29
The Happy End Problem

29.1 The Problem

During the winter of 1932–1933, two young friends, mathematics student Paul Erdős, age 19, and chemistry student George (György) Szekeres, 21, solved the problem posed by their youthful lady friend Esther Klein, 22, but did not send it to a journal for a year and a half. When Erdős finally sent this joint paper for publication, he chose J. E. L. Brouwer's journal *Compositio Mathematica*, where it appeared in 1935 [ES1].

Erdős and Szekeres were first to demonstrate the power and striking beauty of the Ramsey Principle when they solved this problem. Do not miss G. Szekeres' story of this momentous solution later in this chapter. In the process of working with Erdős on the problem, Szekeres actually rediscovered the Finite Ramsey Principle before the authors ran into the 1930 Ramsey publication [Ram2].

Erdős–Szekeres's Theorem 29.1 [ES1] For any positive integer $n \geq 3$ there is an integer m_0 such that any set of at least m_0 points in the plane in general position[2] contains n points that form a convex polygon.

To prove Erdős–Szekeres's Theorem, we need two tools.

Tool 29.2 (Esther Klein, Winter 1932–1933) Any 5 points in the plane in general position contain 4 points that form a convex quadrilateral.

In fact, in anticipation of the proof of Erdős–Szekeres's Theorem, it makes sense to introduce an appropriate notation $ES(n)$ for the Erdős–Szekeres function. For a positive integer n, $ES(n)$ will stand for the minimal number such that any $ES(n)$ points in the plane in general position contain n points that form a convex n-gon. Esther Klein's result can be written as

Result 29.3 (Esther Klein). $ES(4) = 5$.

[2] That is, no three points lie on a line.

A. Soifer, *The Mathematical Coloring Book,*
DOI 10.1007/978-0-387-74642-5_29,

Condensed Proof (Use paper and pencil.) Surely, $ES(4) > 4$. Given 5 points in the plane in general position, consider their convex hull H.[3] If H is a quadrilateral or a pentagon, we are done. If H is a triangle, the line determined by the two given points a, b inside H does not intersect one of the triangle H's sides de. We get a convex quadrilateral formed by the points a, b, d, and e. ∎

Tool 29.4 (P. Erdős and G. Szekeres, [ES1]) Let $n \geq 3$ be a positive integer. Then n points in the plane form a convex polygon if and only if every 4 of them form a convex quadrilateral.

According to Paul Erdős, two members of his circle E. Makai and Paul Turán established (but never published) one more exact value of $ES(n)$:

Result 29.5 (E. Makai and P. Turán). $ES(5) = 9$.

Erdős mentioned the authorship of this result numerous times in his problem papers. However, I know of only one instance when he elaborated on it. During the first of the two March 1989 lectures Paul gave at the University of Colorado at Colorado Springs, I learned that Makai and Turán found proofs *independently*. Paul said that Makai proof was lengthy, and shared with us Turán's short Olympiad-like proof. Turán starts along Esther Klein's lines by looking at the convex hull of the given 9 points. Let me stop right here and allow you the pleasure of finding a proof on your own.

We are now ready to prove Erdős–Szekeres's Theorem asserting the existence of the function $ES(n)$.

Proof of Theorem 29.1 (*P. Erdős and G. Szekeres*) Let $n \geq 3$ be a positive integer. By the Ramsey Principle 28.8 (we set $r = 4$ and $k = 2$) there is an integer $m_0 = \mathrm{R}(4, n, 2)$ such that if $m > m_0$ and all four-element subsets of an m-element set S_m are colored in two colors, then S_m contains a n-element subset S_n such that all four-element subsets of S_n are assigned the same color.

Now let S_m be a set of m points in the plane in general position. We color a four-element subset of S_m red if it forms a convex quadrilateral, and blue if it forms a concave (i.e., non-convex) quadrilateral. Thus, all four-element subsets of S_m are colored red and blue. Hence, S_m contains an n-element subset S_n such that all four-element subsets of S_n are assigned the same color. This color cannot be blue, because in view of Tool 29.2 any five or more element set contains a red four-element subset! Therefore, all four-element subsets of S_n are colored red, i.e., they form convex quadrilaterals. By Tool 29.4, S_n forms a convex n-gon. ∎

I must show you a beautiful alternative proof of Erdős–Szekeres's Theorem 29.1, especially since it was found by an undergraduate student, Michael Tarsi of Israel. He missed the class when the Erdős–Szekeres solution was presented, and had to

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[3] Convex hull of a set S is the minimal convex polygon that contains S. If you pound a nail in every point of S, then a tight rubber band around all nails would produce the convex hull.

come up with his own proof under the gun of the exam! Tarsi recalls (e-mail to me of December 12, 2006):

> Back in 1972, I took the written final exam of an undergraduate Combinatorics course at the Technion – Israel Institute of Technology, Haifa, Israel. Due to personal circumstances, I had barely attended school during that year and missed most lectures of that particular course. The so-called Erdős-Szekeres Theorem was presented and proved in class, and we have been asked to repeat the proof as part of the exam. Having seen the statement for the first time, I was forced to develop my own little proof.
>
> Our teacher in that course, the late Professor Mordechai Levin, had published the story as an article, I cannot recall the journal's name, the word 'Gazette' was there and it dealt with Mathematical Education.
>
> I was born in Prague (Czechoslovakia at that time) in 1948, but was raised and grew up in Israel since 1949. Currently I am a professor of Computer Science at Tel-Aviv University, Israel.

Proof of Theorem 29.1 *by Michael Tarsi.* Let $n \geq 3$ be a positive integer. By the Ramsey Principle 28.8 ($r = 3$ and $k = 2$) there is an integer $m_0 = R(3, n, 2)$ such that, if $m > m_0$ and all three-element subsets of an m-element subset S_m are colored in two colors, then S_m contains an n-element subset S_n such that all three-element subsets of S_n are assigned the same color.

Let now S_m be a set of m points in the plane in general position labeled with integers $1,\ 2,\ \ldots,\ m$.

We color a three-element set $\{i, j, k\}$, where $i < j < k$, red if we travel from i to j to k in a clockwise direction, and blue if counterclockwise. By the above, S_m contains an n-element subset S_n such that all three-element subsets of S_n are assigned the same color, i.e., have the same orientation. But this means precisely that S_n forms a convex n-gon! ▪

In their celebrated paper [ES1], P. Erdős and G. Szekeres also discovered the Monotone Subsequence Theorem.

A sequence $a_1,\ a_2,\ \ldots,\ a_k$ of real numbers is called *monotone* if it is increasing, i.e., $a_1 \leq a_2 \leq \ldots \leq a_k$, or decreasing, i.e., $a_1 \geq a_2 \geq \ldots \geq a_k$ (we use weak versions of these definitions that allow equalities of consecutive terms).

Erdős–Szekeres's Monotone Subsequence Theorem 29.6 [ES1] Any sequence of $n^2 + 1$ real numbers contains a monotone subsequence of $n + 1$ numbers.

I would like to show here how the Ramsey Principle proves such a statement with, of course, much worse upper bound than $n^2 + 1$. I haven't seen this argument in literature before.

Problem 29.7 Any long enough sequence of real numbers contains a monotone subsequence of $n + 1$ numbers.

Solution. Take a sequence S of $m = R(2, n + 1, 2)$ numbers $a_1,\ a_2,\ \ldots,\ a_m$. Color a two-element subsequence $\{a_i,\ a_j\},\ i < j$ red if $a_i \leq a_j$, and blue if $a_i > a_j$. By the Ramsey Principle, there is an $(n + 1)$-element subsequence S_1 with every two-element subsequence of the same color. But this subsequence is monotone! ▪

In [ES1] P. Erdős and G. Szekeres generalize Theorem 29.6 as follows:

Erdős–Szekeres's Monotone Subsequence Theorem 29.8 Any sequence S: $a_1, a_2, \ldots, a_r$ of $r > mn$ real numbers contains a decreasing subsequence of more than m terms or an increasing subsequence of more than n terms.

A quarter of a century later, in 1959, A. Seidenberg of the University of California, Berkeley, found a brilliant "one-line" proof of Theorem 29.8, thus giving it a true Olympiad-like appeal.

Proof of Theorem 29.8 *by A. Seidenberg* [Sei] Assume that the sequence S : $a_1, a_2, \ldots, a_r$ of $r > mn$ real numbers has no decreasing subsequence of more than m terms. To each a_i assign a pair of numbers (m_i, n_i), where m_i is the largest number of terms of a decreasing subsequence beginning with a_i and n_i the largest number of terms of an increasing subsequence beginning with a_i. This correspondence is an injection, i.e., distinct pairs correspond to distinct terms $a_i, a_j, i < j$. Indeed, if $a_i \le a_j$ then $n_i \ge n_j + 1$, and if $a_i > a_j$ then $m_i \ge m_j + 1$.

We get $r > mn$ distinct pairs (m_i, n_i), they are our pigeons, and m possible values (they are our pigeonholes) for m_i, since $1 \le m_i \le m$. By the Pigeonhole Principle, there are at least $n + 1$ pairs (m_0, n_i) with the same first coordinate m_0. Terms a_i corresponding to these pairs (m_0, n_i) form an increasing subsequence! ∎

Erdős and Szekeres note that the result of their Theorem 29.8 is best possible:

Problem 29.9 ([ES1]) Construct a sequence of mn real numbers such that it has no decreasing subsequence of more than m terms *and* no increasing subsequence of more than n terms.

Proof Here is a sequence of mn terms that does the job:

$$m,\ m-1,\ \ldots 1;\ 2m,\ 2m-1,\ \ldots,\ m+1;\ \ldots;$$
$$nm,\ nm-1,\ \ldots,\ (n-1)\,m+1.$$ ∎

H. Burkill and Leon Mirsky in their 1973 paper [BM] observe that the Monotone Subsequence Theorem holds for countable sequences as well.

Countable Monotone Subsequence Theorem 29.10 [BM]. Any countable sequence S : $a_1, a_2, \ldots, a_r, \ldots$ of real numbers contains an infinite increasing subsequence or an infinite strictly decreasing subsequence.

Hint: Color the two-element subsets of S in two colors. ∎

The authors "note in passing that the same type of argument enables us to show" the following cute result (without a proof):

Curvature Preserving Subsequence Theorem 29.11 [BM]. Every countable sequence S possesses an infinite subsequence which is convex or concave.

Hint: Recall Michael Tarsi's proof of Erdős–Szekeres Theorem above, and color the three-element subsets of S in two colors! ∎

The results of this section reminded me of the celebrated Helly Theorem.

Helly's Theorem 29.12 Let $F_1 \ \ldots, \ F_m$ be convex figures in n-dimensional space R^n. If each $n + 1$ of these figures have a common point, then the intersection $F_1 \cap \ldots \cap F_m$ is non-empty.

In particular, for $n = 2$ we get Helly's Theorem for the plane.

Helly's Theorem for the Plane 29.13 A finite family $F_1 \ \ldots, \ F_m$ of convex figures is given in the plane. If every three of them have a non-empty intersection, then the intersection $F_1 \cap_{\ldots} \cap F_m$ of all of these figures is non-empty as well.

The structure of the Helly Theorem appears to me similar to the one of Theorem 29.1. This is why I believe that the Helly Theorem and its numerous beautiful variations are a fertile ground for applications of the powerful tool, the Finite Ramsey Principle 28.8. To the best of my—and Branko Grünbaum's—knowledge this marriage of Helly and Ramsey has not been noticed before. To illustrate it, I have created a sample problem. Its result is not important, but the method may lead you to discovering new theorems.

Problem 29.14 Let m be a large enough positive integer ($m \geq R(3, \ 111, \ 2)$ to be precise), and $F_1, \ \ldots, \ F_m$ be convex figures in the plane. If among every 37 figures there are 3 figures with a point in common, then there are 111 figures with a point in common.

Hint: The fact that $37 \times 3 = 111$ has absolutely nothing to do with solution: the statement of Problem 29.14 remains true if we replace 37 and 111 by arbitrary positive integers l and n, respectively, as long as $l \leq n$.

Solution: Let $m \geq R\,(3, 111, 2)$, and $F_1, \ F_2, \ \ldots, \ F_m$ be convex figures in the plane. Consider the set $S = \{F_1 F_2, \ \ldots, \ F_m\}$. We color a three-element subset $\{F_i, F_j, F_k\}$ of S red if $F_i \cap F_j \cap F_k \neq \emptyset$, and blue otherwise. By the Finite Ramsey Principle 28.8, there is a 111-element subset S_1 of S such that all its three-element subsets are assigned the same color. Which color can it be? Surely not blue, for among every 37 figures there are 3 figures with a point in common, thus forming a red three-element subset. Thus, all three-element subsets of S_1 are red. Therefore, by the Helly Theorem 29.13 the intersection of all 111 figures of the set S_1 is non-empty. ∎

29.2 The Story Behind the Problem

On Paul Erdős's 60th birthday, his lifelong friend George (György) Szekeres gave Paul and us all a present of magnificent reminiscences, allowing us a glimpse into Erdős and Szekeres's first joint paper [ES1]and the emergence of a unique group of unknown young Jewish Hungarian mathematicians in Budapest, many of whom were destined to a great mathematical future. To my request to reproduce these remarkable reminiscences, George Szekeres answered in the March 5, 1992 letter:

> Dear Alexander, . . . Of course, as far as I am concerned, you may quote anything you like (or see fit) from my old reminiscences in “The Art of Counting”. . . But of course it may be different with MIT Press, that you have to sort out with them.

György Szekeres and Esther Klein, Bükk Mountains, Northern Hungary, 1938 (shortly after their 1937 marriage), provided by George Szekeres

I am grateful to George Szekeres and the MIT Press for their kind permission to reproduce George’s memoirs here. His Reminiscences are sad and humorous at the same time, and warm above all. György Szekeres recalls [Szek]:

It is not altogether easy to give a faithful account of events which took place forty years ago, and I am quite aware of the pitfalls of such an undertaking. I shall attempt to describe the genesis of this paper, and the part each of us played in it, as I saw it then and as it lived on in my memory.

For me there is a bit more to it than merely reviving the nostalgic past. Paul Erdős, when referring to the proof of Ramsey’s theorem and the bounds for Ramsey numbers given in the paper, often attributed it to me personally (e.g., in [E42.06]), and he obviously attached some importance to this unusual step of pinpointing authorship in a joint paper. At the same time the authorship of the "second proof" was never clearly identified.

I used to have a feeling of mild discomfort about this until an amusing incident some years ago reassured me that perhaps I should not worry about it too much. A distinguished British mathematician gave a lunch-hour talk to students at Imperial College on Dirichlet's box principle, and as I happened to be with Imperial, I went along. One of his illustrations of the principle was a beautiful proof by Besicovitch of Paul's theorem (2nd proof in [ES1]), and he attributed the theorem itself to "Erdős and someone whose name I cannot remember." After the talk I revealed to him the identity of Paul's coauthor (incidentally also a former coauthor of the speaker) but assured him that no historical injustice had been committed as my part in the theorem was less than ε.

Paul Erdős, early 1930s, Budapest

The origins of the paper go back to the early 1930s. We had a very close circle of young mathematicians, foremost among them Erdős, Turán, and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the 1930s, a vicious world war and our scattering to the four corners of the world. I myself was an "outsider," studying chemical engineering at the Technical University, but often joined the mathematicians at weekend excursions in the charming hill country around Budapest and (in summer) at open air meetings on the benches of the city park.

Paul, then still a young student but already with a few victories in his bag, was always full of problems and his sayings were already a legend. He used to address us in the same fashion as we would sign our names under an article and this habit became universal among us; even today I often call old members of the circle by a distortion of their initials.

"Szekeres Gy., open up your wise mind." This was Paul's customary invitation—or was it an order?—to listen to a proof or a problem of his. Our discussions centered around mathematics, personal gossip, and politics. It was the beginning of a desperate era in Europe. Most of us in the circle belonged to that singular ethnic group of European society which drew its cultural heritage from Heinrich Heine and Gustav Mahler, Karl Marx and Cantor, Einstein and Freud, later to become the principal target of Hitler's fury. Budapest had an exceptionally large Jewish population, well over 200,000, almost a quarter of the total. They were an easily identifiable group speaking an inimitable jargon of their own and driven by a strong urge to congregate under the pressures of society. Many of us had leftist tendencies, following the simple reasoning that our problems can only be solved on a global, international scale and socialism was the only political philosophy that offered such a solution. Being a leftist had its dangers and Paul was quick to spread the news when one of our members got into trouble: "A. L. is studying the theorem of Jordan." It meant that following a political police action A. L. has just verified that the interior of a prison cell is not in the same component as the exterior. I have a dim recollection that this is how I first heard about the Jordan curve theorem.

Apart from political oppression, the Budapest Jews experienced cultural persecution long before anyone had heard the name of Hitler. The notorious "numerus clausu" was operating at the Hungarian Universities from 1920 onwards, allowing only 5% of the total student intake to be Jewish. As a consequence, many of the brightest and most purposeful students left the country to study elsewhere, mostly in Germany, Czechoslovakia, Switzerland, and France. They formed the nucleus of that remarkable influx of Hungarian mathematicians and physicists into the United States, which later played such an important role in the fateful happenings towards the conclusion of the second world war.

For those of us who succeeded in getting into one of the home universities, life was troublesome and the outlook bleak. Jewish students were often beaten up and humiliated by organized student gangs and it was inconceivable that any of us, be he as gifted as Paul, would find employment in academic life. I myself was in a slightly better position as I studied chemical engineering and therefore resigned to go into industrial employment, but for the others even a high school teaching position seemed to be out of reach.

Paul moved to Manchester soon after his Ph.D. at Professor Mordell's invitation, and began his wanderings which eventually took him to almost every mathematical corner of the world. But in the winter of 1932/1933 he was still a student; I had just received my chemical degree and, with no job in sight, I was able to attend the mathematical meetings with greater regularity than during my student years. It was at one of these meetings that a talented girl member of our circle, Esther Klein (later to become Esther Szekeres), fresh from a one-semester stay in Göttingen, came up with a curious problem: given 5 points in the plane, prove that there are 4 which form a convex quadrilateral. In later years this problem frequently appeared in student's competitions, also in the *American Mathematical Monthly* (53(1946)462, Problem

E740). Paul took up the problem eagerly and a generalization soon emerged: is it true that out of $2^{n-2}+1$ points in the plane one can always select n points so that they form a convex n-sided polygon? I have no clear recollection how the generalization actually came about; in the paper we attributed it to Esther, but she assures me that Paul had much more to do with it. We soon realized that a simple-minded argument would not do and there was a feeling of excitement that a new type of geometrical problem emerged from our circle which we were only too eager to solve. For me the fact that it came from Epszi (Paul's nickname for Esther, short for ε) added a strong incentive to be the first with a solution and after a few weeks I was able to confront Paul with a triumphant "E. P., open up your wise mind." What I really found was Ramsey's theorem from which it easily followed that there exists a number $N < \infty$ such that out of N points in the plane it is possible to select n points which form a convex n-gon. Of course at that time none of us knew about Ramsey. It was a genuinely combinatorial argument and it gave for N an absurdly large value, nowhere near the suspected 2^{n-2}. Soon afterwards Paul produced his well-known "second proof" which was independent of Ramsey and gave a much more realistic value for N; this is how a joint paper came into being.

I do not remember now why it took us so long (a year and a half) to submit the paper to the *Compositio*. These were troubled times and we had a great many worries. I took up employment in a small industrial town, some 120 kms from Budapest, and in the following year Paul moved to Manchester; it was from there that he submitted the paper.

I am sure that this paper had a strong influence on both of us. Paul with his deep insight recognized the possibilities of a vast unexplored territory and opened up a new world of combinatorial set theory and combinatorial geometry. For me it was the final proof (if I needed any) that my destiny lay with mathematics, but I had to wait for another 15 years before I got my first mathematical appointment in Adelaide. I never returned to Ramsey again.

Paul's method contained implicitly that $N > 2^{n-2}$, and this result appeared some 35 years later [ES2] in a joint paper, after Paul's first visit to Australia. The problem is still not completely settled and no one yet has improved on Paul's value of

$$N = \binom{2n-4}{n-2} + 1.$$

Of course we firmly believe that $N = 2^{n-2} + 1$ is the correct value. ■

These moving memories prompted me to ask for more. George Szekeres replied on November 30, 1992:

> Dear Sasha, ... Marta Svéd rang me some time ago from Adelaide, reminding me of an article that I was supposed to write about the old Budapest times... From a distance of 60 years, as I approach 82, these events have long lost their "romantic" freshness... My memories of those times are altogether fading away into the remote past, even if they are occasionally refreshed on my visits to Budapest. (I will certainly be there to celebrate Paul's 80-th birthday.)

The following year George did come to Hungary, and we met for dinner during the conference dedicated to Paul Erdős's 80th birthday, when George Szekeres and Esther Klein shared with me unique memories of Tibor Gallai, a key Budapest group member. See them in Chapter 42, dedicated to the Gallai Theorem.

29.3 Progress on the Happy End Problem

In May 1960, when Paul Erdős visited George Szekeres in Adelaide, they improved the lower bound of the Happy End problem [ES2].

Lower Bound 29.15 (Erdős and Szekeres [ES2]). $2^{n-2} \leq ES(n)$, where $ES(n)$ is the Erdős–Szekeres function, i.e., the smallest integer such that any $ES(n)$ points in general position contain a convex n-gon.[4]

It is fascinating how sure Erdős and Szekeres were of their conjecture. In one of his last, posthumously published problem papers [E97.18], Erdős attached the prize and modestly attributed the conjecture to Szekeres: "I would certainly pay $500 for a proof of Szekeres's conjecture."

Erdős–Szekeres Happy End $500 Conjecture 29.16

$$ES(n) = 2^{n-2} + 1.$$

Their confidence is surprising[5] because the foundation for the conjecture was very thin, just results 29.2 and 29.5:

$$ES(4) = 5,$$
$$ES(5) = 9.$$

Computing exact values of the Erdős–Szekeres function $ES(n)$ proved to be a very difficult matter. It took over 70 years to make the next step. In 2006, George Szekeres (posthumously) and Lindsay Peters, with the assistance of Brendan McKay and heavy computing, have established one more exact value in the paper [SP] written "In memory of Paul Erdős":

Result 29.17 (G. Szekeres and L. Peters [SP]). $ES(6) = 17$.

In his latest surveys [Gra7], [Gra8],[6] Ronald L. Graham is offering $1000 for the first proof, or disproof, of the Erdős–Szekeres Happy End Conjecture 29.16.

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[4] Erdős and Szekeres actually proved a strict inequality.

[5] In fact, Paul Erdős repeated $500 offer for the proof of the conjecture in [E97.21], but offered there "only 100 dollars for a disproof."

[6] I thank Ron Graham for kindly providing the preprints.

George Szekeres was, of course, correct when he wrote in his 1973 reminiscences above that their 1935 upper bound

$$ES(n) \leq \binom{2n-4}{n-2} + 1$$

had not been improved. In fact, it withstood all attempts of improvement until 1997 when Fan Chung and Ronald L. Graham [CG] willed it down by 1 point to

$$ES(n) \leq \binom{2n-4}{n-2}.$$

In the process, Chung and Graham offered a fresh approach which started an explosion of improvements. First it was improved by Daniel J. Kleitman and Lior Pachter [KP] to

$$ES(n) \leq \binom{2n-4}{n-2} + 7 - 2n.$$

Then came Géza Tòth and Pavel Valtr [TV1] with

$$ES(n) \leq \binom{2n-5}{n-2} + 2.$$

These developments happened so swiftly that all three above papers appeared in the same 1998 issue of *Discrete Computational Geometry*! In 2005 Tòth and Valtr came again [TV2] with the best known today upper bound

$$ES(n) \leq \binom{2n-5}{n-2} + 1,$$

which is about half of the original Erdős–Szekeres upper bound.

Paul Erdős's trains of thought are infinite—they never end, and each problem gives birth to a new problem, or problems. The Happy End Problem is not an exception. Paul writes about the AfterMath of the Happy End Problem with his vintage humor and warmth [E83.03]:

> Now there is the following variant which I noticed when I was once visiting the Szekeres in 1976 in Sydney, the following variant which is of some interest I think. It goes as follows. $n(k)$ is derived as follows, if it exists. It is the smallest integer with the following property. If you have $n(k)$ points in the plane, no three on a line, then you can always find a convex k-gon with the additional restriction that it doesn't contain a point in the interior. You know this goes beyond the theorem of Esther, I not only require that the k points should form a convex k-gon, I also require that this convex k-gon should contain none of the [given] points in its interior. And surprisingly enough this gives a lot of new difficulties. For example it is trivial that $n(4)$ is again 5, that is no problem. Because if your have a convex quadrilateral, if no point is inside we are happy; if from the five points one of them is inside you draw the diagonal (AC, Fig. 29.1):

And you join these (AE, EC) and now this convex quadrilateral ($AECD$) contains none of the points. And if you have four points and the fifth point is inside then you take this quadrilateral. This is convex again and has no point in the inside. And Harborth proved that $n(5) = 10$. $f(5)$ was 9 in Esther Klein's problem but here $n(5)$ is 10. He dedicated his paper to my memory when I became an archeological discovery. When you are 65 you become an archeological discovery. Now, nobody has proved that $n(6)$ exists. That you can give, for every t, t points in the plane, no three on a line and such that every convex hexagon contains at least one of the points in its interior. It's perfectly possible that can do that. Now Harborth suggested that maybe $n(6)$ exists but $n(7)$ doesn't. Now I don't know the answer here.

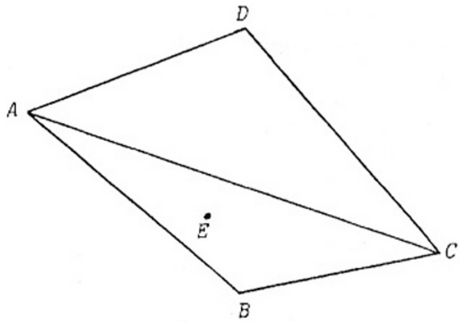

Fig. 29. 1

Indeed, in 1978 Heiko Harborth [Harb] of Braunschweig Technical University, Germany, proved that $n(5) = 10$. In 1983 J. D. Horton [Hort] of the University of New Brunswick, Canada, proved Harborth's conjecture that $n(t)$ does not exist for $t \geq 7$. This left a mystifying gap that is alive and well today:

Open Problem 29.18 Does $n(6)$ exist? If yes, find its value.

This new rich train of thought now includes many cars. I would like to share with you my favorite, the beautiful 2005 result by Adrian Dumitrescu of the University of Wisconsin-Milwaukee.

Dumitrescu's Theorem 29.19 [Dum].[7] For each finite sequence $h_0, h_1, \ldots, h_k$, with $h_i \geq 3$ $(i = 0, \ldots, k)$ there is an integer $N = N(h_0, h_1, \ldots, h_k)$ such that any set S of at least N points in general position in the plane contains either

an empty convex h_0-gon (i.e., a convex h_0-gon that contains no points of S in its interior)

or

k convex polygons $P_1, P_1, \ldots, P_1$, where P_i is an h_i-gon such that P_i strictly contains P_{i+1} in its interior for $i = 1, \ldots, k-1$.

[7] Adrian mistakenly credits 1975 Erdős's paper [E72.25] with the birth of the problem about empty convex polygons. In the cited story Erdős clearly dates it to his 1976 visit of the Szekereses.

29.4 The Happy End Players Leave the Stage as Shakespearian Heroes

Paul Erdős named it *The Happy End Problem*. He explained the name often in his talks. On June 4, 1992 in Kalamazoo I took notes of his talk:

> I call it The Happy End Problem. Esther captured George, and they lived happily ever after in Australia. The poor things are even older than me.

This paper also convinced George Szekeres to become a mathematician. For Paul Erdős the paper had a happy end too: it became one of his early mathematical gems and Paul's first of the numerous contributions to and leadership of the Ramsey Theory and, as Szekeres put it, of "a new world of combinatorial set theory and combinatorial geometry."

The personages of The Happy End Problem appear to me like heroes of Shakespeare's plays. Paul, very much like *Tempest's* Prospero, gave up all his property, including books, to be free. George and Esther were so close that they ended their lives together, like Romeo and Juliet. In the late summer 2005 e-mail, Tony Guttmann conveyed to the world the sad news from Adelaide:

> George and Esther Szekeres both died on Sunday morning [August 28, 2005]. George, 94, had been quite ill for the last 2–3 days, barely conscious, and died first. Esther, 95, died an hour later. George was one of the heroes of Australian mathematics, and, in her own way, Esther was one of the heroines.

I always wanted to know the membership in this amazing Budapest group. On May 28, 2000, during a dinner in the restaurant of the Rydges North Sydney Hotel at 54 McLaren Street,[8] I asked George Szekeres and Esther Klein to name the members of their group, so to speak the Choir of the Happy End Production. Esther produced, signed and dated the following list of young participants, of which according to her "half a dozen usually met":

Paul Erdős, Tibor Grünwald (Gallai), Géza Grünwald (Gergör), Esther Klein (Szekeres), Lily Székely (Sag), George (György) Szekeres, Paul Turán, Martha Wachsberger (Svéd), and Endre Vázsonyi.[9]

George Szekeres also told me that night "my student and I proved Esther's Conjecture for 17 with the use of computer." "Which computer did you use?" asked I. "I don't care how a pencil is made," answered George.

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[8] Esther wrote the list on the letterhead of the hotel.

[9] Mikós Ság and László Molnár occasionally joined the group too.