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De Bruijn–Erdős's Theorem and Its History

26.1 De Bruijn–Erdős's Compactness Theorem[5]

They were both young. On August 4, 1947 the 34-year-old Paul Erdős, in a letter to the 29-year-old Nicolaas Govert de Bruijn of Delft, The Netherlands, offered the following conjecture [E47/8/4ltr]:

Let G be an infinite graph. Any finite subset of it is the sum of k independent sets (two vertices are independent if they are not connected). Then G is the sum of k independent sets.

Paul added in parentheses "I can only prove it if $k = 2$". In his 5-page August 18, 1947 reply [Bru1], de Bruijn reformulated the Erdős conjecture in a way that is very familiar to us today:

Theorem: Let G be an infinite graph, any finite subgraph of which can be k-colored (that means that the nodes are coloured with k different colours, such that the two connected nodes have different colours). Then G can be k-coloured.

Following a nearly three-page long transfinite induction proof of the "Theorem," de Bruijn observed [Bru1]:

> I am sorry that this proof takes so much paper; its idea, however, is simple. Perhaps, you do not call it a proof at all, because it contains "Well ordering", but we can hardly expect to get along without that.

This was an insightful observation, for de Bruijn and Erdős relied on the Axiom of Choice or equivalent (like Well-Ordering Principle or Zorn's Lemma) very heavily. When in early 2004 Professor de Bruijn received from me a reprint of Shelah–Soifer 2003 paper (to be discussed in Chapter 46) which analyzed what happens with the de Bruijn–Erdős Theorem in the absence of the Axiom of Choice, de Bruijn replied to me on January 27, 2004 as follows [Bru7]:

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[5] I am infinitely grateful to N.G. de Bruijn for providing me with copies of his correspondence with Paul Erdös.

A. Soifer, *The Mathematical Coloring Book*,
DOI 10.1007/978-0-387-74642-5_26,

> About the axiom of choice, I remember a conversation with Erdős, during a walk around 1954. I told him that I hated the axiom of choice, and that I wanted to do analysis without it, maybe except for the countable case. He was surprised, and said: but you were always so good at it. Indeed, I had loved transfinite induction, just because it worked exactly the same way as ordinary induction.

This invaluable de Bruijn's e-mail also contained the conclusion of the story of the de Bruijn–Erdős Theorem [Bru5]:

> Erdős and I did not take any steps to publish the k-coloring theorem. In 1951 I met Erdős in London, and from there we went together by train to Aberdeen, which took a full day. It was during that train ride that he told me about the topological proof of the k-coloring theorem. Not long after that, he wrote it up and submitted it for publication. I do not think I had substantial influence on that version.

Let us look at a proof of this celebrated theorem, which we have formulated without proof and used in chapter 5.

De Bruijn–Erdős's Compactness Theorem 26.1 ([BE2], 1951). An infinite graph G is k-colorable if and only if every finite subgraph of G is k-colorable.[6]

In what follows, we will need a few definitions from set theory.

Given a set A; any subset R of the so-called Cartesian product $A \times A = \{(a_1, a_2) : a_1, a_2 \in A\}$ is called a *binary relation* on A. We write $a_1 R a_2$ to indicate that the ordered pair (a_1, a_2) is an element of R.

Poset, or *partially ordered set*, is a set A together with a particularly "nice" binary relation on it, i.e., a relation that satisfies the following three properties:

1. Reflexivity: $a \le a$ for all $a \in A$;
2. Anti-symmetry: If $a \le b$ and $b \le a$ for any $a, b \in A$, then $a = b$;
3. Transitivity: If $a \le b$ and $b \le c$ for any $a, b, c \in A$, then $a \le c$.

A *chain*, or *totally ordered set*, is a poset that satisfies a fourth property:

4. Comparability: For any $a, b \in A$, either $a \le b$ or $b \le a$.

Let A be a set with a partial ordering $\le$ defined on it, and B a subset of A. An *upper bound* of B is an element $a \in A$ such that $b \le a$ for every $b \in B$.

> Let $\le$ be a partial ordering on a set A, and $B \subseteq A$. Then, we say that $b \in B$ is a *maximal element* of B if there exists no $x \in B$ such that $b \le x$ and $x \ne b$.

In 1935 Max Zorn (1906, Germany-1993, USA) introduced the following important tool, which he called *maximum principle*. (It was shown by Paul J. Campbell that, in fact, a number of famous mathematicians—Hausdorff, Kuratowski, and Brouwer—preceded Zorn, but Zorn's name got as attached to this tool as, say, Amerigo Vespucci's name to America.)

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[6] This theorem requires the Axiom of Choice or equivalent.

Zorn's Lemma 26.2 If S is any non-empty partially ordered set in which every chain has an upper bound, then S has a maximal element.

During the summer of 2005, I supervised at the University of Colorado, a research month of Dmytro (Mitya) Karabash, who had just completed his freshman year at Columbia University, and asked to come and work with me. One of my assignments for him was to prove the de Bruijn–Erdős Theorem 26.1, and then to write the solution as well. After going through several revisions, Mitya produced a fine proof, which follows here, slightly edited by me.[7]

Proof of Theorem 26.1 by D. Karabash: We say that G has the property P and write $P(G)$ if every finite subgraph of G is k-colorable. For a graph G we write $G = (V, E)$, where V is the vertex set and E is the edge set of G. Now let S be the set of all graphs with the property Pwhich are obtained from G by an addition of edges, i.e., $S = \{(V, F)|E \subseteq F \text{ and } P(V, F)\}$.

Let S be partially ordered by the inclusion of edge sets. Observe that for every chain A_i in S, its union $A = (V, \bigcup_i E(A_i))$ is also in S (here $E(A_i)$ stands for the edge set of the graph A_i). Indeed, every finite subgraph F of A must be contained in some A_i (because F is finite) and therefore F is k-colorable. Since A has property P, A is in S, as desired.

We have just proved that in S every chain has an upper bound. Therefore, by *Zorn's Lemma*, S contains a maximal element, call it M. Since M is in S, M has property P; since M *is* maximal, no edges can be added to M without violating property P.

We will now prove that *non-adjacency* (here to be denoted by the symbol $\neg adj$) is an equivalence relation on M, i.e., for every $a,\ b,\ c \in V(M)$, if $a \neg adj\ b$ and $b \neg adj\ c$, then $a \neg adj\ c$. Let us consider all finite subgraphs of M that contain a and b, and all k-colorings on them. Since $a \neg adj\ b$, there must be a subgraph M_{ab} for which the colors of a and b are the same for all k-colorings of this subgraph, for otherwise we could add the edge ab to M with preservation of property P and attain a contradiction to M being a maximal element of S. Construct a subgraph M_{bc} similarly. The subgraph $M_{ab} \cup M_{bc}$ is finite and thus k-colorable. $M_{ab} \cup M_{bc}$ contains subgraphs M_{ab} and M_{bc}, therefore by construction of M_{ab} and M_{bc}, any coloring of $M_{ab} \cup M_{bc}$ must have pairs (a, b) and (b, c) colored in the same color. Thus, a and c have the same color for all k-colorings of the subgraph $M_{ab} \cup M_{bc}$ and therefore a is not adjacent to c.

From the fact that the non-adjacency is an equivalence relation on M, we conclude that the edge-complement M' of M is made of some number of disjoin complete graphs K_i because in M' *adjacency* is an equivalence relation. Therefore $a \in K_i,\ b \in K_j,\ i \neq j$ implies $a \neg adj\ b$ in M' or equivalently $a\ adj\ b$ in M.

[7] You can also read the original proof in [BE2]; a nice proof by L. Pósa in the fine book [Lov2] by László Lovász; and a clear insightful proof of the countable case in the best introductory book to Ramsey Theory [Gra2] by Ronald L. Graham.

Suppose there is more than k disjoint complete subgraphs K_i in M'. Then pick $k+1$ vertices, all from distinct $V(K_i)$. Since all of the vertices are located in distinct $V(K_i)$, they must all be pairwise non-adjacent in M' and thus form a complete graph M_{k+1} on $k+1$ vertices in M. We obtained a finite subgraph M_{k+1} of M which is not k-colorable, in contradiction to M having property P. Therefore, M' consists of at most k complete subgraphs $V(K_i)$, $i = 1, \ldots, k$. Now we can color each subgraph $V(K_i)$ in a different color. Since no two vertices of an $V(K_i)$ are adjacent in M, this is a proper k-coloring. Since G is a subgraph of M, G is k-colorable, as desired. ∎

Corollary 26.3 Compactness Theorem 5.1 is true.

The proof of Theorem 26.1 is much more powerful than you may think. It works not only for graphs, but even for their important generalization—hypergraphs. Permit me to burden you with a few definitions.

As you recall from chapter 12, a *graph* $G = G(V, E)$ is a non-empty set V (of vertices) together with a family E of 2-element subsets (edges) of V. If we relax the latter condition, we will end up with a hypergraph.

A *hypergraph* $H = H(V, E)$ is a non-empty set V (of *vertices*) together with a family E of subsets (*edges*) of V each containing *at least two* elements. Thus, an edge e of H is a subset of V; its elements are naturally called *vertices of the edge e* (or *vertices incident with e*).

Let n be a positive integer. We would say that a hypergraph H is *n-colored*, if each vertex of H is assigned one of the given n colors. If *all* vertices of an edge e are assigned the same color, we call e a *monochromatic edge*.

The *chromatic number* $\chi(H)$ of a hypergraph H is the smallest number of colors n for which there is an n-coloring of H without monochromatic edges.

A hypergraph $H_1 = H_1(V_1, E_1)$ is called a *subhypergraph* of a hypergraph $H = H(V, E)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Compactness Theorem for Hypergraphs 26.4 The chromatic number $\chi(H)$ of a hypergraph H is equal to the maximum chromatic number of its finite subhypergraphs.

Proof Repeat word-by-word the proof of Theorem 26.2 (just replace "graph" by "hypergraph"). ∎

26.2 Nicolaas Govert de Bruijn

Ever since 1995, I have exchanged numerous e-mail messages—and sometimes letters—with the Dutch mathematician N. G. de Bruijn. His elegant humor, openness in expressing views even on controversial issues, and his eyewitness accounts of post W.W.II events in Holland made this correspondence most fascinating and enjoyable for me. We also shared interest in finding out who created the conjecture on monochromatic arithmetic progressions, which was proven by B. L. van der Waerden (see chapter 34 for the answer). Yet, for years I have been asking Professor de Bruijn to share with me his autobiography to no avail. For a long while, I did not even know what "N. G." stood for. On October 29, 2005, I tried to be a bit more specific in my e-mail. I wrote:

> May I ask you to describe your life – and any participation in political affairs – during the occupation, May 1940–1945, and during the first post war years, 1945 up to your Sep-1952 appointment to replace Van der Waerden at Amsterdam?

De Bruijn understood my maneuver, but provided the desired reply on November 1, 2005 [Bru12]:

> You are asking for an autobiography in a nutshell.
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> I was born in 1918 [on July 9th in Den Haag], so I just left elementary school in 1930 when the great depression broke out. I managed to finish secondary school education in 4 years (the standard was 5 or 6). After that, I could not get any job, and could not get any financial support for university education. I used my next two years (1934–1936) to study mathematics from books, without any teacher. I passed the examinations that qualified me as a mathematics teacher in all secondary schools in the Netherlands. But there weren't any jobs. Yet I had some success: I could get a small loan that enabled me to study mathematics and physics at Leiden University. In the academic year 1936–1937 I attended courses in physics and astronomy, and in 1937–1938 courses in mathematics on the master's degree level. That was all the university education I had. The most inspiring mathematician in those days at Leiden was H. D. Kloosterman.
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> In 1939 I was so lucky to get an assistantship at Delft Technical University. It didn't pay very much, but it left me plenty of time to get involved in various kinds of mathematical research. It was quite an inspiring environment, and actually it was the only place in the Netherlands that employed mathematical assistants (Delft had about 8 or 9 of them). In 1940 the country was occupied, and from then on the main problem was to avoid being drawn into forced labour in Germany. In that respect my assistantship was a good shelter for quite some time.
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> All the time I lived with my parents in The Hague, not so safe as it seemed. We were hiding a Jewish refugee (a German boy, a few years younger than me), who assisted my brother in producing and distributing forbidden radio material, like antennas that made it possible to eliminate the heavy bleep-bleep-bleep that the Germans used in the wavelengths of the British Radio. And later, when radios were forbidden altogether, my brother built miniature radios, hidden in old encyclopedia volumes. All this activity ended somewhere in the beginning of 1944 when our house was raided by the *Sicherheitspolizei*. My brother and his Jewish assistant where taken into custody, but by some strange coincidence they came back the next day. Nevertheless they had to leave to a safer place, where both of them survived the war. A few months later, I got my first real job. It was at the famous Philips Physical Labs at Eindhoven. The factory worked more or less for German war production, just like most factories in the country, but the laboratories could just do what they always did.
>
> Four months later, Eindhoven was occupied by the allied armies, in their move towards the battle of Arnhem. From then on we were cut off from the rest of the country, where people had a very bad time.
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> So this was about my life during the war. Compared to others, I had been quite lucky. I had even managed to get my doctorate at the [Calvinist] Free University, Amsterdam [March 1943], just a few weeks before all universities in the country

were definitely closed (Leiden University had already been closed in 1940, because of demonstrations against the dismissal of Jewish professors).

In 1946 I got a professorship at Delft Technical University. I had to do quite elementary teaching, leaving me free to do quite some research, mainly in analytical number theory. It got me into correspondence with Erdős, and around 1948 he visited us at Delft.

In 1951 I made a mathematical trip abroad for the first time in my life. There I had contact with Erdős too. We had a long train ride together from London to Edinburgh.

In 1952 I got that [Van der Waerden's] professorship at Amsterdam, at that time the mathematical Mecca of the Netherlands. I stayed there until 1960, when I got my professorship at Eindhoven Technological University, where I retired in 1984. After that, I always kept a place to work there.

I think this is all you wanted to know.

In fact, on November 1, 2005 I asked for a few additional details:

I know you are one of the most modest men. Yet, I would think you were not just an observer when your family hid a Jewish boy, and your brother did activities not appreciated by the occupiers. Would you be so kind to share with me your role is these activities during 1940–1945? What were the names of your brother and his Jewish-German assistant? What was the difference in age between you and your brother?

Two days later, my questions were answered [Bru13]:

I hardly ever participated in my brother's activities. At most three times I delivered an antenna or a radio to some stranger. My brother was a year and a half older than I. His name was Johan.

The Jewish boy's name was Ernest (Ernst) Goldstern. He was born 24 December 1923 (in Muenchen, I believe). His family came to Holland in the late 1930's, where Ernst just completed his secondary school education in Amsterdam. He lived with us in The Hague from 1940 to 1944. I helped him to study advanced mathematics, which he could use after the war. He went into Electrical Engineering and got his degree in Delft. He died 19 January 1993. Johan died in 1996.

On July 9, 2008 N. G. de Bruijn is turning 90—Happy Birthday, Nicolaas!