9
Measurable Chromatic Number of the Plane

9.1 Definitions

As you know, the *length* of a segment $[a,b]$, $a < b$, on the line E^1 is defined as $b - a$. *Area* A of a rectangle $R = [a_1, b_1] \times [a_2, b_2]$, $a_i < b_i$ on the plane E^2 is defined as $A = (b_1 - a_1)(b_2 - a_2)$. The French mathematician Henri Léon Lebesgue (1875–1941) generalized the notion of area to a vast class of plane sets. In place of area, he used the term *measure*. For a set S in the plane, we define its *outer measure* $\mu^*(S)$ as follows:

$$\mu^*(S) = \inf \sum_i A(R_i), \tag{9.1}$$

with the infimum taken over all coverings of S by a countable sequence $\{R_i\}$ of rectangles. When the infimum exists, S is said to be *Lebesgue-measurable* or – since we consider here no other measures—*measurable* set—if for any set B in the plane, $\mu^*(B) = \mu^*(B \cap S) + \mu^*(B \backslash S)$. For a measurable set S, its measure is defined by $\mu(S) = \mu^*(S)$.

Any rectangle is measurable, and its measure coincides with its area. It is shown in every measure theory text that all closed sets and all open sets are measurable. Giuseppe Vitali (1875–1932) was first to show that in the standard system of axioms ZFC for set theory (Zermelo–Fraenkel system plus the Axiom of Choice), there are non-measurable subsets of the set R of real numbers.

We will use the same definition (9.1) for Lebesgue measure on the line E^1, when the infimum is naturally taken over all covering sequences $\{R_i\}$ of segments. For measure of S on the line we will use the symbol $l(S)$. Generalization of the notion of measure to n-dimensional Euclidean space E^n is straight forward; here we will use the symbol $\mu_n(S)$. In particular, for $n = 2$, we will omit the subscript and simply write $\mu(S)$.

9.2 Lower Bound for Measurable Chromatic Number of the Plane

While a graduate student in Great Britain, Kenneth J. Falconer proved the following important result [Fal]:

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Falconer's Theorem 9.1 Let $R^2 = \bigcup_{i=1}^{4} A_i$ be a covering of the plane by four disjoint measurable sets. Then one of the sets A_i realizes distance 1.

In other words, the measurable chromatic number χ_m of the plane is equal to 5, 6, or 7.

I found his 1981 publication [Fal1] to be too concise and not self-contained for the result that I viewed as very important. Accordingly, I asked Kenneth Falconer, currently a professor and dean at the University of St. Andrews in Scotland, for a more detailed and self-contained exposition. In February 2005, I received Kenneth's manuscript, hand-written especially for this book, which I am delighted to share with you.

Before we prove his result, we need to get armed with some basic definitions and tools of the measure theory.

A non-empty collection $\beth$ of subsets of E^2 is called *σ-field*, if $\beth$ is closed under taking complements and countable unions, i.e.,

*) if $A \in \beth$, then $E^2 \backslash A \in \beth$; and

**) if $A_1, A_2, \ldots, A_n, \ldots \in \beth$, then $\bigcup_{i=1}^{\infty} A_i \in \beth$.

Exercise 9.2 Show that any σ-field $\beth$ is closed under countable intersection and set difference. Also, show that $\beth$ contains the empty set $\emptyset$ and the whole space E^2.

It is shown in all measure theory textbooks that the collection of all measurable sets is a σ-field. The intersection of all σ-fields containing the closed sets is a σ-field containing the closed sets, the minimal such σ-field with respect to inclusion. Its elements are called *Borel sets*. Since closed sets are measurable and the collection of all measurable sets is a σ-field, it follows that all Borel sets are measurable.

(Observe that in place of the plane E^2 we can consider the line E^1 or an n-dimensional Euclidean space E^n, and define their Borel sets.)

The following notations will be helpful:

$C(x, r)$ – Circle with center at x and radius r;
$B(x, r)$ – Circular disk (or ball) with center at x and radius r.

For a measurable set S and a point x, we define the *Lebesgue density*, or simply *density*, of S at x as follows:

$$D(S, x) = \lim_{x \to 0} \frac{\mu(S \cap B(x, r))}{\mu(B(x, r))},$$

where $\mu\,(B(x, r))$ is, of course, equal to πr^2.

Lebesgue Density Theorem (LDT) 9.3 For a measurable set $S \subset E^2$, the density $D(S, x)$ exists and equals 1 if $x \in S$ and 0 if $x \in R^2 \backslash S$, except for a set of points x of measure 0.

For a measurable set A, denote

$$\tilde{A} = \{x \in A : D(A, x) = 1\}.$$

Then due to LDT, we get $\mu(\tilde{A} \triangle A) = 0$, i.e., $\tilde{A}$ is 'almost the same' as A.[27] Observe also that $\mu\,(S \cap B(x, r))$ is a continuous function of x for $r > 0$; therefore, $\tilde{A}$ is a Borel set.

We will define the *density boundary* of a set A as follows:

$$\partial A = \{x : D(A, x) \neq 0, 1 \text{ or does not exist}\}.$$

By LDT,

$$\mu(\partial A) = 0.$$

You can find on your own or read in [Cro] the proof of the following tool:

Tool 9.4 For a measurable set $A \subset R^2$, such that both $\mu(A) > 0$ and $\mu(R^2 \backslash A) > 0$, we have $\partial A \neq \emptyset$.

Tool 9.5 If $R^2 = \bigcup_{i=1}^{4} A_i$ is a covering of the plane by four disjoint measurable sets, then $\bigcup_{i=1}^{4} \tilde{A}_i$ is a disjoint union with the complement $\mathfrak{M} \equiv \bigcup_{i=1}^{4} \partial A_i$.

Proof follows from Tool 9.4 and the observation that if $x \in \partial A_i$ then also $x \in \partial A_j$ for some $j \neq i$. ∎

The next tool claims the existence of two concentric circles with the common center in $\mathfrak{M}$, which intersect $\mathfrak{M}$ in length 0.

Tool 9.6 Let $\mathfrak{M}$ be as in Tool 9.5; there exists $x \in \mathfrak{M}$ such that

$$l(C(x, 1) \cap \mathfrak{M}) = l\left(C(x, \sqrt{3}) \cap \mathfrak{M}\right) = 0.$$

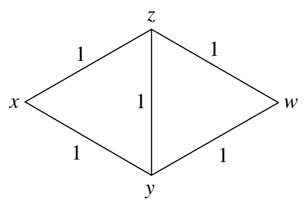

Fig. 9.1

[27] Here $A \triangle B$ stands for the symmetric difference of these two sets, i.e., $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

I will omit the proof, but include Falconer's insight: "The point of this lemma is that if we place the "double equilateral triangle" [Fig. 9.1] of side 1 in almost all orientations with a vertex at x, the point x essentially has "2 colors" in any coloring of the plane, and other points just one color. (Note $|xw| = \sqrt{3}$.)"

Tool 9.7 Let $R^2 = \bigcup_{i=1}^{4} A_i$ be a covering of the plane by four disjoint measurable sets, none of which realizes distance 1. Let $x \in \mathfrak{M}$ as in Tool 9.6, say without loss of generality $x \in \partial A_1$ and $x \in \partial A_2$. Then $l\left(C(x, \sqrt{3})\backslash(\tilde{A}_1 \cup \tilde{A}_2\right) = 0$.

Proof Since $x \in \partial A_1$ and $x \in \partial A_2$, there exists $\varepsilon > 0$ such that

(1) $\varepsilon < \frac{\mu(A_1 \cap B(x,r))}{\pi r^2} < 1 - \varepsilon$ for some arbitrarily small r, and

(2) $\varepsilon < \frac{\mu(A_2 \cap B(x,r))}{\pi r^2} < 1 - \varepsilon$ for some arbitrarily small r.

Consider the diamond (Fig. 9.1) consisting of two unit equilateral triangles xyz and yzw, where x is the point fixed in the statement of this tool, and $y,\ z,\ w \notin \mathfrak{M}$ (this happens for almost all orientations of the diamond, by Tool 9.6). Thus suppose $y \in \tilde{A}_{i(y)},\ z \in \tilde{A}_{i(z)},\ w \in \tilde{A}_{i(w)}$, where $i(y),\ i(z),\ i(w) \in \{1,\ 2,\ 3,\ 4\}$. For sufficiently small r, say $r < r_0$, we get:

(3) $1 - \frac{\varepsilon}{4} < \frac{\mu\left(A_{i(y)} \cap B(y,r)\right)}{\pi r^2} \leq 1$;

(4) $1 - \frac{\varepsilon}{4} < \frac{\mu\left(A_{i(z)} \cap B(z,r)\right)}{\pi r^2} \leq 1$;

(5) $1 - \frac{\varepsilon}{4} < \frac{\mu\left(A_{i(w)} \cap B(w,r)\right)}{\pi r^2} \leq 1$.

We can now choose $r < r_0$ such that (1) holds (as well as (3), (4), (5)). Let v be a vector going from the origin to a point in $B(0, r)$ and consider translation of the diamond x, y, z, w through v, i.e., to the diamond $x + v,\ y + v,\ z + v,\ w + v$. Now (1), (3), (4), (5) imply that

$$\frac{1}{\pi r^2}\mu\left(\left\{v \in B(0,r) : x + v \in A_1, y + v \in A_{i(y)}, z + v \in A_{i(z)}, w + v \in A_{i(w)}\right\}\right)$$
$$> \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} > 0.$$

Thus, we can choose $v \in B(0, r)$ such that $x + v \in A_1,\ y + v \in A_{i(y)},\ z + v \in A_{i(z)},\ w + v \in A_{i(w)}$. Since by our assumption none of the sets $A_i, i = 1, 2, 3, 4$ realizes distance 1, we conclude (by looking at the translated diamond) that $1 \neq i(y),\ 1 \neq i(z), i(y) \neq i(z),\ i(z) \neq i(w)$, and $i(w) \neq i(y)$.

The same argument, using (2), (3), (4), (5) produces $2 \neq i(y), 2 \neq i(z), i(y) \neq i(z),\ i(z) \neq i(w)$, and $i(w) \neq i(y)$. Therefore, $i(y),\ i(z)$ are 3 and 4 in some order, and thus $i(w) = 1$ or 2, i.e., $w \in \tilde{A}_1$ or $w \in \tilde{A}_2$.

By Tool 9.6, this holds for almost every orientation of the diamond. Since $|xw| = \sqrt{3}$, we conclude that for almost all $w \in C(x, \sqrt{3})$, we get $w \in \tilde{A}_1$ or $w \in \tilde{A}_2$. Thus, $l\left(C(x, \sqrt{3})\backslash\left(\tilde{A}_1 \cup \tilde{A}_2\right)\right) = 0$, as required. ∎

Tool 9.8 Let C be a circle of radius $r > \frac{1}{2}$ and let E_1, E_2 be disjoint measurable subsets of C such that $l\,(C \backslash (E_1 \cup E_2) = 0$. Then if $\varphi = 2\sin^{-1}\left(\frac{1}{2r}\right)$ is an irrational multiple of π, either E_1 or E_2 contains a pair of points distance 1 apart.

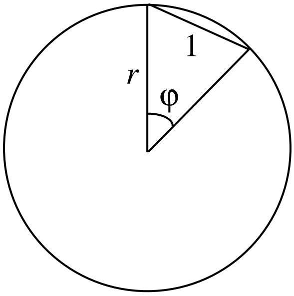

Fig. 9.2

Proof Assume that neither E_1 or E_2 contains a pair of points distance 1 apart. Parameterize C (Fig. 9.2) by angle θ (mod 2π).

Let $l(E_1) > 0$, then by LDT, there is θ and $\varepsilon > 0$ such that

$$l(E_1 \cap (\theta - \varepsilon, \theta + \varepsilon)) > \frac{3}{4} 2\varepsilon.$$

Let θ_1 be an angle. Since φ is an irrational multiple of π, there is a positive integer n such that

$$|\theta_1 - (2n\varphi + \theta)| < \frac{1}{4}\varepsilon \text{ (mod } 2\,\pi).$$

Since neither E_1 or E_2 contain a pair of points distance 1 apart, we get (with angles counted mod 2π):

$$l\,(E_1 \cap (\theta + k\varphi - \varepsilon, \theta + k\varphi + \varepsilon)) = l\,(E_1 \cap (\theta - \varepsilon, \theta + \varepsilon)) \text{ for even } k, \text{ and}$$
$$l\,(E_1 \cap (\theta + k\varphi - \varepsilon, \theta + k\varphi + \varepsilon)) = 2\varepsilon - l\,(E_1 \cap (\theta - \varepsilon, \theta + \varepsilon)) \text{ for odd } k.$$

In particular, $l\,(E_1 \cap (\theta + 2n\varphi - \varepsilon, \theta + 2n\varphi + \varepsilon)) > \frac{3}{4} 2\varepsilon$, thus

$$l\,(E_1 \cap (\theta_1 - \varepsilon, \theta + \varepsilon)) > \frac{3}{4} 2\varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \varepsilon.$$

Hence for all θ_1,

$$\frac{l\,(E_1 \cap (\theta_1 - \varepsilon, \theta + \varepsilon))}{2\varepsilon} \geq \frac{1}{2},$$

and by LDT $l\left(C\backslash E_1\right) = 0$. This means that E_1 is almost all of C, and therefore contains a pair of points until distance apart, a contradiction. ∎

Surprisingly, we need a tool from abstract algebra, or number theory.

Tool 9.9 For any positive integer m, $\left(1 - i\sqrt{11}\right)^{2m} \neq (-12)^m$.

Proof It suffices to note that $Q\left(\sqrt{-11}\right)$ is an Euclidean quadratic field, therefore, its integer ring $Z\left(\sqrt{-11}\right)$ (with units $+1/-1$) has unique factorization. (See Chapters 7 and 8 in the standard abstract algebra textbook [DF] for a proof).

I believe that an alternative proof is possible: it should be not hard to show that the left side cannot be an integer for any m. ∎

Now we are ready to prove Falconer's Theorem 9.1.

Proof of Falconer's Theorem 9.10 Let $R^2 = \bigcup_{i=1}^{4} A_i$ be a covering of the plane by four disjoint measurable sets, none of which realizes distance 1. Due to Tool 9.6, there is $x \in \mathfrak{M}$ such that $l\left(C(x, \sqrt{3})\backslash(\tilde{A}_1 \cup \tilde{A}_2\right) = 0$. Taking $E_1 = \tilde{A}_1$, $E_2 = \tilde{A}_2$ and $r = \sqrt{3}$, we get, the desired result by Tool 9.8—if only we can prove that $\varphi = \sin^{-1}\left(\frac{1}{2\sqrt{3}}\right)$ is an irrational multiple of π. We have $\sin\theta = \frac{1}{2\sqrt{3}}$; $\cos\theta = \frac{\sqrt{11}}{2\sqrt{3}}$. Assume $m\theta$ is an integer multiple of 2π for some integral $2m$. Then

$$\left(\frac{\sqrt{11}}{2\sqrt{3}} + i\frac{1}{2\sqrt{3}}\right)^{2m} = 1$$

or

$$\left(1 - i\sqrt{11}\right)^{2m} = (-12)^m.$$

We are done, as the last equality contradicts Tool 9.9. ∎

9.3 Kenneth J. Falconer

I am always interested in learning about the life and personality of the author whose result impressed me, aren't you! Accordingly, I asked Kenneth to tell me about himself and his life. The following account comes from his September 30, 2005, e-mail to me.

> I was born on 25th January 1952 at Hampton Court on the outskirts of London (at a maternity hospital some 100 metres from the gates of the famous Palace). This was two weeks before Queen Elizabeth II came to the throne and when food rationing was still in place. My father had served in India for 6 years during the war while my mother brought up my brother, 12 years my senior, during the London blitz. My parents

were both school teachers, specializing in English, my brother studied history before becoming a Church of England minister, and I was very much the 'black sheep' of the family, having a passionate interest in mathematics and science from an early age. . .

I gained a scholarship to Corpus Christi College, Cambridge to read mathematics and after doing well in the Mathematical Tripos I continued in Cambridge as a research student, supervised by Hallard Croft. I worked mainly on problems in Euclidean geometry, particularly on convexity and of tomography (the mathematics of the brain scanner) and obtained my PhD in 1977.

I had the good fortune to obtain a Research Fellowship at Corpus Christi College, where I continued to study geometrical problems, including the fascinating problem of the chromatic number of the plane, showing in particular that the chromatic number of a measurable colouring of the plane was at least 5. Also around this time I worked on generalizations of the Kakeya problem (the construction of plane sets of zero area containing a line segment in every direction). Thus I encountered Besicovitch's beautiful idea of thinking of such sets as duals of what are now termed 'fractals', with directional and area properties corresponding to certain projections of the fractals. This led to my 'digital sundial' construction – a subset of R^3 with prescribed projections in (almost) all directions. . .

In 1980 I moved to Bristol University as a Lecturer, where the presence of theoretical physicist Michael Berry, and analyst John Marstrand were great stimulii. Here I started to work on geometric measure theory, or fractal geometry, in particular looking at properties of Hausdorff measures and dimensions, and projections and intersections of fractals. . .

It became clear to me that much of the classical work of Besicovitch and his School on the geometry of sets and measures had been forgotten, and in 1985 I published my first book 'The Geometry of Fractal Sets' to provide a more up to date and accessible treatment. This was around the time that fractals were taking the world by storm, following Mandelbrot's conceptually foundational work publicised in his book 'The Fractal Geometry of Nature' which unified the mathematics and the scientific applications of fractals. My book led to requests for another at a level more suited to postgraduate and advanced undergraduate students and in 1990 I published 'Fractal geometry – Mathematical Foundations and Applications' which has been widely used in courses and by researchers, and has been referred to at conferences as 'the book from which we all learnt our fractal mathematics'. A sequel 'Techniques in Fractal Geometry' followed in 1998. In collaboration with Hallard Croft and Richard Guy, I also authored 'Unsolved Problems in Geometry', a collection of easy to state unsolved geometrical problems. Happily (also sadly!) many of the problems in the book are no longer unsolved!. . .

In 1993 I was appointed Professor at the University of St Andrews in Scotland, where I have been ever since. Although St Andrews is a small town famous largely for its golf, the University has a thriving mathematics department, in particular for analysis and combinatorial algebra, to say nothing of its renowned History of Mathematics web site. I became Head of the School of Mathematics and Statistics in 2001, with the inevitable detrimental effect on research time. I was elected a Fellow of the Royal Society of Edinburgh in 1998, and to the Council of the London Mathematical Society in 2000. . .

My main leisure activity is long distance walking and hillwalking. I have climbed all 543 mountains in Britain over 2500 feet high. I am a keen member of the Long Distance Walkers Association, having been Editor of their magazine 'Strider' from 1986–91 and Chairman from 2000–03. I have completed the last 21 of the LDWA's annual hundred mile non-stop cross-country walks in times ranging from 26 to 32 hours.