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Ramsey Theory Before Ramsey: Schur's Coloring Solution of a Colored Problem and Its Generalizations

32.1 Schur's Masterpiece

Probably no one remembered—if anyone ever noticed—Hilbert's 1892 lemma by the time the second Ramseyan type result appeared in 1916 as a little noticed assertion in number theory. Its author was Issai Schur.

Our interest here lies in the result he obtained during 1913–1916 when he worked at the University of Bonn as the successor to Felix Hausdorff.[2] There he wrote his pioneering paper [Sch]: *Über die Kongruenz* $x^m + y^m \equiv z^m$ (mod.p). In it Schur offered another proof of a theorem by Leonard Eugene Dickson from [Dic1], who was trying to prove Fermat's Last Theorem. For use in his proof, Schur created, as he put it, "a very simple lemma, which belongs more to combinatorics than to number theory."

Nobody then asked questions of the kind Issai Schur posed and solved in his 1916 paper [Sch]. Consequently, nobody appreciated this result much when it was published. Now it shines as one of the most beautiful, classic theorems of mathematics. Its setting is positive integers colored in finitely many colors. The beautiful solution I am going to present utilizes coloring as well. I have got to tell you how I received this solution (see [Soi9] for more details).

In August 1989 I taught at the International Summer Institute in Long Island, New York. A fine international contingent of gifted high school students for the first time included a group from the Soviet Union. Some members of this group turned out to be Mathematics Olympiads "professionals," winners of the Soviet Union National Mathematical Olympiads in Mathematics and in Physics. There was nothing in the Olympiad genre that they did not know or could not solve. I offered them and everyone else an introduction to certain areas of combinatorial geometry.

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[2] Both Alfred Brauer [Bra2] and Walter Ledermann [Led] reported 1911 as the time when Schur became an *Extraordinarius* in Bonn, while Schur's daughter Mrs. Hilde Abelin-Schur [Abe1] gave me 1913 as the time her family moved to Bonn. The Humboldt University's Archive contains personnel forms (Archive of Humboldt University at Berlin, document UK Sch 342, Bd.I, Bl.25) filled up by Issai Schur himself, from which we learn that he worked at the University of Bonn from April 21, 1913 until April 1, 1916, when he returned to Berlin.

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We quickly reached the forefront of mathematics, full of open problems. Students shared with me their favorite problems and solutions as well. Boris Dubrov from Minsk, Belarus, told me about a visit to Moscow by the American mathematician Ronald L. Graham. During his interview with the Russian mathematics magazine *Kvant*, Graham mentioned a beautiful problem that dealt with 2-colored positive integers. Boris generalized the problem to n-coloring, strengthened the result and proved it all! He gave me this generalized problem for the Colorado Mathematical Olympiad.

This problem was the celebrated Schur Theorem of 1916, rediscovered by Boris, with his own proof that was more beautiful than Schur's original proof, but which was already known. Paul Erdős received this proof from Vera T. Sós, and included it in his talk at the 1970 International Congress of Mathematicians in Nice, France [E71.13]. Chances of receiving a solution of such a problem during the Olympiad were very slim. Yet, the symbolism of a Soviet kid offering an astonishingly beautiful problem (and solution!) to his American peers was so great that I decided to include this problem as an additional Problem 6 (Colorado Mathematical Olympiad usually offers 5 problems).

Schur's Theorem 32.1 ([Sch]) For any positive integer n there is an integer $S(n)$ such that any n-coloring of the initial positive integers array $[S(n)]$ contains integers a, b, c of the same color such that $a + b = c$.

In this case we call a, b, c a *monochromatic solution* of the equation $x + y = z$. In fact, Schur proved by induction that $S(n) = n!e$ would work.[3]

Proof of Schur's Theorem Let all positive integers be colored in n colors $c_1, c_2, \ldots, c_n$. Due to Problem 27.13, there is $S(n)$ such that any n-coloring of edges of the complete graph $K_{S(n)}$ contains a monochromatic triangle K_3.

Construct a complete graph $K_{S(n)}$ with its vertices labeled with integers from the initial integers array $[S(n)] = \{1, 2, \ldots, S(n)\}$. Now color the edges of $K_{S(n)}$ in n colors as follows: let i and j, $(i > j)$, be two vertices of $K_{S(n)}$, color the edge ij in precisely the color of the integer $i - j$ (remember, all positive integers were colored in n colors!). We get a complete graph $K_{S(n)}$ whose edges are colored in n colors. By Problem 27.13, $K_{S(n)}$ contains a triangle ijk, $i > j > k$, whose all three edges ij, jk, and ik are colored in the same color (Fig. 32.1).

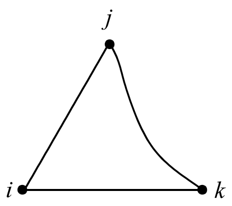

Fig. 32.1

[3] Here e stands for the base of natural logarithms.

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle *ijk* are colored in the same color, the integers a, b, and c are colored in the same color in the original coloring of the integers (this is how we colored the edges of $K_{s(n)}$). In addition, we have the following equality:

$$a + b = (i - j) + (j - k) = i - k = c$$

We are done! ■

The result of the Schur Theorem can be strengthened by an additional clever trick in the proof.

Strong Version of Schur's Theorem 32.2 For any positive integer n there is an integer $S^*(n)$ such that any n-coloring of the initial positive integers array $[S^*(n)]$ contains distinct integers *a, b, c* of the same color such that $a + b = c$.

Proof Let all positive integers be colored in n colors $c_1, c_2, \ldots, c_n$. We add n more colors $c'_1, c'_2, \ldots, c'_n$ different from the original n colors and construct a complete graph $K_{S(2n)}$ with the set of positive integers $\{1, 2, \ldots, S(2n)\}$ labeling its vertices (See the definition of $S(2n)$ in the proof of Theorem 32.1). Now we are going to color the edges of $K_{S(2n)}$ in $2n$ colors.

Let i and j, $(i > j)$, be two vertices of $K_{S(2n)}$, and c_p be the color in which the integer $i - j$ is colored, $1 \leq p \leq n$ (remember, all positive integers are colored in n colors $c_1, c_2, \ldots, c_n$). Then we color the edge *ij* in color c_p if the number $\left\lfloor \frac{i}{i-j} \right\rfloor$ is even, and in color c'_p if the number $\left\lfloor \frac{i}{i-j} \right\rfloor$ is odd (for a real number r, the symbol $\lfloor r \rfloor$, as usual, denotes the largest integer not exceeding r).

We get a complete graph $K_{S(2n)}$ whose edges are colored in $2n$ colors. By Problem 27.13, $K_{S(2n)}$ contains a triangle *ijk*, $i > j > k$, whose all three edges *ij, jk*, and *ik* are colored in the same color (Fig. 32.1).

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle *ijk* are colored in the same color, from the definition of coloring of edges of $K_{S(2n)}$ it follows that in the original coloring of positive integers, the integers a, b, and c were colored in the same color. In addition we have

$$a + b = (i - j) + (j - k) = i - k = c.$$

We are almost done. We only need to show (our additional pledge!) that the numbers *a, b, c* are all distinct. In fact, it suffices to show that $a \neq b$. Assume the opposite: $a = b$ and c_p is the color in which the number $a = b = i - j = j - k$ is colored. But then

$$\left\lfloor \frac{i}{i-j} \right\rfloor = \left\lfloor 1 + \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{j-k} \right\rfloor,$$

i.e., the numbers $\left\lfloor \frac{i}{i-j} \right\rfloor$ and $\left\lfloor \frac{j}{j-k} \right\rfloor$ have different parity, thus the edges *ij* and *jk* of the triangle *ijk* must have been colored in different colors. This contradiction to the fact that all three edges of the triangle *ijk* have the same color proves that $a \neq b$. Theorem 32.2 is proven. ■

32.2 Generalized Schur

It is fitting that the Schur Theorem was generalized by one of Schur's best students—Richard Rado. Rado calls a linear equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad (^*)$$

regular, if for any positive integer r, no matter how all positive integers are colored in r colors, there is a monochromatic solution of the equation (*). As before, we say that a solution $x_1,\ x_2, \ldots, x_n$ is *monochromatic*, if all numbers $x_1,\ x_2, \ldots, x_n$ are colored in the same color.

For example, the Schur Theorem 32.1 proves precisely that the equation $x + y - z = 0$ is regular. In 1933 Richard Rado, among other results, found the following criterion:

Rado's Theorem 32.3 (A particular case of [Rad1]) Let E be a linear equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$, where all $a_1,\ a_2, \ldots, a_n$ are integers. Then E is regular if and only if some non-empty subset of the coefficients a_i sums up to zero.

For example the equation $x_1 + 3x_2 - 2x_3 + x_4 + 10x_5 = 0$ is regular because $1 + 3 - 2 = 0$.

Problem 32.4 (*trivial*) Schur Theorem 32.1 follows from Rado's Theorem.

Richard Rado found regularity criteria for systems of homogeneous equations as well. His fundamental contributions to and influence on Ramsey Theory is hard to overestimate. I have just given you a taste of his theorems here. For more of Rado's results read his papers [Rad1], [Rad2], and others, and the monograph [GRS2]. Instead of a formal biographical data, I prefer to include here a few passages about Richard Rado (1906, Berlin—1989, Henley-on-Thames, Oxfordshire) written by someone who knew Rado very well—Paul Erdős—from the latter's paper *My joint work with Richard Rado* [E87.12]:

> I first became aware of Richard Rado's existence in 1933 when his important paper *Studien zur Kombinatorik* [Rado's Ph.D. thesis under Issai Schur] [Rad1] [4] appeared. I thought a great deal about the many fascinating and deep unsolved problems stated in this paper but I never succeeded to obtain any significant results here and since I have to report here about our joint work I will mostly ignore these questions. Our joint work extends to more than 50 years; we wrote 18 joint papers, several of them jointly with A. Hajnal, three with E. Milner, one with F. Galvin, one with Chao Ko, and we have a book on partition calculus with A. Hajnal and A. Mate. Our most important work is

[4] Two years later, Rado obtained his second Ph.D. degree at Cambridge under G. H. Hardy.

undoubtedly in set theory and, in particular, the creation of the partition calculus. The term partition calculus is, of course, due to Rado. Without him, I often would have been content in stating only special cases. We started this work in earnest in 1950 when I was at University College and Richard in King's College. We completed a fairly systematic study of this subject in 1956, but soon after this we started to collaborate with A. Hajnal, and by 1965 we published our GTP (Giant Triple Paper - this terminology was invented by Hajnal) which, I hope, will outlive the authors by a long time. I would like to write by centuries if the reader does not consider this as too immodest...

I started to correspond with Richard in late 1933 or early 1934 when he was a German [Jewish] refugee in Cambridge. We first met on October 1, 1934 when I first arrived in Cambridge from Budapest. Davenport and Richard met me at the railroad station in Cambridge and we immediately went to Trinity College and had our first long mathematical discussion...

Actually our first joint paper was done with Chao Ko and was essentially finished in 1938. Curiously enough it was published only in 1961. One of the reasons for the delay was that at that time there was relatively little interest in combinatorics. Also, in 1938, Ko returned to China, I went to Princeton and Rado stayed in England. I think we should have published the paper in 1938. This paper "Intersection theorems for systems of finite sets" became perhaps our most quoted result.

It is noteworthy to notice how differently people see the same fact. For Richard Rado, Schur's Theorem was about monochromatic solutions of a homogeneous linear equation $x + y - z = 0$, and so Rado generalized the Schur Theorem to a vast class of homogeneous linear equations (Rado's Theorem 32.3) and systems of homogeneous linear equations [Rad1]. Three other mathematicians saw Schur's Theorem quite differently. This group consisted of Jon Folkman, a young Rand Corporation scientist; Jon Henry Sanders, the last Ph.D. student of the legendary Norwegian graph theorist Øystein Ore at Yale (B.A. 1964 Princeton University; Ph.D. 1968, Yale University); and Vladimir I. Arnautov, a 30-year-old Moldavian topological ring theorist. For the three, the Schur Theorem spoke about monochromatic sets of symmetric sums

$$\{a_1, a_2, a_1 + a_2\} = \left\{ \sum_{i=1,2} \varepsilon_i a_i : \varepsilon_i = 0, 1; \varepsilon_1 \varepsilon_2 \neq 0 \right\}.$$

Consequently, the three proved a different —from Rado's kind —Schur's Theorem generalization and paved the way for further important developments. I see therefore no choice at all but to name the following fine theorem by its three inventors. This may surprise those of you accustomed to different attributions. I will address these concerns later in this chapter.

Arnautov–Folkman–Sander's Theorem 32.5 ([San], [Arn]) For any positive integers m and n there exists an integer $AFS(m, n)$ such that any m-coloring of the initial integers array $[AFS(m, n)]$ contains an n-element subset $S \subset [AFS(m, n)]$ such that the set $\left\{ \sum_{x \in F} x : \varnothing \neq F \subseteq S \right\}$ is monochromatic.

Problem 32.6 (*trivial*) Show that both Hilbert's Theorem 31.1 and Schur's Theorem 32.1 follow from Arnautov–Folkman–Sander's Theorem 32.5.

In their important 1971 paper [GR1] Ron Graham and Bruce Rothschild, having vastly generalized some theorems of the Ramsey Theory (beyond the scope of this book), conjectured that the word "finite" in reference to the subset S in Arnautov–Folkman–Sander's Theorem 32.5 can be omitted. Paul Erdős gave a high praise to their conjecture at his 1971 talk in Fort Collins, Colorado, published in 1973 [E73.21]:

> Graham and Rothschild ask the following beautiful question: split the integers into two classes. Is there always an infinite sequence so that all the finite sums $\sum \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1 (not all $\varepsilon_i = 0$) all belong to the same class? ... This problem seems very difficult.

Surprisingly, the proof in the positive came in soon. In the paper submitted in 1972 and published in 1974 [Hin], Neil Hindman proved Graham–Rothschild's conjecture. While Graham–Rothschild's conjecture asked for an initial generalization for two colors (probably to test waters before diving into the general case), Hindman proved the result for any finite number of colors, thus fully generalizing Arnautov–Folkman–Sander's Theorem.

Hindman's Theorem 32.7 (Hindman [Hin]) For any positive integer n any n-coloring of the set of positive integers N contains an infinite subset $S \subseteq N$ such that the set $\left\{ \sum_{x \in F} x : \varnothing \neq F \subset S; |F| < \aleph_0 \right\}$ is monochromatic.

Let us now go back and establish the most appropriate credit for Theorem 32.5. It is called Folkman–Rado–Sanders' Theorem in [GRS1], [Gra2] and [EG]; and Folkman's Theorem in [Gra1] and [GRS2]. Most of other authors have simply copied attribution from these works. Which credit is most justified? In one publication only [Gra2], Ronald L. Graham gives the date of Jon Folkman's personal communication to Graham: 1965. In one publication only [Gra1], in 1981 Graham publishes Folkman's proof that uses Schur–Baudet–Van der Waerden's Theorem (see Chapters 33 and 35). Thus, Folkman merits credit. In the standard text on Ramsey Theory [GRS2], I find an argument for giving credit to Folkman alone, disagreeing with the first edition [GRS1] of the same book:

> Although the result was proved independently by several mathematicians, we choose to honor the memory of our friend Jon Folkman by associating his name with the result.

Jon H. Folkman left this world tragically in 1969 at the age of 31. He was full of great promise. Sympathy and grief of his friends is understandable and noble. Yet, do we, mathematicians, have the liberty to award credits? In this case, how can we deny Jon Henry Sanders credit, when Sanders' independent authorship is absolutely clear and undisputed (he could not have been privy to the mentioned above personal communication)? Sanders formulates and proves Theorem 32.5 in his 1968 Ph.D. dissertation [San]. Moreover, Sanders proves it in a different way from Folkman: he does not use Schur–Baudet–Van der Waerden's Theorem, but instead generalizes Ramsey's Theorem to what he calls in his dissertation "Iterated Ramsey Theorem" [San, pp. 3–4].

Vladimir I. Arnautov's discovery is even more striking. His paper is much closer in style to that of Schur's classic 1916 paper, where Schur's Theorem appears as a useful tool, "a very simple lemma," and is immediately used for obtaining a number-theoretic result, related to Fermat's Last Theorem. Arnautov formulates and proves Theorem 32.5, but treats it as a useful tool and calls it simply "lemma 2" (in the proof of lemma 2, he uses Schur–Baudet–Van der Warden's Theorem). He then uses lemma 2 and other Ramseyan tools to prove that every (not necessarily associative) countable ring allows a non-discrete topology. This brilliant paper was submitted to *Doklady Akademii Nauk USSR* on August 22, 1969, and on September 2, 1969 was recommended for publication by the celebrated topologist Pavel S. Aleksandrov.[5] We have no choice but to savor the pleasure of associating Aknautov's name with Theorem 32.5.

What about Rado, one may ask? As Graham–Rothschild–Spencer [GRS2] observe, Theorem 32.5 "may be derived as a corollary of Rado's theorem [Rad1] by elementary, albeit non-trivial, methods."[6] In my opinion, this is an insufficient reason to attach Rado's name to Theorem 32.5. Arnautov, Folkman, and Sanders envisioned a generalization in the direction different from that of Rado, and paved the way for Graham–Rothschild's conjecture proved by Hindman. In fact, Erdős came to the same conclusion in 1973 [E73.21] when he put Rado's name in parentheses (Erdős did not know about Arnautov's paper, or he would have definitely added him to the authors of Theorem 32.5):

> Sanders and Folkman proved the following result (which also follows from earlier results of Rado [Rad1]).

32.3 Non-linear Regular Equations

A number of mathematicians studied regularity of non-linear equations. The following problem was posed by Ronald L. Graham and Paul Erdős circa 1975 (Graham estimates it as "has been opened for over 30 years" in his 2005 talk published as [Gra7]), and still remains open today, as Graham reports in [Gra7], [Gra8], where he offers $250 for the first solution:

Open $250 Problem 32.8 (R. L. Graham and P. Erdős, 1975) Determine whether the Pythagorean equation $x^2 + y^2 = z^2$ is partition regular, i.e., whether for any positive integer k, any k-coloring of the set of positive integers contains a non-trivial monochromatic solution x, y, z of the equation.

"There is actually very little data (in either direction) to know which way to guess," Graham remarks [Gra7], [Gra8]. However, I recall the following story.

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5 *Doklady* published only papers by full and corresponding members of the Academy. A non-member's paper had to be recommended for publication by a full member of the Academy.

6 Theorem 32.5 also follows from Graham and Rothschild's results published in 1971 [GR1].

In May of 1993 in a Budapest hotel, right after Paul Erdős's 80th-Birthday Conference in Keszthely, Hungary, Hanno Lefmann from Bielefeld University, Germany, told me that he and Arie Bialostocki from the University of Idaho, Moscow, generated by computer, with an assistance of a student, a coloring of positive integers from 1 to over 60,000 in two colors that forbade monochromatic solutions x, y, z of the equation $x^2 + y^2 = z^2$. This could be a basis for conjecturing a negative answer to Problem 32.8, but of course the problem remains open and is still awaiting new approaches.

Let us roll back a few years. Inspired by the old K. F. Roth's conjecture (published by Erdős already in 1961 [E61.22], Problem 16, p. 230), Paul Erdős, András Sárközy, and Vera T. Sós proved in 1989 a number of results and posed a number of conjectures [ESS]. I would like to present here one of each.

Erdős–Sárközy–Sós's Theorem 32.9 [ESS, Theorem 3] Any k-coloring of the positive integers, $k \leq 3$, contains a monochromatic pair x, y such that $x + y = z^2$, for infinitely many integers z.

The authors then posed a conjecture:

Erdős–Sárközy–Sós's Conjecture 32.10 [ESS, Problem 2] Let $f(x)$ be a polynomial of integer coefficients, such that $f(a)$ is even for some integer a. Is it true that for any k-coloring of positive integers for some b (for infinitely many b) the equation $x + y = f(b)$ has a monochromatic solution with $x \neq y$?

On the first reading you may be surprised by the condition on $f(a)$ to be even for some integer a. However, you could, easily construct a counterexample to the Erdős–Sárközy–Sós's Conjecture 32.10 if this condition were not satisfied. Indeed, let $f(x) = 2x^2 + 1$, and color the integers in two colors, one for even integers and another for the odd.

In 2006 Ayman Khalfalah, professor of engineering in Alexandria, Egypt, and Endre Szemerédi [KSz] generalized Theorem 32.9 to all k.

Khalfalah–Szemerédi's Theorem 32.11 [KSz] For any positive integer k there exists $N(k)$, such that any k-coloring of the initial segment of positive integers $[N(k)]$ contain a monochromatic pair x, y such that $x + y = z^2$, for an integer z.

Khalfalah and Szemerédi also proved Conjecture 32.10.

Khalfalah–Szemerédi's Generalized Theorem 32.12 [KSz] Given a positive integer k and a polynomial with integer coefficients $f(x)$ such that $f(a)$ is even for some a, there exists $N(k)$, such that any k-coloring of the initial segment of positive integers $[N(k)]$ contains a monochromatic pair $x, y, x \neq y$, such that $x + y = f(z)$, for some integer z.

Endre Szemerédi is a witty speaker, with humor reminiscent to that of Paul Erdős, which he displayed on April 4, 2007 when he presented these results at the Discrete Mathematics Seminar at Princeton-Math.

While these results were a step forward, non-linear regular equations remain a little studied vast area of Ramsey Theory. It deserves its own Richard Rado!