7

Additional Topics

This chapter includes additional examples of fractals, and hints at parts of the subject that we have not covered. In Sections 7.3 and 7.4, the computation of the fractal dimension requires more than just a simple application of the results of Chap. 6. So these examples show that there is more to the subject than we have seen in this book.

7.1 *Deconstruction

Let K be the attractor for an iterated function system (f_e) of similarities in $\mathbb{R}^d$, and let the ratio list have sim-value s. Then $\dim K \leq \operatorname{Dim} K \leq s$ (Theorem 6.4.10). If the parts $f_e[K]$ are disjoint, or have small overlap (specified by the open set condition), then $\dim K = \operatorname{Dim} K = s$ (Theorem 6.5.4). But if there is too much overlap, then the dimension could be strictly smaller than s. Sometimes the fractal dimension can be still computed by "deconstructing" the attractor—interpreting it in a manner different from the one provided by the iterated function system.

Recall the Li's lace fractal on p. 84. It is made up of blocks of two kinds, P, Q, as described on p. 126. From the decompostions of the two sets, we get $P \supseteq Q$. Note that Fig. 4.3.3 not only describes the iterated function system, but also shows that the graph open set condition is satisfied. So we have $\dim P = \dim Q = \operatorname{Dim} P = \operatorname{Dim} Q = s$, where $s = \log(\sqrt{2}+2)/\log 2 \approx 1.7716$ is the sim-value for the M-W graph shown in Fig. 4.3.4(b).†

But that was not the way in which this fractal was originally defined. Example (c) in Jun Li's thesis [43] is described using an iterated function system in the plane consisting of four similarities:

* Optional section.

† Answer for Exercise 4.3.11.

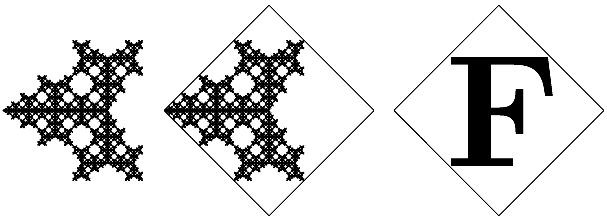


Fig. 7.1.1. A fractal and a surrounding square

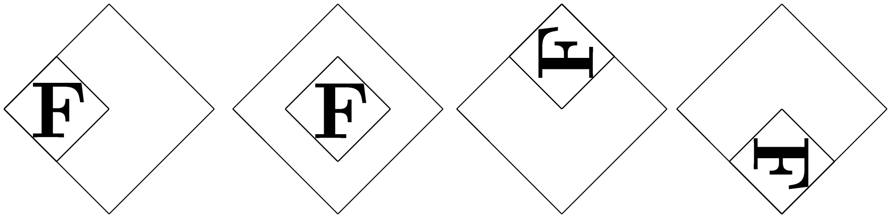


Fig. 7.1.2. Four images

$$f_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad f_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

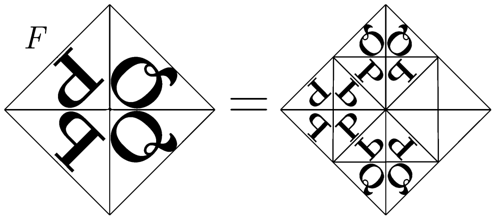
$$f_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \qquad f_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The attractor is shown in Fig. 7.1.1, and the iterated function system is illustrated in Fig. 7.1.2.

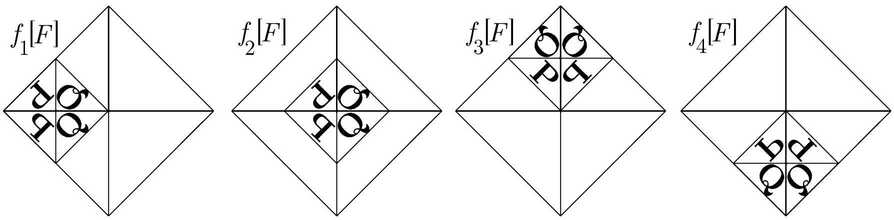
Deconstruction. We claim that if P, Q are the sets described on p. 126, and four of these are assembled into a square F as shown,

then F is the attractor of the iterated function system 7.1.2. But we know that the attractor is unique (Theorem 4.1.3), so we need only prove the self-referential equation $F = f_1[F] \cup f_2[F] \cup f_3[F] \cup f_4[F]$. We will prove this using pictures, which can sometimes be more convincing than words.

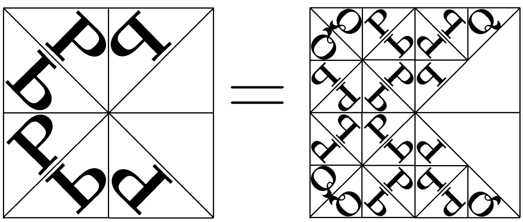
We should show that



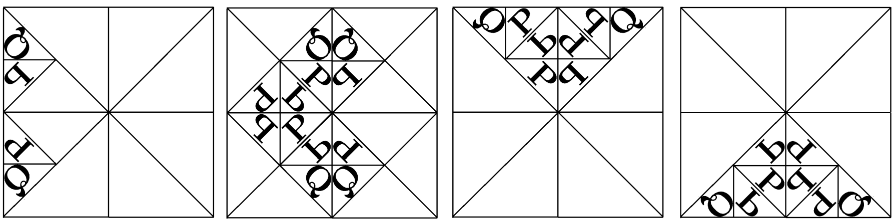
is the union of four sets:



The outer triangles match as required. The triangles in $f_2[F]$ do not align with the others, so to see how the union behaves we require another level of subdivision. We need to prove that the inner square



is the union of four sets:



Everything now matches, taking into account $P \cup P = P$ and $P \cup Q = P$.

In this example, then, the iterated function system 7.1.1 has sim-value 2, but the attractor it defines has fractal dimension $\log(\sqrt{2}+2)/\log 2 \approx 1.7716$.

More Self-Similar Sets with Overlap

Now we consider a "similarity dimension" example with overlap in $\mathbb{R}$, and leave the details to the reader. We will use the ratio list $(1/5, 1/5, 1/5)$. Let a, b, c be three real numbers. Consider the three dilations f_1, f_2, f_3 of $\mathbb{R}$ with fixed points a, b, c, respectively. For certain choices of the points a, b, c, this realization satisfies the open set condition, and the invariant fractal K has Hausdorff dimension equal to the sim-value of the ratio list. For certain other choices of a, b, c (such as two or three of them coincident) the Hausdorff dimension of K is not equal to the dimension of the ratio list. There is, nevertheless, always an inequality.

Let us normalize things by assuming $a = 0$, $c = 1$, $0 < b < 1$. (Any choice of three distinct points can be reduced to this case.) All three of the maps send $[0, 1]$ into itself, so the invariant set K is a subset of $[0, 1]$; in fact K may be constructed in the usual way by the contraction mapping theorem starting with $[0, 1]$.

Exercise 7.1.3. For what values of b are the three images of the open interval $(0, 1)$ disjoint?

Exercise 7.1.4. Compute the Hausdorff dimension when $b = 1/10$.

Exercise 7.1.5. Compute the Hausdorff dimension when $b = 1/5$.

There is a result of Falconer [24] that is relevant in situations like this. In this case it asserts that the Hausdorff dimension of the invariant set K is equal to the similarity dimension $\log 3/\log 5$ for *almost all* choices of $b \in [0, 1]$. That is, the set of all $b \in [0, 1]$ for which $\dim K = \log 3/\log 5$ fails is a set of Lebesgue measure 0.

Exercise 7.1.6. Give an example of an iterated function system of similarities in $\mathbb{R}^d$ where the Hausdorff dimension of the invariant set coincides with the similarity dimension, but Moran's open set condition fails.

Consider the Barnsley leaf fractal B defined on p. 26. It is the attractor of an iterated function system with three maps, and sim-value $2\log(1+\sqrt{2})/\log 2$. This is > 2, so it is certainly not the fractal dimension of the fractal B itself. Plate 10 shows the set B. The three images are in three colors cyan, magenta, and yellow. Where cyan and magenta overlap, it is blue. Where all three overlap it is black.

Exercise 7.1.7. Deconstruct Barnsley's leaf, and determine its fractal dimension.

7.2 *Self-Affine Sets

The idea of an iterated function system makes good sense even when the maps are not similarities. One possibility that comes up often involves affine maps. The invariant set is then said to be ***self-affine***. In the general self-affine case the evaluation of the Hausdorff dimension is not completely understood. It has even been argued [45] that the Hausdorff dimension is not the proper dimension to use at all. We will present a few examples in this section.

A Self-Affine Dust

As a reference, take the ***unit square*** in $\mathbb{R}^2$:

$$S = \{\,(x,y) :\ 0 \le x \le 1, 0 \le y \le 1\,\}\,.$$

The images of S under the two maps will be two rectangles:

$$R_1 = \{\,(x,y) :\ 0 \le x \le 1/2, 0 \le y \le 2/3\,\}$$
$$R_2 = \{\,(x,y) :\ 1/2 \le x \le 1, 1/3 \le y \le 1\,\}\,.$$

The function f_1 is an affine map of $\mathbb{R}^2$ onto itself, and sends the vertices of S to the corresponding vertices of R_1. The function f_2 is an affine map of $\mathbb{R}^2$ onto itself, and sends the vertices of S to the corresponding vertices of R_2.

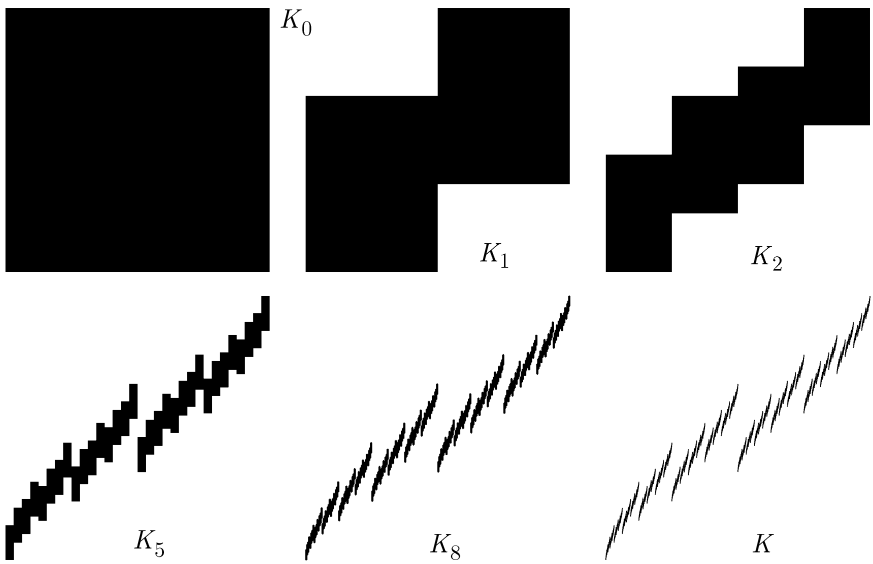


Fig. 7.2.1. Self-Affine Dust

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* Optional section.

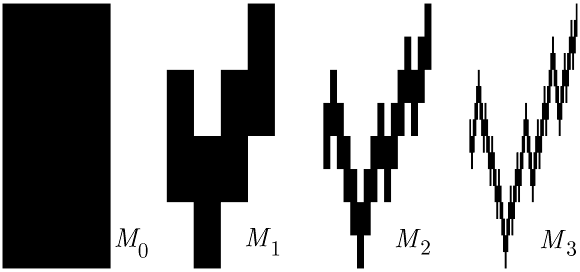

Fig. 7.2.4. Kiesswetter's curve

Exercise 7.2.2. There is a unique compact nonempty set $K \subseteq \mathbb{R}^2$ such that $K = f_1[K] \cup f_2[K]$.

Exercise 7.2.3. Compute the Hausdorff dimension of the set K.

Kiesswetter's Curve

This is illustrated in two different ways. The set can be decomposed into four subsets, which are affine images of the whole thing. The four affine maps may be written in matrix notation. A point (x, y) in the plane is identified with a 2×1 column matrix.

$$\begin{aligned}
f_1 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1/4 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
f_2 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix} \\
f_3 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \\
f_4 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/2 \end{bmatrix} .
\end{aligned}$$

The first construction starts with the rectangle $M_0 = [0, 1] \times [-1, 1]$, and at each stage replaces the current set M_n with $M_{n+1} = f_1[M_n] \cup f_2[M_n] \cup f_3[M_n] \cup f_4[M_n]$. Because each of the maps f_j sends M_0 to a subset of M_0, this results in a decreasing sequence of compact sets. Kiesswetter's curve is the intersection $\bigcap_{n \in \mathbb{N}} M_n$.

The second construction starts with the line segment from $(0, 0)$ to $(1, 1)$, and makes the same transformation as before. Since $f_j\big((1, 1)\big) = f_{j+1}\big((0, 0)\big)$ for $j = 1, 2, 3$, these sets are all polygons. They are graphs of a sequence of continuous functions defined on $[0, 1]$; this sequence converges uniformly. The limit g is called ***Kiesswetter's function***. Its graph $G = \{\, (x, y) : \ y = g(x) \,\}$ is Kiesswetter's curve.

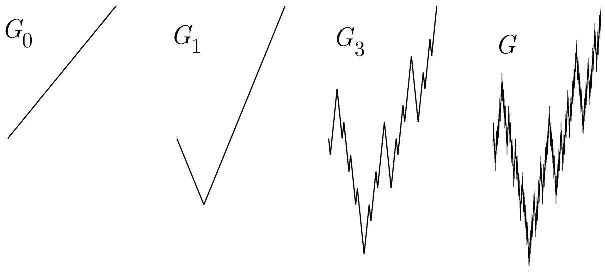

Fig. 7.2.5. Kiesswetter's curve

Exercise 7.2.6. Let g be Kiesswetter's function. Then for any integers $k \geq 0$ and $0 \leq j < 2^k$, prove

$$\left| g\left(\frac{j}{4^k}\right) - g\left(\frac{j+1}{4^k}\right) \right| = \frac{1}{2^k}.$$

Exercise 7.2.7. Kiesswetter's curve is the graph of a continuous but nowhere differentiable function $g\colon [0,1] \to \mathbb{R}$.

Exercise 7.2.8. Find the Hausdorff and packing dimensions of Kiesswetter's curve.

Besicovitch–Ursell Functions

Besicovitch and Ursell investigated the dimension of the graphs of non-differentiable functions. The most famous examples of these functions, dating back to Weierstrass, have a form

$$f(x) = \sum_{k=0}^{\infty} a_k \sin(b_k x),$$

for appropriate choices of a_k and b_k. A simpler variant was used by Besicovitch and Ursell, which will now be described.

Define a "sawtooth" function $g\colon \mathbb{R} \to \mathbb{R}$ by:

$$\begin{aligned} g(x) &= x && \text{for } -1/2 \leq x \leq 1/2 \\ g(x) &= 1 - x && \text{for } 1/2 \leq x \leq 3/2 \\ g(x+2) &= g(x) && \text{for all } x. \end{aligned}$$

If $0 < a < 1$, the ***Besicovitch–Ursell*** function with parameter a is:

$$f(x) = \sum_{k=0}^{\infty} a^k g(2^k x).$$

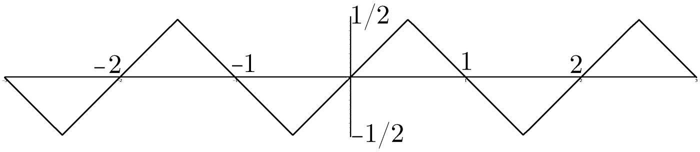

Fig. 7.2.9. Sawtooth function

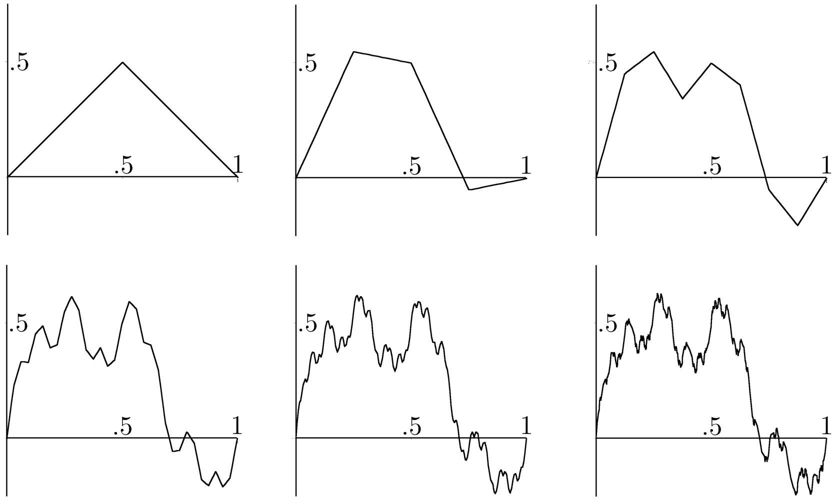

Fig. 7.2.10. Partial sums $\sum_{k=0}^{n} a^k g(2^k x)$ with $n = 0, 1, 2, 5, 6, 12$

Partial sums $\sum_{k=0}^{n} a^k g(2^k x)$ of the series are illustrated in Fig. 7.2.10, with $a = 0.6$. The pictures show only $0 \le x \le 1$, but the rest of the graph is simply related to this part.

Exercise 7.2.11. The function $f(x)$ exists and is continuous.

Exercise 7.2.12. Is the graph of f the invariant set for some iterated function system?

Pictures for various values of a are shown in Fig. 7.2.15.

Exercise 7.2.13. For what values of a is f a Lipschitz function?

Exercise 7.2.14. Compute the Hausdorff dimension of the graph of the Besicovitch-Ursell function with parameter $a = 3/5$.

Hironaka's Curve

Pictured (Fig. 7.2.16) are some approximations to ***Hironaka's curve***. The first approximation consists of two vertical line segments, one unit long, one

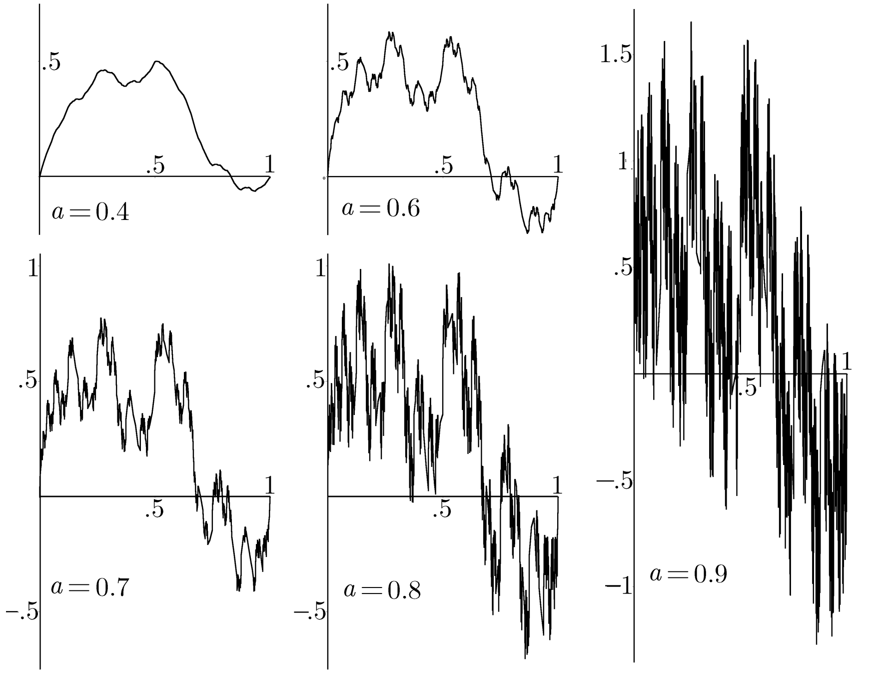

Fig. 7.2.15. Besicovitch-Ursell Functions

unit apart. For each subsequent approximation, additional line segments are added. The length of the new line segments is decreased by a factor of 1/2 at each stage. The distance between the line segments is decreased by a factor

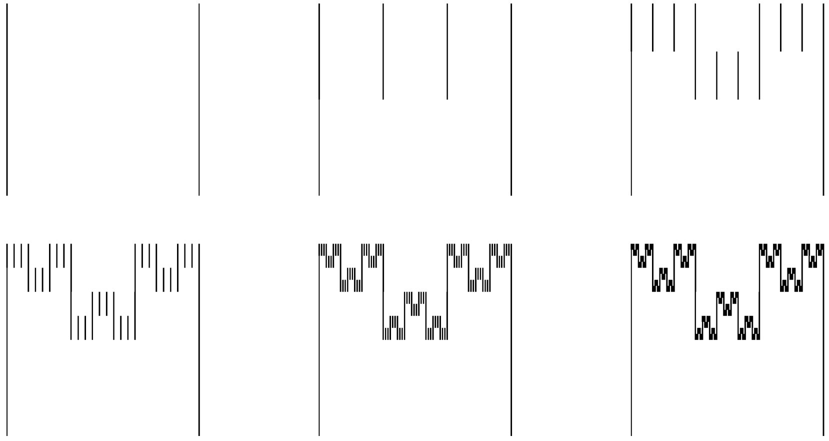

Fig. 7.2.16. Hironaka's curve

of 1/3 at each stage. The position of the line segments is determined by the pattern illustrated. Hironaka's curve is the limit set.

Exercise 7.2.17. Find topological and Hausdorff dimensions for Hironaka's curve.

Number Systems

Here is a way to generalize the "number systems" of Sect. 1.6. Elements of $\mathbb{R}^d$ should be identified with $d \times 1$ column matrices. Let D be a finite set in $\mathbb{R}^d$, including 0, and let B be a $d \times d$ matrix. What conditions should B satisfy so that all of the following vectors exist?

$$\sum_{j=1}^{\infty} B^j a_j,$$

where the "digits" $a_j \in D$. The set F of all these vectors is the invariant set of an iterated function system of affine maps.

7.3 *Self-Conformal

An affine transformation that is not a similarity changes distances by different ratios in different directions. Here we will talk about non-affine transformations that change distances by the same ratio in all directions, but only in the limit near a point.

Let S be a metric space, let $f\colon S \to S$ be a transformation, let $r > 0$, and let $a \in S$. We say that f is ***conformal*** at a with ratio r if:

$$\lim_{\substack{x,y \to a \\ x \neq y}} \frac{\varrho\big(f(x), f(y)\big)}{\varrho(x,y)} = r.$$

More technically stated: for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in S$, if $\varrho(x,a) < \delta$ and $\varrho(y,a) < \delta$, then

$$(1-\varepsilon) r \varrho(x.y) \leq \varrho\big(f(x), f(y)\big) \leq (1+\varepsilon) r \varrho(x,y).$$

We say that f is conformal on a set E if a is conformal at every point of E, but not necessarily with the same ratio.

Proposition 7.3.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuously differentiable, let $a \in \mathbb{R}$, and assume $f'(a) \neq 0$. Then f is conformal at a with ratio $|f'(a)|$.*

---

* Optional section.

Proof. Let $\varepsilon > 0$ be given. Then since f' is continuous, there is $\delta > 0$ so that if $|x - a| < \delta$, then

$$(1-\varepsilon)|f'(a)| < |f'(x)| < (1+\varepsilon)|f'(a)|.$$

Now let x, y satisfy $|x-a| < \delta$, $|y-a| < \delta$, and $x \neq y$. Applying the Mean Value Theorem on the interval from x to y, we conclude there is z between x and y so that

$$\frac{f(x)-f(y)}{x-y} = f'(z).$$

Now $|z-a| < \delta$, so we have

$$\frac{|f(x)-f(y)|}{|x-y|} = |f'(z)| < (1+\varepsilon)|f'(a)|,$$

so that

$$|f(x)-f(y)| < (1+\varepsilon)|f'(a)|\,|x-y|.$$

Similarly,

$$|f(x)-f(y)| > (1-\varepsilon)|f'(a)|\,|x-y|.$$

Thus f is conformal at a with ratio $|f'(a)|$. □

Is continuity of the derivative required?

Exercise 7.3.2. Give an example where $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at a point a with $f'(a) \neq 0$, but f is not conformal at a.

If you have studied multi-dimensional calculus, you can attempt the next exercise.

Exercise 7.3.3. Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable, and let $a \in \mathbb{R}^d$. If the derivative $Df(a)$, interpreted as as $d \times d$ matrix, defines a similarity on $\mathbb{R}^d$, then f is conformal at a, and the ratio of f at a is the same as the ratio of the similarity $Df(a)$.

In Euclidean space $\mathbb{R}^d$, examples of conformal maps (where they are defined) are: translation, rotation, reflection in a hyperplane, reflection in a sphere. In particular, in $\mathbb{R}^2$, reflection in a circle (p. 28) is conformal. Of course the ratio is not the same everywhere.

In the mathematical subject of complex analysis, you can find this:

Proposition 7.3.4. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be continuously differentiable in the complex sense, let $a \in \mathbb{C}$, and assume $f'(a) \neq 0$. Then f is conformal at a with ratio $|f'(a)|$.*

If a function f (defined for complex numbers z) has the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$, then f is called a ***linear fractional transformation***. If $ad - bc = 0$, then f is constant, so we will assume $ad - bc \neq 0$. Then f is defined everywhere in $\mathbb{C}$ except $z = -b/a$, and

$$f'(z) = \frac{ad-bc}{(cz+d)^2}$$

is never zero. So f is conformal. An important property of a linear fractional transformation is that it maps circles to circles (provided a line is considered to be a circle).

The attractor of an iterated function system consisting of conformal maps is known as a ***self-conformal*** set. Of course self-similar sets are self-conformal. Pharaoh's breastplate (p. 30) is self-conformal but not self-similar.

Appolonian Gasket

Figure 7.3.5 shows a subset of the plane. The first approximation is obtained by taking three mutually tangent circles with radius 1. The set C_0 is the region enclosed by three arcs (including the arcs themselves). Each approximation will consist of some regions bounded by three mutually tangent circular arcs. To obtain C_{k+1}, remove from each region of C_k the circle in the region tangent to all three of the arcs. (The boundary of the circle remains.) The Appolonian gasket is the "limit" (intersection) of the sets C_k.

The gasket is is self-coinformal. It is not self-similar or self-affine.

Exercise 7.3.6. The Appolonian gasket is an invariant set for an iterated function system of linear fractional transformations.

Exercise 7.3.7. Discuss the topological dimension and fractal dimension of the Appolonian gasket.

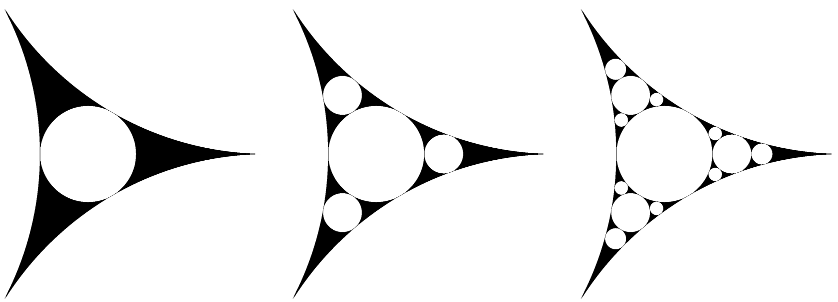

Fig. 7.3.5. Appolonian Gasket

Julia Set

Recall the Julia set described on p. 28 for the function $\varphi(z) = z^2 + c$, where $c = -0.15 + 0.72\mathrm{i}$. The two branches f_0, f_1 of the function $\sqrt{z-c}$ are not continuous on the set J that we construct, so it is not so simply interpreted as the attractor of an iterated function system.

The Julia set J is the union of two sets U and $L = -U$. See Fig. 7.3.8. There are choices of inverse maps $f_0(z), f_1(z) = -f_0(z)$ for φ that are continuous on U and choices of inverse maps $g_0(z), g_1(z) = -g_0(z)$ for φ that are continuous on L so that

$$U = f_0[U] \cup g_0[L], \qquad V = f_1[U] \cup g_1[L].$$

See Fig. 7.3.9. The point c does not belong to J, and $\sqrt{z-c}$ is conformal except at the point c. All four maps are conformal on their appropriate domains, since they are continuous branches of $\sqrt{z-c}$. Sets U and L are isometric; U is the attractor of the iterated function system consisting of conformal maps $f_0(z), g_0(-z)$. So U is ***self-conformal***.

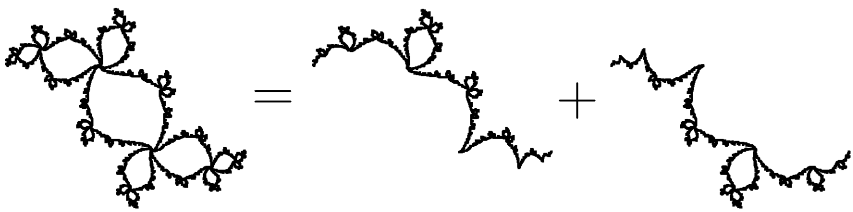

Fig. 7.3.8. J made up of U and L

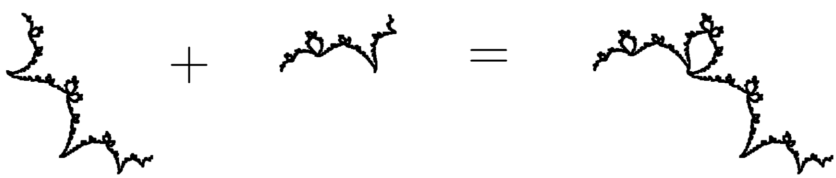

Fig. 7.3.9. $f_0[U]$ and $g_0[L]$ make up U

7.4 *A Multifractal

Most of the fractals that have been considered in this book are closed sets (or even compact sets). We will discuss now an example that is not closed. Mandelbrot calls it the ***Besicovitch fractal***. It was studied by Besicovitch and Eggleston; more recently it occurs in the physics literature in connection with "multifractals" or "fractal measures". The proof will require some knowledge of probability theory, however.

Given $x \in [0,1]$, consider its binary expansion, $x = \sum_{i=1}^{\infty} a_i 2^{-i}$, where each a_i is 0 or 1. We are interested in the frequency of the occurrence of the digit 0. More precisely, let $K_n^{(0)}(x)$ be the number of 0's and $K_n^{(1)}(x)$ the number of 1's occurring among the first n digits, $(a_1, a_2, \cdots, a_n)$. The ***frequencies*** in question are

$$F^{(0)}(x) = \lim_{n\to\infty} \frac{K_n^{(0)}(x)}{n},$$
$$F^{(1)}(x) = \lim_{n\to\infty} \frac{K_n^{(1)}(x)}{n}.$$

(Of course, the limits in question exist for only some $x \in [0,1]$.)

Fix a number p, with $0 < p < 1$. We are interested in the set

$$S_p = \left\{ x \in [0,1] : F^{(0)}(x) \text{ exists, and } F^{(0)}(x) = p \right\}.$$

We will compute the Hausdorff dimension of the set S_p. If we write $q = 1 - p$, so that $F^{(0)}(x) = p$ implies $F^{(1)}(x) = q$, then we will show that

$$\dim S_p = \frac{-p \log p - q \log q}{\log 2}.$$

The proof will use a string model, as usual. But it will also use the "strong law of large numbers", an important result from probability theory.

Before we turn to the proof, let us consider the set S_p more carefully. Note that $[0,1]$ is not equal to $\bigcup_{0\le p\le 1} S_p$, since the limit $F^{(0)}(x)$ does not exist for many x.

I will next prove that the set S_p is a Borel set. (This is the first example we have seen where measurability is not immediately obvious.) First, given a, b, n, the set

$$\left\{ x : a \le K_n^{(0)}(x) \le b \right\}$$

* Optional section.

is a Borel set, since it consists of a finite number of intervals of length 2^{-n}. Then

$$S_p = \left\{ x : \lim_{n\to\infty} \frac{K_n^{(0)}(x)}{n} = p \right\}$$
$$= \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \left\{ x : p - \frac{1}{k} \leq \frac{K_n^{(0)}(x)}{n} \leq p + \frac{1}{k} \right\},$$

so S_p is a Borel set.

Next, note that if x and y agree except in the first k digits, then $F^{(0)}(x) = F^{(0)}(y)$. So any open interval in $[0,1]$ intersects S_p. That is, S_p is dense in $[0,1]$. Certainly $S_p \neq [0,1]$, so of course S_p is not closed.

If the digits of x are all shifted to the right, and a new digit is added on the left, then the frequencies are unchanged. So S_p exhibits a natural self-similarity: If $x \in [0,1]$, then $F^{(0)}(x) = F^{(0)}(x/2) = F^{(0)}(1/2 + x/2)$. Thus the two similarities

$$f_0(x) = \frac{x}{2},$$
$$f_1(x) = \frac{x+1}{2},$$

have the property

$$S_p = f_0[S_p] \cup f_1[S_p],$$

with no overlap. The similarity dimension of the iterated function system (f_0, f_1) is 1. The conclusion is: similarity dimension may be misleading for non-closed sets.

Theorem 7.4.1. *The Hausdorff dimension of the set S_p is*

$$s = \frac{-p\log p - q\log q}{\log 2}.$$

Proof. Let $E = \{\mathtt{0}, \mathtt{1}\}$ be our two-letter alphabet, and recall the "base 2" model map $h\colon E^{(\omega)} \to [0,1]$ defined on p. 14. Then $h[E^{(\omega)}] = [0,1]$. Also, $\operatorname{diam}[\alpha] = \operatorname{diam} h\big[[\alpha]\big] = 2^{-n}$ if $\alpha \in E^{(n)}$. We define frequencies for strings in the same way as for numbers: For $\alpha \in E^{(*)}$, let $K^{(0)}(\alpha)$ be the number of $\mathtt{0}$'s in α, let $K^{(1)}(\alpha)$ be the number of $\mathtt{1}$'s in α. For $\sigma \in E^{(\omega)}$ let

$$F^{(0)}(\sigma) = \lim_{n\to\infty} \frac{K^{(0)}(\sigma{\restriction}n)}{n},$$
$$F^{(1)}(\sigma) = \lim_{n\to\infty} \frac{K^{(1)}(\sigma{\restriction}n)}{n}.$$

These limits are defined for some strings $\sigma \in E^{(\omega)}$, and not for others. Let

$$T_p = \left\{ \sigma \in E^{(\omega)} : F^{(0)}(\sigma) = p \right\}.$$

Then clearly $S_p = h[T_p]$.

Now consider a measure $\mathcal{M}_p$ defined on $E^{(\omega)}$ as follows. Let $\alpha \in E^{(n)}$. If $k = K^{(1)}(\alpha)$, $n - k = K^{(0)}(\alpha)$, let $w_\alpha = p^{n-k} q^k$. Then $w_\alpha = w_{\mathtt{0}} + w_{\mathtt{1}}$, so these numbers define a metric measure $\mathcal{M}_p$ on $E^{(\omega)}$ with $\mathcal{M}_p\big([\alpha]\big) = w_\alpha$ for all $\alpha \in E^{(*)}$.

Now we require the result from probability theory. According to the measure $\mathcal{M}_p$ just defined, the "digits" of σ constitute independent Bernoulli trials, with probability p of outcome $\mathtt{0}$ and probability $q = 1 - p$ of outcome $\mathtt{1}$. So by the strong law of large numbers (for example, [7, Example 6.1]), we have

$$\mathcal{M}_p(T_p) = 1, \qquad \text{or, equivalently,} \qquad \mathcal{M}_p(E^{(\omega)} \setminus T_p) = 0.$$

We will take the case $p < 1/2$. The case $p > 1/2$ is similar, and the case $p = 1/2$ is the usual measure $\mathcal{M}_{1/2}$ and dimension 1 computed before (Proposition 6.3.1).

We begin with the upper bound, $\dim S_p \leq s$. Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ satisfy $2^{-N} < \varepsilon$. Let $q' < q$. We will show that $\dim S_p \leq s'$, where $s' = (-q' \log q - (1 - q') \log p)/\log 2$.

Consider the set $G \subseteq E^{(*)}$ defined as follows: if $\alpha \in E^{(n)}$, and $k = K^{(1)}(\alpha)$, then $\alpha \in G$ iff $k/n > q'$. For such α, we have $\operatorname{diam}[\alpha] = 2^{-n}$ and

$$\begin{aligned}
\mathcal{M}_p\big([\alpha]\big) &= p^{n-k} q^k = p^n \left(\frac{q}{p}\right)^k \\
&> p^n \left(\frac{q}{p}\right)^{q'n} = p^{(1-q')n} q^{q'n} \\
&= \left(2^{-n}\right)^{s'} = \left(\operatorname{diam}[\alpha]\right)^{s'}.
\end{aligned}$$

Let G' be the set of all $\alpha \in G$ with length $|\alpha| \geq N$ but $\alpha{\upharpoonright}n \notin G$ for $N \leq n < |\alpha|$. That is, α belongs to G, but no ancestors of α (except possibly ancestors before generation N) belong to G. If $\sigma \in T_p$, then $\lim_{n\to\infty} K^{(1)}(\sigma{\upharpoonright}n)/n = q > q'$, so for some $n \geq N$ we have $\sigma{\upharpoonright}n \in G$, and therefore for some $n \geq N$ we have $\sigma{\upharpoonright}n \in G'$. So

$$\{\, [\alpha] : \alpha \in G' \,\}$$

is a disjoint cover of T_p. But

$$\begin{aligned}
\sum_{\alpha \in G'} \left(\operatorname{diam}[\alpha]\right)^{s'} &< \sum_{\alpha \in G'} \mathcal{M}_p\big([\alpha]\big) \\
&= \mathcal{M}_p\left(\bigcup_{\alpha \in G'} [\alpha]\right) \leq 1.
\end{aligned}$$

Therefore $\overline{\mathcal{H}}^{s'}_{\varepsilon}(T_p) \le 1$. Let $\varepsilon \to 0$ to conclude $\mathcal{H}^{s'}(T_p) \le 1$, and therefore $\dim T_p \le s'$. Now when $q' \to q$ we have $s' \to s$, so $\dim T_p \le s$.

Now the model map h has bounded decrease, so $\dim S_p \le s$. (Or, cover S_p with the sets $h\big[[\alpha]\big], \alpha \in G'$.)

Next I must prove the lower bound, $\dim S_p \ge s$. Let $q' > q$, and define $s' = (-q' \log q - (1-q') \log p)/\log 2$. I will show $\dim S_p \ge s'$. Now

$$\mathcal{M}_p \left\{ \sigma \in E^{(\omega)} : \lim \frac{K^{(1)}(\sigma \restriction n)}{n} = q \right\} = \mathcal{M}_p(T_p) = 1,$$

and therefore

$$\mathcal{M}_p \left\{ \sigma : \text{ there exists } N \in \mathbb{N} \text{ such that for all } n \ge N,\ \frac{K^{(1)}(\sigma \restriction n)}{n} < q' \right\} = 1.$$

So by countable additivity,

$$\lim_{N \to \infty} \mathcal{M}_p \left\{ \sigma : \sup_{n \ge N} \frac{K^{(1)}(\sigma \restriction n)}{n} < q' \right\} = 1.$$

Choose N so that $\mathcal{M}_p(F) > 1/2$, where

$$F = \left\{ \sigma : \sup_{n \ge N} \frac{K^{(1)}(\sigma \restriction n)}{n} < q' \right\}.$$

Let $\varepsilon = 2^{-N}$.

Suppose $\mathcal{A}$ is a countable cover of S_p by sets A with $\operatorname{diam} A \le \varepsilon$. First, we reduce to a cover by intervals of the form $h\big[[\alpha]\big]$. Each set $A \in \mathcal{A}$ is covered by (at most) three of the intervals $h\big[[\alpha]\big]$, where the length $|\alpha|$ is the integer n with $2^{-n} < \operatorname{diam} A \le 2^{-n+1}$. Let $G \subseteq E^{(*)}$ be the set of all these α. (We may assume that the sets $[\alpha]$ are disjoint, since if two of them intersect, then one is a subset of the other, so we may delete the smaller one.) Thus

$$\sum_{\alpha \in G} \big(\operatorname{diam}[\alpha]\big)^{s'} < 3 \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^{s'}$$

and $T_p \subseteq \bigcup_{\alpha \in G} [\alpha]$, so $\mathcal{M}_p\big(\bigcup_{\alpha \in G}[\alpha]\big) = 1$.

For $\alpha \in G$ we have $|\alpha| \ge N$. If $[\alpha] \cap F \ne \varnothing$, $|\alpha| = n$ and $K^{(1)}(\alpha) = k$, then

$$\begin{aligned} \mathcal{M}_p\big([\alpha]\big) &= p^{n-k} q^k = p^n \left(\frac{q}{p}\right)^k \\ &< p^n \left(\frac{q}{p}\right)^{q'n} = p^{(1-q')n} q^{q'n} \\ &= \big(2^{-n}\big)^{s'} = \big(\operatorname{diam}[\alpha]\big)^{s'}. \end{aligned}$$

Now if $G' = \{\, \alpha \in G : \ [\alpha] \cap F \neq \varnothing \,\}$, then

$$\begin{aligned} \frac{1}{2} &< \mathcal{M}_p(F) \le \mathcal{M}_p\left(\bigcup_{\alpha \in G'} [\alpha] \right) \\ &= \sum_{\alpha \in G'} \mathcal{M}_p\big([\alpha]\big) < \sum_{\alpha \in G'} \big(\operatorname{diam}[\alpha]\big)^{s'} \\ &\le \sum_{\alpha \in G} \big(\operatorname{diam}[\alpha]\big)^{s'} < 3 \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^{s'}. \end{aligned}$$

Now $\mathcal{A}$ is any cover of S_p by sets of diameter $\le \varepsilon$, so $\overline{\mathcal{H}}_{\varepsilon}^{s'}(S_p) > 1/6$. Therefore $\mathcal{H}^{s'}(S_p) > 1/6$, so $\dim S_p \ge s'$. Now let $q' \to q$ to obtain $\dim S_p \ge s$. □

Exercise 7.4.2. Compute the box dimension (the lower entropy index) of the set S_p.

Exercise 7.4.3. Compute the packing dimension $\operatorname{Dim} S_p$.

Exercise 7.4.4. Let E be a finite alphabet, let $\mathcal{M}$ be a metric measure on the space $E^{(\omega)}$ of infinite strings, and let ϱ be a metric on $E^{(\omega)}$. Suppose t is a positive real number, and let

$$S = \left\{ \sigma \in E^{(\omega)} : \ \lim_{n \to \infty} \frac{\log \mathcal{M}\big([\sigma{\restriction}n]\big)}{\log \operatorname{diam}[\sigma{\restriction}n]} = t \right\}.$$

If $0 < \mathcal{M}(S) < \infty$, does it follow that $\dim S = t$?

7.5 *A Superfractal

Examples called "Kline curves" were included in the first edition of this book. It was included as extra material that could be assigned to students for independent investigation. The Kline curves provide examples of parametric curves in the plane where the Lipschitz classes of the two coordinate functions and the fractal dimension of the curve itself can be controlled independently. Kline's paper [40] was published in 1945.

Around 2005, motivated by their study of fractal methods for picture generation, Barnsley, Hutchinson, and Stenflo developed the ***superfractal*** formalism (see Barnsley's book [4]). Unexpectedly, the Kline curves give us an interesting example of a superfractal.

* Optional section.

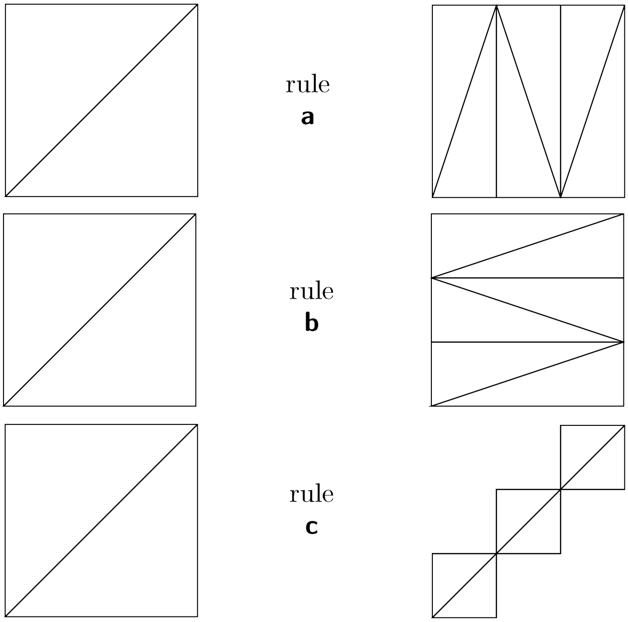

Fig. 7.5.1. Kline rules

Kline curves

The Kline curves are subsets of the plane $\mathbb{R}^2$. They are constructed using approximation by "Kline polygons". We begin with the line segment from the point $(0,0)$ to the point $(1,1)$; it is the diagonal of the rectangle (actually a square) $[0,1] \times [0,1]$. There are three rules used to build more complicated Kline polygons. Each of them replaces each of the line segments by three line segments. Rule a is implemented by subdividing the horizontal dimension of the containing rectangle in thirds, and replacing the diagonal by a three-part zig-zag, as illustrated. Rule b is implemented by subdividing the vertical dimension of the containing rectangle in thirds, and replacing the diagonal by a three-part zig-zag. Rule c is implemented by subdividing both the horizontal and vertical dimensions by three, and replacing the line segment by three parts of itself, inside the three diagonal subrectangles.

Each Kline polygon is obtained by applying these three rules in some order. Each finite string built from the alphabet $\{\mathsf{a},\mathsf{b},\mathsf{c}\}$ may be considered a "program" for the construction of a polygon. Several examples are illustrated in Fig. 7.5.3. We will write Kline$[\alpha]$ for the Kline polygon corresponding to the string α.

Now let $\sigma \in \{\mathsf{a},\mathsf{b},\mathsf{c}\}^{(\omega)}$ be an infinite string. The ***Kline curve*** Kline$[\sigma]$ is the limit of the Kline polygons Kline$[\sigma{\restriction}k]$ as k increases.

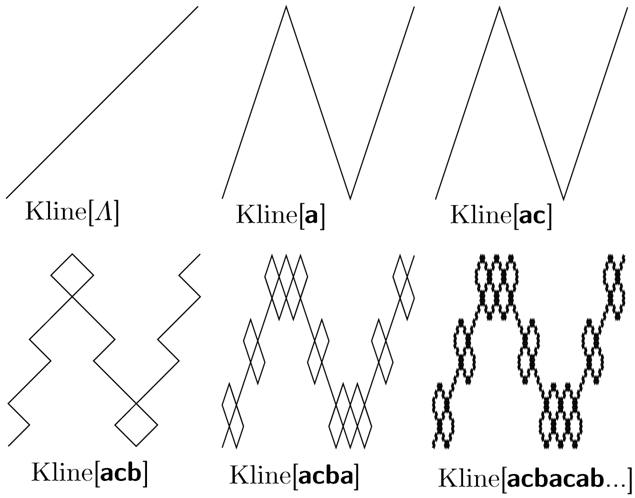

Fig. 7.5.3. Kline polygons, Kline curve

Exercise 7.5.2. If σ is an infinite string from the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, then Kline$[\sigma\restriction k]$ converges in the Hausdorff metric.

What is the justification of the use of the word "curve"? The "natural" parameterization of a polygon Kline$[\alpha]$ with 3^n segments of equal length ($n = |\alpha|$) is obtained by subdividing $[0, 1]$ into 3^n subintervals of equal length, and mapping each of the subintervals affinely onto the corresponding segment of the polygon.

Exercise 7.5.4. Determine necessary and sufficient conditions on the string σ for the natural parameterizations of the polygons Kline$[\sigma\restriction n]$ to converge uniformly to a parameterization of the Kline curve Kline$[\sigma]$ (which will again be called the natural parameterization).

There are two periodic strings σ that deserve special mention. They are cases where Kline$[\sigma]$ specializes to curves which we have seen before. For the constant string $\mathsf{ccc}\cdots$, the Kline curve is a line segment. It has Cov = dim = Dim = 1. For the period-two string $\mathsf{abab}\cdots$, the Kline curve is the Peano space-filling curve (p. 70). It has Cov = dim = Dim = 2. Clearly for a general string σ, the topological dimension Cov Kline$[\sigma]$ is either 1 or 2. And the fractal dimension satisfies $1 \le \dim \text{Kline}[\sigma] \le 2$.

Exercise 7.5.5. Prove necessary and sufficient conditions on the string σ for Cov Kline$[\sigma] = 1$.

Exercise 7.5.6. Let $\sigma \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^{(\omega)}$ be the program for a Kline curve. For $n \in \mathbb{N}$, let $a_n(\sigma)$ be the number of times the letter a occurs in the restriction

$\sigma\restriction n$. Similarly, let $b_n(\sigma)$ be the number of times the letter b occurs and $c_n(\sigma)$ the number of times the letter c occurs. Assume the limits

$$\alpha = \lim_{n\to\infty} \frac{a_n(\sigma)}{n}, \qquad \beta = \lim_{n\to\infty} \frac{b_n(\sigma)}{n}, \qquad \gamma = \lim_{n\to\infty} \frac{c_n(\sigma)}{n}$$

exist and $\alpha \geq \beta$. Show that the Hausdorff dimension of Kline$[\sigma]$ is

$$\frac{2\alpha+\gamma}{\alpha+\gamma} = 2 - \frac{\gamma}{\alpha+\gamma} = 2 - \frac{\gamma}{1-\beta}.$$

Exercise 7.5.7. Discuss the packing dimension of a Kline curve.

Exercise 7.5.8. Let $\sigma \in \{\mathsf{a},\mathsf{b},\mathsf{c}\}^{(\omega)}$, assume α, β, γ exist as in Exercise 7.5.6, and assume $1 > \alpha \geq \beta > 0$. Let $\big(\varphi(t), \psi(t)\big)$, $t \in [0,1]$, be the natural parameterization of Kline$[\sigma]$. Show that

$$\varphi \in \operatorname{Lip}\left(\frac{1}{1-\alpha}\right), \qquad \psi \in \operatorname{Lip}\left(\frac{1}{1-\beta}\right).$$

SuperIFS

Let us review the definition of a hyperbolic iterated function system and its attractor. We have a complete metric space S. We have a finite index set, an alphabet, E. For each letter $e \in E$ we have a contractive Lipschitz map $f_e\colon S \to S$. The data of the iterated function system let us define a map $F\colon \mathbb{H}(S) \to \mathbb{H}(S)$ by

$$F(A) = \bigcup_{e\in E} f_e[A]. \tag{1}$$

Sometimes we use the same letter (here F) to refer either to the iterated function system $(f_e)_{e\in E}$ itself or to the corresponding map (1) of the hyperspace. The attractor for the iterated function system F is the fixed point, the unique $K \in \mathbb{H}(S)$ satisfying the self-referential equation $F(K) = K$. The attractor K may be described using a string model. The addressing function $h\colon E^{(\omega)} \to S$ defined on the string space $E^{(\omega)}$ is defined by

$$h(\sigma) = \lim_n f_{\sigma\restriction n}(x),$$

where the limit is independent of the point $x \in S$. The range of h is K.

A superIFS is, roughly speaking, an IFS where the space is a hyperspace $\mathbb{H}(S)$ and the maps are themselves IFSs on S. The space S itself is where we are interested in describing sets, but there are two layers of data used to do it.

A more precise description: Let S be a complete metric space. Let E be a finite set, an alphabet. For each $e \in E$, let F_e be a hyperbolic IFS on the space S, so that the corresponding map $F_e\colon \mathbb{H}(S) \to \mathbb{H}(S)$ is a contractive Lipschitz map. (The IFSs F_e all act on the same space S, and each one has an alphabet and set of maps. Their alphabets may or may not be the same as each other, and probably are not the same as the master alphabet E.) An IFS $(F_e)_{e\in E}$ constructed in this way is known as a ***superIFS*** on S. So we define a map $\mathbf{F}\colon \mathbb{H}(\mathbb{H}(S)) \to \mathbb{H}(\mathbb{H}(S))$ by

$$\mathbf{F}(\mathcal{A}) = \bigcup_{e\in E} F_e[\mathcal{A}]. \tag{2}$$

Sometimes we use the same letter (here $\mathbf{F}$) to refer either to the SuperIFS $(F_e)_{e\in E}$ itself or to the corresponding map (2) of the hyperhyperspace. And there is an attractor: a unique $\mathcal{K} \in \mathbb{H}(\mathbb{H}(S))$ satisfying the self-referential equation $\mathbf{F}(\mathcal{K}) = \mathcal{K}$. This is called a ***superfractal.**** There is, as usual, a string model. For any string $\sigma \in E^{(\omega)}$ define

$$\mathbf{h}(\sigma) = \lim_n F_{\sigma\restriction n}(A).$$

This limit is taken according to the Hausdorff metric in $\mathbb{H}(S)$, and it does not depend on the starting set $A \in \mathbb{H}(S)$. So $\mathbf{h}\colon E^{(\omega)} \to S$ is continuous, and its range is the superfractal $\mathcal{K}$.

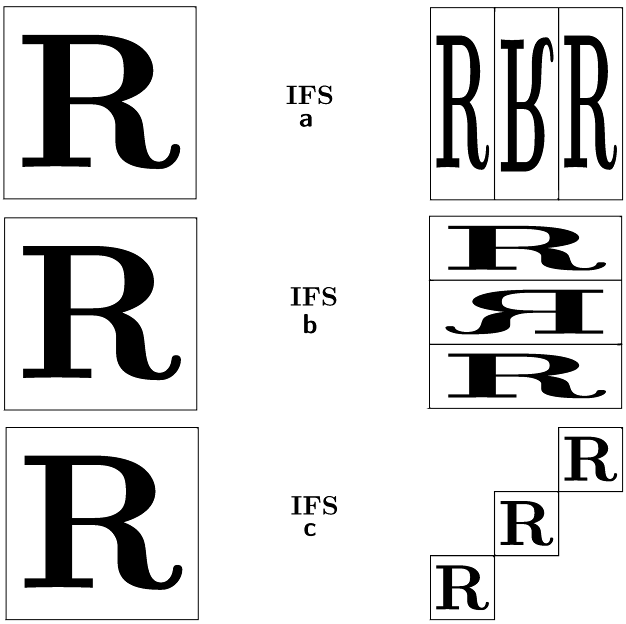

Fig. 7.5.9. Kline IFSs

* In [4] this is called a "1-variable superfractal", and that book also discusses V-variable superfractals.

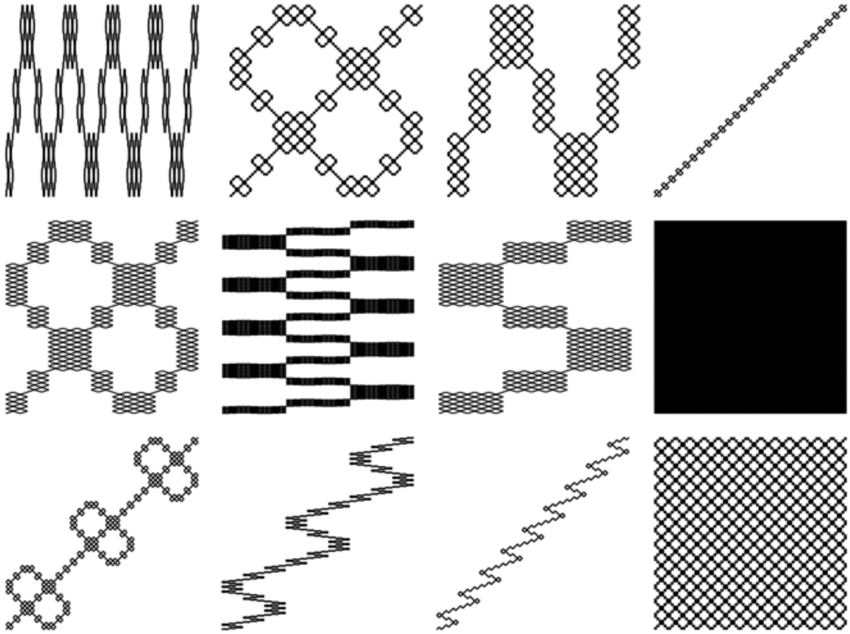

Fig. 7.5.10. Kline superfractal

The Kline curve construction described above is an example of a superfractal. Space S is the unit square. The master alphabet is $E = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$. The three IFSs are shown in Fig. 7.5.9. The superfractal $\mathcal{K}$ attractor is made up of all the Kline curves. The addressing function is $\sigma \mapsto \text{Kline}[\sigma]$. Some Kline curves are shown in Fig. 7.5.10. A family resemblance among the images reflects the fact that they all arise from a single superfractal $\mathcal{K}$.

See Barnsley [4, Chap. 5] and the references there for additional material on superfractals.

7.6 *Remarks

Robert Strichartz took the modern literary term "deconstruction" for use with iterated function systems. Deconstruction of a literary or philosophical text may mean finding meanings that were not intended by the original author. So deconstruction of the invariant set of an IFS means decomposing it in a way different from the one provided by the original iterated function system.

Karl Kiesswetter's curve is from [39]. It was proposed as a particularly elementary example of a continuous but nowhere differentiable function.

Theorem 7.4.1 is due to A. S. Besicovitch [5, Part II] and H. G. Eggleston [21]. Another proof is given by Patrick Billingsley [6, Section 14].

Comments on the Exercises

Exercise 7.1.4. The open set condition is satisfied, but not by the open set $(0, 1)$.

Exercise 7.1.5. The set K is contained in $[0, 1]$. Consider 4 parts of the set:

$$\begin{aligned} A &= K \cap [4/5, 1] \\ B &= K \cap [1/5, 4/5] \\ C &= K \cap [4/25, 1/5] \\ D &= K \cap [0, 4/25], \end{aligned}$$

and observe that $(1/5)A \subseteq C$. Show that the graph similarity obeys Fig. 7.6.1, and the open set condition is satisfied for the corresponding realization. The dimension is $\log\big((\sqrt{5}+3)/2\big)/\log 5 \approx 0.59799$; compare it to the upper bound obtained from the the ratio list $(1/5, 1/5, 1/5)$, namely $\log 3/\log 5 \approx 0.6826$.

Exercise 7.1.7. This is more like a class project than a homework assignment. Of course there are many possible answers. My deconstruction involves four isosceles right triangluar blocks A, D, H, K and their reflections A', D', H', K' with the graph iterated function system and open set condition shown in Fig. 7.6.2. The fractal dimensions are ≈ 1.92926.

Exercise 7.2.7. If g is differentiable at a point a, and $x_n \le a \le y_n$, $\lim x_n = a = \lim y_n$, $x_n < y_n$, then

$$\lim_{n\to\infty} \frac{g(y_n) - g(x_n)}{y_n - x_n} = g'(a).$$

This is false by Exercise 7.2.6.

Exercise 7.2.8: 3/2.

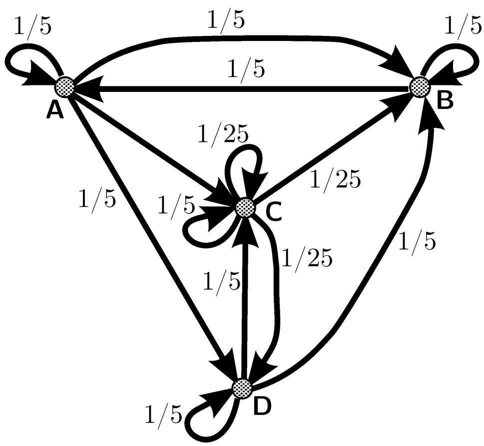

Fig. 7.6.1. A graph

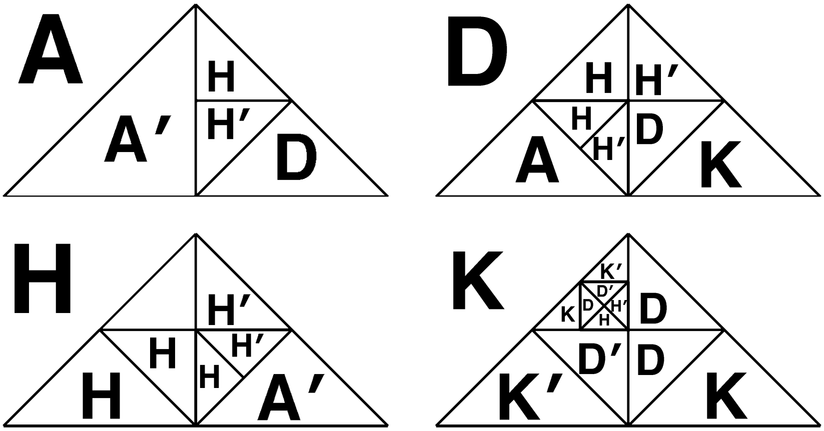

Fig. 7.6.2. Leaf deconstruction

Exercise 7.2.14: [5, Part V].
Exercise 7.2.17: [49].
Exercise 7.3.7: [23, Section 8.4].
Exercise 7.4.4: [6, Section 14].
Exercise 7.5.6: [40].

Life is a fractal in Hilbert space.
—Rudy Rucker, *Mind Tools*

I am a strange loop.
—Douglas Hofstadter