Chapter 3 THE SIMPLEX METHOD

The idea of the simplex method is to proceed from one basic feasible solution (that is, one extreme point) of the constraint set of a problem in standard form to another, in such a way as to continually decrease the value of the objective function until a minimum is reached. The results of Chapter 2 assure us that it is sufficient to consider only basic feasible solutions in our search for an optimal feasible solution. This chapter demonstrates that an efficient method for moving among basic solutions to the minimum can be constructed.

In the first five sections of this chapter the simplex machinery is developed from a careful examination of the system of linear equations that defines the constraints and the basic feasible solutions of the system. This approach, which focuses on individual variables and their relation to the system, is probably the simplest, but unfortunately is not easily expressed in compact form. In the last few sections of the chapter, the simplex method is viewed from a matrix theoretic approach, which focuses on all variables together. This more sophisticated viewpoint leads to a compact notational representation, increased insight into the simplex process, and to alternative methods for implementation.

3.1 PIVOTS

To obtain a firm grasp of the simplex procedure, it is essential that one first understand the process of pivoting in a set of simultaneous linear equations. There are two dual interpretations of the pivot procedure.

First Interpretation

Consider the set of simultaneous linear equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdot \qquad\qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad\qquad \cdot \\ \cdot \qquad\qquad\qquad\qquad\qquad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{array} \tag{1}$$

where $m \leqslant n$. In matrix form we write this as

$$\mathbf{Ax} = \mathbf{b}. \tag{2}$$

In the space E^n we interpret this as a collection of m linear relations that must be satisfied by a vector $\mathbf{x}$. Thus denoting by $\mathbf{a}^i$ the ith row of $\mathbf{A}$ we may express (1) as:

$$\begin{array}{c} \mathbf{a}^1\mathbf{x} = b_1 \\ \mathbf{a}^2\mathbf{x} = b_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{a}^m\mathbf{x} = b_m. \end{array} \tag{3}$$

This corresponds to the most natural interpretation of (1) as a set of m equations.

If $m < n$ and the equations are linearly independent, then there is not a unique solution but a whole linear variety of solutions (see Appendix B). A unique solution results, however, if $n - m$ additional independent linear equations are adjoined. For example, we might specify $n - m$ equations of the form $\mathbf{e}^k\mathbf{x} = 0$, where $\mathbf{e}^k$ is the kth unit vector (which is equivalent to $x_k = 0$), in which case we obtain a basic solution to (1). Different basic solutions are obtained by imposing different additional equations of this special form.

If the equations (3) are linearly independent, we may replace a given equation by any nonzero multiple of itself plus any linear combination of the other equations in the system. This leads to the well-known Gaussian reduction schemes, whereby multiples of equations are systematically subtracted from one another to yield either a triangular or canonical form. It is well known, and easily proved, that if the first m columns of $\mathbf{A}$ are linearly independent, the system (1) can, by a sequence of such multiplications and subtractions, be converted to the following *canonical form*:

$$\begin{array}{llllll} x_1 & & + y_{1,m+1}x_{m+1} + y_{1,m+2}x_{m+2} + \cdots & + y_{1,n}x_n = y_{10} \\ & x_2 & + y_{2,m+1}x_{m+1} + y_{2,m+2}x_{m+2} + \cdots & + y_{2,n}x_n = y_{20} \\ & & \cdot & \cdot \\ & & \cdot & \cdot \\ & & \cdot & \cdot \\ & \quad x_m & + y_{m,m+1}x_{m+1} + \quad \cdots & + y_{m,n}x_n = y_{m0}. \end{array} \tag{4}$$

Corresponding to this canonical representation of the system, the variables x_1, $x_2, \ldots, x_m$ are called *basic* and the other variables are *nonbasic*. The corresponding basic solution is then:

$$x_1 = y_{10}, \quad x_2 = y_{20}, \ldots, x_m = y_{m0}, \quad x_{m+1} = 0, \ldots, x_n = 0,$$

or in vector form: $\mathbf{x} = (\mathbf{y}_0, \mathbf{0})$ where $\mathbf{y}_0$ is m-dimensional and $\mathbf{0}$ is the $(n-m)$-dimensional zero vector.

Actually, we relax our definition somewhat and consider a system to be in *canonical form* if, among the n variables, there are m basic ones with the property that each appears in only one equation, its coefficient in that equation is unity, and no two of these m variables appear in any one equation. This is equivalent to saying that a system is in canonical form if by some reordering of the equations and the variables it takes the form (4).

Also it is customary, from the dictates of economy, to represent the system (4) by its corresponding array of coefficients or *tableau*:

$$\begin{array}{ccccccccc}
1 & 0 & \cdots & 0 & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1n} & y_{10} \\
0 & 1 & \cdots & 0 & y_{2,m+1} & y_{2,m+2} & \cdots & y_{2n} & y_{20} \\
0 & 0 & \cdots & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{mn} & y_{m0}
\end{array}$$

The question solved by pivoting is this: given a system in canonical form, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic; what is the new canonical form corresponding to the new set of basic variables? The procedure is quite simple. Suppose in the canonical system (4) we wish to replace the basic variable x_p, $1 \leqslant p \leqslant m$, by the nonbasic variable x_q. This can be done if and only if y_{pq} is nonzero; it is accomplished by dividing row p by y_{pq} to get a unit coefficient for x_q in the pth equation, and then subtracting suitable multiples of row p from each of the other rows in order to get a zero coefficient for x_q in all other equations. This transforms the qth column of the tableau so that it is zero except in its pth entry (which is unity) and does not affect the columns of the other basic variables. Denoting the coefficients of the new system in canonical form by y'_{ij}, we have explicitly

$$\begin{cases} y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, & i \neq p \\ y'_{pj} = \frac{y_{pj}}{y_{pq}}. \end{cases} \tag{5}$$

Equations (5) are the pivot equations that arise frequently in linear programming. The element y_{pq} in the original system is said to be the *pivot element*.

Example 1. Consider the system in canonical form:

$$\begin{aligned} x_1 \qquad\quad + \ x_4 + \ x_5 - x_6 &= \ \ 5 \\ x_2 \quad + 2x_4 - 3x_5 + x_6 &= \ \ 3 \\ x_3 - \ x_4 + 2x_5 - x_6 &= -1. \end{aligned}$$

Let us find the basic solution having basic variables x_4, x_5, x_6. We set up the coefficient array below:

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 1 & 0 & 0 & 1 & 1 & -1 & 5 \\ 0 & 1 & 0 & 2 & -3 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 & -1 \end{array}$$

The circle indicated is our first pivot element and corresponds to the replacement of x_1 by x_4 as a basic variable. After pivoting we obtain the array

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 1 & 0 & 0 & 1 & 1 & -1 & 5 \\ -2 & 1 & 0 & 0 & -5 & 3 & -7 \\ 1 & 0 & 1 & 0 & 3 & -2 & 4 \end{array}$$

and again we have circled the next pivot element indicating our intention to replace x_2 by x_5. We then obtain

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 3/5 & 1/5 & 0 & 1 & 0 & -2/5 & 18/5 \\ 2/5 & -1/5 & 0 & 0 & 1 & -3/5 & 7/5 \\ -1/5 & 3/5 & 1 & 0 & 0 & -1/5 & -1/5 \end{array}$$

Continuing, there results

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 1 & -1 & -2 & 1 & 0 & 0 & 4 \\ 1 & -2 & -3 & 0 & 1 & 0 & 2 \\ 1 & -3 & -5 & 0 & 0 & 1 & 1 \end{array}$$

From this last canonical form we obtain the new basic solution

$$x_4 = 4, \qquad x_5 = 2, \qquad x_6 = 1.$$

Second Interpretation

The set of simultaneous equations represented by (1) and (2) can be interpreted in E^m as a vector equation. Denoting the columns of $\mathbf{A}$ by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ we write (1) as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}. \tag{6}$$

In this interpretation we seek to express $\mathbf{b}$ as a linear combination of the $\mathbf{a}_i$'s.

If $m < n$ and the vectors $\mathbf{a}_i$ span E^m then there is not a unique solution but a whole family of solutions. The vector $\mathbf{b}$ has a unique representation, however, as a linear combination of a given linearly independent subset of these vectors. The corresponding solution with $n - m$ x_i variables set equal to zero is a basic solution to (1).

Suppose now that we start with a system in the canonical form corresponding to the tableau

$$\begin{array}{cccccccccc}
1 & 0 & \cdots & 0 & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1n} & y_{10} \\
0 & 1 & \cdot & 0 & y_{2,m+1} & y_{2,m+2} & \cdots & y_{2n} & y_{20} \\
0 & 0 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{mn} & y_{m0}
\end{array} \tag{7}$$

In this case the first m vectors form a basis. Furthermore, every other vector represented in the tableau can be expressed as a linear combination of these basis vectors by simply reading the coefficients down the corresponding column. Thus

$$\mathbf{a}_j = y_{1j}\mathbf{a}_1 + y_{2j}\mathbf{a}_2 + \cdots + y_{mj}\mathbf{a}_m. \tag{8}$$

The tableau can be interpreted as giving the representations of the vectors $\mathbf{a}_j$ in terms of the basis; the jth column of the tableau is the representation for the vector $\mathbf{a}_j$. In particular, the expression for $\mathbf{b}$ in terms of the basis is given in the last column.

We now consider the operation of replacing one member of the basis by another vector not already in the basis. Suppose for example we wish to replace the basis vector $\mathbf{a}_p$, $1 \leqslant p \leqslant m$, by the vector $\mathbf{a}_q$. Provided that the first m vectors with $\mathbf{a}_p$ replaced by $\mathbf{a}_q$ are linearly independent these vectors constitute a basis and every vector can be expressed as a linear combination of this new basis. To find the new representations of the vectors we must update the tableau. The linear independence condition holds if and only if $y_{pq} \neq 0$.

Any vector $\mathbf{a}_j$ can be expressed in terms of the old array through (8). For $\mathbf{a}_q$ we have

$$\mathbf{a}_q = \sum_{\substack{i=1 \\ i \neq p}}^{m} y_{iq}\mathbf{a}_i + y_{pq}\mathbf{a}_p$$

from which we may solve for $\mathbf{a}_p$,

$$\mathbf{a}_p = \frac{1}{y_{pq}}\mathbf{a}_q - \sum_{\substack{i=1 \\ i \neq p}}^{m} \frac{y_{iq}}{y_{pq}}\mathbf{a}_i. \tag{9}$$

Substituting (9) into (8) we obtain:

$$\mathbf{a}_j = \sum_{\substack{i=1 \\ i \neq p}}^{m} \left(y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \right) \mathbf{a}_i + \frac{y_{pj}}{y_{pq}}\mathbf{a}_q. \tag{10}$$

Denoting the coefficients of the new tableau, which gives the linear combinations, by y'_{ij} we obtain immediately from (10)

$$\begin{cases} y'_{ij} = y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj}, & i \neq p \\ y'_{pj} = \frac{y_{pj}}{y_{pq}}. \end{cases} \tag{11}$$

These formulae are identical to (5).

If a system of equations is not originally given in canonical form, we may put it into canonical form by adjoining the m unit vectors to the tableau and, starting with these vectors as the basis, successively replace each of them with columns of $\mathbf{A}$ using the pivot operation.

Example 2. Suppose we wish to solve the simultaneous equations

$$\begin{aligned} x_1 + \ \ x_2 - x_3 &= \ \ 5 \\ 2x_1 - 3x_2 + x_3 &= \ \ 3 \\ -x_1 + 2x_2 - x_3 &= -1. \end{aligned}$$

To obtain an original basis, we form the augmented tableau

$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{b}$
1	0	0	1	1	−1	5
0	1	0	2	−3	1	3
0	0	1	−1	2	−1	−1

and replace $\mathbf{e}_1$ by $\mathbf{a}_1$, $\mathbf{e}_2$ by $\mathbf{a}_2$, and $\mathbf{e}_3$ by $\mathbf{a}_3$. The required operations are identical to those of Example 1.

3.2 ADJACENT EXTREME POINTS

In Chapter 2 it was discovered that it is only necessary to consider basic feasible solutions to the system

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned} \tag{12}$$

when solving a linear program, and in the previous section it was demonstrated that the pivot operation can generate a new basic solution from an old one by replacing one basic variable by a nonbasic variable. It is clear, however, that although the pivot operation takes one basic solution into another, the nonnegativity of the solution will not in general be preserved. Special conditions must be satisfied in order that a pivot operation maintain feasibility. In this section we show how it is possible to select pivots so that we may transfer from one basic feasible solution to another.

We show that although it is not possible to arbitrarily specify the pair of variables whose roles are to be interchanged and expect to maintain the nonnegativity condition, it is possible to arbitrarily specify which nonbasic variable is to become basic and then determine which basic variable should become nonbasic. As is conventional, we base our derivation on the vector interpretation of the linear equations although the dual interpretation could alternatively be used.

Nondegeneracy Assumption

Many arguments in linear programming are substantially simplified upon the introduction of the following.

Nondegeneracy assumption: Every basic feasible solution of (12) is a nondegenerate basic feasible solution.

This assumption is invoked throughout our development of the simplex method, since when it does not hold the simplex method can break down if it is not suitably amended. The assumption, however, should be regarded as one made primarily for convenience, since all arguments can be extended to include degeneracy, and the simplex method itself can be easily modified to account for it.

Determination of Vector to Leave Basis

Suppose we have the basic feasible solution $\mathbf{x} = (x_1,\ x_2, \ldots, x_m, 0, 0, \ldots, 0)$ or, equivalently, the representation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}. \tag{13}$$

Under the nondegeneracy assumption, $x_i > 0,\ i = 1, 2, \ldots, m$. Suppose also that we have decided to bring into the representation the vector $\mathbf{a}_q$, $q > m$. We have available a representation of $\mathbf{a}_q$ in terms of the current basis

$$\mathbf{a}_q = y_{1q}\mathbf{a}_1 + y_{2q}\mathbf{a}_2 + \cdots + y_{mq}\mathbf{a}_m. \tag{14}$$

Multiplying (14) by a variable $\varepsilon \geqslant 0$ and subtracting from (13), we have

$$(x_1 - \varepsilon y_{1q})\,\mathbf{a}_1 + (x_2 - \varepsilon y_{2q})\,\mathbf{a}_2 + \cdots + (x_m - \varepsilon y_{mq})\,\mathbf{a}_m + \varepsilon\mathbf{a}_q = \mathbf{b}. \tag{15}$$

Thus, for any $\varepsilon \geqslant 0$ (15) gives $\mathbf{b}$ as a linear combination of at most $m+1$ vectors. For $\varepsilon = 0$ we have the old basic feasible solution. As ε is increased from zero, the coefficient of $\mathbf{a}_q$ increases, and it is clear that for small enough ε, (15) gives a feasible but nonbasic solution. The coefficients of the other vectors will either increase or decrease linearly as ε is increased. If any decrease, we may set ε equal to the value corresponding to the first place where one (or more) of the coefficients vanishes. That is

$$\varepsilon = \min_i \left\{x_i/y_{iq} : y_{iq} > 0\right\}. \tag{16}$$

In this case we have a new basic feasible solution, with the vector $\mathbf{a}_q$ replacing the vector $\mathbf{a}_p$ where p corresponds to the minimizing index in (16). If the minimum in (16) is achieved by more than a single index i, then the new solution is degenerate and any of the vectors with zero component can be regarded as the one which left the basis.

If none of the y_{iq}'s are positive, then all coefficients in the representation (15) increase (or remain constant) as ε is increased, and no new basic feasible solution is obtained. We observe, however, that in this case, where none of the y_{iq}'s are positive, there are feasible solutions to (12) having arbitrarily large coefficients. This means that the set K of feasible solutions to (12) is unbounded, and this special case, as we shall see, is of special significance in the simplex procedure.

In summary, we have deduced that given a basic feasible solution and an arbitrary vector $\mathbf{a}_q$, there is either a new basic feasible solution having $\mathbf{a}_q$ in its basis and one of the original vectors removed, or the set of feasible solutions is unbounded.

Let us consider how the calculation of this section can be displayed in our tableau. We assume that corresponding to the constraints

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{x} \geqslant \mathbf{0},$$

we have a tableau of the form

$$\begin{array}{cccccccccc}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_m & \mathbf{a}_{m+1} & \mathbf{a}_{m+2} & \cdots & \mathbf{a}_n & \mathbf{b} \\
1 & 0 & 0 & \cdots & 0 & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1n} & y_{10} \\
0 & 1 & 0 & & 0 & y_{2,m+1} & y_{2,m+2} & & \cdot & y_{20} \\
0 & 0 & 1 & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & \cdot & & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{mn} & y_{m0}
\end{array} \tag{17}$$

This tableau may be the result of several pivot operations applied to the original tableau, but in any event, it represents a solution with basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. We assume that $y_{10}, y_{20}, \ldots, y_{m0}$ are nonnegative, so that the corresponding basic solution $x_1 = y_{10}, x_2 = y_{20}, \ldots, x_m = y_{m0}$ is feasible. We wish to bring into the basis the vector $\mathbf{a}_q$, $q > m$, and maintain feasibility. In order to determine which element in the qth column to use as pivot (and hence which vector in the basis will leave), we use (16) and compute the ratios $x_i/y_{iq} = y_{i0}/y_{iq}$, $i = 1, 2, \ldots, m$, select the smallest nonnegative ratio, and pivot on the corresponding y_{iq}.

Example 3. Consider the system

$$\begin{array}{cccccccc}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \\
1 & 0 & 0 & 2 & 4 & 6 & 4 \\
0 & 1 & 0 & 1 & 2 & 3 & 3 \\
0 & 0 & 1 & -1 & 2 & 1 & 1
\end{array}$$

which has basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ yielding a basic feasible solution $\mathbf{x} = (4, 3, 1, 0, 0, 0)$. Suppose we elect to bring $\mathbf{a}_4$ into the basis. To determine which element in the fourth column is the appropriate pivot, we compute the three ratios:

$$4/2 = 2, \quad 3/1 = 3, \quad 1/-1 = -1$$

and select the smallest nonnegative one. This gives 2 as the pivot element. The new tableau is

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
1/2	0	0	1	2	3	2
−1/2	1	0	0	0	0	1
1/2	0	1	0	4	4	3

with corresponding basic feasible solution $\mathbf{x} = (0, 1, 3, 2, 0, 0)$.

Our derivation of the method for selecting the pivot in a given column that will yield a new feasible solution has been based on the vector interpretation of the equation $\mathbf{Ax} = \mathbf{b}$. An alternative derivation can be constructed by considering the dual approach that is based on the rows of the tableau rather than the columns. Briefly, the argument runs like this: if we decide to pivot on y_{pq}, then we first divide the pth row by the pivot element y_{pq} to change it to unity. In order that the new y_{p0} remain positive, it is clear that we must have $y_{pq} > 0$. Next we subtract multiples of the pth row from each other row in order to obtain zeros in the qth column. In this process the new elements in the last column must remain nonnegative—if the pivot was properly selected. The full operation is to subtract, from the ith row, y_{iq}/y_{pq} times the pth row. This yields a new solution obtained directly from the last column:

$$x_i' = x_i - \frac{y_{iq}}{y_{pq}} x_p.$$

For this to remain nonnegative, it follows that $x_p/y_{pq} \leqslant x_i/y_{iq}$, and hence again we are led to the conclusion that we select p as the index i minimizing x_i/y_{iq}.

Geometrical Interpretations

Corresponding to the two interpretations of pivoting and extreme points, developed algebraically, are two geometrical interpretations. The first is in *activity space*, the space where $\mathbf{x}$ is represented. This is perhaps the most natural space to consider, and it was used in Section 2.5. Here the feasible region is shown directly as a convex set, and basic feasible solutions are extreme points. Adjacent extreme points are points that lie on a common edge.

The second geometrical interpretation is in *requirements space*, the space where the columns of $\mathbf{A}$ and $\mathbf{b}$ are represented. The fundamental relation is

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n = \mathbf{b}.$$

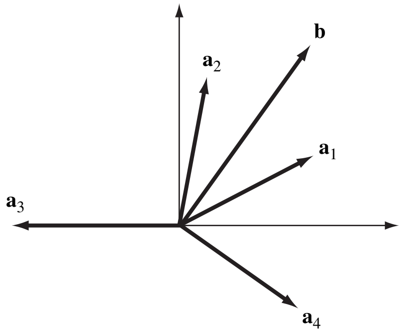


Fig. 3.1 Constraint representation in requirements space

An example for $m = 2$, $n = 4$ is shown in Fig. 3.1. A feasible solution defines a representation of $\mathbf{b}$ as a positive combination of the $\mathbf{a}_1$'s. A basic feasible solution will use only m positive weights. In the figure a basic feasible solution can be constructed with positive weights on $\mathbf{a}_1$ and $\mathbf{a}_2$ because $\mathbf{b}$ lies between them. A basic feasible solution cannot be constructed with positive weights on $\mathbf{a}_1$ and $\mathbf{a}_4$. Suppose we start with $\mathbf{a}_1$ and $\mathbf{a}_2$ as the initial basis. Then an adjacent basis is found by bringing in some other vector. If $\mathbf{a}_3$ is brought in, then clearly $\mathbf{a}_2$ must go out. On the other hand, if $\mathbf{a}_4$ is brought in, $\mathbf{a}_1$ must go out.

3.3 DETERMINING A MINIMUM FEASIBLE SOLUTION

In the last section we showed how it is possible to pivot from one basic feasible solution to another (or determine that the solution set is unbounded) by arbitrarily selecting a column to pivot on and then appropriately selecting the pivot in that column. The idea of the simplex method is to select the column so that the resulting new basic feasible solution will yield a lower value to the objective function than the previous one. This then provides the final link in the simplex procedure. By an elementary calculation, which is derived below, it is possible to determine which vector should enter the basis so that the objective value is reduced, and by another simple calculation, derived in the previous section, it is possible to then determine which vector should leave in order to maintain feasibility.

Suppose we have a basic feasible solution

$$(\mathbf{x_B}, \mathbf{0}) = (y_{10}, y_{20}, \ldots, y_{m0}, 0, 0, \ldots, 0)$$

together with a tableau having an identity matrix appearing in the first m columns as shown below:

$$
\begin{array}{ccccccccc}
\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m & \mathbf{a}_{m+1} & \cdots & \mathbf{a}_n & \mathbf{b} \\
1 & 0 & & 0 & y_{1,m+1} & \cdots & y_{1n} & y_{10} \\
0 & 1 & & 0 & y_{2,m+1} & \cdots & y_{20} & y_{20} \\
\cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & & 1 & y_{m,m+1} & \cdots & y_{mn} & y_{m0}
\end{array}
\tag{18}
$$

The value of the objective function corresponding to any solution $\mathbf{x}$ is

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n, \tag{19}$$

and hence for the basic solution, the corresponding value is

$$z_0 = \mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}}, \tag{20}$$

where $\mathbf{c}_{\mathbf{B}}^T = [c_1, c_2, \ldots, c_m]$.

Although it is natural to use the basic solution $(\mathbf{x}_{\mathbf{B}}, \mathbf{0})$ when we have the tableau (18), it is clear that if arbitrary values are assigned to $x_{m+1}, x_{m+2}, \ldots, x_n$, we can easily solve for the remaining variables as

$$
\begin{aligned}
x_1 &= y_{10} - \sum_{j=m+1}^{n} y_{1j} x_j \\
x_2 &= y_{20} - \sum_{j=m+1}^{n} y_{2j} x_j \\
&\cdot \\
&\cdot \\
&\cdot \\
x_m &= y_{m0} - \sum_{j=m+1}^{n} y_{mj} x_j .
\end{aligned}
\tag{21}
$$

Using (21) we may eliminate $x_1, x_2, \ldots, x_m$ from the general formula (19). Doing this we obtain

$$
\begin{aligned}
z = \mathbf{c}^T \mathbf{x} = z_0 &+ (c_{m+1} - z_{m+1})\, x_{m+1} \\
&+ (c_{m+2} - z_{m+2})\, x_{m+2} + \cdots + (c_n - z_n)\, x_n
\end{aligned}
\tag{22}
$$

where

$$z_j = y_{1j} c_1 + y_{2j} c_2 + \cdots + y_{mj} c_m, \quad m+1 \leqslant j \leqslant n, \tag{23}$$

which is the fundamental relation required to determine the pivot column. The important point is that this equation gives the values of the objective function z for any solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ in terms of the variables $x_{m+1}, \ldots, x_n$. From it we can determine if there is any advantage in changing the basic solution by introducing one of the nonbasic variables. For example, if $c_j - z_j$ is negative for some j, $m+1 \leqslant j \leqslant n$, then increasing x_j from zero to some positive value would decrease the total cost, and therefore would yield a better solution. The formulae (22) and (23) automatically take into account the changes that would be required in the values of the basic variables $x_1, x_2, \ldots, x_m$ to accommodate the change in x_j.

Let us derive these relations from a different viewpoint. Let $\mathbf{y}_i$ be the ith column of the tableau. Then any solution satisfies

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m = \mathbf{y}_0 - x_{m+1}\mathbf{y}_{m+1} - x_{m+2}\mathbf{y}_{m+2} - \cdots - x_n\mathbf{y}_n.$$

Taking the inner product of this vector equation with $\mathbf{c}_\mathbf{B}^T$, we have

$$\sum_{i=1}^{m} c_i x_i = \mathbf{c}_\mathbf{B}^T\mathbf{y}_0 - \sum_{j=m+1}^{n} z_j x_j,$$

where $z_j = \mathbf{c}_\mathbf{B}^T\mathbf{y}_j$. Thus, adding $\sum_{j=m+1}^{n} c_j x_j$ to both sides,

$$\mathbf{c}^T\mathbf{x} = z_0 + \sum_{j=m+1}^{n} (c_j - z_j)\, x_j \tag{24}$$

as before.

We now state the condition for improvement, which follows easily from the above observation, as a theorem.

Theorem. (Improvement of basic feasible solution). *Given a nondegenerate basic feasible solution with corresponding objective value z_0, suppose that for some j there holds $c_j - z_j < 0$. Then there is a feasible solution with objective value $z < z_0$. If the column $\mathbf{a}_j$ can be substituted for some vector in the original basis to yield a new basic feasible solution, this new solution will have $z < z_0$. If $\mathbf{a}_j$ cannot be substituted to yield a basic feasible solution, then the solution set K is unbounded and the objective function can be made arbitrarily small (toward minus infinity).*

Proof. The result is an immediate consequence of the previous discussion. Let $(x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0)$ be the basic feasible solution with objective value z_0 and suppose $c_{m+1} - z_{m+1} < 0$. Then, in any case, new feasible solutions can be constructed of the form $(x'_1, x'_2, \ldots, x'_m, x'_{m+1}, 0, 0, \ldots, 0)$ with $x'_{m+1} > 0$. Substituting this solution in (22) we have

$$z - z_0 = (c_{m+1} - z_{m+1})\, x'_{m+1} < 0,$$

and hence $z < z_0$ for any such solution. It is clear that we desire to make x'_{m+1} as large as possible. As x'_{m+1} is increased, the other components increase, remain constant, or decrease. Thus x'_{m+1} can be increased until one $x'_i = 0$, $i \leqslant m$, in which case we obtain a new basic feasible solution, or if none of the x'_i's decrease, x'_{m+1} can be increased without bound indicating an unbounded solution set and an objective value without lower bound. ▌

We see that if at any stage $c_j - z_j < 0$ for some j, it is possible to make x_j positive and decrease the objective function. The final question remaining is whether $c_j - z_j \geqslant 0$ for all j implies optimality.

Optimality Condition Theorem. *If for some basic feasible solution $c_j - z_j \geqslant 0$ for all j, then that solution is optimal.*

Proof. This follows immediately from (22), since any other feasible solution must have $x_i \geqslant 0$ for all i, and hence the value z of the objective will satisfy $z - z_0 \geqslant 0$. ▌

Since the constants $c_j - z_j$ play such a central role in the development of the simplex method, it is convenient to introduce the somewhat abbreviated notation $r_j = c_j - z_j$ and refer to the r_j's as the *relative cost coefficients* or, alternatively, the *reduced cost coefficients* (both terms occur in common usage). These coefficients measure the cost of a variable relative to a given basis. (For notational convenience we extend the definition of relative cost coefficients to basic variables as well; the relative cost coefficient of a basic variable is zero.)

We conclude this section by giving an economic interpretation of the relative cost coefficients. Let us agree to interpret the linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} \\ & \ \ \mathbf{x} \geqslant \mathbf{0} \end{array}$$

as a diet problem (see Section 2.2) where the nutritional requirements must be met exactly. A column of **A** gives the nutritional equivalent of a unit of a particular food. With a given basis consisting of, say, the first m columns of **A**, the corresponding simplex tableau shows how any food (or more precisely, the nutritional content of any food) can be constructed as a combination of foods in the basis. For instance, if carrots are not in the basis we can, using the description given by the tableau, construct a *synthetic* carrot which is nutritionally equivalent to a carrot, by an appropriate combination of the foods in the basis.

In considering whether or not the solution represented by the current basis is optimal, we consider a certain food not in the basis—say carrots—and determine if it would be advantageous to bring it into the basis. This is very easily determined by examining the cost of carrots as compared with the cost of synthetic carrots. If carrots are food j, then the unit cost of carrots is c_j. The cost of a unit of synthetic carrots is, on the other hand,

$$z_j = \sum_{i=1}^{m} c_i y_{ij}.$$

If $r_j = c_j - z_j < 0$, it is advantageous to use real carrots in place of synthetic carrots, and carrots should be brought into the basis.

In general each z_j can be thought of as the price of a unit of the column $\mathbf{a}_j$ when constructed from the current basis. The difference between this synthetic price and the direct price of that column determines whether that column should enter the basis.

3.4 COMPUTATIONAL PROCEDURE—SIMPLEX METHOD

In previous sections the theory, and indeed much of the technique, necessary for the detailed development of the simplex method has been established. It is only necessary to put it all together and illustrate it with examples.

In this section we assume that we begin with a basic feasible solution and that the tableau corresponding to $\mathbf{Ax} = \mathbf{b}$ is in the canonical form for this solution. Methods for obtaining this first basic feasible solution, when one is not obvious, are described in the next section.

In addition to beginning with the array $\mathbf{Ax} = \mathbf{b}$ expressed in canonical form corresponding to a basic feasible solution, we append a row at the bottom consisting of the relative cost coefficients and the negative of the current cost. The result is a *simplex tableau.*

Thus, if we assume the basic variables are (in order) $x_1, x_2, \ldots, x_m$, the simplex tableau takes the initial form shown in Fig. 3.2.

The basic solution corresponding to this tableau is

$$x_i = \begin{cases} y_{i0} & 0 \leqslant i \leqslant m \\ 0 & m+1 \leqslant i \leqslant n \end{cases}$$

which we have assumed is feasible, that is, $y_{i0} \geqslant 0$, $i = 1, 2, \ldots, m$. The corresponding value of the objective function is z_0.

$\mathbf{a}_1$	$\mathbf{a}_2$	$\cdots$	$\mathbf{a}_m$	$\mathbf{a}_{m+1}$	$\mathbf{a}_{m+2}$	$\cdots$	$\mathbf{a}_j$	$\cdots$	$\mathbf{a}_n$	$\mathbf{b}$
1	0	$\cdots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\cdots$	y_{1j}	$\cdots$	y_{1n}	y_{10}
0	1		·	·	·		·		·	·
·	·		·	·	·		·		·	·
·	·		·	·	·		·		·	·
0	0		·	$y_{i,m+1}$	$y_{i,m+2}$	$\cdots$	y_{ij}	$\cdots$	y_{in}	y_{i0}
·	·		·	·	·		·		·	·
·	·		·	·	·		·		·	·
0	0		1	$y_{m,m+1}$	$y_{m,m+2}$	$\cdots$	y_{mj}	$\cdots$	y_{mn}	y_{m0}
0	0	$\cdots$	0	r_{m+1}	r_{m+2}	$\cdots$	r_j	$\cdots$	r_n	$-z_0$

Fig. 3.2 Canonical simplex tableau

The relative cost coefficients r_j indicate whether the value of the objective will increase or decrease if x_j is pivoted into the solution. If these coefficients are all nonnegative, then the indicated solution is optimal. If some of them are negative, an improvement can be made (assuming nondegeneracy) by bringing the corresponding component into the solution. When more than one of the relative cost coefficients is negative, any one of them may be selected to determine in which column to pivot. Common practice is to select the most negative value. (See Exercise 13 for further discussion of this point.)

Some more discussion of the relative cost coefficients and the last row of the tableau is warranted. We may regard z as an additional variable and

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n - z = 0$$

as another equation. A basic solution to the augmented system will have $m+1$ basic variables, but we can require that z be one of them. For this reason it is not necessary to add a column corresponding to z, since it would always be $(0, 0, \ldots, 0, 1)$. Thus, initially, a last row consisting of the c_i's and a right-hand side of zero can be appended to the standard array to represent this additional equation. Using standard pivot operations, the elements in this row corresponding to basic variables can be reduced to zero. This is equivalent to transforming the additional equation to the form

$$r_{m+1}x_{m+1} + r_{m+2}x_{m+2} + \cdots + r_nx_n - z = -z_0. \tag{25}$$

This must be equivalent to (24), and hence the r_j's obtained are the relative cost coefficients. Thus, the last row can be treated operationally like any other row: just start with c_j's and reduce the terms corresponding to basic variables to zero by row operations.

After a column q is selected in which to pivot, the final selection of the pivot element is made by computing the ratio y_{i0}/y_{iq} for the positive elements y_{iq}, $i = 1, 2, \ldots, m$, of the qth column and selecting the element p yielding the minimum ratio. Pivoting on this element will maintain feasibility as well as (assuming nondegeneracy) decrease the value of the objective function. If there are ties, any element yielding the minimum can be used. If there are no nonnegative elements in the column, the problem is unbounded. After updating the entire tableau with y_{pq} as pivot and transforming the last row in the same manner as all other rows (except row q), we obtain a new tableau in canonical form. The new value of the objective function again appears in the lower right-hand corner of the tableau.

The simplex algorithm can be summarized by the following steps:

Step 0. Form a tableau as in Fig. 3.2 corresponding to a basic feasible solution. The relative cost coefficients can be found by row reduction.

Step 1. If each $r_j \geqslant 0$, stop; the current basic feasible solution is optimal.

Step 2. Select q such that $r_q < 0$ to determine which nonbasic variable is to become basic.

Step 3. Calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$, $i = 1, 2, \ldots, m$. If no $y_{iq} > 0$, stop; the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio.

Step 4. Pivot on the pqth element, updating all rows including the last. Return to Step 1.

Proof that the algorithm solves the problem (again assuming nondegeneracy) is essentially established by our previous development. The process terminates only if optimality is achieved or unboundedness is discovered. If neither condition is discovered at a given basic solution, then the objective is strictly decreased. Since there are only a finite number of possible basic feasible solutions, and no basis repeats because of the strictly decreasing objective, the algorithm must reach a basis satisfying one of the two terminating conditions.

Example 1. Maximize $3x_1 + x_2 + 3x_3$ subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leqslant 2 \\ x_1 + 2x_2 + 3x_3 &\leqslant 5 \\ 2x_1 + 2x_2 + x_3 &\leqslant 6 \\ x_1 \geqslant 0, \quad x_2 \geqslant 0, \ x_3 &\geqslant 0. \end{aligned}$$

To transform the problem into standard form so that the simplex procedure can be applied, we change the maximization to minimization by multiplying the objective function by minus one, and introduce three nonnegative slack variables x_4, x_5, x_6. We then have the initial tableau

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	5
$\mathbf{r}^T$	−3	−1	−3	0	0	0	0

First tableau

The problem is already in canonical form with the three slack variables serving as the basic variables. We have at this point $r_j = c_j - z_j = c_j$, since the costs of the slacks are zero. Application of the criterion for selecting a column in which to pivot shows that any of the first three columns would yield an improved solution. In each of these columns the appropriate pivot element is determined by computing the ratios y_{i0}/y_{ij} and selecting the smallest positive one. The three allowable pivots are all circled on the tableau. It is only necessary to determine one allowable pivot, and normally we would not bother to calculate them all. For hand calculation on problems of this size, however, we may wish to examine the allowable pivots and select one that will minimize (at least in the short run) the amount of division required. Thus for this example we select 1.

2	1	1	1	0	0	2
−3	0	1	−2	1	0	1
−2	0	−1	−2	0	1	2
−1	0	−2	1	0	0	2

Second tableau

We note that the objective function—we are using the negative of the original one—has decreased from zero to minus two. Again we pivot on 1.

5	1	0	3	−1	0	1
−3	0	1	−2	1	0	1
−5	0	0	−4	1	1	3
−7	0	0	−3	2	0	4

Third tableau

The value of the objective function has now decreased to minus four and we may pivot in either the first or fourth column. We select 5 .

1	1/5	0	3/5	−1/5	0	1/5
0	3/5	1	−1/5	2/5	0	8/5
0	1	0	−1	0	1	4
0	7/5	0	6/5	3/5	0	27/5

Fourth tableau

Since the last row has no negative elements, we conclude that the solution corresponding to the fourth tableau is optimal. Thus $x_1 = 1/5$, $x_2 = 0$, $x_3 = 8/5$, $x_4 = 0$, $x_5 = 0$, $x_6 = 4$ is the optimal solution with a corresponding value of the (negative) objective of $-(27/5)$.

Degeneracy

It is possible that in the course of the simplex procedure, degenerate basic feasible solutions may occur. Often they can be handled as a nondegenerate basic feasible solution. However, it is possible that after a new column q is selected to enter the basis, the minimum of the ratios y_{i0}/y_{iq} may be zero, implying that the zero-valued basic variable is the one to go out. This means that the new variable x_q will come in at zero value, the objective will not decrease, and the new basic feasible solution will also be degenerate. Conceivably, this process could continue for a series of steps until, finally, the original degenerate solution is again obtained. The result is a *cycle* that could be repeated indefinitely.

Methods have been developed to avoid such cycles (see Exercises 15–17 for a full discussion of one of them, which is based on perturbing the problem slightly so that zero-valued variables are actually small positive values, and Exercise 32 for Bland's rule, which is simpler). In practice, however, such procedures are found to be unnecessary. When degenerate solutions are encountered, the simplex procedure generally does not enter a cycle. However, anticycling procedures are simple, and many codes incorporate such a procedure for the sake of safety.

3.5 ARTIFICIAL VARIABLES

A basic feasible solution is sometimes immediately available for linear programs. For example, in problems with constraints of the form

$$\begin{aligned} \mathbf{Ax} &\leqslant \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned} \tag{26}$$

with $\mathbf{b} \geqslant \mathbf{0}$, a basic feasible solution to the corresponding standard form of the problem is provided by the slack variables. This provides a means for initiating the simplex procedure. The example in the last section was of this type. An initial basic feasible solution is not always apparent for other types of linear programs, however, and it is necessary to develop a means for determining one so that the simplex method can be initiated. Interestingly (and fortunately), an auxiliary linear program and corresponding application of the simplex method can be used to determine the required initial solution.

By elementary straightforward operations the constraints of a linear programming problem can always be expressed in the form

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geqslant \mathbf{0} \end{aligned} \tag{27}$$

with $\mathbf{b} \geqslant \mathbf{0}$. In order to find a solution to (27) consider the (artificial) minimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} y_i \\ \text{subject to} \quad & \mathbf{Ax} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{y} \geqslant \mathbf{0} \end{aligned} \tag{28}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ is a vector of artificial variables. If there is a feasible solution to (27), then it is clear that (28) has a minimum value of zero with $\mathbf{y} = \mathbf{0}$. If (27) has no feasible solution, then the minimum value of (28) is greater than zero.

Now (28) is itself a linear program in the variables **x, y**, and the system is already in canonical form with basic feasible solution $\mathbf{y} = \mathbf{b}$. If (28) is solved using the simplex technique, a basic feasible solution is obtained at each step. If the minimum value of (28) is zero, then the final basic solution will have all $y_i = 0$, and hence barring degeneracy, the final solution will have no y_i variables basic. If in the final solution some y_i are both zero and basic, indicating a degenerate solution, these basic variables can be exchanged for nonbasic x_i variables (again at zero level) to yield a basic feasible solution involving x variables only. (However, the situation is more complex if **A** is not of full rank. See Exercise 21.)

Example 1. Find a basic feasible solution to

$$2x_1 + x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + x_3 = 3$$
$$x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 \geqslant 0.$$

We introduce artificial variables $x_4 \geqslant 0$, $x_5 \geqslant 0$ and an objective function $x_4 + x_5$. The initial tableau is

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
	2	1	2	1	0	4
	3	3	1	0	1	3
$\mathbf{c}^T$	0	0	0	1	1	0

Initial tableau

A basic feasible solution to the expanded system is given by the artificial variables. To initiate the simplex procedure we must update the last row so that it has zero components under the basic variables. This yields:

	2	1	2	1	0	4
	3	3	1	0	1	3
$\mathbf{r}^T$	−5	−4	−3	0	0	−7

First tableau

Pivoting in the column having the most negative bottom row component as indicated, we obtain:

0	−1	4/3	1	−2/3	2
1	1	1/3	0	1/3	1
0	1	−4/3	0	5/3	−2

Second tableau

In the second tableau there is only one choice for pivot, and it leads to the final tableau shown.

0	−3/4	1	3/4	−1/2	3/2
1	5/4	0	−1/4	1/2	1/2
0	0	0	1	1	0

Final tableau

Both of the artificial variables have been driven out of the basis, thus reducing the value of the objective function to zero and leading to the basic feasible solution to the original problem

$$x_1 = 1/2, \quad x_2 = 0, \quad x_3 = 3/2.$$

Using artificial variables, we attack a general linear programming problem by use of the *two-phase method*. This method consists simply of a *phase I* in which artificial variables are introduced as above and a basic feasible solution is found (or it is determined that no feasible solutions exist); and a *phase II* in which, using the basic feasible solution resulting from phase I, the original objective function is minimized. During phase II the artificial variables and the objective function of phase I are omitted. Of course, in phase I artificial variables need be introduced only in those equations that do not contain slack variables.

Example 2. Consider the problem

$$\begin{aligned} \text{minimize} \quad & 4x_1 + x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{aligned}$$

There is no basic feasible solution apparent, so we use the two-phase method. The first phase was done in Example 1 for these constraints, so we shall not repeat it here. We give only the final tableau with the columns corresponding to the artificial variables deleted, since they are not used in phase II. We use the new cost function in place of the old one. Temporarily writing $\mathbf{c}^T$ in the bottom row we have

	x_1	x_2	x_3	$\mathbf{b}$
	0	$-3/4$	1	3/2
	1	5/4	0	1/2
$\mathbf{c}^T$	4	1	1	0

Initial tableau

Transforming the last row so that zeros appear in the basic columns, we have

0	$-3/4$	1	3/2
1	5/4	0	1/2
0	$-13/4$	0	$-7/2$

First tableau

3/5	0	1	9/5
4/5	1	0	2/5
13/5	0	0	$-11/5$

Second tableau

and hence the optimal solution is $x_1 = 0$, $x_2 = 2/5$, $x_3 = 9/5$.

Example 3. (A free variable problem).

$$\begin{array}{ll}
\text{minimize} & -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\
\text{subject to} & -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\
& -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\
& -x_1 + x_2 + x_3 + 2x_4 + x_5 = 4 \\
\end{array}$$

$$x_1 \text{ free}, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0, \quad x_5 \geqslant 0.$$

Since x_1 is free, it can be eliminated, as described in Chapter 2, by solving for x_1 in terms of the other variables from the first equation and substituting everywhere else. This can all be done with the simplex tableau as follows:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
	-1	1	2	1	2	7
	-1	2	3	1	1	6
	-1	1	1	2	1	4
$\mathbf{c}^T$	-2	4	7	1	5	0

Initial tableau

We select any nonzero element in the first column to pivot on—this will eliminate x_1.

1	-1	-2	-1	-2	-7
0	1	1	0	-1	-1
0	0	-1	1	-1	-3
0	2	3	-1	1	-14

Equivalent problem

We now save the first row for future reference, but our linear program only involves the sub-tableau indicated. There is no obvious basic feasible solution for this problem, so we introduce artificial variables x_6 and x_7.

	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{b}$
	-1	-1	0	1	1	0	1
	0	1	-1	1	0	1	3
$\mathbf{c}^T$	0	0	0	0	1	1	0

Initial tableau for phase I

Transforming the last row appropriately we obtain

	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{b}$
	-1	-1	0	1	1	0	1
	0	1	-1	1	0	1	3
$\mathbf{r}^T$	1	0	1	-2	0	0	-4

First tableau—phase I

x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{b}$
−1	−1	0	1	1	0	1
1	2	−1	0	−1	1	2
−1	−2	1	0	2	0	−2

Second tableau—phase I

0	1	−1	1	0	1	3
1	2	−1	0	−1	1	2
0	0	0	0	1	1	0

Final tableau—phase I

Now we go back to the equivalent reduced problem

	x_2	x_3	x_4	x_5	$\mathbf{b}$
	0	1	−1	1	3
	1	2	−1	0	2
$\mathbf{c}^T$	2	3	−1	1	−14

Initial tableau—phase II

Transforming the last row appropriately we proceed with:

0	1	−1	1	3
1	2	−1	0	2
0	−2	2	0	−21

First tableau—phase II

−1/2	0	−1/2	1	2
1/2	1	−1/2	0	1
1	0	1	0	−19

Final tableau—phase II

The solution $x_3 = 1$, $x_5 = 2$ can be inserted in the expression for x_1 giving

$$x_1 = -7 + 2 \cdot 1 + 2 \cdot 2 = -1;$$

thus the final solution is

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 - 2.$$

3.6 MATRIX FORM OF THE SIMPLEX METHOD

Although the elementary pivot transformations associated with the simplex method are in many respects most easily discernible in the tableau format, with attention focused on the individual elements, there is much insight to be gained by studying a matrix interpretation of the procedure. The vector–matrix relationships that exist between the various rows and columns of the tableau lead, however, not only to increased understanding but also, in a rather direct way, to the *revised simplex procedure* which in many cases can result in considerable computational advantage. The matrix formulation is also a natural setting for the discussion of dual linear programs and other topics related to linear programming.

A preliminary observation in the development is that the tableau at any point in the simplex procedure can be determined solely by a knowledge of which variables are basic. As before we denote by $\mathbf{B}$ the submatrix of the original $\mathbf{A}$ matrix consisting of the m columns of $\mathbf{A}$ corresponding to the basic variables. These columns are linearly independent and hence the columns of $\mathbf{B}$ form a basis for E^m. We refer to $\mathbf{B}$ as the basis matrix.

As usual, let us assume that $\mathbf{B}$ consists of the first m columns of $\mathbf{A}$. Then by partitioning $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{c}^T$ as

$$\mathbf{A} = [\mathbf{B},\ \mathbf{D}]$$
$$\mathbf{x} = (\mathbf{x_B},\ \mathbf{x_D}), \quad \mathbf{c}^T = \left[\mathbf{c}_\mathbf{B}^T,\ \mathbf{c}_\mathbf{D}^T\right],$$

the standard linear program becomes

$$\begin{aligned} &\text{minimize} \quad \mathbf{c}_\mathbf{B}^T\mathbf{x_B} + \mathbf{c}_\mathbf{D}^T\mathbf{x_D} \\ &\text{subject to} \quad \mathbf{B}\mathbf{x_B} + \mathbf{D}\mathbf{x_D} = \mathbf{b} \\ &\qquad \mathbf{x_B} \geqslant \mathbf{0}, \quad \mathbf{x_D} \geqslant \mathbf{0}. \end{aligned} \tag{29}$$

The basic solution, which we assume is also feasible, corresponding to the basis $\mathbf{B}$ is $\mathbf{x} = (\mathbf{x_B}, \mathbf{0})$ where $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$. The basic solution results from setting $\mathbf{x_D} = \mathbf{0}$. However, for any value of $\mathbf{x_D}$ the necessary value of $\mathbf{x_B}$ can be computed from (29) as

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{D}\mathbf{x_D}, \tag{30}$$

and this general expression when substituted in the cost function yields

$$\begin{aligned} z &= \mathbf{c}_\mathbf{B}^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{D}\mathbf{x_D}) + \mathbf{c}_\mathbf{D}^T\mathbf{x_D} \\ &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} + \left(\mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D}\right)\mathbf{x_D}, \end{aligned} \tag{31}$$

which expresses the cost of any solution to (29) in terms of $\mathbf{x_D}$. Thus

$$\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D} \tag{32}$$

is the relative cost vector (for nonbasic variables). It is the components of this vector that are used to determine which vector to bring into the basis.

Having derived the vector expression for the relative cost it is now possible to write the simplex tableau in matrix form. The initial tableau takes the form

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c}^T & 0 \end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{B} & \mathbf{D} & \mathbf{b} \\ \hline \mathbf{c}_{\mathbf{B}}^T & \mathbf{c}_{\mathbf{D}}^T & 0 \end{array}\right], \tag{33}$$

which is not in general in canonical form and does not correspond to a point in the simplex procedure. If the matrix $\mathbf{B}$ is used as a basis, then the corresponding tableau becomes

$$\mathbf{T} = \left[\begin{array}{c|c|c} \mathbf{I} & \mathbf{B}^{-1}\mathbf{D} & \mathbf{B}^{-1}\mathbf{b} \\ \hline \mathbf{0} & \mathbf{c}_{\mathbf{D}}^T - \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{D} & -\mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{b} \end{array}\right], \tag{34}$$

which is the matrix form we desire.

3.7 THE REVISED SIMPLEX METHOD

Extensive experience with the simplex procedure applied to problems from various fields, and having various values of n and m, has indicated that the method can be expected to converge to an optimum solution in about m, or perhaps $3m/2$, pivot operations. (Except in the worst case. See Chapter 5.) Thus, particularly if m is much smaller than n, that is, if the matrix $\mathbf{A}$ has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization.

Since the other columns are not explicitly used, it appears that the work expended in calculating the elements in these columns after each pivot is, in some sense, wasted effort. The revised simplex method is a scheme for ordering the computations required of the simplex method so that unnecessary calculations are avoided. In fact, even if pivoting is eventually required in all columns, but m is small compared to n, the revised simplex method can frequently save computational effort.

The revised form of the simplex method is this: Given the inverse $\mathbf{B}^{-1}$ of a current basis, and the current solution $\mathbf{x}_{\mathbf{B}} = \mathbf{y}_0 = \mathbf{B}^{-1}\mathbf{b}$,

Step 1. Calculate the current relative cost coefficients $\mathbf{r}_{\mathbf{D}}^T = \mathbf{c}_{\mathbf{D}}^T - \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}\mathbf{D}$. This can best be done by first calculating $\boldsymbol{\lambda}^T = \mathbf{c}_{\mathbf{B}}^T\mathbf{B}^{-1}$ and then the relative cost vector $\mathbf{r}_{\mathbf{D}}^T = \mathbf{c}_{\mathbf{D}}^T - \boldsymbol{\lambda}^T\mathbf{D}$. If $\mathbf{r}_{\mathbf{D}} \geqslant \mathbf{0}$ stop; the current solution is optimal.

Step 2. Determine which vector $\mathbf{a}_q$ is to enter the basis by selecting the most negative cost coefficient; and calculate $\mathbf{y}_q = \mathbf{B}^{-1}\mathbf{a}_q$ which gives the vector $\mathbf{a}_q$ expressed in terms of the current basis.

Step 3. If no $y_{iq} > 0$, stop; the problem is unbounded. Otherwise, calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$ to determine which vector is to leave the basis.

Step 4. Update $\mathbf{B}^{-1}$ and the current solution $\mathbf{B}^{-1}\mathbf{b}$. Return to Step 1.

Updating of $\mathbf{B}^{-1}$ is accomplished by the usual pivot operations applied to an array consisting of $\mathbf{B}^{-1}$ and $\mathbf{y}_q$, where the pivot is the appropriate element in $\mathbf{y}_q$. Of course $\mathbf{B}^{-1}\mathbf{b}$ may be updated at the same time by adjoining it as another column.

To begin the procedure one requires, as always, an initial basic feasible solution and, in this case, the inverse of the initial basis. In most problems the initial basis (and hence also its inverse) is an identity matrix, resulting either from slack or surplus variables or from artificial variables. The inverse of any initial basis can, however, be explicitly calculated in order to initiate the revised simplex procedure.

To illustrate the method and to indicate how the computations and storage can be handled, we consider an example.

Example 1. We solve again Example 1 of Section 3.4. The vectors are listed once for reference

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
2	1	1	1	0	0	2
1	2	3	0	1	0	5
2	2	1	0	0	1	6

and the objective function is determined by

$$\mathbf{c}^T = [-3, -1, -3, 0, 0, 0].$$

We start with an initial basic feasible solution and corresponding $\mathbf{B}^{-1}$ as shown in the tableau below

Variable	$\mathbf{B}^{-1}$			$\mathbf{x_B}$
4	1	0	0	2
5	0	1	0	5
6	0	0	1	6

We compute

$$\boldsymbol{\lambda}^T = [0, 0, 0]\,\mathbf{B}^{-1} = [0, 0, 0]$$

and then

$$\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \boldsymbol{\lambda}^T\mathbf{D} = [-3, -1, -3].$$

We decide to bring $\mathbf{a}_2$ into the basis (violating the rule of selecting the most negative relative cost in order to simplify the hand calculation). Its current representation is found by multiplying by $\mathbf{B}^{-1}$; thus we have

Variable	$\mathbf{B}^{-1}$			$\mathbf{x}_\mathbf{B}$	$\mathbf{y}_2$
4	1	0	0	2	1
5	0	1	0	5	2
6	0	0	1	6	2

After computing the ratios in the usual manner, we select the pivot indicated. The updated tableau becomes

Variable	$\mathbf{B}^{-1}$			$\mathbf{x}_\mathbf{B}$
2	1	0	0	2
5	−2	1	0	1
6	−2	0	1	2

then

$$\boldsymbol{\lambda}^T = [-1, 0, 0]\,\mathbf{B}^{-1} = [-1, 0, 0]$$

$$r_1 = -1, \;\; r_3 = -2, \;\; r_4 = 1.$$

We select $\mathbf{a}_3$ to enter. We have the tableau

Variable	$\mathbf{B}^{-1}$			$\mathbf{x}_\mathbf{B}$	$\mathbf{y}_3$
2	1	0	0	2	1
5	−2	1	0	1	1
6	−2	0	1	2	−1

Using the pivot indicated we obtain

Variable	$\mathbf{B}^{-1}$			$\mathbf{x}_\mathbf{B}$
2	3	−1	0	1
3	−2	1	0	1
6	−4	1	1	3

Now

$$\boldsymbol{\lambda}^T = [-1, -3, 0]\,\mathbf{B}^{-1} = [3, -2, 0],$$

and

$$r_1 = -7, \quad r_4 = -3, \quad r_5 = 2.$$

We select $\mathbf{a}_1$ to enter the basis. We have the tableau

$$\begin{array}{c|ccc|c|c}
\text{Variable} & & \mathbf{B}^{-1} & & \mathbf{x_B} & \mathbf{y}_1 \\
2 & 3 & -1 & 0 & 1 & 5 \\
3 & -2 & 1 & 0 & 1 & -3 \\
6 & -4 & 1 & 1 & 3 & -5
\end{array}$$

Using the pivot indicated we obtain

$$\begin{array}{c|ccc|c}
\text{Variable} & & \mathbf{B}^{-1} & & \mathbf{x_B} \\
1 & 3/5 & -1/5 & 0 & 1/5 \\
3 & -1/5 & 2/5 & 0 & 8/5 \\
6 & -1 & 0 & 1 & 4
\end{array}$$

Now

$$\boldsymbol{\lambda}^T = [-3, -3, 0]\,\mathbf{B}^{-1} = [-6/5, -3/5, 0],$$

and

$$r_2 = 7/5, \quad r_4 = 6/5, \quad r_5 = 3/5.$$

Since the r_i's are all nonnegative, we conclude that the solution $\mathbf{x} = (1/5, 0, 8/5, 0, 0, 4)$ is optimal.

*3.8 THE SIMPLEX METHOD AND LU DECOMPOSITION

We may go one step further in the matrix interpretation of the simplex method and note that execution of a single simplex cycle is not explicitly dependent on having $\mathbf{B}^{-1}$ but rather on the ability to solve linear systems with $\mathbf{B}$ as the coefficient matrix. Thus, the revised simplex method stated at the beginning of Section 3.7 can be restated as: Given the current basis $\mathbf{B}$,

Step 1. Calculate the current solution $\mathbf{x_B} = \mathbf{y}_0$ satisfying $\mathbf{B}\mathbf{y}_0 = \mathbf{b}$.

Step 2. Solve $\boldsymbol{\lambda}^T\mathbf{B} = \mathbf{c}_\mathbf{B}^T$, and set $\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \boldsymbol{\lambda}^T\mathbf{D}$. If $\mathbf{r_D} \geqslant \mathbf{0}$, stop; the current solution is optimal.

Step 3. Determine which vector $\mathbf{a}_q$ is to enter the basis by selecting the most negative relative cost coefficient, and solve $\mathbf{B}\mathbf{y}_q = \mathbf{a}_q$.

Step 4. If no $y_{iq} > 0$, stop; the problem is unbounded. Otherwise, calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$ and select the smallest nonnegative one to determine which vector is to leave the basis.

Step 5. Update $\mathbf{B}$. Return to Step 1.

In this form it is apparent that there is no explicit need for having $\mathbf{B}^{-1}$, but rather it is only necessary to solve three systems of equations, two involving the matrix $\mathbf{B}$ and one (the one for $\boldsymbol{\lambda}$) involving $\mathbf{B}^T$. In previous sections these three equations were solved, as the method progressed, by the pivoting operations. From the viewpoints of efficiency and numerical stability, however, this pivoting procedure is not as effective as the method of Gaussian elimination for general systems of linear equations (see Appendix C), and it therefore seems appropriate to investigate the possibility of adapting the numerically superior method of Gaussian elimination to the simplex method. The result is a version of the revised simplex method that possesses better numerical stability than other methods, and which for large-scale problems can offer tremendous storage advantages.

We concentrate on the problem of solving the linear systems

$$\mathbf{B}\mathbf{y}_0 = \mathbf{b}, \quad \boldsymbol{\lambda}^T\mathbf{B} = \mathbf{c}_{\mathbf{B}}^T, \quad \mathbf{B}\mathbf{y}_q = \mathbf{a}_q \tag{35}$$

that are required by a single step of the simplex method. Suppose $\mathbf{B}$ has been decomposed into the form $\mathbf{B} = \mathbf{L}\mathbf{U}$ where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix.† Then each of the linear systems (35) can be solved by solving two triangular systems. Since solving in this fashion is simple, knowledge of $\mathbf{L}$ and $\mathbf{U}$ is as good as knowledge of $\mathbf{B}^{-1}$.

Next, we show how the $\mathbf{LU}$ decomposition of $\mathbf{B}$ can be updated when a single basis vector is changed. At the beginning of the simplex cycle suppose $\mathbf{B}$ has the form

$$\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m].$$

At the end of the cycle we have the new basis

$$\overline{\mathbf{B}} = \left[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_m, \mathbf{a}_q\right],$$

where it should be noted that when $\mathbf{a}_k$ is dropped all subsequent vectors are shifted to the left, and the new vector $\mathbf{a}_q$ is appended on the right. This procedure leads to a fairly simple updating technique.

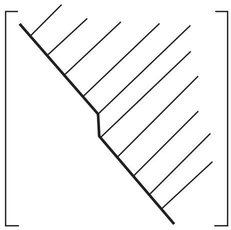
We have

$$\begin{aligned}\mathbf{L}^{-1}\overline{\mathbf{B}} &= \left[\mathbf{L}^{-1}\mathbf{a}_1, \mathbf{L}^{-1}\mathbf{a}_2, \ldots, \mathbf{L}^{-1}\mathbf{a}_{k-1}, \mathbf{L}^{-1}\mathbf{a}_{k+1}, \ldots, \mathbf{L}^{-1}\mathbf{a}_m, \mathbf{L}^{-1}\mathbf{a}_q\right]\\ &= \left[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}, \ldots, \mathbf{u}_m, \mathbf{L}^{-1}\mathbf{a}_q\right] = \overline{\mathbf{H}},\end{aligned}$$

†For simplicity, we are assuming that no row interchanges are required to produce the $\mathbf{LU}$ decomposition. This assumption can be relaxed, but both the notation and the method itself become somewhat more complex. In practice row interchanges are introduced to preserve accuracy or sparsity.

where the $\mathbf{u}_i$'s are the columns of $\mathbf{U}$. The matrix $\overline{\mathbf{H}}$ takes the form

$\overline{\mathbf{H}} =$



with zeros below the main diagonal in the first $k-1$ columns, and zeros below the element immediately under the diagonal in all other columns. The matrix $\overline{\mathbf{H}}$ itself can be constructed without additional computation, since the $\mathbf{u}_i$'s are known and $\mathbf{L}^{-1}\mathbf{a}_q$ is a by-product in the computation of $\mathbf{y}_q$.

$\overline{\mathbf{H}}$ can be reduced to upper triangular form by using Gaussian elimination to zero out the subdiagonal elements. Thus the upper triangular matrix $\overline{\mathbf{U}}$ can be obtained from $\overline{\mathbf{H}}$ by application of a series of transformations, each having the form

$$\mathbf{M}_i = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & 1 & & & & \\ & & & & m_i & 1 & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & 1 \end{bmatrix} \tag{36}$$

for $i = k, k+1, \ldots, m-1$. The matrix $\overline{\mathbf{U}}$ becomes

$$\overline{\mathbf{U}} = \mathbf{M}_{m-1}\mathbf{M}_{m-2}\ldots\mathbf{M}_k\overline{\mathbf{H}}. \tag{37}$$

We then have

$$\overline{\mathbf{B}} = \mathbf{L}\overline{\mathbf{H}} = \mathbf{L}\mathbf{M}_k^{-1}\ \mathbf{M}_{k+1}^{-1}\ \ldots\ \mathbf{M}_{m-1}^{-1}\ \overline{\mathbf{U}}, \tag{38}$$

and thus evaluating

$$\overline{\mathbf{L}} = \mathbf{L}\mathbf{M}_k^{-1}\ \ldots\ \mathbf{M}_{m-1}^{-1}, \tag{39}$$

we obtain the decomposition

$$\overline{\mathbf{B}} = \overline{\mathbf{L}}\overline{\mathbf{U}}. \tag{40}$$

Since $\mathbf{M}_i^{-1}$ is simply $\mathbf{M}_i$ with the sign of the off-diagonal term reversed, evaluation of $\overline{\mathbf{L}}$ is straightforward.

There are numerous variations of this basic idea. The elementary transformations (36) can be carried rather than explicitly evaluating $\mathbf{L}$, the $\mathbf{LU}$ decomposition can be periodically reevaluated, and row and column interchanges can be handled in such a way as to maximize stability or minimize the density of the decomposition. Some of these extensions are discussed in the references at the end of the chapter.

3.9 DECOMPOSITION

Large linear programming problems usually have some special structural form that can (and should) be exploited to develop efficient computational procedures. One common structure is where there are a number of separate activity areas that are linked through common resource constraints. An example is provided by a multidivisional firm attempting to minimize the total cost of its operations. The divisions of the firm must each meet internal requirements that do not interact with the constraints of other divisions; but in addition there are common resources that must be shared among divisions and thereby represent linking constraints.

A problem of this form can be solved by the Dantzig–Wolfe decomposition method described in this section. The method is an iterative process where at each step a number of separate subproblems are solved. The subproblems are themselves linear programs within the separate areas (or within divisions in the example of the firm). The objective functions of these subproblems are varied from iteration to iteration and are determined by a separate calculation based on the results of the previous iteration. This action coordinates the individual subproblems so that, ultimately, the solution to the overall problem is solved. The method can be derived as a special version of the revised simplex method, where the subproblems correspond to evaluation of reduced cost coefficients for the main problem.

To describe the method we consider the linear program in standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned} \tag{41}$$

Suppose, for purposes of this entire section, that the $\mathbf{A}$ matrix has the special "block-angular" structure:

$$\mathbf{A} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \cdots & \mathbf{L}_N \\ \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_N \end{bmatrix} \tag{42}$$

By partitioning the vectors $\mathbf{x}, \mathbf{c}^T$, and $\mathbf{b}$ consistent with this partition of $\mathbf{A}$, the problem can be rewritten as

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{N} \mathbf{c}_i^T \mathbf{x}_i \\
\text{subject to} \quad & \sum_{i=1}^{N} \mathbf{L}_i \mathbf{x}_i = \mathbf{b}_0 \\
& \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i \\
& \mathbf{x}_i \geqslant \mathbf{0}, \qquad i = 1, \ldots, N.
\end{aligned}
\tag{43}
$$

This may be viewed as a problem of minimizing the total cost of N different linear programs that are independent except for the first constraint, which is a linking constraint of, say, dimension m.

Each of the subproblems is of the form

$$
\begin{aligned}
\text{minimize} \quad & \mathbf{c}_i^T \mathbf{x}_i \\
\text{subject to} \quad & \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i \\
& \mathbf{x}_i \geqslant \mathbf{0}.
\end{aligned}
\tag{44}
$$

The constraint set for the ith subproblem is $S_i = \{\mathbf{x}_i : \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i, \mathbf{x}_i \geqslant \mathbf{0}\}$. As for any linear program, this constraint set S_i is a polytope and can be expressed as the intersection of a finite number of closed half-spaces. There is no guarantee that each S_i is bounded, even if the original linear program (41) has a bounded constraint set. We shall assume for simplicity, however, that each of the polytopes S_i, $i = 1, \ldots, N$ is indeed bounded and hence is a polyhedron. One may guarantee that this assumption is satisfied by placing artificial (large) upper bounds on each $\mathbf{x}_i$.

Under the boundedness assumption, each polyhedron S_i consists entirely of points that are convex combinations of its extreme points. Thus, if the extreme points of S_i are $\{\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{iK_i}\}$, then any point $\mathbf{x}_i \in S_i$ can be expressed in the form

$$
\begin{aligned}
& \mathbf{x}_i = \sum_{j=1}^{K_i} \alpha_{ij} \mathbf{x}_{ij}, \\
\text{where} \quad & \sum_{j=1}^{K_i} \alpha_{ij} = 1 \\
\text{and} \quad & \alpha_{ij} \geqslant 0, \quad j = 1, \ldots, K_i.
\end{aligned}
\tag{45}
$$

The α_{ij}'s are the weighting coefficients of the extreme points.

We now convert the original linear program to an equivalent *master problem*, of which the objective is to find the optimal weighting coefficients for each polyhedron, S_i. Corresponding to each extreme point $\mathbf{x}_{ij}$ in S_i, define $p_{ij} = \mathbf{c}_i^T \mathbf{x}_{ij}$ and $\mathbf{q}_{ij} = \mathbf{L}_i \mathbf{x}_{ij}$.

Clearly p_{ij} is the equivalent cost of the extreme point $\mathbf{x}_{ij}$, and $\mathbf{q}_{ij}$ is its equivalent activity vector in the linking constraints.

Then the original linear program (41) is equivalent, using (45), to the *master problem*:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N}\sum_{j=1}^{K_i} p_{ij}\alpha_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{N}\sum_{j=1}^{K_i} \mathbf{q}_{ij}\alpha_{ij} = \mathbf{b}_0 \\ & \left.\begin{aligned} & \sum_{j=1}^{K_i} \alpha_{ij} = 1 \\ & \alpha_{ij} \geqslant 0, \quad j = 1, \ldots, K_i \end{aligned}\right\} i = 1, \ldots, N. \end{aligned} \tag{46}$$

This master problem has variables

$$\boldsymbol{\alpha} = \left(\alpha_{11}, \ldots, \alpha_{1K_1}, \alpha_{21}, \ldots, \alpha_{2K_2}, \ldots, \alpha_{N1}, \ldots, \alpha_{NK_N}\right)$$

and can be expressed more compactly as

$$\begin{aligned} \text{minimize} \quad & \mathbf{p}^T\boldsymbol{\alpha} \\ \text{subject to} \quad & \mathbf{Q}\boldsymbol{\alpha} = \mathbf{g} \\ & \boldsymbol{\alpha} \geqslant \mathbf{0}, \end{aligned} \tag{47}$$

where $\mathbf{g}^T = [\mathbf{b}_0^T, 1, 1, \ldots, 1]$; the element of $\mathbf{p}$ associated with α_{ij} is p_{ij}; and the column of $\mathbf{Q}$ associated with α_{ij} is

$$\begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix},$$

with $\mathbf{e}_i$ denoting the ith unit vector in E^N.

Suppose that at some stage of the revised simplex method for the master problem we know the basis $\mathbf{B}$ and corresponding simplex multipliers $\boldsymbol{\lambda}^T = \mathbf{p}_{\mathbf{B}}^T\mathbf{B}^{-1}$. The corresponding relative cost vector is $\mathbf{r}_{\mathbf{D}}^T = \mathbf{c}_{\mathbf{D}}^T - \boldsymbol{\lambda}^T\mathbf{D}$, having components

$$r_{ij} = p_{ij} - \boldsymbol{\lambda}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix}. \tag{48}$$

It is not necessary to calculate all the r_{ij}'s; it is only necessary to determine the minimal r_{ij}. If the minimal value is nonnegative, the current solution is optimal and the process terminates. If, on the other hand, the minimal element is negative, the corresponding column should enter the basis.

The search for the minimal element in (48) is normally made with respect to nonbasic columns only. The search can be formally extended to include basic columns as well, however, since for basic elements

$$p_{ij} - \boldsymbol{\lambda}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix} = 0.$$

The extra zero values do not influence the subsequent procedure, since a new column will enter only if the minimal value is less than zero.

We therefore define r^* as the minimum relative cost coefficient for *all* possible basis vectors. That is,

$$r^* = \underset{i\in\{1,\ldots,N\}}{\text{minimum}} \left\{ r_1^* = \underset{j\in\{1,\ldots,K_i\}}{\text{minimum}} \{p_{ij} - \boldsymbol{\lambda}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix}\} \right\}.$$

Using the definitions of p_{ij} and $\mathbf{q}_{ij}$, this becomes

$$r_i^* = \underset{j\in\{1,\ldots,K_i\}}{\text{minimum}} \left\{ \mathbf{c}_i^T \mathbf{x}_{ij} - \boldsymbol{\lambda}_0^T \mathbf{L}_i \mathbf{x}_{ij} - \boldsymbol{\lambda}_{m+i} \right\}, \tag{49}$$

where $\boldsymbol{\lambda}_0$ is the vector made up of the first m elements of $\boldsymbol{\lambda}$, m being the number of rows of $\mathbf{L}_i$ (the number of linking constraints in (43)).

The minimization problem in (49) is actually solved by the ith *subproblem*:

$$\begin{aligned} \text{minimize} \quad & (\mathbf{c}_i^T - \boldsymbol{\lambda}_0^T \mathbf{L}_i)\mathbf{x}_i \\ \text{subject to} \quad & \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i \\ & \mathbf{x}_i \geqslant \mathbf{0}. \end{aligned} \tag{50}$$

This follows from the fact that λ_{m+i} is independent of the extreme point index j (since $\boldsymbol{\lambda}$ is fixed during the determination of the r_i's), and that the solution of (50) must be that extreme point of S_i, say $\mathbf{x}_{ik}$, of minimum cost, using the adjusted cost coefficients $\mathbf{c}_i^T - \boldsymbol{\lambda}_0^T \mathbf{L}_i$.

Thus, an algorithm for this special version of the revised simplex method applied to the master problem is the following: Given a basis $\mathbf{B}$

Step 1. Calculate the current basic solution $\mathbf{x}_\mathbf{B}$, and solve $\boldsymbol{\lambda}^T \mathbf{B} = \mathbf{c}_\mathbf{B}^T$ for $\boldsymbol{\lambda}$.

Step 2. For each $i = 1, 2, \ldots, N$, determine the optimal solution $\mathbf{x}_i^*$ of the ith subproblem (50) and calculate

$$r_i^* = \left(\mathbf{c}_i^T - \boldsymbol{\lambda}_0^T \mathbf{L}_i\right) \mathbf{x}_i^* - \boldsymbol{\lambda}_{m+i}. \tag{51}$$

If all $r_i^* > 0$, stop; the current solution is optimal.

Step 3. Determine which column is to enter the basis by selecting the minimal r_i^*.

Step 4. Update the basis of the master problem as usual.

This algorithm has an interesting economic interpretation in the context of a multidivisional firm minimizing its total cost of operations as described earlier. Division i's activities are internally constrained by $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$, and the common resources $\mathbf{b}_0$ impose linking constraints. At Step 1 of the algorithm, the firm's central management formulates its current master plan, which is perhaps suboptimal, and announces a new set of prices that each division must use to revise its recommended strategy at Step 2. In particular, $-\boldsymbol{\lambda}_0$ reflects the new prices that higher management has placed on the common resources. The division that reports the greatest rate of potential cost improvement has its recommendations incorporated in the new master plan at Step 3, and the process is repeated. If no cost improvement is possible, central management settles on the current master plan.

Example 2. Consider the problem

$$
\begin{array}{ll}
\text{minimize} & -x_1 - 2x_2 - 4y_1 - 3y_2 \\
\text{subject to} & x_1 + x_2 + 2y_1 \leqslant 4 \\
& x_2 + y_1 + y_2 \leqslant 3 \\
& 2x_1 + x_2 \leqslant 4 \\
& x_1 + x_2 \leqslant 2 \\
& y_1 + y_2 \leqslant 2 \\
& 3y_1 + 2y_2 \leqslant 5 \\
\end{array}
$$
$$
x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad y_1 \geqslant 0, \quad y_2 \geqslant 0.
$$

The decomposition algorithm can be applied by introducing slack variables and identifying the first two constraints as linking constraints. Rather than using double subscripts, the primary variables of the subsystems are taken to be $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$.

Initialization. Any vector $(\mathbf{x, y})$ of the master problem must be of the form

$$
\mathbf{x} = \sum_{i=1}^{I} \alpha_i \mathbf{x}_i, \quad \mathbf{y} = \sum_{j=1}^{J} \beta_j \mathbf{y}_j,
$$

where $\mathbf{x}_i$ and $\mathbf{y}_j$ are extreme points of the subsystems, and

$$
\sum_{i=1}^{J} \alpha_i = 1, \quad \sum_{j=1}^{J} \beta_j = 1, \quad \alpha_i \geqslant 0, \quad \beta_j \geqslant 0.
$$

Therefore the master problem is

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{I} p_i \alpha_i + \sum_{j=1}^{J} t_j \beta_j \\
\text{subject to} & \displaystyle\sum_{i=1}^{I} \alpha_i \mathbf{L}_1 \mathbf{x}_i + \sum_{j=1}^{J} \beta_j \mathbf{L}_2 \mathbf{y}_j + \mathbf{s} = \mathbf{b}
\end{array}
$$

$$\sum_{i=1}^{I} \alpha_i = 1, \qquad \alpha_i \geqslant 0, \quad i = 1, 2, \ldots, I$$

$$\sum_{j=1}^{j} \beta_j = 1, \qquad \beta_j \geqslant 0, \quad j = 1, 2, \ldots, J,$$

where p_i is the cost of $\mathbf{x}_i$, t_j is the cost of $\mathbf{y}_j$, and where $\mathbf{s} = (s_1, s_2)$ is a vector of slack variables for the linking constraints. This problem corresponds to (47).

A starting basic feasible solution is $\mathbf{s} = \mathbf{b}$, $\alpha_1 = 1$, $\beta_1 = 1$, where $\mathbf{x}_1 = \mathbf{0}$, $\mathbf{y}_1 = \mathbf{0}$ are extreme points of the subsystems. The corresponding starting basis is $\mathbf{B} = \mathbf{I}$ and, accordingly, the initial tableau for the revised simplex method for the master problem is

Variable	$\mathbf{B}^{-1}$				Value
s_1	1	0	0	0	4
s_2	0	1	0	0	3
α_1	0	0	1	0	1
β_1	0	0	0	1	1

Then $\boldsymbol{\lambda}^T = [0, 0, 0, 0]\,\mathbf{B}^{-1} = [0, 0, 0, 0]$.

Iteration 1. The relative cost coefficients are found by solving the subproblems defined by (50). The first is

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 \\ \text{subject to} & 2x_1 + x_2 \leqslant 4 \\ & x_1 + x_2 \leqslant 2 \\ & x_1 \geqslant 0, \qquad x_2 \geqslant 0. \end{array}$$

This problem can be solved easily (by the simplex method or by inspection). The solution is $\mathbf{x} = (0, 2)$, with $r_1 = -4$.

The second subsystem is solved correspondingly. The solution is $\mathbf{y} = (1, 1)$ with $r_2 = -7$.

It follows from Step 2 of the general algorithm that $r^* = -7$. We let $\mathbf{y}_2 = (1, 1)$ and bring β_2 into the basis of the master problem.

Master Iteration. The new column to enter the basis is

$$\begin{bmatrix} \mathbf{L}_2\mathbf{y}_2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and since the current basis is $\mathbf{B} = \mathbf{I}$, the new tableau is

Variable	$\mathbf{B}^{-1}$				Value	New column
s_1	1	0	0	0	4	2
s_2	0	1	0	0	3	2
α_1	0	0	1	0	1	0
β_1	0	0	0	1	1	1

which after pivoting leads to

Variable	$\mathbf{B}^{-1}$				Value
s_1	1	0	0	−2	2
s_2	0	1	0	−2	1
α_1	0	0	1	0	1
β_2	0	0	0	1	1

Since $t_2 = \mathbf{c}_2^T\,\mathbf{y}_2 = -7$, we find

$$\boldsymbol{\lambda} = [0\ 0\ 0\ \ -7]\ \mathbf{B}^{-1} = [0\ 0\ 0\ \ -7].$$

Iteration 2. Since $\boldsymbol{\lambda}_0$, which comprises the first two components of $\boldsymbol{\lambda}$, has not changed, the subproblems remain the same, but now according to (51), $r^* = -4$ and α_2 should be brought into the basis, where $\mathbf{x}_2 = (0, 2)$.

Master Iteration. The new column to enter the basis is

$$\begin{bmatrix} \mathbf{L}_1\mathbf{x}_2 \\ \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

This must be multiplied by $\mathbf{B}^{-1}$ to obtain its representation in terms of the current basis (but the representation does not change it in this case). The master tableau is then updated as follows:

Variable	$\mathbf{B}^{-1}$				Value	New column
s_1	1	0	0	−2	2	2
s_2	0	1	0	−2	1	2
α_1	0	0	1	0	1	1
β_2	0	0	0	1	1	0

Variable	$\mathbf{B}^{-1}$				Value
0. s_1	1	−1	0	0	1
α_2	0	1/2	0	−1	1/2
α_1	0	−1/2	1	1	1/2
β_2	0	0	0	1	1

Since $p_2 = -4$, we have

$$\boldsymbol{\lambda}^T = [0, -4, 0, -7]\,\mathbf{B}^{-1} = [0, -2, 0, -3]\,.$$

Iteration 3. The subsystem's problems are now

$$\begin{array}{ll} \text{minimize} & -x_1 \\ \text{subject to} & 2x_1 + x_2 \leqslant 4 \\ & x_1 + x_2 \leqslant 2 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & -2y_1 - y_2 + 3 \\ \text{subject to} & y_1 + y_2 \leqslant 2 \\ & 3y_1 + 2y_2 \leqslant 5 \\ & y_1 \geqslant 0, \quad y_2 \geqslant 0. \end{array}$$

It follows that $\mathbf{x}_3 = (2, 0)$ and α_3 should be brought into the basis.
Master Iteration. Proceeding as usual, we obtain the new tableau and new $\boldsymbol{\lambda}$ as follows.

Variable	$\mathbf{B}^{-1}$				Value	
s_1	1	−1	0	0	1	2
α_2	0	1/2	0	−1	1/2	0
α_1	0	−1/2	1	1	1/2	1
β_2	0	0	0	1	1/2	0
s_1	1	0	−2	−2	0	
α_2	0	1/2	0	−1	1/2	
α_3	0	−1/2	1	1	1/2	
β_3	0	0	0	1	1	

$$\boldsymbol{\lambda}^T = [0, -4, -2, -7]\,\mathbf{B}^{-1} = [0, -1, -2, -5]$$

The subproblems now have objectives $-x_1-x_2+2$ and $-3y_1-2y_2+5$, respectively, which both have minimum values of zero. Thus the current solution is optimal. The solution is $(1/2)\mathbf{x}_2+(1/2)\mathbf{x}_3+\mathbf{y}_2$, or equivalently, $x_1=1$, $x_2=1$, $y_1=1$, $y_2=1$.

3.10 SUMMARY

The simplex method is founded on the fact that the optimal value of a linear program, if finite, is always attained at a basic feasible solution. Using this foundation there are two ways in which to visualize the simplex process. The first is to view the process as one of continuous change. One starts with a basic feasible solution and imagines that some nonbasic variable is increased slowly from zero. As the value of this variable is increased, the values of the current basic variables are continuously adjusted so that the overall vector continues to satisfy the system of linear equality constraints. The change in the objective function due to a unit change in this nonbasic variable, taking into account the corresponding required changes in the values of the basic variables, is the relative cost coefficient associated with the nonbasic variable. If this coefficient is negative, then the objective value will be continuously improved as the value of this nonbasic variable is increased, and therefore one increases the variable as far as possible, to the point where further increase would violate feasibility. At this point the value of one of the basic variables is zero, and that variable is declared nonbasic, while the nonbasic variable that was increased is declared basic.

The other viewpoint is more discrete in nature. Realizing that only basic feasible solutions need be considered, various bases are selected and the corresponding basic solutions are calculated by solving the associated set of linear equations. The logic for the systematic selection of new bases again involves the relative cost coefficients and, of course, is derived largely from the first, continuous, viewpoint.

3.11 EXERCISES

1. Using pivoting, solve the simultaneous equations

$$\begin{aligned} 3x_1+2x_2&=5\\ 5x_1+\ \ x_2&=9. \end{aligned}$$

2. Using pivoting, solve the simultaneous equations

$$\begin{aligned} x_1+2x_2+x_3&=7\\ 2x_1-x_2+2x_3&=6\\ x_1+x_2+3x_3&=12. \end{aligned}$$

3. Solve the equations in Exercise 2 by Gaussian elimination as described in Appendix C.

4. Suppose $\mathbf{B}$ is an $m \times m$ square nonsingular matrix, and let the tableau $\mathbf{T}$ be constructed, $\mathbf{T} = [\mathbf{I}, \mathbf{B}]$ where $\mathbf{I}$ is the $m \times m$ identity matrix. Suppose that pivot operations are performed on this tableau so that it takes the form $[\mathbf{C}, \mathbf{I}]$. Show that $\mathbf{C} = \mathbf{B}^{-1}$.

5. Show that if the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are a basis in E^m, the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{p-1}, \mathbf{a}_q, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_m$ also are a basis if and only if $y_{pq} \neq 0$, where y_{pq} is defined by the tableau (7).

6. If $r_j > 0$ for every j corresponding to a variable x_j that is not basic, show that the corresponding basic feasible solution is the unique optimal solution.

7. Show that a degenerate basic feasible solution may be optimal without satisfying $r_j \geqslant 0$ for all j.

8. a) Using the simplex procedure, solve

$$\begin{aligned} \text{maximize} \quad & -x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 \leqslant 2 \\ & x_1 + x_2 \leqslant 6 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{aligned}$$

b) Draw a graphical representation of the problem in x_1, x_2 space and indicate the path of the simplex steps.

c) Repeat for the problem

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \leqslant 1 \\ & x_1 - x_2 \leqslant 1 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{aligned}$$

9. Using the simplex procedure, solve the spare-parts manufacturer's problem (Exercise 4, Chapter 2).

10. Using the simplex procedure, solve

$$\begin{array}{ll} \text{minimize} & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} & x_1 + 3x_2 \qquad\quad + x_4 \leqslant 4 \\ & 2x_1 + x_2 \qquad\qquad\quad \leqslant 3 \\ & \qquad x_2 + 4x_3 + x_4 \leqslant 3 \\ & x_1 \geqslant 0 \qquad i = 1, 2, 3, 4. \end{array}$$

11. For the linear program of Exercise 10

a) How much can the elements of $\mathbf{b} = (4, 3, 3)$ be changed without changing the optimal basis?

b) How much can the elements of $\mathbf{c} = (2, 4, 1, 1)$ be changed without changing the optimal basis?

c) What happens to the optimal cost for small changes in $\mathbf{b}$?
d) What happens to the optimal cost for small changes in $\mathbf{c}$?

12. Consider the problem

$$\begin{array}{ll} \text{minimize} & x_1 - 3x_2 - 0.4x_3 \\ \text{subject to} & 3x_1 - x_2 + 2x_3 \leqslant 7 \\ & -2x_1 + 4x_2 \leqslant 12 \\ & -4x_1 + 3x_2 + 3x_3 \leqslant 14 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{array}$$

a) Find an optimal solution.
b) How many optimal basic feasible solutions are there?
c) Show that if $c_4 + \frac{1}{3}a_{14} + \frac{4}{5}a_{24} \geqslant 0$, then another activity x_4 can be introduced with cost coefficient c_1 and activity vector (a_{14}, a_{24}, a_{34}) without changing the optimal solution.

13. Rather than select the variable corresponding to the most negative relative cost coefficient as the variable to enter the basis, it has been suggested that a better criterion would be to select that variable which, when pivoted in, will produce the greatest improvement in the objective function. Show that this criterion leads to selecting the variable x_k corresponding to the index k minimizing $\max\limits_{i, y_{ik}>0} r_k y_{i0}/y_{ik}$.

14. In the ordinary simplex method one new vector is brought into the basis and one removed at every step. Consider the possibility of bringing two new vectors into the basis and removing two at each stage. Develop a complete procedure that operates in this fashion.

15. *Degeneracy*. If a basic feasible solution is degenerate, it is then theoretically possible that a sequence of degenerate basic feasible solutions will be generated that endlessly cycles without making progress. It is the purpose of this exercise and the next two to develop a technique that can be applied to the simplex method to avoid this *cycling*.
    Corresponding to the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ define the perturbed system $\mathbf{Ax} = \mathbf{b}(\varepsilon)$ where $\mathbf{b}(\varepsilon) = \mathbf{b} + \varepsilon \mathbf{a}_1 + \varepsilon^2 \mathbf{a}_2 + \cdots + \varepsilon^n \mathbf{a}_n$, $\varepsilon > 0$. Show that if there is a basic feasible solution (possibly degenerate) to the unperturbed system with basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m]$, then corresponding to the same basis, there is a nondegenerate basic feasible solution to the perturbed system for some range of $\varepsilon > 0$.

16. Show that corresponding to any basic feasible solution to the perturbed system of Exercise 15, which is nondegenerate for some range of $\varepsilon > 0$, and to a vector $\mathbf{a}_k$ not in the basis, there is a unique vector $\mathbf{a}_i$ in the basis which when replaced by $\mathbf{a}_k$ leads to a basic feasible solution; and that solution is nondegenerate for a range of $\varepsilon > 0$.

17. Show that the tableau associated with a basic feasible solution of the perturbed system of Exercise 15, and which is nondegenerate for a range of $\varepsilon > 0$, is identical with that of the unperturbed system except in the column under $\mathbf{b}(\varepsilon)$. Show how the proper pivot in a given column to preserve feasibility of the perturbed system can be determined from the tableau of the unperturbed system. Conclude that the simplex method will avoid cycling if whenever there is a choice in the pivot element of a column k, arising from a tie in the minimum of y_{i0}/y_{ik} among the elements $i \in I_0$, the tie is resolved by finding

the minimum of y_{i1}/y_{ik}, $i \in I_0$. If there still remainties among elements $i \in I$, the process is repeated with y_{i2}/y_{ik}, etc., until there is a unique element.

18. Using the two-phase simplex procedure solve

a)
$$
\begin{array}{ll}
\text{minimize} & -3x_1 + x_2 + 3x_3 - x_4 \\
\text{subject to} & x_1 + 2x_2 - \ \ x_3 + \ \ x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - \ \ x_2 + 2x_3 - \ \ x_4 = 6 \\
& x_1 \geqslant 0, \quad i = 1, 2, 3, 4.
\end{array}
$$

b)
$$
\begin{array}{ll}
\text{minimize} & x_1 + 6x_2 - \ \ 7x_3 + \ \ x_4 + 5x_5 \\
\text{subject to} & 5x_1 - 4x_2 + 13x_3 - 2x_4 + \ \ x_5 = 20 \\
& x_1 - \ \ x_2 + \ \ 5x_3 - \ \ x_4 + \ \ x_5 = 8 \\
& x_1 \geqslant 0, \qquad i = 1, 2, 3.4, 5.
\end{array}
$$

19. Solve the oil refinery problem (Exercise 3, Chapter 2).

20. Show that in the phase I procedure of a problem that has feasible solutions, if an artificial variable becomes nonbasic, it need never again be made basic. Thus, when an artificial variable becomes nonbasic its column can be eliminated from future tableaus.

21. Suppose the phase I procedure is applied to the system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$, and that the resulting tableau (ignoring the cost row) has the form

$$
\begin{array}{|ccc|c|c|ccc|c|}
x_1 & x_2 \cdots & x_k & x_{k+1} \cdots x_n & y_1 \ y_2 \cdots y_k & y_{k+1} & \cdots & y_m & \\
\hline
1 & & & & & 0 & \cdots & 0 & \bar{b}_1 \\
& 1 & & & & 0 & \cdots & 0 & \vdots \\
& & & \mathbf{R_1} & \mathbf{S}_1 & \vdots & & & \\
& & 1 & & & 0 & \cdots & 0 & \bar{b}_k \\
\hline
0 & 0 \cdots & 0 & & & 1 & & & 0 \\
\vdots & & & \mathbf{R}_2 & \mathbf{S}_2 & & 1 & & \vdots \\
0 & \cdots & 0 & & & & & 1 & 0 \\
\hline
\end{array}
$$

This corresponds to having $m - k$ basic artificial variables at zero level.

a) Show that any nonzero element in $\mathbf{R}_2$ can be used as a pivot to eliminate a basic artificial variable, thus yielding a similar tableau but with k increased by one.

b) Suppose that the process in (a) has been repeated to the point where $\mathbf{R}_2 = \mathbf{0}$. Show that the original system is redundant, and show how phase II may proceed by eliminating the bottom rows.

c) Use the above method to solve the linear program

$$
\begin{array}{ll}
\text{minimize} & 2x_1 + 6x_2 + x_3 + \ \ x_4 \\
\text{subject to} & x_1 + 2x_2 \qquad\ \ + \ \ x_4 = 6 \\
& x_1 + 2x_2 + x_3 + \ \ x_4 = 7 \\
& x_1 + 3x_2 - x_3 + 2x_4 = 7 \\
& x_1 + \ \ x_2 + x_3 \qquad\quad\ = 5 \\
& x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0.
\end{array}
$$

22. Find a basic feasible solution to

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 3 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 12 \\ x_1 + 4x_2 + 2x_3 + x_4 &= 9 \\ x_1 \geqslant 0, \qquad i &= 1, 2, 3, 4. \end{aligned}$$

23. Consider the system of linear inequalities $\mathbf{Ax} \geqslant \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$ with $\mathbf{b} \geqslant \mathbf{0}$. This system can be transformed to standard form by the introduction of m surplus variables so that it becomes $\mathbf{Ax} - \mathbf{y} = \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{y} \geqslant \mathbf{0}$. Let $b_k = \max_i b_i$ and consider the new system in standard form obtained by adding the kth row to the negative of every other row. Show that the new system requires the addition of only a single artificial variable to obtain an initial basic feasible solution.
Use this technique to find a basic feasible solution to the system.

$$\begin{aligned} x_1 + 2x_2 + x_3 &\geqslant 4 \\ 2x_1 + x_2 + x_3 &\geqslant 5 \\ 2x_1 + 3x_2 + 2x_3 &\geqslant 6 \\ x_i \geqslant 0, \quad i &= 1, 2, 3. \end{aligned}$$

24. It is possible to combine the two phases of the two-phase method into a single procedure by the *big–M method*. Given the linear program in standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned}$$

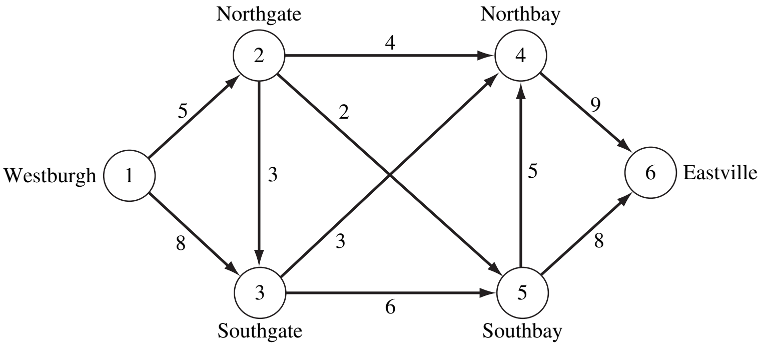
one forms the approximating problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} + M\sum_{i=1}^{m} y_i \\ \text{subject to} \quad & \mathbf{Ax} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \\ & \mathbf{y} \geqslant \mathbf{0}. \end{aligned}$$

In this problem $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ is a vector of artificial variables and M is a large constant. The term $M\sum_{i=1}^{m} y_i$ serves as a penalty term for nonzero y_i's.
If this problem is solved by the simplex method, show the following:

a) If an optimal solution is found with $\mathbf{y} = \mathbf{0}$, then the corresponding $\mathbf{x}$ is an optimal basic feasible solution to the original problem.
b) If for every $M > 0$ an optimal solution is found with $\mathbf{y} \neq \mathbf{0}$, then the original problem is infeasible.
c) If for every $M > 0$ the approximating problem is unbounded, then the original problem is either unbounded or infeasible.
d) Suppose now that the original problem has a finite optimal value $V(\infty)$. Let $V(M)$ be the optimal value of the approximating problem. Show that $V(M) \leqslant V(\infty)$.
e) Show that for $M_1 \leqslant M_2$ we have $V(M_1) \leqslant V(M_2)$.
f) Show that there is a value M_0 such that for $M \geqslant M_0$, $V(M) = V(\infty)$, and hence conclude that the big–M method will produce the right solution for large enough values of M.

25. A certain telephone company would like to determine the maximum number of long-distance calls from Westburgh to Eastville that it can handle at any one time. The company has cables linking these cities via several intermediary cities as follows:



Each cable can handle a maximum number of calls simultaneously as indicated in the figure. For example, the number of calls routed from Westburgh to Northgate cannot exceed five at any one time. A call from Westburgh to Eastville can be routed through any other city, as long as there is a cable available that is not currently being used to its capacity. In addition to determining the maximum number of calls from Westburgh to Eastville, the company would, of course, like to know the optimal routing of these calls. Assume calls can be routed only in the directions indicated by the arrows.

a) Formulate the above problem as a linear programming problem with upper bounds. (*Hint*: Denote by x_{ij} the number of calls routed from city i to city j.)
b) Find the solution by inspection of the graph.

26. Using the revised simplex method find a basic feasible solution to

$$\begin{aligned} x_1 + 2x_2 - \ \ x_3 + \ \ x_4 &= 3 \\ 2x_1 + 4x_2 + \ \ x_3 + 2x_4 &= 12 \\ x_1 + 4x_2 + 2x_3 + \ \ x_4 &= 9 \\ x_1 \geqslant 0, \quad i = 1, 2, 3, 4. & \end{aligned}$$

27. The following tableau is an intermediate stage in the solution of a minimization problem:

	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$	$\mathbf{y}_0$
	1	2/3	0	0	4/3	0	4
	0	−7/3	3	1	−2/3	0	2
	0	−2/3	−2	0	2/3	1	2
$\mathbf{r}^T$	0	8/3	−11	0	4/3	0	−8

a) Determine the next pivot element.

b) Given that the inverse of the current basis is

$$\mathbf{B}^{-1} = [\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_6]^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

and the corresponding cost coefficients are

$$\mathbf{c}_{\mathbf{B}}^T = (c_1, c_4, c_6) = (-1, -3, 1),$$

find the original problem.

28. In many applications of linear programming it may be sufficient, for practical purposes, to obtain a solution for which the value of the objective function is within a predetermined tolerance ε from the minimum value z^*. Stopping the simplex algorithm at such a solution rather than searching for the true minimum may considerably reduce the computations.

    a) Consider a linear programming problem for which the sum of the variables is known to be bounded above by s. Let z_0 denote the current value of the objective function at some stage of the simplex algorithm, $(c_j - z_j)$ the corresponding relative cost coefficients, and

    $$M = \max_j \left(z_j - c_j\right).$$

    Show that if $M \leqslant \varepsilon/s$, then $z_0 - z^* \leqslant \varepsilon$.
    b) Consider the transportation problem described in Section 2.2 (Example 2). Assuming this problem is solved by the simplex method and it is sufficient to obtain a solution within ε tolerance from the optimal value of the objective function, specify a stopping criterion for the algorithm in terms of ε and the parameters of the problem.

29. Work out an extension of **LU** decomposition, as described in Appendix C, when row interchanges are introduced.

30. Work out the details of **LU** decomposition applied to the simplex method when row interchanges are required.

31. *Anticycling Rule*. A remarkably simple procedure for avoiding cycling was developed by Bland, and we discuss it here.
    Bland's Rule. *In the simplex method*:

    a) *Select the column to enter the basis by* $j = \min\{j : r_j < 0\}$; *that is, select the lowest-indexed favorable column.*
    b) *In case ties occur in the criterion for determining which column is to leave the basis, select the one with lowest index.*

    We can prove by contradiction that the use of Bland's rule prohibits cycling. Suppose that cycling occurs. During the cycle a finite number of columns enter and leave the basis. Each of these columns enters at level zero, and the cost function does not change.

Delete all rows and columns that do not contain pivots during a cycle, obtaining a new linear program that also cycles. Assume that this reduced linear program has m rows and n columns. Consider the solution stage where column n is about to leave the basis, being replaced by column p. The corresponding tableau is as follows (where the entries shown are explained below):

$$
\begin{array}{cccccc}
\mathbf{a}_1 & \cdots & \mathbf{a}_p & \cdots & \mathbf{a}_n & \mathbf{b} \\
 & & \leqslant 0 & & 0 & 0 \\
 & & \leqslant 0 & & 0 & 0 \\
 & & \vdots & & \vdots & \vdots \\
 & & > 0 & & 1 & 0 \\
\hline
\mathbf{c}^T & & < 0 & & 0 & 0
\end{array}
$$

Without loss of generality, we assume that the current basis consists of the last m columns. In fact, we may define the reduced linear program in terms of this tableau, calling the current coefficient array $\mathbf{A}$ and the current relative cost vector $\mathbf{c}$. In this tableau we pivot on a_{mp}, so $a_{mp} > 0$. By Part b) of Bland's rule, $\mathbf{a}_n$ can leave the basis only if there are no ties in the ratio test, and since $\mathbf{b} = \mathbf{0}$ because all rows are in the cycle, it follows that $a_{ip} \leqslant 0$ for all $i \neq m$.

Now consider the situation when column n is about to reenter the basis. Part a) of Bland's rule ensures that $r_n < 0$ and $r_i \geqslant 0$ for all $i \neq n$. Apply the formula $r_i = c_i - \boldsymbol{\lambda}^T \mathbf{a}_i$ to the last m columns to show that each component of $\boldsymbol{\lambda}$ except λ_m is nonpositive; and $\lambda_m > 0$. Then use this to show that $r_p = c_p - \boldsymbol{\lambda}^T \mathbf{a}_p < c_p < 0$, contradicting $r_p \geqslant 0$.

32. Use the Dantzig–Wolfe decomposition method to solve

$$
\begin{array}{ll}
\text{minimize} & -4x_1 - x_2 - 3x_3 - 2x_4 \\
\text{subject to} & 2x_1 + 2x_2 + x_3 + 2x_4 \leqslant 6 \\
 & x_2 + 2x_3 + 3x_4 \leqslant 4 \\
 & 2x_1 + x_2 \leqslant 5 \\
 & x_2 \leqslant 1 \\
 & -x_3 + 2x_4 \leqslant 2 \\
 & x_3 + 2x_4 \leqslant 6 \\
 & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0.
\end{array}
$$

REFERENCES

3.1–3.7 All of this is now standard material contained in most courses in linear programming. See the references cited at the end of Chapter 2. For the original work in this area, see Dantzig [D2] for development of the simplex method; Orden [O2] for the artificial basis technique; Dantzig, Orden and Wolfe [D8], Orchard-Hays [O1], and Dantzig [D4] for the revised simplex method; and Charnes and Lemke [C3] and Dantzig [D5] for upper bounds. The synthetic carrot interpretation is due to Gale [G2].

3.8 The idea of using **LU** decomposition for the simplex method is due to Bartels and Golub [B2]. See also Bartels [B1]. For a nice simple introduction to Gaussian elimination, see Forsythe and Moler [F15]. For an expository treatment of modern computer implementation issues of linear programming, see Murtagh [M9].

3.9 For a more comprehensive description of the Dantzig and Wolfe [D11] decomposition method, see Dantzig [D6].

3.11 The degeneracy technique discussed in Exercises 15–17 is due to Charnes [C2]. The anticycling method of Exercise 35 is due to Bland [B19].
For the state of the art in Simplex solvers see Bixby [B18]