30
The Man behind the Theory: Frank Plumpton Ramsey

I verified harmony by algebra.
Only then, experienced in science,
I dared to surrender to the bliss of creative dream.
– Aleksandr Pushkin, *Mozart and Salieri*

Knowledge is a correspondence between idea and fact.
– Frank Plumpton Ramsey

30.1 Frank Plumpton Ramsey and the Origin of the Term "Ramsey Theory"

Who was "Ramsey," the man behind the theory named for him by others?

Let us start with the introduction to Ramsey's collected works [Ram3], assembled and edited right after his passing in 1930 by Ramsey's friend and disciple Richard Bevan Braithwaite, then Fellow of King's College and later the Knightbridge Professor of Philosophy at the University of Cambridge, who opens as follows:

> Frank Plumpton Ramsey was born on 22nd February, 1903, and died on 19th January 1930 [a jaundice attack prompted by an unsuccessful surgery]. The son of the President of Magdalene, he spent nearly all his life in Cambridge, where he was successively Scholar of Trinity, Fellow of King's [at 21], and Lecturer in Mathematics in the University [at 23]. His death at the height of his powers deprives Cambridge of one of its intellectual glories and contemporary philosophy of one of its profoundest thinkers. Though mathematical teaching was Ramsey's profession, philosophy was his vocation.

The celebrated British philosopher, Cambridge "Professor of Mental Philosophy and Logic" and Fellow of Trinity College, George Edward Moore wrote the preface for the book [Ram3]:

> He [Ramsey] was an extraordinarily clear thinker: no-one could avoid more easily than he the sort of confusions of thought to which even the best philosophers are liable, and he was capable of apprehending clearly and observing consistently, the subtlest

A. Soifer, *The Mathematical Coloring Book*,
DOI 10.1007/978-0-387-74642-5_30,

> distinctions. He had, moreover, an exceptional power of drawing conclusions from a complicated set of facts: he could see what followed from them all taken together, or at least what might follow, in cases where others could draw no conclusions whatsoever. And, with all this, he produced the impression of also possessing the soundest common sense: his subtlety and ingenuity did not lead him, as it seems to have led some philosophers, to deny obvious facts. He had, moreover, so it seemed to me, an excellent sense of proportion: he could see which problems were the most fundamental, and it was these in which he was most interested and which he was most anxious to solve. For all these reasons, and perhaps for others as well, I almost always felt, with regard to any subject that we discussed, that he understood it much better than I did, and where (as was often the case) he failed to convince me, I generally thought the probability was that he was right and I was wrong, and that my failure to agree with him was due to lack of mental power on my part.

Indeed, Ramsey's philosophical essays impress me immensely by their depth, clarity, and common sense—a combination that reminds me the great Michel de Montaigne. Here is my favorite quotation from Ramsey [Ram5, p. 53]:

> *Knowledge is a correspondence between idea and fact.*

Frank P. Ramsey's parents were Arthur Stanley Ramsey and Agnes Mary Wilson. In addition to Magdalene College's presidency, Arthur S. Ramsey was a tutor in mathematics. Frank was the oldest of four children, he had two sisters and a brother, Arthur Michael Ramsey, who much later became The Most Reverend Michael Ramsey, Archbishop of Canterbury (1961–1974). In 1925, Frank P. Ramsey married Lettice C. Baker, and their marriage produced two daughters. It is surprising to find in one family two brothers, Michael, the head of the Church of England and Frank, "a militant atheist," as Lettice described her husband.

The great economist John Maynard Keynes (1883–1946), who was then a Fellow of King's College and a close friend of Frank Ramsey, writes in March 1930 about Ramsey's contribution to economics [Key]:

> He [Ramsey] has left behind him in print (apart from his philosophical papers) only two witnesses of his powers – his papers published in the *Economic Journal* on "A Contribution to the Theory of Taxation" in March 1927, and on "A Mathematical Theory of Saving" in December 1928. The latter of these is, I think, one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject.

Keynes also draws for us a portrait of Ramsey the man (ibid.):

> His bulky Johnsonian frame, his spontaneous gurgling laugh, the simplicity of feelings and reactions, half-alarming sometimes and occasionally almost cruel in their directness and literalness, his honesty of mind and heart, his modesty, and the amazing, easy efficiency of the intellectual machine which ground away behind his wide temples and broad, smiling face, have been taken from us at the height of their excellence and before their harvest of work and life could be gathered in.

This portrait reminds me of Frank Ramsey's joking about his size while favoring human emotion over all issues of the universe (February 28, 1925):

> Where I seem to differ from some of my friends is in attaching little importance to physical size. I do not feel the least humble before the vastness of the heavens. The stars may be large, but they cannot think of love; and those are qualities which impress me far more than the size does. I take no credit for weighing nearly seventeen stone.

Frank Plumpton Ramsey, aged 18. Reproduced by kind permission of the Provost and Scholars of King's College, Cambridge

By kind permission of the Provost and Scholars of King's College, Cambridge, I can share with you two photographs of the gentle giant, Frank Plumpton Ramsey. As Jacqueline Cox, Modern Archivist of King's College Library advises in her November 21, 1991 letter, "Both photographs come from the J. M. Keynes Papers (ref. JMK B/4). The first is a portrait of him at age the 18 in 1921 [page 283]. The second [page 284] shows him sitting on the ground in the open air reading a book at the age 25 in 1928. The photographers are not indicated, but in the case of the second photograph a note records that it was taken in the Austrian Tyrol in August 1928."

Considering his short life, Ramsey produced an enormous amount of work in logic, foundations of mathematics, mathematics, probability, economics, decision theory, cognitive psychology, semantics, and of course philosophy. Ramsey manuscripts, held in the Hillman Library of the University of Pittsburg, fill seven boxes and number about 1500 pages[10] [Ram5]. Probability fare is worthy of our attention. In his February 27, 1978 BBC radio broadcast (reprinted as an article [Mel] in 1995), Emeritus Professor of Philosophy at Cambridge D. H. Mellor explains:

> The economist John Maynard Keynes, to whom Braithwaite introduced Ramsey in 1921, published his *Treatise on Probability* in August of that year. . .It did not satisfy Ramsey, whose objections to it – some of them published before he was nineteen – were so cogent and comprehensible that Keynes himself abandoned it.

Frank Plumpton Ramsey, aged 25, Austrian Tyrol, August, 1928. Reproduced by kind permission of the Provost and Scholars of King's College, Cambridge

In fact, the Princeton Professor Emeritus of both Mathematics and Economics Harold W. Kuhn tells me that Keynes decided against continuing with mathematics because Ramsey was so much superior in it. Mellor continues:

> In this paper [Ram4], after criticizing Keynes, Ramsey went on to produce his own theory. This starts from the fact that people's actions are largely determined by what they believe and what they desire – and by strength of those beliefs and desires. The strength of people's beliefs is measured by the so-called 'subjective probability' they

[10] In *A Tribute to Frank P. Ramsey* [Har2], Frank Harary writes: "At her home, she [Mrs. Lettice Ramsey, the widow] showed me box upon box of notes and papers of Frank Ramsey and invited me to pore through them. As they dealt mostly with philosophy, I had to decline." As "a tribute," could Prof. Harary have shown more interest and curiosity?

attach to events. . .Subjective utility measures the strength of people's desires just as subjective probability measures the strength of their beliefs.

The problem is how to separate these two components of people's actions. . . One of the things Ramsey's paper did was to show how to extract people's subjective utilities and probabilities from the choices they make between different gambles; and by doing so it laid the foundations for the serious use of these concepts in economics and statistics as well as in philosophy.

It took a long time, however, from this 1926 paper of Ramsey's to bear fruit. Only after the publication in 1944 of a now classic book [NM] by John von Neumann and Oskar Morgenstern, *The Theory of Games and Economic Behavior*, did utility theory begin to catch on and be applied in modern decision theory and games theory. And for many years no one realized how much of it had been anticipated in Ramsey's 1926 paper.

I am looking at the classic 1944 book [NM] Mellor mentions above, written by the two celebrated Institute for Advanced Study and Princeton University members, John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977) respectively, and at its later editions (Fine Library of Princeton-Math is very good). The authors cite many colleagues in the book: Daniel Bernoulli, Dedekind, Kronecker, D. Hilbert, F. Hausdorff, E. Zermelo, G. Birkhoff, E. Borel, W. Burnside, C. Carathéodory, W. Heisenberg, A. Speiser – and even Euclid. One missing name that merits credit the most is that of Frank P. Ramsey. Harold W. Kuhn tells me that in a 1953 letter he asked von Neumann why the latter gave no credit to Ramsey for inventing subjective probability. Indeed, this question and von Neumann's answer are reflected in H. W. Kuhn and A. W. Tucker's 1958 memorial article about von Neumann [KT, pp. 107–108]:

Interest in this problem as posed [measuring "moral worth" of money] was first shown by F. P. Ramsey [Ram4] who went beyond Bernoulli in that he defined utility operationally in terms of individual behavior. (Once von Neumann was asked [by H. W. Kuhn] why he did not refer to the work of Ramsey, which might have been known to someone conversant with the field of logic. He replied that after Gödel published his papers on undecidability and the incompleteness of logic, he did not read another paper in symbolic logic.[11]

Ramsey's priority was discovered and acknowledged in print by others. In already mentioned D. H. Mellor's broadcast, the philosopher of probability Richard Carl Jeffrey (1926–2002; Ph.D. Princeton 1957; Professor of Philosophy at Princeton 1974–1999) says:

It was when Leonard Savage, statistician, was working on his book on subjective probability theory, and he wished to find out what if anything the philosophers had to say on the subject, he went to Ramsey article [Ram4] and read it, and he found that what

[11] Indeed, von Neumann and Morgenstern probably did not expect Ramsey to publish on a topic far away from the foundations, such as economics, and thus might not have known about Ramsey's pioneering work by the time of the first 1944 edition of their celebrated book. However, new editions, which came out in 1947, 1953, 1961, etc., did not give Ramsey a credit either.

> he [Ramsey] had done was to a great extend fairly describable as rediscovering another aspect of Ramsey's work in that article – the foundations of the theory of subjective probability. It was Savage's book, *The Foundations of Statistics*, that was published in 1954, that made subjectivism a respectable sort of doctrine for serious statistician to maintain; and the remarkable thing is that Ramsey in this little paper to the Moral Sciences Club in 1926 has done all of that already.

Indeed, Leonard Jimmie Savage (1917–1971) writes in 1954 [Sav, pp. 96–97]:

> Ramsey improves on Bernoulli in that he defines utility operationally in terms of the behavior of a person constrained by certain postulates...
>
> Why should not the range, the variance, and the skewness, not to mention countless other features, of the distribution of some function join with the expected value in determining preference? The question was answered by the construction of Ramsey and again by that of von Neumann and Morgenstern.

Richard C. Jeffrey writes [Jef, p. 35]:

> This method of measurement [of desirability] was discovered by F. P. Ramsey and rediscovered by von Neumann and Morgenstern, through whose work it came to play its current role in economics and statistics.

More importantly, most of his 1965 book *The Logic of Decision* [Jef] is based on Ramsey's ideas, while one Chapter is simply called *Ramsey's Theory*.

Ramsey's first mathematical paper, *Mathematical Logic* [Ram1] appeared in 1926 in the midst of the *Grundlagenstreit* (Crisis in the Foundations), the confrontation between the two giants, David Hilbert and L. E. J. Brouwer, over the foundations of mathematics. Ramsey, who always addressed the most important issues of his day did not shy away from this one either. However, he did not, take either side. Ramsey did not agree with the intuitionist approach:

> Weyl has changed his view and become a follower of Brouwer, the leader of what is called the intuitionist school, whose chief doctrine is the denial of the Law of Excluded Middle, that every proposition is either true or false. This is denied apparently because it is thought impossible to know such a thing *a priori*, and equally impossible to know it by experience... *Brouwer would refuse to agree that either it was raining or it was not raining, unless he had looked to see*.

Ramsey did not support Hilbert either:

> I must say something of the system of Hilbert and his followers, which is designed to put an end to such skepticism once and for all. This is to be done by regarding higher mathematics as the manipulation of meaningless symbols according to fixed rules. We start with certain symbols called axioms: from these we can derive others by substituting certain symbols called constants for others called variables, and by proceeding from the pair of formulae p, if p then q to the formula q.
>
> Mathematics proper is thus regarded as a sort of game, played with meaningless marks on paper rather like noughts and crosses; but besides this there will be another subject called metamathematics, which is not meaningless, but consists of real assertions about mathematics, telling us what this or that formula can or cannot be obtained from the axioms according to the rules of deduction...

> Now, whatever else a mathematician is doing, he is certainly making marks on paper, and so this point of view consists of nothing but the truth; but it is hard to suppose it is the whole truth. There must be some reason for the choice of axioms. . .Again, it may be asked whether it is really possible to prove that the axioms do not lead to contradiction, since nothing can be proved unless some principles are taken for granted and assumed to lead to contradiction.

Summing up both Hilbert and Brouwer–Weyl approaches, Ramsey concluded:

> We see then that these authorities, great as they are the differences between them, are agreed that mathematical analysis as originally taught cannot be regarded as a body of truth, but is either false or at best a meaningless game with marks on paper.

What was a mathematician to do? Ramsey was in favor of using the Axiom of Infinity. "As to how to carry the matter further, I have no suggestion to make; all I hope is to have made it clear that the subject is very difficult," wrote Ramsey in the end. (4 years later Ramsey would take a finitist view of rejecting the existence of any actual infinity.)

Ramsey came back with a specific approach in his second mathematical paper *On a Problem of Formal Logic* [Ram2], submitted on November 28, 1928, and published posthumously in 1930. This paper gives a clear and unambiguous start to what was later named the *Ramsey Theory*. What is the aim of this work? Fortunately, Ramsey answers this question right in the beginning of this paper:

> This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.

Indeed, Ramsey solves the problem in the special case, as he promises. However, little does he—or for that matter anyone else—expect that the next year, in 1931 another young genius, the 25-year-old Kurt Gödel will shock the mathematical world by publishing the (Second) Incompleteness Theorem [Göd1] that shows that Hilbert–Ackermann's *Entscheidungsproblem*, "the leading problem of mathematical logic" as Ramsey calls it, cannot have a solution in general case. Ramsey continues:

> The theorems which we actually require concern finite classes only, but we shall begin with a similar theorem about infinite classes which is easier to prove and gives a simple example of the method of argument.

Yes, the infinite case here—as often happens—is easier than the finite, but is very well worth of the presentation (in fact, the finite case follows from the infinite by the de Bruijn–Erdős Compactness Theorem, as we have seen in Chapter 28). Later in the paper, Ramsey also observes that his infinite case requires the use of the Axiom of Choice:

> Whenever universe is infinite we shall have to assume the axiom of selection.

In fact, some 40 years later, in 1969 Eugene M. Kleinberg [Kle] will prove that Ramsey's Theorem is independent from ***ZF***, the Zermelo–Fraenkel set theory. (More precisely, if ***ZF*** is consistent, then Ramsey's theorem is not provable in ***ZF***.)

As we have seen in Chapter 28, Frank P. Ramsey realizes— and clearly states— that his new pioneering method and his "theorems on combinations have an independent interest." Indeed, Ramsey's theorems deliver the principles and the foundation to a new field of mathematics, the *Ramsey Theory*. Now, this requires a certain clarification.

Three Ramsey Theory results appeared before Frank P. Ramsey erected its foundation, and is the reason why I combine these three early results under the name *Ramsey Theory before Ramsey*. They are Hilbert's Theorem of 1892, Schur's Theorem of 1916, and Baudet–Schur–Van der Waerden's Theorem of 1927. These classic results, which we will discuss in great detail in the next part, discovered particular properties of colored integers or colored spaces in particular circumstances. These theorems contributed real "meat" to the Ramsey Theory, real applications of the Ramsey Principle to particular contexts before Ramsey even formulated it!

Ramsey's amazing logical and philosophical gifts allowed him to abstract the idea from any particular context, to formulate his theorems as a *method*, a *principle* of the new theory—a great achievement indeed. Surely, Ramsey fully deserves his name to be placed on the new theory, whose principle he so clearly formulated and proved, but could anyone point out to who and when coined the term *Ramsey Theory*?

We have already seen *Ramsey's Theory of Decision* in Richard C. Jeffrey's 1965 book [Jef]. But we are after *The Ramsey Theory*, a new and flourishing branch of combinatorial mathematics. On July 21, 1995, I posed the question to the leader of the Ramsey Theory, Ronald L. Graham. Here is our brief exchange of the day:

Dear Ron:
Who and when coined the name "Ramsey Theory"?
Yours, Sasha

Sasha,
Beats me! Who first used the term Galois theory?
Ron

On January 22, 1996, I asked Ron again, and received another concise reply the same day:

Dear Sasha,
I would imagine that Motzkin may have used the term Ramsey Theory in the 60's. You might check with Bruce Rothschild at UCLA who should know.

Still the same day I received a reply from Bruce Rothschild:

Dear Alexander,

> This is a good question, to which I have no real answer. I do not recall Motzkin using the phrase,[12] though he might have. I also don't recall hearing Rota use it when I was at MIT in the late 60's. My best recollection is that I began using the term informally along with Ron sometime in the very early '70's... But I could be way off here.

Frank Harary was less concise. On February 19, 1996, during a conference in Baton Rouge, Louisiana, he gave me a multi-page statement (you saw it in its entirety in Chapter 27), showing that Frank Harary and Václav Chvátal were the first to introduce the term *Generalized Ramsey Theory for Graphs* in their series of papers that started in 1972. I am looking at the first paper [CH] of the series: Chvátal, Václav, and Harary, Frank, *Generalized Ramsey theory for graphs*. The authors generalize the notion of the Ramsey number by including in the study graphs other than complete graphs. By doing so Harary and Chvatal open a new, now flourishing chapter, *Graph Ramsey Theory*. However, *The Ramsey Theory* as we understand it today stands for so much broader a body of knowledge, including Schur's, Van der Waerden's, and Hales-Jewett's Theorems that it does not completely fit inside Graph Theory. Thus, my search for the true birth of the name continued.

One 1971 survey [GR2], by Ronald L. Graham and Bruce L. Rothschild show a clear realization that a new theory has been born and needs an appropriate new name. Following a recitation of Ramsey's Theorem and Schur's Theorem, the authors write:

> These two theorems are typical of what we shall call a *Ramsey theorem* and a *Schur theorem*, respectively. In this paper we will survey a number of more general Ramsey and Schur theorems which have appeared in the past 40 years. It will be seen that quite a few of these results are rather closely related, e.g., van der Waerden's theorem on arithmetic progressions [Wae2], [Khi4], Rado's work on regularity and systems of linear equations [Rad1], [Rad2], the results of Hales and Jewett [HJ] and others [Garsia, personal communication] on arrays of points and Rota's conjectured analogue of Ramsey's Theorem for finite vector spaces, as well as the original theorems of Ramsey and Schur.

Yes, I agree that the new theory was created by 1971, and the choice of its name was between two deserving candidates: *The Schur Theory*, in honor of the main early contributor Issai Schur and his School (Schur's work was continued by his students Alfred Brauer and Richard Rado); and *The Ramsey Theory*, in honor of Frank P. Ramsey who formulated the principles of the new theory. Soon Graham and Rothschild arrived at the decision, and in their 1974 survey made the first published announcement of their choice [GR3]:

> Recently a number of striking new results have been proved in an area becoming known as RAMSEY THEORY. It is our purpose here to describe some of these. Ramsey Theory is a part of combinatorial mathematics dealing with assertions of a certain type, which we will indicate below. Among the earliest theorem of thus type

[12] Motzkin did not use "Ramsey Theory" in his 1960s articles, as I have verified shortly after.

are RAMSEY's theorem, of course, VAN DER WAERDEN's theorem on arithmetic progressions and SCHUR's theorem on solutions of $x + y = z$.

It seems that *The Ramsey Theory* has been shaping throughout the 1970s, and the central engine of this process was new results and the above mentioned surveys. In 1980 the long life of the name was assured when it appeared as the title of the book *Ramsey Theory* [GRS1] by three of the leading researchers of the field, Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer. A decade later, the authors produced the second, updated edition [GRS2]. This book has not only assured the acceptance of the name—it has become the standard text in the new field of mathematics. It still remains the standard bearer today.[13]

Now is the time to share a bit of information about the co-creators of the term *Ramsey Theory*, who of course contributed much more than just the name.

Bruce Lee Rothschild was born on August 26, 1941 in Los Angeles. Following his B.S. degree from the California Institute of Technology in 1963, he earned a Ph.D. degree from Yale in 1967 with the thesis *A Generalization of Ramsey's Theorem and a Conjecture of Rota*, supervised by the legendary Norwegian graph theorist Øystein Ore (1899–1968). After 2 years 1967–1969 at MIT, Bruce became a professor at the University of California, Los Angeles, where he continues his work today. In 1972 Graham, Rothschild, and Leeb shared the Polya Prize of SIAM with Hales and Jewett.

Ronald Lewis Graham was born on October 31, 1935 in Taft, California. In 1962, he earned his Ph. D. degree from the University if California, Berkeley with the thesis *On Finite Sums of Rational Numbers*, supervised by Derrick Lehmer. Following decades as Director of Mathematical Sciences at Bell Laboratories, Ron moved South and West, and is now Irwin and Joan Jacobs Professor in the Department of Computer Science and Engineering at the University of California, San Diego. There is much to be said about this unique individual, who besides publishing well over 300 papers and several books, served as President of the American Mathematical Society (AMS), President of the Mathematical Association of America, and since 1996 is the Treasurer of the National Academy of Sciences. In 2003, AMS awarded Graham Steele Prize for lifetime achievement. I can attest to Ron's supreme elegance and depth as author and lecturer, and limitless energy in promoting the Ramsey Theory.

This certainly does not cover Ron's excellence in juggling ("juggling is a metaphor," he likes to say), fluency in Mandarin, friendship with Paul Erdős, etc. See all those on "Ronald Graham's special page" created by Ron's wife and well-known mathematician in her own rights Fan Chung at http://math.ucsd.edu/~fan/ron/.

Ron maintained a room for Paul Erdős in his New Jersey house, and took care of Erdős's finances. In the whole world, Ron knew best where Paul Erdős was on any given day, although as Ron's December 20, 1993 e-mail shows, even his knowledge was imperfect:

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[13] I have only one problem with this beautiful book: today it sells for a whopping $199 at Wiley, its publisher, and on Amazon.com.

Sasha,
Erdős is staying with me for a while. During the night he is at (908)322–4111. During the day it's anyone's guess where he will be!
Ron

Now that Paul passed on, Ron has graciously taken upon himself to keep the tradition going by paying Erdős's prizes for first solutions of Erdős's problems, but not all of them. So, all those interested in making a living by solving Erdős's problems, pay attention to the small print in Ron's e-mail of February 12, 2007:

Hi Sasha,
I am willing to pay all the prizes offered by Paul that are listed in the book that Fan and I wrote: Erdos on Graphs: His Legacy of Unsolved Problems. These we have checked. The others (e.g., in number theory or set theory) are not (automatically) part of the offer. I did have to pay $100 last year (the first time for a problem in this book that was solved) to Jacques Verstraete in Canada!

Best regards,
Ron Graham

30.2 Reflections on Ramsey and Economics, by Harold W. Kuhn

Why is it that sometimes people work together for decades and still remain strangers, while in other instances friendship arrives at the first sight? This is a question for psychologists to ponder. I should only observe that Harold W. Kuhn and I instantly became friends in early 2003, when I arrived in Princeton-Math., just as in 1988 when an instant friendship linked Paul Erdős and I. It has always been intellectually stimulating to discuss any subject with Harold, from mathematics to the cinema of Michelangelo Antonioni, and from African Art to Pierre Bonnard's drawings.

Harold William Kuhn was born in Santa Monica, California on July 29, 1925. Following his B.S. degree in 1947 from the California Institute of Technology, he earned a Ph.D. degree from Princeton University in 1950, while also serving as Henry B. Fine Instructor in the Mathematics Department, 1949–1950. Following a professorship at Bryn Mawr, 1952–1958, Harold has been a Professor of Mathematical Economics at Princeton's two departments, Mathematics and Economics, becoming Emeritus in 1995. His honors include presidency of the Society for Industrial and Applied Mathematics (1954–55), service as Executive Secretary of the Division of Mathematics of the National Research Council (1957–1960), John von Neumann Theory Prize of the Operation Research Society of America (1982; jointly with David Gale and A. W. Tucker), and Guggenheim Fellowship (1982). It was Harold Kuhn who nominated John F. Nash Jr. for Nobel Prize (awarded in 1994).

In the fall of 2006, upon my return to Princeton University, I asked myself, who could best evaluate Frank P. Ramsey's works on economics? It would take an expert on mathematics and economics. Harold was the only choice, and he most generously agreed. Best of all, Harold wrote a triptych about F. P. Ramsey, John von Neumann and John F. Nash Jr., especially for this book. In all that follows in this section, the podium—shall I say, the pages—belong to Harold W. Kuhn. ▪

Although mathematics became the *lingua franca* of 20th century economics, only a handful of mathematicians have exerted a direct and lasting influence on the subject. They surely include Frank Plumpton Ramsey, John von Neumann, and John Forbes Nash Jr. The similarities and differences in their life trajectories are striking. Ramsey died at 26 years of age after an exploratory liver operation following a bout of jaundice, while Nash's most productive period ended when he fell prey to schizophrenia at the age of 30. Von Neumann's original work on game theory and growth models was done before he was 30 years old. For all three, the work in economics appears as a sideline. Ramsey's friend and biographer, Richard Braithwaite has written: "Though mathematical teaching was Ramsey's profession, philosophy was his vocation," without mentioning his contributions to economics at all or including the three papers on economics in the posthumous "complete" works that Braithwaite edited. The contributions of von Neumann to mathematical economics is but one chapter in the seven chapters comprising the memorial issue devoted to von Neumann's research and published as a special issue of the Bulletin of the American Mathematical Society. Regarding Nash, John Milnor considered "...Nash's [Nobel Economics] prize work [to be] an ingenious but not surprising application of well-known methods, while his subsequent mathematical work was much more rich and important."

Ramsey, von Neumann, and Nash came from very different backgrounds and had very different relationships to the economics and the economists of their day. Ramsey, an intimate friend of Bertrand Russell and Wittgenstein, was a Cambridge man by birth. He appears to have been interested in economics from the age of 16 and wrote his first published piece on economics at 18. He had close personal and professional contacts with such well-known economists as John Maynard Keynes, Arthur Pigou, Piero Sraffa, and Roy Harrod. He served as an advisor to the Economic Journal, where Keynes took his counsel most seriously. He was well acquainted with the trends in economic theory of his day.

Von Neumann, the scion of a Jewish banking family in Budapest, had a wide circle of intellectual friends from Budapest, Berlin, and Vienna that included economists such as William Fellner (who was a friend from gymnasium days) and Lord Nicholas Kaldor, who gave von Neumann a reading list in contemporary economics in the 1920s, and who arranged for an English translation of von Neumann's growth model to be published in the Review of Economic Studies in 1945. Thus there is ample evidence that von Neumann was well-informed of the state of economics throughout his life.

The case of John Nash, who grew up in the coal mining and railroad town of Bluefield, West Virginia, is very different. When he came to Princeton to do graduate work in mathematics at the age of 20, he had taken one undergraduate course in economics (on International trade) at Carnegie Tech, taught by an Austrian émigré, Bert Hoselitz. His major contribution on bargaining, which appears to have had its origin in this course, has two boys (Bill and Jack) trading objects such as a whip, a bat, a ball, and a knife. This was the work of a teenager. There is no evidence that Nash had read any contemporary economist outside the required readings of his one undergraduate course. Of course, later in his life, in the period when he was on the faculty at the Massachusetts Institute of Technology, he had contact with Paul Samuelson and Robert Solow, Nobel Prize winners in Economics, who knew of his work in game theory. Nash's only later excursion into economics is a theory of "ideal money," an idea that appears to have been anticipated in part by Friederich Hayek.

Now that game theory has become part of the economist's tool kit, anyone who takes an introductory economics course learns about the contributions of von Neumann and Nash. Ramsey's work, however, is less well-known and the principal reason for this note is to give the reader an appreciation for the contributions of Ramsey to economics. Between the ages of 18 and 29, Ramsey wrote four papers, which we shall discuss in detail below.

(A) "The Douglas Proposals," *The Cambridge Magazine*, Vol. XI, No. 1, January 1922, pp. 74–76.

Ramsey's first work related to economics (A) was published when he was 18. He was no common 18-year-old; here is how Keynes described him: "From a very early age, about 16 I think, his precocious mind was intensely interested in economic problems." The Cambridge Magazine was edited by C. K. Ogden, a Fellow of Magdalene College where Ramsey's father was President, from 1912 to 1922. Ramsey and Ogden met while Ramsey was still a student in his public school, Winchester, and Ogden persuaded him to study the then much-discussed social credit proposals of a certain Major Douglas. I. A. Richards recalled the upshot: "Soon after he'd done the Douglas credit thing, you know, A. S. Ramsey, his father, called up Ogden and said 'What have you been doing to Frank?', and Ogden said 'What's he been doing?'. 'Oh he's written a paper on Douglas Credit which would have won him a Fellowship in any University anywhere in the world instantly. It's a new branch of mathematics'."

Who was this Major Douglas? Briefly, he was one of those crackpots who exist on the fringe of academic economics and whose theories promise a redistribution of wealth that appealed to a large part of the public (including, in Douglas's case, Ezra Pound and T. S. Eliot). Like many of those offering a panacea for the Great Depression, he was also an anti-Semite who invoked the theses expounded in the Protocols of the Elders of Zion in defense of his economic theories.

What was Major Douglas's heresy that Ramsey demolished? It is centered on the so-called $A + B$ "theorem" (called by Keynes "mere mystification"). In producing a good, price is made up of two parts of the cost paid out by the producer: A equals the amount paid out for raw materials and overhead and B equals the sums paid out in wages, salaries, and dividends. According to Douglas, the amount B, paid to the consumers, is never sufficient to buy all of the good, whose cost (and price) is $A+B$. Therefore, the state should make up the difference through "social credit."

Ramsey first provides a verbal argument that shows that, in a stationary state, the total rate of distribution of purchasing power (taking into account payments originating in intermediate goods) equals the rate of flow of costs of consumable goods. He then writes:

"...it is possible, using some complicated mathematics to show that the ratio is unity under much wider conditions which allow for changes in the quantity of production, in the rate of wages, in the productivity of labor, and in the national wealth." The "complicated mathematics," other than Ramsey's curiously rigid set of modeling assumptions, consists of the use of "integration by parts," a technique taught to every beginning student of the calculus.

(B) "A Contribution to the Theory of Taxation," *The Economic Journal*, Vol. XXXVII, March 1927, pp. 47–61.

The young Ramsey assisted A. C. Pigou, who was the successor to Alfred Marshall in the chair of Political Economy at Cambridge, on a number of occasions beginning before 1926. After providing Pigou with a mathematical proposition and examples for two articles, one on credit and one on unemployment, Ramsey assisted Pigou with changes in the third edition of *The Economics of Welfare*, published in 1929. However, it appears that Ramsey's work on taxation (B) was inspired by questions raised in Pigou's *A Study in Public Finance*.

The problem posed by Ramsey in (B) was to find an optimal system of taxation of n commodities so as to raise a given quantity of revenue. For Ramsey in (B), "optimal" means minimizing aggregate sacrifice. Using this objective function, he shows that the production of each commodity should be reduced in the same proportion, thus a system of differential taxation. The mathematics employed is rather standard, namely, optimization under equality constraints using Lagrange multipliers which was taught to mathematicians of this period by treatises such as de la Vallee Poussin's *Cours d'Analyse*. The treatment is careful for the period and Ramsey includes a number of examples of potential applications of his results. Of particular interest is a discussion of the application of income tax to savings, a subject that I believe was part of a larger research agenda that Ramsey had formulated.

(C) "A Mathematical Theory of Saving," *The Economic Journal*, Vol. XXXVIII, December 1928, pp. 543–549.

Papers (B) and (C) were published in the Economic Journal which Keynes controlled with an iron hand. Keynes wrote of (C) that it "is, I think, one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject." The article (C) is concerned with the derivation of optimal saving programs under a variety of conditions. Samuelson captures the spirit of the paper in the society in which it was created when he wrote: "Frank Ramsey, living in a happier age and being a Cambridge philosopher assumed society would last forever and seek to maximize the utility of its consumption over all infinite time." A major stumbling block immediately presents itself in that the "utility of its consumption over all infinite time" is an improper integral which, in general, will not have a finite maximum value. Ramsey proposed an elegant device to get around this problem. He assumed that there was a maximum amount of attainable utility (called "bliss") and, instead of maximizing the improper integral he minimized the deviation from bliss over the infinite horizon.

Ramsey then derives a result that is easy to express in common English, namely: "The optimal rate of saving multiplied by the marginal utility of consumption should always equal the difference between bliss and the actual rate of utility enjoyed." The paper contains a derivation of this result by simple verbal reasoning provided by Keynes (which does not apply to the most general cases considered by Ramsey but which does give the non-mathematically adept, the feeling of "understanding the result"). Contemporary mathematical economists will instantly recognize the problem as one to which the calculus of variations applies and, indeed, over 30 years after Ramsey wrote (C) such techniques took over the theoretical models of growth. We can say with real justice that Ramsey was "ahead of his time."

Recently, three economic historians (D. A. Collard, M. Gaspard, and P. C. Duarte) have put forth a very persuasive theory (based largely on unpublished notes of Ramsey that are archived at the University of Pittsburg) that Ramsey's two papers on taxation and savings were not isolated works of a mathematician answering questions put to him by economists but were rather part of an over-arching research program that Ramsey had clearly in mind. If this plausible theory is true, it makes his early death even more tragic.

(D) "Truth and Probability," in R. B. Braithwaite (ed.), *The Foundations of Mathematics and Other Logical Essays*, London: Routledge and Kegan Paul, 1931, pp. 156–198. Reprinted in H. E. Kyburg and H. E. Smokler (eds.) *Studies in Subjective Probability*, New York: Wiley 1964, pp. 61–92.

In modeling the decisions of an individual who chooses an alternative from a set of uncertain outcomes, it has long been the tradition to introduce a numerical function to measure the objective of the individual involved. When von Neumann first formulated "the most favorable result" for a player in a strategic game, he identified "the most favorable result" with "the greatest expected monetary value," remarking that this or some similar assumption was necessary in order to apply the methods of probability theory. While doing so, he was well aware of the objections to the principal of maximizing expected winnings as a prescription for behavior, but wished to concentrate on other problems. The St. Petersburg paradox illustrates in clear terms the fact that the principle of maximizing expected winnings does not reflect the actual preferences of many people.

To resolve this paradox, Daniel Bernoulli suggested that people do not follow monetary value as an index for preferences but rather the "moral worth" of the money. He then proposed a quite serviceable function to measure the moral worth of an amount of money, namely, its logarithm. Whatever the defects of this function as a universal measure of preferences, and they are many, it raises the question of the existence of a numerical index which will reflect accurately the choices of an individual in situations of risk. Interest in this problem was first shown by Ramsey in (D) in which he defined utility operationally in terms of individual behavior. As Mellor has written: "In this paper (D), after criticizing Keynes, Ramsey went on to produce his own theory. This starts from the fact that people's actions are largely determined by what they believe and what they desire, and by strength of those beliefs and desires. The strength of people's beliefs is measured by the so-called subjective probability' they attach to events. . .Subjective utility measures the strength of people's desires just as subjective probability measures the strength of their beliefs. The problem is how to separate these two components of people's actions. One of the things Ramsey's paper did was to show how to extract people's subjective utilities and probabilities from the choices they make between different gambles; and by doing so it laid the foundations for the serious use of these concepts in economics and statistics as well as in philosophy."

The bible of game theory, *The Theory of Games and Economic Behavior* by von Neumann and Morgenstern which confronts similar problems contains no reference to the work of Ramsey. When von Neumann was queried about this omission, he explained it by saying that, after Goedel published his papers on undecidability and the incompleteness of logic, he did not read another paper in symbolic logic. Although his excuse is strengthened by the fact that (D) first appeared in the volume that Braithwaite edited after Ramsey's death, no such excuse exists for Morgenstern, when he wrote *"Some Reflections on Utility"* in 1979 and cites two articles by J. Pfanzagl while overlooking Ramsey's paper (D) and Savage's *The Foundations of Statistics*.

Aside from Ramsey's paper on Major Douglas, which was an exemplary mathematical model refuting errant nonsense, he has clear precedence in four major themes of 20th century economics. The paper on taxation (B) was a source for both public finance theorists and for monetary economists who have characterized inflation as a tax on money holdings and have formulated optimal inflation policies as optimal taxation schemes. The paper on savings (C) has become the touchstone for economists working on growth. The fourth area is the theory of expected utility and decisions under risk which has used in an essential way Ramsey's insights on subjective probability in (D).

I have been a friend of John Nash since he arrived in Princeton in 1948. I knew John von Neumann from 1948 until his death in 1957. I very much regret not having known Frank Ramsey. Given the modernity of his work, it is hard to grasp the fact that he died over 77 years ago.