11
Rational Coloring

I would like to mention here one more direction of assault on the chromatic number of the plane. By placing Cartesian coordinates on the plane E^2, we get an algebraic representation of the plane as the set of all ordered pairs (x, y) with coordinates x and y from the set R of real numbers, with the distance between two points defined as usual:

$$E^2 = \{(x, y) : x, y \in R\}. \tag{11.1}$$

Since by De Bruijn–Erdős's Theorem 5.1 it suffices to deal with finite subsets of R^2, we can surely restrict the coordinates in (11.1) to some subset of R. The problem is, which subset should we choose?

A set A is called *countable* if there is a one-to-one correspondence between A and the set of positive integers N.

For any set C, we define C^2 as the set of all ordered pairs (c_1, c_2), where c_1 and c_2 are elements of C:

$$C^2 = \{(c_1, c_2) : c_1, c_2 \in C\}.$$

Open Problem 11.1 Find a countable subset C of the set of real numbers R such that the chromatic number $\chi(C^2)$ is equal to the chromatic number $\chi(E^2)$ of the plane.

The set Q of all rational numbers would not work, as Douglas R. Woodall showed in 1973.

Chromatic Number of Q^2 11.2 (D. R. Woodall, [Woo1])

$$\chi(Q^2) = 2.$$

Proof by D. R. Woodall ([Woo1]): We need to color the points of the rational plane Q^2, i.e., the set of ordered pairs (r_1, r_2), where r_1 and r_2 are rational numbers. We partition Q^2 into disjoint classes as follows: we put two pairs (r_1, r_2), and (q_1, q_2)

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into the same class if and only if both $r_1 - q_1$ and $r_2 - q_2$ have odd denominators when written in their lowest terms (an integer n is written in its lowest terms as $\frac{n}{1}$).

This partition of Q^2 into subsets has an important property: if the distance between two points of Q^2 is 1, then both points belong to the same subset of the partition! Indeed, let the distance between $(r_1,\ r_2)$, and $(q_1,\ q_2)$ be equal to 1. This means precisely that

$$\sqrt{(r_1 - q_1)^2 + (r_2 - q_2)^2} = 1,$$

i.e.,

$$(r_1 - q_1)^2 + (r_2 - q_2)^2 = 1$$

Let $r_1 - q_1 = \frac{a}{b}$ and $r_2 - q_2 = \frac{c}{d}$ be these differences written in their lowest terms. We have

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 1$$

i.e.,

$$a^2d^2 + b^2c^2 = b^2d^2.$$

Therefore, b and d must be both odd (can you see why?), i.e., by our definition above, $(r_1,\ r_2)$, and $(q_1,\ q_2)$ must belong to the same subset.

Since any class of our partition can be obtained from any other class of the partition by a translation (can you prove this?), it suffices for us to color just one class, and extend the coloring to the whole Q^2 by translations. Let us color the class that contains the point (0,0). This class consists of the points $(r_1,\ r_2)$, where in their lowest terms the denominators of both $r_1,\ r_2$ are odd (can you see why?). We color red the points of the form $\left(\frac{o}{o}, \frac{o}{o}\right)$ and $\left(\frac{e}{o}, \frac{e}{o}\right)$, and color blue the points of the form $\left(\frac{o}{o}, \frac{e}{o}\right)$ and $\left(\frac{e}{o}, \frac{o}{o}\right)$, where o stands for an odd number and e for an even number. In this coloring, two points of the same color may not be distance 1 apart (prove this on your own). ■

Then there came a "legendary unpublished manuscript," as Peter D. Johnson, Jr. referred [Joh8] to the paper by Miro Benda, then with the University of Washington, and Micha Perles, then with the Hebrew University, Jerusalem. The widely circulated and admired manuscript was called *Colorings of Metric Spaces*. Peter Johnson tells its story on the pages of *Geombinatorics* [Joh8]:

> The original manuscript of "Colorings...," from which some copies were made and circulated (and then copies were made of the copies, etc.), was typed in Brazil in 1976. I might have gotten my first or second generation copy in 1977.... The paper was a veritable treasure trove of ideas, approaches, and results, marvelously informative and inspiring.

> During the early and mid 1980s "Colorings..." was mentioned at a steady rhythm, in my experience, at conferences and during visits. I don't remember who said what about it, or when (except for a clear memory of Joseph Zaks mentioning it, at the University of Waterloo, probably in 1987), but it must surely win the all-time prize for name recognition in the "unpublished manuscript" category.

Johnson's story served as an introduction and homage to the conversion of the unpublished manuscript into the Benda–Perles publication [BP] in *Geombinatorics*' January 2000 issue.

This paper, dreamed up in the fall of 1975 over a series of lunches the two authors shared in Seattle, created a new, algebraic approach to the chromatic number of the plane problem. Moreover, it formulated a number of open problems, not directly connected to the chromatic number of the plane, problems that gave algebraic chromatic investigations their own identity. Let us take a look at a few of their results and problems. First of all, Benda and Perles prove (independently; apparently they did not know about the Woodall's paper) Woodall's result 11.2 about the chromatic number of the rational plane. They are a few years too late to coauthor the result, but their analysis allows an insight into the algebraic structure that we do not find in Woodall's paper. They then use this insight to establish more sophisticated results and the structure of the rational spaces they study.

Chromatic Number of $\boldsymbol{Q^3}$ 11.3 (Benda & Perles [BP])

$$\chi(Q^3) = 2.$$

Chromatic Number of $\boldsymbol{Q^4}$ 11.4 (Benda & Perles [BP])

$$\chi(Q^4) = 4.$$

Benda and Perles then pose important problems.

Open Problem 11.5 (Benda & Perles [BP]) Find $\chi(Q^5)$ and, in general, $\chi(Q^n)$.

Open Problem 11.6 (Benda & Perles [BP]) Find the chromatic number of $Q^2(\sqrt{2})$ and, in general, of any algebraic extension of Q^2.

This direction was developed by Peter D. Johnson, Jr. from Auburn University [Joh1], [Joh2], [Joh3], [Joh4], [Joh5] and [Joh6]; Joseph Zaks from the University of Haifa, Israel [Zak1], [Zak2], [Zak4], [Zak6], [Zak7]; Klaus Fischer from George Mason University [Fis1], [Fis2]; Kiran B. Chilakamarri [Chi1], [Chi2], [Chi4]; Douglas Jungreis, Michael Reid, and David Witte ([JRW]); and Timothy Chow ([Cho]). In fact, Peter Johnson has published in 2006 in *Geombinatorics* "A Tentative History and Compendium" of this direction of research inquiry [Joh9]. I refer the reader to this survey and works cited there for many exciting results of this algebraic direction.

In the recent years Matthias Mann from Germany entered the scene and discovered partial solutions of Problem 11.5, which he published in *Geombinatorics* [Man1].

Lower Bound for $\boldsymbol{Q^5}$ 11.7 (Mann [Man1])

$$\chi(Q^5) \geq 7.$$

This jump from $\chi(Q^4) = 4$ explains the difficulty in finding $\chi(Q^5)$, exact value of which remains open. Matthias then found a few more important lower bounds [Man2].

Lower Bounds for Q^6, Q^7 and Q^8 11.8 (Mann [Man2])

$$\chi(Q^6) \geq 10;$$
$$\chi(Q^7) \geq 13;$$
$$\chi(Q^8) \geq 16.$$

In reply to my request, Matthias Mann wrote about himself on January 4, 2007:

As I have not spent much time on Unit Distance-Graphs since the last article in Geombinatorics 2003, I do not have any news concerning this topic. To summarize, I found the following chromatic numbers:

Q^5 >= 7
Q^6 >= 10
Q^7 >= 13
Q^8 >= 16

The result for Q^8 improved the upper bounds for the dimensions 9–13.

For the Q^7 I think that I found a graph with chromatic number 14, but up to now I cannot prove this result because I do not trust the results of the computer in this case.

Now something about me: I was born on May 12th 1972 and studied mathematics at the University of Bielefeld, Germany from 1995–2000. I wrote my Diploma-thesis (the "Dipl.-Math." is the old German equivalent to the Master) in 2000. It was supervised by Eckhard Steffen, who has worked on edge-colorings. I had the opportunity to choose the topic of my thesis freely, so I red the book "Graph Coloring Problems" by Tommy Jensen and Bjarne Toft (Wiley Interscience 1995) and was very interested in the article about the Hadwiger-Nelson-Problem, and found the restriction to rational spaces even more interesting. After reading articles of Zaks and Chilakamarri (a lot of them in *Geombinatorics*), I started to work on the problem with algorithms.

Unfortunately, I had no opportunity to write a Ph.D.-thesis about unit distance-graphs, so I started work as an information technology consultant in 2000.

In the previous chapter, you have already met Josef Cibulka, a first year graduate student at Charles University in Prague. His essay submitted to *Geombinatorics* on March 31, 2008, a month after this book was sent off to Springer, contained new lower bounds for the chromatic numbers of rational spaces, improving Mann's results:

Newest Lower Bounds for Q^5 and Q^7 11.9 (Cibulka, [Cib])

$$\chi(Q^5) \geq 8;$$
$$\chi(Q^7) \geq 15.$$

Cibulka's paper will be published in the October 2008 issue XVIII(2) of *Geombinatorics*.

We started this chapter with Woodall's 2-coloring of the rational plane (result 11.2). I would like to finish with it, as a musical composition requires. This Woodall's coloring has been used in July 2007 by the Australian undergraduate student Michael Payne to construct a wonderful example of a unit distance graph—do not miss it in Chapter 46!