Chapter 4 DUALITY

Associated with every linear program, and intimately related to it, is a corresponding dual linear program. Both programs are constructed from the same underlying cost and constraint coefficients but in such a way that if one of these problems is one of minimization the other is one of maximization, and the optimal values of the corresponding objective functions, if finite, are equal. The variables of the dual problem can be interpreted as prices associated with the constraints of the original (primal) problem, and through this association it is possible to give an economically meaningful characterization to the dual whenever there is such a characterization for the primal.

The variables of the dual problem are also intimately related to the calculation of the relative cost coefficients in the simplex method. Thus, a study of duality sharpens our understanding of the simplex procedure and motivates certain alternative solution methods. Indeed, the simultaneous consideration of a problem from both the primal and dual viewpoints often provides significant computational advantage as well as economic insight.

4.1 DUAL LINEAR PROGRAMS

In this section we define the dual program that is associated with a given linear program. Initially, we depart from our usual strategy of considering programs in standard form, since the duality relationship is most symmetric for programs expressed solely in terms of inequalities. Specifically then, we define duality through the pair of programs displayed below.

$$\begin{array}{llll} & \textit{Primal} & & \textit{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} & \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathrm{c}^T \\ & \mathbf{x} \geqslant 0 & & \boldsymbol{\lambda} \geqslant 0 \end{array} \tag{1}$$

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{x}$ is an n-dimensional column vector, $\mathbf{b}$ is an n-dimensional column vector, $\mathbf{c}^T$ is an n-dimensional row vector, and $\boldsymbol{\lambda}^T$ is an m-dimensional row vector. The vector $\mathbf{x}$ is the variable of the primal program, and $\boldsymbol{\lambda}$ is the variable of the dual program.

The pair of programs (1) is called the *symmetric form* of duality and, as explained below, can be used to define the dual of any linear program. It is important to note that the role of primal and dual can be reversed. Thus, studying in detail the process by which the dual is obtained from the primal: interchange of cost and constraint vectors, transposition of coefficient matrix, reversal of constraint inequalities, and change of minimization to maximization; we see that this same process applied to the dual yields the primal. Put another way, if the dual is transformed, by multiplying the objective and the constraints by minus unity, so that it has the structure of the primal (but is still expressed in terms of $\boldsymbol{\lambda}$), its corresponding dual will be equivalent to the original primal.

The dual of any linear program can be found by converting the program to the form of the primal shown above. For example, given a linear program in standard form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0}, \end{array}$$

we write it in the equivalent form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} \\ & -\mathbf{A}\mathbf{x} \geqslant -\mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0}, \end{array}$$

which is in the form of the primal of (1) but with coefficient matrix $\begin{bmatrix} \mathbf{A} \\ --- \\ -\mathbf{A} \end{bmatrix}$. Using a dual vector partitioned as $(\mathbf{u}, \mathbf{v})$, the corresponding dual is

$$\begin{array}{ll} \text{minimize} & \mathbf{u}^T\mathbf{b} - \mathbf{v}^T\mathbf{b} \\ \text{subject to} & \mathbf{u}^T\mathbf{A} - \mathbf{v}^T\mathbf{A} \leqslant \mathbf{c}^T \\ & \qquad \mathbf{u} \geqslant \mathbf{0} \\ & \qquad \mathbf{v} \geqslant \mathbf{0}. \end{array}$$

Letting $\boldsymbol{\lambda} = \mathbf{u} - \mathbf{v}$ we may simplify the representation of the dual program so that we obtain the pair of problems displayed below:

$$\begin{array}{llll} & \textit{Primal} & & \textit{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T. \\ & \quad \mathbf{x} \geqslant \mathbf{0} & & \end{array} \tag{2}$$

This is the *asymmetric form* of the duality relation. In this form the dual vector $\boldsymbol{\lambda}$ (which is really a composite of $\mathbf{u}$ and $\mathbf{v}$) is not restricted to be nonnegative.

Similar transformations can be worked out for any linear program to first get the primal in the form (1), calculate the dual, and then simplify the dual to account for special structure.

In general, if some of the linear inequalities in the primal (1) are changed to equality, the corresponding components of $\boldsymbol{\lambda}$ in the dual become free variables. If some of the components of $\mathbf{x}$ in the primal are free variables, then the corresponding inequalities in $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$ are changed to equality in the dual. We mention again that these are not arbitrary rules but are direct consequences of the original definition and the equivalence of various forms of linear programs.

Example 1 (Dual of the diet problem). The diet problem, Example 1, Section 2.2, was the problem faced by a dietician trying to select a combination of foods to meet certain nutritional requirements at minimum cost. This problem has the form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0} \end{array}$$

and hence can be regarded as the primal program of the symmetric pair above. We describe an interpretation of the dual problem.

Imagine a pharmaceutical company that produces in pill form each of the nutrients considered important by the dietician. The pharmaceutical company tries to convince the dietician to buy pills, and thereby supply the nutrients directly rather than through purchase of various foods. The problem faced by the drug company is that of determining positive unit prices $\lambda_1, \lambda_2, \ldots, \lambda_m$ for the nutrients so as to maximize revenue while at the same time being competitive with real food. To be competitive with real food, the cost of a unit of food i made synthetically from pure nutrients bought from the druggist must be no greater than c_i, the market price of the food. Thus, denoting by $\mathbf{a}_i$ the ith food, the company must satisfy $\boldsymbol{\lambda}^T\mathbf{a}_i \leqslant c_i$ for each i. In matrix form this is equivalent to $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$. Since b_j units of the jth nutrient will be purchased, the problem of the druggist is

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T \\ & \quad \boldsymbol{\lambda} \geqslant \mathbf{0}, \end{array}$$

which is the dual problem.

Example 2 (Dual of the transportation problem). The transportation problem, Example 2, Section 2.2, is the problem, faced by a manufacturer, of selecting the pattern of product shipments between several fixed origins and destinations so as to minimize transportation cost while satisfying demand. Referring to (6) and (7) of Chapter 2, the problem is in standard form, and hence the asymmetric version of the duality relation applies. There is a dual variable for each constraint. In this case we denote the variables $u_i, i = 1, 2, \ldots, m$ for (6) and $v_j, j = 1, 2, \ldots, n$ for (7). Accordingly, the dual is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\ \text{subject to} & u_i + v_j \leqslant c_{ij}, \qquad i = 1, 2, \ldots, m, \\ & \qquad\qquad\qquad j = 1, 2, \ldots, n. \end{array}$$

To interpret the dual problem, we imagine an entrepreneur who, feeling that he can ship more efficiently, comes to the manufacturer with the offer to buy his product at the plant sites (origins) and sell it at the warehouses (destinations). The product price that is to be used in these transactions varies from point to point, and is determined by the entrepreneur in advance. He must choose these prices, of course, so that his offer will be attractive to the manufacturer.

The entrepreneur, then, must select prices $-u_1, -u_2, \ldots, -u_m$ for the m origins and $v_1, v_2, \ldots, v_n$ for the n destinations. To be competitive with usual transportation modes, his prices must satisfy $u_i + v_j \leqslant c_{ij}$ for all i, j, since $u_i + v_j$ represents the net amount the manufacturer must pay to sell a unit of product at origin i and buy it back again at destination j. Subject to this constraint, the entrepreneur will adjust his prices to maximize his revenue. Thus, his problem is as given above.

4.2 THE DUALITY THEOREM

To this point the relation between the primal and dual programs has been simply a formal one based on what might appear as an arbitrary definition. In this section, however, the deeper connection between a program and its dual, as expressed by the Duality Theorem, is derived.

The proof of the Duality Theorem given in this section relies on the Separating Hyperplane Theorem (Appendix B) and is therefore somewhat more advanced than previous arguments. It is given here so that the most general form of the Duality Theorem is established directly. An alternative approach is to use the theory of the simplex method to derive the duality result. A simplified version of this alternative approach is given in the next section.

Throughout this section we consider the primal program in standard form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \;\;\mathbf{x} \geqslant \mathbf{0} \end{array} \tag{3}$$

and its corresponding dual

$$\begin{array}{ll} \text{minimize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T. \end{array} \tag{4}$$

In this section it is *not* assumed that $\mathbf{A}$ is necessarily of full rank. The following lemma is easily established and gives us an important relation between the two problems.

Fig. 4.1 Relation of primal and dual values

Lemma 1. (Weak Duality Lemma). *If* $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *are feasible for (3) and (4), respectively, then* $\mathbf{c}^T\mathbf{x} \geqslant \boldsymbol{\lambda}^T\mathbf{b}$.

Proof. We have

$$\boldsymbol{\lambda}^T\mathbf{b} = \boldsymbol{\lambda}^T\mathbf{A}\mathbf{x} \leqslant \mathbf{c}^T\mathbf{x},$$

the last inequality being valid since $\mathbf{x} \geqslant \mathbf{0}$ and $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$. ▌

This lemma shows that a feasible vector to either problem yields a bound on the value of the other problem. The values associated with the primal are all larger than the values associated with the dual as illustrated in Fig. 4.1. Since the primal seeks a minimum and the dual seeks a maximum, each seeks to reach the other. From this we have an important corollary.

Corollary. *If* $\mathbf{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are feasible for (3) and (4), respectively, and if* $\mathbf{c}^T\mathbf{x}_0 = \boldsymbol{\lambda}_0^T\mathbf{b}$, *then* $\mathbf{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are optimal for their respective problems.*

The above corollary shows that if a pair of feasible vectors can be found to the primal and dual programs with equal objective values, then these are both optimal. The Duality Theorem of linear programming states that the converse is also true, and that, in fact, the two regions in Fig. 4.1 actually have a common point; there is no "gap."

Duality Theorem of Linear Programming. *If either of the problems (3) or (4) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other problem has no feasible solution.*

Proof. We note first that the second statement is an immediate consequence of Lemma 1. For if the primal is unbounded and $\boldsymbol{\lambda}$ is feasible for the dual, we must have $\boldsymbol{\lambda}^T\mathbf{b} \leqslant -M$ for arbitrarily large M, which is clearly impossible.

Second we note that although the primal and dual are not stated in symmetric form it is sufficient, in proving the first statement, to assume that the primal has a finite optimal solution and then show that the dual has a solution with the same value. This follows because either problem can be converted to standard form and because the roles of primal and dual are reversible.

Suppose (3) has a finite optimal solution with value z_0. In the space E^{m+1} define the convex set

$$C = \left\{(r, \mathbf{w}) : r = tz_0 - \mathbf{c}^T\mathbf{x}, \mathbf{w} = t\mathbf{b} - \mathbf{A}\mathbf{x}, \mathbf{x} \geqslant \mathbf{0}, t \geqslant 0\right\}.$$

It is easily verified that C is in fact a closed convex cone. We show that the point $(1, \mathbf{0})$ is not in C. If $\mathbf{w} = t_0\mathbf{b} - \mathbf{A}\mathbf{x}_0 = \mathbf{0}$ with $t_0 > 0$, $\mathbf{x}_0 \geqslant \mathbf{0}$, then $\mathbf{x} = \mathbf{x}_0/t_0$ is feasible for (3) and hence $r/t_0 = z_0 - \mathbf{c}^T\mathbf{x} \leqslant \mathbf{0}$; which means $r \leqslant \mathbf{0}$. If $\mathbf{w} = -\mathbf{A}\mathbf{x}_0 = \mathbf{0}$ with $\mathbf{x}_0 \geqslant \mathbf{0}$ and $\mathbf{c}^T\mathbf{x}_0 = -1$, and if $\mathbf{x}$ is any feasible solution to (3), then $\mathbf{x} + \alpha\mathbf{x}_0$ is feasible for any $\alpha \geqslant 0$ and gives arbitrarily small objective values as α is increased. This contradicts our assumption on the existence of a finite optimum and thus we conclude that no such $\mathbf{x}_0$ exists. Hence $(1, \mathbf{0}) \notin C$.

Now since C is a closed convex set, there is by Theorem 1, Section B.3, a hyperplane separating (1, $\mathbf{0}$) and C. Thus there is a nonzero vector $[s, \boldsymbol{\lambda}] \in E^{m+1}$ and a constant c such that

$$s < c = \inf\left\{sr + \boldsymbol{\lambda}^T\mathbf{w} : (r, \mathbf{w}) \in C\right\}.$$

Now since C is a cone, it follows that $c \geqslant 0$. For if there were $(r, \mathbf{w}) \in C$ such that $sr + \boldsymbol{\lambda}^T\mathbf{w} < 0$, then $\alpha(r, \mathbf{w})$ for large α would violate the hyperplane inequality. On the other hand, since $(0, \mathbf{0}) \in C$ we must have $c \leqslant 0$. Thus $c = 0$. As a consequence $s < 0$, and without loss of generality we may assume $s = -1$.

We have to this point established the existence of $\boldsymbol{\lambda} \in E^m$ such that

$$-r + \boldsymbol{\lambda}^T\mathbf{w} \geqslant 0$$

for all $(r, \mathbf{w}) \in C$. Equivalently, using the definition of C,

$$\left(\mathbf{c} - \boldsymbol{\lambda}^T\mathbf{A}\right)\mathbf{x} - tz_0 + t\boldsymbol{\lambda}^T\mathbf{b} \geqslant 0$$

for all $\mathbf{x} \geqslant \mathbf{0}$, $t \geqslant 0$. Setting $t = 0$ yields $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$, which says $\boldsymbol{\lambda}$ is feasible for the dual. Setting $\mathbf{x} = \mathbf{0}$ and $t = 1$ yields $\boldsymbol{\lambda}^T\mathbf{b} \geqslant z_0$, which in view of Lemma 1 and its corollary shows that $\boldsymbol{\lambda}$ is optimal for the dual. ▮

4.3 RELATIONS TO THE SIMPLEX PROCEDURE

In this section the Duality Theorem is proved by making explicit use of the characteristics of the simplex procedure. As a result of this proof it becomes clear that once the primal is solved by the simplex procedure a solution to the dual is readily obtainable.

Suppose that for the linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0}, \end{array} \tag{5}$$

we have the optimal basic feasible solution $\mathbf{x} = (\mathbf{x}_\mathbf{B}, \mathbf{0})$ with corresponding basis $\mathbf{B}$. We shall determine a solution of the dual program

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T \end{array} \tag{6}$$

in terms of $\mathbf{B}$.

We partition $\mathbf{A}$ as $\mathbf{A} = [\mathbf{B}, \mathbf{D}]$. Since the basic feasible solution $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$ is optimal, the relative cost vector $\mathbf{r}$ must be nonnegative in each component. From Section 3.7 we have

$$\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T \mathbf{B}^{-1}\mathbf{D},$$

and since $\mathbf{r_D}$ is nonnegative in each component we have $\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D} \leqslant \mathbf{c}_\mathbf{D}^T$.

Now define $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$. We show that this choice of $\boldsymbol{\lambda}$ solves the dual problem. We have

$$\boldsymbol{\lambda}^T\mathbf{A} = \left[\boldsymbol{\lambda}^T\mathbf{B}, \boldsymbol{\lambda}^T\mathbf{D}\right] = \left[\mathbf{c}_\mathbf{B}^T, \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D}\right] \leqslant \left[\mathbf{c}_\mathbf{B}^T, \mathbf{c}_\mathbf{D}^T\right] = \mathbf{c}^T.$$

Thus since $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$, $\boldsymbol{\lambda}$ is feasible for the dual. On the other hand,

$$\boldsymbol{\lambda}^T\mathbf{b} = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_\mathbf{B}^T\mathbf{x}_\mathbf{B},$$

and thus the value of the dual objective function for this $\boldsymbol{\lambda}$ is equal to the value of the primal problem. This, in view of Lemma 1, Section 4.2, establishes the optimality of $\boldsymbol{\lambda}$ for the dual. The above discussion yields an alternative derivation of the main portion of the Duality Theorem.

Theorem. *Let the linear program (5) have an optimal basic feasible solution corresponding to the basis* $\mathbf{B}$. *Then the vector* $\boldsymbol{\lambda}$ *satisfying* $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$ *is an optimal solution to the dual program (6). The optimal values of both problems are equal.*

We turn now to a discussion of how the solution of the dual can be obtained directly from the final simplex tableau of the primal. Suppose that embedded in the original matrix $\mathbf{A}$ is an $m \times m$ identity matrix. This will be the case if, for example, m slack variables are employed to convert inequalities to equalities. Then in the final tableau the matrix $\mathbf{B}^{-1}$ appears where the identity appeared in the beginning. Furthermore, in the last row the components corresponding to this identity matrix will be $\mathbf{c}_\mathbf{I}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$, where $\mathbf{c_I}$ is the m-vector representing the cost coefficients of the variables corresponding to the columns of the original identity matrix. Thus by subtracting these cost coefficients from the corresponding elements in the last row, the negative of the solution $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$ to the dual is obtained. In particular, if, as is the case with slack variables, $\mathbf{c_I} = \mathbf{0}$, then the elements in the last row under $\mathbf{B}^{-1}$ are equal to the negative of components of the solution to the dual.

Example. Consider the primal program

$$\begin{array}{lrl} \text{minimize} & -x_1 - 4x_2 - 3x_3 & \\ \text{subject to} & 2x_1 + 2x_2 + x_3 & \leqslant 4 \\ & x_1 + 2x_2 + 2x_3 & \leqslant 6 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. & \end{array}$$

This can be solved by introducing slack variables and using the simplex procedure. The appropriate sequence of tableaus is given below without explanation.

2	2	1	1	0	4
1	2	2	0	1	6
−1	−4	−3	0	0	0

1	1	1/2	1/2	0	2
−1	0	1	−1	1	2
3	0	−1	2	0	8

3/2	1	0	1	−1/2	1
−1	0	1	−1	1	2
2	0	0	1	1	10

The optimal solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 2$. The corresponding dual program is

$$\begin{array}{ll} \text{maximize} & 4\lambda_1 + 6\lambda_2 \\ \text{subject to} & 2\lambda_1 + \lambda_2 \leqslant -1 \\ & 2\lambda_1 + 2\lambda_2 \leqslant -4 \\ & \lambda_1 + 2\lambda_2 \leqslant -3 \\ & \lambda_1 \leqslant 0, \quad \lambda_2 \leqslant 0. \end{array}$$

The optimal solution to the dual is obtained directly from the last row of the simplex tableau under the columns where the identity appeared in the first tableau: $\lambda_1 = -1$, $\lambda_2 = -1$.

Geometric Interpretation

The duality relations can be viewed in terms of the dual interpretations of linear constraints emphasized in Chapter 3. Consider a linear program in standard form. For sake of concreteness we consider the problem

$$\begin{array}{ll} \text{minimize} & 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ \text{subject to} & 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ & x_1 + 3x_2 - x_4 = 2 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0. \end{array}$$

The columns of the constraints are represented in requirements space in Fig. 4.2. A basic solution represents construction of $\mathbf{b}$ with positive weights on two of the $\mathbf{a}_i$'s. The dual problem is

$$\begin{array}{ll} \text{maximize} & 2\lambda_1 + 2\lambda_2 \\ \text{subject to} & 3\lambda_1 + \lambda_2 \leqslant 18 \\ & \lambda_1 + 3\lambda_2 \leqslant 12 \\ & -2\lambda_1 \leqslant 2 \\ & \lambda_1 - \lambda_2 \leqslant 6. \end{array}$$

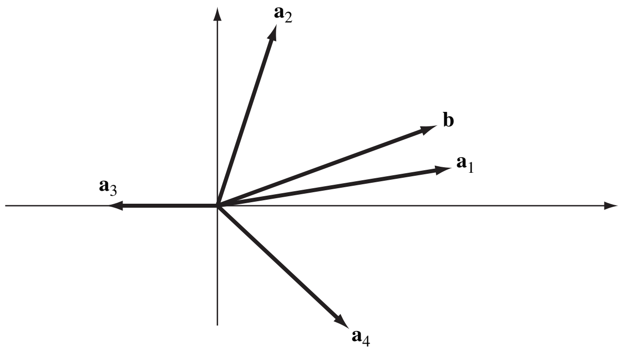

Fig. 4.2 The primal requirements space

The dual problem is shown geometrically in Fig. 4.3. Each column $\mathbf{a}_i$ of the primal defines a constraint of the dual as a half-space whose boundary is orthogonal to that column vector and is located at a point determined by c_i. The dual objective is maximized at an extreme point of the dual feasible region. At this point exactly two dual constraints are active. These active constraints correspond to an optimal basis of the primal. In fact, the vector defining the dual objective is a positive linear combination of the vectors. In the specific example, $\mathbf{b}$ is a positive combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. The weights in this combination are the x_i's in the solution of the primal.

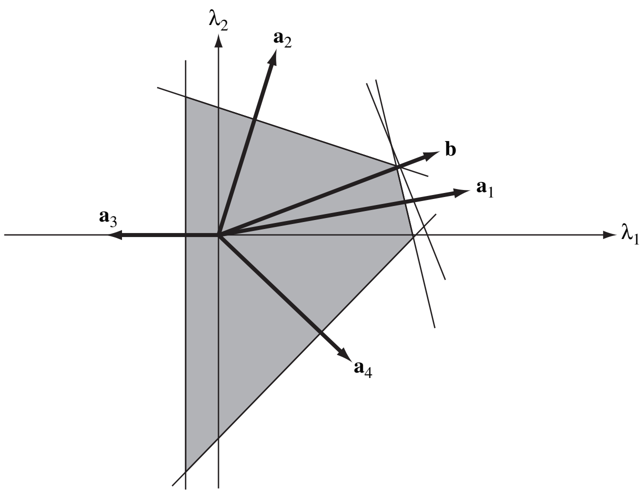

Fig. 4.3 The dual in activity space

Simplex Multipliers

We conclude this section by giving an economic interpretation of the relation between the simplex basis and the vector $\boldsymbol{\lambda}$. At any point in the simplex procedure we may form the vector $\boldsymbol{\lambda}$ satisfying $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$. This vector is not a solution to the dual unless $\mathbf{B}$ is an optimal basis for the primal, but nevertheless, it has an economic interpretation. Furthermore, as we have seen in the development of the revised simplex method, this $\boldsymbol{\lambda}$ vector can be used at every step to calculate the relative cost coefficients. For this reason $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$, corresponding to any basis, is often called the vector of *simplex multipliers*.

Let us pursue the economic interpretation of these simplex multipliers. As usual, denote the columns of $\mathbf{A}$ by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ the m unit vectors in E^m. The components of the $\mathbf{a}_i$'s and $\mathbf{b}$ tell how to construct these vectors from the $\mathbf{e}_i$'s.

Given any basis $\mathbf{B}$, however, consisting of m columns of $\mathbf{A}$, any other vector can be constructed (synthetically) as a linear combination of these basis vectors. If there is a unit cost c_i associated with each basis vector $\mathbf{a}_i$, then the cost of a (synthetic) vector constructed from the basis can be calculated as the corresponding linear combination of the c_i's associated with the basis. In particular, the cost of the jth unit vector, $\mathbf{e}_j$, when constructed from the basis $\mathbf{B}$, is λ_j, the jth component of $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$. Thus the λ_j's can be interpreted as synthetic prices of the unit vectors.

Now, any vector can be expressed in terms of the basis $\mathbf{B}$ in two steps: (i) express the unit vectors in terms of the basis, and then (ii) express the desired vector as a linear combination of unit vectors. The corresponding synthetic cost of a vector constructed from the basis $\mathbf{B}$ can correspondingly be computed directly by: (i) finding the synthetic price of the unit vectors, and then (ii) using these prices to evaluate the cost of the linear combination of unit vectors. Thus, the simplex multipliers can be used to quickly evaluate the synthetic cost of any vector that is expressed in terms of the unit vectors. The difference between the true cost of this vector and the synthetic cost is the relative cost. The process of calculating the synthetic cost of a vector, with respect to a given basis, by using the simplex multipliers is sometimes referred to as *pricing out* the vector.

Optimality of the primal corresponds to the situation where every vector $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is cheaper when constructed from the basis than when purchased directly at its own price. Thus we have $\boldsymbol{\lambda}^T\mathbf{a}_i \leqslant c_i$ for $i = 1, 2, \ldots, n$ or equivalently $\boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T$.

4.4 SENSITIVITY AND COMPLEMENTARY SLACKNESS

The optimal values of the dual variables in a linear program can, as we have seen, be interpreted as prices. In this section this interpretation is explored in further detail.

Sensitivity

Suppose in the linear program

$$\begin{array}{ll}\text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0},\end{array} \tag{7}$$

the optimal basis is $\mathbf{B}$ with corresponding solution $(\mathbf{x}_\mathbf{B}, \mathbf{0})$, where $\mathbf{x}_\mathbf{B} = \mathbf{B}^{-1}\mathbf{b}$. A solution to the corresponding dual is $\boldsymbol{\lambda}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$.

Now, assuming nondegeneracy, small changes in the vector $\mathbf{b}$ will not cause the optimal basis to change. Thus for $\mathbf{b} + \boldsymbol{\Delta}\mathbf{b}$ the optimal solution is

$$\mathbf{x} = (\mathbf{x}_\mathbf{B} + \boldsymbol{\Delta}\mathbf{x}_\mathbf{B}, \mathbf{0}),$$

where $\boldsymbol{\Delta}\mathbf{x}_\mathbf{B} = \mathbf{B}^{-1}\boldsymbol{\Delta}\mathbf{b}$. Thus the corresponding increment in the cost function is

$$\boldsymbol{\Delta}z = \mathbf{c}_\mathbf{B}^T\boldsymbol{\Delta}\mathbf{x}_\mathbf{B} = \boldsymbol{\lambda}^T\boldsymbol{\Delta}\mathbf{b}. \tag{8}$$

This equation shows that $\boldsymbol{\lambda}$ gives the sensitivity of the optimal cost with respect to small changes in the vector $\mathbf{b}$. In other words, if a new program were solved with $\mathbf{b}$ changed to $\mathbf{b} + \boldsymbol{\Delta}\mathbf{b}$, the change in the optimal value of the objective function would be $\boldsymbol{\lambda}^T\boldsymbol{\Delta}\mathbf{b}$.

This interpretation of the dual vector $\boldsymbol{\lambda}$ is intimately related to its interpretation as a vector of simplex multipliers. Since λ_j is the price of the unit vector $\mathbf{e}_j$ when constructed from the basis $\mathbf{B}$, it directly measures the change in cost due to a change in the jth component of the vector $\mathbf{b}$. Thus, λ_j may equivalently be considered as the *marginal price* of the component b_j, since if b_j is changed to $b_j + \Delta b_j$ the value of the optimal solution changes by $\lambda_j \Delta b_j$.

If the linear program is interpreted as a diet problem, for instance, then λ_j is the maximum price per unit that the dietician would be willing to pay for a small amount of the jth nutrient, because decreasing the amount of nutrient that must be supplied by food will reduce the food bill by λ_j dollars per unit. If, as another example, the linear program is interpreted as the problem faced by a manufacturer who must select levels $x_1, x_2, \ldots, x_n$ of n production activities in order to meet certain required levels of output $b_1, b_2, \ldots, b_m$ while minimizing production costs, the λ_i's are the marginal prices of the outputs. They show directly how much the production cost varies if a small change is made in the output levels.

Complementary Slackness

The optimal solutions to primal and dual programs satisfy an additional relation that has an economic interpretation. This relation can be stated for any pair of dual linear programs, but we state it here only for the asymmetric and the symmetric pairs defined in Section 4.1.

Theorem 1 (Complementary slackness—asymmetric form). *Let* $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *be feasible solutions for the primal and dual programs, respectively, in the pair (2). A necessary and sufficient condition that they both be optimal solutions is that*† *for all i*

i) $x_i > 0 \Rightarrow \boldsymbol{\lambda}^T \mathbf{a}_i = c_i$
ii) $x_i = 0 \Leftarrow \boldsymbol{\lambda}^T \mathbf{a}_i < c_i$.

Proof. If the stated conditions hold, then clearly $(\boldsymbol{\lambda}^T \mathbf{A} - \mathbf{c}^T)\mathbf{x} = 0$. Thus $\boldsymbol{\lambda}^T \mathbf{b} = \mathbf{c}^T \mathbf{x}$, and by the corollary to Lemma 1, Section 4.2, the two solutions are optimal. Conversely, if the two solutions are optimal, it must hold, by the Duality Theorem, that $\boldsymbol{\lambda}^T \mathbf{b} = \mathbf{c}^T \mathbf{x}$ and hence that $(\boldsymbol{\lambda}^T \mathbf{A} - \mathbf{c}^T)\mathbf{x} = 0$. Since each component of $\mathbf{x}$ is nonnegative and each component of $\boldsymbol{\lambda}^T \mathbf{A} - \mathbf{c}^T$ is nonpositive, the conditions (i) and (ii) must hold. ▮

Theorem 2 (Complementary slackness—symmetric form). *Let* $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *be feasible solutions for the primal and dual programs, respectively, in the pair (1). A necessary and sufficient condition that they both be optimal solutions is that for all i and j*

i) $x_i > 0 \Rightarrow \boldsymbol{\lambda}^T \mathbf{a}_i = c_i$
ii) $x_i = 0 \Leftarrow \boldsymbol{\lambda}^T \mathbf{a}_i < c_i$
iii) $\lambda_j > 0 \Rightarrow \mathbf{a}^j \mathbf{x} = b_j$
iv) $\lambda_j = 0 \Leftarrow \mathbf{a}^j \mathbf{x} > b_j$,

(where $\mathbf{a}^j$ *is the jth row of* $\mathbf{A}$*).*

Proof. This follows by transforming the previous theorem. ▮

The complementary slackness conditions have a rather obvious economic interpretation. Thinking in terms of the diet problem, for example, which is the primal part of a symmetric pair of dual problems, suppose that the optimal diet supplies more than b_j units of the jth nutrient. This means that the dietician would be unwilling to pay anything for small quantities of that nutrient, since availability of it would not reduce the cost of the optimal diet. This, in view of our previous interpretation of λ_j as a marginal price, implies $\lambda_j = 0$ which is (iv) of Theorem 2. The other conditions have similar interpretations which the reader can work out.

*4.5 THE DUAL SIMPLEX METHOD

Often there is available a basic solution to a linear program which is not feasible but which prices out optimally; that is, the simplex multipliers are feasible for the dual problem. In the simplex tableau this situation corresponds to having no negative elements in the bottom row but an infeasible basic solution. Such a situation may arise, for example, if a solution to a certain linear programming problem is

†The symbol $\Rightarrow$ means "implies" and $\Leftarrow$ means "is implied by."

calculated and then a new problem is constructed by changing the vector $\mathbf{b}$. In such situations a basic feasible solution to the dual is available and hence it is desirable to pivot in such a way as to optimize the dual.

Rather than constructing a tableau for the dual problem (which, if the primal is in standard form; involves m free variables and n nonnegative slack variables), it is more efficient to work on the dual from the primal tableau. The complete technique based on this idea is the dual simplex method. In terms of the primal problem, it operates by maintaining the optimality condition of the last row while working toward feasibility. In terms of the dual problem, however, it maintains feasibility while working toward optimality.

Given the linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned} \tag{9}$$

suppose a basis $\mathbf{B}$ is known such that $\boldsymbol{\lambda}$ defined by $\boldsymbol{\lambda}^T = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ is feasible for the dual. In this case we say that the corresponding basic solution to the primal, $\mathbf{x}_{\mathbf{B}} = \mathbf{B}^{-1}\mathbf{b}$, is *dual feasible*. If $\mathbf{x}_B \geqslant \mathbf{0}$ then this solution is also primal feasible and hence optimal.

The given vector $\boldsymbol{\lambda}$ is feasible for the dual and thus satisfies $\boldsymbol{\lambda}^T\mathbf{a}_j \leqslant c_j$, for $j = 1, 2, \ldots, n$. Indeed, assuming as usual that the basis is the first m columns of $\mathbf{A}$, there is equality

$$\boldsymbol{\lambda}^T\mathbf{a}_j = c_j, \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{10a}$$

and (barring degeneracy in the dual) there is inequality

$$\boldsymbol{\lambda}^T\mathbf{a}_j < c_j, \quad \text{for} \quad j = m+1, \ldots, n. \tag{10b}$$

To develop one cycle of the dual simplex method, we find a new vector $\bar{\boldsymbol{\lambda}}$ such that one of the equalities becomes an inequality and one of the inequalities becomes equality, while at the same time increasing the value of the dual objective function. The m equalities in the new solution then determine a new basis.

Denote the ith row of $\mathbf{B}^{-1}$ by $\mathbf{u}^i$. Then for

$$\bar{\boldsymbol{\lambda}}^T = \boldsymbol{\lambda}^T - \varepsilon\mathbf{u}^i, \tag{11}$$

we have $\bar{\boldsymbol{\lambda}}^T\mathbf{a}_j = \boldsymbol{\lambda}^T\mathbf{a}_j - \varepsilon\mathbf{u}^i\mathbf{a}_j$. Thus, recalling that $z_j = \boldsymbol{\lambda}^T\mathbf{a}_j$ and noting that $\mathbf{u}^i\mathbf{a}_j = y_{ij}$, the ijth element of the tableau, we have

$$\bar{\boldsymbol{\lambda}}^T\mathbf{a}_j = c_j, \qquad j = 1, 2, \ldots, m, \quad i \neq j \tag{12a}$$

$$\bar{\boldsymbol{\lambda}}^T\mathbf{a}_i = c_i - \varepsilon \tag{12b}$$

$$\bar{\boldsymbol{\lambda}}^T\mathbf{a}_j = z_j - \varepsilon y_{ij}, \qquad j = m+1, \quad m+2, \ldots, n. \tag{12c}$$

Also,

$$\bar{\boldsymbol{\lambda}}^T\mathbf{b} = \boldsymbol{\lambda}^T\mathbf{b} - \varepsilon x_{\mathbf{B}i}. \tag{13}$$

These last equations lead directly to the algorithm:

Step 1. Given a dual feasible basic solution $\mathbf{x_B}$, if $\mathbf{x_B} \geqslant \mathbf{0}$ the solution is optimal. If $\mathbf{x_B}$ is not nonnegative, select an index i such that the ith component of $\mathbf{x_B}$, $x_{\mathbf{B}i} < 0$.

Step 2. If all $y_{ij} \geqslant 0$, $j = 1, 2, \ldots, n$, then the dual has no maximum (this follows since by (12) $\bar{\boldsymbol{\lambda}}$ is feasible for all $\varepsilon > 0$). If $y_{ij} < 0$ for some j, then let

$$\varepsilon_0 = \frac{z_k - c_k}{y_{ik}} = \min_j \left\{ \frac{z_j - c_j}{y_{ij}} : y_{ij} < 0 \right\}. \tag{14}$$

Step 3. Form a new basis $\mathbf{B}$ by replacing $\mathbf{a}_i$ by $\mathbf{a}_k$. Using this basis determine the corresponding basic dual feasible solution $\mathbf{x_B}$ and return to Step 1.

The proof that the algorithm converges to the optimal solution is similar in its details to the proof for the primal simplex procedure. The essential observations are: (a) from the choice of k in (14) and from (12a, b, c) the new solution will again be dual feasible; (b) by (13) and the choice $x_{\mathbf{B}_i} < 0$, the value of the dual objective will increase; (c) the procedure cannot terminate at a nonoptimum point; and (d) since there are only a finite number of bases, the optimum must be achieved in a finite number of steps.

Example. A form of problem arising frequently is that of minimizing a positive combination of positive variables subject to a series of "greater than" type inequalities having positive coefficients. Such problems are natural candidates for application of the dual simplex procedure. The classical diet problem is of this type as is the simple example below.

$$\begin{array}{ll}
\text{minimize} & 3x_1 + 4x_2 + 5x_3 \\
\text{subject to} & x_i + 2x_2 + 3x_3 \geqslant 5 \\
& 2x_1 + 2x_2 + x_3 \geqslant 6 \\
& x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0.
\end{array}$$

By introducing surplus variables and by changing the sign of the inequalities we obtain the initial tableau

$$\begin{array}{rrrrrr}
-1 & -2 & -3 & 1 & 0 & -5 \\
-2 & -2 & -1 & 0 & 1 & -6 \\
3 & 4 & 5 & 0 & 0 & 0
\end{array}$$

Initial tableau

The basis corresponds to a dual feasible solution since all of the $c_j - z_j$'s are nonnegative. We select any $\mathbf{x}_{\mathbf{B}_i} < 0$, say $x_5 = -6$, to remove from the set of basic variables. To find the appropriate pivot element in the second row we compute the ratios $(z_j - c_j)/y_{2j}$ and select the minimum positive ratio. This yields the pivot indicated. Continuing, the remaining tableaus are

0	−1	−5/2	1	−1/2	−2
1	1	1/2	0	−1/2	3
0	1	7/2	0	3/2	9

Second tableau

0	1	5/2	−1	1/2	2
1	0	−2	1	−1	1
0	0	1	1	1	11

Final tableau

The third tableau yields a feasible solution to the primal which must be optimal. Thus the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 0$.

*4.6 THE PRIMAL–DUAL ALGORITHM

In this section a procedure is described for solving linear programming problems by working simultaneously on the primal and the dual problems. The procedure begins with a feasible solution to the dual that is improved at each step by optimizing an *associated restricted primal* problem. As the method progresses it can be regarded as striving to achieve the complementary slackness conditions for optimality. Originally, the primal–dual method was developed for solving a special kind of linear program arising in network flow problems, and it continues to be the most efficient procedure for these problems. (For general linear programs the dual simplex method is most frequently used). In this section we describe the generalized version of the algorithm and point out an interesting economic interpretation of it. We consider the program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{0} \end{array} \tag{15}$$

and the corresponding dual program

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T. \end{array} \tag{16}$$

Given a feasible solution $\boldsymbol{\lambda}$ to the dual, define the subset P of $1, 2, \ldots, n$ by $i \in P$ if $\boldsymbol{\lambda}^T\mathbf{a}_i = c_i$ where $\mathbf{a}_i$ is the ith column of $\mathbf{A}$. Thus, since $\boldsymbol{\lambda}$ is dual feasible, it follows that $i \notin P$ implies $\boldsymbol{\lambda}^T \mathbf{a}_i < c_i$. Now corresponding to $\boldsymbol{\lambda}$ and P, we define the *associated restricted primal* problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T \mathbf{y} \\ \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \quad\quad \mathbf{x} \geqslant \mathbf{0}, \qquad x_i = 0 \quad \text{for} \quad i \notin P \\ & \quad\quad \mathbf{y} \geqslant \mathbf{0}, \end{array} \tag{17}$$

where $\mathbf{1}$ denotes the m-vector $(1, 1, \ldots, 1)$.

The dual of this associated restricted primal is called the *associated restricted dual*. It is

$$\begin{array}{ll} \text{maximize} & \mathbf{u}^T \mathbf{b} \\ \text{subject to} & \mathbf{u}^T \mathbf{a}_i \leqslant \mathbf{0}, \qquad i \notin P \\ & \quad\quad \mathbf{u} \leqslant \mathbf{1}. \end{array} \tag{18}$$

The condition for optimality of the primal–dual method is expressed in the following theorem.

Primal–Dual Optimality Theorem. *Suppose that $\boldsymbol{\lambda}$ is feasible for the dual and that $\mathbf{x}$ and $\mathbf{y} = \mathbf{0}$ is feasible (and of course optimal) for the associated restricted primal. Then $\mathbf{x}$ and $\boldsymbol{\lambda}$ are optimal for the original primal and dual programs, respectively.*

Proof. Clearly $\mathbf{x}$ is feasible for the primal. Also we have $\mathbf{c}^T \mathbf{x} = \boldsymbol{\lambda}^T \mathbf{A}\mathbf{x}$, because $\boldsymbol{\lambda}^T \mathbf{A}$ is identical to $\mathbf{c}^T$ on the components corresponding to nonzero elements of $\mathbf{x}$. Thus $\mathbf{c}^T \mathbf{x} = \boldsymbol{\lambda}^T \mathbf{A}\mathbf{x} = \boldsymbol{\lambda}^T \mathbf{b}$ and optimality follows from Lemma 1, Section 4.2.

The primal–dual method starts with a feasible solution to the dual and then optimizes the associated restricted primal. If the optimal solution to this associated restricted primal is not feasible for the primal, the feasible solution to the dual is improved and a new associated restricted primal is determined. Here are the details:

Step 1. Given a feasible solution $\boldsymbol{\lambda}_0$ to the dual program (16), determine the associated restricted primal according to (17).

Step 2. Optimize the associated restricted primal. If the minimal value of this problem is zero, the corresponding solution is optimal for the original primal by the Primal–Dual Optimality Theorem.

Step 3. If the minimal value of the associated restricted primal is strictly positive, obtain from the final simplex tableau of the restricted primal, the solution $\mathbf{u}_0$ of the associated restricted dual (18). If there is no j for which $\mathbf{u}_0^T \mathbf{a}_j > 0$ conclude the primal has no feasible solutions. If, on the other hand, for at least one j, $\mathbf{u}_0^T \mathbf{a}_j > 0$, define the new dual feasible vector

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \varepsilon_0 \mathbf{u}_0$$

where

$$\varepsilon_0 = \frac{c_k - \boldsymbol{\lambda}_0^T \mathbf{a}_k}{\mathbf{u}_0^T \mathbf{a}_k} = \min_j \left\{ \frac{c_j - \boldsymbol{\lambda}_0^T \mathbf{a}_j}{\mathbf{u}_0^T \mathbf{a}_j} : \mathbf{u}_0^T \mathbf{a}_j > 0 \right\}.$$

Now go back to Step 1 using this $\boldsymbol{\lambda}$.

To prove convergence of this method a few simple observations and explanations must be made. First we verify the statement made in Step 3 that $\mathbf{u}_0^T \mathbf{a}_j \leqslant 0$ for all j implies that the primal has no feasible solution. The vector $\boldsymbol{\lambda}_\varepsilon = \boldsymbol{\lambda}_0 + \varepsilon \mathbf{u}_0$ is feasible for the dual problem for all positive ε, since $\mathbf{u}_0^T \mathbf{A} \leqslant \mathbf{0}$. In addition, $\boldsymbol{\lambda}_\varepsilon^T \mathbf{b} = \boldsymbol{\lambda}_0^T \mathbf{b} + \varepsilon \mathbf{u}_0^T \mathbf{b}$ and, since $\mathbf{u}_0^T \mathbf{b} = \mathbf{1}^T \mathbf{y} > 0$, we see that as ε is increased we obtain an unbounded solution to the dual. In view of the Duality Theorem, this implies that there is no feasible solution to the primal.

Next suppose that in Step 3, for at least one j, $\mathbf{u}_0^T \mathbf{a}_j > 0$. Again we define the family of vectors $\boldsymbol{\lambda}_\varepsilon = \boldsymbol{\lambda}_0 + \varepsilon \mathbf{u}_0$. Since $\mathbf{u}_0$ is a solution to (18) we have $\mathbf{u}_0^T \mathbf{a}_i \leqslant 0$ for $i \in P$, and hence for small positive ε the vector $\boldsymbol{\lambda}_\varepsilon$ is feasible for the dual. We increase ε to the first point where one of inequalities $\boldsymbol{\lambda}_\varepsilon^T \mathbf{a}_j < c_j$, $j \notin P$ becomes an equality. This determines $\varepsilon_0 > 0$ and k. The new $\boldsymbol{\lambda}$ vector corresponds to an increased value of the dual objective $\boldsymbol{\lambda}^T \mathbf{b} = \boldsymbol{\lambda}_0^T \mathbf{b} + \varepsilon \mathbf{u}_0^T \mathbf{b}$. In addition, the corresponding new set P now includes the index k. Any other index i that corresponded to a positive value of x_i in the associated restricted primal is in the new set P, because by complementary slackness $\mathbf{u}_0^T \mathbf{a}_i = 0$ for such an i and thus $\boldsymbol{\lambda}^T \mathbf{a}_i = \boldsymbol{\lambda}_0^T \mathbf{a}_i + \varepsilon_0 \mathbf{u}_0^T \mathbf{a}_i = c_i$. This means that the old optimal solution is feasible for the new associated restricted primal and that $\mathbf{a}_k$ can be pivoted into the basis. Since $\mathbf{u}_0^T \mathbf{a}_k > 0$, pivoting in $\mathbf{a}_k$ will decrease the value of the associated restricted primal.

In summary, it has been shown that at each step either an improvement in the associated primal is made or an infeasibility condition is detected. Assuming nondegeneracy, this implies that no basis of the associated primal is repeated—and since there are only a finite number of possible bases, the solution is reached in a finite number of steps.

The primal–dual algorithm can be given an interesting interpretation in terms of the manufacturing problem in Example 3, Section 2.2. Suppose we own a facility that is capable of engaging in n different production activities each of which produces various amounts of m commodities. Each activity i can be operated at any level $x_i \geqslant 0$, but when operated at the unity level the ith activity costs c_i dollars and yields the m commodities in the amounts specified by the m-vector $\mathbf{a}_i$. Assuming linearity of the production facility, if we are given a vector $\mathbf{b}$ describing output requirements of the m commodities, and we wish to produce these at minimum cost, ours is the primal problem.

Imagine that an entrepreneur *not knowing* the value of our requirements vector $\mathbf{b}$ decides to sell us these requirements directly. He assigns a price vector $\boldsymbol{\lambda}_0$ to these requirements such that $\boldsymbol{\lambda}_0^T \mathbf{A} \leqslant \mathbf{c}$. In this way his prices are competitive with our production activities, and he can assure us that purchasing directly from him is no more costly than engaging activities. As owner of the production facilities we are reluctant to abandon our production enterprise but, on the other hand, we deem it not frugal to engage an activity whose output can be duplicated by direct purchase for lower cost. Therefore, we decide to engage only activities that cannot be duplicated cheaper, and at the same time we attempt to minimize the total business volume given the entrepreneur. Ours is the associated restricted primal problem.

Upon receiving our order, the greedy entrepreneur decides to modify his prices in such a manner as to keep them competitive with our activities but increase the cost of our order. As a reasonable and simple approach he seeks new prices of the form

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \varepsilon \mathbf{u}_0,$$

where he selects $\mathbf{u}_0$ as the solution to

$$\begin{array}{lll} \text{maximize} & \mathbf{u}^T\mathbf{y} & \\ \text{subject to} & \mathbf{u}^T\mathbf{a}_i \leqslant \mathbf{0}, & i \in P \\ & \mathbf{u} \leqslant \mathbf{1}. & \end{array}$$

The first set of constraints is to maintain competitiveness of his new price vector for small ε, while the second set is an arbitrary bound imposed to keep this subproblem bounded. It is easily shown that the solution $\mathbf{u}_0$ to this problem is identical to the solution of the associated dual (18). After determining the maximum ε to maintain feasibility, he announces his new prices.

At this point, rather than concede to the price adjustment, we recalculate the new minimum volume order based on the new prices. As the greedy (and shortsighted) entrepreneur continues to change his prices in an attempt to maximize profit he eventually finds he has reduced his business to zero! At that point we have, with his help, solved the original primal problem.

Example. To illustrate the primal–dual method and indicate how it can be implemented through use of the tableau format consider the following problem:

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 = 3 \\ & 2x_1 + x_2 + 3x_3 = 5 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{array}$$

Because all of the coefficients in the objective function are nonnegative, $\boldsymbol{\lambda} = (0, 0)$ is a feasible vector for the dual. We lay out the simplex tableau shown below

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	·	·	$\mathbf{b}$
	1	1	2	1	0	3
	2	1	3	0	1	5
	−3	−2	−5	0	0	−8
$c_i - \boldsymbol{\lambda}^T\mathbf{a}_i \rightarrow$	2	1	4	·	·	·

First tableau

To form this tableau we have adjoined artificial variables in the usual manner. The third row gives the relative cost coefficients of the associated primal problem—the same as the row that would be used in a phase I procedure. In the fourth row are listed the $c_i - \boldsymbol{\lambda}^T \mathbf{a}_i$'s for the current $\boldsymbol{\lambda}$. The allowable columns in the associated restricted primal are determined by the zeros in this last row.

Since there are no zeros in the last row, no progress can be made in the associated restricted primal and hence the original solution $x_1 = x_2 = x_3 = 0$, $y_1 = 3$, $y_2 = 5$ is optimal for this $\boldsymbol{\lambda}$. The solution $\mathbf{u}_0$ to the associated restricted dual is $\mathbf{u}_0 = (1, 1)$, and the numbers $-\mathbf{u}_0^T \mathbf{a}_i$, $i = 1, 2, 3$ are equal to the first three elements in the third row. Thus, we compute the three ratios $\frac{2}{3}, \frac{1}{2}, \frac{4}{5}$ from which we find $\varepsilon_0 = \frac{1}{2}$. The new values for the fourth row are now found by adding ε_0 times the (first three) elements of the third row to the fourth row.

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	·	·	$\mathbf{b}$
1	1	2	1	0	3
2	1	3	0	1	5
−3	−2	−5	0	0	−8
1/2	0	3/2	·	·	·

Second tableau

Minimizing the new associated restricted primal by pivoting as indicated we obtain

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	·	·	$\mathbf{b}$
1	1	2	1	0	3
1	0	1	−1	1	2
−1	0	−1	2	0	−2
−1/2	0	3/2	·	·	·

Now we again calculate the ratios $\frac{1}{2}, \frac{3}{2}$ obtaining $\varepsilon_0 = \frac{1}{2}$, and add this multiple of the third row to the fourth row to obtain the next tableau.

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	·	·	$\mathbf{b}$
1	1	2	1	0	3
1	0	1	−1	1	2
−1	0	−1	2	0	−2
0	0	1	·	·	·

Third tableau

Optimizing the new restricted primal we obtain the tableau:

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	·	·	$\mathbf{b}$
0	1	1	2	−1	1
1	0	1	−1	1	2
0	0	0	1	1	0
0	0	1	·	·	·

Final tableau

Having obtained feasibility in the primal, we conclude that the solution is also optimal: $x_1 = 2$, $x_2 = 1$, $x_3 = 0$.

*4.7 REDUCTION OF LINEAR INEQUALITIES

Linear programming is in part the study of linear inequalities, and each progressive stage of linear programming theory adds to our understanding of this important fundamental mathematical structure. Development of the simplex method, for example, provided by means of artificial variables a procedure for solving such systems. Duality theory provides additional insight and additional techniques for dealing with linear inequalities.

Consider a system of linear inequalities in standard form

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0}, \end{aligned} \tag{19}$$

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ is a constant nonzero m-vector, and $\mathbf{x}$ is a variable n-vector. Any point $\mathbf{x}$ satisfying these conditions is called a *solution*. The set of solutions is denoted by S.

It is the set S that is of primary interest in most problems involving systems of inequalities—the inequalities themselves acting merely to provide a description of S. Alternative systems having the same solution set S are, from this viewpoint, equivalent. In many cases, therefore, the system of linear inequalities originally used to define S may not be the simplest, and it may be possible to find another system having fewer inequalities or fewer variables while defining the same solution set S. It is this general issue that is explored in this section.

Redundant Equations

One way that a system of linear inequalities can sometimes be simplified is by the elimination of redundant equations. This leads to a new equivalent system having the same number of variables but fewer equations.

Definition. Corresponding to the system of linear inequalities

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0}, \end{aligned} \tag{19}$$

we say the system has *redundant equations* if there is a nonzero $\boldsymbol{\lambda} \in E^m$ satisfying

$$\begin{aligned} \boldsymbol{\lambda}^T\mathbf{A} &= \mathbf{0} \\ \boldsymbol{\lambda}^T\mathbf{b} &= 0. \end{aligned} \tag{20}$$

This definition is equivalent, as the reader is aware, to the statement that a system of equations is redundant if one of the equations can be expressed as a linear combination of the others. In most of our previous analysis we have assumed, for simplicity, that such redundant equations were not present in our given system or that they were eliminated prior to further computation. Indeed, such redundancy presents no real computational difficulty, since redundant equations are detected and can be eliminated during application of the phase I procedure for determining a basic feasible solution. Note, however, the hint of duality even in this elementary concept.

Null Variables

Definition. Corresponding to the system of linear inequalities

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0}, \end{aligned} \tag{21}$$

a variable x_i is said to be a *null variable* if $x_i = 0$ in every solution.

It is clear that if it were known that a variable x_i were a null variable, then the solution set S could be equivalently described by the system of linear inequalities obtained from (21) by deleting the ith column of $\mathbf{A}$, deleting the inequality $x_i \geqslant 0$, and adjoining the equality $x_i = 0$. This yields an obvious simplification in the description of the solutions set S. It is perhaps not so obvious how null variables can be identified.

Example. As a simple example of how null variables may appear consider the system

$$\begin{aligned} &2x_1 + 3x_2 + 4x_3 + 4x_4 = 6 \\ &x_1 \;\;+\;\; x_2 + 2x_3 + \;\; x_4 = 3 \\ &x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0. \end{aligned}$$

By subtracting twice the second equation from the first we obtain

$$x_2 + 2x_4 = 0.$$

Since the x_i's must all be nonnegative, it follows immediately that x_2 and x_4 are zero in any solution. Thus x_2 and x_4 are null variables.

Generalizing from the above example it is clear that if a linear combination of the equations can be found such that the right-hand side is zero while the coefficients on the left side are all either zero or positive, then the variables corresponding to the positive coefficients in this equation are null variables. In other words, if from the original system it is possible to combine equations so as to yield

$$\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n = 0$$

with $\xi_i \geqslant 0$, $i = 1, 2, \ldots, n$, then $\xi_i > 0$ implies that x_i is a null variable.

The above elementary observations clearly can be used to identify null variables in some cases. A more surprising result is that the technique described above can be used to identify all null variables. The proof of this fact is based on the Duality Theorem.

Null Value Theorem. *If S is not empty, the variable x_i is a null variable in the system (21) if and only if there is a nonzero vector $\boldsymbol{\lambda} \in E^m$ such that*

$$\begin{aligned} \boldsymbol{\lambda}^T \mathbf{A} &\geq \mathbf{0} \\ \boldsymbol{\lambda}^T \mathbf{b} &= 0. \end{aligned} \tag{22}$$

and the ith component of $\boldsymbol{\lambda}^T \mathbf{A}$ is strictly positive.

Proof. The "if" part follows immediately from the discussion above. To prove the "only if" part, suppose that x_i is a null variable, and suppose that S is not empty. Consider the program

$$\begin{aligned} \text{minimize} \quad & -\mathbf{e}^i \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned}$$

where $\mathbf{e}^i$ is the ith unit row vector. By our hypotheses, there is a feasible solution and the optimal value is zero. By the Duality Theorem the dual program

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{\lambda}^T \mathbf{b} \\ \text{subject to} \quad & \boldsymbol{\lambda}^T \mathbf{A} \leqslant -\mathbf{e}^i \end{aligned}$$

is also feasible and has optimal value zero. Thus there is a $\boldsymbol{\lambda}$ with

$$\begin{aligned} \boldsymbol{\lambda}^T \mathbf{A} &\leqslant -\mathbf{e}^i \\ \boldsymbol{\lambda}^T \mathbf{b} &= 0. \end{aligned}$$

Changing the sign of $\boldsymbol{\lambda}$ proves the theorem.

Nonextremal Variables

Example 1. Consider the system of linear inequalities

$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 4 \\ 2x_1 + x_2 + 3x_3 &= 6 \\ x_1 \geqslant 0, \qquad x_2 \geqslant 0, \qquad & x_3 \geqslant 0. \end{aligned} \tag{23}$$

By subtracting the second equation from the first and rearranging, we obtain

$$x_1 = 2 + 2x_2 + x_3. \tag{24}$$

From this we observe that since x_2 and x_3 are nonnegative, the value of x_1 is greater than or equal to 2 in any solution to the equalities. This means that the inequality $x_1 \geqslant 0$ can be dropped from the original set, and x_1 can be treated as a free variable even though the remaining inequalities actually do not allow complete freedom. Hence x_1 can be replaced everywhere by (24) in the original system (23) leading to

$$\begin{gathered} 5x_2 + 5x_3 = 2 \\ x_2 \geqslant 0, \qquad x_3 \geqslant 0 \\ x_1 = 2 + 2x_2 + x_3. \end{gathered} \tag{25}$$

The first two lines of (25) represent a system of linear inequalities in standard form with one less variable and one less equation than the original system. The last equation is a simple linear equation from which x_1 is determined by a solution to the smaller system of inequalities.

This example illustrates and motivates the concept of a nonextremal variable. As illustrated, the identification of such nonextremal variables results in a significant simplification of a system of linear inequalities.

Definition. A variable x_i in the system of linear inequalities

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned} \tag{26}$$

is *nonextremal* if the inequality $x_i \geqslant 0$ in (26) is redundant.

A nonextremal variable can be treated as a free variable, and thus can be eliminated from the system by using one equation to define that variable in terms of the other variables. The result is a new system having one less variable and one less equation. Solutions to the original system can be obtained from solutions to the new system by substituting into the expression for the value of the free variable.

It is clear that if, as in the example, a linear combination of the equations in the system can be found that implies that x_i is nonnegative if all other variables are nonnegative, then x_i is nonextremal. That the converse of this statement is also true is perhaps not so obvious. Again the proof of this is based on the Duality Theorem.

Nonextremal Variable Theorem. *If S is not empty, the variable x_j is a nonextremal variable for the system (26) if and only if there is $\boldsymbol{\lambda} \in E^m$ and $\mathbf{d} \in E^n$ such that*

$$\boldsymbol{\lambda}^T \mathbf{A} = \mathbf{d}^T, \tag{27}$$

where

$$d_j = -1, \qquad d_i \geqslant 0 \quad \text{for} \quad i \neq j;$$

and such that

$$\boldsymbol{\lambda}^T \mathbf{b} = -\beta, \tag{28}$$

for some $\beta \geqslant 0$.

Proof. The "if" part of the result is trivial, since forming the corresponding linear combination of the equations in (28) yields

$$x_j = \beta + d_1 x_1 + \cdots + d_{j-1} x_{j-1} + d_{j+1} x_{j+1} + \cdots + d_n x_n,$$

which implies that x_j is nonextremal.

To prove the "only if" part, let $\mathbf{a}_i$, $i = 1, 2, \ldots, n$ denote the ith column of $\mathbf{A}$. Let us assume that the solution set S is nonempty and that x_j is nonextremal. Consider the linear program

$$\begin{array}{ll} \text{minimize} & x_j \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & x_i \geqslant 0, \quad i \neq j. \end{array} \tag{29}$$

By hypothesis the minimum value is nonnegative, say it is $\beta \geqslant 0$. Then by the Duality Theorem the value of the dual program

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^T \mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T \mathbf{a}_i \leqslant 0, \qquad i \neq j \\ & \boldsymbol{\lambda}^T \mathbf{a}_j = 1 \end{array}$$

is also β. Taking the negative of the optimal solution to the dual yields the desired result. ▌

Nonextremal variables occur frequently in systems of linear inequalities. It can be shown, for instance, that every system having three nonnegative variables and two (independent) equations can be reduced to two non-negative variables and one equation.

Applications

Each of the reduction concepts can be applied by searching for a $\boldsymbol{\lambda}$ satisfying an appropriate system of linear inequalities. This can be done by application of the simplex method. Thus, the theorems above translate into systematic procedures for reducing a system.

The reduction methods described in this section can be applied to any linear program in an effort to simplify the representation of the feasible region. Of course, for the purpose of simply solving a given linear program the reduction process is not particularly worthwhile. However, when considering a large problem that will be solved many times with different objective functions, or a problem with linear constraints but a nonlinear objective, the reduction procedure can be valuable.

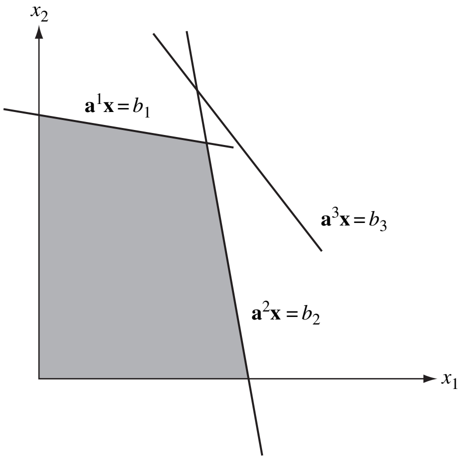

Fig. 4.4 Redundant inequality

One interesting area of application is the elimination of redundant inequality constraints. Consider the region shown in Fig. 4.4 defined by the nonnegativity constraint and three other linear inequalities. The system can be expressed as

$$\mathbf{a}^1\mathbf{x} \leqslant b_1, \qquad \mathbf{a}^2\mathbf{x} \leqslant b_2, \qquad \mathbf{a}^3\mathbf{x} \leqslant b_3, \qquad \mathbf{x} \geqslant \mathbf{0}, \tag{30}$$

which in standard form is

$$\mathbf{a}^1\mathbf{x} + y_1 = b_1, \quad \mathbf{a}^2\mathbf{x} + y_2 = b_2, \quad \mathbf{a}^3\mathbf{x} + y_3 = b_3, \quad \mathbf{x} \geqslant \mathbf{0}, \quad \mathbf{y} \geqslant \mathbf{0}. \tag{31}$$

The third constraint is, as seen from the figure, redundant and can be eliminated without changing the solution set. In the standard form (31) this is reflected in the fact that y_3 is nonextremal and hence it, together with the third constraint, can be eliminated. This special example generalizes, of course, to higher dimensional problems involving many inequalities where, in general, redundant inequalities show up as having nonextremal slack variables. The detection and elimination of such redundant inequalities can be helpful in the cutting-plane methods (discussed in Chapter 14) where inequalities are continually appended to a system as the method progresses.

4.8 EXERCISES

1. Verify in detail that the dual of a linear program is the original problem.

2. Show that if a linear inequality in a linear program is changed to equality, the corresponding dual variable becomes free.

3. Find the dual of

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \quad \mathbf{x} \geqslant \mathbf{a} \\ \text{where} & \quad \mathbf{a} \geqslant \mathbf{0}. \end{array}$$

4. Show that in the transportation problem the linear equality constraints are not linearly independent, and that in an optimal solution to the dual problem the dual variables are not unique. Generalize this observation to any linear program having redundant equality constraints.

5. Construct an example of a primal problem that has no feasible solutions and whose corresponding dual also has no feasible solutions.

6. Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ be an n-vector. Prove that $\mathbf{A}\mathbf{x} \leqslant \mathbf{0}$ implies $\mathbf{c}^T\mathbf{x} \leqslant \mathbf{0}$ if and only if $\mathbf{c}^T = \boldsymbol{\lambda}^T\mathbf{A}$ for some $\boldsymbol{\lambda} \geqslant \mathbf{0}$. Give a geometric interpretation of the result.

7. There is in general a strong connection between the theories of optimization and free competition, which is illustrated by an idealized model of activity location. Suppose there are n economic activities (various factories, homes, stores, etc.) that are to be individually located on n distinct parcels of land. If activity i is located on parcel j that activity can yield s_{ij} units (dollars) of value.
If the assignment of activities to land parcels is made by a central authority, it might be made in such a way as to maximize the total value generated. In other words, the assignment would be made so as to maximize $\sum_i \sum_j s_{ij}x_{ij}$ where

$$x_{ij} = \begin{cases} 1 & \text{if activity } i \text{ is assigned to parcel } j \\ 0 & \text{otherwise.} \end{cases}$$

More explicitly this approach leads to the optimization problem

$$\begin{array}{lll} \text{maximize} & \sum\limits_i \sum\limits_j s_{ij}x_{ij} & \\ \text{subject to} & \sum\limits_j x_{ij} = 1, & i = 1, 2, \ldots, n \\ & \sum\limits_i x_{ij} = 1, & j = 1, 2, \ldots, n \\ & \quad x_{ij} \geqslant 0, & x_{ij} = 0 \text{ or } 1. \end{array}$$

Actually, it can be shown that the final requirement ($x_{ij} = 0$ or 1) is automatically satisfied at any extreme point of the set defined by the other constraints, so that in fact the optimal assignment can be found by using the simplex method of linear programming.

If one considers the problem from the viewpoint of free competition, it is assumed that, rather than a central authority determining the assignment, the individual activities bid for the land and thereby establish prices.

a) Show that there exists a set of activity prices $p_i, i = 1, 2, \ldots, n$ and land prices $q_j, j = 1, 2, \ldots, n$ such that

$$p_i + q_j \geqslant s_{ij}, \qquad i = 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, n$$

with equality holding if in an optimal assignment activity i is assigned to parcel j.

b) Show that Part (a) implies that if activity i is optimally assigned to parcel j and if j' is any other parcel

$$s_{ij} - q_j \geqslant s_{ij'} - q_{j'}.$$

Give an economic interpretation of this result and explain the relation between free competition and optimality in this context.

c) Assuming that each s_{ij} is positive, show that the prices can all be assumed to be nonnegative.

8. *Game theory* is in part related to linear programming theory. Consider the game in which player X may select any one of m moves, and player Y may select any one of n moves. If X selects i and Y selects j, then X wins an amount a_{ij} from Y. The game is repeated many times. Player X develops a *mixed* strategy where the various moves are played according to probabilities represented by the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, where $x_1 \geqslant 0, i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$. Likewise Y develops a mixed strategy $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, where $y_i \geqslant 0, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} y_i = 1$. The average payoff to X is then $\mathbf{P}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A} \mathbf{y}$.

a) Suppose X selects $\mathbf{x}$ as the solution to the linear program

$$\begin{array}{lll}
\text{maximize} & A & \\
\text{subject to} & \sum_{i=1}^{m} x_i = 1 & \\
 & \sum_{i=1}^{m} x_i a_{ij} \geqslant A, & j = 1, 2, \ldots, n \\
 & x_i \geqslant 0, & i = 1, 2, \ldots, m.
\end{array}$$

Show that X is guaranteed a payoff of at least A no matter what $\mathbf{y}$ is chosen by Y.

b) Show that the dual of the problem above is

$$\begin{array}{lll}
\text{minimize} & B & \\
\text{subject to} & \sum_{j=1}^{n} y_j = 1 & \\
 & \sum_{j=1}^{n} a_{ij} y_j \leqslant B, & i = 1, 2, \ldots, m \\
 & y_j \geqslant 0, & j = 1, 2, \ldots, n.
\end{array}$$

c) Prove that $\max A = \min B$. (The common value is called the *value* of the game.)

d) Consider the "matching" game. Each player selects heads or tails. If the choices match, X wins \$1 from Y; if they do not match, Y wins \$1 from X. Find the value of this game and the optimal mixed strategies.

e) Repeat Part (d) for the game where each player selects either 1, 2, or 3. The player with the highest number wins \$1 unless that number is exactly 1 higher than the other player's number, in which case he loses \$3. When the numbers are equal there is no payoff.

9. Consider the primal linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}. \end{array}$$

Suppose that this program and its dual are feasible. Let $\boldsymbol{\lambda}$ be a known optimal solution to the dual.

a) If the kth equation of the primal is multiplied by $\mu \neq 0$, determine an optimal solution $\mathbf{w}$ to the dual of this new problem.
b) Suppose that, in the original primal, we add μ times the kth equation to the rth equation. What is an optimal solution $\mathbf{w}$ to the corresponding dual problem?
c) Suppose, in the original primal, we add μ times the kth row of $\mathbf{A}$ to $\mathbf{c}$. What is an optimal solution to the corresponding dual problem?

10. Consider the linear program (P) of the form

$$\begin{array}{ll} \text{minimize} & \mathbf{q}^T\mathbf{z} \\ \text{subject to} & \mathbf{M}\mathbf{z} \geq -\mathbf{q} \\ & \mathbf{z} \geq \mathbf{0} \end{array}$$

in which the matrix $\mathbf{M}$ is *skew symmetric*; that is, $\mathbf{M} = -\mathbf{M}^T$.

(a) Show that problem (P) and its dual are the same.
(b) A problem of the kind in part (a) is said to be *self-dual*. An example of a self-dual problem has

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}.$$

Give an interpretation of the problem with this data.
(c) Show that a self-dual linear program has an optimal solution if and only if it is feasible.

11. A company may manufacture n different products, each of which uses various amounts of m limited resources. Each unit of product i yields a profit of c_i dollars and uses a_{ji} units of the jth resource. The available amount of the jth resource is b_j. To maximize profit the company selects the quantities x_i to be manufactured of each product by solving

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}. \end{array}$$

The unit profits c_i already take into account the variable cost associated with manufacturing each unit. In addition to that cost, the company incurs a fixed overhead H, and for accounting purposes it wants to allocate this overhead to each of its products. In other words, it wants to adjust the unit profits so as to account for the overhead. Such an overhead allocation scheme must satisfy two conditions: (1) Since H is fixed regardless of the product mix, the overhead allocation scheme must not alter the optimal solution, (2) All the overhead must be allocated; that is, the optimal value of the objective with the modified cost coefficients must be H dollars lower than z—the original optimal value of the objective.

a) Consider the allocation scheme in which the unit profits are modified according to $\hat{\mathbf{c}}^T = \mathbf{c}^T - r\boldsymbol{\lambda}_0^T\mathbf{A}$, where $\boldsymbol{\lambda}_0$ is the optimal solution to the original dual and $r = H/z_0$ (assume $H \leqslant z_0$).

i) Show that the optimal $\mathbf{x}$ for the modified problem is the same as that for the original problem, and the new dual solution is $\hat{\boldsymbol{\lambda}}_0 = (1-r)\boldsymbol{\lambda}_0$.
ii) Show that this approach fully allocates H.

b) Suppose that the overhead can be traced to each of the resource constraints. Let $H_i \geqslant 0$ be the amount of overhead associated with the ith resource, where $\sum_{i=1}^{m} H_i \leqslant z_0$ and $r_i = H_i/b_i \leqslant \lambda_i^0$ for $i = 1, \ldots, m$. Based on this information, an allocation scheme has been proposed where the unit profits are modified such that $\hat{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{r}^T\mathbf{A}$.

i) Show that the optimal $\mathbf{x}$ for this modified problem is the same as that for the original problem, and the corresponding dual solution is $\hat{\boldsymbol{\lambda}}_0 = \boldsymbol{\lambda}_0 - \mathbf{r}$.
ii) Show that this scheme fully allocates H.

12. Solve the linear inequalities

$$
\begin{aligned}
-2x_1 + 2x_2 &\leqslant -1 \\
2x_1 - x_2 &\leqslant 2 \\
- 4x_2 &\leqslant 3 \\
-15x_1 - 12x_2 &\leqslant -2 \\
12x_1 + 20x_2 &\leqslant -1.
\end{aligned}
$$

Note that x_1 and x_2 are *not* restricted to be positive. Solve this problem by considering the problem of maximizing $0 \cdot x_1 + 0 \cdot x_2$ subject to these constraints, taking the dual and using the simplex method.

13. a) Using the simplex method solve

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 - x_2 \\
\text{subject to} \quad & 2x_1 - x_2 - x_3 \geqslant 3 \\
& x_1 - x_2 + x_3 \geqslant 2 \\
& x_i \geqslant 0, \quad i = 1, 2, 3.
\end{aligned}
$$

(*Hint*: Note that $x_1 = 2$ gives a feasible solution.)
b) What is the dual problem and its optimal solution?

14. a) Using the simplex method solve

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + 3x_2 + 2x_3 + 2x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 + 2x_4 = 3 \\
& x_1 + x_2 + 2x_3 + 4x_4 = 5 \\
& x_i \geqslant 0, \qquad i = 1, 2, 3, 4.
\end{aligned}
$$

b) Using the work done in Part (a) and the dual simplex method, solve the same problem but with the right-hand sides of the equations changed to 8 and 7 respectively.

15. For the problem

$$\begin{array}{ll} \text{minimize} & 5x_1+3x_2 \\ \text{subject to} & 2x_1-x_2+4x_3 \leqslant 4 \\ & x_1+x_2+2x_3 \leqslant 5 \\ & 2x_1-x_2+x_3 \geqslant 1 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0; \end{array}$$

a) Using a single pivot operation with pivot element 1, find a feasible solution.
b) Using the simplex method, solve the problem.
c) What is the dual problem?
d) What is the solution to the dual?

16. Solve the following problem by the dual simplex method:

$$\begin{array}{ll} \text{minimize} & -7x_1+7x_2-2x_3-x_4-6x_5 \\ \text{subject to} & 3x_1-x_2+x_3-2x_4 = -3 \\ & 2x_1+x_2+x_4+x_5 = 4 \\ & -x_1+3x_2-3x_4+x_6 = 12 \\ \text{and} & x_i \geqslant 0, \quad i=1,\ldots,6. \end{array}$$

17. Given the linear programming problem in standard form (3) suppose a basis **B** and the corresponding (not necessarily feasible) primal and dual basic solutions **x** and $\boldsymbol{\lambda}$ are known. Assume that at least one relative cost coefficient $c_i-\boldsymbol{\lambda}^T\mathbf{a}_i$ is negative. Consider the auxiliary problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x}=\mathbf{b} \\ & \sum_{i\in T} x_i+y=M \\ & \mathbf{x} \geqslant \mathbf{0}, \quad y \geqslant 0, \end{array}$$

where $T=\{i : c_i-\boldsymbol{\lambda}^T\mathbf{a}_i<0\}$, y is a slack variable, and M is a large positive constant. Show that if k is the index corresponding to the most negative relative cost coefficient in the original solution, then $(\boldsymbol{\lambda}, c_k-\boldsymbol{\lambda}^T\mathbf{a}_k)$ is dual feasible for the auxiliary problem. Based on this observation, develop a big–M artificial constraint method for the dual simplex method. (Refer to Exercise 24, Chapter 3.)

18. A textile firm is capable of producing three products—x_1, x_2, x_3. Its production plan for next month must satisfy the constraints

$$\begin{array}{c} x_1+2x_2+2x_3 \leqslant 12 \\ 2x_1+4x_2+x_3 \leqslant f \\ x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{array}$$

The first constraint is determined by equipment availability and is fixed. The second constraint is determined by the availability of cotton. The net profits of the products are 2, 3, and 3, respectively, exclusive of the cost of cotton and fixed costs.

a) Find the shadow price λ_2 of the cotton input as a function of f. (*Hint*: Use the dual simplex method.) Plot $\lambda_2(f)$ and the net profit $z(f)$ exclusive of the cost for cotton.

b) The firm may purchase cotton on the open market at a price of 1/6. However, it may acquire a limited amount at a price of 1/12 from a major supplier that it purchases from frequently. Determine the net profit of the firm $\pi(s)$ as a function of s.

19. Consider the problem

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 = 3 \\ & 2x_1 + x_2 + 3x_3 = 5 \\ & x_i \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{array}$$

a) What is the dual problem?

b) Note that $\boldsymbol{\lambda} = (1, 0)$ is feasible for the dual. Starting with this $\boldsymbol{\lambda}$, solve the primal using the primal–dual algorithm.

20. Show that in the associated restricted dual of the primal–dual method the objective $\boldsymbol{\lambda}^T\mathbf{b}$ can be replaced by $\boldsymbol{\lambda}^T\mathbf{y}$.

21. Given the system of linear inequalities (19), what is implied by the existence of a $\boldsymbol{\lambda}$ satisfying $\boldsymbol{\lambda}^T\mathbf{A} = \mathbf{0}$, $\boldsymbol{\lambda}^T\mathbf{b} \neq 0$?

22. Suppose a system of linear inequalities possesses null variables. Show that when the null variables are eliminated, by setting them identically to zero, the resulting system will have redundant equations. Verify this for the example in Section 4.7.

23. Prove that any system of linear inequalities in standard form having two equations and three variables can be reduced.

24. Show that if a system of linear inequalities in standard form has a nondegenerate basic feasible solution, the corresponding nonbasic variables are extremal.

25. Eliminate the null variables in the system

$$\begin{array}{r} 2x_1 + x_2 - x_3 + x_4 + x_5 = 2 \\ -x_1 + 2x_2 + x_3 + 2x_4 + x_5 = -1 \\ -x_1 - x_2 \qquad - 3x_4 + 2x_5 = -1 \end{array}$$

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0, \quad x_5 \geqslant 0.$$

26. Reduce to minimal size

$$\begin{array}{r} x_1 + x_2 + 2x_3 + x_4 + x_5 = 6 \\ 3x_2 + x_3 + 5x_4 + 4x_5 = 4 \\ x_1 + x_2 - x_3 + 2x_4 + 2x_5 = 3 \end{array}$$

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0, \quad x_5 \geqslant 0.$$

REFERENCES

4.1–4.4 Again most of the material in this chapter is now quite standard. See the references of Chapter 2. A particularly careful discussion of duality can be found in Simonnard [S6].

4.5 The dual simplex method is due to Lemke [L4].

4.6 The general primal–dual algorithm is due to Dantzig, Ford and Fulkerson [D7]. See also Ford and Fulkerson [F13]. The economic interpretation given in this section is apparently novel.

4.7 The concepts of reduction are due to Shefi [S5], who has developed a complete theory in this area. For more details along the lines presented here, see Luenberger [L15].