6

Fractal Dimension

Next we come to the fractal dimensions: in particular the Hausdorff dimension and the packing dimension. The surprising feature for these dimensions is that they need not be integers: they can be fractions. The Hausdorff dimension is the one singled out by Mandelbrot when he defined "fractal". The Hausdorff and packing dimensions are perhaps a bit more difficult to define than some of the other kinds of fractal dimension. But in recent years it has become clear that they are the most useful of the fractal dimensions.

6.1 Hausdorff Measure

Let S be a metric space. Consider a positive real number s, the candidate for the dimension. The ***s-dimensional Hausdorff outer measure*** is the method II outer measure defined from the set function $\mathbf{c}_s(A) = (\operatorname{diam} A)^s$. It is written $\overline{\mathcal{H}}^s$. The restriction to the measurable sets is called ***s-dimensional Hausdorff measure***, and written $\mathcal{H}^s$. Since $\overline{\mathcal{H}}^s$ is constructed by method II, it is a metric outer measure. So all Borel sets are measurable (in particular, all open sets, closed sets, compact sets).

Recall that the Method I theorem gives a more explicit construction: A family $\mathcal{A}$ of subsets of S is called a ***countable cover*** of a set F iff

$$F \subseteq \bigcup_{A \in \mathcal{A}} A,$$

and $\mathcal{A}$ is a countable (possibly even finite) family of sets. Let ε be a positive number (presumably very small). The cover $\mathcal{A}$ is an ***ε-cover*** iff $\operatorname{diam} A \leq \varepsilon$ for all $A \in \mathcal{A}$. Define

$$\overline{\mathcal{H}}^s_\varepsilon(F) = \inf \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^s,$$

where the infimum is over all countable ε-covers $\mathcal{A}$ of the set F. (By convention, $\inf \varnothing = \infty$.) A computation shows that when ε gets smaller, $\overline{\mathcal{H}}^s_\varepsilon(F)$ gets larger. Finally:

$$\overline{\mathcal{H}}^s(F) = \lim_{\varepsilon \to 0} \overline{\mathcal{H}}^s_\varepsilon(F) = \sup_{\varepsilon > 0} \overline{\mathcal{H}}^s_\varepsilon(F)$$

is the s-dimensional Hausdorff outer measure of the set F. Figures 6.1.1 and 6.1.2 illustrate some of the ideas behind the definition.

There are variants in the definition of the Hausdorff measure that are sometimes useful. (i) Since the closure of a set has the same diameter as the set itself, we may use only closed sets in the covers $\mathcal{A}$. The class of closed sets is a (method II) reduced cover class for $\mathcal{H}^s$. (ii) If A is any set, it is contained in an open set with diameter as close as I like to the diameter of A. The class of open sets is a reduced cover class for $\mathcal{H}^s$. (iii) Any set of diameter r is contained in a closed ball of radius r (and diameter $\leq 2r$). The collection of open balls is a reduced cover class with factor 2^s for $\mathcal{H}^s$. (iv) In Euclidean space $\mathbb{R}^d$, the convex hull of any set has the same diameter as the set. The

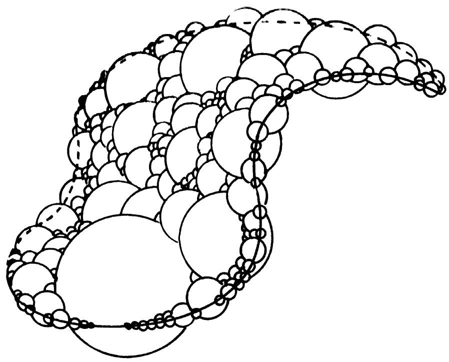

Fig. 6.1.1. The Hausdorff measure (area) of a piece of surface A is approximated by the cross-sections of little balls which cover it. (From [52])

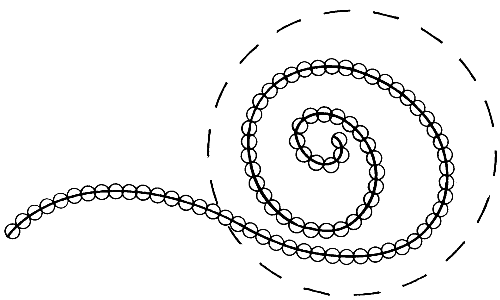

Fig. 6.1.2. One must cover by *small* sets to compute length accurately. Here the length of the spiral is well-estimated by the sum of the diameters of the tiny balls, but grossly under-estimated by the diameter of the huge ball. (From [52])

collection of convex sets is a reduced cover class for $\mathcal{H}^s$. (v) If a set K is compact, then every open cover of K has a finite subcover, so to compute the Hausdorff measure of a compact set K, we may use finite covers $\mathcal{A}$. (vi) If we replace a set in a cover $\mathcal{A}$ of a set F by a subset of itself, so that the result is still a cover of F, the sum

$$\sum_{A\in\mathcal{A}} (\operatorname{diam} A)^s$$

only becomes smaller. So if $F \subseteq T \subseteq S$, the value of $\overline{\mathcal{H}}^s_\varepsilon(F)$ when F is considered to be a subset of T is the same as when F is considered to be a subset of S. In particular, we may assume (if it is convenient) that the sets used in the covers $\mathcal{A}$ of the set F are subsets of F.

Exercise 6.1.3. If F is a finite set, then $\mathcal{H}^s(F) = 0$ for all $s > 0$.

Theorem 6.1.4. *In the metric space $\mathbb{R}$, the one-dimensional Hausdorff measure $\mathcal{H}^1$ coincides with the Lebesgue measure $\mathcal{L}$.*

Proof. If $A \subseteq \mathbb{R}$ has finite diameter r, then $\sup A - \inf A = r$, so A is contained in a closed interval I with length r, and $\overline{\mathcal{L}}(A) \le \overline{\mathcal{L}}(I) = r$. But by the Method I theorem (5.2.2), $\overline{\mathcal{H}}^1_\varepsilon$ is the largest outer measure $\overline{\mathcal{M}}$ satisfying $\overline{\mathcal{M}}(A) \le \operatorname{diam} A$ for all sets A with diameter less than ε. So $\overline{\mathcal{H}}^1_\varepsilon(F) \ge \overline{\mathcal{L}}(F)$ for all F. Therefore $\overline{\mathcal{H}}^1(F) \ge \overline{\mathcal{L}}(F)$.

Now if $[a, b)$ is a half-open interval and $\varepsilon > 0$, we may find points $a = x_0 < x_1 < \cdots < x_n = b$ with $x_j - x_{j-1} < \varepsilon$ for all j. Then $[a, b)$ is covered by the countable collection $\{\, [x_{j-1}, x_j] : 1 \le j \le n \,\}$, and

$$\sum_{j=1}^{n} \operatorname{diam}[x_{j-1}, x_j] = \sum_{j=1}^{n} (x_j - x_{j-1}) = b - a.$$

Therefore $\overline{\mathcal{H}}^1_\varepsilon\big([a, b)\big) \le b - a$. But by the Method I theorem, $\overline{\mathcal{L}}$ is the largest outer measure satisfying $\overline{\mathcal{L}}\big([a, b)\big) \le b - a$ for all half-open intervals $[a, b)$. Therefore $\overline{\mathcal{L}}(F) \ge \overline{\mathcal{H}}^1(F)$. for all F.

The two outer measures $\overline{\mathcal{L}}$ and $\overline{\mathcal{H}}^1$ coincide. The measurable sets in each case are given by the criterion of Carathéodory, so the measures $\mathcal{L}$ and $\mathcal{H}^1$ also coincide. □

For a "zero-dimensional" Hausdorff measure, we can use the set function $\mathbf{c}_0$ defined by $\mathbf{c}_0(A) = 1$ for $A \ne \varnothing$ and $\mathbf{c}_0(\varnothing) = 0$.

Exercise 6.1.5. With this definition, $\mathcal{H}^0(A) = n$ if A has n elements, and $\mathcal{H}^0(A) = \infty$ if A is infinite.

Hausdorff Dimension

How does the Hausdorff measure $\mathcal{H}^s(F)$ behave as a function of s for a given set F? An easy calculation shows that when s increases, $\mathcal{H}^s(F)$ decreases. But much more is true.

Theorem 6.1.6. *Let F be a Borel set. Let $0 < s < t$. If $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. If $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$.*

Proof. If $\operatorname{diam} A \le \varepsilon$, then

$$\overline{\mathcal{H}}^t_\varepsilon(A) \le (\operatorname{diam} A)^t \le \varepsilon^{t-s}(\operatorname{diam} A)^s.$$

Therefore by the Method I theorem, $\overline{\mathcal{H}}^t_\varepsilon(F) \le \varepsilon^{t-s}\overline{\mathcal{H}}^s_\varepsilon(F)$ for all F. Now if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) \le \lim_{\varepsilon\to 0} \varepsilon^{t-s}\,\overline{\mathcal{H}}^s_\varepsilon(F) = 0 \times \mathcal{H}^s(F) = 0$. The second assertion is the contrapositive. □

This means that, for a given set F, there is a unique "critical value" $s_0 \in [0,\infty]$ such that:

$$\begin{aligned} \mathcal{H}^s(F) &= \infty \qquad &&\text{for all } s < s_0; \\ \mathcal{H}^s(F) &= 0 \qquad &&\text{for all } s > s_0. \end{aligned}$$

This value s_0 is called the ***Hausdorff dimension*** of the set F. We will write $s_0 = \dim F$. Of course, it is possible that $\mathcal{H}^s(F) = 0$ for all $s > 0$; in that case $\dim F = 0$. In the same way, it is possible that $\mathcal{H}^s(F) = \infty$ for all s; in that case $\dim F = \infty$.

This idea of dimension is an abstraction of what we already know from elementary geometry. If A is a nice smooth rectifiable curve, then its length is a useful way to measure its size; but its "area" and "volume" are 0. The dimensions 2 and 3 are too large to help in measuring the size of A. If B is the surface of a sphere, then its area is positive and finite. We can say its "length" is infinite (for example, since it contains curves that are as long as we like which spiral around); its "volume" is 0, since it is contained in a solid shperical shell whose thickness is as small as we like. So for the set B, the dimension 1 is too small, the dimension 3 is too large, and the dimension 2 is just right. The s-dimensional Hausdorff measure give us a way of measuring the size of a set for dimensions s other than the integers $1, 2, 3, \cdots$.

Theorem 6.1.7. *Let A, B be Borel sets.*

(1) *If $A \subseteq B$, then* $\dim A \le \dim B$.
(2) $\dim(A \cup B) = \max\{\dim A, \dim B\}$.

Proof. (1) Suppose $A \subseteq B$. If $s > \dim B$, then $\mathcal{H}^s(A) \le \mathcal{H}^s(B) = 0$. Therefore $\dim A \le s$. This is true for all $s > \dim B$, so $\dim A \le \dim B$.

(2) Let $s > \max\{\dim A, \dim B\}$. Then $s > \dim A$, so $\mathcal{H}^s(A) = 0$. Similarly, $\mathcal{H}^s(B) = 0$. Then $\mathcal{H}^s(A\cup B) \le \mathcal{H}^s(A) + \mathcal{H}^s(B) = 0$. Therefore $\dim(A\cup B) \le s$. This is true for all $s > \max\{\dim A, \dim B\}$, so we have $\dim(A \cup B) \le \max\{\dim A, \dim B\}$. By (1), $\dim(A \cup B) \ge \max\{\dim A, \dim B\}$. □

Exercise 6.1.8. Suppose $A_1, A_2, \cdots$ are Borel sets. Is it true that

$$\dim \bigcup_{k\in\mathbb{N}} A_k = \sup_k \dim A_k?$$

Theorem 6.1.9. *Let $f\colon S \to T$ be a similarity with ratio $r > 0$, let s be a positive real number, and let $F \subseteq S$ be a set. Then $\overline{\mathcal{H}}^s(f[F]) = r^s \overline{\mathcal{H}}^s(F)$. So $\dim f[F] = \dim F$.*

Proof. We may assume that $T = f[S]$. Then f has an inverse f^{-1}. A set $A \subseteq S$ satisfies $\operatorname{diam} f[A] = r \operatorname{diam} A$. Therefore $(\operatorname{diam} f[A])^s = r^s (\operatorname{diam} A)^s$. By the Method I theorem (applied twice), $\overline{\mathcal{H}}^s_{r\varepsilon}(f[F]) = r^s \overline{\mathcal{H}}^s_{\varepsilon}(F)$. Therefore $\overline{\mathcal{H}}^s(f[F]) = r^s \overline{\mathcal{H}}^s(F)$ and $\dim f[F] = \dim F$. □

Exercise 6.1.10. Suppose $f\colon S \to T$ is a function. Let $A \subseteq S$ be a Borel set. Prove or disprove: (1) If f is Lipschitz, then

$$\dim f[A] \le \dim A.$$

(2) If f is inverse Lipschitz, then

$$\dim f[A] \ge \dim A.$$

Exercise 6.1.11. Suppose S is a metric space and $\dim S < \infty$. Does it follow that S is separable?

6.2 Packing Measure

In this section we define the packing measures and the packing dimension.

Mandelbrot says that his definition for "fractal" ($\operatorname{Cov} S < \dim S$) is too broad, in that it admits "true geometric chaos". The sets that are of interest for applications (and in mathematics) are generally not the most general set, with few special properties. So it may be useful to restrict the term "fractal" so that the sets meeting the conditions have useful properties. One possible way to do this has been proposed by James Taylor. He proposed to apply the term "fractal" to (Borel) sets where the packing dimension is equal to the Hausdorff dimension.

Motivations

Before we formulate the definition of the packing measures, let us discuss some of the reasons for the definition, and why it has the form given.

Hausdorff measure is based on "covering" of a set. The set E to be measured is covered by small sets A_i. We attempt to make the covering "efficient" by minimizing

$$\sum_{i \in \mathbb{N}} \mathbf{c}(A_i),$$

subject to the constraint that the sets A_i cover E. When this sum is smaller, the cover of E by $\{A_i\}$ is considered to be more efficient.

Another possibility for "measuring" the set E is to do it by packing rather than covering. We want to put disjoint sets A_i inside E. We attempt to make this packing "efficient" by maximizing

$$\sum_{i\in\mathbb{N}} \mathbf{c}(A_i),$$

subject to the constraint that the sets A_i are disjoint subsets of E. When this sum is larger, the packing $\{A_i\}$ is considered to be more efficient.

For fractal measures, the set function $\mathbf{c}$ should be of the form $(\operatorname{diam} A)^s$, where $s > 0$ is the dimension we are interested in. But this leads to certain undesirable features if taken at face value. For example, in the plane, what if we pack a square as in Fig. 6.2.1(a)? By making the sets A_i narrow enough, we can make the sum

$$\sum_{i\in\mathbb{N}} (\operatorname{diam} A_i)^s,$$

as large as we like.

The way to avoid this is to pack only by sets of a special type. For example, in $\mathbb{R}$, packings with intervals cannot be beat. In Euclidean space, the choice is often to pack with cubes. In order to get a definition that applies in a general metric space, we will pack with balls.

Packing a set E with balls $A_i \subseteq E$ is fine when E is an open set, but other sets may contain no balls at all. So we drop the requirement that the balls be contained in E. But to make sure the packing measures the set E we require instead that the centers of the balls lie in E.

Let S be a metric space, $x \in S$ and $r > 0$. Recall the notation

$$B_r(x) = \{\, y \in S : \varrho(x,y) < r \,\}, \qquad \overline{B}_r(x) = \{\, y \in S : \varrho(x,y) \leq r \,\}.$$

We will pack with closed balls. But open balls could be used just as well in our setting.

For two balls $\overline{B}_r(x), \overline{B}_s(y)$ in Euclidean space, we know that they are disjoint, $\overline{B}_r(x) \cap \overline{B}_s(y) = \varnothing$, if and only if $\varrho(x,y) > r + s$. In metric space other than Euclidean space, this equivalence may fail. But we do know that if $\varrho(x,y) > r+s$, then $\overline{B}_r(x) \cap \overline{B}_s(y) = \varnothing$. We will use $\varrho(x,y) > r+s$ for our definition of "packing".

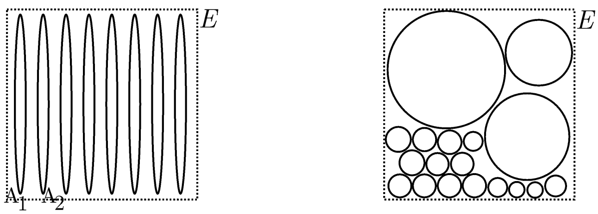

Fig. 6.2.1. (a) Packing with any sets (b) packing with balls

In Euclidean space, two balls are equal, $\overline{B}_r(x) = \overline{B}_s(y)$, if and only if $x = y$ and $r = s$. In metric space other than Euclidean space, this equivalence may fail. For example, in an ultrametric space, every point of a ball is a center. This is a reason for our use of "constituents" rather than balls in the definition.

In Euclidean space, the diameter of the ball $\overline{B}_r(x)$ is $2r$. In metric space other than Euclidean space, this may not be true. For example, in an ultrametric space, $\operatorname{diam} \overline{B}_r(x) \le r$. We will use the set function $(2r)^s$ for our "radius-based" packing measure rather than the "diameter-based" option of $(\operatorname{diam} \overline{B}_r(x))^s$.

In some texts—including the first edition of this one—one or more of the above choices may be reversed. As noted, in Euclidean space this makes no difference.

Definition

Let S be a metric space. A ***constituent*** in S is a pair (x, r), where $x \in S$ and $r > 0$. We think of the constutuent (x, r) as standing for the closed ball $\overline{B}_r(x)$. We may even call x the "center" and r the "radius" of the constituent (x, r).

Let $E \subseteq S$. A ***packing*** of E is a countable collection Π of constituents, such that: (a) for all $(x, r) \in \Pi$, we have $x \in E$; (b) for all $(x, r), (y, s) \in \Pi$ with $(x, r) \neq (y, s)$, we have $\varrho(x, y) > r + s$.

For $\delta > 0$, we say that a packing Π is ***δ-fine*** iff for all $(x, r) \in \Pi$ we have $r \le \delta$. Let $F \subseteq S$, and let $\delta, s > 0$. Define

$$\widetilde{\mathcal{P}}^s_\delta(F) = \sup \sum_{(x,r)\in\Pi} (2r)^s,$$

where the supremum is over all δ-fine packings Π of F. Note: because of the sup involved, we may restrict this to *finite* packings Π only.

When δ decreases to 0, the value $\widetilde{\mathcal{P}}^s_\delta(F)$ decreases, so we define

$$\widetilde{\mathcal{P}}^s_0(F) = \lim_{\delta\to 0} \widetilde{\mathcal{P}}^s_\delta(F) = \inf_{\delta>0} \widetilde{\mathcal{P}}^s_\delta(F).$$

When we have done this, we get a family $\left(\widetilde{\mathcal{P}}^s_0\right)$ of set functions indexed by s. As before, there is a critical value:

Exercise 6.2.2. Let F be a set in a metric space. There is $s_0 \in [0, \infty]$ such that

$$\begin{aligned} \widetilde{\mathcal{P}}^s_0(F) &= \infty \qquad \text{for all } s < s_0; \\ \widetilde{\mathcal{P}}^s_0(F) &= 0 \qquad \text{for all } s > s_0. \end{aligned}$$

This critical value s_0 will be called the ***packing index*** of the set F. However, the set functions $\widetilde{\mathcal{P}}^s_0$ are not really what we want. They are *not* outer measures. This is not unexpected, since the process used to construct them is not method II. Here is an illustration showing that $\widetilde{\mathcal{P}}_0^{1/2}$ fails to be an outer measure on $\mathbb{R}$:

Proposition 6.2.3. *Let K be the compact set $\{0, 1, 1/2, 1/3, 1/4, 1/5, \cdots\} \subseteq \mathbb{R}$. Then $\widetilde{\mathcal{P}}_0^{1/2}(K) > 0$.*

Proof. Let $k \in \mathbb{N}$ be odd, let $\varepsilon = 2^{-k}$, and let $n = 2^{(k-1)/2}$. Then

$$\frac{1}{n-1} - \frac{1}{n} > \frac{1}{n^2} = 2\varepsilon,$$

so the constituents with radius ε and centers $1, 1/2, 1/3, \cdots, 1/n$ form a packing. (That is, the balls with radius ε and centers $1, 1/2, 1/3, \cdots, 1/n$ are disjoint.) So

$$\widetilde{\mathcal{P}}_\varepsilon^{1/2}(K) \geq n\,(2\varepsilon)^{1/2} = 1,$$

and therefore $\widetilde{\mathcal{P}}_0^{1/2}(K) \geq 1$. □

For many purposes it is unreasonable to claim that this countable set K has positive dimension. We know a good way (method I) to get an outer measure from a set function. So we apply method I to the set function $\widetilde{\mathcal{P}}_0^s$:

$$\overline{\mathcal{P}}^s(E) = \inf \sum_{C \in \mathcal{C}} \widetilde{\mathcal{P}}_0^s(C),$$

where the inf is over all countable covers $\mathcal{C}$ of the set E.

Theorem 6.2.4 (The closure theorem). *If C is a set and $\overline{C}$ is its closure, then $\widetilde{\mathcal{P}}_0^s(C) = \widetilde{\mathcal{P}}_0^s(\overline{C})$.*

Proof. Any packing of C is automatically a packing of $\overline{C}$. This shows that $\widetilde{\mathcal{P}}_\delta^s(C) \leq \widetilde{\mathcal{P}}_\delta^s(\overline{C})$ for all δ and thus $\widetilde{\mathcal{P}}_0^s(C) \leq \widetilde{\mathcal{P}}_0^s(\overline{C})$.

Conversely, let $\delta > 0$ and let Π be a finite δ-fine packing of $\overline{C}$. Write $\Pi = \{(x_1, r_1), \cdots, (x_n, r_n)\}$. For any $i \neq j$, we have $\varrho(x_i, x_j) - r_i - r_j > 0$, and there are only finitely many pairs i, j, so there is $\varepsilon > 0$ with $\varrho(x_i, x_j) - r_i - r_j > \varepsilon$ for all $i \neq j$. Now for each i, the point x_i belongs to the closure of C, so there is $y_i \in C$ with $\varrho(x_i, y_i) < \varepsilon/2$. But then $\Pi' = \{(y_1, r_1), \cdots, (y_n, r_n)\}$ is a packing of C, still δ-fine, and it has the same value $\sum (2r_i)^s$ as the packing Π. Therefore we get $\widetilde{\mathcal{P}}_\delta^s(C) \geq \widetilde{\mathcal{P}}_\delta^s(\overline{C})$ for all δ and thus $\widetilde{\mathcal{P}}_0^s(C) \geq \widetilde{\mathcal{P}}_0^s(\overline{C})$. □

The class of closed sets is a reduced cover class for $\overline{\mathcal{P}}^s$:

Corollary 6.2.5. *Let $E \subseteq S$. Then*

$$\overline{\mathcal{P}}^s(E) = \inf \sum_{C \in \mathcal{C}} \widetilde{\mathcal{P}}_0^s(C),$$

where the inf *is over all countable covers $\mathcal{C}$ of the set E by closed sets.*

Lemma 6.2.6. *Let $A, B \subseteq S$. Then $\widetilde{\mathcal{P}}^s_0(A \cup B) \le \widetilde{\mathcal{P}}^s_0(A) + \widetilde{\mathcal{P}}^s_0(B)$. If $\operatorname{dist}(A, B) > 0$, then $\widetilde{\mathcal{P}}^s_0(A \cup B) = \widetilde{\mathcal{P}}^s_0(A) + \widetilde{\mathcal{P}}^s_0(B)$.*

Proof. Let $\delta > 0$ be given. Let Π be a δ-fine packing of $A \cup B$. Then Π is the disjoint union of

$$\Pi_1 = \{\, (x, r) \in \Pi : x \in A \,\} \quad \text{and} \quad \Pi_2 = \{\, (x, r) \in \Pi : x \notin A \,\}.$$

But Π_1 is a δ-fine packing of A and Π_2 is a δ-fine packing of B. So

$$\sum_{(x,r)\in\Pi} (2r)^s = \sum_{(x,r)\in\Pi_1} (2r)^s + \sum_{(x,r)\in\Pi_2} (2r)^s \le \widetilde{\mathcal{P}}^s_\delta(A) + \widetilde{\mathcal{P}}^s_\delta(B).$$

Take the supremum over all δ-fine packings Π to get $\widetilde{\mathcal{P}}^s_\delta(A \cup B) \le \widetilde{\mathcal{P}}^s_\delta(A) + \widetilde{\mathcal{P}}^s_\delta(B)$. Let $\delta \to 0$ to get $\widetilde{\mathcal{P}}^s_0(A \cup B) \le \widetilde{\mathcal{P}}^s_0(A) + \widetilde{\mathcal{P}}^s_0(B)$.

Let $\operatorname{dist}(A, B) = \varepsilon > 0$. Then if $\delta < \varepsilon/2$, any δ-fine packing of $A \cup B$ is the disjoint union of a δ-fine packing of A and a δ-fine packing of B. And conversely, the union of a δ-fine packing of A and a δ-fine packing of B is a δ-fine packing of $A \cup B$. So $\widetilde{\mathcal{P}}^s_\delta(A \cup B) = \widetilde{\mathcal{P}}^s_\delta(A) + \widetilde{\mathcal{P}}^s_\delta(B)$. Let $\delta \to 0$ to get $\widetilde{\mathcal{P}}^s_0(A \cup B) = \widetilde{\mathcal{P}}^s_0(A) + \widetilde{\mathcal{P}}^s_0(B)$. □

Theorem 6.2.7. *The set function $\overline{\mathcal{P}}^s$ is a metric outer measure.*

Proof. The only packing of the empty set is the empty packing, and an empty sum has the value 0. Therefore $\widetilde{\mathcal{P}}^s_\delta(\varnothing) = 0$ for all $\delta > 0$ and $\widetilde{\mathcal{P}}^s_0(\varnothing) = 0$. The empty set can be covered $\varnothing \subseteq \bigcup_{n\in\mathbb{N}} E_n$, where $E_n = \varnothing$ for all n, so $\overline{\mathcal{P}}^s(\varnothing) = 0$.

If $A \subseteq B$, and $B \subseteq \bigcup_{n\in\mathbb{N}} E_n$, then also $A \subseteq \bigcup_{n\in\mathbb{N}} E_n$, so $\overline{\mathcal{P}}^s(A) \le \overline{\mathcal{P}}^s(B)$.

Suppose $A = \bigcup_{i\in\mathbb{N}} A_i$. We must show $\overline{\mathcal{P}}^s(A) \le \sum_{i=1}^\infty \overline{\mathcal{P}}^s(A_i)$. If $\sum_i \overline{\mathcal{P}}^s(A_i)$ diverges, then there is nothing to do, so assume $\sum_{i=1}^\infty \overline{\mathcal{P}}^s(A_i) < \infty$. Let $\varepsilon > 0$ be given. For each i, there exist sets E_{ni}, $n \in \mathbb{N}$, so that $A_i \subseteq \bigcup_n E_{ni}$ and $\sum_n \widetilde{\mathcal{P}}^s_0(E_{ni}) < \overline{\mathcal{P}}^s(A_i) + \varepsilon/2^i$. Then $A \subseteq \bigcup_i \bigcup_n E_{ni}$ is a countable cover of A, so

$$\overline{\mathcal{P}}^s(A) \le \sum_i \sum_n \widetilde{\mathcal{P}}^s_0(E_{ni}) < \sum_i \left(\overline{\mathcal{P}}^s(A_i) + \frac{\varepsilon}{2^i} \right) = \left(\sum_i \overline{\mathcal{P}}^s(A_i) \right) + \varepsilon.$$

This holds for any $\varepsilon > 0$, so $\overline{\mathcal{P}}^s(A) \le \sum_i \overline{\mathcal{P}}^s(A_i)$.

The metric property follows from Lemma 6.2.6. □

The restriction of $\overline{\mathcal{P}}^s$ to the measurable sets is a measure, called the ***s-dimensional packing measure***, and written $\mathcal{P}^s$. As usual there is a critical value for s:

Exercise 6.2.8. Let F be a Borel set in a metric space. There is $s_0 \in [0, \infty]$ such that

$$\begin{aligned} \mathcal{P}^s(F) &= \infty \qquad \text{for all } s < s_0; \\ \mathcal{P}^s(F) &= 0 \qquad \text{for all } s > s_0. \end{aligned}$$

This value s_0 is called the ***packing dimension*** of the set F. We will write $s_0 = \operatorname{Dim} F$. It is a more reasonable quantity than the packing index defined above.

Elementary Properties

The packing dimension has many of the same properties as the Hausdorff dimension.

Theorem 6.2.9. *Let A, B be Borel sets.*

(1) *If $A \subseteq B$, then* $\operatorname{Dim} A \le \operatorname{Dim} B$.
(2) $\operatorname{Dim}(A \cup B) = \max\{\operatorname{Dim} A, \operatorname{Dim} B\}$.

Proof. (1) Assume $A \subseteq B$. Let $s > \operatorname{Dim} B$. then $\overline{\mathcal{P}}^s(B) = 0$. Therefore $\overline{\mathcal{P}}^s(A) = 0$. This shows $\operatorname{Dim} A \le s$. This holds for all $s > \operatorname{Dim} B$, so $\operatorname{Dim} A \le \operatorname{Dim} B$.

(2) By (1), $\operatorname{Dim}(A \cup B) \ge \operatorname{Dim} A$ and $\operatorname{Dim}(A \cup B) \ge \operatorname{Dim} B$. Therefore $\operatorname{Dim}(A \cup B) \ge \max\{\operatorname{Dim} A, \operatorname{Dim} B\}$. If $s > \max\{\operatorname{Dim} A, \operatorname{Dim} B\}$, then $\overline{\mathcal{P}}^s(A) = 0$ and $\overline{\mathcal{P}}^s(B) = 0$. So by subadditivity, $\overline{\mathcal{P}}^s(A \cup B) = 0$. This shows $\operatorname{Dim}(A \cup B) \le s$. It holds for all $s > \max\{\operatorname{Dim} A, \operatorname{Dim} B\}$, so $\operatorname{Dim}(A \cup B) \le \max\{\operatorname{Dim} A, \operatorname{Dim} B\}$. ☐

Exercise 6.2.10. Suppose $A_1, A_2, \cdots$ are Borel sets. Is it true that

$$\operatorname{Dim} \bigcup_{k \in \mathbb{N}} A_k = \sup_k \operatorname{Dim} A_k?$$

Theorem 6.2.11. *Let $f\colon S \to T$ be a similarity with ratio $r > 0$, let s be a positive real number, and let $E \subseteq S$ be a set. Then $\overline{\mathcal{P}}^s(f[E]) = r^s \overline{\mathcal{P}}^s(E)$. So* $\operatorname{Dim} f[E] = \operatorname{Dim} E$.

Proof. Let Π be a δ-fine packing of F. Then $\{\, (f(x), rt) : (x,t) \in \Pi \,\}$ is an $r\delta$-fine packing of $f[F]$. So

$$\widetilde{\mathcal{P}}^s_{r\delta}\big(f[F]\big) \ge \sum_{(x,t)\in\Pi} (2rt)^s = r^s \sum_{(x,t)\in\Pi} (2t)^s.$$

This holds for all δ-fine packings of F, so

$$\widetilde{\mathcal{P}}^s_{r\delta}\big(f[F]\big) \ge r^s \widetilde{\mathcal{P}}^s_\delta\big(F\big).$$

Let $\delta \to 0$ to get

$$\widetilde{\mathcal{P}}^s_0\big(f[F]\big) \geq r^s \widetilde{\mathcal{P}}^s_0\big(F\big).$$

This holds for all subsets $F \subseteq S$.

Since $r > 0$, the map f is one-to-one, and maps F onto $f[F]$. Every $r\delta$-fine packing of $f[F]$ is of the form $\big\{ \big(f(x), rt\big) : (x,t) \in \Pi \big\}$ for some δ-fine packing of F. So the estimates hold in reverse, and we conclude

$$\widetilde{\mathcal{P}}^s_0\big(f[F]\big) = r^s \widetilde{\mathcal{P}}^s_0\big(F\big).$$

Now if $E \subseteq \bigcup_i A_i$ is a countable cover of a set E, then $f[E] \subseteq \bigcup_i f[A_i]$ is a cover of the image set $f[E]$. So

$$\sum_i \widetilde{\mathcal{P}}^s_0(A_i) = r^s \sum_i \widetilde{\mathcal{P}}^s_0\big(f[A_i]\big) \geq r^s \overline{\mathcal{P}}^s\big(f[E]\big).$$

This holds for all covers of E, so $\overline{\mathcal{P}}^s\big(E\big) \geq r^s \overline{\mathcal{P}}^s\big(f[E]\big)$.

If $f[E] \subseteq \bigcup B_i$ is a countable cover of $f[E]$, let $A_i = f^{-1}[B_i]$, so that $E \subseteq \bigcup A_i$ is a cover of E. Note $f[A_i] \subseteq B_i$. Now

$$r^s \sum_i \widetilde{\mathcal{P}}^s_0(B_i) \geq r^s \sum_i \widetilde{\mathcal{P}}^s_0(f[A_i]) = \sum_i \widetilde{\mathcal{P}}^s_0(A_i) \geq \overline{\mathcal{P}}^s(E).$$

This holds for all covers of $f[E]$, so $r^s \overline{\mathcal{P}}^s\big(f[E]\big) \geq \overline{\mathcal{P}}^s(E)$.

Therefore, we have $\overline{\mathcal{P}}^s(E) = r^s \overline{\mathcal{P}}^s\big(f[E]\big)$. And $\operatorname{Dim} f[E] = \operatorname{Dim} E$. □

Exercise 6.2.12. Suppose $f\colon S \to T$ is a function. Let $A \subseteq S$ be a Borel set. Prove or disprove: (1) If f is Lipschitz, then

$$\operatorname{Dim} f[A] \leq \operatorname{Dim} A.$$

(2) If f is inverse Lipschitz, then

$$\operatorname{Dim} f[A] \geq \operatorname{Dim} A.$$

Proposition 6.2.13. *In the metric space $\mathbb{R}$, the one-dimensional packing measure $\mathcal{P}^1$ coincides with Lebesgue measure $\mathcal{L}$.*

Proof. First consider a half-open interval, $[a,b)$. If Π is a finite packing of $[a,b)$, write $\Pi = \{(x_1,r_1),(x_2,r_2),\cdots,(x_n,r_n)\}$ with $x_1 < x_2 < \cdots < x_n$. Then all the balls $\overline{B}_{r_i}(x_i)$ are contained in the interval $[a-r_1, b+r_n]$ and are disjoint. By the additivity of Lebesgue measure (and the fact that intervals are Lebesgue measurable sets), we have

$$\sum_{i=1}^{n} (2r_i) \leq b - a + r_1 + r_n.$$

If Π is δ-fine, then

$$\sum_{i=1}^{n}(2r_i) \le b - a + 2\delta.$$

Take the supremum on all δ-fine packings of $[a, b)$ to conclude

$$\widetilde{\mathcal{P}}^1_\delta\big([a, b)\big) \le b - a + 2\delta.$$

Let $\delta \to 0$ to get

$$\widetilde{\mathcal{P}}^1_0\big([a, b)\big) \le b - a.$$

On the other hand, given $\delta > 0$, choose n with $(b-a)/n < \delta$, then we can pack $[a, b)$ with n balls all of radius $r_i = (b-a)/(2n)$. So $\widetilde{\mathcal{P}}^1_\delta\big([a, b)\big) \ge b - a$. Take the limit to get $\widetilde{\mathcal{P}}^1_0\big([a, b)\big) \ge b - a$. Therefore $\widetilde{\mathcal{P}}^1_0\big([a, b)\big) = b - a$.

Now consider a finite disjoint union of half-open intervals $[a, b)$. If two of them are adjacent (the right endpoint of one is the left endpoint of the other), then they may be combined into a single interval. If two of them are not adjacent, then there is a gap of positive length between them. So by Lemma 6.2.6 we have $\widetilde{\mathcal{P}}^1_0(V) = \mathcal{L}(V)$ for all such finite disjoint unions. This holds in particular for the dyadic ring $\mathcal{R}$ defined on p. 150.

Fix a large $N > 0$ and consider $\mathcal{L}$ and $\overline{\mathcal{P}}^1$ for subsets of $[-N, N]$. We claim that $\mathcal{R}$ is a reduced cover class for $\overline{\mathcal{P}}^1$. Given any closed set $F \subseteq [-N, N]$ and any $\varepsilon > 0$, there is $V \in \mathcal{R}$ with $V \supseteq F$ and $\mathcal{L}(V \setminus F) < \varepsilon/2$; then there is $U \in \mathcal{R}$ with $U \supseteq V \setminus F$ and $\mathcal{L}(U \setminus (V \setminus F)) < \varepsilon/2$. Then

$$\begin{aligned}\widetilde{\mathcal{P}}^1_0(V) &\le \widetilde{\mathcal{P}}^1_0(F) + \widetilde{\mathcal{P}}^1_0(V \setminus F) \le \widetilde{\mathcal{P}}^1_0(F) + \widetilde{\mathcal{P}}^1_0(U)\\ &\le \widetilde{\mathcal{P}}^1_0(F) + \mathcal{L}(U) \le \widetilde{\mathcal{P}}^1_0(F) + \varepsilon.\end{aligned}$$

Now the closed sets form a reduced cover class for $\overline{\mathcal{P}}^1$, so this shows that the dyadic ring $\mathcal{R}$ also forms a reduced cover class for $\overline{\mathcal{P}}^1$ in $[-N, N]$.

We have seen that $\widetilde{\mathcal{P}}^1_0$ and $\mathcal{L}$ agree on $\mathcal{R}$, so their method I measures also agree: $\overline{\mathcal{L}} = \overline{\mathcal{P}}^1$ for subsets of $[-N, N]$. For a general subset A of $\mathbb{R}$, take the increasing limit of $[-N, N] \cap A$. So $\overline{\mathcal{L}} = \overline{\mathcal{P}}^1$. □

The packing dimension is related to the Hausdorff dimension.*

Proposition 6.2.14. *Let S be a metric space and $F \subseteq S$ a Borel set. Then $\mathcal{H}^s(F) \le 2^s \mathcal{P}^s(F)$ and $\dim F \le \operatorname{Dim} F$.*

Proof. I first show that $\overline{\mathcal{H}}^s_{4\varepsilon}(F) \le 2^s \widetilde{\mathcal{P}}^s_\varepsilon(F)$. Now if $\widetilde{\mathcal{P}}^s_\varepsilon(F) = \infty$, then this is clear. So suppose $\widetilde{\mathcal{P}}^s_\varepsilon(F) < \infty$. If there were an *infinite* packing of F with all radii equal to ε, then $\widetilde{\mathcal{P}}^s_\varepsilon(F) = \infty$. So there is a maximal finite packing $\{(x_1, \varepsilon), (x_2, \varepsilon), \cdots, (x_n, \varepsilon)\}$ of F. Then $\widetilde{\mathcal{P}}^s_\varepsilon(F) \ge n(2\varepsilon)^s$. By the maximality,

* In the first edition, this was stated only in Euclidean space—one of the drawbacks of the diameter-based definition of the packing measure.

for any $x \in F$, there is some i between 1 and n with $\varrho(x, x_i) \leq 2\varepsilon$. So the collection $\{ \overline{B}_{2\varepsilon}(x_i) : 1 \leq i \leq n \}$ covers F, and

$$\overline{\mathcal{H}}^s_{4\varepsilon}(F) \leq \sum_{i=1}^{n} \big(\operatorname{diam} \overline{B}_{2\varepsilon}(x_i) \big)^s \leq n(4\varepsilon)^s = 2^s n (2\varepsilon)^s \leq 2^s \widetilde{\mathcal{P}}^s_\varepsilon(F).$$

Therefore $\overline{\mathcal{H}}^s_{4\varepsilon}(F) \leq 2^s \widetilde{\mathcal{P}}^s_\varepsilon(F)$.

Now take the limit as $\varepsilon \to 0$ and conclude $\mathcal{H}^s(F) \leq 2^s \widetilde{\mathcal{P}}^s_0(F)$. So by the Method I theorem, $\mathcal{H}^s(F) \leq 2^s \mathcal{P}^s(F)$.

Now if $s < \dim F$, then $\mathcal{H}^s(F) = \infty$, so $\mathcal{P}^s(F) = \infty$, and therefore $s \leq \operatorname{Dim} F$. We therefore conclude that $\dim F \leq \operatorname{Dim} F$. □

A set $F \subseteq \mathbb{R}^d$ is a ***fractal*** (in the sense of Taylor) iff $\dim F = \operatorname{Dim} F$.

6.3 Examples

According to Mandelbrot's definition, a fractal is a set A with $\operatorname{Cov} A < \dim A$. According to Taylor's definition, a fractal is a set A with $\dim A = \operatorname{Dim} A$. In order for these definitions to be useful, we will have to be able to compute the dimensions involved. In some cases this is not easy to do.

In this section, we will do a few examples directly from the definitions. We will carry out the calculations in great detail. In Sect. 6.4 we will discuss self-similar sets in general.

Binary Tree

Here is our first official example of a fractal. We computed $\operatorname{ind}\{0,1\}^{(\omega)} = 0$ in Theorem 3.4.4. For $\{0,1\}^{(\omega)}$ we will use the metric $\varrho_{1/2}$ defined on p. 44 and the measure $\mathcal{M}_{1/2}$ defined on p. 160. Recall the notation $[\alpha]$ for cylinders from p. 13.

Proposition 6.3.1. *Let $E = \{0,1\}$ be a two-letter alphabet, let $E^{(\omega)}$ be the space of all infinite strings using E, and let $\varrho_{1/2}$ be the metric for $E^{(\omega)}$. Then $\mathcal{H}^1 = \mathcal{M}_{1/2}$ and $\dim E^{(\omega)} = 1$.*

Proof. To prove that $\mathcal{H}^1 = \mathcal{M}_{1/2}$, we will use two applications of the Method I theorem.

If a set $A \subseteq E^{(\omega)}$ has positive diameter, then (Proposition 2.6.7) there is a string $\alpha \in E^{(*)}$ with $A \subseteq [\alpha]$ and $\operatorname{diam} A = \operatorname{diam}[\alpha]$. So $\mathcal{M}_{1/2}(A) \leq \mathcal{M}_{1/2}([\alpha]) = \operatorname{diam}[\alpha] = \operatorname{diam} A$. But $\overline{\mathcal{H}}^1_\varepsilon$ is the largest outer measure with $\overline{\mathcal{H}}^1_\varepsilon(A) \leq \operatorname{diam} A$ for all sets A of diameter $\leq \varepsilon$. So $\overline{\mathcal{M}}_{1/2} \leq \overline{\mathcal{H}}^1_\varepsilon$. This is true for all $\varepsilon > 0$, so $\overline{\mathcal{M}}_{1/2} \leq \overline{\mathcal{H}}^1$.

On the other hand, let $\alpha \in E^{(*)}$ be a finite string, and $\varepsilon > 0$. There is n so large that $2^{-n} < \varepsilon$ and $n \geq |\alpha|$, the length of α. Then the basic open set $[\alpha]$ is the disjoint union of all sets $[\beta]$, where $\beta \geq \alpha$ and $|\beta| = n$. There are $2^{n-|\alpha|}$ of these sets. Then

$$\overline{\mathcal{H}}^1_\varepsilon\big([\alpha]\big) \leq \sum_{\substack{\beta \geq \alpha \\ |\beta|=n}} \operatorname{diam}[\beta] = \sum_{\substack{\beta \geq \alpha \\ |\beta|=n}} 2^{-n} = 2^{-|\alpha|}.$$

But $\overline{\mathcal{M}}_{1/2}$ is the largest outer measure with $\overline{\mathcal{M}}_{1/2}([\alpha]) \leq 2^{-|\alpha|}$ for all $\alpha \in E^{(*)}$. So $\overline{\mathcal{H}}^1_\varepsilon \leq \overline{\mathcal{M}}_{1/2}$, and thus $\overline{\mathcal{H}}^1 \leq \overline{\mathcal{M}}_{1/2}$.

Therefore $\overline{\mathcal{H}}^1 = \overline{\mathcal{M}}_{1/2}$. The measurable sets in both cases are given by the criterion of Carathéodory, so $\mathcal{H}^1 = \mathcal{M}_{1/2}$.

Now since $0 < \mathcal{H}^1(E^{(\omega)}) < \infty$, we conclude that $\dim E^{(\omega)} = 1$. □

So we know that $\{0, 1\}^{(\omega)}$ is a fractal in the sense of Mandelbrot. It is also a fractal in the sense of Taylor:

Proposition 6.3.2. *Let $E = \{0, 1\}$ be a two-letter alphabet, let $E^{(\omega)}$ be the space of all infinite strings using E, and let $\varrho_{1/2}$ be the metric for $E^{(\omega)}$. Then $\mathcal{P}^1 = 4\mathcal{M}_{1/2}$ and $\operatorname{Dim} E^{(\omega)} = 1$.*

Proof. The outer measure $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{1/2}$ is the largest outer measure such that $\overline{\mathcal{M}}\big([\alpha]\big) \leq 2^{-|\alpha|}$ for all $\alpha \in E^{(*)}$. In fact, $\overline{\mathcal{M}}\big([\alpha]\big) = 2^{-|\alpha|}$.

We will describe the balls in $E^{(\omega)}$. Let $\sigma \in E^{(\omega)}$ and let r satisfy $0 < r < 1$. There is a unique $n \in \mathbb{N}$ with $2^{-n} \leq r < 2^{-n+1}$. The prefix $\alpha = \sigma{\restriction}n$ of length n defines a cylinder $[\alpha]$. I claim that $\overline{B}_r(\sigma) = [\alpha]$. To see this, note that any string $\tau \in [\alpha]$ agrees with σ at least for the first n letters, so $\varrho(\sigma, \tau) \leq 2^{-n} \leq r$. And any string $\tau \notin [\alpha]$ disagrees with σ somewhere in the first n letters, so the longest common prefix is shorter than n, and thus $\varrho(\sigma, \tau) \geq 2^{-n+1} > r$. The measure of the ball is $\mathcal{M}\big(\overline{B}_r(\sigma)\big) = \mathcal{M}\big([\alpha]\big) = 2^{-n}$, so $\mathcal{M}\big(\overline{B}_r(\sigma)\big) \leq r$, $\mathcal{M}\big(\overline{B}_r(\sigma)\big) > r/2$, and $r < 2\mathcal{M}\big(\overline{B}_r(\sigma)\big)$.

(a) First we prove $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) \leq 4$. Let $\delta > 0$, and let Π be a δ-packing of $E^{(\omega)}$. Then the corresponding closed balls $\big\{\, \overline{B}_r(\sigma) : (\sigma, r) \in \Pi \,\big\}$ are disjoint. So

$$\sum_{(\sigma,r)\in\Pi} (2r) \leq 4 \sum_{(\sigma,r)\in\Pi} \mathcal{M}\big(\overline{B}_r(\sigma)\big) \leq 4.$$

This is true for all δ-packings, so $\widetilde{\mathcal{P}}^1_\delta(E^{(\omega)}) \leq 4$. This holds for all $\delta > 0$, so $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) \leq 4$.

Because $E^{(\omega)}$ is a one-element cover of itself, we have also $\mathcal{P}^1(E^{(\omega)}) \leq 4$.

(b) Now we prove* $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) \geq 4$. Fix $N \in \mathbb{N}$, $N \geq 2$. Write $\eta = 1 - 2^{-N}$, so $0 < \eta < 1$. Note $(1 + 2^{-N})\eta = 1 - 2^{-2N} < 1$. Let $\delta > 0$ be given. We will construct a δ-fine packing Π. Choose $M \in \mathbb{N}$ so that $2^{-M} < \delta/2$. Define

* A technical note: packing by balls all the same size is *not* the most efficient packing in this case!

$$N_0 = M, N_1 = M + N, \cdots, N_k = M + kN, \cdots.$$

For $k = 0, 1, \cdots$, let Π_k be the set of all constituents (σ, r) where $r = 2^{-N_k+1}\eta$ and the string σ, counting from the beginning, has letter $\mathsf{0}$ in locations $N_0, N_1, \cdots, N_{k-1}$, letter $\mathsf{1}$ in location N_k and all $\mathsf{0}$s beyond location N_k. Pictorially,

for Π_0

$$\underbrace{xx\cdots x\mathsf{1}}_{M}\,\mathsf{000}\cdots,$$

for Π_1

$$\underbrace{xx\cdots x\mathsf{0}}_{M}\,\underbrace{xx\cdots x\mathsf{1}}_{N}\,\mathsf{000}\cdots,$$

and in general for Π_k

$$\underbrace{xx\cdots x\mathsf{0}}_{M}\,\underbrace{xx\cdots x\mathsf{0}}_{N}\,\underbrace{xx\cdots x\mathsf{0}}_{N}\cdots\,\underbrace{xx\cdots x\mathsf{0}}_{N}\,\underbrace{xx\cdots x\mathsf{1}}_{N}\,\mathsf{000}\cdots,$$

where there are k blocks of length N. In the the locations x, arbitrary letters are allowed. The number of elements in Π_k is determined by the number of locations where the letter may be freely chosen, so Π_k has $2^{M-1+k(N-1)} = 2^{N_k-k-1}$ elements.

Let $\Pi = \bigcup_k \Pi_k$. We claim that Π is a packing. Let $(\sigma, r), (\sigma', r') \in \Pi$. We must show $\varrho(\sigma, \sigma') > r + r'$. First suppose that $(\sigma, r), (\sigma', r')$ are in the same Π_k. So $r = r' = 2^{-N_k+1}\eta$. Strings σ, σ' differ somewhere in the first $N_k - 1$ places, so their longest common prefix has length at most $N_k - 2$, and $\varrho(\sigma, \sigma') \geq 2^{-N_k+2}$. On the other hand, $r + r' = 2^{-N_k+1}\eta + 2^{-N_k+1}\eta = 2^{-N_k+2}\eta < 2^{-N_k+2}$, as required. Now suppose $(\sigma, r) \in \Pi_k$, $(\sigma', r') \in \Pi_{k'}$, $k' > k$. Strings σ, σ' differ in location N_k, so their longest common prefix has length at most $N_k - 1$, so $\varrho(\sigma, \sigma') \geq 2^{-N_k+1}$. And $r + r' = 2^{-N_k+1}\eta + 2^{-N_{k'}+1}\eta \leq 2^{-N_k+1}(1 + 2^{-N})\eta < 2^{-N_k+1}$ as required.

Note that the packing Π is δ-fine, since for any k, the radius $2^{-N_k+1}\eta \leq 2^{-M+1}\eta < 2^{-M+1} \leq \delta$. Now compute

$$\begin{aligned}
\widetilde{\mathcal{P}}^1_\delta(E^{(\omega)}) &\geq \sum_{(\sigma,r)\in\Pi} (2r) = \sum_{k=0}^{\infty} \sum_{(\sigma,r)\in\Pi_k} (2r) \\
&= \sum_{k=0}^{\infty} 2^{N_k-k-1} 2\eta 2^{-N_k+1} = 2\eta \sum_{k=0}^{\infty} 2^{-k} = 4\eta.
\end{aligned}$$

This holds for all $\delta > 0$, so $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) \geq 4\eta$. Now let $N \to \infty$ so that $\eta \to 1$, to get $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) \geq 4$.

From (a) and (b), we have $\widetilde{\mathcal{P}}^1_0(E^{(\omega)}) = 4 = 4\mathcal{M}(E^{(\omega)})$.

(c) Next: if $[\alpha]$ is a cylinder, then $\widetilde{\mathcal{P}}^1_0\big([\alpha]\big) = 4 \times 2^{-|\alpha|}$. The right shift $\sigma \mapsto \alpha\sigma$ is a similarity with ratio $2^{-|\alpha|}$, so this follows from the case already proved. Then we have also $\mathcal{P}^1\big([\alpha]\big) \le 4\mathcal{M}\big([\alpha]\big)$. The method I outer measure $\overline{\mathcal{M}}$ is the largest outer measure such that $\overline{\mathcal{M}}\big([\alpha]\big) \le 2^{-|\alpha|}$ for all cylinders. So we conclude $\overline{\mathcal{P}}^1 \le 4\overline{\mathcal{M}}$.

The clopen sets in $E^{(\omega)}$ are exactly the finite disjoint unions of cylinders $[\alpha]$. Two disjoint cylinders have positive distance. Therefore, by Lemma 6.2.6, for any clopen set V, we have $\widetilde{\mathcal{P}}^1_0(V) = 4\mathcal{M}(V)$.

(d) Now we claim that the class of clopen sets is a reduced cover class for $\overline{\mathcal{P}}^1$. Let F be a closed set and $\varepsilon > 0$. Now $\overline{\mathcal{M}}$ is a metric outer measure, so by Proposition 5.4.3, there is an open set $V \supseteq F$ with $\mathcal{M}(U \setminus F) < \varepsilon/4$; while V is a union of cylinders, so by compactness we may replace it by a finite union. Then applying Lemma 6.2.6 we have

$$\widetilde{\mathcal{P}}^1_0(V) \le \widetilde{\mathcal{P}}^1_0(F) + \widetilde{\mathcal{P}}^1_0(V \setminus F) \le \widetilde{\mathcal{P}}^1_0(F) + 4\mathcal{M}(V \setminus F) \le \widetilde{\mathcal{P}}^1_0(F) + \varepsilon.$$

So the collection of clopen sets is a reduced cover class. This means, in the application of method I to define $\overline{\mathcal{P}}^1$, we may use only covers by clopen sets. But we have $\widetilde{\mathcal{P}}^1_0(V) = 4\mathcal{M}(V)$ for all clopen sets V, and $\overline{\mathcal{P}}^1$ is the largest outer measure such that $\overline{\mathcal{P}}^1(V) \le \widetilde{\mathcal{P}}^1_0(V)$ for all clopen V. So $4\overline{\mathcal{M}} \le \overline{\mathcal{P}}^1$.

Thus we get $\overline{\mathcal{P}}^1 = 4\overline{\mathcal{M}}$. Then in particular, $0 < \mathcal{P}^1\big(E^{(\omega)}\big) < \infty$, so $\operatorname{Dim} E^{(\omega)} = 1$. □

The Line

Next is our first official example of a non-fractal. We proved $\operatorname{Cov}\mathbb{R} = 1$ in Theorem 3.2.15.

Proposition 6.3.3. *The Hausdorff dimension of the line* $\mathbb{R}$ *is* 1.

Proof. By Theorem 6.1.4, we have $\mathcal{H}^1\big([0,1]\big) = \mathcal{L}\big([0,1]\big) = 1$. Therefore $\dim[0,1] = 1$. Now $[0,1] \subseteq \mathbb{R}$, so $\dim\mathbb{R} \ge \dim[0,1] = 1$. If $s > 1$, then $\mathcal{H}^s([0,1]) = 0$. The intervals $[n, n+1]$ are isometric to $[0,1]$, so it follows that $\mathcal{H}^s([n,n+1]) = 0$. Therefore

$$\mathcal{H}^s(\mathbb{R}) \le \sum_{n=-\infty}^{\infty} \mathcal{H}^s([n,n+1]) = 0.$$

This means that $\dim\mathbb{R} \le s$. But this is true for any $s > 1$, so $\dim\mathbb{R} \le 1$. Therefore we have seen that $\dim\mathbb{R} = 1$. □

Exercise 6.3.4. The packing dimension of the line $\mathbb{R}$ is 1.

Lebesgue Measure vs. Hausdorff Measure

Since the Lebesgue measure was useful in computing $\dim\mathbb{R}$, it is easy to guess that $\mathcal{L}^2$ is useful in computing $\dim\mathbb{R}^2$.

Proposition 6.3.5. *The Hausdorff dimension of two-dimensional Euclidean space* $\mathbb{R}^2$ *is* 2.

Proof. Consider the (half-open) unit square $Q = [0,1)\times[0,1)$. It is covered by n^2 small squares with side $1/n$, so if $\varepsilon \geq \sqrt{2}/n$, we have $\overline{\mathcal{H}}^2_\varepsilon(Q) \leq n^2(\sqrt{2}/n)^2 = 2$. Therefore $\mathcal{H}^2(Q) \leq 2$, so $\dim Q \leq 2$.

On the other hand, if $\mathcal{A}$ is any cover of Q by closed sets, then (since any set A of diameter r is contained in a closed square Q_A with side $\leq r$),

$$\begin{aligned}\sum_{A\in\mathcal{A}} (\operatorname{diam} A)^2 &\geq \sum_{A\in\mathcal{A}} \mathcal{L}^2(Q_A)\\ &\geq \mathcal{L}^2\left(\bigcup_{A\in\mathcal{A}} Q_A\right)\\ &\geq \mathcal{L}^2(Q) = 1.\end{aligned}$$

Therefore $\mathcal{H}^2(Q) \geq 1$, so $\dim Q \geq 2$.

For $\mathbb{R}^2$, since $Q \subseteq \mathbb{R}^2$, we have $\dim \mathbb{R}^2 \geq \dim Q = 2$. If $s > 2$, then $\mathcal{H}^s(Q) = 0$; but $\mathbb{R}^2$ can be covered by a countable collection $\{\, Q_n : n \in \mathbb{N}\,\}$ of squares of side 1, so $\mathcal{H}^s(\mathbb{R}^2) \leq \sum_n \mathcal{H}^s(Q_n) = 0$. This shows that $\dim \mathbb{R}^2 \leq s$. Therefore $\dim \mathbb{R}^2 \leq 2$. □

Note that the proof showed $0 < \mathcal{H}^2(Q) < \infty$, where Q is the unit square.

What is the relation between the two measures $\mathcal{L}^2$ and $\mathcal{H}^2$ on $\mathbb{R}^2$? In fact, one of them is just a constant multiple of the other.

Theorem 6.3.6. *There is a positive constant c such that*

$$\mathcal{H}^2(B) = c\,\mathcal{L}^2(B)$$

for all Borel sets $B \subseteq \mathbb{R}^2$.

Proof. Let $Q = [0,1)\times[0,1)$ be the unit square. Let $c = \mathcal{H}^2(Q)$. (We have seen that $1 \leq c \leq 2$.) First, if $B = rQ = [0,r)\times[0,r)$, then $\mathcal{H}^2(B) = r^2\,\mathcal{H}^2(Q) = r^2 c = c\mathcal{L}^2(B)$. Next, the same is true for a translate of such a square.

Both measures are metric measures, and these squares are Borel sets. So we have $\mathcal{H}^2(V) = c\mathcal{L}^2(V)$ for any finite disjoint union of squares. In particular, this holds for V belonging to the dyadic ring $\mathcal{R}$ (see p. 155).

Now consider $\mathcal{H}^2$ and $\mathcal{L}^2$ restricted to subsets of a large square $K = [-N,N]\times[-N,N]$. The dyadic ring $\mathcal{R}$ is a reduced measure class for $\mathcal{L}^2$ on K. And $\mathcal{H}^2(V) = c\mathcal{L}^2(V)$ it follows that $\mathcal{R}$ is also a reduced measure class for $\mathcal{H}^2$ on K. Since $\mathcal{H}^2$ and $\mathcal{L}^2$ agree on $\mathcal{R}$, their method I extensions also agree. Thus $\mathcal{L}^2$ and $\mathcal{H}^2$ agree on K.

By countable additivity, $\mathcal{H}^2$ and $\mathcal{L}^2$ agree on the whole plane $\mathbb{R}^2$. □

Let d be a positive integer. The same method may be used to prove that there exists a positive constant c_d such that $\mathcal{H}^d(B) = c_d\,\mathcal{L}^d(B)$ for all Borel sets $B \subseteq \mathbb{R}^d$.

Exercise 6.3.7. If $B \subseteq \mathbb{R}^d$, then $\dim B \leq d$. If B contains an open ball, then $\dim B = d$.

Arc Length

Let $f\colon [0,1] \to S$ be a continuous curve in S. The ***arc length*** of the curve is

$$\sup \sum_{i=1}^{n} \varrho\big(f(x_{i-1}), f(x_i)\big),$$

where the supremum is over all finite subdivisions

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

of the interval $[0,1]$. If the arc length is finite, then we say that the curve is ***rectifiable***.

Theorem 6.3.8. *Let $f\colon [0,1] \to S$ be a continuous curve, let l be its arc length, and write $C = f\big[[0,1]\big]$.*

(a) *$l \geq \mathcal{H}^1(C)$;*
(b) *If f is one-to-one, then $l = \mathcal{H}^1(C)$.*

Proof. (a) Let $\varepsilon > 0$. Now f is uniformly continuous (Theorem 2.3.21), so there is $\delta > 0$ such that $\varrho\big(f(x), f(y)\big) < \varepsilon$ whenever $|x - y| < \delta$. Choose a subdivision

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

of $[0,1]$ with $|x_i - x_{i-1}| < \delta$ for all i. Then the sets

$$A_i = f\Big[[x_{i-1}, x_i]\Big]$$

cover C. (But $\operatorname{diam} A_i$ may not be $\varrho\big(f(x_{i-1}), f(x_i)\big)$.) By the compactness of $[x_{i-1}, x_i]$, there exist y_i, z_i with $x_{i-1} \leq y_i < z_i \leq x_i$ such that $\operatorname{diam} A_i = \varrho\big(f(y_i), f(z_i)\big)$. Now we may use the subdivision

$$0 \leq y_1 \leq z_1 \leq y_2 \leq z_2 \leq \cdots \leq y_n \leq z_n \leq 1$$

to estimate the length. So

$$l \geq \sum_{i=1}^{n} \varrho\big(f(y_i), f(z_i)\big) = \sum_{i=1}^{n} \operatorname{diam} A_i \geq \overline{\mathcal{H}}^1_\varepsilon(C).$$

Now let $\varepsilon \to 0$ to obtain $l \geq \mathcal{H}^1(C)$.

(b) First, I claim that if $0 \leq a < b \leq 1$, then $\mathcal{H}^1\big(f\big[[a,b]\big]\big) \geq \varrho\big(f(a), f(b)\big)$. To see this, consider the function $h\colon f\big[[a,b]\big] \to \mathbb{R}$ defined by $h(u) = \varrho\big(f(a), u\big)$. Now h is continuous, and h has values $h(a) = 0$ and $h(b) = \varrho\big(f(a), f(b)\big)$, so by the intermediate value theorem, applied to the continuous function $h \circ f\colon [a,b] \to \mathbb{R}$, we know that h also has all values between. Now h satisfies the Lipschitz condition $|h(u) - h(v)| \leq \varrho(u,v)$, and we have

$$\begin{aligned}\mathcal{H}^1\Big(f\big[[a,b]\big]\Big) &\geq \mathcal{H}^1\Big(h\big[f\big[[a,b]\big]\big]\Big) \\ &\geq \mathcal{H}^1\Big(\big[0, \varrho\big(f(a), f(b)\big)\big]\Big) \\ &= \varrho\big(f(a), f(b)\big).\end{aligned}$$

This proves the claim.

Now we apply this inequality. If we have a subdivision

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

of $[0,1]$, then the set $f\big[[x_{i-1}, x_i)\big] = f\big[[x_{i-1}, x_i]\big] \setminus \big\{f(x_i)\big\}$ is the difference of two compact sets, hence measurable. The sets $f\big[[x_{i-1}, x_i)\big]$ are disjoint, since f is one-to-one. So

$$\begin{aligned}\sum_{i=1}^{n} \varrho\big(f(x_{i-1}), f(x_i)\big) &\leq \sum_{i=1}^{n} \mathcal{H}^1\Big(f\big[[x_{i-1}, x_i)\big]\Big) \\ &= \mathcal{H}^1\left(\bigcup_{i=1}^{n} f\big[[x_{i-1}, x_i)\big]\right) \\ &= \mathcal{H}^1\Big(f\big[[0,1)\big]\Big) \leq \mathcal{H}^1(C).\end{aligned}$$

This is true for all subdivisions, so $l \leq \mathcal{H}^1(C)$. □

Exercise 6.3.9. What is the relation between the surface area (of a surface in $\mathbb{R}^3$) and its two-dimensional Hausdorff measure?

Fractal Dimension vs. Topological Dimension

We will see in the next section that it is possible for the Hausdorff dimension $\dim F$ to have a non-integer value. But it is not completely unrelated to the topological dimension.

Theorem 6.3.10. *Let S be a metric space. Then* $\operatorname{Cov} S \leq \dim S$*.*

A complete proof of this result can be found, for example, in [18, Sect. 3.1]. (The proof uses Lebesgue integration, which we have avoided in this book.) Here, we will prove it for compact spaces:

Theorem 6.3.11. *Let S be a compact metric space. Then* $\operatorname{Cov} S \leq \dim S$*.*

Proof. Let $n = \operatorname{Cov} S$. This means that $\operatorname{Cov} S \leq n-1$ is false. So there exist open sets $U_1, U_2, \cdots, U_{n+1}$ such that $\bigcup_{i=1}^{n+1} U_i = S$, but for any closed sets $F_i \subseteq U_i$ with $\bigcup_{i=1}^{n+1} F_i = S$, we must have $\bigcap_{i=1}^{n+1} F_i \neq \varnothing$.

Define functions on S as follows:

$$d_i(x) = \operatorname{dist}(x, S \setminus U_i), \qquad 1 \le i \le n+1$$
$$d(x) = d_1(x) + d_2(x) + \cdots + d_{n+1}(x).$$

The functions are continuous—in fact, Lipschitz:

$$|d_i(x) - d_i(y)| \le \varrho(x,y),$$
$$|d(x) - d(y)| \le (n+1)\varrho(x,y).$$

Since the sets U_i cover S, we have $d(x) > 0$ for all x. So since S is compact, there exist positive constants a, b such that $a \le d(x) \le b$ for all $x \in S$. Now define $h\colon S \to \mathbb{R}^{n+1}$ by

$$h(x) = \left(\frac{d_1(x)}{d(x)}, \frac{d_2(x)}{d(x)}, \cdots, \frac{d_{n+1}(x)}{d(x)} \right).$$

The function h is Lipschitz:

$$\begin{aligned} \left| \frac{d_i(x)}{d(x)} - \frac{d_i(y)}{d(y)} \right| &= \frac{|d(x)d_i(y) - d(y)d_i(x)|}{d(x)d(y)} \\ &\le \frac{d(x)\,|d_i(y) - d_i(x)| + d_i(x)\,|d(x) - d(y)|}{d(x)d(y)} \\ &\le \frac{b(n+2)}{a^2}\,\varrho(x,y), \end{aligned}$$

and therefore

$$|h(x) - h(y)| \le \frac{b(n+1)(n+2)}{a^2}\,\varrho(x,y).$$

Now I claim that $h[S]$ includes the simplex

$$T = \left\{ (t_1, t_2, \cdots, t_{n+1}) \in \mathbb{R}^{n+1} : t_i > 0, \sum_{i=1}^{n+1} t_i = 1 \right\}.$$

Given $(t_1, t_2, \cdots, t_{n+1}) \in T$, consider the sets

$$F_i = \left\{ x : \frac{d_i(x)}{d(x)} \ge t_i \right\}.$$

Then F_i is closed, $F_i \subseteq U_i$, and $\bigcup_{i=1}^{n+1} F_i = S$ since $\sum_i d_i(x)/d(x) = 1$. So we know by hypothesis that $\bigcap_{i=1}^{n+1} F_i \ne \varnothing$. That is, there exists a point $x \in S$ with $d_i(x)/d(x) \ge t_i$ for all i. But since $\sum_i d_i(x)/d(x) = 1$ we have $d_i(x)/d(x) = t_i$ for all i. That is, $h(x) = (t_1, t_2, \cdots t_{n+1})$. So $h[S] \supseteq T$.

Now T is isometric to an open set in $\mathbb{R}^n$. By Theorem 6.1.7 and Exercise 6.1.10, we have $\dim S \ge \dim T = n$. □

Exercise 6.3.12. Let S be a metric space. If $\operatorname{Cov} S \geq 1$, then $\dim S \geq 1$.

6.4 Self-Similarity

Self-similarity is one of the easiest ways to produce examples of fractals. This section deals with the question of when the similarity dimension can be used to compute the Hausdorff dimension. When the two coincide we have a desirable situation: the similarity dimension is easy to compute, and the Hausdorff dimension is more generally applicable and has many useful properties.

Let $(r_1, r_2, \cdots, r_n)$ be a contracting ratio list. Let $(f_1, f_2, \cdots, f_n)$ be an iterated function system of similarities realizing the ratio list in a complete metric space S. Let s be the sim-value for the iterated function system. Let K is the invariant set for the iterated function system. Of course, K is a measurable set, since it is compact. Does it follow that $\dim K = s$? In general, the answer is no. There is always an inequality $\dim K \leq s$. But simple examples show that if there is "too much" overlap among the pieces $f_i[K]$, then $\dim K < s$ is possible.

String Models

Hausdorff and packing measures are often easy to compute for the string models we use. Or if not easy to compute exactly, easy to estimate. Often estimates are good enough, since to compute the fractal dimensions dim and Dim it is enough to know merely whether $\mathcal{H}^s$ or $\mathcal{P}^s$ is positive or finite.

When computing $\mathcal{H}^s$ or $\mathcal{P}^s$ in our string spaces, we often already have a candidate measure $\mathcal{M}$. This helps in the computation.

Lemma 6.4.1. *Let E be a finite alphabet, let $E^{(\omega)}$ be the space of all infinite strings constructed from E. Let $s > 0$ and let $\overline{\mathcal{M}}$ be a finite metric outer measure on $E^{(\omega)}$.* (i) *If $\widetilde{\mathcal{P}}^s_0\big([\alpha]\big) = \mathcal{M}\big([\alpha]\big)$ for all $\alpha \in E^{(*)}$, then $\overline{\mathcal{P}}^s = \overline{\mathcal{M}}$.* (ii) *If $\widetilde{\mathcal{P}}^s_0\big([\alpha]\big) \leq \mathcal{M}\big([\alpha]\big)$ for all $\alpha \in E^{(*)}$, then $\overline{\mathcal{P}}^s \leq \overline{\mathcal{M}}$.*

Proof. The balls in $E^{(\omega)}$ are the cylinders $[\alpha]$. The clopen sets in $E^{(\omega)}$ are the finite disjoint unions of cylinders. Write $\mathcal{R}$ for the class of clopen sets in $E^{(\omega)}$. Note that $\mathcal{R}$ is an algebra of sets. By Lemma 6.2.6, we have $\widetilde{\mathcal{P}}^s_0(V) = \mathcal{M}(V)$ for all $V \in \mathcal{R}$ in case (i) and $\widetilde{\mathcal{P}}^s_0(V) \leq \mathcal{M}(V)$ in case (ii).

Let $F \subseteq E^{(\omega)}$ be closed, and let $\varepsilon > 0$. Then there is an open set $U \supseteq F$ with $\mathcal{M}(U \setminus F) < \varepsilon$. The open set U is a union of cylinders, so by compactness of F there is a finite union V of cylinders with $F \subseteq V \subseteq U$. (Alternatively, think of this as the fact that $E^{(\omega)}$ is zero-dimensional.) We conclude (as in the proof of Proposition 6.3.2) that $\widetilde{\mathcal{P}}^s_0(V) \leq \widetilde{\mathcal{P}}^s_0(F) + \varepsilon$. So $\mathcal{R}$ is a reduced cover class for $\mathcal{P}^s$. For any set $A \subseteq E^{(\omega)}$ we have

$$\begin{aligned}\overline{\mathcal{P}}^s(A) &= \sup\left\{ \sum_n \widetilde{\mathcal{P}}_0^s(F_n) :\ A \subseteq \bigcup_n F_n,\ F_n \text{ closed} \right\} \\ &= \sup\left\{ \sum_n \widetilde{\mathcal{P}}_0^s(V_n) :\ A \subseteq \bigcup_n V_n,\ V_n \in \mathcal{R} \right\} \\ &= \sup\left\{ \sum_n \mathcal{M}(V_n) :\ A \subseteq \bigcup_n V_n,\ V_n \in \mathcal{R} \right\} = \overline{\mathcal{M}}(A)\end{aligned}$$

in case (i) and inequality $\le$ in case (ii). □

Exercise 6.4.2. Let E be a finite alphabet, let $E^{(\omega)}$ be the space of all infinite strings constructed from E. Let $s > 0$ and let $\overline{\mathcal{M}}$ be a finite metric outer measure on $E^{(\omega)}$. If $\overline{\mathcal{H}}^s\big([\alpha]\big) = \mathcal{M}\big([\alpha]\big)$ for all $\alpha \in E^{(*)}$, then $\overline{\mathcal{H}}^s = \overline{\mathcal{M}}$. If $\overline{\mathcal{H}}^s\big([\alpha]\big) \le \mathcal{M}\big([\alpha]\big)$ for all $\alpha \in E^{(*)}$, then $\overline{\mathcal{H}}^s \le \overline{\mathcal{M}}$.

The Natural Measure

Begin with a contracting ratio list $(r_1, r_2, \cdots, r_n)$, with $n > 1$. Then the sim-value s associated with it is the unique positive number s satisfying

$$\sum_{i=1}^n r_i^s = 1.$$

Let E be an n-letter alphabet, and let $E^{(\omega)}$ be the string model. The metric ϱ on E is defined so that the right shifts realize the given ratio list. We define $r(\alpha)$ recursively, starting with the empty string Λ, by:

$$\begin{aligned} r(\Lambda) &= 1, \\ r(\alpha e) &= r(\alpha)\, r_e, \end{aligned}$$

then define ϱ so that $\mathrm{diam}[\alpha] = r(\alpha)$.

We will also need a measure defined to fit the ratio list. The basis is the equation defining the sim-value s:

$$\sum_{i=1}^n r_i^s = 1.$$

It follows from this that

$$\sum_{i=1}^n \big(r(\alpha) r_i\big)^s = r(\alpha)^s.$$

That is, the expression $r(\alpha)^s$ satisfies the additivity condition for a metric outer measure (Theorem 5.5.4). The measure $\mathcal{M}$ in question is defined on the string space $E^{(\omega)}$, and satisfies $\mathcal{M}([\alpha]) = r(\alpha)^s$ for all α. Of course, it is no coincidence that s was chosen so that $\mathcal{M}([\alpha]) = (\mathrm{diam}[\alpha])^s$.

Theorem 6.4.3. *Let $E^{(\omega)}$ have metric ϱ and measure $\mathcal{M}$ defined from ratio list (r_e) with sim-value s. Then* (a) $\mathcal{H}^s = \mathcal{M}$, (b) *there is a constant $c > 0$ so that $\mathcal{P}^s = c\mathcal{M}$, and thus* (c) $\dim E^{(\omega)} = \operatorname{Dim} E^{(\omega)} = s$.

Proof. Write $r_{\max} = \max_e r_e$ and $r_{\min} = \min_e r_e$.

(a) If a set $A \subseteq E^{(\omega)}$ has positive diameter, then (Proposition 2.6.7) there is a string $\alpha \in E^{(*)}$ with $A \subseteq [\alpha]$ and $\operatorname{diam} A = \operatorname{diam}[\alpha]$. So $\mathcal{M}(A) \leq \mathcal{M}([\alpha]) = (\operatorname{diam}[\alpha])^s = (\operatorname{diam} A)^s$. But $\overline{\mathcal{H}}^s_\varepsilon$ is the largest outer measure with $\overline{\mathcal{H}}^s_\varepsilon(A) \leq (\operatorname{diam} A)^s$ for all sets A of diameter $\leq \varepsilon$. So $\overline{\mathcal{M}} \leq \overline{\mathcal{H}}^s_\varepsilon$. This is true for all $\varepsilon > 0$, so $\overline{\mathcal{M}} \leq \overline{\mathcal{H}}^s$.

On the other hand, let $\alpha \in E^{(*)}$ be a finite string, and $\varepsilon > 0$. There is n so large that $r^n_{\max} < \varepsilon$, $n \geq |\alpha|$, and so $r(\beta) < \varepsilon$ for all $\beta \in E^{(n)}$. The cylinder $[\alpha]$ is the disjoint union of all sets $[\beta]$, where $\beta \geq \alpha$ and $|\beta| = n$. Then

$$\overline{\mathcal{H}}^s_\varepsilon\big([\alpha]\big) \leq \sum_{\substack{\beta \geq \alpha \\ |\beta| = n}} \big(\operatorname{diam}[\beta]\big)^s = \sum_{\substack{\beta \geq \alpha \\ |\beta| = n}} \mathcal{M}\big([\beta]\big) = \mathcal{M}\big([\alpha]\big).$$

Let $\varepsilon \to 0$ to get $\overline{\mathcal{H}}^s\big([\alpha]\big) \leq \mathcal{M}\big([\alpha]\big)$. Therefore $\overline{\mathcal{H}}^s \leq \overline{\mathcal{M}}$.

(b) Now consider the packing measure. We will show that $c = \widetilde{\mathcal{P}}^s_0\big(E^{(\omega)}\big)$ satisfies the condition. Note that

$$\widetilde{\mathcal{P}}^s_0\big(E^{(\omega)}\big) \geq \mathcal{P}^s\big(E^{(\omega)}\big) > 0$$

by part (a) and Proposition 6.2.14.

We will describe the balls in $E^{(\omega)}$. Let $\sigma \in E^{(\omega)}$ and $0 < t < 1$. Consider the ball $\overline{B}_t(\sigma)$. The ratios $r(\sigma{\restriction}n)$ go to 0 as $n \to \infty$, and $r(\Lambda) = 1$. So there is a unique n with $r(\sigma{\restriction}n) \leq t < r(\sigma{\restriction}(n-1))$. Then as in the proof of Proposition 6.3.2 we have $\overline{B}_t(\sigma) = [\alpha]$ where $\alpha = \sigma{\restriction}n$. And $\mathcal{M}\big(\overline{B}_t(\sigma)\big) = \mathcal{M}\big([\alpha]\big) = r(\alpha)^s$. Estimate from above: $\mathcal{M}\big(\overline{B}_t(\sigma)\big) = r(\alpha)^s \leq t^s$. Estimate from below: $\mathcal{M}\big(\overline{B}_t(\sigma)\big) = r(\sigma{\restriction}n)^s \geq \big(r_{\min}\, r(\sigma{\restriction}(n-1))\big)^s > (r_{\min}/2)^s\,(2t)^s$.

Now let Π be a packing of $E^{(\omega)}$. The corresponding balls $\overline{B}_t(\sigma), (\sigma,t) \in \Pi$, are disjoint. So

$$\sum_{(\sigma,t)\in\Pi} (2t)^s < \frac{2^s}{r^s_{\min}} \sum_{(\sigma,t)\in\Pi} \mathcal{M}\big(\overline{B}_t(\sigma)\big) \leq \frac{2^s}{r^s_{\min}}.$$

This holds for all δ-fine packings, so $\widetilde{\mathcal{P}}^s_\delta(E^{(\omega)}) \leq 2^s/r^s_{\min}$. This holds for all $\delta > 0$, so $\widetilde{\mathcal{P}}^s_0(E^{(\omega)}) \leq 2^s/r^s_{\min} < \infty$.

Thus $c = \widetilde{\mathcal{P}}^s_0(E^{(\omega)})$ is positive and finite. Now if $\alpha \in E^{(*)}$, then the right shift $\sigma \mapsto \alpha\sigma$ is a similarity with ratio $r(\alpha)$, and it maps $E^{(\omega)}$ onto $[\alpha]$. Therefore $\widetilde{\mathcal{P}}^s_0\big([\alpha]\big) = r(\alpha)^s\, c = c\,\mathcal{M}\big([\alpha]\big)$. And thus, by Lemma 6.4.1(i), we have $\mathcal{P}^s = c\,\mathcal{M}$. ☐

Exercise 6.4.4. Let s be any positive real number. There is a metric space S with $\dim S = s$.

Exercise 6.4.5. Prove versions of Lemma 6.4.1, Exercise 6.4.2, and Theorem 6.4.3 for the path spaces $E_v^{(\omega)}$ defined by a directed multigraph (V, E, i, t).

Exercise 6.4.6. Compute the value of the constant c in Theorem 6.4.3(b).

Cantor Dust

Let us consider the fractal dimension of the triadic Cantor dust (defined on p. 2). The ratio list for this set is $(1/3, 1/3)$. The string model is the set $E^{(\omega)}$ of infinite strings from the alphabet $E = \{0, 1\}$, together with the metric $\varrho_{1/3}$. The two similarities on the model space are the right shifts, say θ_0 and θ_1, defined as follows:

$$\theta_0(\sigma) = 0\sigma$$
$$\theta_1(\sigma) = 1\sigma.$$

Thus (θ_0, θ_1) is a realization of the ratio list $(1/3, 1/3)$, with invariant set $E^{(\omega)}$.

Proposition 6.4.7. *The Hausdorff dimension and packing dimension for* $E^{(\omega)}$ *with metric* $\varrho_{1/3}$ *are both* $\log 2/\log 3$.

Proof. For the sim-value, solve $2\,(1/3)^s = 1$ for s to get $s = \log 2/\log 3$. By Theorem 6.4.3 we have $\dim = \operatorname{Dim} = s$. □

Corollary 6.4.8. *The Cantor dust has Hausdorff dimension and packing dimension* $\log 2/\log 3$.

Proof. A lipeomorphism preserves the Hausdorff dimension (Exercise 6.1.10) and the packing dimension (Exercise 6.2.12). The addressing function h from $E^{(\omega)}$ onto the triadic Cantor dust C is a lipeomorphism (Proposition 2.6.3). □

Sierpiński gasket

Next we discuss a slightly more difficult example, the Sierpiński gasket (see p. 8).

Let S be the Sierpiński gasket. It is the invariant set for an iterated function system with ratio list $(1/2, 1/2, 1/2)$. Let $s\ \ [= \log 3/\log 2]$ be the sim-value of the ratio list. Let $E = \{\mathsf{L}, \mathsf{U}, \mathsf{R}\}$ be the appropriate three-letter alphabet. Next we describe the natural metric and measure defined on $E^{(\omega)}$ from the ratio list.

Let ϱ be the metric on $E^{(\omega)}$ for the ratio list $(1/2, 1/2, 1/2)$. That is, ϱ is defined so that $\operatorname{diam}[\alpha] = 2^{-|\alpha|}$ for all $\alpha \in E^{(*)}$. Then the right shifts realize the ratio list:

$$\varrho(\mathsf{L}\sigma, \mathsf{L}\tau) = \frac{1}{2}\,\varrho(\sigma, \tau),$$
$$\varrho(\mathsf{U}\sigma, \mathsf{U}\tau) = \frac{1}{2}\,\varrho(\sigma, \tau),$$
$$\varrho(\mathsf{R}\sigma, \mathsf{R}\tau) = \frac{1}{2}\,\varrho(\sigma, \tau).$$

The measure $\mathcal{M}$ is specified by $\mathcal{M}([\alpha]) = 3^{-|\alpha|}$. Each node in $E^{(*)}$ has exactly 3 children, so these numbers satisfy the required additivity (Theorem 5.5.4). The fact to notice is this:

$$\mathcal{M}([\alpha]) = \big(\operatorname{diam}[\alpha]\big)^s$$

for all $\alpha \in E^{(*)}$, where $s = \log 3/\log 2$. By Theorem 6.4.3, the Hausdorff dimension of the string space $E^{(\omega)}$ is $s = \log 3/\log 2$.

The dimension calculation for the string model will be used to help with the dimension calculation of the Sierpiński gasket S itself.

Let $h\colon E^{(\omega)} \to \mathbb{R}^2$ be the addressing function that sends $E^{(\omega)}$ onto the gasket S. If the iterated function system in $\mathbb{R}^2$ is $(f_{\mathsf{L}}, f_{\mathsf{U}}, f_{\mathsf{R}})$, then

$$\begin{aligned} h(\mathsf{L}\sigma) &= f_{\mathsf{L}}\big(h(\sigma)\big),\\ h(\mathsf{U}\sigma) &= f_{\mathsf{U}}\big(h(\sigma)\big),\\ h(\mathsf{R}\sigma) &= f_{\mathsf{R}}\big(h(\sigma)\big). \end{aligned}$$

Proposition 6.4.9. *The Sierpiński gasket has Hausdorff dimension and packing dimension at most* $\log 3/\log 2$.

Proof. The addressing function h is Lipschitz (Exercise 4.2.1). By Exercise 6.1.10, we have $\dim S \le \operatorname{Dim} S \le \operatorname{Dim} E^{(\omega)} = \log 3/\log 2$. □

For the general iterated function system, the upper bound is proved in the same way.

Theorem 6.4.10. *Let $(r_e)_{e\in E}$ be a contracting ratio list. Let s be its sim-value, and let $(f_e)_{e\in E}$ be a realization in a complete metric space S. Let K be the invariant set. Then* $\dim K \le \operatorname{Dim} K \le s$.

Proof. The string model $E^{(\omega)}$ with the natural metric ϱ has $\operatorname{Dim} E^{(\omega)} = s$ (Theorem 6.4.3). The addressing function $h\colon E^{(\omega)} \to K$ is Lipschitz. Therefore $\operatorname{Dim} K \le s$. □

Lower Bound

The addressing function for the Sierpiński gasket is not inverse Lipschitz. In fact, it is not even one-to-one. (This is the answer to Exercise 4.2.2.) So we will need a bit more effort to prove the lower bound for the fractal dimension of S. Pay attention to the ingredients of the proof, since they will be used again for the general case. To simplify the notation, we will write $\mathsf{L}(x)$ in place of $f_{\mathsf{L}}(x)$, and similarly for the other two letters, and write $\alpha(x)$ for a finite string α.

Proposition 6.4.11. *The Sierpiński gasket S has Hausdorff dimension equal to the similarity dimension* $\log 3/\log 2$.

Proof. Let V be the interior of the first triangle S_0 approximating the Sierpiński gasket S. Then $\mathcal{L}^2(V) = \sqrt{3}/4$, and if $|\alpha| = |\beta|$, $\alpha \neq \beta$, then $\alpha[V] \cap \beta[V] = \varnothing$. Also, $h\big[[\alpha]\big] = \overline{\alpha[V]} \cap S$. The set S_k approximating S is the union

$$\bigcup_{\alpha \in E^{(k)}} \overline{\alpha[V]}.$$

Given a set $A \subseteq S$, let k be the positive integer satisfying

$$2^{-k} < \operatorname{diam} A \leq 2^{-k+1}.$$

Let

$$T = \Big\{ \alpha \in E^{(k)} : \overline{\alpha[V]} \cap A \neq \varnothing \Big\}.$$

I claim that T has at most 100 elements. Let m be the number of elements in T. A set $\alpha[V]$ is the image of V under a similarity with ratio 2^{-k}, so it has area

$$\mathcal{L}^2\big(\alpha[V]\big) = 4^{-k}\frac{\sqrt{3}}{4}.$$

The sets $\alpha[V]$ with $\alpha \in T$ are all disjoint. If x is a point of A, then all of the elements of all of the sets $\alpha[V]$ with $\alpha \in T$ are within distance $\operatorname{diam} A + 2^{-k} \leq 3 \cdot 2^{-k}$ of x. So m disjoint sets of area $4^{-k}\sqrt{3}/4$ are contained in the ball with center x and radius $3 \cdot 2^{-k}$. Therefore

$$m 4^{-k}\frac{\sqrt{3}}{4} \leq \pi(3 \cdot 2^{-k})^2.$$

Solving for m, we get $m \leq 36\pi/\sqrt{3}$, which is smaller than 100.

Next I claim $\mathcal{M}\big(h^{-1}[A]\big) \leq 100\,(\operatorname{diam} A)^s$ for all Borel sets $A \subseteq S$. Given A, let k and T be as above. Then $A \subseteq \bigcup_{\alpha \in T} \overline{\alpha[V]}$, so $h^{-1}[A] \subseteq \bigcup_{\alpha \in T} [\alpha]$. Therefore

$$\begin{aligned}\mathcal{M}\big(h^{-1}[A]\big) &\leq \sum_{\alpha \in T} \mathcal{M}([\alpha]) \\ &\leq 100 \times 3^{-k} = 100\,(2^{-k})^s \\ &\leq 100\,(\operatorname{diam} A)^s.\end{aligned}$$

By the Method I theorem, $\mathcal{M}\big(h^{-1}[A]\big) \leq 100, \mathcal{H}^s(A)$ for all Borel sets A. So $1 \leq 100, \mathcal{H}^s(S)$, and therefore $\dim S \geq s$. □

Exercise 6.4.12. Improve the estimate 100.

6.5 The Open Set Condition

Let $(r_1, \cdots, r_n)$ be a contracting ratio list with dimension s. Let $(f_1, \cdots, f_n)$ be a corresponding iterated function system of similarities in $\mathbb{R}^d$. Suppose K is the invariant set for the iterated function system. Write s for the sim-value.

In general it is not true that $\dim K = s$. For example, consider the iterated function system$(f_{\mathsf{L}}, f_{\mathsf{U}}, f_{\mathsf{R}})$ for the Sierpiński gasket, realizing the ratio list $(1/2, 1/2, 1/2)$. Now the iterated function system$(f_{\mathsf{L}}, f_{\mathsf{L}}, f_{\mathsf{U}}, f_{\mathsf{R}})$ has the same invariant set, of course, but it realizes the longer ratio list $(1/2, 1/2, 1/2, 1/2)$. The Hausdorff dimension of the invariant set K is $\log 3/\log 2$, but the sim-value of the iterated function system is 2.

Of course, the problem is that the first two images $f_{\mathsf{L}}[K]$ and $f_{\mathsf{L}}[K]$ overlap too much. Now we might require that the images do not overlap at all, as in the Cantor dust. But that would rule out many of the most interesting examples, such as the Sierpiński gasket itself, where the overlap sets like $f_{\mathsf{L}}[K] \cap f_{\mathsf{U}}[K]$ are nonempty.

We do have inequality between the Hausdorff dimension, packing dimension, and similarity dimension. If s is the similarity dimension, then the string model has packing dimension s and the addressing function is Lipschitz, so $\dim K \le \operatorname{Dim} K \le s$.

Lower Bound

Now we turn to the "lower bound" proof. That is, we want to show $\dim K \ge$ something or $\operatorname{Dim} K \ge$ something. Generally we do this by showing $\mathcal{H}^s(K) > 0$ or $\mathcal{P}^s(K) > 0$.

The iterated function system $(f_1, f_2, \cdots, f_n)$ satisfies ***Moran's open set condition*** iff there exists a nonempty open set U for which we have $f_i[U] \cap f_j[U] = \varnothing$ for $i \ne j$ and $U \supseteq f_i[U]$ for all i. Such an open set U will be called a ***Moran open set*** for the iterated function system.

For example, consider the Cantor dust. The similarities are

$$f_0(x) = \frac{x}{3},$$
$$f_1(x) = \frac{x+2}{3}.$$

The open set $U = (0, 1)$ is a Moran open set: the two images are $(0, 1/3)$ and $(2/3, 1)$, which are disjoint and contained in U.

Or, consider the Sierpiński gasket (Fig. 1.2.1). The interior* U of the large triangle S_0 is a Moran open set. The three images are three small triangles, contained in U, and disjoint.

For a third example, consider the Koch curve (Fig. 1.5.1). The interior of the triangle L_0 is a Moran open set.

The fourth example to consider is Heighway's dragon. This time the open set condition is not quite as trivial. The interior U of Heighway's dragon itself will serve. The fact that the two images are contained in U is a consequence of the fact that Heighway's dragon itself is the invariant set of the iterated function system. The fact that the two images are disjoint is a consequence of the

* The interior of a set consists of all the interior points of the set.

fact that the approximating polygon never crosses itself (Proposition 1.5.7). The verification requires some work, but it is left to the reader:

Exercise 6.5.1. The interior of Heighway's dragon is a Moran open set for the iterated function system realizing Heighway's dragon.

In one case, the open set condition is easily verified:

Exercise 6.5.2. If the invariant set K for an iterated function system $\{f_i\}$ satisfies $f_i[K] \cap f_j[K] = \varnothing$ for $i \neq j$, then Moran's open set condition is satisfied.

The proof of the lower bound will proceed following the same technique as Proposition 6.4.11. The area is replaced by the d-dimensional volume, namely $\mathcal{L}^d$. You may find it instructive to compare this argument with the proof of the special case in Proposition 6.4.11.

Let E be an alphabet with n letters. Write the ratio list as $(r_e)_{e\in E}$ and the iterated function system as $(f_e)_{e\in E}$. To simplify the notation, we will write $e(x)$ in place of $f_e(x)$, and similarly $\alpha(x)$ for a finite string α. With this notation, the model map $h\colon E^{(\omega)} \to \mathbb{R}^d$ satisfies $h(e\alpha) = e\big(h(\alpha)\big)$ for $\alpha \in E^{(*)}$ and $e \in E$.

The open set condition implies that $\alpha[U] \cap \beta[U] = \varnothing$ for two strings $\alpha, \beta \in E^{(*)}$ unless one is an initial segment of the other. If α is a string with length $k \geq 1$, we will write α^- for the parent of α; that is: $\alpha^- = \alpha{\restriction}(k-1)$.

Lemma 6.5.3. *Let $(r_e)_{e\in E}$ be a contracting ratio list. Let s be its sim-value, and let $(f_e)_{e\in E}$ be a realization in $\mathbb{R}^d$. Let K be the invariant set. Let U be a Moran open set for (f_e). Then there is a constant $c > 0$ such that: if $A \subseteq K$, then the set*

$$T = \left\{ \alpha \in E^{(*)} : \overline{\alpha[U]} \cap A \neq \varnothing,\ \operatorname{diam} \alpha[U] < \operatorname{diam} A \leq \operatorname{diam} \alpha^-[U] \right\}$$

has at most c elements.

Proof. As α ranges over T, the sets $\alpha[U]$ are disjoint, since no such α is an initial segment of another. The map f_α is a similarity with ratio equal to $\operatorname{diam}[\alpha]$, so if w is the diameter of U, then $w \operatorname{diam}[\alpha]$ is the diameter of $\alpha[U]$. Write $r_{\min} = \min r_e$. Then

$$\begin{aligned} \operatorname{diam} \alpha[U] &= w \operatorname{diam}[\alpha] \geq w r_{\min} \operatorname{diam}[\alpha^-] \\ &= r_{\min} \operatorname{diam} \alpha^-[U] \geq r_{\min} \operatorname{diam} A. \end{aligned}$$

If $p = \mathcal{L}^d(U)$ is the volume of U, then the volume of $\alpha[U]$ is

$$\mathcal{L}^d\big(\alpha[U]\big) = p \left(\frac{\operatorname{diam} \alpha[U]}{\operatorname{diam} U} \right)^d \geq \frac{p r_{\min}^d}{w^d} (\operatorname{diam} A)^d$$

If x is a point of A, then every point of every set $\alpha[U]$ for $\alpha \in T$ is within distance $\operatorname{diam} A + \operatorname{diam} \alpha[U] \leq 2 \operatorname{diam} A$ of x. If m is the number of elements of T, then we have m disjoint sets $\alpha[U]$, all with volume at least $(pr_{\min}^d / w^d)(\operatorname{diam} A)^d$, contained within a ball of radius $2 \operatorname{diam} A$. So if $t = \mathcal{L}^d(B_1(0))$ is the volume of the unit ball, we have

$$\frac{mpr_{\min}^d}{w^d} (\operatorname{diam} A)^d \leq t(2 \operatorname{diam} A)^d.$$

Solving for m yields

$$m \leq \frac{tw^d 2^d}{pr_{\min}^d}.$$

Summary: We may use the constant $c = tw^d 2^d / pr_{\min}^d$, where $r = \min r_e$, t is the volume of the unit ball, p is the volume of the Moran open set U, and w is the diameter of U. □

Theorem 6.5.4. *Let $(r_e)_{e \in E}$ be a contracting ratio list. Let s be its sim-value, and let $(f_e)_{e \in E}$ be a realization in $\mathbb{R}^d$. Let K be the invariant set. If Moran's open set condition is satisfied, then* $\dim K = s$.

Proof. Let c be a constant as in the lemma. I claim there is a positive constant b so that for any Borel set $A \subseteq K$, we have

$$\mathcal{M}\big(h^{-1}[A]\big) \leq b\, (\operatorname{diam} A)^s.$$

Let U be a Moran open set, and let $w = \operatorname{diam} U$. Given A, let T be as in the lemma. So $A \subseteq \overline{\bigcup_{\alpha \in T} \alpha[U]}$, and $h^{-1}[A] \subseteq \bigcup_{\alpha \in T} [\alpha]$. If $\alpha \in T$, then $\mathcal{M}([\alpha]) = (\operatorname{diam}[\alpha])^s = ((1/w) \operatorname{diam} \alpha[U])^s \leq (1/w^s)(\operatorname{diam} A)^s$. Therefore

$$\begin{aligned} \mathcal{M}\big(h^{-1}[A]\big) &\leq \sum_{\alpha \in T} \mathcal{M}([\alpha]) \\ &\leq c(1/w^s)(\operatorname{diam} A)^s. \end{aligned}$$

So $b = c/w^s$ will work.

Therefore, by the Method I theorem, $1 = \mathcal{M}\big(h^{-1}[K]\big) \leq b\mathcal{H}^s(K)$, so we have $\dim K \geq s$. □

The proof given above clearly used the properties of Lebesgue measure in $\mathbb{R}^d$. What happens in other metric spaces? Readers who know about some exotic metric spaces may like to attempt this:

Exercise 6.5.5. Let S be a complete metric space (other than $\mathbb{R}^d$). Let $(f_1, f_2, \cdots, f_n)$ be a realization in S of a contracting ratio list $(r_1, r_2, \cdots, r_n)$ with dimension s. Let K be the invariant set. Suppose Moran's open set condition is satisfied. Does it follow that $\dim K = s$?

Exercise 6.5.6. Let $(r_1, r_2, \cdots, r_n)$ be a contracting ratio list with dimension s. Let $(f_1, f_2, \cdots, f_n)$ be an iterated function system consisting not of similarities, but of maps $f_i \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\varrho\big(f_i(x), f_i(y)\big) \geq r_i \, \varrho(x, y).$$

Suppose the open set condition holds, and suppose there is an invariant set K. Does it follow that $\dim K \geq s$?

Heighway Dragon Boundary

Heighway's Dragon (p. 20) is a set P in the plane with nonempty interior (a space-filling curve) which tiles the plane (p. 74). We have $\operatorname{Cov} P = \dim P = \operatorname{Dim} P = 2$, so it is not a fractal in the sense of Mandelbrot. We now have the tools to analyze the boundary of P. It turns out that ∂P is a fractal. So the tile P is an example of what we have called a fractile.

Recall the discussion on p. 74 showing that P tiles the plane. Let us continue with this line of reasoning. In Plate 16 we see a black segment from A to B that produces a sequence P_n of polygons that converge to P, the black tile in Plate 17.

First note that the point A in P_0 also belongs to all P_n and is a boundary point of P, since point A lies not only in the black tile P, but also in the brown, gray, and red tiles. Similarly point B is a boundary point of P. We will write $\partial P = U \cup V$ as follows. Set U is the portion of the boundary to the left of curve AB—that is, the points that belong not only to the black tile, but also to at least one of the brown, blue, or yellow tiles. Set V is the portion of the boundary to the right of curve AB—that is, the points that belong not only to the black tile, but also to at least one of the red, green, or cyan tiles. (In fact, the cyan tile never meets the black tile, as we can see by looking at P_1 in Plate 16.) No other tiles can touch the black tile, because the plane has topological dimension 2, so they would have to cross one of the curves shown to reach the black tile.

Consider set U. According to P_1 in Plate 16, the "midpoint" C of P is a boundary point, since it lies in both the black and blue tiles. The portion of U between A and C is a copy of U shrunk by factor $1/\sqrt{2}$. The portion of U between B and C is a copy of V shrunk by factor $1/\sqrt{2}$.

Now consider set V. Look at P_2 in Plate 16. The "three-quarter" point D of P is a boundary point, because it lies in both the black and red tiles. The portion of V between B and D is a copy of U shrunk by factor $1/2$. The portion of V between D and A is the boundary between black and red; looking at it from the red point of view, we see it is a copy of U shrunk by factor $1/2$.

Set V is made up of two copies of U, so $\dim U = \dim V$ and $\operatorname{Dim} U = \operatorname{Dim} V$. Set U is made up of one copy of U (shrunk by factor $1/\sqrt{2}$) and one copy of V (shrunk by factor $1/\sqrt{2}$). That copy of V is made up of two copies of U each shrunk by a further factor of 1/2. So the complete decomposition has U made up of three copies of itself, with ratio list $(2^{-1/2}, 2^{-3/2}, 2^{-3/2})$.

This ratio list has sim-value $s \approx 1.52$ given by $s = 2\log\lambda/\log 2$, where $\lambda \approx 1.6956$ is a solution of $\lambda^3 - \lambda^2 - 2 = 0$. In order to claim that this value s is also the Hausdorff and packing dimensions of U, we need an open set condition. Define an open set G as follows: in Plate 16 start with the four segments: black, yellow, blue, brown. The curve U lies in the union of the four tiles they produce: see the black, yellow, blue, brown in Plate 17. The open set G will be the interior of this union. Set G is shown in blue in Plate 18, with U shown in yellow. The images of G under the three maps that make up the iterated function system for U are shown in the second picture. The large red set is an image of G shrunk by factor $1/\sqrt{2}$. The cyan and green sets are images of G shrunk by factor $2^{-3/2}$. These (open) images are disjoint, since the tiles generated from different segments in Plate 16 have disjoint interiors. The three images descend from the edges bordering the like-colored squares shown in P_3 of Plate 18. This completes our description of the open set condition.

Eisenstein Boundary

Let us consider next the set F of "fractions" for the Eisenstein number system (Fig. 1.6.11). The base is $b = -2$, and the digit set consists of 0, 1, $A = \omega$, and $B = \omega^2$. The set F may be done in the same way as the twindragon (p. 33). But let us proceed in a more direct way. The first set L_0 is just the point 0. The next set L_1 consists of the four points $(.0)_{-2}$, $(.1)_{-2}$, $(.A)_{-2}$, and $(.B)_{-2}$. The set L_2 contains 16 points, all that can be represented with two digits in this system. The illustrations in Fig. 6.5.7 show these approximations. The set F obtained this time has (of course) similarity dimension 2.

To see that F is a "fractile" we need to analyze its boundary. The Eisenstein boundary is made up of six congruent parts, as shown in Fig. 6.5.8. The individual parts are self-similar. Each consists of three copies of itself, shrunk by factor 1/2. See Fig. 6.5.9(a).

Drawings in Logo can be done. When it is done as a curve, we need to take into account that some parts are drawn backward and/or reflected.

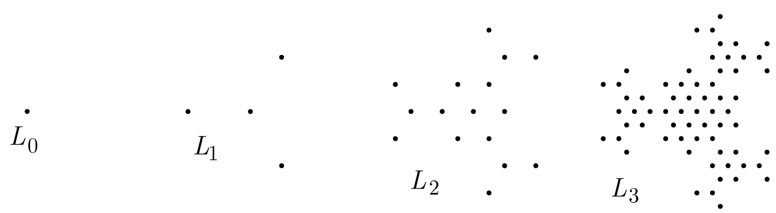

Fig. 6.5.7. Eisenstein fractions

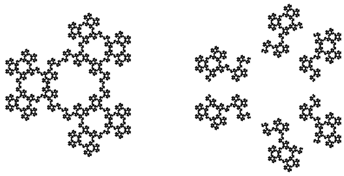

Fig. 6.5.8. The Eisenstein boundary consists of six congruent parts

```
to EBforward "depth "size "parity
  if :depth = 0 [forward :size stop] [
    left (60 * :parity)
    EBbackward (:depth - 1) (:size / 2) (-:parity)
    right (120 * :parity)
    EBforward (:depth - 1) (:size / 2) :parity
    left (60 * :parity)
    EBforward (:depth - 1) (:size / 2) (-:parity)]
end
to EBbackward "depth "size "parity
  if :depth = 0 [forward :size stop] [
    EBbackward (:depth - 1) (:size / 2) (-:parity)
    right (60 * :parity)
    EBbackward (:depth - 1) (:size / 2) :parity
    left (120 * :parity)
    EBforward (:depth - 1) (:size / 2) (-:parity)
    right (60 * :parity)]
end
```

The iterated function system and the open set condition for this curve are both illustrated by Fig. 6.5.9(b)(c). So we may conclude that the Eisenstein boundary has fractal dimension s (both Hausdorff and packing) satisfying $3\ (1/2)^s = 1$, so $s = \log 3/\log 2$. This is > 1, so this boundary is a fractal. That is, F is a fractile.

Examples for the Reader

Here is another dragon curve (Fig. 6.5.10, Plate 11), known as the ***terdragon***.

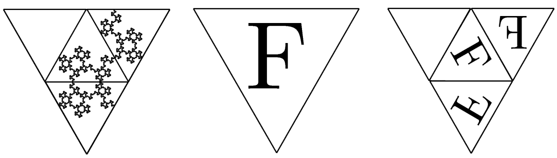

Fig. 6.5.9. (a) Decomposition (b)(c) Iterated function system and OSC

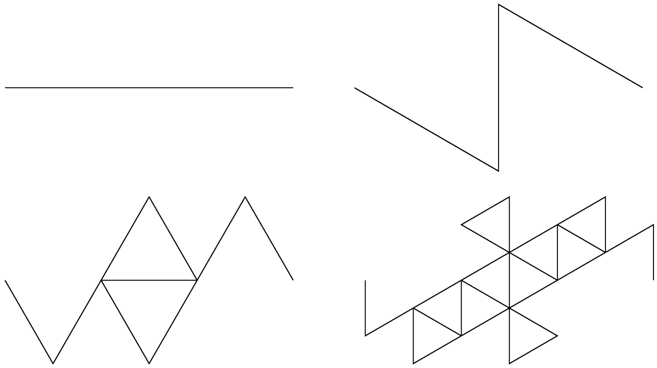

Fig. 6.5.10. Terdragon

```
make "shrink 1 / sqrt 3
to ter :depth :size
  if :depth = 0 [forward :size stop] [
    right 30
    ter :depth - 1 :size * :shrink
    left 120
    ter :depth - 1 :size * :shrink
    right 120
    ter :depth - 1 :size * :shrink
    left 30]
end
```

This is a space-filling curve. (Six copies of the terdragon exactly fit around a point; Plate 12.) It is not a fractal. But what about its boundary? Is it a "fractile"?

Exercise 6.5.11. Compute the Hausdorff and packing dimensions for the boundary of the terdragon.

A variant of the dragon that constructs the McWorter pentigree (p. 24) is shown in Fig. 6.5.12. Five copies of the limit set fit together to form a certain "dendrite". This will be called the ***pentadendrite***. (Fig. 6.5.13).

Exercise 6.5.14. Compute the topological, Hausdorff, and packing dimensions of the pentadendrite.

The set of "fractions" for the number system with base $-1 + \mathrm{i}$ and digit set $\{0, 1\}$ is the twindragon (Fig. 1.6.8).

Exercise 6.5.15. Compute the Hausdorff and packing dimensions of the boundary of the twindragon.

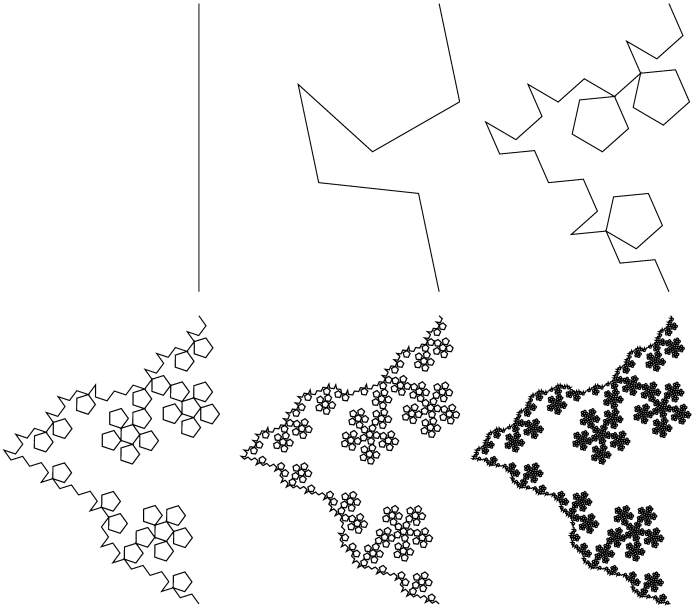

Fig. 6.5.12. Pentadendrite construction

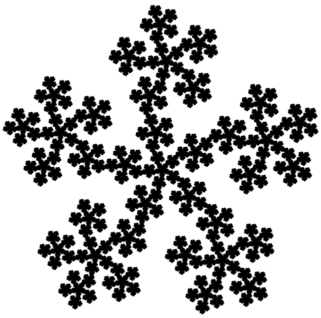

Fig. 6.5.13. Complete pentadendrite

Exercise 6.5.16. Compute the Hausdorff and packing dimensions for the limit of the sets constructed by the following program (p. 116). Warning: it is not self-similar.

```
to Schmidt :depth :size
  if :depth = 0 [stop] [
    repeat 3 [
      forward :size
      Schmidt :depth - 1 :size / 2
      right 120] ]
end
```

Another example of the same kind is the "I" fractal, p. 135.

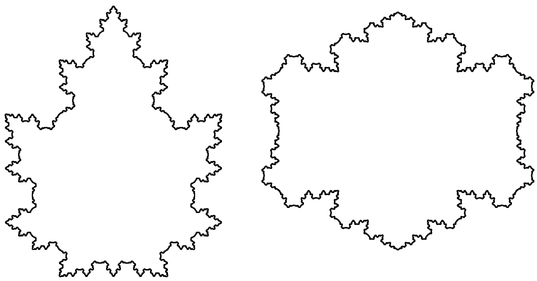

Fig. 6.5.17. Leaf outline; leaf lake

Barnsley Leaf Outline

Recall Barnsley's leaf B (p. 26). Because of the overlap among the parts of the iterated function system, the open set condition fails, so the methods of this section cannot compute its dimension (as far as I know). Discussion of overlap is in Sect. 7.1. But there are some related dimensions that can be done now. When all of the surrounded areas of B are filled in, we get a solid leaf with fractal boundary. Call that boundary the "leaf outline" J. Or consider one of the small regions surrounded by the leaf: call that a "leaf lake" K.

We will do a "deconstruction" of the leaf in Sect. 7.1 and conclude that there is a set H so that B consists of 8 copies of H (and its reflection H') arranged as shown in Fig. 6.5.18: four copies of the tile H on the right, and four copies of the reflected tile H' on the left.

Accordingly, the outline J is made up of 8 copies of the "upper left" edge of H (when oriented as shown). Tile H is deconstructed as in Fig. 6.5.19(a); the portion in the lower right is irrelevant for us now. Thus the segment of the leaf outline obeys an iterated function system also represented by 6.5.19(a). The picture also provides a Moran open set. The ratio list is $(2^{-1}, 2^{-3/2}, 2^{-3/2})$. The fractal dimension is $-2\log r/\log 2 \approx 1.21076$, where $r \approx 0.657298$ is a solution of the cubic $r^2 + 2r^3 = 1$.

6.6 Graph Self-Similarity

Next we consider evaluation of the Hausdorff dimension connected with graph self-similar sets.

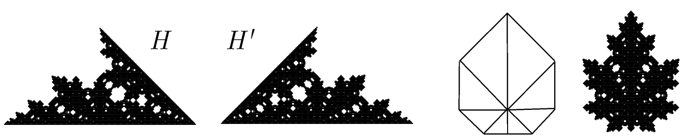

Fig. 6.5.18. Deconstruction of Barnsley's leaf

Fig. 6.5.19. (a) Decomposition of H (b) Self-similarity of the segment

The Two-Part Dust

We begin with a simple example, the ***two-part dust***. It has been "rigged" so that the calculation of the dimension is easier than the general case. The Mauldin–Williams graph is as illustrated in Fig. 6.6.1. Here is a description of the realization in $\mathbb{R}^2$ that will be considered. The map a has ratio $1/2$, fixed point $(0,0)$, and rotation 30 degrees counterclockwise. The map b has ratio $1/4$, fixed point $(1,0)$, and rotation 60 degrees clockwise. The map c has ratio $1/2$, fixed point $(0,0)$, and rotation 90 degrees counterclockwise. The map d has ratio $3/4$, fixed point $(1,0)$, and rotation 120 degrees clockwise.

As we know, there is a unique pair of nonempty compact sets $U, V \subseteq \mathbb{R}^2$ satisfying

$$U = \mathsf{a}[U] \cup \mathsf{b}[V]$$
$$V = \mathsf{c}[U] \cup \mathsf{d}[V].$$

This pair of sets is the ***two-part dust***. A sequence of approximations is pictured in Fig. 6.6.2. They converge in the Hausdorff metric. We may start with any two nonempty compact sets U_0 and V_0 in $\mathbb{R}^2$. In this case, both have been chosen as the line segment from the point $(0,0)$ to the point $(1,0)$. Then further approximations are defined recursively:

$$U_{n+1} = \mathsf{a}[U_n] \cup \mathsf{b}[V_n],$$
$$V_{n+1} = \mathsf{c}[U_n] \cup \mathsf{d}[V_n].$$

The sequence (U_n) converges in the Hausdorff metric to a nonempty compact set U, and the sequence (V_n) converges in the Hausdorff metric to a nonempty compact set V. This pair of sets is the required invariant list.

Here is the Logo program for the pictures.

```
; two-part dust
to U :depth :size
```

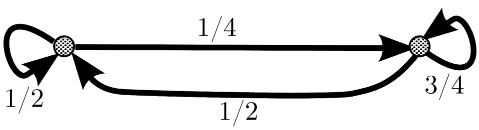

Fig. 6.6.1. Graph for the two-part dust

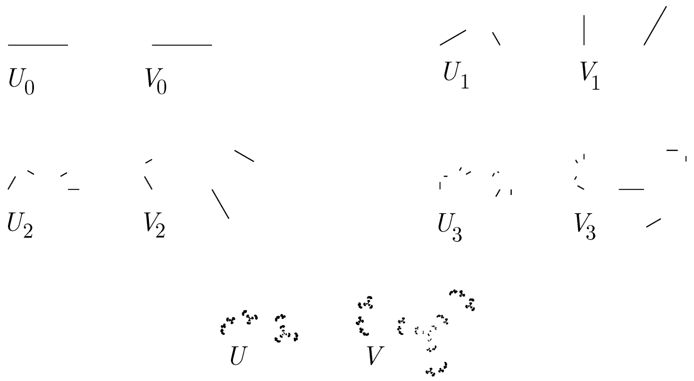

Fig. 6.6.2. Two-part dust

```
  if :depth = 0 [forward :size stop] [
    left 30
    U :depth - 1 :size / 2
    penup
        back :size / 2
        right 30
        forward :size
        right 60
        back :size / 4
    pendown
    V :depth - 1 :size / 4 left 60]
end
to V :depth :size
  if :depth=0 [forward :size stop] [
    left 90
    U :depth - 1 :size / 2
    penup
        back :size / 2
        right 90
        forward :size
        right 120
        back :size * 0.75
    pendown
    V :depth - 1 :size * 0.75
    left 120]
end
```

We are interested in computing the Hausdorff (and packing) dimensions of the sets U and V. Since each of the sets is similar to a subset of the other, their dimensions must be the same. As usual, we begin by computing the dimension of the path models corresponding to the Mauldin–Williams graph.

We will need to use the ***Perron numbers*** of the graph. In this case, the Perron numbers (one for each node) are $q_{\mathsf{U}} = 1/3$ and $q_{\mathsf{V}} = 2/3$. The important facts about these numbers are: they are positive and they satisfy equations

$$\begin{aligned} q_{\mathsf{U}} &= r(\mathsf{a})q_{\mathsf{U}} + r(\mathsf{b})q_{\mathsf{V}}, \\ q_{\mathsf{V}} &= r(\mathsf{c})q_{\mathsf{U}} + r(\mathsf{d})q_{\mathsf{V}}. \end{aligned} \tag{1}$$

By Proposition 4.3.16, the graph has* sim-value 1. I want to show that U and V have Hausdorff dimension 1. We will use the Perron numbers to assign diameters to the nodes of the path forest. Two ultrametrics ϱ, one on each of the two path spaces $E_{\mathsf{U}}^{(\omega)}$, $E_{\mathsf{V}}^{(\omega)}$, will be defined so that the diameters of the basic open sets $[\alpha]$ are as follows: Begin with $\operatorname{diam}[\Lambda_{\mathsf{U}}] = q_{\mathsf{U}}$ and $\operatorname{diam}[\Lambda_{\mathsf{V}}] = q_{\mathsf{V}}$. If α is a path and e is an edge with $t(e) = i(\alpha)$, then the diameter for the string $e\alpha$ is $\operatorname{diam}([e\alpha]) = r(e)\ \operatorname{diam}([\alpha])$.

Next, we will use the same numbers to define measures.

There is a metric measure $\mathcal{M}$ on each of the path spaces $E_v^{(\omega)}$ such that $\mathcal{M}([\alpha]) = \operatorname{diam}([\alpha])$ for all finite paths α. The additivity condition (Exercise 5.5.5) is true by equations (1). Of course we can repeat the steps from Lemma 6.4.1 and Exercise 6.4.2 to conclude that the path spaces $E_{\mathsf{U}}^{(\omega)}$, $E_{\mathsf{V}}^{(\omega)}$ both have Hausdorff and packing dimension 1. (Therefore, we say that U and V have [graph] similarity dimension 1.)

But we are more interested in the Hausdorff and packing dimensions of U and V. We will need an "open set condition". A little experimentation with a graphics program reveals that this may be satisfied by the two sets pictured in Fig. 6.6.3. The two sets are a rectangle and an irregular hexagon. (The images of these sets under the maps are appropriately disjoint and contained in the appropriate sets.) The dimension is now easy to check.

Exercise 6.6.4. Let U and V be the two parts of the two part dust. Show that the addressing functions

$$\begin{aligned} h_{\mathsf{U}} &\colon E_{\mathsf{U}}^{(\omega)} \to \mathbb{R}^2 \\ h_{\mathsf{V}} &\colon E_{\mathsf{V}}^{(\omega)} \to \mathbb{R}^2 \end{aligned}$$

are lipeomorphisms.

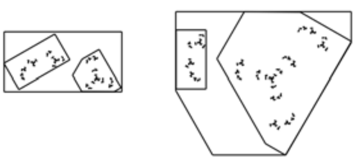

Fig. 6.6.3. Open set condition

* Solution to Exercise 4.3.14.

As consequences of this we have $\dim U = \operatorname{Dim} U = \dim E_{\mathsf{U}}^{(\omega)} = 1$ and $\dim V = \operatorname{Dim} V = \dim E_{\mathsf{V}}^{(\omega)} = 1$.

Perron Numbers

To compute the Hausdorff dimension for the other examples with graph self-similarity, we need only find the proper sort of "Perron numbers" in those cases. (It will not be quite as simple as the rigged example above if the dimension is not 1.)

Consider a Mauldin-Williams graph (V, E, i, t, r). We will consider only the case when the graph is strongly connected (p. 80). This will mean that when the invariant set list is found, each of the sets will be similar to a subset of each of the others. So they will all have the same fractal dimension.

We are interested in assigning metrics to the spaces $E_v^{(\omega)}$ of strings. (There is one space for each tree in the path forest.) The realization consists, as usual, of the right shifts. For an edge $e \in E_{uv}$, the function θ_e defined by

$$\theta_e(\sigma) = e\sigma$$

maps $E_v^{(\omega)}$ to $E_u^{(\omega)}$. The metrics should be chosen in such a way that θ_e is a similarity with ratio $r(e)$. We are also interested in defining measures (one for each space $E_v^{(\omega)}$) that will make the computation of the Hausdorff dimension easy.

In order to do this, we need the proper Perron numbers. If s is a positive real number, then s-dimensional ***Perron numbers*** for the graph are positive numbers q_v, one for each vertex $v \in V$, such that

$$q_u^s = \sum_{\substack{v \in V \\ e \in E_{uv}}} r(e)^s \, q_v^s$$

for all $u \in V$.

There is exactly one positive number s such that s-dimensional Perron numbers exist. This unique number s will be called the ***sim-value*** of the Mauldin-Williams graph. The existence and uniqueness of the sim-value were proved for the case of a 2 node graph in Sect 4.3. For the general graph, the proof requires some linear algebra. See Theorem 6.9.5.

We can proceed even without the proof of this result: if we can find Perron numbers, then we will be able to do the computations. When the set V of nodes is small, finding Perron numbers can often be done by trial and error.

Fractal Dimension

Now that all of the ingredients have been specified, we may proceed to analyze the Hausdorff and packing dimensions in this case. Suppose that (V, E, i, t, r)

is a strongly connected, contracting Mauldin-Williams graph. Let $s > 0$ be such that s-dimensional Perron numbers q_v exist. We will suppose that the graph is strictly contracting so that $r(e) < 1$ for all e. We will compute the dimension for the path model. There is one path space $E_v^{(\omega)}$ for each node $v \in V$.

First we need metrics for the path spaces. We want the right shifts to realize similarities with the ratios assigned by the Mauldin-Williams graph. For each finite path α, let $r(\alpha)$ be the product of the numbers $r(e)$, for all the edges e in α. For $\alpha \in E_{uv}^{(*)}$, we want the diameter of $[\alpha]$ to be $r(\alpha)q_v$.

Ultrametrics ϱ exist with these diameters. (One for each space $E_v^{(*)}$.) They satisfy

$$\varrho(e\sigma, e\tau) = r(e)\, \varrho(\sigma, \tau)$$

for $\sigma, \tau \in E_v^{(*)}$ and $e \in E_{uv}$.

Next we want to define measures on the path spaces. (The measures will all be called $\mathcal{M}$.) Because of the equations satisfied by the Perron numbers, we see that the values $(\operatorname{diam}[\alpha])^s$ satisfy the additivity condition (Theorem 5.5.4), namely

$$(\operatorname{diam}[\alpha])^s = \sum_{i(e)=t(\alpha)} (\operatorname{diam}[\alpha e])^s.$$

So there exists a metric measure on each of the spaces $E_v^{(\omega)}$ satisfying $\mathcal{M}([\alpha]) = (\operatorname{diam}[\alpha])^s$ for all $\alpha \in E_v^{(*)}$.

We can easily find an upper bound for the packing dimension. This is done in much the same way as has been done in previous cases. By Lemma 6.4.1, there is a positve constant c such that $\mathcal{P}^s(E_v^{(\omega)}) = c\mathcal{M}(E_v^{(\omega)}) = cq_v^s$. And by Exercise 6.4.2, $\mathcal{H}^s(E_v^{(\omega)}) = \mathcal{M}(E_v^{(\omega)}) = q_v^s$. But $0 < q_v^s < \infty$, so $\dim E_v^{(\omega)} = \operatorname{Dim} E_v^{(\omega)} = s$.

Once we know the Hausdorff and packing dimensions of the path spaces, we can try to apply it to the sets in $\mathbb{R}^d$ that we are really interested in. Let $(f_e)_{e \in E}$ be an iterated function system realizing the Mauldin-Williams graph (V, E, i, t, r) in $\mathbb{R}^d$. Let $(K_v)_{v \in V}$ be the unique invariant list of nonempty compact sets. As usual, we may construct the addressing functions

$$h_v \colon E_v^{(\omega)} \to \mathbb{R}^d,$$

one for each $v \in V$, such that

$$h_u(e\sigma) = f_e(h_v(\sigma)),$$

for $\sigma \in E_v^{(*)}$ and $e \in E_{uv}$. Then $h_v[E_v^{(\omega)}] = K_v$ for $v \in V$. These are Lipschitz maps, so the upper bound for the fractal dimensions is easy: $\dim K_v \le \operatorname{Dim} K_v \le \operatorname{Dim} E_v^{(\omega)} = s$.

6.7 Graph Open Set Condition

For the lower bound, we need to limit the overlap. We will use an open set condition.

Definition 6.7.1. If (f_e) is a realization of (V, E, i, t, r) in $\mathbb{R}^d$, then we say it satisfies the ***open set condition*** iff there exist nonempty open sets U_v, one for each $v \in V$, with

$$U_u \supseteq f_e[U_v]$$

for all $u, v \in V$ and $e \in E_{uv}$; and

$$f_e[U_v] \cap f_{e'}[U_{v'}] = \varnothing$$

for all $u, v, v' \in V$, $e \in E_{uv}$, $e' \in E_{uv'}$ with $e \neq e'$.

Now the argument proceeds as before. Fix a node $v \in V$. I want to show that $\dim K_v \geq s$. As before, if α is a finite (nonempty) string, write α^- for its parent. Also, we will use the notation $e(x)$ for $f_e(x)$, and similarly for strings: $\alpha(x)$.

First, I claim that there is a constant $c > 0$ such that: if $A \subseteq K_v$, then the set

$$T = \Big\{ \alpha \in E_v^{(*)} : \overline{\alpha[U_{t(\alpha)}]} \cap A \neq \varnothing, \\ \operatorname{diam} \alpha[U_{t(\alpha)}] < \operatorname{diam} A \leq \operatorname{diam} \alpha^-[U_{t(\alpha^-)}] \Big\}$$

has at most c elements.

Writing

$$w_{\max} = \max_{u \in V} \operatorname{diam} U_u, \qquad w_{\min} = \min_{u \in V} \operatorname{diam} U_u, \qquad r_{\min} = \min_{e \in E} r_e,$$

we have for $\alpha \in T$:

$$\begin{aligned} \operatorname{diam} \alpha[U_{t(\alpha)}] &= r(\alpha) \operatorname{diam} U_{t(\alpha)} \geq w_{\min} r_{\min}\, r(\alpha^-) \\ &\geq \frac{w_{\min} r_{\min}}{w_{\max}} \operatorname{diam} \alpha^-[U_{t(\alpha^-)}] \geq \frac{w_{\min} r_{\min}}{w_{\max}} \operatorname{diam} A. \end{aligned}$$

Now if $p = \min_{u \in V} \mathcal{L}^d(U_u)$, we have the volume calculation

$$\mathcal{L}^d(\alpha[U_{t(\alpha)}]) \geq p \left(\frac{\operatorname{diam} \alpha[U_{t(\alpha)}]}{\operatorname{diam} U_{t(\alpha)}} \right)^d \geq p \left(\frac{w_{\min} r_{\min}}{w^{*2}} \right)^d (\operatorname{diam} A)^d.$$

Now if $x \in A$, then every point of every set $\alpha[U_{t(\alpha)}]$ for $\alpha \in T$ is within distance $\operatorname{diam} A + \operatorname{diam} \alpha[U_{t(\alpha)}] < 2 \operatorname{diam} A$ of x. The sets $\alpha[U_{t(\alpha)}]$ are disjoint, so if T has m elements, then there are m disjoint sets, with volume at least $p(w_{\min} r_{\min}/w^{*2})^d (\operatorname{diam} A)^d$ inside a ball with radius $2 \operatorname{diam} A$. If $t = \mathcal{L}^d\big(B_1(0)\big)$, we have

$$mp \left(\frac{w_{\min} r_{\min}}{w^{*2}} \right)^d (\operatorname{diam} A)^d \leq t(2 \operatorname{diam} A)^d.$$

Solving for m, we get

$$m \le \frac{t}{p}\left(\frac{2w^{*2}}{w_{\min} r_{\min}}\right)^d.$$

Next, I claim that there is a constant $b > 0$ so that for any Borel set $A \subseteq K_v$, we have

$$\mathcal{M}\big(h_v^{-1}[A]\big) \le b(\operatorname{diam} A)^s.$$

Given A, let T be as before. Write $q = \max_{u\in V} q_u$. Then $h_v^{-1}[A] \subseteq \bigcup_{\alpha\in T}[\alpha]$. If $\alpha \in T$, then

$$\mathcal{M}\big([\alpha]\big) \le r(\alpha)^s\, q^s \le q^s \left(\frac{\operatorname{diam} \alpha[U_{t(\alpha)}]}{w_{\min}}\right)^s \le \frac{q^s(\operatorname{diam} A)^s}{w_{\min}^s}.$$

Therefore

$$\mathcal{M}\big(h_v^{-1}[A]\big) \le \sum_{\alpha\in T} \mathcal{M}\big([\alpha]\big) \le \frac{cq^s}{w_{\min}^s}(\operatorname{diam} A)^s.$$

Then we conclude from the Method I theorem that $\mathcal{M}(h_v^{-1}[A]) \le b\mathcal{H}^s(A)$ for all Borel sets A. In particular,

$$\mathcal{H}^s(K_v) \ge \frac{\mathcal{M}(h_v^{-1}[K_v])}{b} = \frac{\mathcal{M}\big(E_v^{(\omega)}\big)}{b} = \frac{q_v^s}{b} > 0.$$

Therefore $\dim K_v \ge s$. And of course $\operatorname{Dim} K_v \ge \dim K_v$.

We have established the result:

Theorem 6.7.2. *Let (V, E, i, t, r) be a strongly connected contracting Mauldin-Williams graph describing the graph self-similarity of a list $(K_v)_{v\in V}$ of nonempty compact sets in $\mathbb{R}^d$. Let $s > 0$ be such that s-dimensional Perron numbers exist. Then $\dim K_v \le \operatorname{Dim} K_v \le s$ for all v. If, in addition, the realization satisfies the open set condition, then $\dim K_v = \operatorname{Dim} K_v = s$.*

Exercise 6.7.3. Let (V, E, i, t, r) be a strongly connected contracting Mauldin-Williams graph. Let $(f_e)_{e\in E}$ be a family of maps on $\mathbb{R}^d$ satisfying

$$\varrho(f_e(x), f_e(y)) \le r(e)\varrho(x, y).$$

Formulate the proper analog of Theorem 6.7.2.

Exercise 6.7.4. Let (V, E, i, t, r) be a Mauldin-Williams graph. Let $(f_e)_{e\in E}$ be a family of maps on $\mathbb{R}^d$ satisfying $\varrho(f_e(x), f_e(y)) \ge r(e)\varrho(x, y)$. Formulate the proper analog of Theorem 6.7.2.

Exercise 6.7.5. Discuss Hausdorff dimension for graph self-similar sets with Mauldin-Williams graph not strongly connected.

Li Lake

The graph self-similar set called Li's Lace is seen on p. 84. Its description is on p. 126 shows it is is made up of isosceles right trianglar blocks of two kinds, called P and Q. An open space completely surrounded by the fractal is called a ***lake***. The boundary of one lake is shown in Fig. 6.7.6(b). This boundary can also be described in the language of graph self-similarity. Fig. 6.7.7 identifies two boundary parts A and B in tile P. The lake boundary is made up of eight parts A.

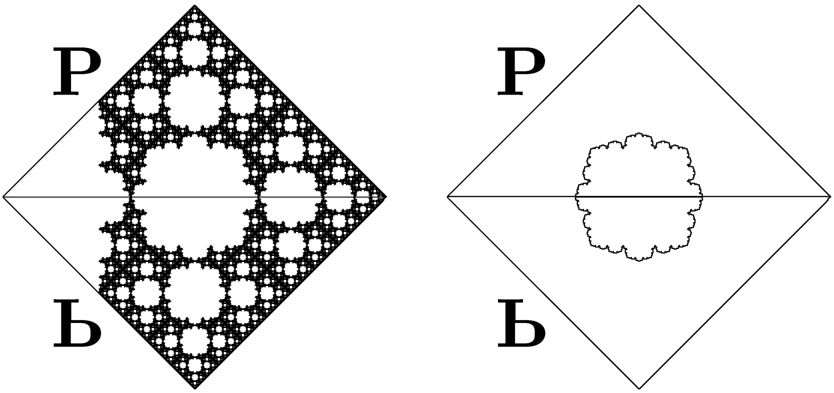

Fig. 6.7.6. (a) Two tiles P (b) Lake boundary

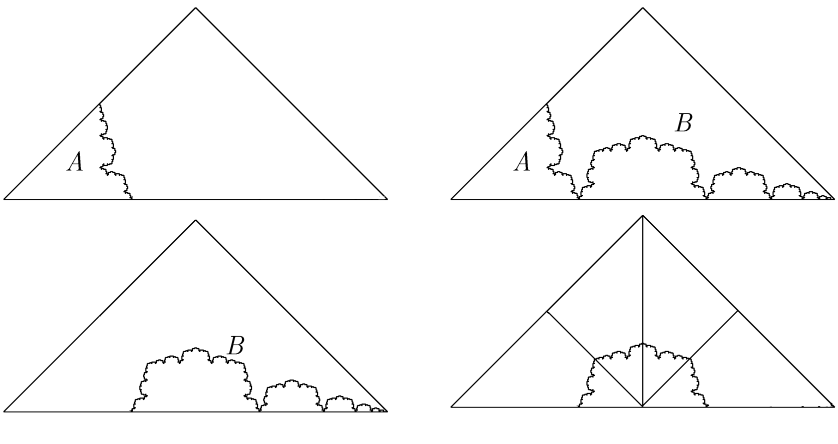

Fig. 6.7.7. (a) Parts A and B (b) Four parts A = half of boundary

Exercise 6.7.8. The parts A and B may be described using the descriptions of P and Q from Fig. 4.3.3. Set A is made up of two copies of A shrunk by factor 1/4 and one copy of B shrunk by factor 1/4. Set B is made up of four copies of A shrunk by factor 1/2 and one copy of B shrunk by factor 1/2. Compute the corresponding sim-value for sets A and B. Verify the OSC to conclude this is the Hausdorff and packing dimension for the lake boundary.

Pentigree Outline

Recall the construction on p. 25 of the second form of McWorter's pentigree. The ***pentigree outline*** is what we get if we fill in all the "lakes", and take the boundary of the result (Fig. 6.7.9, see also p. XIII).

Here is the program used to draw Fig. 6.7.9.

```
; pentigree outline
make "shrink (3 + sqrt 5)/2
to pent :depth :size
    repeat 5 [A :depth :size]
end
to A :depth :size
  if :depth = 0 [forward :size right 72 stop] [
    B :depth - 1 :size / :shrink
    A :depth - 1 :size / :shrink
    A :depth - 1 :size / :shrink
    BR :depth - 1 :size / :shrink]
end
to B :depth :size
  if :depth = 0 [forward :size left 36 stop] [
    C :depth - 1 :size / :shrink
    A :depth - 1 :size / :shrink
    BR :depth - 1 :size / :shrink]
end
to BR :depth :size
```

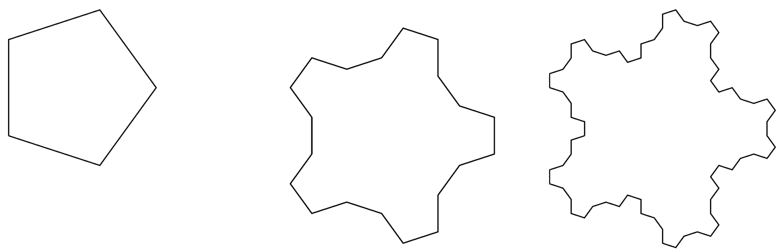

Fig. 6.7.9. Pentigree outline

```
    if :depth = 0 [forward :size left 36 stop] [
      B :depth - 1 :size / :shrink
      A :depth - 1 :size / :shrink
      C :depth - 1 :size / :shrink]
  end
  to C :depth :size
    if :depth = 0 [forward :size left 72 stop] [
      B :depth - 1 :size / :shrink
      BR :depth - 1 :size / :shrink]
  end
```

Exercise 6.7.10. Does this really converge to the outline of McWorter's pentigree?

Exercise 6.7.11. Determine the Mauldin-Williams graph describing the self-similarity of curve A.

Exercise 6.7.12. Compute the Hausdorff dimension of the pentigree outline.

Number Systems

Let b be a complex number, $|b| > 1$, and let D be a finite set of complex numbers, including 0. We are interested in the numbers that can be represented in the form

$$\sum_{j=1}^{\infty} a_j b^{-j}.$$

In some cases, the set of representations may be restricted to allow only certain combinations of digits. Consider $b = 3$ and $D = \{0, 1, 2\}$. Let F be the set of all numbers x of the form

$$x = \sum_{j=1}^{\infty} a_j b^{-j},$$

where each a_j is in the set D, and such that $a_j + a_{j+1} \leq 2$ for all j. This set is graph self-similar.

Let $F(d_1)$ be the set of numbers where the representation has $a_1 = d_1$. Let $F(d_1, d_2)$ be those numbers where the representation has $a_1 = d_1$ and $a_2 = d_2$. Then $F = F(0) \cup F(1) \cup F(2)$. The set $F(0)$ is similar to F, with ratio 1/3. The set $F(1) = F(1,0) \cup F(1,1)$, since $F(1,2) = \varnothing$. But $F(1,0)$ is similar to F with ratio 1/9 and $F(1,1)$ is similar to $F(1)$ with ratio 1/3. Finally, $F(2) = F(2,0)$ is similar to F with ratio 1/9. The graph is shown in Fig. 6.7.13.

Exercise 6.7.14. Compute the Hausdorff dimension of the set F.

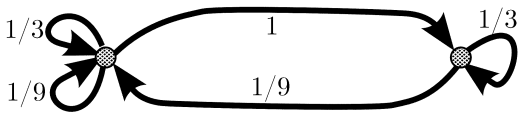

Fig. 6.7.13. Graph

Exercise 6.7.15. Let $b = (1+\sqrt{5})/2$ and $D = \{0,1\}$. Let F be the set of numbers of the form

$$\sum_{j=1}^{\infty} a_j b^{-j},$$

where each $a_j \in D$, and two consecutive digits 1 are not allowed. Describe the set F.

Topological Dimension

The addressing function, or model map, which has been developed here for the purpose of computing the fractal dimension of a [graph] self-similar set, can also sometimes be used for the topological dimension as well. The addressing functions $h_v \colon E_v^{(\omega)} \to K$ are continuous and surjective. The spaces $E_v^{(\omega)}$ are compact. When the overlap is small, the characterization of topological dimension of Theorem 3.4.19 is often applicable. Let us do some examples.

The addressing function for the Cantor dust is one-to-one. Therefore the Cantor dust is zero-dimensional.

The addressing function for the Sierpiński gasket maps at most 2 strings to each point. Therefore the Sierpiński gasket has small inductive dimension ≤ 1. It contains line segments, so it has dimension exactly 1.

Exercise 6.7.16. Show that McWorter's pentigree has small inductive dimension 1.

If you have not solved Exercise 1.6.5 yet, you can now do so painlessly.

Theorem 3.4.19 yields only an inequality. The addressing function for the Menger sponge (using the construction suggested in Fig. 4.1.9) is 4-to-1 at some points, so we obtain the uninteresting result that the topological dimension is ≤ 3. In fact, the topological dimension is 1. Is there another way to produce the Menger sponge as a self-similar set such that the model map is only at most 2-to-one?

6.8 *Other Fractal Dimensions

According to Mandelbrot, a ***fractal*** is a set S with $\dim S > \operatorname{Cov} S$. We will consider (for the moment) only nonempty compact sets in Euclidean space.

* Optional section.

Mandelbrot expressed dissatisfaction with the definition for two reasons: (1) “borderline fractals” are excluded; and (2) “true geometric chaos” is included.

Borderline Fractal

What might be meant by a “borderline fractal”? This will be a set K with the usual features of fractals that we have seen often, but where $\dim K = \operatorname{Cov} K$ anyway. To illustrate this, we will consider a curve, with a dragon-like construction. We begin with a sequence of positive numbers w_k, with $w_0 = 1$, $w_k > w_{k+1} > w_k/2$, and $\lim_{k\to\infty} w_k = 0$. The first set is a line segment P_0 with length $w_0 = 1$. If the polygon P_k has been constructed, consisting of many line segments of length w_k, then to construct P_{k+1}, we replace each of those line segments by two segments of length w_{k+1}. (It is possible to do this, and still have a polygon, since $w_{k+1} > w_k/2$.) If $w_k \to 0$ fast enough, we can avoid having the curve cross itself (even in the limit) by alternating between sides of the curve, as shown in the illustration. Then the limit will be homeomorphic to an interval, and therefore have small inductive dimension 1.

If we choose w_k that satisfy $w_k^s = 1/2^k$, for some $s > 1$, then this is a self-similar dragon curve that we have seen before. In the binary tree, if α is a finite string of length k, then when we use the metric and measure appropriate for the tree, we have $\operatorname{diam}[\alpha] = w_k$ and $\mathcal{M}([\alpha]) = 1/2^k = (\operatorname{diam}[\alpha])^s$. The usual calculation shows that the Hausdorff dimension for the curve will be s.

But suppose we have w_k that satisfy

$$\frac{w_k}{\log(1/w_k)} = \frac{1}{2^k}.$$

This means that w_k goes to zero more rapidly than $(1/2^k)^{1/s}$ for any $s > 1$, but more slowly than $1/2^k$ itself.

Exercise 6.8.2. When

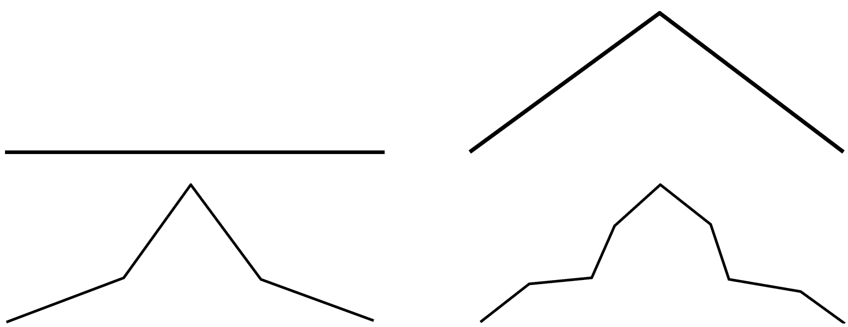

Fig. 6.8.1. Borderline dragon

$$\frac{w_k}{\log(1/w_k)} = \frac{1}{2^k},$$

the topological, Hausdorff, and packing dimensions are all 1.

Why would we call this a "borderline fractal"?

Box Dimensions

We will next discuss some other fractal dimensions known as box dimensions. We will begin with $\mathbb{R}^2$ for simplicity. But first consider a variant of the Hausdorff dimension.

If $r > 0$, then the ***square net*** of side r consists of all squares of the form

$$A = \big[(m-1)r, mr\big) \times \big[(n-1)r, nr\big),$$

where $m, n \in \mathbb{Z}$. Write $\mathcal{S}_r$ for this set of squares. So the plane $\mathbb{R}^2$ is the disjoint union of this countable collection of squares. Write $\mathcal{S} = \bigcup_{r>0} \mathcal{S}_r$.

For $s > 0$, consider the method II outer measure $\overline{\mathcal{M}}^s$ defined using the set function $\mathbf{c}\colon \mathcal{S} \to [0,\infty)$ defined by: $\mathbf{c}(A) = r^s$ if $A \in \mathcal{S}_r$. Now any set $B \subseteq \mathbb{R}^2$ of diameter r is contained in the union of at most 4 sets of $\mathcal{S}_r$. On the other hand, a square with side r has diameter $\sqrt{2}\,r$. This means that

$$2^{-s/2}\,\overline{\mathcal{H}}^s(F) \le \overline{\mathcal{M}}^s(F) \le 4\,\overline{\mathcal{H}}^s(F).$$

Therefore $s_0 = \dim F$ is the critical value for which

$$\begin{aligned} \overline{\mathcal{M}}^s(F) &= \infty \qquad \text{for all } s < s_0; \\ \overline{\mathcal{M}}^s(F) &= 0 \qquad \text{for all } s > s_0. \end{aligned}$$

Some calculations involving the Hausdorff dimension are simplified by using this alternative to the definition (for example [23, Chapt. 5]).

Now we discuss another variant. Fix a number $r > 0$, and cover only by sets of $\mathcal{S}_r$; then let $r \to 0$. It should be emphasized that this is not method II. Now if $\mathcal{A} \subseteq \mathcal{S}_r$ covers a set F, then $\sum_{A\in\mathcal{A}} r^s$ is just Nr^s, where N is the number of elements of $\mathcal{A}$. So the definition may be phrased as follows. Let $N_r(F)$ be the number of sets of the square net $\mathcal{S}_r$ that intersect F. Define

$$\begin{aligned} \widetilde{\mathcal{K}}^s_r(F) &= N_r(F)\, r^s, \\ \widetilde{\mathcal{K}}^s(F) &= \liminf_{r\to 0} \widetilde{\mathcal{K}}^s_r(F). \end{aligned}$$

As usual there is a critical value for s.

Exercise 6.8.3. Let

$$s_0 = \liminf_{r\to 0} \frac{\log N_r(F)}{\log(1/r)}.$$

Then

$$\widetilde{\mathcal{K}}^s(F) = \infty \qquad \text{for all } s < s_0;$$
$$\widetilde{\mathcal{K}}^s(F) = 0 \qquad \text{for all } s > s_0.$$

The critical value s_0 will be called the ***lower box dimension*** or ***lower box-counting dimension*** or ***lower entropy index*** of F. We will write $\underline{\dim}_{\mathrm{B}} F$. The set functions $\widetilde{\mathcal{K}}^s$ have the same shortcoming as $\widetilde{\mathcal{P}}^s$:

Exercise 6.8.4. There is a countable compact set K with positive lower box dimension.

Because of this undesirable property, we can apply Method I to $\widetilde{\mathcal{K}}^s$ to get a metric outer measure. Or we can modify the dimension directly. The ***modified lower box dimension*** is

$$\underline{\dim}_{\mathrm{MB}} F = \inf \sup_i \underline{\dim}_{\mathrm{B}} F_i$$

where the infimum is over all countable covers $F \subseteq \bigcup_{i \in \mathbb{N}} F_i$ of F.

A variant is obtained by replacing lim inf with lim sup. The set functions are then

$$\limsup_{r \to 0} \widetilde{\mathcal{K}}^s_r(F).$$

The critical value for s, called the the ***upper box dimension***,* is given by

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{r \to 0} \frac{\log N_r(F)}{\log(1/r)}.$$

Again the set function is not an outer measure, and $\overline{\dim}_{\mathrm{B}}$ is not countably stable, so define the ***modified upper box dimension*** by

$$\overline{\dim}_{\mathrm{MB}} F = \inf \sup_i \overline{\dim}_{\mathrm{B}} F_i$$

where the infimum is over all countable covers $F \subseteq \bigcup_{i \in \mathbb{N}} F_i$ of F.

Exercise 6.8.5. Let $F \subseteq \mathbb{R}^2$. Then

$$\dim F \le \underline{\dim}_{\mathrm{B}} F \le \overline{\dim}_{\mathrm{B}} F \le \text{packing index of } F,$$
$$\dim F \le \underline{\dim}_{\mathrm{MB}} F \le \overline{\dim}_{\mathrm{MB}} F \le \operatorname{Dim} F.$$

So if F is a fractal in the sense of Taylor, these four fractal dimensions all coincide. (In fact, we will see below that $\overline{\dim}_{\mathrm{MB}} F = \operatorname{Dim} F$.)

Exercise 6.8.6. Define the set functions $\widetilde{\mathcal{K}}^s$ and $\overline{\mathcal{K}}^s$ in the space $\mathbb{R}^d$. Prove analogs of Exercise 6.8.5.

* Barnsley [3] uses the term "fractal dimension" for this value.

Boxes do not make sense in a general metric space, but a box dimension can be defined there anyway. Let S be a metric space, let $F \subseteq S$, and let $r > 0$. Define $\dot{N}_r(F)$ to be the maximum number of disjoint closed balls with radius r and center in F. (Actually, instead of "disjoint" we can take the sense used for the packing measure: the centers x_i satisfy $\varrho(x_i, x_j) > 2r$ for all $i \neq j$.)

Proposition 6.8.7. *Let* $F \subseteq \mathbb{R}^2$. *Then*

$$\liminf_{r\to 0} \frac{\log N_r(F)}{\log(1/r)} = \liminf_{r\to 0} \frac{\log \dot{N}_r(F)}{\log(1/r)}$$
$$\limsup_{r\to 0} \frac{\log N_r(F)}{\log(1/r)} = \limsup_{r\to 0} \frac{\log \dot{N}_r(F)}{\log(1/r)}$$

Proof. Let $\delta > 0$. There are $\dot{N}_\delta(F)$ disjoint balls with radius δ and center in F. No two of those centers have distance $\leq 2\delta$, so no two of those centers are in the same square of $\mathcal{S}_{\delta\sqrt{2}}$. So $\dot{N}_\delta(F) \leq N_{\delta\sqrt{2}}(F)$.

Let $r > 0$. In each square of $\mathcal{S}_r$ that meets F, choose one point of F, and use it as the center of a closed ball of radius $r/4$. Any such ball intersects at most 4 squares of $\mathcal{S}_r$, and $N_r(F) \leq 4\dot{N}_{r/4}(E)$.

Therefore, for any $r > 0$ we have

$$\frac{\log N_r(F)}{\log(1/r)} \leq \frac{\log(4\dot{N}_{r/4}(F))}{\log((1/4)(4/r))} = \frac{\log \dot{N}_{r/4}(F) + \log 4}{\log(4/r) - \log 4}$$
$$\frac{\log N_r(F)}{\log(1/r)} \geq \frac{\log \dot{N}_{r/\sqrt{2}}(F)}{\log((1/\sqrt{2})(\sqrt{2}/r))} = \frac{\log \dot{N}_{r/\sqrt{2}}(F)}{\log(\sqrt{2}/r) - \log\sqrt{2}}.$$

But as $r \to 0$, we have $\log(1/r) \to \infty$, so we get the lim sup and lim inf results claimed. ☐

In a metric space S, define the upper and lower box dimension using $\dot{N}$ in place of N. Then define the upper and lower modified box dimensions as before. It doesn't have "boxes" in the defintion any more. Sometimes $\overline{\dim}_{\mathrm{B}}$ or $\underline{\dim}_{\mathrm{B}}$ may be called ***Bouligand dimension*** or ***Minkowski dimension*** instead.

Proposition 6.8.8. *Let* S *be a metric space and* $F \subseteq S$. *Then* $\overline{\dim}_{\mathrm{MB}} F = \operatorname{Dim} F$.

Proof. First we claim: $\operatorname{Dim} F \leq \overline{\dim}_{\mathrm{B}} F$. Let $t < s < \operatorname{Dim} F$. Then $\overline{\mathcal{P}}^s(F) = \infty$, so $\widetilde{\mathcal{P}}^s_0(F) = \infty$ and $\widetilde{\mathcal{P}}^s_\delta(F) = \infty$ for all $\delta > 0$. Now let δ be given with $0 < \delta < 1$. Then there is a δ-fine packing Π of F with $\sum_{(x,r)\in\Pi} r^s > 1$. For $k \in \mathbb{N}$, let n_k be the number of $(x,r) \in \Pi$ with $2^{-k-1} < r \leq 2^{-k}$. Then there is some k with $n_k > 2^{kt}(1 - 2^{t-2})$, since if not we would have

$$1 < \sum_{(x,r)\in\Pi} r^s \leq \sum_{k=0}^{\infty} n_k 2^{-ks} \leq (1 - 2^{t-s}) \sum_{k=0}^{\infty} 2^{k(t-s)} = 1.$$

For that k, we get n_k disjoint balls centered in F with radius 2^{-k-1}, so

$$\dot{N}_{2^{-k-1}}(F) \geq n_k > 2^{kt}(1 - 2^{t-2}),$$
$$\frac{\log \dot{N}_{2^{-k-1}}(F)}{\log(2^{k+1})} \geq \frac{kt \log 2 + \log(1 - 2^{t-s})}{(k+1)\log 2}.$$

Now the right-hand side goes to t as $k \to \infty$, and for any δ there is k as given with $2^{-k} < \delta$, so we conclude that $\overline{\dim}_{\mathrm{B}} F \geq t$. But t was any value $< \operatorname{Dim} F$, so in fact $\overline{\dim}_{\mathrm{B}} F \geq \operatorname{Dim} F$.

Next we claim $\operatorname{Dim} F \leq \overline{\dim}_{\mathrm{MB}} F$. Let $F \subseteq \bigcup_i F_i$ be a countable cover of F. Then

$$\operatorname{Dim} F \leq \sup_i \operatorname{Dim} F_i \leq \sup_i \overline{\dim}_{\mathrm{B}} F_i.$$

Take the infimum over all such covers to get $\operatorname{Dim} F \leq \overline{\dim}_{\mathrm{MB}} F$.

And finally we claim $\overline{\dim}_{\mathrm{MB}} F \leq \operatorname{Dim} F$. Let $s > \operatorname{Dim} F$. Then $\overline{\mathcal{P}}^s(F) = 0$, so there is a countable cover $F \subseteq \bigcup F_i$ with $\sum_i \widetilde{\mathcal{P}}^s_0(F_i) < \infty$ for all i. Fix an i. There is $\delta > 0$ with $\widetilde{\mathcal{P}}^s_\delta(F_i) < \infty$. Then since $\dot{N}_\delta(F_i)\delta^s \leq \widetilde{\mathcal{P}}^s_\delta(F_i)$, it remains bounded as $\delta \to 0$, so $\overline{\dim}_{\mathrm{B}} F_i \leq s$. This is true for all i, so $\sup_i \overline{\dim}_{\mathrm{B}} F_i \leq s$. Take the infimum on all covers to get $\overline{\dim}_{\mathrm{MB}} F \leq s$. This is true for all $s > \operatorname{Dim} F$, so $\overline{\dim}_{\mathrm{MB}} F \leq \operatorname{Dim} F$. □

Lipschitz Condition of Order *p*

Let S, T be metric spaces, and let $p > 0$. We say that a function $f\colon S \to T$ satisfies a ***Lipschitz condition*** of order p if there is a constant M so that for all $x, y \in S$,

$$\varrho\big(f(x), f(y)\big) \leq M \varrho(x, y)^p. \tag{1}$$

We may say $f \in \operatorname{Lip}(p)$. This is also called a ***Hölder condition*** of order p.

Proposition 6.8.9. *Let* $f\colon S \to T$, $f \in \operatorname{Lip}(p)$. *Then* (a) $p \dim f[S] \leq \dim S$ *and* (b) $p \operatorname{Dim} f[S] \leq \operatorname{Dim} S$.

Proof. Let M satisfy (1). (a) If $A \subseteq S$, then $\operatorname{diam} f[A] \leq M\,(\operatorname{diam} A)^p$. Let $\delta > 0, s > 0$, and let $\{A_n\}$ be a cover of S by sets of diameter $\leq \delta$. Then $\{f[A_n]\}$ is a cover of $f[S]$ by sets of diameter $\leq M\delta^p$. So

$$\overline{\mathcal{H}}^s_{M\delta^p}\big(f[S]\big) \leq \sum_{n\in\mathbb{N}} \big(\operatorname{diam} f[A_n]\big)^s \leq M^s \sum_{n\in\mathbb{N}} \big(\operatorname{diam} A_n\big)^{ps}.$$

Taking the infimum over all δ-covers, we get

$$\overline{\mathcal{H}}^s_{M\delta^p}\big(f[S]\big) \leq M^s \overline{\mathcal{H}}^{ps}_\delta(S).$$

Taking the limit as $\delta \to 0$, we get

$$\overline{\mathcal{H}}^s\big(f[S]\big) \leq M^s \overline{\mathcal{H}}^{ps}(S).$$

If $s > (1/p)\dim S$, then $\overline{\mathcal{H}}^{ps}(S) = 0$ so $\overline{\mathcal{H}}^{s}(f[S]) = 0$, which means $s \geq \dim f[S]$. Therefore $\dim f[S] \leq (1/p)\dim S$ as required.

(b) Let $\delta > 0$. Define $\delta' = M\delta^p$. Then $\delta' \to 0$ as $\delta \to 0$. Also, $\log(2/\delta') = p\log(2/\delta) + C$ for a certain constant C. Now let $E \subseteq S$. If $u_1, \cdots, u_N \in f[E]$ have $\varrho(u_i, u_j) \geq \delta'$ for all $i \neq j$, then there exist $x_i \in E$ with $f(x_i) = u_i$ and $\varrho(x_i, x_j) \geq \delta$ for all $i \neq j$. Therefore

$$\dot{N}_{\delta/2}(E) \geq \dot{N}_{\delta'/2}\big(f[E]\big),$$

and thus $\overline{\dim}_{\mathrm{B}} E \geq p\,\overline{\dim}_{\mathrm{B}} f[E]$.

Let $S = \bigcup_i E_i$ be a countable cover. Then

$$\sup_i \overline{\dim}_{\mathrm{B}} E_i \geq p \sup_i \overline{\dim}_{\mathrm{B}} f[E_i] \geq p\,\overline{\dim}_{\mathrm{MB}} f[S].$$

Take the infimum over all covers to get $\overline{\dim}_{\mathrm{MB}} S \geq p\,\overline{\dim}_{\mathrm{MB}} f[S]$. And by Prop. 6.8.8, $\overline{\dim}_{\mathrm{MB}} = \mathrm{Dim}$. □

Theorem 6.8.10. *Let $0 < p \leq 1$. Suppose $f\colon [a,b] \to \mathbb{R}$ satisfies a Lipschitz condition of order p. Then the graph*

$$G = \big\{\, \big(x, f(x)\big) : x \in [a,b] \,\big\}$$

satisfies $\dim G \leq 2 - p$.

Proof. We may assume that $[a,b] = [0,1]$. Divide $[0,1]$ into n sub-intervals of length $1/n$. On each of these intervals f can vary by no more than $M(1/n)^p$. Thus, the part of the graph over one of the sub-intervals can be covered by no more than $Mn^{1-p} + 1$ squares of side $1/n$. Thus

$$\overline{\mathcal{H}}^{s}_{\sqrt{2}/n}(G) \leq n(Mn^{1-p} + 1)\left(\frac{\sqrt{2}}{n}\right)^{s} = M2^{s/2}n^{2-p-s} + 2^{s/2}n^{1-s}.$$

If $s = 2 - p$, then this shows $\mathcal{H}^s(G) \leq M2^{(2-p)/2} + 2^{(2-p)/2}$, so $\dim G \leq s = 2 - p$. □

Besicovitch and Ursell [5, part V] gave examples of functions satisfying a Lipschitz condition of order p (and no better) where $\dim G = 2 - p$ and other examples where $\dim G < 2 - p$.

6.9 *Remarks

Felix Hausdorff [32] formulated the concepts that we call today the Hausdorff measures and the Hausdorff dimension. Almost all of the early work on the subject was done by A. S. Besicovitch [5]. Mandelbrot therefore uses the term "Hausdorff–Besicovitch dimension".

In fact, Hausdorff proposed a much more general class of measures than the ones discussed here. For example, he proposed using functions of the diameter other than a power: for example, the function $h(x) = x^s\big(1/\log(1/x)\big)^t$ corresponds to the construction in 6.8.2 when $s = t = 1$. He also proposed using characteristics of the covering sets other than the diameter. The seminal paper [32] is required reading for the aspiring expert on fractals. Computation of the Hausdorff dimension using self-similarity appears even in Hausdorff's paper. It was carefully worked out by P. A. P. Moran [51] for subsets of $\mathbb{R}$, and by John Hutchinson [36] for subsets of $\mathbb{R}^d$. The open set condition is used in both of these papers.

The similarity dimension agrees with the Hausdorff and packing dimensions also for iterated function systems in (complete separable) metric spaces other than Euclidean space. But the open set condition must—in general [59]—be replaced by a ***strong open set condition***. In addition to the properties listed on p. 149, the closure $\overline{U}$ of the open set U must intersect the attractor K.

The packing measure was introduced by Claude Tricot (but see [34, Exercise (10.51), p. 145]), and advocated by Taylor & Tricot [61], Saint Raymond & Tricot [60], and Taylor [62]. Fractal dimensions in addition to those defined here can be found in [44, p. 357*ff*] and [45].

The generalization of self-similarity that we have called "graph self-similarity" has a complicated history. The version that is used here is based on the work of R. Daniel Mauldin and S. C. Williams [48]. The "two-part dust" was invented explicitly to illustrate the computation of the Hausdorff dimension for graph self-similar sets.

The pentadendrite was shown to me by my colleague W. A. McWorter. The terdragon comes from Chandler Davis and Donald Knuth [13].

Topological vs. Hausdorff Dimension

In Theorem 6.3.11, $\operatorname{Cov} S \le \dim S$ was proved only for some spaces S, such as compact spaces. In fact, it is true for any metric space S. A complete proof is in [18, Sect. 3.1]. As noted, that proof used Lebesgue integration. Recently, a proof without integration was published by M. G. Charalambous [9]. The covering dimension of a space S is a topological property of the space. That is, if S is homeomorphic to T, then $\operatorname{Cov} S = \operatorname{Cov} T$. The Hausdorff dimension is not a topological property. The spaces $(E^{(\omega)}, \varrho_r)$, where $E = \{0, 1\}$ is a two-letter alphabet, are all homeomorphic, but the Hausdorff dimension varies as r varies. We know that $\operatorname{Cov} S \le \dim S$. In fact, $\operatorname{Cov} S$ is the largest topologically invariant lower bound for $\dim S$:

Theorem 6.9.1. *Let S be a separable metric space. Then*

$$\operatorname{Cov} S = \inf \{ \dim T : T \textit{ is homeomorphic to } S \} .$$

I will prove here only the simplest case:

Proposition 6.9.2. *Let S be a separable zero-dimensional metric space. Then*

$$0 = \inf \{ \dim T : T \text{ is homeomorphic to } S \} .$$

Proof. First, S is homeomorphic to a subset T of the string space $\{0,1\}^{(\omega)}$, by Theorem 3.4.4. With metric ϱ_r, the space $\{0,1\}^{(\omega)}$ has Hausdorff dimension $\log 2/ \log(1/r)$. But $\lim_{r\to 0} \log 2/ \log(1/r) = 0$. □

The general case may be proved in a similar way [35, Theorem VII 5]. For example, the Menger sponge is a universal 1-dimensional space, so metric spaces homeomorphic to the Menger sponge, but with Hausdorff dimension close to 1 should be exhibited. The approximation shown in Fig. 6.9.3 suggests the idea. It is self-affine, rather than self-similar, so our methods of computation will not evaluate its Hausdorff dimension, however.

Two-Dimensional Lebesgue Measure Compared to Two-Dimensional Hausdorff Measure

According to Theorem 6.3.6 there is a positive constant c such that $\mathcal{H}^2(B) = c\,\mathcal{L}^2(B)$. We will show here that $c = 4/\pi$.

For the lower bound on $\mathcal{H}^2$, we need an interesting fact from two-dimensional geometry: Among all sets with a given diameter, the disk has the largest area. That is, if A is a set with diameter t, then $\overline{\mathcal{L}}^2(A) \leq \pi t^2/4$.* The proof requires some knowledge concerning convexity in two dimensions. First, A has the same diameter as its convex hull, so we may assume that A is convex. Similarly we may assume that A is closed. Choose any boundary point of A; let it be the origin of coordinates. A has a support line there,

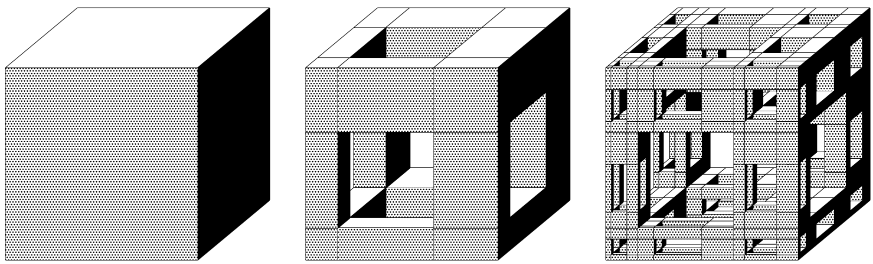

Fig. 6.9.3. Homeomorph of the Menger Sponge

* The corresponding fact for higher dimensions can be proved from Steiner's symmetrization construction. See, for example, [21, p. 107].

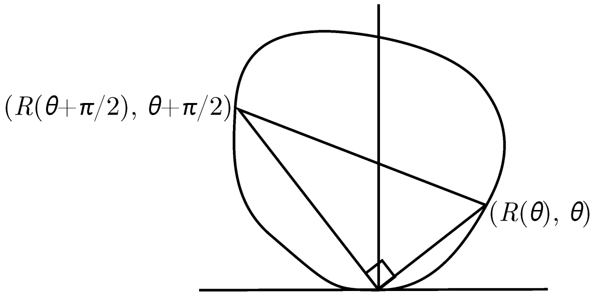

Fig. 6.9.4. Polar Coordinates

let it be the x-axis. Then the set A is given in polar coordinates (r, θ) by equations

$$0 \leq r \leq R(\theta), \quad 0 \leq \theta \leq \pi,$$

for some function R. Now A has diameter t. So the distance between the polar points $(R(\theta), \theta)$ and $(R(\theta + \pi/2), \theta + \pi/2)$ is at most t (Fig. 6.9.4). By the Pythagorean theorem, we may conclude

$$R(\theta)^2 + R(\theta + \pi/2)^2 \leq t^2.$$

Then the area may be computed in polar coordinates:

$$\begin{aligned}
\int_0^\pi \frac{R(\theta)^2}{2}\, \mathrm{d}\theta &= \int_0^{\pi/2} \frac{R(\theta)^2}{2}\, \mathrm{d}\theta + \int_{\pi/2}^{\pi} \frac{R(\theta)^2}{2}\, \mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{R(\theta)^2 + R(\theta + \pi/2)^2}{2}\, \mathrm{d}\theta \\
&\leq \int_0^{\pi/2} \frac{t^2}{2}\, \mathrm{d}\theta = \frac{\pi t^2}{4}.
\end{aligned}$$

So: a set $A \subseteq \mathbb{R}^2$ with diameter t has area at most $\pi t^2/4$. Then the argument given in Theorem 6.3.6, with $a = 4/\pi$, will show that $(4/\pi)\mathcal{L}^2(B) \leq \mathcal{H}^2(B)$ for any Borel set B.

For the upper bound, we use the Vitali covering theorem [23, Theorem 1.10] or [18, Theorem 1.3.7]. Let $b = \mathcal{H}^2(Q)$, where Q is the open unit square. Now a disk can be approximated inside and outside by little squares, so we have (by the argument in the proof of Theorem 6.3.6) $\mathcal{H}^2(B) = b\,\mathcal{L}^2(B)$ for all disks B. The collection of all closed disks with diameter $< \varepsilon$ and contained in the square Q satisfies the hypothesis of the Vitali theorem, so there is a countable disjoint set $\{\, B_i : i \in \mathbb{N} \,\}$ of them with $\mathcal{L}^2(Q \setminus \bigcup_{i \in \mathbb{N}} B_i) = 0$. But then, by the inequality $\mathcal{H}^2(B) \leq b\,\mathcal{L}^2(B)$, we know that $\mathcal{H}^2(Q \setminus \bigcup_{i \in \mathbb{N}} B_i)$ is also 0, so $\mathcal{H}^2_\varepsilon(Q \setminus \bigcup_{i \in \mathbb{N}} B_i) = 0$. Now

$$\begin{aligned}\mathcal{H}^2_\varepsilon\left(\bigcup_{i\in\mathbb{N}} B_i\right) &\le \sum_{i=1}^\infty (\operatorname{diam} B_i)^2 \\ &= \frac{4}{\pi}\sum_{i=1}^\infty \frac{\pi}{4}(\operatorname{diam} B_i)^2 \\ &= \frac{4}{\pi}\sum_{i=1}^\infty \mathcal{L}^2(B_i) \\ &= \frac{4}{\pi}\mathcal{L}^2(Q) = \frac{4}{\pi}.\end{aligned}$$

Therefore $b = \mathcal{H}^2(Q) = \mathcal{H}^2(\bigcup_{i\in\mathbb{N}} B_i) \le 4/\pi$.

So we have exactly $\mathcal{H}^2(B) = (4/\pi)\mathcal{L}^2(B)$ for all Borel sets B.

The same result is true in $\mathbb{R}^d$, namely $\mathcal{H}^d(B) = c_d\,\mathcal{L}^d(B)$, where c_d is the appropriate constant $1/\mathcal{L}^d(B_{1/2}(0))$.

The Sim-Value of a Mauldin–Williams Graph

The sim-value of a Mauldin–Williams graph exists and is unique. The proof of this fact will be given here. It requires some knowledge of linear algebra. In particular, it requires information from the Perron–Frobenius theorem (stated below).

Let A be a square matrix. The ***spectral radius*** of A is the maximum of the absolute values of all of the complex eigenvalues of A. We will write $\operatorname{sp\,rad} A$ for the spectral radius of A.

We will use some additional notation: $A \ge 0$ means all of the entries of A are nonnegative, and $A > 0$ means all of the entries of A are positive; $A \ge B$ means $A - B \ge 0$, and $A > B$ means $A - B > 0$. The matrix $A \ge 0$ is called ***reducible*** iff the rows and columns can be permuted (by the same permutation) so that A has the form

$$A = \begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square matrices (with at least one row each), and O is a rectangular matrix of zeros. If A is not reducible, then it is ***irreducible***. A column matrix with all entries 0 is $\mathbf{0}$, and a column matrix with all entries 1 is $\mathbf{1}$.

Here is (part of) the Perron-Frobenius theorem. See [27, Chap. XIII] for a proof.

Theorem 6.9.5. *Let $A \ge 0$ be an irreducible square matrix, and let $\lambda \in \mathbb{R}$. Then:*

(1) *If $\lambda = \operatorname{sp\,rad} A$, then there is a column matrix $\mathbf{x} > \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$.*
(2) *If there is a nonzero column matrix $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} = \lambda\mathbf{x}$, then $\lambda = \operatorname{sp\,rad} A$.*

(3) *If there is a nonzero column matrix* $\mathbf{x} \geq \mathbf{0}$ *with* $A\mathbf{x} < \lambda\mathbf{x}$*, then* $\lambda > \operatorname{sp\,rad} A$.
(4) *If there is a nonzero column matrix* $\mathbf{x} \geq \mathbf{0}$ *with* $A\mathbf{x} > \lambda\mathbf{x}$*, then* $\lambda < \operatorname{sp\,rad} A$.

Now we are in a position to prove that the dimension of a strongly connected, strictly contracting Mauldin-Williams graph exists and is unique.

Theorem 6.9.6. *Let* (V, E, i, t, r) *be a strongly connected, strictly contracting Mauldin–Williams graph. There is a unique number* $s \geq 0$ *such that positive numbers* q_v *exist satisfying*

$$q_u^s = \sum_{\substack{v \in V \\ e \in E_{uv}}} r(e)^s \, q_v^s$$

for all $u \in V$.

Proof. We will be using matrices with the rows (and columns) labeled by V. For each pair $u, v \in V$, and $s \geq 0$, let

$$A_{uv}(s) = \sum_{e \in E_{uv}} r(e)^s.$$

Let $A(s)$ be the matrix with entry $A_{uv}(s)$ in row u column v. Let $\Phi(s) = \operatorname{sp\,rad} A(s)$ be the spectral radius of the matrix $A(s)$. Now the matrix $A(s)$ has nonnegative entries. The entry $A_{uv}(s)$ is positive if and only if E_{uv} is not empty. Since the graph is strongly connected, the matrices $A(s)$ are irreducible. I will prove: (a) s-dimensional Perron numbers exist if and only if $\Phi(s) = 1$, and (b) the equation $\Phi(s) = 1$ has a unique solution in $[0, \infty)$.

First, suppose that s-dimensional Perron numbers exist, so that

$$q_u^s = \sum_{\substack{v \in V \\ e \in E_{uv}}} r(e)^s \, q_v^s$$

for all $u \in V$. Thus, if the column matrix $\mathbf{x}$ has entries q_v^s, then $\mathbf{x} > \mathbf{0}$ and $A(s)\mathbf{x} = \mathbf{x}$, so by the Perron-Frobenius theorem, 1 is the spectral radius of $A(s)$.

Conversely, suppose that $1 = \operatorname{sp\,rad} A(s)$. Then by the Perron-Frobenius theorem, there is a column matrix $\mathbf{x} > \mathbf{0}$ with $A(s)\mathbf{x} = \mathbf{x}$. If we write x_v for the entries of $\mathbf{x}$, then the numbers $q_v = x_v^{1/s}$ will be s-dimensional Perron numbers.

Next, I claim that the function Φ is continuous. Certainly the entries $A_{uv}(s)$ of the matrix are continuous functions of s. Fix a number s_0. Let $\mathbf{x} > \mathbf{0}$ be the Perron-Frobenius eigenvector: $A(s_0)\mathbf{x} = \Phi(s_0)\mathbf{x}$. Let the entries of $\mathbf{x}$ be x_v. Define positive numbers a, b by $a = \min x_v$, $b = \max x_v$. Suppose V has n elements, so the matrices are $n \times n$. Let $\varepsilon > 0$ be given. By the continuity of the entries A_{uv}, there exists $\delta > 0$ so that if $|s - s_0| < \delta$, then

$$|A_{uv}(s) - A_{uv}(s_0)| < \frac{a\varepsilon}{nb}$$

for all u, v. Now we have

$$\begin{aligned}\sum_v A_{uv}(s)x_v &= \sum_v A_{uv}(s_0)x_v + \sum_v \big(A_{uv}(s) - A_{uv}(s_0)\big)x_v \\ &\le \Phi(s_0)x_u + n\,\frac{a\varepsilon}{nb}\,b \le \big(\Phi(s_0) + \varepsilon\big)x_u.\end{aligned}$$

Therefore, by the Perron–Frobenius theorem, $\Phi(s) = \operatorname{sp\,rad} A(s) \le \Phi(s_0) + \varepsilon$. Similarly $\Phi(s) \ge \Phi(s_0) - \varepsilon$. This shows that Φ is continuous.

Since the graph is strongly connected, each row has at least one nonzero entry. So for each u there is v with $A_{uv}(0) \ge 1$. Therefore $A(0)\mathbf{1} \ge \mathbf{1}$, so that $\Phi(0) \ge 1$. The entries $A_{uv}(s) \to 0$ as $s \to \infty$, so for large enough s, we have $A_{uv}(s) \le 1/(2n)$ for all u, v, so that $A(s)\mathbf{1} \le (1/2)\mathbf{1}$, and thus $\Phi(s) \le 1/2$. Now by the intermediate value theorem, there is a solution s to the equation $\Phi(s) = 1$.

Finally, to prove the uniqueness, we will show that Φ is strictly decreasing. The derivative of A_{uv} is ≤ 0, and in fact < 0 unless A_{uv} is identically 0. Each row has at least one nonzero entry, so if $s > s_0$ and $\mathbf{x}$ is the Perron-Frobenius eigenvector for $A(s_0)$, we have $A(s)\mathbf{x} < A(s_0)\mathbf{x} = \Phi(s_0)\mathbf{x}$. So $\Phi(s) < \Phi(s_0)$. Therefore the function Φ is strictly decreasing. □

Exercise 6.9.7. Let (V, E, i, t, r) be a contracting, strongly connected Mauldin–Williams graph. Are the conclusions of Theorem 6.9.6 still correct?

To compute the dimension of a strictly contracting, strongly connected Mauldin–Williams graph, we would ordinarily find the numbers s such that 1 is an eigenvalue of the matrix $A(s)$. If that s is unique, it is the dimension. If not, then we find for each s the corresponding eigenvector for $A(s)$; only one of the values s will admit an eigenvector with all entries positive.

Remarks on the Exercises

Exercise 6.1.10: $f\colon S \to T$ is Lipschitz, and $A \subseteq S$. Say $\varrho\big(f(x), f(y)\big) \le b\varrho(x, y)$. If $\mathcal{D}$ is a countable cover of A by sets with diameter at most ε, then $\mathcal{D}' = \{\, f[D] : D \in \mathcal{D}\,\}$ is a countable cover of $f[A]$ by sets with diameter at most $b\varepsilon$. Now

$$\sum_{D\in\mathcal{D}} (\operatorname{diam} f[D])^s \le b^s \sum_{D\in\mathcal{D}} (\operatorname{diam} D)^s,$$

so $\overline{\mathcal{H}}^s_{b\varepsilon}(f[A]) \le b^s\, \overline{\mathcal{H}}^s_{\varepsilon}(A)$. Therefore $\dim f[A] \le \dim A$. The case of inverse Lipschitz is similar.

Exercise 6.3.12: Suppose $\operatorname{Cov} S \ge 1$. Then S does not have a base for the open sets consisting of clopen sets. So there is a point $a \in S$ and $\varepsilon > 0$ such that for $0 < r < \varepsilon$, the ball $B_r(a)$ is not clopen. The function $h\colon S \to \mathbb{R}$

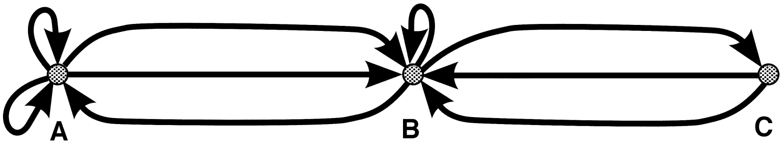

Fig. 6.9.8. All edges have value $(3-\sqrt{5})/2$

defined by $h(x) = \varrho(x,a)$ satisfies $|h(x) - h(y)| \le \varrho(x,y)$. Its range includes the interval $(0,\varepsilon)$. So we have $\dim S \ge \dim h[S] \ge \dim(0,\varepsilon) = 1$.

For Exercise 6.5.11: The terdragon boundary is made up of two copies of the 120-degree dragon of Fig. 1.5.8. The open set condition (Plate 13) is satisfied by an open set the shape of the filled-in fudgeflake (Fig. 1.5.8); it may be thought of as the union of three terdragons. Exercise 6.7.11. Fig. 6.9.8.

Exercise 6.7.12. 1.22.

Exercise 6.7.5. [48].

Exercise 6.7.14. The graph of Fig. 4.3.13. This is a special case of the situation considered in [17].

The fractile lines of the sandstone.
—*Scribner's Magazine*, April, 1893
(quoted in the *Oxford English Dictionary*)