number λ such that

$$\langle x - Tx, x - p\rangle \geq \lambda\|x - Tx\|^2 \text{ for all } x \in C \text{ and } p \in F(T).$$

If C is a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ is a mapping that satisfies condition (A), $I - T$ is demiclosed at zero, and $\{x_n\}$ is a sequence in C generated by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \in \mathbb{N}$ with $0 < a < \alpha_n \leq b < 1$, show that $\{x_n\}$ converges weakly to an element of $F(T)$.

6.8 Let C be a nonempty convex subset of a Banach space X and $T_i : C \to C$ $(i = 1, 2, \cdots, k)$ nonexpansive mappings. Let

$$S = \alpha_0 + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k,$$

where $\alpha_i \geq 0, \alpha_0 > 0$ and $\sum_{i=1}^{k} \alpha_i = 1$. If $\{x_n\}$ is a bounded sequence in C defined by

$$x_{n+1} = Sx_n, \quad n \in \mathbb{N},$$

show that $x_n - Sx_n \to 0$ as $n \to \infty$.

6.9 Let X be a Banach space that satisfies the Opial condition and C a weakly compact convex subset of X. Let T_i $(i = 1, 2, \cdots, k)$ and $\{x_n\}$ be as in Exercise 6.8. Show that $\{x_n\}$ converges weakly to a fixed point of S.

6.10 Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i : i = 1, 2, \cdots, k\}$ be k asymptotically quasi-nonexpansive self-mappings of C, i.e., $\|T_i^n x - q_i\| \le k_{in}\|x - q_i\|$ for all $x \in C, q_i \in F(T_i), i \in \{1, 2, 3, \cdots, k\}$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_0 \in C, \{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, and $\sum_{n=1}^\infty (k_{in} - 1) < \infty$ for all $i \in \{1, 2, 3, \cdots, k\}$. Show that the implicity iterative sequence $\{x_n\}$ generated by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_i^m x_n, \quad n \in \mathbb{N},$$

where $n = (m-1)k+i, i \in \{1, 2, 3, \cdots, k\}$ converges strongly to a common fixed point if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Chapter 7

Strong Convergence Theorems

In this chapter, we prove convergence theorems for approximants of self-mappings and non-self mappings in Banach spaces. We also study a Halpern's type iteration process for approximation of fixed points of nonexpansive mappings in a Banach space with a uniformly Gâteaux differentiable norm.

7.1 Convergence of approximants of self-mappings

In this section, we study strong convergence of approximants of nonexpansive and asymptotically nonexpansive type self-mappings in Banach spaces.

First, we establish a fundamental strong convergence theorem for nonexpansive mappings in a Hilbert space.

Theorem 7.1.1 (Browder's convergence theorem) – *Let C be a nonempty closed convex bounded subset of a Hilbert space H. Let u be an element in C and $G_t : C \to C$, $t \in (0,1)$ the family of mappings defined by*

$$G_t x = (1-t)u + tTx, \quad x \in C.$$

Then the following hold:

(a) There is exactly one fixed point x_t of G_t, i.e.,

$$x_t = (1-t)u + tTx_t. \tag{7.1}$$

(b) The path $\{x_t\}$ converges strongly to Pu as $t \to 1$, where P is the metric projection mapping from C onto $F(T)$.

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Proof. (a) Note for each $t \in (0,1), G_t$ is a contraction mapping of C into itself. Hence G_t has a unique fixed point x_t in C.

(b) Because $F(T)$ is a nonempty closed convex subset of C, there exists an element $u_0 \in F(T)$ that is the nearest point of u. By boundedness of $\{x_t\}$, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup z \in C$. Write $x_{t_n} = x_n$. Because $x_n - Tx_n \to 0$, it follows that $z = Tz$. Indeed, for $z \neq Tz$

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - z\| &< \limsup_{n\to\infty} \|x_n - Tz\| \\ &\leq \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|Tx_n - Tz\|) \\ &\leq \limsup_{n\to\infty} \|x_n - z\|, \end{aligned}$$

a contradiction, because H has the Opial condition. Observe that

$$(1-t_n)x_n + t_n(x_n - Tx_n) = (1-t_n)u$$

and

$$(1-t_n)u_0 + t_n(u_0 - Tu_0) = (1-t_n)u_0.$$

Subtracting and taking the inner product of the difference with $x_n - u_0$, we get

$$\begin{aligned} (1-t_n)\langle x_n - u_0, x_n - u_0\rangle &+ t_n\langle Ux_n - Uu_0, x_n - u_0\rangle \\ &= (1-t_n)\langle u - u_0, x_n - u_0\rangle, \end{aligned}$$

where $U = I - T$. Because $U = I - T$ is monotone, $\langle Ux_n - Uu_0, x_n - u_0\rangle \geq 0$, it follows that

$$\|x_n - u_0\|^2 \leq \langle u - u_0, x_n - u_0\rangle \text{ for all } n \in \mathbb{N}.$$

Because $u_0 \in F(T)$ is the nearest point to u,

$$\langle u - u_0, z - u_0\rangle \leq 0,$$

which gives

$$\begin{aligned} \|x_n - u_0\|^2 &\leq \langle u - u_0, x_n - u_0\rangle \\ &= \langle u - u_0, x_n - z\rangle + \langle u - u_0, z - u_0\rangle \\ &\leq \langle u - u_0, x_n - z\rangle. \end{aligned}$$

Thus, from $x_n \rightharpoonup z$, we obtain $x_n \to u_0$ as $n \to \infty$. We show that $x_t \to u_0$ as $t \to 1$, i.e., u_0 is the only strong cluster point of $\{x_t\}$. Suppose, for contradiction, that $\{x_{t_{n'}}\}$ is another subsequence of $\{x_t\}$ such that $x_{t_{n'}} \to v \neq u_0$ as $n' \to \infty$. Set $x_{n'} := x_{t_{n'}}$. Because $x_{n'} - Tx_{n'} \to 0$, it follows that $v \in F(T)$. From (7.1), we have

$$x_t - Tx_t = (1-t)(u - Tx_t). \tag{7.2}$$

Because for $y \in F(T)$

$$\begin{aligned}\langle x_t - Tx_t, x_t - y\rangle &= \langle x_t - Ty + Ty - Tx_t, x_t - y\rangle \\ &= \|x_t - y\|^2 - \langle Tx_t - Ty, x_t - y\rangle \\ &\geq 0,\end{aligned}$$

this gives from (7.2) that $\langle u - Tx_t, x_t - y\rangle \geq 0$. Thus, $\langle x_t - u, x_t - y\rangle \leq 0$ for all $t \in (0,1)$ and $y \in F(T)$. It follows that

$$\langle u_0 - u, u_0 - v\rangle \leq 0 \text{ and } \langle v - u, v - u_0\rangle \leq 0,$$

which imply that $u_0 = v$, a contradiction. Therefore, $\{x_t\}$ converges strongly to Pu, where P is metric projection mapping from C onto $F(T)$. ∎

We now prove strong convergence of path $\{x_t\}$ in a more general situation.

Proposition 7.1.2 *Let C be a nonempty subset of a Banach space X and $T : C \to X$ a pseudocontractive mapping such that for some $u \in C$, the equation*

$$x = (1-t)u + tTx \tag{7.3}$$

has a unique solution x_t in C for each $t \in (0,1)$. If $F(T) \neq \emptyset$, there exists $j(x_t - v) \in J(x_t - v)$ such that

$$\langle x_t - u, j(x_t - v)\rangle \leq 0 \textit{ for all } v \in F(T) \textit{ and } t \in (0,1).$$

Proof. From (7.3) we have

$$x_t - Tx_t = (1-t)(u - Tx_t) \text{ for all } t \in (0,1).$$

For $y \in F(T)$, there exists $j(x_t - y) \in J(x_t - y)$ such that

$$\begin{aligned}\langle x_t - Tx_t, j(x_t - y)\rangle &= \langle x_t - Ty + Ty - Tx_t, j(x_t - y)\rangle \\ &= \|x_t - y\|^2 - \langle Tx_t - Ty, j(x_t - y)\rangle \\ &\geq 0,\end{aligned}$$

which implies that

$$\langle u - Tx_t, j(x_t - y)\rangle \geq 0.$$

It follows from (7.3) that

$$\langle x_t - u, j(x_t - y)\rangle \leq 0 \text{ for all } y \in F(T) \text{ and } t \in (0,1).$$ ∎

Theorem 7.1.3 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to X$ a demicontinuous pseudocontractive mapping such that for some $u \in C$, the equation defined by (7.3) has a unique solution x_t in C for each $t \in (0,1)$. If the path $\{x_t\}$ is bounded, then it converges strongly to a fixed point of T as $t \to 1$.*

Proof. Because $\{x_t\}$ is bounded, $\{Tx_t\}$ is bounded by (7.3) and

$$\|x_t - Tx_t\| = (1-t)\|u - Tx_t\| \leq (1-t)\mathrm{diam}(\{u - Tx_t\}) \to 0.$$

Because X is reflexive and $\{x_t\}$ is bounded, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup v$ as $t_n \to 1$. Write $x_{t_n} := x_n$. Because $(t^{-1}-1)x_t = (t^{-1}-1)u + Tx_t - x_t$, it follows that

$$\begin{aligned} \langle (t_n^{-1}-1)x_n &- (t_m^{-1}-1)x_m, J(x_n - x_m)\rangle \\ &= (t_n^{-1} - t_m^{-1})\langle u, J(x_n - x_m)\rangle \\ &\quad + \langle Tx_n - Tx_m - (x_n - x_m), J(x_n - x_m)\rangle \\ &\leq (t_n^{-1} - t_m^{-1})\langle u, J(x_n - x_m)\rangle \text{ for all } n, m \in \mathbb{N}. \end{aligned}$$

Taking the limit as $m \to \infty$, we obtain

$$\langle (t_n^{-1}-1)x_n, J(x_n - v)\rangle \leq (t_n^{-1}-1)\langle u, J(x_n - v)\rangle,$$

and thus,

$$\langle x_n - u, J(x_n - v)\rangle \leq 0.$$

Hence

$$\|x_n - v\|^2 = \langle x_n - v, J(x_n - v)\rangle = \langle x_n - u, J(x_n - v)\rangle + \langle u - v, J(x_n - v)\rangle.$$

Therefore, $x_n \to v$ as $n \to \infty$. Because $Tx_n \to v$ by $x_n - Tx_n \to 0$, it follows from the demicontinuity of T that $v \in F(T)$.

We show that v is the only strong cluster point of $\{x_t\}$. Suppose, for contradiction, that $\{x_{t_{n'}}\}$ is another subsequence of $\{x_t\}$ such that $x_{t_{n'}} \to w \ (\neq v)$ as $t_{n'} \to 1$. It can be easily seen that $w = Tw$. Thus, from Proposition 7.1.2, we have

$$\langle x_{t_n} - u, J(x_n - w)\rangle \leq 0 \text{ and } \langle x_{t_{n'}} - u, J(x_{t_{n'}} - v)\rangle \leq 0$$

which imply that

$$\langle v - u, J(v - w)\rangle \leq 0 \text{ and } \langle w - u, J(w - v)\rangle \leq 0.$$

Hence

$$\|u - w\|^2 = \langle v - w, J(v - w)\rangle = \langle v - u, J(v - w)\rangle + \langle u - w, J(v - w)\rangle \leq 0,$$

a contradiction. Therefore, $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$. ∎

Corollary 7.1.4 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to X$ a nonexpansive mapping such that for some $u \in C$, the equation (7.3) has a unique solution x_t in C for each $t \in (0,1)$. If the path $\{x_t\}$ is bounded, then it converges strongly to a fixed point of T as $t \to 1$.*

Applying Theorem 7.1.3, we obtain

Theorem 7.1.5 *Let X be a reflexive Banach space with a weakly continuous duality mapping, C a nonempty closed convex bounded subset of X, u an element in C, and $T : C \to C$ a continuous pseudocontractive mapping. Then the following hold:*

(a) For each $t \in (0,1)$, there exists exactly one $x_t \in C$ such that

$$x_t = (1-t)u + tTx_t. \tag{7.4}$$

(b) $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$.

Proof. (a) For each $t \in (0,1)$, define $G_t : C \to C$ by

$$G_t x = (1-t)u + tTx, \; x \in C.$$

Then G_t is well defined because $u \in C$ and $T(C) \subset C$. Because for each $t \in (0,1)$, G_t is strongly pseudocontractive, it follows from Corollary 5.7.15 that G_t has exactly one fixed point $x_t \in C$.

(b) It follows from Theorem 7.1.3. ∎

Corollary 7.1.6 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$, C a nonempty closed convex bounded subset of X, and $T : C \to C$ a continuous pseudocontractive mapping. Then $F(T)$ is a sunny nonexpansive retract of C.*

Proof. For each $u \in C$, by Theorem 7.1.5, there is a unique path $\{x_t\}$ defined by (7.4) such that $\lim_{t\to 1} x_t = v \in F(T)$. Then there exists a mapping P from C onto $F(T)$ defined by $Pu = \lim_{t\to 1} x_t$, as u is an arbitrary element of C.

Because

$$\langle x_t - u, J(x_t - y)\rangle \leq 0 \text{ for all } y \in F(T) \text{ and } t \in (0,1),$$

this implies that

$$\langle Pu - u, J(Pu - y)\rangle \leq 0 \text{ for all } u \in C, y \in F(T).$$

Therefore, by Proposition 2.10.21, P is the sunny nonexpansive retraction from C onto $F(T)$. ∎

Next, we study a strong convergence theorem for the following more general class of mappings:

Definition 7.1.7 *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. Then T is said to be asymptotically pseudocontractive if for each $n \in \mathbb{N}$ and $x, y \in C$, there exist a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that $\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2$.*

We note that every asymptotically nonexpansive mapping is asymptotically pseudocontractive, but the converse is not true. In fact, if T is asymptotically nonexpansive with domain $Dom(T)$ and sequence $\{k_n\}$, then for each $n \in \mathbb{N}$ and $x, y \in Dom(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq \|T^n x - T^n y\|\|x-y\| \leq k_n\|x-y\|^2.$$

Theorem 7.1.8 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to C$ a demicontinuous asymptotically pseudocontractive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of nonnegative numbers in $(0,1)$ such that $t_n \to 1$ and $\lim_{n\to\infty}(k_n-1)/(1-t_n) = 0$. Let $\{x_n\}$ be a bounded sequence in C with $x_n - Tx_n \to 0$ such that*

$$x_n = (1-t_n)u + t_n T^n x_n \text{ for all } n \in \mathbb{N}. \tag{7.5}$$

If $I - T$ is demiclosed at zero, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From (7.5), we have

$$x_n - T^n x_n = (1-t_n)(u - T^n x_n) \text{ and } t_n(u - T^n x_n) = u - x_n.$$

Thus, whenever $y \in F(T)$, we have

$$\begin{aligned}(1-t_n)\langle u - T^n x_n, J(x_n - y)\rangle &= \langle x_n - T^n x_n, J(x_n-y)\rangle \\ &= \langle x_n - y + y - T^n x_n, J(x_n - y)\rangle \\ &= \|x_n - y\|^2 - \langle T^n x_n - T^n y, J(x_n - y)\rangle \\ &\geq -(k_n - 1)\|x_n - y\|^2,\end{aligned}$$

which yields

$$\langle x_n - u, J(x_n - y)\rangle \leq \frac{k_n - 1}{1-t_n}\|x_n - y\|^2 \leq \frac{k_n-1}{1-t_n}K \tag{7.6}$$

for some $K \geq 0$.

Because X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Because $I - T$ is demiclosed at zero, $v = Tv$. Hence

$$\begin{aligned}\|x_{n_i} - v\|^2 &= \langle x_{n_i} - v, J(x_{n_i} - v)\rangle \\ &= \langle x_{n_i} - u, J(x_{n_i} - v)\rangle + \langle u - v, J(x_{n_i} - v)\rangle \\ &\leq \frac{k_{n_i} - 1}{1 - t_{n_i}}K + \langle u - v, J(x_{n_i} - v)\rangle.\end{aligned}$$

From $J(x_{n_i} - v) \to^* 0$ and $(k_{n_i} - 1)/(1 - t_{n_i}) \to 0$, we get $x_{n_i} \to v$.

We now show that v is only strong cluster point of $\{x_n\}$. Suppose, for contradiction, that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ such that $x_{n_j} \to w \in C$. Because $x_{n_j} - Tx_{n_j} \to 0$, it follows that $Tx_{n_j} \to w$. By demicontinuity of T, we have that $Tx_{n_k} \rightharpoonup Tw$. Hence $Tw = w$. From (7.6), we have

$$\langle v-u, J(v-w)\rangle \leq 0 \text{ and } \langle w-u, J(w-v)\rangle \leq 0,$$

which imply that

$$\|v-w\|^2 = \langle v-w, J(u-w)\rangle = \langle v-u, J(v-w)\rangle + \langle u-w, J(v-w)\rangle \leq 0,$$

a contradiction. Therefore, $\{x_n\}$ converges strongly to a fixed point of T. ∎

Corollary 7.1.9 *Let X be a reflexive Banach space with a weakly continuous duality mapping $J : X \to X^*$. Let C be a nonempty closed subset of X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of real numbers in $(0,1)$ such that $t_n \to 1$ and $\lim_{n\to\infty}(k_n-1)/(1-t_n) = 0$. Let $\{x_n\}$ be a bounded sequence in C with $x_n - Tx_n \to 0$ such that $x_n = (1-t_n)u + t_nT^nx_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

The following result is very useful for strong convergence of AFPS of self-mappings as well as non-self mappings.

Theorem 7.1.10 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, $T : C \to X$ a demicontinuous mapping with $F(T) \neq \emptyset$, and $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0,1)$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}^+$ with $\lim_{n\to\infty}\alpha_n = 0$ and $\{x_n\}$ a bounded sequence in C such that $x_n - Tx_n \to 0$ as $n \to \infty$ and*

$$\langle x_n - Ax_n, J(x_n-p)\rangle \leq \alpha_n\|x_n-p\|^2 \text{ for all } n \in \mathbb{N} \text{ and } p \in F(T). \qquad (7.7)$$

Suppose the set $M = \{x \in C : LIM_n\|x_n - x\|^2 = \inf_{y\in C} LIM\|x_n-y\|^2\}$ contains a fixed point of T, where LIM is a Banach limit. Then $\{x_n\}$ converges strongly to an element of $M \cap F(T)$.

Proof. By Theorem 2.9.11, M is a nonempty closed convex and bounded set. By assumption, T has a fixed point in M. Denote such a fixed point by v. It follows from Corollary 2.9.13 that

$$LIM_n\langle z, J(x_n-v)\rangle \leq 0 \text{ for all } x \in C.$$

In particular,

$$LIM_n\langle Av - v, J(x_n-v)\rangle \leq 0. \qquad (7.8)$$

From (7.7), we obtain

$$LIM_n\langle x_n - Ax_n, J(x_n - v\rangle \leq 0. \tag{7.9}$$

Combining (7.8) and (7.9), we have

$$\begin{aligned} LIM_n\|x_n - v\|^2 &= LIM_n[\langle x_n - Ax_n, J(x_n - v)\rangle + \langle Ax_n - Av, J(x_n - v)\rangle \\ &\quad +\langle Av - v, J(x_n - v)\rangle] \\ &\leq kLIM_n\|x_n - v\|^2, \end{aligned}$$

i.e., $(1-k)LIM_n\|x_n - v\|^2 \leq 0$. Therefore, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges strongly to v. To complete the proof, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to z$ as $j \to \infty$. Because $x_{n_j} - Tx_{n_j} \to 0$, it follows that $Tx_{n_j} \to z$. By demicontinuity of T, we have that $Tz = z$. From (7.7), we have

$$\langle v - Av, J(v - z)\rangle \leq 0 \text{ and } \langle z - Az, J(z - v)\rangle \leq 0.$$

Hence $z = v$. This proves that $\{x_n\}$ converges strongly to v. ∎

Corollary 7.1.11 *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X, and $T : C \to X$ a demicontinuous mapping with $F(T) \neq \emptyset$. Let u be an element in C, $\{\alpha_n\}$ a sequence in $\mathbb{R}^+$ with $\lim\limits_{n\to\infty} \alpha_n = 0$, and $\{x_n\}$ a bounded sequence in C such that $x_n - Tx_n \to 0$ as $n \to \infty$ and*

$$\langle x_n - u, J(x_n - p)\rangle \leq \alpha_n\|x_n - p\|^2 \text{ for all } n \in \mathbb{N} \text{ and all } p \in F(T).$$

Suppose the set $M = \{x \in C : LIM_n\|x_n - x\|^2 = \inf\limits_{y\in C} LIM\|x_n - y\|^2\}$ contains a fixed point of T, where LIM is a Banach limit. Then $\{x_n\}$ converges strongly to an element of $F(T)$.

We now prove a notable strong convergence theorem for nonexpansive mappings in a uniformly smooth Banach space.

Theorem 7.1.12 (Reich's convergence theorem) – *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, x an element in C, $T : C \to C$ a nonexpansive mapping, and $G_t : C \to C$, $t \in (0,1)$, the family of mappings defined by $G_t(x) = (1-t)x + tTG_t(x)$. If T has a fixed point, then for each $x \in C$, $\lim\limits_{t\to 1} G_t(x)$ exists and is a fixed point of T.*

Proof. Let $\{t_n\}$ be a sequence of real numbers in $(0,1)$ such that $t_n \to 1$. Set $x_n := G_{t_n}(x)$. Because $F(T) \neq \emptyset$, it follows that $\{x_n\}$ is bounded and $x_n - Tx_n \to 0$ as $n \to \infty$. Then the set M defined by (2.32) is a nonempty closed convex bounded T-invariant subset of C (see Proposition 6.1.3). Note every uniformly smooth Banach space is reflexive and has normal structure. Hence every closed convex bounded set of X has fixed point property. Thus,

T has a fixed point in M. Observe that $\{x_n\}$ satisfies (7.7) with $\alpha_n = 0$ for all $n \in \mathbb{N}$ (see Proposition 7.1.2). It follows from Corollary 7.1.11 that $\{x_n\}$ converges strongly to an element of $F(T)$. ∎

Applying Corollary 7.1.11, we obtain

Theorem 7.1.13 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let u be an element in C and $\{t_n\}$ a sequence of real numbers in $(0,1)$ such that $t_n \to 1$ and $(k_n - 1)/(1 - t_n) \to 0$. Then the following hold:*

(a) There exists exactly one point $x_n \in C$ such that

$$x_n = (1 - t_n)u + t_n T^n x_n, \quad n \in \mathbb{N}.$$

(b) If $\{x_n\}$ is a bounded AFPS of T and $M = \{x \in C : LIM_n \|x_n - x\|^2 = \inf_{y \in C} LIM_n \|x_n - y\|^2\}$ contains a fixed point of T, then $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof. (a) Because $\lim_{n\to\infty} (k_n - 1)/(1 - t_n) = 0$, then there exists a sufficiently large natural number n_0 such that $k_n t_n < 1$ for all $n \geq n_0$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n x = (1 - t_n)u + t_n T^n x, \quad x \in C.$$

Because for each $n \geq n_0$, T_n is contraction, there exists exactly one fixed point $x_n \in C$ of T_n. We may assume that $x_n = u$ for all $n = 1, 2, \cdots, n_0 - 1$. Then

$$x_n = (1 - t_n)u + t_n T^n x_n \text{ for all } n \in \mathbb{N}.$$

(b) As in the proof of Theorem 7.1.8, it can be easily seen that $\{x_n\}$ satisfies the inequality (7.6). Note that M is a nonempty closed convex bounded set. Moreover, T has a fixed point in M by assumption.

Observe that
(i) (7.7) is satisfied with $\alpha_n = (k_n - 1)/(1 - t_n) \to 0$ as $n \to \infty$,
(ii) T has a fixed point in M,
(iii) $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Hence this part follows from Corollary 7.1.11. ∎

The following proposition shows that for a bounded AFPS, the set M satisfies the property (P) defined by (5.52).

Proposition 7.1.14 *Let C be a nonempty closed convex bounded subset of a reflexive Banach space X and $T : C \to C$ asymptotically nonexpansive. Let $\{x_n\}$ be an AFPS. Then the set M satisfies property (P).*

Proof. By Theorem 2.9.11, M is a nonempty closed convex bounded subset of C. Let $x \in M$. Because $\{T^m x\}$ is bounded in C, there exists a subsequence $\{T^{m_j} x\}$ of $\{T^m x\}$ such that $T^{m_j} x \rightharpoonup u \in C$. Let k_n be the Lipschitz constant of T^n. By w-lsc of the function $\varphi(z) = LIM_n \|x_n - z\|^2$, we have

$$\begin{aligned}
\varphi(u) &= \liminf_{j\to\infty} \varphi(T^{m_j} x) \\
&\leq \limsup_{m\to\infty} \varphi(T^m x) \\
&= \limsup_{m\to\infty} (LIM_n \|x_n - T^m x\|^2) \\
&\leq \limsup_{m\to\infty} (LIM_n (\|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{m-1} x_n - T^m x_n\| \\
&\quad + \|T^m x_n - T^m x\|)^2) \\
&\leq \limsup_{m\to\infty} (LIM_n (k_m \|x_n - x\|))^2 \\
&= \varphi(x) = \inf_{z\in M} \varphi(z).
\end{aligned}$$

Thus, $u \in M$. Therefore, M has property (P). ∎

Applying Theorem 5.5.8 and Proposition 7.1.14, we obtain

Theorem 7.1.15 *Let C be a nonempty closed convex bounded subset of a uniformly smooth Banach space X and $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$. Let $u \in C$ and $\{t_n\}$ a sequence in $(0,1)$ such that $t_n \to 1$ and $(k_n - 1)/(1 - t_n) \to 0$. Suppose the sequence $\{x_n\}$ defined by (7.5) is an AFPS of T. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Proposition 7.1.14, the set M has property (P). It follows from Theorem 5.5.8 that T has a fixed point in M. Therefore, by Theorem 7.1.13, $\{x_n\}$ converges strongly to an element of $F(T)$. ∎

7.2 Convergence of approximants of non-self mappings

In this section, we discuss strong convergence of approximants of non-self nonexpansive mappings.

The following theorem is an extension of Browder's strong convergence theorem for non-self nonexpansive mappings with unbounded domain.

Theorem 7.2.1 (Singh and Watson's convergence theorem) – *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a nonexpansive mapping such that $T(\partial C) \subseteq C$ and $T(C)$ is bounded. Let u be an element in C and define $G_t : C \to H$ by*

$$G_t x = (1-t)u + tTx, \quad x \in C \text{ and } t \in (0,1).$$

Let $x_t = G_t x_t$. Then $\{x_t\}$ converges strongly to v as $t \to 1$, where v is the fixed point of T closest to u.

Proof. Note $F(T)$ is nonempty by Theorem 5.2.25. Then for any $y \in F(T)$, we have

$$\|x_t - y\| \leq \|u - y\| \text{ for all } t \in (0,1),$$

so $\{x_t\}$ is bounded. By boundedness of $\{Tx_t\}$, we obtain that

$$\|x_t - Tx_t\| \leq (1-t) \sup_{t \in (0,1)} \|u - Tx_t\| \to 0 \text{ as } t \to 1.$$

Because H is reflexive, $\{x_t\}$ has a weakly convergent subsequence. Let $\{x_{t_n}\}$ be subsequence of $\{x_t\}$ such that $x_{t_n} \rightharpoonup z$ as $t_n \to 1$. Write $x_n = x_{t_n}$. Because $I - T$ is demiclosed at zero, $z \in F(T)$. Because $F(T)$ is a nonempty closed convex set in C by Corollary 5.2.29, there exists a unique point $v \in F(T)$ that is closest to u, i.e., $v \in F(T)$ is the nearest point projection of u. Now, for $y \in F(T)$, we have

$$\begin{aligned} \|x_t - u + t(u-y)\|^2 &= t^2 \|Tx_t - y\|^2 \\ &\leq t^2 \|x_t - y\|^2 = t^2 \|x_t - u + u - y\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|x_t - u\|^2 &+ t^2\|u-y\|^2 + 2t\langle x_t - u, u - y\rangle = \|x_t - u + t(u-y)\|^2 \\ &\leq t^2(\|x_t - u\|^2 + \|u-y\|^2 + 2\langle x_t - u, u - y\rangle). \end{aligned}$$

It follows that

$$\|x_t - u\|^2 \leq \frac{2t}{1+t}\langle x_t - u, y - u\rangle \leq \langle x_t - u, y - u\rangle \leq \|x_t - u\| \cdot \|y - u\|.$$

Hence $\|x_t - u\| \leq \|y - u\|$. By w-lsc of the norm of H,

$$\|z - u\| \leq \liminf_{n \to \infty} \|x_n - u\| \leq \|y - u\| \text{ for all } y \in F(T).$$

But v is the nearest point projection of u. Therefore, $z = v$ is the unique element in $F(T)$ that is the nearest point projection of u. This shows that $x_n \rightharpoonup v$ as $n \to \infty$. It remains to show that the convergence is strong. Because

$$\|x_n - u\|^2 = \|x_n - v + v - u\|^2 = \|x_n - v\|^2 + \|u - v\|^2 + 2\langle x_n - v, v - u\rangle,$$

this implies that

$$\begin{aligned} \|x_n - v\|^2 &= \|x_n - u\|^2 - \|u - v\|^2 - 2\langle x_n - v, v - u\rangle \\ &\leq -2\langle x_n - v, v - u\rangle \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{x_t\}$ converges strongly to v. ∎

We now establish a strong convergence theorem for non-self mappings in a Banach space.

Theorem 7.2.2 *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, u an element in C, and $T : C \to X$ a weakly inward nonexpansive mapping with $F(T) \neq \emptyset$. Suppose for $t \in (0,1)$, the contraction $G_t : C \to X$ defined by*

$$G_t x = (1-t)u + tTx, \;\; x \in C \tag{7.10}$$

has a unique fixed point $x_t \in C$. Then $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$.

Proof. Because $F(T)$ is nonempty, then $\{x_t\}$ is bounded. In fact, we have

$$\|x_t - v\| \leq \|u - v\| \text{ for all } v \in F(T) \text{ and } t \in (0,1).$$

We now show that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 1$. To this end, let $\{t_n\}$ be a sequence of real numbers in (0,1) such that $t_n \to 1$ as $n \to \infty$. Set $x_n := x_{t_n}$. Then we can define $\varphi : C \to [0,\infty)$ by $\varphi(x) = LIM_n\|x_n - x\|^2$. Then the set M defined by (2.32) is a nonempty closed convex bounded subset of C. Because

$$\|x_n - Tx_n\| = (1-t_n)\|Tx_n - u\| \to 0 \text{ as } n \to \infty, \tag{7.11}$$

it follows that for $x \in M$

$$\begin{aligned} \varphi(Tx) &= LIM_n\|x_n - Tx\|^2 \\ &\leq LIM_n\|Tx_n - Tx\|^2 \\ &\leq LIM_n\|x_n - x\|^2 = \varphi(x). \end{aligned} \tag{7.12}$$

By Theorem 2.9.11, M consists of one point, say z. We now show that this z is a fixed point of T. Because T is weakly inward, there are some $v_n \in C$ and $\lambda_n \geq 0$ such that

$$w_n := z + \lambda_n(v_n - z) \to Tz \text{ strongly.}$$

If $\lambda_n \leq 1$ for infinitely many n and for these n, then we have $w_n \in C$ and hence $Tz \in C$. Thus, we have $Tz = z$ by (7.12). So, we may assume $\lambda_n > 1$ for all sufficiently large n. We then write $v_n = r_n w_n + (1-r_n)z$, where $r_n = \lambda_n^{-1}$. Suppose $r_n \to 1$. Then $v_n \to Tz$ and hence $Tz \in C$. By (7.12), we have $Tz = z$. So, without loss of generality, we may assume $r_n \leq a < 1$. By Theorem 2.8.17, there exists a continuous increasing function $g = g_r : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|),$$

for all $x, y \in B_r[0]$ and $\lambda \in [0,1]$, where $B_r[0]$ (the closed ball centered at 0 and with radius r) is big enough so that $B_r[0]$ contains z and $\{w_n\}$. It follows that

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r[0]$ and $\lambda \in [0,1]$. Because $v_n \in C$, we obtain that

$$\begin{aligned}\varphi(z) &\leq \varphi(v_n)\\ &\leq r_n\varphi(w_n) + (1-r_n)\varphi(z) - r_n(1-r_n)g(\|w_n - z\|)\end{aligned}$$

and hence

$$\begin{aligned}(1-a)g(\|w_n - z\|) &\leq (1-r_n)g(\|w_n - z\|)\\ &\leq \varphi(w_n) - \varphi(z).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}(1-a)g(\|Tz - z\|) &\leq \varphi(Tz) - \varphi(z)\\ &\leq 0. \qquad\qquad (by\ (7.12))\end{aligned}$$

Therefore, $Tz = z$, i.e., z is a fixed point of T. Observe that

(i) $x_n - Tx_n \to 0$ by (7.11),

(ii) (7.7) is satisfied with $\alpha_n = 0$,

(iii) the set M contains a fixed point z of T.

By Corollary 7.1.11, we conclude that $\{x_t\}$ converges strongly to z as $t \to 1$. ∎

7.3 Convergence of Halpern iteration process

In Chapter 6, we have seen that the Mann and S-iteration processes are weakly convergent for nonexpansive mappings even in uniformly convex Banach spaces. The purpose of this section is to develop an iteration process so that it can generate a strongly convergent sequence in a Banach space.

Definition 7.3.1 *Let C be a nonempty convex subset of a linear space X and $T : C \to C$ a mapping. Let $u \in C$ and $\{\alpha_n\}$ a sequence in $[0,1]$. Then a sequence $\{x_n\}$ in C defined by*

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0 \end{cases} \tag{7.13}$$

is called the Halpern iteration.

We now prove the main convergence theorem of this section.

Theorem 7.3.2 *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in C$ and $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ that satisfies*

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{7.14}$$

Suppose that $\{z_t\}$ converges strongly to $z \in F(T)$ as $t \to 1$, where for $t \in (0,1)$, z_t is a unique element of C that satisfies $z_t = (1-t)u + tTz_t$. Then the sequence $\{x_n\}$ defined by (7.13) converges strongly to z.

Proof. Because $F(T) \neq \emptyset$, it follows that $\{x_n\}$ and $\{Tx_n\}$ are bounded. Set $K := \sup\{\|u\| + \|Tx_n\| : n \in \mathbb{N}\}$. From (7.13), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}|(\|u\| + \|Tx_{n-1}\|) + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|K + (1-\alpha_n)\|x_n - x_{n-1}\|. \end{aligned}$$

Hence for $m, n \in \mathbb{N}$, we have

$$\begin{aligned} &\|x_{n+m+1} - x_{n+m}\| \\ &\leq \Big(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\Big)K + \Big(\prod_{k=m}^{n+m-1} (1-\alpha_{k+1})\Big)\|x_{m+1} - x_m\| \\ &\leq \Big(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\Big)K + exp\Big(-\sum_{k=m}^{n+m-1} \alpha_{k+1}\Big)\|x_{m+1} - x_m\|. \end{aligned}$$

So the boundedness of $\{x_n\}$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$ yield

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq \Big(\sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|\Big)K$$

for all $m \in \mathbb{N}$. Because $\sum_{k=0}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Notice

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Tx_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Let LIM be a Banach limit. Then, we get

$$LIM_n\|x_n - Tz_t\|^2 \leq LIM_n\|x_n - z_t\|^2.$$

Because $t(x_n - Tz_t) = (x_n - z_t) - (1-t)(x_n - u)$, we have

$$\begin{aligned} t^2\|x_n - Tz_t\|^2 &\geq \|x_n - z_t\|^2 - 2(1-t)\langle x_n - u, J(x_n - z_t)\rangle \\ &= (2t-1)\|x_n - z_t\|^2 + 2(1-t)\langle u - z_t, J(x_n - z_t)\rangle \end{aligned}$$

for all $n \in \mathbb{N}$. These inequalities yield

$$\frac{1-t}{2}LIM_n\|x_n - z_t\|^2 \geq LIM_n\langle u - z_t, J(x_n - z_t)\rangle.$$

Letting t go to 1, we get

$$0 \geq LIM_n\langle u - z, J(x_n - z)\rangle,$$

because X has uniformly Gâteaux differentiable norm. Because $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} |\langle u - z, J(x_{n+1} - z)\rangle - \langle u - z, J(x_n - z)\rangle| = 0.$$

Hence by Proposition 2.9.7, we obtain

$$\limsup_{n\to\infty} \langle u - z, J(x_n - z)\rangle \le 0. \tag{7.15}$$

Because $(1 - \alpha_n)(Tx_n - z) = (x_{n+1} - z) - \alpha_n(u - z)$, we have

$$\|(1 - \alpha_n)(Tx_n - z)\|^2 \ge \|x_{n+1} - z\|^2 - 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle,$$

it follows that

$$\|x_{n+1} - z\|^2 \le (1 - \alpha_n)\|x_n - z\|^2 + 2(1 - (1 - \alpha_n))\langle u - z, J(x_{n+1} - z)\rangle$$

for each $n \in \mathbb{N}$. Let $\varepsilon > 0$. From (7.15), there exists $n_0 \in \mathbb{N}$ such that

$$\langle u - z, J(x_n - z)\rangle \le \varepsilon/2 \text{ for all } n \ge n_0.$$

Then we have

$$\|x_{n+n_0} - z\|^2 \le \left(\prod_{k=n_0}^{n+n_0-1}(1 - \alpha_k)\right)\|x_{n_0} - z\|^2 + \left(1 - \prod_{k=n_0}^{n+n_0-1}(1 - \alpha_k)\right)\varepsilon$$

for all $n \in \mathbb{N}$. By the condition $\sum_{k=0}^{\infty}\alpha_k = \infty$, we have

$$\limsup_{n\to\infty}\|x_n - z\|^2 = \limsup_{n\to\infty}\|x_{n+n_0} - z\|^2 \le \varepsilon.$$

Therefore, $\{x_n\}$ converges strongly to z, because ε is an arbitrary positive real number. ∎

Corollary 7.3.3 *Let C be a nonempty closed convex subset of a uniformly smooth Banach space and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in C$ and $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ satisfying (7.14). Then the sequence $\{x_n\}$ defined by (7.13) converges strongly to a fixed point of T.*

Bibliographic Notes and Remarks

The main results presented in Section 7.1 are proved in Browder [29], Lim and Xu [98], Morales and Jung [114], Reich [126], and Takahashi and Ueda [158].

Theorem 7.2.1 is due to Singh and Watson [149]. The strong convergence of approximants of nonexpansive non-self mappings can be found in Jung and Kim [78], Xu [165], and Xu and Yin [168]. Theorem 7.3.2 follows from Shioji and Takahashi [144]. Such strong convergence results have been recently generalized by viscosity approximation method (see Moudafi [111], Xu [167]).

Exercises

7.1 Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ a contraction mapping. Let $\{x_n\}$ be the sequence defined by the scheme

$$x_n = \frac{1}{1+\varepsilon_n} T x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f x_n,$$

where ε_n is a sequence $(0,1)$ with $\varepsilon_n \to 0$. Show that $\{x_n\}$ converges strongly to the unique solution of the variational inequality:

$$\langle (I-f)\tilde{x}, \tilde{x} - x \rangle \leq 0 \text{ for all } x \in F(T).$$

7.2 Let H be a Hilbert space, C a closed convex subset of H, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ a contraction. Let $\{x_n\}$ be a sequence in C defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n) T x_n + \alpha_n f(x_n), \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfies

$(H1)$ $\alpha_n \to 0$;

$(H2)$ $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(H3)$ either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1$.

Show that under the hypotheses $(H1) \sim (H3)$, $x_n \to \tilde{x}$, where $\tilde{x}$ is the unique solution of the variational inequality:

$$\langle (I-f)\tilde{x}, \tilde{x} - x \rangle \leq 0 \text{ for all } x \in F(T).$$

7.3 Let C be a nonempty closed convex subset of a uniformly smooth Banach space X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If Π_C is the set of all contractions on C, show that the path $\{x_t\}$ defined by

$$x_t = t f x_t + (1-t) T x_t, \quad t \in (0,1), f \in \Pi_C,$$

converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) = \lim_{t\to 0^+} x_t, \quad f \in \Pi_C,$$

show that $Q(f)$ solves the variational inequality:

$$\langle (I-f)Q(f), J(Q(f) - v) \rangle \leq 0, \quad f \in \Pi_C \text{ and } v \in F(T).$$

7.4 Let C be a nonempty closed convex subset of a Banach space X. Let $A : C \to C$ be a continuous strongly pseudocontractive with constant $k \in [0, 1)$ and $T : C \to C$ a continuous pseudocontractive mapping. Show that

(a) for each $t \in (0, 1)$, there exists unique solution $x_t \in C$ of equation

$$x = tAx + (1 - t)Tx.$$

(b) Moreover, if v is a fixed point of T, then for each $t \in (0, 1)$, there exists $j(x_t - v) \in J(x_t - v)$ such that

$$\langle x_t - Ax_t, j(x_t - v)\rangle \leq 0;$$

(c) $\{x_t\}$ is bounded.

7.5 Let C be a nonempty closed convex subset of a Banach space X that has a uniformly Gâteaux differentiable norm and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. For a fixed $\delta \in (0, 1)$, define $S : C \to C$ by

$$Sx := (1 - \delta)x + \delta Tx$$

for all $x \in C$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of C that satisfies

$$z_t = tu + (1 - t)Tz_t$$

for arbitrary $u \in C$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ that satisfies the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For arbitrary $x_0 \in C$, let the sequence $\{x_n\}$ be defined iteratively by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n.$$

Show that $\{x_n\}$ converges strongly to a fixed point of T.

Chapter 8

Applications of Fixed Point Theorems

The aim of this chapter is to sketch applications of fixed point theorems in attractors of hyperbolic IFS, best approximation theory, operator equations, variational inequalities, and variational inclusions.

8.1 Attractors of the IFS

Definition 8.1.1 *A (hyperbolic) iterated function system consists of a complete metric space* (X, d) *together with a finite set of contraction mappings* $T_n : X \to X$*, with respective Lipschitz constants* k_n*, for* $n = 1, 2, \cdots, N$*.*

The abbreviation "IFS" is used for "iterated function system". The notation for the IFS is $\{X; T_n, n = 1, 2, \cdots, N\}$ and its Lipschitz constant is $k = max\{k_n : n = 1, 2, \cdots, N\}$.

Let $\{X; T_n, n = 1, 2, \cdots, N\}$ be a hyperbolic iterated function system with Lipschitz constant k. Then the fixed point of the mapping $W : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$W(B) = \cup_{n=1}^{N} T_n(B) \text{ for all } B \in \mathcal{K}(X),$$

is called the *attractor of the IFS*.

The following propositions tell us how to make a contraction mapping on $(\mathcal{K}(X), H)$ out of a contraction mapping on a metric space (X, d).

Proposition 8.1.2 *Let* (X, d) *be a metric space and* $T : X \to X$ *a continuous mapping. Then* T *maps* $\mathcal{K}(X)$ *into itself.*

Proof. Let C be a nonempty compact subset of X. Then clearly $T(C) = \{Tx : x \in C\}$ is nonempty. We show that $T(C)$ is compact. Let $\{y_n = Tx_n\}$ be a sequence in C. Because C is compact, there is a subsequence $\{x_{n_i}\}$ that

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converges to a point in $\hat{x} \in C$. The continuity of T implies that $\{y_{n_i} = Tx_{n_i}\}$ is a subsequence of $\{y_n\}$ that converges to $\hat{y} = T\hat{x} \in T(C)$. ∎

Proposition 8.1.3 *Let $T : X \to X$ be a contraction mapping on the metric space (X, d) with Lipschitz constant k. Then $T : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by*

$$T(B) = \{Tx : x \in B\} \text{ for all } B \in \mathcal{K}(X)$$

is a contraction mapping on $(\mathcal{K}(X), H)$ with Lipschitz constant k.

Proof. Because $T : X \to X$ is continuous, it follows from Proposition 8.1.2 that T maps $\mathcal{K}(X)$ into itself. Now let $B, C \in \mathcal{K}(X)$. Then

$$\begin{aligned} \delta(T(B), T(C)) &= \sup\{\inf\{d(Tx, Ty) : y \in C\} : x \in B\} \\ &\leq \sup\{\inf\{kd(x, y) : y \in C\} : x \in B\} = k \cdot \delta(B, C). \end{aligned}$$

Similarly, $\delta(T(C),\ T(B)) \leq k \cdot \delta(C, B)$. Hence

$$\begin{aligned} H(T(B), T(C) &= \max\{\delta(T(B), T(C), \delta(T(C), T(B))\} \\ &\leq k \max\{\delta(B, C), \delta(C, B)\} \\ &\leq k \cdot H(B, C). \quad ∎ \end{aligned}$$

The next proposition provides an important method for combining contraction mappings on $(\mathcal{K}(X), H)$ to produce new contraction mappings on $(\mathcal{K}(X), H)$.

Proposition 8.1.4 *Let (X, d) be a metric space. Let $\{T_n : n = 1, 2, \cdots, N\}$ be contraction mappings on $(\mathcal{K}(X), H)$. Let the Lipschitz constant for T_n be denoted by k_n for each n. Define $W : \mathcal{K}(X) \to \mathcal{K}(X)$ by*

$$W(B) = T_1(B) \cup T_2(B) \cup \cdots \cup T_N(B) = \cup_{n=1}^N T_n(B) \text{ for all } B \in \mathcal{K}(X).$$

Then W is a contraction mapping with Lipschitz constant $k = \max\{k_n : n = 1, 2, \cdots, N\}$.

Proof. We show the claim for $N = 2$. An inductive argument then completes the proof. Let $B, C \in \mathcal{K}(X)$. We have

$$\begin{aligned} H(W(B), W(C)) &= H(T_1(B) \cup T_2(B), T_1(C) \cup T_2(C)) \\ &\leq \max\{H(T_1(B), T_1(C)), H(T_2(B), T_2(C))\} \\ &\quad \text{(by Proposition 4.2.5)} \\ &\leq \max\{k_1 H(B, C), k_2 H(B, C)\} \leq k\ H(B, C). \quad ∎ \end{aligned}$$

The following theorem gives the main result concerning a hyperbolic IFS.

Theorem 8.1.5 *Let $\{X; T_n, n = 1, 2, \cdots, N\}$ be a hyperbolic iterated function system with Lipschitz constant k. Then the mapping $W : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by*

$$W(B) = \cup_{n=1}^N T_n(B) \text{ for all } B \in \mathcal{K}(X),$$

is a contraction mapping on the complete metric space $(\mathcal{K}(X), H)$ *with Lipschitz constant* k, *i.e.,*

$$H(W(B), W(C)) \leq kH(B, C) \text{ for all } B, C \in \mathcal{K}(X).$$

Its unique fixed point $A \in \mathcal{K}(X)$ *satisfies*

$$A = W(A) = \cup_{n=1}^{N} T_n(A)$$

and is given by $A = \lim_{n\to\infty} W^n(B)$ *for any* $B \in \mathcal{K}(X)$.

Proof. By Proposition 8.1.4, $W : \mathcal{K}(X) \to \mathcal{K}(X)$ is a contraction mapping. Hence the result follows from the Banach contraction principle. ∎

8.2 Best approximation theory

Recall that when C is a nonempty subset of a normed space X, the set of best approximation to $x \in X$ from C is

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\},$$

where P_C is the metric projection from X into 2^C.

We begin with the fundamental result concerning invariance best approximation.

Theorem 8.2.1 ***(Brosowski's theorem)*** – *Let* X *be a Banach space and* $T : X \to X$ *a nonexpansive mapping with a fixed point* $\overline{x} \in X$. *Let* C *be a nonempty subset of* X *such that* $T(C) \subseteq C$. *Suppose* $P_C(\overline{x})$ *is a nonempty compact convex subset of* C. *Then* T *has a fixed point in* $P_C(\overline{x})$.

Proof. First, we show that $T : P_C(\overline{x}) \to P_C(\overline{x})$. Let $y \in P_C(\overline{x})$. Then

$$\|Ty - \overline{x}\| \leq \|y - \overline{x}\| = \inf_{z\in C} \|\overline{x} - z\|$$

implying that $Ty \in P_C(\overline{x})$.

Let $\{a_n\}$ be a sequence in $(0, 1)$ such that $a_n \to 1$. Define $T_n : P_C(x) \to P_C(\overline{x})$ by

$$T_n x = (1 - a_n)u + a_n Tx \text{ for all } x \in P_C(\overline{x}).$$

By the Banach contraction principle, each T_n has a unique fixed point, say x_n. Because $P_C(\overline{x})$ is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to z \in P_C(\overline{x})$. Because $\|x_n - Tx_n\| \to$ as $n \to \infty$, it follows that $z \in F(T)$. ∎

The following result is an improvement of Brosowski's theorem.

Theorem 8.2.2 *Let X be a Banach space and $T : X \to X$ a nonexpansive mapping with a fixed point $\overline{x} \in X$. Let C be a nonempty subset of X such that $T(\partial C) \subseteq C$. Suppose $P_C(\overline{x})$ is weakly compact and star-shaped. Assume that*

(i) T is nonexpansive on $P_C(\overline{x})$,

(ii) $\|Tx - T\overline{x}\| \leq \|x - \overline{x}\|$ for all $x \in P_C(\overline{x})$,

(iii) $I - T$ is demiclosed on $P_C(\overline{x})$.

Then $P_C(\overline{x}) \cap F(T) \neq \emptyset$.

Proof. First, we show that T is a self-mapping on $P_C(\overline{x})$. Let $y \in P_C(\overline{x})$. Theorem 2.10.10 implies that $y \in \partial C$. As $T(\partial C) \subseteq C$, so $Ty \in C$. Because $T\overline{x} = \overline{x}$ and T is nonexpansive, we have

$$\|Ty - \overline{x}\| = \|Ty - T\overline{x}\| \leq \|y - \overline{x}\| = \inf_{z \in C} \|\overline{x} - z\|.$$

Because $Ty \in C$, it follows that $Ty \in P_C(\overline{x})$. Hence T is a self-mapping on $P_C(\overline{x})$. Now, let p be the star-center of $P_C(\overline{x})$ and $\{t_n\}$ a sequence of real numbers in $(0,1)$ with $t_n \to 1$. Define $T_n : P_C(\overline{x}) \to P_C(\overline{x})$ by

$$T_n x = (1 - t_n)p + t_n Tx, \quad x \in P_C(\overline{x}).$$

For each $n \in \mathbb{N}$, T_n is a contraction, so there exists exactly one fixed point x_n of T_n.

Now

$$\|x_n - Tx_n\| \quad = \quad (1 - t_n)\|Tx_n - p\| \leq (1 - t_n)K \to 0 \text{ as } n \to \infty$$

for some $K \geq 0$. Because $P_C(\overline{x})$ is weakly compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in P_C(\overline{x})$. Because $I - T$ is demiclosed on $P_C(\overline{x})$ and $x_{n_i} - Tx_{n_i} \to 0$ as $i \to \infty$, it follows that $z \in F(T)$. Therefore, $P_C(\overline{x}) \cap F(T) \neq \emptyset$. ∎

8.3 Solutions of operator equations

In this section, we study applications of fixed point theorems to solutions of operator equations in Banach spaces.

We give an application of the Browder-Göhde-Kirk's fixed point theorem for the existence of solutions of the operator equation $x - Tx = f$.

Theorem 8.3.1 *Let X be a uniformly convex Banach space X, f an element in X, and $T : X \to X$ a nonexpansive mapping. Then the operator equation*

$$x - Tx = f \tag{8.1}$$

has a solution x if and only if for any $x_0 \in X$, the sequence of Picard iterates $\{x_n\}$ in X defined by $x_{n+1} = Tx_n + f$, $n \in \mathbb{N}_0$ is bounded.

Proof. Let T_f be the mapping from X into X given by

$$T_f(u) = Tu + f.$$

Then u is a solution of (8.1) if and only if u is a fixed point of T_f. It is easy to see that T_f is nonexpansive. Suppose T_f has a fixed point $u \in X$. Then

$$\|x_{n+1} - u\| \leq \|x_n - u\| \text{ for all } n \in \mathbb{N}.$$

Hence $\{x_n\}$ is bounded.

Conversely, suppose that $\{x_n\}$ is bounded. Let $d = diam(\{x_n\})$ and

$$B_d[x] = \{y \in X : \|x - y\| \leq d\} \text{ for each } x \in X.$$

Set $C_n := \cap_{i \geq n} B_d[x_i]$. Then C_n is a nonempty convex set for each n, and $T_f(C_n) \subset C_{n+1}$. Let C be the closure of the union of C_n for $n \in \mathbb{N}$, i.e., $C = \overline{\cup_{n \in \mathbb{N}} C_n}$. Because C_n increases with n, C is closed convex bounded subset of X. Because T_f maps C into itself, T_f has a fixed point in C by the Browder-Göhde-Kirk theorem. ∎

Next we show that if T is nonexpansive, then the Mann iteration converges strongly to the solution of the operator equation $x + Tx = f$.

Theorem 8.3.2 *Let H be a Hilbert space and $T : H \to H$ a monotonic non-expansive operator. For $f \in H$, define $S : H \to H$ by*

$$Sx = -Tx + f, \quad x \in H. \tag{8.2}$$

Then we have the following:

(a) The Mann iteration $\{x_n\}$ defined by

$$x_{n+1} = M(x_n, \alpha_n, S) \text{ with } \alpha_n \in [0,1], \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty,$$

converges strongly to the unique solution $x = v$ of the operator equation

$$x + Tx = f. \tag{8.3}$$

(b) If $\alpha_n = (n+1)^{-1}$, then $\|x_{n+1} - v\| \leq (n+1)^{-1/2}\|x_1 - v\|$ for all $n \in \mathbb{N}$.

Proof. From (8.2), we have

$$\|Sx - Sy\| = \|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H.$$

The solution v of operator equation (8.3) is a fixed point of the nonexpansive operator S. By the monotonicity of T,

$$\langle Sx - Sy, x - y\rangle = -\langle Tx - Ty, x - y\rangle \leq 0 \text{ for all } x, y \in H.$$

Hence

$$\begin{aligned}\|x_{n+1}-v\|^2 &= \|(1-\alpha_n)(x_n-v)+\alpha_n(Sx_n-Sv)\|^2\\ &= (1-\alpha_n)^2\|x_n-v\|^2+2\alpha_n(1-\alpha_n)\langle Sx_n-Sv, x_n-v\rangle\\ &\quad +\alpha_n^2\|Sx_n-Sv\|^2\\ &\le (1-\alpha_n)^2\|x_n-v\|^2+\alpha_n^2\|Sx_n-Sv\|^2 \quad (\text{as } \alpha_n(1-\alpha_n)\ge 0)\\ &\le (1-\alpha_n)^2\|x_n-v\|^2+\alpha_n^2\|x_n-v\|^2, \qquad (8.4)\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-v\|^2 &\le [1-2\alpha_n(1-\alpha_n)]\|x_n-v\|^2\\ &\le \prod_{i=1}^{n}[1-2\alpha_i(1-\alpha_i)]\|x_1-v\|^2. \qquad (8.5)\end{aligned}$$

Because $\sum_{i=1}^{\infty}\alpha_i(1-\alpha_i)=\infty$, it follows that

$$\prod_{i=1}^{n}[1-2\alpha_i(1-\alpha_i)] \le exp\bigg(-2\sum_{i=1}^{n}\alpha_i(1-\alpha_i)\bigg) \to 0 \text{ as } n\to\infty.$$

Hence from (8.5), $\{x_n\}$ converges strongly to the unique solution of the operator equation (8.3).

(b) Note

$$\begin{aligned}\|x_{n+1}-v\| &= \|(1-\alpha_n)(x_n-v)+\alpha_n(Sx_n-v)\|\\ &\le (1-\alpha_n)\|x_n-v\|+\alpha_n\|Sx_n-v\|\\ &\le \|x_n-v\|\le\cdots\le\|x_1-v\| \text{ for all } n\in\mathbb{N}.\end{aligned}$$

Hence from (8.4), we have

$$\|x_{n+1}-v\|^2 \le \left(\frac{n}{n+1}\right)^2\|x_n-v\|^2+\left(\frac{1}{n+1}\right)^2\|x_n-v\|^2,$$

which gives that

$$(n+1)^2\|x_{n+1}-v\|^2-n^2\|x_n-v\|^2 \le \|x_1-v\|^2.$$

Summing from $n=1$ to m, we get

$$(m+1)^2\|x_{m+1}-v\|^2-\|x_1-v\|^2 \le m\|x_1-v\|^2.$$

Hence

$$\|x_{m+1}-v\| \le \frac{1}{\sqrt{m+1}}\|x_1-v\| \text{ for all } m\in\mathbb{N}. \quad \blacksquare$$

8.4 Differential and integral equations

Let $f(x, y)$ be a continuous real-valued function on $[a, b] \times [c, d]$. The Cauchy initial value problem is to find a continuous differentiable function y on $[a, b]$ satisfying the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{8.6}$$

Consider the Banach space $C[a, b]$ of continuous real-valued functions with supremum norm defined by $\|y\| = \sup\{|y(x)| : x \in [a, b]\}$.

Integrating (8.6), we obtain an integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{8.7}$$

The problem (8.6) is equivalent the problem solving the integral equation (8.7).

We define an integral operator $T : C[a, b] \to C[a, b]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Thus, a solution of Cauchy initial value problem (8.6) corresponds with a fixed point of T. One may easily check that if T is contraction, then the problem (8.6) has a unique solution.

Now our purpose is to impose certain conditions on f under which the integral operator T is Lipschitzian with $\sigma(T) < 1$.

Theorem 8.4.1 *Let $f(x, y)$ be a continuous function of $Dom(f) = [a, b] \times [c, d]$ such that f is Lipschitzian with respect to y, i.e., there exists $L > 0$ such that*

$$|f(x, u) - f(x, v)| \leq L|u - v| \text{ for all } u, v \in [c, d] \text{ and for } x \in [a, b].$$

Suppose $(x_0, y_0) \in int(Dom(f))$. Then for sufficiently small $h > 0$, there exists a unique solution of the problem (8.6).

Proof. Let $M = \sup\{|f(x, y)| : x, y \in Dom(f)\}$ and choose $h > 0$ such that $Lh < 1$ and $[x_0 - h, x_0 + h] \subseteq [a, b]$. Set

$$C := \{y \in C[x_0 - h, x_0 + h] : |y(x) - y_0| \leq Mh\}.$$

Then C is a closed subset of the complete metric space $C[x_0 - h, x_0 + h]$ and hence C is complete. Note $T : C \to C$ is a contraction mapping. Indeed, for $x \in [x_0 - h, x_0 + h]$ and two continuous functions $y_1, y_2 \in C$, we have

$$\begin{aligned} \|Ty_1 - Ty_2\| &= \left\| \int_{x_0}^{x} f(x, y_1) - f(x, y_2)dt \right\| \\ &\leq |x - x_0| \sup_{s \in [x_0 - h, x_0 + h]} L|y_1(s) - y_2(s)| \\ &\leq Lh\|y_1 - y_2\|. \end{aligned}$$

Therefore, T has a unique fixed point implying that the problem (8.6) has a unique solution. ∎

Now, consider the Fredholm integral equation for an unknown function $y : [a,b] \to \mathbb{R}$ $(-\infty < a < b < \infty)$:

$$y(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt, \tag{8.8}$$

where

$$k(x,t) \text{ is continuous on } [a,b] \times [a,b]$$

and

$$f(x) \text{ is continuous on } [a,b].$$

Consider the Banach space $X = C[a,b]$ of continuous real-valued functions with supremum norm $\|y\| = \sup\{|y(x)| : x \in [a,b]\}$ and define an operator $T : C[a,b] \to C[a,b]$ by

$$Ty(x) = f(x) + \lambda \int_a^b k(x,t)y(t)dt. \tag{8.9}$$

Thus, a solution of Fredholm integral equation (8.8) is a fixed point of T.

We now impose a restriction on the real number λ such that T becomes a contraction.

Theorem 8.4.2 *Let $K(x,t)$ be a continuous function on $[a,b] \times [a,b]$ with $M = \sup\{|k(x,t)| : x,t \in [a,b]\}$, f a continuous function on $[a,b]$, and λ a real number such that $M(b-a)|\lambda| < 1$. Then the Fredholm integral equation (8.8) has a unique solution.*

Proof. It is sufficient to show that the mapping T defined by (8.9) is a contraction. For two continuous functions $y_1, y_2 \in C[a,b]$, we have

$$\begin{aligned}
\|Ty_1 - Ty_2\| &= \sup_{x\in[a,b]} |\lambda|| \int_a^b k(x,t)[y_1(t) - y_2(t)]dt| \\
&\leq |\lambda| \sup_{x\in[a,b]} \int_a^b |k(x,t)||y_1(t) - y_2(t)|dt \\
&\leq |\lambda| M \int_a^b \sup_{t\in[a,b]} |y_1(t) - y_2(t)|dt \\
&= |\lambda| M \|y_1 - y_2\| \int_a^b dt. \quad ∎
\end{aligned}$$

8.5 Variational inequality

Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C, and $A : C \to C$ a mapping. We consider the following variational inequality $VI_D(C, I - A)$:

$$\text{to find a } z \in D \text{ such that } \langle (I - A)z, J(z - y)\rangle \leq 0 \text{ for all } y \in D,$$

where J is the duality mapping from X into X^*.

The set of solutions of the variational inequality $VI_D(C, I - A)$ is denoted by $\Omega_D(I - A)$, i.e.,

$$\Omega_D(I - A) = \{z \in C : \langle (I - A)z, J(z - y)\rangle \leq 0 \text{ for all } y \in D\}.$$

Proposition 8.5.1 *Let C be a nonempty convex subset of a smooth Banach space X. Let $A : C \to C$ be strongly pseudocontractive with constant $k \in [0, 1)$. Then variational inequality $VI_D(C, I - A)$ has at most one solution.*

Proof. Let x^* and y^* be two distinct solutions of $VI_D(C, I - A)$. Then

$$\langle x^* - Ax^*, J(x^* - y^*)\rangle \leq 0$$

and

$$\langle y^* - Ay^*, J(y^* - x^*)\rangle \leq 0.$$

Adding these inequalities, we get

$$\langle x^* - y^* - (Ax^* - Ay^*), J(x^* - y^*)\rangle \leq 0,$$

which implies that

$$\begin{aligned} \|x^* - y^*\|^2 &\leq \langle Ax^* - Ay^*, J(x^* - y^*)\rangle \\ &\leq k\|x^* - y^*\|^2, \end{aligned}$$

a contradiction. Therefore, $x^* = y^*$. ∎

Proposition 8.5.2 *Let C be a nonempty convex subset of a smooth Banach space X and D a nonempty subset of C. Let $A : C \to C$ be a mapping and let P be the sunny nonexpansive retraction from C onto D. Then following are equivalent:*

(a) z is a fixed point of PA.

(b) z is a solution of variational inequality $VI_D(C, I - A)$.

Proof. Let $x \in C$ and $x_0 \in D$. Then from Proposition 2.10.21, we have

$$x_0 = Px \text{ if and only if } \langle x_0 - x, J(x_0 - y)\rangle \leq 0 \text{ for all } y \in D. \tag{8.10}$$

For $z \in C$, we obtain from (8.10) that

$$z = Pfz \text{ if and only if } \langle z - fz, J(z - y)\rangle \leq 0 \text{ for all } y \in D.$$ ∎

Theorem 8.5.3 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex bounded subset of X, $A : C \to C$ a continuous strongly pseudocontractive mapping with constant $k \in [0, 1)$, and $T : C \to X$ a weakly inward continuous pseudocontractive mapping. Suppose every nonempty closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then we have the following:*

(a) for each $t \in (0, 1)$, there exists a path $\{x_t\}$ in C defined by

$$x_t = (1 - t)Ax_t + tTx_t, \tag{8.11}$$

(b) $\{x_t\}$ converges strongly to $z \in F(T)$ as $t \to 1$,

(c) z is the unique solution of the variational inequality $VI_{F(T)}(C, I - A)$.

Proof. (a) For each $t \in (0, 1)$, define the mapping $G_t : C \to X$ by

$$G_t^A x = (1 - t)Ax + tTx, \quad x \in C.$$

Note for each $t \in (0, 1)$, G_t^A is weakly inward continuous strongly pseudocontractive. By Corollary 5.7.14, G_t^A has exactly one fixed point x_t in C.

(b) Because the mapping $(2I - T)$ has a nonexpansive inverse g, then g maps C into itself. Note $x_t - Tx_t \to 0$ as $t \to 1$ implies that $x_t - g(x_t) \to 0$ as $t \to 1$. Write $x_n = x_{t_n}$. Then, we have

$$x_n - Tx_n \to 0 \text{ as } n \to \infty. \tag{8.12}$$

Because $\{x_n\}$ is bounded, we define $\varphi : C \to \mathbb{R}^+$ by $\varphi(z) = LIM_n \|x_n - z\|^2, z \in C$. As in proof of Theorem 7.1.12, the set M defined by (2.32) is a nonempty closed convex bounded g-invariant subset of C and hence the nonexpansive mapping g has a fixed point in M by assumption. Denote such a fixed point by v.

On the other hand, by monotonicity of $I - T$, we have

$$\langle x_t - Tx_t, J(x_t - y)\rangle \geq 0 \text{ for all } y \in F(T).$$

From (8.11), we have $x_t - Tx_t = \frac{1-t}{t}(Ax_t - x_t)$. Thus,

$$\langle x_t - Ax_t, J(x_t - y)\rangle \leq 0 \text{ for all } y \in F(T). \tag{8.13}$$

Clearly, the sequence $\{x_n\}$ satisfies (7.7). Therefore, by Theorem 7.1.10, $\{x_n\}$ converges strongly to the fixed point v.

We finally prove that the path $\{x_t\}$ converges strongly. Toward this end, we assume that $\{t_{n'}\}$ is another subsequence in $(0, 1)$ such that $x_{t_{n'}} \to v'$ as $t_{n'} \to 1$. By (8.12), we obtain $v' \in F(T)$. From (8.13), we have that

$$\langle v - Av, J(v - v')\rangle \leq 0 \text{ and } \langle v' - Av', J(v' - v)\rangle \leq 0.$$

We must have $v = v'$. Therefore, $\{x_t\}$ converges strongly to $v \in F(T)$.

(c) Because $x_t \to v \in F(T)$ as $t \to 1$, it follows from (8.13) that

$$\langle v - Av, J(v - y)\rangle \leq 0 \text{ for all } y \in F(T). \qquad \blacksquare$$

Corollary 8.5.4 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $A : C \to C$ a contraction mapping with Lipschitz constant $k \in [0, 1)$, and $T : C \to C$ a nonexpansive with $F(T) \neq \emptyset$. Suppose every nonempty closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then conclusions of Theorem 8.5.3 hold.*

Proof. It suffices to show that the path $\{x_t\}$ defined by (8.11) is bounded. Let $y \in F(T)$. Then

$$\begin{aligned} \|x_t - y\| &\leq (1-t)\|Ax_t - y\| + t\|Tx_t - y\| \\ &\leq (1-t)(\|Ax_t - Ay\| + \|Ay - y\|) + t\|Tx_t - y\| \\ &\leq (1-t)(k\|x_t - y\| + \|Ay - y\|) + t\|x_t - y\| \end{aligned}$$

which implies that

$$\|x_t - y\| \leq \frac{1}{1-k}\|Ay - y\|.$$

Hence $\{x_t\}$ is bounded. ∎

Theorem 8.5.5 *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-mappings. Then $F(T)$ is the sunny nonexpansive retract of C.*

Proof. The proof is followed by Corollary 8.5.4.

8.6 Variational inclusion problem

In this section, we study the existence and uniqueness of solutions and the convergence of the Mann iteration for a variational inclusion problem in a Banach space.

Let A, T be two self-mappings defined on a Banach space X, $g : X \to X^*$ another mapping, and $\varphi : X^* \to (-\infty, \infty]$ a proper convex lower semicontinuous function. Let us consider the following variational inclusion problem:

$$\begin{cases} \text{to find an } u \in X \text{ such that } g(u) \in Dom(\partial\varphi), \\ \langle Tu - Au - f, v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v) \text{ for all } v \in X^*. \end{cases} \tag{8.14}$$

We begin with the following basic result:

Proposition 8.6.1 *Let X be a reflexive Banach space. Then the following are equivalent:*

(a) $x^* \in X$ *is a solution of variational inclusion problem (8.14).*

(b) $x^* \in X$ *is a fixed point of the mapping* $S : X \to 2^X$*:*

$$S(x) = f - (Tx - Ax + \partial\varphi(g(x)) + x,\ x \in X.$$

(c) $x^* \in X$ *is a solution of equation* $f \in Tx - Ax + \partial\varphi(g(x)),\ x \in X.$

Proof. (a) $\Rightarrow$ (c). If x^* is a solution of the variational inclusion problem (8.14), then $g(x^*) \in Dom(\partial\varphi)$ and

$$\langle Tx^* - Ax^* - f, v - g(x^*)\rangle \geq \varphi(g(x^*)) - \varphi(v) \text{ for all } v \in X^*.$$

By the definition of subdifferential of φ, it follows from the above expression that

$$f + Ax^* - Tx^* \in \partial\varphi(g(x^*)). \tag{8.15}$$

This implies that x^* is a solution of equation $f \in Tx - Ax + \partial\varphi(g(x))$.

(c) $\Rightarrow$ (b). Adding x^* to both sides of (8.15), we have

$$x^* \in f - (Tx^* - Ax^* + \partial\varphi(g(x^*))) + x^* = Sx^*. \tag{8.16}$$

This implies that x^* is a fixed point of S in X.

(b) $\Rightarrow$ (a). From (8.16), we have $f - (Tx^* - Ax^*) \in \partial\varphi(g(x^*))$, hence from the definition of $\partial\varphi$, it follows that

$$\varphi(v) - \varphi(g(x^*)) \geq \langle f - (Tx^* - Ax^*), v - g(x^*)\rangle \text{ for all } v \in X^*,$$

i.e.,

$$\langle Tx^* - Ax^* - f, v - g(x^*)\rangle \geq \varphi(g(x^*)) - \varphi(v) \text{ for all } v \in X^*.$$

Thus, x^* is a solution of the variational inclusion problem (8.14). ∎

Theorem 8.6.2 *Let* X *be a uniformly smooth Banach space and let* $T, A : X \to X, g : X \to X^*$ *be three continuous mappings. Let* $\varphi : X^* \to (-\infty, \infty]$ *be a function with a continuous Gâteaux differential* $\partial\varphi$ *and satisfying the following conditions:*

(i) $T - A : X \to X$ *is a strongly accretive mapping with constant* $k \in (0, 1)$*,*

(ii) $\varphi o g : X \to X$ *is accretive.*

For any given $f \in X$*, define a mapping* $S : X \to X$ *by*

$$Sx = f - (Tx - Ax + \partial\varphi(g(x))) + x, \quad x \in X. \tag{8.17}$$

If $S(X)$ *is bounded, then for any given* $x_1 \in X$*, the Mann iterative sequence* $\{x_n\}$ *defined by*

$$x_{n+1} \;=\; M(x_n,\ \alpha_n, S), \quad n \in \mathbb{N}, \tag{8.18}$$

where $\{\alpha_n\}$ *is the sequence in* $[0,1]$ *with the restriction*

$$\alpha_n \to 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \tag{8.19}$$

converges strongly to the unique solution of variational inclusion (8.14).

Proof. First we show that the variational inclusion problem (8.14) has a unique solution.

From conditions $(i) \sim (ii)$ and Proposition 5.7.12, the mapping $T - A + \partial\varphi og : X \to X$ is a strongly accretive continuous mapping with a strongly accretive constant $k \in (0,1)$. Because $T - A + \partial og$ is a continuous strongly accretive mapping, $T - A + \partial\varphi og$ is surjective. Therefore, for any given $f \in X$, the equation $f = (T - A + \partial\varphi og)(x)$ has a solution x^*. Because X is reflexive, by Proposition 8.6.1, x^* is a solution of variational inclusion (8.14), and it is also a fixed point of the self-mapping S defined by (8.17).

We now break the proof into the following three steps:

Step 1: x^* *is the unique solution of (8.14) in* X.

Suppose, for contradiction, $u^* \in X$ is another solution of (8.14). Then, u^* is also a fixed point of S. Hence, we have

$$\begin{aligned}
\|x^* - u^*\|^2 &= \langle x^* - u^*, J(x^* - u^*)\rangle \\
&= \langle Sx^* - Su^*, J(x^* - u^*)\rangle \\
&= \langle f - (T - A + \partial\varphi og)(x^*) + x^* \\
&\quad -(f - (T - A + \partial\varphi og)(u^*) + u^*), J(x^* - u^*)\rangle \\
&= \|x^* - u^*\|^2 - \langle (T - A + \partial\varphi og)(x^*) \\
&\quad -(T - A + \partial\varphi og)(u^*), J(x^* - u^*)\rangle \\
&\le \|x^* - u^*\|^2 - k\|x^* - u^*\|^2,
\end{aligned}$$

a contradiction. Hence $x^* = u^*$.

Step 2: $\{x_n\}$ *is bounded.*

Because $S(X)$ is bounded, let

$$K = \sup\{\|Sx - x^*\| + \|x_1 - x^*\| : x \in X\}. \tag{8.20}$$

Now we show that

$$\|x_n - x^*\| \le K \text{ for all } n \in \mathbb{N}. \tag{8.21}$$

In fact, for $n = 1$ it follows from (8.20) that $\|x_1 - x^*\| \le K$. Suppose (8.21) is true for $n = k \ge 1$, then for $n = k + 1$, we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|(1 - \alpha_k)(x_k - x^*) + \alpha_k(Sx_k - x^*)\| \\
&\le (1 - \alpha_k)\|x_k - x^*\| + \alpha_k\|Sx_k - x^*\| \\
&\le K,
\end{aligned}$$

Hence $\{x_n\}$ is bounded.

Step 3: $x_n \to x^*$.

From (8.18) and Proposition 2.4.6(b), we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)(x_n-x^*)+\alpha_n(Sx_n-x^*)\|^2\\ &\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\langle Sx_n-x^*, J(x_{n+1}-x^*)\rangle\\ &= (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\langle Sx_n-x^*, J(x_n-x^*)\rangle\\ &\quad +2\alpha_n\langle Sx_n-x^*, J(x_{n+1}-x^*)-J(x_n-x^*)\rangle. \qquad (8.22)\end{aligned}$$

Observe that

$$\begin{aligned}\langle Sx_n-x^*, J(x_n-x^*)\rangle &= \langle f-(T-A+\partial\varphi og)(x_n)+x_n\\ &\quad -(f-(T-A+\partial\varphi og)(x^*)-x^*), J(x_n-x^*)\rangle\\ &= \|x_n-x^*\|^2-\langle (T-A+\partial\varphi og)(x_n)\\ &\quad -(T-A+\partial\varphi og))(x^*), J(x_n-x^*)\rangle\\ &\le (1-k)\|x_n-x^*\|^2.\end{aligned}$$

Set $\beta_n := |\langle Sx_n-x^*, J(x_{n+1}-x^*)-J(x_n-x^*)\rangle|$. Then

$$\beta_n \le K\|J(x_{n+1}-x^*)-J(x_n-x^*)\|_*.$$

Observe that

$$\begin{aligned}x_{n+1}-x^*-(x_n-x^*) &= x_{n+1}-x_n\\ &= \alpha_n(Sx_n-x_n). \qquad (8.23)\end{aligned}$$

Again because $\{x_n\}, \{Sx_n\}$ are bounded, and $\alpha_n \to 0$, hence, from (8.23), we have

$$x_{n+1}-x^*-(x_n-x^*) \to 0 \text{ as } n\to\infty.$$

By the uniform continuity of J, $\|J(x_{n+1}-x^*)-J(x_n-x^*)\|_* \to 0$. Thus, we have

$$\beta_n \to 0. \qquad (8.24)$$

So from (8.22), (8.23), and (8.24), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le [(1-\alpha_n)^2+2\alpha_n(1-k)]\|x_n-x^*\|^2+2\alpha_n\beta_n\\ &= (1+\alpha_n^2-2\alpha_n k)\|x_n-x^*\|^2+2\alpha_n\beta_n\\ &= [1-\alpha_n k+\alpha_n(\alpha_n-k)]\|x_n-x^*\|^2+2\alpha_n\beta_n. \qquad (8.25)\end{aligned}$$

Because $\alpha_n \to 0$, there exists an $n_0 \in \mathbb{N}$ such that $\alpha_n < k$ for $n \ge n_0$. Hence for any $n \ge n_0$, from (8.25) we have

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n k)\|x_n-x^*\|^2+2\alpha_n\beta_n. \qquad (8.26)$$

Let $a_n = \|x_n-x^*\|^2, t_n = \alpha_n k, b_n = 2\alpha_n\beta_n$. Therefore, by Lemma 6.1.6, $x_n \to x^*$ as $n\to\infty$. ∎

Bibliographic Notes and Remarks

The results presented in Section 8.1 may be found in Barnsly [12]. Theorem 8.2.1 is due to Brosowski [24]. Theorem 8.2.2 follows from Carbone [34]. General best approximation theory is well described in Singh, Watson, and Srivastava [150].

The results described in Section 8.3 are stated in Browder and Petryshyn [30] and Dotson [48]. Theorems 8.4.1 and 8.4.2 can be found in standard books (Goebel and Kirk [59], Kreyszig [94], and Zeidler [171]).

The existence and approximation of solutions of variational inclusion problem (8.14) were studied in Chang [37].

Exercises

8.1 Let C be a closed convex cone in a Hilbert space H and $T : C \to H$ a mapping. Show that the following are equivalent:

(a) Find $\overline{x} \in C$ such that

$$T\overline{x} \in C^* \text{ (dual cone) and } \langle \overline{x}, T\overline{x} \rangle = 0. \tag{C.P.}$$

(b) Find $\overline{x} \in C$ such that $g\overline{x} = \overline{x}$, where $g : C \to C$ is defined by $gx = P_C(x - \rho Tx)$, and $\rho > 0$ is a real number.

8.2 Let C be a closed convex cone of a Hilbert space H. Show that the complementarity problem (C.P.) has a solution if and only if $T(x) = P_C x - TP_C x$ for $x \in H$ has a fixed point in H. If $x_0 = Tx_0$, show that $\overline{x} = P_C x_0$ is a solution of the complementarity problem.

8.3 Let C be a nonempty compact convex subset of a normed pace X and $T : C \to X$ a continuous mapping. Show that there exists a point $u \in C$ such that $\|u - Tu\| = d(Tu, C)$.

8.4 Let $K(s,t)$ and $w(s,t)$ be continuous real functions on the unit square $[0,1]^2$, and let $v(s)$ be a continuous real function on $[0,1]$. Suppose that

$$|w(s,t_1) - w(s,t_2)| \leq N|t_1 - t_2| \text{ for all } 0 \leq t_1, t_2, s \leq 1.$$

Show that there is a unique continuous real function $y(s)$ on $[0,1]$ such that

$$y(s) = v(s) + \int_0^s K(s,t)w(t, y(t))dt.$$

8.5 Let $K(s,t,u)$ be a continuous function on $0 \leq s,t \leq 1, u \geq 0$ such that

$$|K(x,t,u_1) - K(s,t,u_2)| \leq N(s,t)|u_1 - u_2|,$$

where $N(s,t)$ is a continuous function satisfying

$$\int_0^1 N(s,t)dt \leq k < 1,$$

for every $0 \le s \le 1$. Show that for every $y \in C[0,1]$, there exists a unique function $y \in C[0,1]$ such that

$$y(s) = v(s) + \int_0^1 K(s,t,y(t))dt.$$

Appendix A

A.1 Basic inequalities

Lemma A.1.1 *Let $a, b \in \mathbb{R}^+$ and $2 \le p < \infty$. Then we have the following:*

(a) $a^p + b^p \le (a^2 + b^2)^{p/2}$,

(b) $(a^2 + b^2)^{p/2} \le 2^{(p-2)/2}(a^p + b^p)$.

Proof. We note that both the inequalities hold if either a or b is zero. So we prove the Lemma for $a \ne 0$ and $b \ne 0$.

(a) Because

$$\frac{a^2}{a^2+b^2} \le 1 \text{ and } \frac{b^2}{a^2+b^2} \le 1,$$

we have

$$\begin{aligned} \frac{a^p}{(a^2+b^2)^{p/2}} + \frac{b^p}{(a^2+b^2)^{p/2}} &= \left(\frac{a^2}{a^2+b^2}\right)^{p/2} + \left(\frac{b^2}{a^2+b^2}\right)^{p/2} \\ &\le \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} \qquad \text{(since } p/2 \le 1) \\ &= 1 \end{aligned}$$

(b) It is obvious for $p = 2$. So, assume that $p > 2$. Set $p' := p/2 > 1$ and $q' = p'/(p'-1) = p/(p-2)$. Then $1/p' + 1/q' = 1$. By Holder's inequality, we have

$$\begin{aligned} a^2 + b^2 &\le ((a^2)^{p'} + (b^2)^{p'})^{1/p'}(1^{q'} + 1^{q'})^{1/q'} \\ &= 2^{(p-2)/p}(a^p + b^p)^{2/p}, \end{aligned}$$

which implies that

$$(a^2 + b^2)^{p/2} \le 2^{(p-2)/2}(a^p + b^p). \quad \blacksquare$$

A.2 Partially ordered set

Let $\leq$ be a relation on a nonempty set X. Then the relation $\leq$ is said to be partially ordered if it is

(i) reflexive: $a \leq a$ for all $a \in X$;

(ii) antisymmetric: $a \leq b$ and $b \leq a \Rightarrow a = b$ for some $a, b \in X$;

(iii) transitive: $a \leq b$ and $b \leq c \Rightarrow a \leq c$ for some $a, b, c \in X$.

The ordered pair $(X, \leq)$ is called a partially ordered set.

Totally ordered (or linearly ordered) set - Let $(X, \leq)$ be a partially ordered set. Then a subset S of X is said to be totally ordered or linearly ordered if for all $a, b \in S$ either $a \leq b$ or $b \leq a$, i.e., all the elements of S are comparable.

Infimum and supremum of a set - Let X be a partially ordered set with relation $\leq$ and let S be a nonempty subset of X. Then an element $u \in X$ $(v \in S)$ is said to be an upper bound of S (a lower bound of S) if

$$x \leq u \text{ for all } x \in S \ (v \leq x \text{ for all } x \in S).$$

The least upper bound of S is called the supremum and it is denoted $\sup S$. The greatest lower bound of S is called the infimum and it is denoted by $\inf S$.

Minimum and maximum of a set - Let S be a nonempty subset of a partially ordered set $(X, \leq)$. If $\inf S$ exists and belongs to S, then it is called a minimum of S. Similarly, if $\sup S$ exists and belongs to S, then it is called a maximum of S.

Minimal and maximal elements of a set - Let $(X, \leq)$ be a partially ordered set. An element $m \in X$ is said to be minimal if $x \leq m$ for $x \in X \Rightarrow x = m$. Similarly, an element $m \in X$ is said to maximal if $m \leq x$ for $x \in X \Rightarrow x = m$.

We now state a very useful lemma that is known as Zorn's lemma.

Lemma A.2.1 (Zorn's Lemma) - *Let $(X, \leq)$ be a partially ordered set in which every chain has an upper bound. Then X has a maximal element.*

A.3 Ultrapowers of Banach spaces

Let Λ denote an index set.

Definition A.3.1 *Let $\mathcal{F}$ be a nonempty family of subsets of Λ. Then the family $\mathcal{F}$ is said to be a filter on Λ if*

(F_1) *$\mathcal{F}$ is closed under super set:*

$$A \in \mathcal{F} \text{ and } A \subseteq B \subseteq \Lambda \Rightarrow B \in \mathcal{F},$$

(F_2) *$\mathcal{F}$ is closed under intersection:*

$$A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}.$$

Observation

- 2^{Λ}, the power set of index set Λ, defines a filter.
- A filter $\mathcal{F}$ is proper if $\mathcal{F} \neq 2^{\Lambda}$.

For $i_0 \in \Lambda$, let $\mathcal{F}_{i_0} = \{A \subseteq \Lambda : i_0 \in A\}$. Then a filter of the form $\mathcal{F}_{i_0}$ for some i_0 is called a non-free filter.

Definition A.3.2 *Let $\mathcal{U}$ be a filter on Λ. Then $\mathcal{U}$ is said to be an ultrafilter on Λ if it is maximal with respect to ordering of filters on Λ by inclusions, i.e., if $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{F}$ is a filter on Λ, then $\mathcal{F} = \mathcal{U}$.*

Observation

- If $\mathcal{U}$ is an ultrafilter on Λ, then it is not properly contained in any other filter on Λ.

Definition A.3.3 *Let (X, τ) be a topological space, $\mathcal{U}$ an ultrafilter on Λ and $\{x_i\}_{i \in \Lambda}$ a subset in X. We say*

$$\lim_{\mathcal{U}} x_i (\equiv \tau - \lim_{\mathcal{U}} x_i) = x$$

if for every neighborhood U of x we have $\{i \in \Lambda : x_i \in U\} \in \mathcal{U}$.

Observation

- If X is a Hausdorff topological space, then the limit along $\mathcal{U}$ of the set $\{x_i\}_{i \in \Lambda}$ in X is always unique.
- If $\{x_n\}$ is a bounded sequence in $\mathbb{R}$ and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, then $\lim_{\mathcal{U}} x_n$ exists and
$$\liminf_{n \to \infty} x_n \leq \lim_{\mathcal{U}} x_n \leq \limsup_{n \to \infty} x_n.$$

We now give basic properties of ultrafilter.

Proposition A.3.4 *Let X be a Hausdorff topological vector space and $\mathcal{U}$ an ultrafilter over an index set Λ. If $\{x_i\}_{i \in \Lambda}$ and $\{y_i\}_{i \in \Lambda}$ are two subsets of X such that $\lim_{\mathcal{U}} x_i$ and $\lim_{\mathcal{U}} y_i$ exist,*

$$\lim_{\mathcal{U}} (x_i + y_i) = \lim_{\mathcal{U}} x_i + \lim_{\mathcal{U}} y_i$$

and

$$\lim_{\mathcal{U}} \alpha x_i = \alpha \lim_{\mathcal{U}} x_i \text{ for any scalar } \alpha \in \mathbb{K}.$$

Proposition A.3.5 *Let $\{x_n\}$ be a sequence in a metric space X and $\mathcal{U}$ an ultrafilter (over an index set $\mathbb{N}$) such that $\lim_{\mathcal{U}} x_n = x$. Then there exists a subsequence of $\{x_n\}$ that converges to x.*

Proposition A.3.6 *Let X be a Hausdorff topological space. Then X is compact if and only if $\lim_{\mathcal{U}} x_i$ exists for all $\{x_i\}_{i \in I} \subset X$ and any ultrafilter over Λ.*

Now we are in position to define an ultrapower of a Banach space:

Let X be a Banach space and $\mathcal{U}$ an ultrafilter over an index set Λ. Let $\ell_\infty(X)$ denote the space

$$\{\{y_n\} : y_n \in X \text{ and } \{\|y_n\|\} \in \ell_\infty\}$$

with the norm $\|\{y_n\}\|_{\ell_\infty(X)} := \sup_{n\in\mathbb{N}} \|y_n\|$ and let $\mathcal{N}$ be the closed subspace of $\ell_\infty(X)$

$$\{\{y_n\} : y_n \in X \text{ and } \lim_{\mathcal{U}} \|y_n\| = 0\}.$$

Definition A.3.7 *Let X be a Banach space. Then the Banach space ultrapower of X over $\mathcal{U}$ is defined to be the Banach space quotient*

$$\{X\}_{\mathcal{U}} := \ell_\infty(X)/\mathcal{N}.$$

The norm $\|\cdot\|_{\mathcal{U}}$ in $\{X\}_{\mathcal{U}}$ is the usual quotient norm, i.e., $\|\{x_i\}\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|$.

We now give some useful properties of an ultrapower of Banach spaces.

Proposition A.3.8 *Let X be a Banach space. Then $\{X\}_{\mathcal{U}}$ is a Banach space.*

Proposition A.3.9 *The ultrapower $\{X\}_{\mathcal{U}}$ of a Banach space X contains a subspace isometrically isomorphic to X.*

Observation

- X is a subspace of $\{X\}_{\mathcal{U}}$.
- $\{H\}_{\mathcal{U}}$ is a Hilbert space, i.e.,

$$\|\{x_i\} + \{y_i\}\|_{\mathcal{U}}^2 + \|\{x_i\} - \{y_i\}\|_{\mathcal{U}}^2 = 2\|\{x_i\}\|_{\mathcal{U}}^2 + 2\|\{y_i\}\|_{\mathcal{U}}^2 \text{ for all } \{x_i\}, \{y_i\} \in \{H\}_{\mathcal{U}}.$$

Proposition A.3.10 *The ultrapower $\{X\}_{\mathcal{U}}$ of a Banach space X is finitely representable in X.*

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