8
Chromatic Number of the Plane in Special Circumstances

As you know from Chapters 4 and 6, in 1973, 3 years after Dmitry E. Raiskii, Douglas R. Woodall published the paper [Woo1] on problems related to the chromatic number of the plane. In the paper he gave his own proofs of Raiskii's inequalities of Problems 4.1 and 6.1. In the same paper, Woodall also formulated and attempted to prove a lower bound for the chromatic number of the plane for the special case of map-type coloring of the plane. This was the main result of [Woo1]. However, in 1979 the mathematician from the University of Aberdeen Stephen Phillip Townsend found an error in Woodall's proof, and constructed a counterexample demonstrating that one essential component of Woodall's proof was false. Townsend had also found a proof of this statement, which was much more elaborate than Woodall's unsuccessful attempt.

The intriguing history of this discovery and Townsend's wonderful proof are a better fit in Chapter 24, as a part of our discussion of map coloring—do not overlook them! Here I will formulate an important corollary of Townsend's proof.

Chromatic Number of Map-Colored Plane 8.1 The chromatic number of the plane under map-type coloring is 6 or 7.

Woodall showed that this result implies one more meritorious statement:

Closed Chromatic Number of the Plane 8.2 ([Woo1]). The chromatic number of the plane under coloring with closed monochromatic sets is 6 or 7.

I do not like to use the Greek word "lemma" since there is an appropriate English word "tool" :-). And I would like to offer my readers the following tool from topology to prove on their own. We will use this tool in the proof that follows.

Tool 8.3 If a bounded closed set S does not realize a distance d, then there is $\varepsilon > 0$ such that S does not realize any distance from the segment $[d - \varepsilon,\ d + \varepsilon]$.

Proof of Result 8.2 [Woo1]: Assume that the union of closed sets $A_1,\ A_2, \ldots,\ A_n$ covers the plane and for each i the set A_i does not realize a distance d_i. Place onto the plane a unit square lattice L, and choose an arbitrary closed unit square U of L. Choose also i from the set $\{1,\ 2, \ldots,\ n\}$. Denote by $C(U)_i$ the closed set that contains all points of the plane that are at most distance d_i from a point in U. The set

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$A_i \cap C(U)_i$ is closed and bounded, thus by Tool 8.3 there is $\varepsilon_i(U)$ such that no two points of A_i, at least one of which lies in U, realize any distance from the segment

$$[d_i - \varepsilon_i(U), d_i + \varepsilon_i(U)]. \tag{8.1}$$

Denote by $\varepsilon(U)$ the minimum of $\varepsilon_i(U)$ over all $i = 1, 2, \ldots, n$.

Now for the square U we choose a positive integer $m(U)$ such that

$$\frac{1}{2^{m(U)}}\sqrt{2} < \frac{1}{2}\varepsilon(U). \tag{8.2}$$

On the unit square U we place a square lattice L' of little closed squares u of side $\frac{1}{2^{m(U)}}$. The inequality (8.2) guarantees that the diagonal of u is shorter than half of our epsilon $\varepsilon(U)$.

For each little square u contained in each unit square U of the entire plane, we determine $f(u) = \min\{i : u \cap A_i \neq \emptyset\}$, and then for each $i = 1, 2, \ldots, n$ define the monochromatic color set of our new n-coloring of the plane as follows:

$$B_i = \bigcup_{f(u)=i} u. \tag{8.3}$$

As unions of closed squares u, each B_i is closed, and all B_i together cover the plane. The interiors of these n sets B_i are obviously disjoint. All there is left to prove is that the set B_i does not realize the distance d_i. Indeed, assume that the points b,c of B_i are distance d_i apart. The points b,c belong to little squares u_1,u_2 respectively, each little square of side $\frac{1}{2^{m(U)}}$. Due to the definition (8.3) of B_i, the squares u_1,u_2 contain points a_1,a_2 from A_i respectively. With vertical bars denoting the distance between two points, and by utilizing the inequality (8.2) we get:

$$|b,c| - \varepsilon(U) < |a_1,a_2| < |b,c| + \varepsilon(U),$$

i.e.,

$$d_i - \varepsilon(U) < |a_1,a_2| < d_i + \varepsilon(U),$$

which contradicts (8.1).

Thus, the chromatic number under the conditions of result 8.2 is not smaller than the chromatic number under the conditions of result 8.1. ■

During 1993–1994 a group of three young undergraduate students Nathanial Brown, Nathan Dunfield, and Greg Perry, in a series of three essays, (their first publications,) proved on the pages of *Geombinatorics* [BDP1], [BDP2], [BDP3][26] that a similar result is true for coloring with open monochromatic sets. Now the youngsters are professors of mathematics, Nathan at the University of Illinois at Urbana-Champaign, and Nathanial at Pennsylvania State University.

Open Chromatic Number of the Plane 8.4 (Brown–Dunfield–Perry). The chromatic number of the plane under coloring with open monochromatic sets is 6 or 7.

[26] The important problem book [BMP] mistakenly cites only one of these series of three papers. It also incorrectly states that the authors proved only the lower bound 5, whereas they raised the lower bound to 6.