Chapter 5

Analysis of Tensegrity Dynamics

Throughout this chapter we construct dynamic models in the form of ordinary differential equations for tensegrity structures. We make the following assumptions:

a) Rods are rigid, thin, and long and so rotational motion about the longitudinal axis can be neglected.

b) Strings are massless elastic elements with Hookean (linear) behavior only when in tension.

c) The connectivity of the structure is fixed.

These assumptions reflect tensegrity structures where the rods are massive and stiff, here approximated as rigid, as compared with a network of lightweight, elastic strings. Herein we are motivated by the network approach in [Ske05]. We first study the dynamics of a single rod.

5.1 Vectors and Notation

In dynamics a vector was conceived as an entity that has magnitude and direction in three-dimensional space. This concept was introduced by Gibbs (see [Hug86]). In the more modern linear algebra, the axiomatic definitions of a vector allow the treatment of an n-dimensional space, but the concepts of inner products and outer products in linear algebra do not exactly match the concepts of dot and cross products of the dynamics literature. Such distinctions should be made clear. Let $\vec{r}$ be the label we use to represent a (Gibbs) vector in the three-dimensional (non-relativistic) space. This vector is defined independently from any basis system or frame of reference. While

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the vector is not *defined* by any reference frame, this vector can be *described* in any chosen reference frame. With that in mind we define the following entities.

Definition 5.1 (Dextral Set) *The set of vectors $\vec{e}_i$, i = 1,2,3, form a* dextral set *if the dot products satisfy $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ (where δ_{ij} is a Kronecker delta) and the cross products satisfy $\vec{e}_i \times \vec{e}_j = \vec{e}_k$, where the indices i,j,k form the cyclic permutations, i,j,k = 1,2,3 or 2,3,1, or 3,1,2.*

Definition 5.2 (Vectrix) *Let $\vec{e}_i, i = 1, 2, 3,$ define a* dextral set *of unit vectors fixed in an inertial frame, and define the* vectrix *$\vec{E}$ by $\vec{E} = \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\end{bmatrix}$.*

The item $\vec{E}$ is called a *vectrix*, since it is an 1×3 array of the three horizontally stacked items $\vec{e}_i, i = 1, 2, 3$ (the dextral set). Hence, these arrays $\vec{E}$ contain Gibbs vectors $\vec{e}_i$, so they are not matrices. Neither is the 1×3 item $\vec{E}$ a *vector* in the sense of linear algebra. Hence the label *vectrix*, coined by Peter Hughes [Hug86].

Now consider two reference frames, described by the dextral sets (vectrices $\vec{E}$ and $\vec{X}$), where the coordinate transformation between these two frames is described by the 3×3 direction cosine matrix $\mathbf{X}^E$ (orthonormal) so that $\vec{X} = \vec{E}\mathbf{X}^E$, ${\mathbf{X}^E}^T\mathbf{X}^E = \mathbf{I}_3$. Let the three-dimensional column vectors $\mathbf{r}^X$ and $\mathbf{r}^E$ describe the components of the same vector $\vec{r}$ in the two reference frames $\vec{X}$ and $\vec{E}$, respectively. That is,

$$\vec{r} = \vec{E}\mathbf{r}^E = \vec{X}\mathbf{r}^X. \tag{5.1}$$

Hence, if we wish to describe the relationship between the components of the same vector $\vec{r}$, described in two different reference frames, then

$$\vec{X} = \vec{E}\mathbf{X}^E, \qquad \vec{r} = \vec{X}\mathbf{r}^X = \vec{E}\,\mathbf{X}^E\mathbf{r}^X. \tag{5.2}$$

After all terms in an equation are written in the same basis, then the chosen basis (vectrix $\vec{E}$ in this chapter) can be dropped, yielding

$$\mathbf{r}^E = \mathbf{X}^E\mathbf{r}^X. \tag{5.3}$$

The item labeled $\vec{r}$ is a "Gibbs vector", and the items labeled $\mathbf{r}^X$ and $\mathbf{r}^E$ are "vectors" in the spirit of the linear vector spaces of linear algebra, where we use the notation $\mathbf{r}^X, \mathbf{r}^E \in \mathbb{R}^3$ to denote that the items $\mathbf{r}^X$ and $\mathbf{r}^E$ live in a real three-dimensional space. However, the items $\mathbf{r}^X$ and $\mathbf{r}^E$ provide no useful information unless we have previously specified the frames of reference $\vec{X}$ and $\vec{E}$ for these quantities.

The above discussions on notation is for those familiar with traditional literature on rigid body dynamics. However, unlike many problems in aerospace, where multiple coordinate frames are utilized (one fixed in each body), this chapter uses only one coordinate frame (the inertial frame, described by the vectrix $\vec{E}$) to describe all vectors. Hence, one could then shorten the notation for convenience. Instead of the proper notation of a Gibbs vector $\vec{n}_i = \vec{E}\mathbf{n}_i^E$, we will simplify the notation to $\mathbf{n}_i^E = \mathbf{n}_i$ and write $\vec{n}_i = \vec{E}\mathbf{n}_i$, where $\mathbf{n}_i^T = \begin{bmatrix} n_{i_x}, & n_{i_y}, & n_{i_z} \end{bmatrix}$ describes the components of the vector $\vec{n}_i$ in coordinates $\vec{E}$. Hence the only difference between the Gibbs vector $\vec{n}_i$ and the three-dimensional array of its components $\mathbf{n}_i$ is that the frame is specified a priori, $\vec{E}$, and in this chapter, all other vectors that might be mentioned are referenced to the same frame $\vec{E}$.

Hence, for the given basis $\vec{E}$, the dot product and the components of the cross product of any two Gibbs vectors $\vec{b} = \vec{E}\mathbf{b}$ and $\vec{f} = \vec{E}\mathbf{f}$ can be written as

$$\vec{b} \cdot \vec{f} = \mathbf{b}^T\mathbf{f}, \tag{5.4}$$

$$\vec{b} \times \vec{f} = \vec{E}\tilde{\mathbf{b}}\,\mathbf{f}, \qquad \tilde{\mathbf{b}} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{5.5}$$

Since we are committed to the same reference frame $\vec{E}$ throughout the chapter, we wish not to burden the notation with the explicit notation of the reference frame. So, by a slight abuse of vector notation, in lieu of the more accurate notation of the cross product, $\vec{b} \times \vec{f} = \vec{E}\tilde{\mathbf{b}}\mathbf{f}$, we will simply write $\mathbf{b} \times \mathbf{f} = \tilde{\mathbf{b}}\mathbf{f}$.

5.2 Dynamics of a Single Rigid Rod

We start by defining some important quantities associated with the dynamics of the single rigid rod in three-dimensional space as illustrated in Figure 5.1. This rod has mass $m > 0$ and length $\ell > 0$ with extreme points $\mathbf{n}_j$, $\mathbf{n}_i \in \mathbb{R}^3$, hence $\|\mathbf{n}_j - \mathbf{n}_i\| = \ell$. We often make use of the normalized rod vector

$$\mathbf{b} = \ell^{-1}(\mathbf{n}_j - \mathbf{n}_i), \qquad \|\mathbf{b}\| = 1. \tag{5.6}$$

Any point in the rod can be located by the vector

$$\mathbf{v}(\mu) = \mu\mathbf{n}_j + (1 - \mu)\mathbf{n}_i, \tag{5.7}$$

where $\mu \in [0, 1]$. Let $\rho(\mu) \geq 0$ be a density function defined on the interval $\mu \in [0, 1]$ which describes the mass density along the rod, that is,

$$m = \int_0^1 \rho(\mu)\, d\mu > 0.$$

In this section we describe the position of the rod by means of the configuration vector

$$\mathbf{q} = \begin{pmatrix} \mathbf{r} \\ \mathbf{b} \end{pmatrix} \in \mathbb{R}^6, \tag{5.8}$$

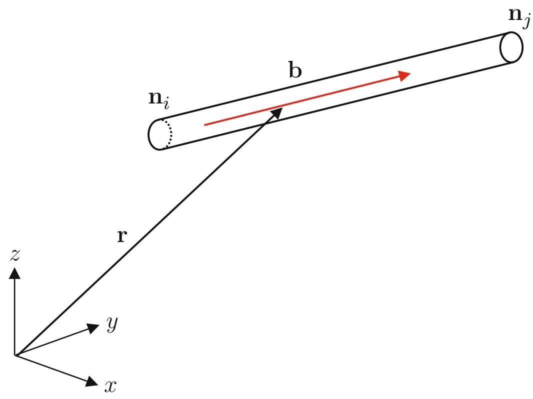

Figure 5.1: Illustration of a rigid rod with its configuration in $\mathbb{R}^3$. The vectors $\mathbf{r}$ and $\mathbf{b}$ describe the translational and rotational position of the rod, respectively

where $\mathbf{r} = \mathbf{v}(\sigma)$, $\sigma \in [0, 1]$, is any fixed point in the rod. Whenever possible $\mathbf{r}$ will be made to coincide with the center of mass of the rod.

Any point in the rod can be equivalently described as a linear function of the configuration vector:

$$\mathbf{v}(\eta) = \mathbf{r} + \eta\,\mathbf{b} = \begin{bmatrix}\mathbf{I}_3 & \eta\mathbf{I}_3\end{bmatrix}\mathbf{q}, \qquad \eta \in [-\sigma\ell, (1-\sigma)\ell]. \tag{5.9}$$

Note that μ and η are related by $\mu = \sigma + \eta/\ell$. Using η we can compute higher order mass moments around $\mathbf{r}$, the next two of which are

$$f(\sigma) = \ell^{-1}\int_{-\sigma\ell}^{(1-\sigma)\ell} \rho(\sigma + \eta/\ell)\,\eta\,d\eta,$$

$$J(\sigma) = \ell^{-1}\int_{-\sigma\ell}^{(1-\sigma)\ell} \rho(\sigma + \eta/\ell)\,\eta^2\,d\eta > 0.$$

Such moments are associated with two important quantities, the *kinetic energy* and the *angular momentum* of the rod. The kinetic energy of the rod is given by the formula

$$T = \frac{1}{2}\int_0^1 \rho(\mu)\,\dot{\mathbf{v}}(\mu)^T\dot{\mathbf{v}}(\mu)\,d\mu = \frac{1}{2}\,\dot{\mathbf{q}}^T(\mathbf{J}(\sigma)\otimes\mathbf{I}_3)\,\dot{\mathbf{q}}, \tag{5.10}$$

$$\mathbf{J}(\sigma) = \begin{bmatrix} m & f(\sigma) \\ f(\sigma) & J(\sigma)\end{bmatrix} \succeq 0. \tag{5.11}$$

We note that a Kronecker product of two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is an $n^2 \times n^2$ matrix composed of $n \times n$ blocks of matrices of the type $A_{ij}\mathbf{B}$.

The angular momentum of the rod about $\mathbf{r}$ is

$$\mathbf{h} = \int_m \rho(\mu)\,(\mathbf{v}(\mu) - \mathbf{r}) \times (\dot{\mathbf{v}}(\mu) - \dot{\mathbf{r}})\,d\mu$$

$$= J(\sigma)\,\mathbf{b} \times \dot{\mathbf{b}}, \tag{5.12}$$

$$= J(\sigma)\,\tilde{\mathbf{b}}\,\dot{\mathbf{b}}, \tag{5.13}$$

where $\tilde{\mathbf{b}}$ denotes a skew-symmetric matrix composed of the three components of the vector $\mathbf{b}$, as defined in (5.4).

The matrix $\mathbf{J}(\sigma)$ is positive semidefinite because $T \geq 0$ for all $\dot{\mathbf{q}}$. Indeed, for most practical mass distribution functions ρ (see next section), matrix $\mathbf{J}$ will be positive definite ($\mathbf{J} \succ 0$), a property that will be used in the next chapters.

Note that if we choose $\sigma = \int_0^1 \rho(\mu)\mu\,d\mu$ so that $\mathbf{r}$ coincides with the center of mass of the rod then $f(\sigma) = 0$. This leads to the well-known decoupling of the kinetic energy in translational and rotational components in a rigid body described by its center of mass. One should choose to describe a rod by its center of mass whenever possible, with the main exception being the case when constraints are present in points of the rod other than the center of mass. We will illustrate this case later in this book. The next example discusses some useful mass distributions and their properties.

Example 5.1

In most parts of this book we consider rods with mass uniformly distributed along the rod, that is,

$$\rho(\mu) = m\,\ell^{-1}.$$

In this case the mass moments f and J are

$$f(\sigma) = \frac{1}{2} m\ell\,(1 - 2\sigma), \qquad J(\sigma) = \frac{1}{3} m\ell^2 \left(1 - 3\sigma + 3\sigma^2\right),$$

which are functions of σ, hence depends on the choice of the fixed point $\mathbf{r}$. Indeed, in this case the center of mass is the center of the bar, i.e., $\sigma = 1/2$ in which case f and J are familiar

$$f(1/2) = 0, \qquad J(1/2) = \frac{1}{12} m\ell^2.$$

Another familiar choice is when $\mathbf{r}$ coincides with one of the extreme points of the rod, say $\mathbf{r} = \mathbf{n}_i$ ($\sigma = 0$) so that

$$f(0) = \frac{1}{2} m\ell, \qquad J(0) = \frac{1}{3} m\ell^2.$$

Interestingly, for any rod with uniform mass distribution matrix $\mathbf{J}$ is positive definite, i.e., $\mathbf{J} \succ 0$, regardless of σ. Indeed, for any $\sigma \in [0,1]$ the function $J(\sigma)$ has two imaginary roots so that

$$J(\sigma) > 0, \qquad m - \frac{f(\sigma)^2}{J(\sigma)} = \frac{m}{4\,(1 - 3\sigma + 3\sigma^2)} > 0 \qquad \text{for all } \sigma \in [0,1].$$

Using the Schur complement [BGFB94] this implies $\mathbf{J} \succ 0$.

The matrix $\mathbf{S} = \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{D}$ is called a *Schur complement* of the matrix $\mathbf{P}$ if either

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{D} & \mathbf{C} \end{bmatrix} \qquad \text{or} \qquad \mathbf{P} = \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}. \tag{5.14}$$

Example 5.2

A mass distribution of interest is that comprised of a number of lumped masses along the rod, i.e.,

$$\rho(\mu) = \sum_{k=1}^{K} m_k \delta(\mu - \mu_k),$$

where $\sum_{k=1}^{K} m_k = m$ and $m_k > 0$, $\mu_k \in [0,1]$ for all $k = 1, \ldots, K$. The quantities f and J, expressed as a function of σ, are

$$f(\sigma) = \ell \sum_{k=1}^{K} m_k(\mu_k - \sigma), \qquad J(\sigma) = \ell^2 \sum_{k=1}^{K} m_k(\mu_k - \sigma)^2.$$

In this case

$$J(\sigma) > 0, \quad m - \frac{f(\sigma)^2}{J(\sigma)} = m - \frac{\left(\sum_{k=1}^{K} m_k(\mu_k - \sigma)\right)^2}{\sum_{k=1}^{K} m_k(\mu_k - \sigma)^2} \geq 0 \quad \text{for all } \sigma \in [0,1].$$

Note that for $K > 1$ and $\mu_k \neq \mu_j$ for at least one $j \neq k$ then $m > f(\sigma)^2/J(\sigma)$. For instance, with $K = 2$

$$m - \frac{f(\sigma)^2}{J(\sigma)} = \frac{\ell^2 m_1 m_2 (\mu_1 - \mu_2)^2}{J(\sigma)} > 0 \qquad \text{for all } \mu_1 \neq \mu_2 \text{ and } \sigma \in [0,1],$$

in which case $\mathbf{J} \succ 0$.

In the absence of constraints to the rod kinematics, such as in class 1 tensegrity structures, we find it convenient to work with a *configuration matrix*

$$\mathbf{Q} = \begin{bmatrix} \mathbf{r} & \mathbf{b} \end{bmatrix} \in \mathbb{R}^{3 \times 2} \tag{5.15}$$

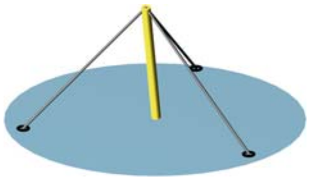

Figure 5.2: A single rod with three strings

as opposed to the configuration vector (5.8). Points on the rod can be described simply as

$$\mathbf{v}(\eta) = \mathbf{Q}\begin{bmatrix}1\\ \eta\end{bmatrix}. \tag{5.16}$$

Compare the above with (5.9).

5.2.1 Nodes as Functions of the Configuration

In dynamics, the node vectors must be expressed as a function of the configuration matrix $\mathbf{Q}$ or the configuration vector $\mathbf{q}$. In the next sections we focus on the configuration matrix $\mathbf{Q}$. The configuration vector $\mathbf{q}$ will be considered when we deal with constrained rods in Section 5.4. One of the major advantages of our approach is that the relationship between the configuration matrix and nodes is linear for all tensegrity structures, as illustrated in the next example.

Example 5.3

Consider the single rod pinned at one end with three strings as illustrated in Figure 5.2. Let $\mathbf{Q}$ be as in (5.15). Because any node $\mathbf{n}_i$ located in a rod is computed through

$$\mathbf{n}_i = \mathbf{Q}\begin{bmatrix}1\\ \eta_i\end{bmatrix}, \qquad i = \{1, 2\},$$

where $\eta_1 = 0$, $\eta_2 = \ell$, we have that

$$\mathbf{N} = \begin{bmatrix}\mathbf{n}_1 & \cdots & \mathbf{n}_5\end{bmatrix} = \mathbf{Q}\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 0 & \ell & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{n}_3 & \mathbf{n}_4 & \mathbf{n}_5\end{bmatrix}.$$

In general we should have

$$\mathbf{N} = \mathbf{Q}\,\mathbf{\Psi}^T + \mathbf{Y}, \qquad \mathbf{N}, \mathbf{Y} \in \mathbb{R}^{3\times n}, \qquad \mathbf{\Psi} \in \mathbb{R}^{n\times 2}, \tag{5.17}$$

where $\mathbf{\Psi} \in \mathbb{R}^{n\times 2}$ and $\mathbf{Y} \in \mathbb{R}^{3\times n}$ are constant. The above expression is valid even when more than one rod is considered (see Section 5.3).

5.2.2 String Forces

Forces on the rod are due to the elongation of strings and ground reactions. For simplicity, we assume that the strings are Hookean, as in Section 2.3, and that they are firmly attached to nodes on the rods or on fixed space coordinates. That is, strings are linear force elements with rest length l_i^0 and stiffness k_i. The force vector of the ith string is

$$\mathbf{t}_i := \begin{cases} 0, & \|\mathbf{s_i}\| < l_i^0, \\ -\kappa_i(\|\mathbf{s_i}\| - l_i^0)\|(\mathbf{s_i}/\|\mathbf{s_i}\|), & \|\mathbf{s_i}\| \geq l_i^0, \end{cases} \tag{5.18}$$

where $\mathbf{s}_i$ is a vector in the direction of the ith string. String vectors are linear functions of the nodes of the structure. As in Sections 2.1 and 2.4, assembling a matrix of string vectors and nodes

$$\mathbf{S} = \begin{bmatrix}\mathbf{s}_1 & \cdots & \mathbf{s}_m\end{bmatrix} \in \mathbb{R}^{3\times m}, \quad \mathbf{N} = \begin{bmatrix}\mathbf{n}_1 & \cdots & \mathbf{n}_n\end{bmatrix} \in \mathbb{R}^{3\times n}, \quad \mathbf{S} = \mathbf{N}\mathbf{C}_S^T,$$

where the vector $\mathbf{n}_k$ denotes the kth node in the structure and the string connectivity matrix $\mathbf{C_S} \in \mathbb{R}^{m\times n}$, it follows that

$$\mathbf{T} = -\mathbf{S}\boldsymbol{\Gamma}, \qquad \mathbf{F} = \begin{bmatrix}\mathbf{f}_1 \cdots \mathbf{f}_n\end{bmatrix} = \mathbf{T}\mathbf{C}_S = -\mathbf{N}\mathbf{C}_S^T\boldsymbol{\Gamma}\mathbf{C}_S, \tag{5.19}$$

where we made use of the diagonal matrix $\boldsymbol{\Gamma}$ which contains the force densities

$$\gamma_i := \max\{0, \quad \kappa_i(\|\mathbf{s_i}\| - l_i^0)\|/\|\mathbf{s_i}\|\} \tag{5.20}$$

on its diagonal, as in Sections 2.2 and 2.4. The matrix $\mathbf{F}$ is the matrix of nodal forces.

5.2.3 Generalized Forces and Torques

Equations of motion will be written in terms of the configuration matrix or vector, whereas the forces in the previous section are functions of the nodes. Hence, one needs to express forces in terms of the configuration matrix or vector coordinates. That is, one needs to compute *generalized forces*. As shown at the end of the chapter and because of linearity of (5.17) the matrix of generalized forces is computed as

$$\mathbf{F_Q} = -(\mathbf{Q}\,\boldsymbol{\Psi}^T + \mathbf{Y})\,\mathbf{C}_S^T\,\boldsymbol{\Gamma}\,\mathbf{C}_S\,\boldsymbol{\Psi}. \tag{5.21}$$

A closer look at (5.21) reveals that

$$\mathbf{F_Q} = \begin{bmatrix}\mathbf{f_r} & \mathbf{f_b}\end{bmatrix}, \qquad \mathbf{f_r} = \sum_{i=1}^{n} \mathbf{f}_i, \qquad \mathbf{f_b} = \sum_{i=1}^{n} \eta_i \mathbf{f}_i,$$

where $\mathbf{f_r}$ is simply the sum of all forces applied to the rods and $\mathbf{f_b}$ is related to the sum of the torques on the rod. Indeed

$$\tilde{\mathbf{b}}\,\mathbf{f_b} = \sum_{i=1}^{n} \tau_i, \qquad \tau_i = \eta_i \tilde{\mathbf{b}}\,\mathbf{f}_i.$$

This fact will be used next.

5.2.4 Equations of Motion

Newtonian approach

In this section let us assume that $\mathbf{r}$ coincides with the location of the center of mass of the rod. Then the translation of the center of mass is governed by the equations of motion

$$m\,\ddot{\mathbf{r}} = \sum_{i=1}^{n} \mathbf{f}_i, \tag{5.22}$$

where $\mathbf{f}_i$ are external forces applied to the rod. Let $\tau = \sum_{i=1}^{n} \tau_i$ be the torques applied to the rod, assuming that the fixed point $\mathbf{r}$ is both the center of mass and center of rotation. Then from Newton's laws $\dot{\mathbf{h}} = \tau$, where $\mathbf{h} = J\,\tilde{\mathbf{b}}\,\dot{\mathbf{b}}$, so that

$$\dot{\mathbf{h}} = J\left(\dot{\tilde{\mathbf{b}}}\,\dot{\mathbf{b}} + \tilde{\mathbf{b}}\,\ddot{\mathbf{b}}\right) = J\,\tilde{\mathbf{b}}\,\ddot{\mathbf{b}}, \tag{5.23}$$

where, for ease of notation, we have omitted the dependence of J on σ. We shall do the same with respect to f from now on. Hence

$$J\,\tilde{\mathbf{b}}\,\ddot{\mathbf{b}} = \sum_{i=1}^{n} \tau_i. \tag{5.24}$$

We must add to these equations a constraint on the length of the rod, $(\ell\mathbf{b})^T(\ell\mathbf{b}) = \ell^2$, or simply $\mathbf{b}^T\mathbf{b} = 1$, as described in (5.6). Differentiating this constraint twice with respect to time yields

$$\mathbf{b}^T\dot{\mathbf{b}} = 0, \qquad \mathbf{b}^T\ddot{\mathbf{b}} + \|\dot{\mathbf{b}}\|^2 = 0. \tag{5.25}$$

Equations (5.23) and (5.25) must be solved simultaneously for $\ddot{\mathbf{b}}$. Note that they are linear in the vector $\ddot{\mathbf{b}}$, yielding the solution for $\ddot{\mathbf{b}}$ (proof at the end of chapter),

$$\ddot{\mathbf{b}} = -(\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2\,\mathbf{b} - \sum_{i=1}^{n} J^{-1}(\tilde{\mathbf{b}}\,\tau_i)/\|\mathbf{b}\|^2. \tag{5.26}$$

We also show at the end of the chapter that with the help of the *projection matrix*

$$\mathbf{P}(\mathbf{b}) := \mathbf{I} - (\mathbf{b}\mathbf{b}^T)/\|\mathbf{b}\|^2, \tag{5.27}$$

we can write

$$\sum_{i=1}^{n} \tilde{\mathbf{b}}\,\tau_i = -\|\mathbf{b}\|^2\mathbf{P}(\mathbf{b})\,\mathbf{f}_{\mathbf{b}}$$

to express the torques in the right-hand side of (5.26) in terms of the generalized force $\mathbf{f}_{\mathbf{b}} = \sum_1^n \eta_i \mathbf{f}_i$ so that,

$$\ddot{\mathbf{b}} = J^{-1}\mathbf{P}(\mathbf{b})\,\mathbf{f}_{\mathbf{b}} - (\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2\,\mathbf{b}. \tag{5.28}$$

Lagrangian approach

In Section 5.4 we will deal with tensegrity structures in which some or all the rods may have kinematic constraints. In such structures it may be advantageous to make $\mathbf{r}$ not coincide with the location of the center of mass of the rod. In such cases, deriving the equations of motion using the momentum approach of the previous section may be unnecessarily complicated. A simpler approach is the use of energy methods, whose full potential we explore in Section 5.4. In the present section we simply rederive the equations of the previous section in order to introduce the reader to energy methods.

Consider the Lagrangian function

$$L = T - V - \frac{J\xi}{2}\left(\mathbf{b}^T\mathbf{b} - 1\right), \tag{5.29}$$

where ξ is the Lagrange multiplier responsible for enforcing the constraint that $\mathbf{b}$ must remain unitary (5.6) and V is some appropriately defined potential function. Assume once again that $\mathbf{r}$ coincides with the location of the center of mass of the rod, i.e., $f = 0$. Following standard derivations as shown at the end of the chapter we arrive at the equations of motion

$$m\,\ddot{\mathbf{r}} = \mathbf{f_r}, \qquad J\,\ddot{\mathbf{b}} = \mathbf{f_b} - J\xi\mathbf{b}, \qquad \mathbf{b}^T\mathbf{b} - 1 = 0, \tag{5.30}$$

where $\mathbf{f_r}$ and $\mathbf{f_b}$ are the vector of generalized forces acting on the rod written in the coordinates $\mathbf{q}$ (see Section 5.2.1).

The difficulty in (5.30) is not solving for $\ddot{\mathbf{b}}$ (which can be done easily because $J > 0$) but avoiding the explicit calculation of the Lagrange multiplier ξ. This can be overcome once again by using the constraint (5.6), as shown in the notes at the end of the chapter, where it is found that

$$\xi = (\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2 + J^{-1}\mathbf{b}^T\mathbf{f_b}/\|\mathbf{b}\|^2. \tag{5.31}$$

Substituting ξ on (5.30) produces the rotational equations of motion

$$\ddot{\mathbf{b}} = J^{-1}\mathbf{P}(\mathbf{b})\,\mathbf{f_b} - (\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2\mathbf{b}, \tag{5.32}$$

where $\mathbf{P}(\mathbf{b})$ is the projection matrix (5.27). Not surprisingly, the above equation is the same as the one previously obtained in (5.28).

5.3 Class 1 Tensegrity Structures

The equations of motion developed in the previous section can be extended to cope with general class 1 tensegrity structures in a fairly straightforward way. Instead of presenting a lengthy and detailed derivation of the equations of motion for general class 1 tensegrity systems, we shall limit ourselves to indicate what are the steps needed to be taken in order to undertake such generalizations based on the material presented so far.

Because in class 1 tensegrity structures no rods touch each other, there exists no extra constraint that should be taken into consideration beyond the ones already considered in Section 5.2. In fact, using the energy approach all that needed to derive equations of motion for a class 1 tensegrity system with K rods is to define the combined Lagrangian

$$L = \sum_{j=1}^{k} L_j,$$

where each L_j is a Lagrangian function written for each rod $j = 1, \ldots, k$ as in (5.29) and following the procedure outlined in Section 5.2.4 for enforcing the individual rod constraints and deriving the equations of motion. With that in mind define the configuration matrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{R} & \mathbf{B} \end{bmatrix} \in \mathbb{R}^{3\times 2k}, \tag{5.33}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_1 & \cdots & \mathbf{r}_k \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{bmatrix} \in \mathbb{R}^{3\times k}. \tag{5.34}$$

Note that in the absence of constraints (5.17) is still valid provided an appropriate matrix $\mathbf{\Psi} \in \mathbb{R}^{n\times 2k}$ is constructed. Likewise, generalized forces are easily computed using (5.21)

$$\mathbf{F_Q} = \begin{bmatrix} \mathbf{F_R} & \mathbf{F_B} \end{bmatrix} \in \mathbb{R}^{3\times 2k}, \tag{5.35}$$

where

$$\mathbf{F_R} = \begin{bmatrix} \mathbf{f}_{\mathbf{r}_1} & \cdots & \mathbf{f}_{\mathbf{r}_k} \end{bmatrix}, \qquad \mathbf{F_B} = \begin{bmatrix} \mathbf{f}_{\mathbf{b}_1} & \cdots & \mathbf{f}_{\mathbf{b}_k} \end{bmatrix} \in \mathbb{R}^{3\times k}. \tag{5.36}$$

The relationship between each column of $\mathbf{F_B}$ and torques is the same as provided in Section 5.2.1.

A surprisingly compact matrix expression for the resulting equations of motion is possible by combining (5.30) and (5.31) as follows:

$$(\ddot{\mathbf{Q}} + \mathbf{Q}\,\mathbf{\Xi})\,\mathbf{M} = \mathbf{F_Q}, \tag{5.37}$$

where

$$\mathbf{M} = \operatorname{diag}[m_1, \ldots, m_k, J_1, \ldots, J_k] \tag{5.38}$$

is a constant matrix and

$$\mathbf{\Xi} = \operatorname{diag}[0, \ldots, 0, \xi_1, \ldots, \xi_k], \tag{5.39}$$

where ξ_j are Lagrange multipliers computed as in (5.31) for each individual bar $\mathbf{b}_j$. The above discussion is summarized in the next theorem.

Theorem 5.1 *Consider an unconstrained class* 1 *tensegrity system with k rigid fixed length rods. Define the* configuration matrix (5.33)

$$\mathbf{Q} = \begin{bmatrix}\mathbf{R} & \mathbf{B}\end{bmatrix} \in \mathbb{R}^{3\times 2k},$$

where the columns of $\mathbf{R}$ *describe the center of mass of the k rods and the columns of* $\mathbf{B}$ *describe the rod vectors. The* $\mathbf{\Psi} \in \mathbb{R}^{n\times 2k}$ *and* $\mathbf{Y} \in \mathbb{R}^{3\times n}$ *are constant matrices that relate the* $n \geq 2k$ *nodes of the structure with the configuration matrix through* (5.17)

$$\mathbf{N} = \mathbf{Q}\,\mathbf{\Psi}^T + \mathbf{Y}, \qquad \mathbf{N}, \mathbf{Y} \in \mathbb{R}^{3\times n}, \qquad \mathbf{\Psi} \in \mathbb{R}^{n\times 2}.$$

The dynamics of such unconstrained class 1 *tensegrity systems satisfy* (5.37)

$$(\ddot{\mathbf{Q}} + \mathbf{Q}\,\mathbf{\Xi})\,\mathbf{M} = \mathbf{F}_{\mathbf{Q}},$$

where

$$\mathbf{M} = \mathrm{diag}[m_1, \dots, m_k, J_1, \dots, J_k], \qquad \mathbf{\Xi} = \mathrm{diag}[0, \dots, 0, \xi_1, \dots, \xi_k].$$

The Lagrange multipliers ξ_i, $i = 1, \dots, k$, *are computed by*

$$\xi_i = (\|\dot{\mathbf{b}}_i\|/\|\mathbf{b}_i\|)^2 + J_i^{-1}\mathbf{b}_i^T\mathbf{f}_{\mathbf{b}_i}/\|\mathbf{b}_i\|^2, \tag{5.40}$$

where $\mathbf{f}_{\mathbf{b}_i}$ *are columns of the matrix* $\mathbf{F}_{\mathbf{B}}$ *which is part of the* matrix of generalized forces

$$\mathbf{F}_{\mathbf{Q}} = \begin{bmatrix}\mathbf{F}_{\mathbf{R}} & \mathbf{F}_{\mathbf{B}}\end{bmatrix} \in \mathbb{R}^{3\times 2k},$$

which is computed by (5.21)

$$\mathbf{F}_{\mathbf{Q}} = [\mathbf{W} - (\mathbf{Q}\,\mathbf{\Psi}^T + \mathbf{Y})\,\mathbf{C}_S^T\,\mathbf{\Gamma}\,\mathbf{C}_S\,]\mathbf{\Psi},$$

where $\mathbf{C}_S$ *is the* string connectivity matrix, *and the external force acting on node* $\mathbf{n}_i$ *is* $\mathbf{w_i}$, *and the matrix of all such external forces is*

$$\mathbf{W} = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_{2k}\end{bmatrix}. \tag{5.41}$$

By parametrizing the configuration in terms of the components of vectors, the usual transcendental nonlinearities involved with the use of angles, angular velocities, and coordinate transformations are avoided. Indeed, the absence of trigonometric functions in this formulation leads to a simplicity in the analytical form of the dynamics. This might facilitate more efficient numerical solutions of the differential equations (simulations) and the design of control laws. Actually, the simplicity of the structure of these equations (5.37) is partly due to the use of the matrix form and partly due to the enlarged space in which the dynamics are described. The actual degrees of freedom for each rod is 5, whereas the model (5.37) has as many equations as required for 6 degrees of freedom for each rod. That is, the equations are a non-minimal realization of the dynamics. The *mathematical structure* of the equations are simple, however. This will allow much easier integration of structure and control design, since the control variables (string force densities) appear linearly, and the simple structure of the nonlinearities can be exploited in later control investigations.

Example 5.4

Consider the tensegrity prism depicted in Figure 5.3. This structure has 6 nodes, 3 rods, and 12 strings. Let the node matrix be

$$\mathbf{N} = \begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 & \mathbf{n}_4 & \mathbf{n}_5 & \mathbf{n}_6\end{bmatrix},$$

where each pair of nodes is a pair of bottom and top nodes on a rod. That is,

$$\mathbf{B} = \begin{bmatrix}\ell_1^{-1}(\mathbf{n}_1 - \mathbf{n}_2) & \ell_2^{-1}(\mathbf{n}_3 - \mathbf{n}_4) & \ell_3^{-1}(\mathbf{n}_5 - \mathbf{n}_6)\end{bmatrix}.$$

Assuming that the mass m_j of the jth rods is uniformly distributed then the center of each rods is its center of mass

$$\mathbf{R} = \frac{1}{2}\begin{bmatrix}\mathbf{n}_1 + \mathbf{n}_2 & \mathbf{n}_3 + \mathbf{n}_4 & \mathbf{n}_5 + \mathbf{n}_6\end{bmatrix}.$$

The nodes can be retrieved from the configuration matrix $\mathbf{Q} = \begin{bmatrix}\mathbf{R} & \mathbf{B}\end{bmatrix}$ through (5.17) with

$$\mathbf{\Psi} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \ell_1/2 & -\ell_1/2 & \ell_2/2 & -\ell_2/2 & \ell_3/2 & -\ell_3/2 \end{bmatrix},$$

and, because of the uniform mass distribution and the choice of $\mathbf{R}$, we have that $f_j = 0$ and

$$J_j = \frac{1}{12} m_j \ell_j^2, \qquad j = \{1, 2, 3\}.$$

The string connectivity is

$$\mathbf{C}_S = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

With this information one can write the equations of motion (5.37).

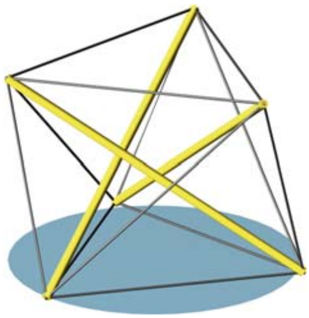

Figure 5.3: A class 1 tensegrity prism with 3 rods and 12 strings

5.4 Constrained Class 1 Tensegrity Structures

We now consider class 1 tensegrity structures in which nodes in some of the rods may have linear kinematic constraints due to its interaction with the environment. Still no rods touch each other. In such cases, it may be advantageous to work with a reduced configuration vector as opposed to our oversized configuration matrix $\mathbf{Q}$, since the latter might not be well defined, as in the next example.

Example 5.5

Let the z-coordinate of node $\mathbf{r} = \mathbf{v}(0)$ of Example 5.3 be constrained to stay at xy-plane, i.e., $\mathbf{r}_z = 0$. Define the *reduced* configuration vector

$$\mathbf{q} = \begin{pmatrix} \mathbf{r}_x \\ \mathbf{r}_y \\ \mathbf{b} \end{pmatrix}.$$

In this case, the relationship between the configuration vector $\mathbf{q}$ and the nodes is of the form

$$\mathbf{n} = \mathbf{\Phi}\,\mathbf{q} + \mathbf{y}. \tag{5.42}$$

Note that when (5.17) holds then (5.42) is obtained from (5.17) by vectorization with $\mathbf{n} = \operatorname{vec}\mathbf{N}$, $\mathbf{y} = \operatorname{vec}\mathbf{Y}$, and $\mathbf{\Phi} = \mathbf{\Psi} \otimes \mathbf{I}_3$. In general, $\mathbf{\Phi} \neq \mathbf{\Psi} \otimes \mathbf{I}_3$, as in the next example. As shown at the end of the chapter, with external forces $\mathbf{w}_i$ added to each node (where the total external vector of forces is $\mathbf{w}$), generalized forces are computed as

$$\mathbf{f}_{\mathbf{q}} = \mathbf{\Phi}^T \mathbf{w} - \mathbf{\Phi}^T \left(\mathbf{C}_S^T\,\mathbf{\Gamma}\,\mathbf{C}_S \otimes \mathbf{I}_3\right)(\mathbf{\Phi}\,\mathbf{q} + \mathbf{y}). \tag{5.43}$$

Example 5.6

Consider the reduced configuration vector of Example 5.5. For the same configuration of strings as in Example 5.3 we have

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{E} & \ell\,\mathbf{I}_3 \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{n}_3 \\ \mathbf{n}_4 \\ \mathbf{n}_5 \end{bmatrix}. \tag{5.44}$$

Note that $\mathbf{\Phi} \neq \mathbf{\Psi} \otimes \mathbf{I}_3$.

5.4.1 Single Constrained Rigid Rod

Let $\mathbf{r} = \mathbf{v}(\sigma)$ for some $\sigma \in [0, 1]$ be a fixed point in the rod which may not coincide with the rod's center of mass and is subject to the linear constraint

$$\mathbf{D}\,\mathbf{r} = \bar{\mathbf{r}}, \tag{5.45}$$

where $r = \text{rank}(\mathbf{D}) < 3$ and $\bar{\mathbf{r}} \in \mathbb{R}^r$ constant. Let $\mathbf{E} \in \mathbb{R}^{3\times(3-r)}$ be an orthonormal matrix, i.e., $\mathbf{E}^T\mathbf{E} = \mathbf{I}$, such that $\mathbf{DE} = 0$. Then all solutions to (5.45) are parametrized by

$$\mathbf{r} = \mathbf{D}^\dagger\bar{\mathbf{r}} + \mathbf{Ez}, \tag{5.46}$$

where $\mathbf{z} \in \mathbb{R}^{3-r}$. Define the *reduced configuration vector*

$$\mathbf{q} = \begin{pmatrix} \mathbf{z} \\ \mathbf{b} \end{pmatrix}. \tag{5.47}$$

Example 5.7

In Example 5.5 we have $\mathbf{r} = \mathbf{v}(0)$ with

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad \bar{\mathbf{r}} = 0.$$

Then

$$\mathbf{D}^\dagger = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0} \end{bmatrix}.$$

The particular case when $r = \text{rank}(\mathbf{D}) = 3$ is handled by defining the reduced configuration vector as

$$\mathbf{q} = \mathbf{b}, \tag{5.48}$$

because $\mathbf{r} = \mathbf{D}^{-1}\bar{\mathbf{r}}$ is simply a constant.

The above discussion also provides clues on how to compute matrices $\boldsymbol{\Phi}$ and $\mathbf{y}$ in Section 5.2.1. For instance, after one has computed $\boldsymbol{\Psi}$ and $\mathbf{Y}$ such that

$$\mathbf{N} = \mathbf{Q}\,\boldsymbol{\Psi}^T + \mathbf{Y},$$

where $\mathbf{Q}$ is the "non-reduced" configuration matrix (5.15), it becomes clear that $\mathbf{n}$ and $\mathbf{q}$ should be related through (5.42) where

$$\boldsymbol{\Phi} = (\boldsymbol{\Psi} \otimes \mathbf{I}_3) \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix}, \qquad \mathbf{y} = \operatorname{vec} \mathbf{Y} + (\boldsymbol{\Psi} \otimes \mathbf{I}_3) \begin{pmatrix} \mathbf{D}^\dagger \bar{\mathbf{r}} \\ \mathbf{0} \end{pmatrix},$$

because

$$\operatorname{vec} \mathbf{Q} = \begin{bmatrix} \mathbf{r} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^\dagger \bar{\mathbf{r}} + \mathbf{E}\,\mathbf{z} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \mathbf{q} + \begin{pmatrix} \mathbf{D}^\dagger \bar{\mathbf{r}} \\ \mathbf{0} \end{pmatrix}.$$

Equations of motion

Since $\mathbf{r}$ may not be the center of mass, the equations of motion are expected to be more complex than the ones seen so far. This is justified by the extra work that is required to handle the constraint (5.6). As shown at the end of the chapter, the equations of motion for a single constrained rod are of the form

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{H}(\mathbf{q})\,\mathbf{f}_{\mathbf{q}}, \tag{5.49}$$

where

$$\mathbf{M}(\mathbf{q}) := \begin{bmatrix} m\,\mathbf{I} - f^2 J^{-1} \mathbf{E}^T \mathbf{P}(\mathbf{b}) \mathbf{E} & \mathbf{0} \\ f J^{-1} \mathbf{P}(\mathbf{b}) \mathbf{E} & \mathbf{I} \end{bmatrix},$$

$$\mathbf{H}(\mathbf{q}) := \begin{bmatrix} \mathbf{I} & -f J^{-1} \mathbf{E}^T \mathbf{P}(\mathbf{b}) \\ \mathbf{0} & J^{-1} \mathbf{P}(\mathbf{b}) \end{bmatrix},$$

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}) := (\|\dot{\mathbf{b}}\| / \|\mathbf{b}\|)^2 \begin{pmatrix} -f\,\mathbf{E}^T \mathbf{b} \\ \mathbf{b} \end{pmatrix}. \tag{5.50}$$

When dealing with equations of motion of the form (5.49) an issue that arises is that of solving for $\ddot{\mathbf{q}}$ as a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$. This is indeed the case in most cases of interest where $\mathbf{J}$ is positive definite due to the following lemma which is proved at the end of the chapter.

Lemma 5.1 *Let* $\mathbf{J}$ *and* $\mathbf{M}(\mathbf{q})$ *be defined by* (5.11) *and* (5.50)*, respectively. If* $\mathbf{J} \succ 0$ *then* $\mathbf{M}(\mathbf{q})$ *is nonsingular for all* $\mathbf{q}$ *such that* $\|\mathbf{b}\| = 1$*.*

5.4.2 General Class 1 Tensegrity Structures

The equations of motion developed in the previous section can be generalized to cope with general constrained class 1 tensegrity structures as done in Section 5.3.

After defining local configuration vectors $\mathbf{q}_j$, $j = 1, \ldots, K$, we can follow the derivations in the previous section and arrive at the system of differential equations of the form (5.49), that is,

$$\mathbf{M}_j(\mathbf{q}_j)\,\ddot{\mathbf{q}}_j + \mathbf{g}_j(\mathbf{q}_j, \dot{\mathbf{q}}_j) = \mathbf{H}_j(\mathbf{q}_j)\,\mathbf{f}_{\mathbf{q}_j}, \qquad j = 1, \ldots, K, \tag{5.51}$$

where $\mathbf{M}_j$, $\mathbf{g}_j$, and $\mathbf{H}_j$ are as defined in (5.50) for the jth rod.

Example 5.8

Consider again the tensegrity prism of Example 5.4 depicted in Figure 5.3. As before, the mass of the three rods is assumed to be uniformly distributed but this time $\mathbf{r}_j = \mathbf{v}(0)$, for all $j = \{1, 2, 3\}$,

$$\sigma_j = 0, \qquad \Longrightarrow \qquad f_j = \frac{1}{2} m_j \ell_j, \qquad J_j = \frac{1}{3} m_j \ell_j, \qquad j = \{1, 2, 3\},$$

that is, the vectors $\mathbf{r}_j$ all point to one extreme node of each rod. Now set $\mathbf{r}_1$ to be the origin ($\mathbf{r}_1 = \mathbf{0}$) and consider that nodes $\mathbf{r}_2$ and $\mathbf{r}_3$ be constrained as in (5.45) and (5.46) with

$$\bar{\mathbf{r}}_2 = 0, \qquad \mathbf{D}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\bar{\mathbf{r}}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{D}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{E}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The above matrices reflect the fact that the z-coordinate of the bottom node of the second rod ($\mathbf{r}_2$) is set to zero, i.e., it is free to move only in the xy-plane; and the x- and z-coordinates of the bottom node of the third rod ($\mathbf{r}_3$) are set to zero, i.e., it can move only on the y-axis. This set of six constraints eliminates the six rigid body modes of the structure. The configuration vector of the system $\mathbf{q} \in \mathbb{R}^{12}$ is then $\mathbf{q} = (\mathbf{b}_1, \mathbf{z}_2, \mathbf{b}_2, \mathbf{z}_3, \mathbf{b}_3)$.

With the data on the above example one can construct the equations of motion (5.51) with the exception of the generalized force vector $\mathbf{f}_{\mathbf{q}_j}$, which should be computed using (5.43) and the data in the following example.

Example 5.9

For the same tensegrity prism depicted in Figure 5.3 of Examples 5.4 and 5.8 let the node matrix $\mathbf{N} \in \mathbb{R}^{3\times 5}$ be

$$\mathbf{N} = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 & \mathbf{n}_4 & \mathbf{n}_5 \end{bmatrix},$$

with connectivity matrix

$$\mathbf{C}_S = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \in \mathbb{R}^{12\times 5},$$

and the matrix

$$\Phi = \begin{bmatrix} \ell_1\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 & \ell_2\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_3 & \ell_3\mathbf{I} \end{bmatrix} \in \mathbb{R}^{15\times 12}.$$

Vector $\mathbf{y} \in \mathbb{R}^{15}$ is equal to zero.

5.5 Chapter Summary

The equations of motion for any tensegrity system composed of rods and strings are provided in simple form, to make computation and control design easier. One might argue that the absence of simple equations for the dynamics of tensegrity systems has been a limiting factor to the acceptance of tensegrity in engineering practice.

Axially loaded elements (rods and strings) are used throughout. Two kinds of constraints are treated in the dynamics. The length of rods are constant, and position of any node may be fixed. The main contributions of the chapter include both energy and Newtonian approaches, constrained and unconstrained systems, non-minimal realizations of the constrained dynamics, and finally a new matrix form of the equations in Theorem 5.1.

To obtain equations that are efficient for dynamic simulation, with constraints, there are many debates about which method is more efficient. Here the energy and the Newtonian derivations produce the same equations. This is done without using the classical angular velocity vector, since in our case where a 5 DOF system is modeled by 6 DOF (non-minimal equations), the angular velocity about the long axis of a rod is undefined. By using the vector along the rod as a generalized coordinate, the final equations are devoid of the transcendental functions that complicate the form of the dynamics. Putting these equations in matrix form allow the mathematical structure of the equations to be extremely simple. The motivation for seeking simple structure of the equations is the hope that control laws can be found to exploit the known simple structure of the dynamic model. This hope is high enough, we believe, to justify the non-minimality of the equations. Quite often in mathematics the minimal number of equations are often the most complex in form.

The constraints are treated with and without the use of Lagrange multipliers. Without Lagrange multipliers it is shown that the correct equations are obtained by linear algebra, to obtain a least squares solution which enforces the constraints.

The distribution of members and forces are characterized as networks, where efficient matrix methods simplify the description of forces and connections. A connectivity matrix is introduced that characterizes the topology of all rod to string connections. These network equations, together with a simple characterization of the dynamics of a rigid body, allow efficient forms for the final equations.

By using force densities as the input variable (later to be the control variable) the final equations of motion for the general nonlinear tensegrity system has a bilinear structure (equations are nonlinear in the generalized coordinates, but linear in the string force densities). This will offer great advantage in control design.

5.6 Advanced Material

5.6.1 Dynamics of a Single Rigid Rod

Most of the quantities defined in Section 5.2 can be visualized directly from Figure 5.1, from where (5.9) follows. The kinetic energy formula comes from (5.9) after expanding

$$T = \frac{1}{2\ell}\int_{-\sigma\ell}^{(1-\sigma)\ell} \rho(\sigma+\eta/\ell)\,(\dot{\mathbf{r}}^T\dot{\mathbf{r}} + 2\eta\,\dot{\mathbf{r}}^T\dot{\mathbf{b}} + \eta^2\,\dot{\mathbf{b}}^T\dot{\mathbf{b}})\,d\eta = \frac{1}{2}\,\dot{\mathbf{q}}^T(\mathbf{J}(\sigma)\otimes\mathbf{I})\,\dot{\mathbf{q}}.$$

Likewise, the angular momentum formula follows from

$$\mathbf{h} = \ell^{-1}\int_{\sigma\ell}^{(1-\sigma)\ell} \rho(\sigma+\eta/\ell)\,\eta^2\,(\tilde{\mathbf{b}}\,\dot{\mathbf{b}})\,d\eta = J\,\tilde{\mathbf{b}}\,\dot{\mathbf{b}}.$$

Generalized forces and torques

The matrix of generalized forces (5.21) is obtained after analyzing the work produced by the matrix of nodal forces $\mathbf{W}+\mathbf{F}$, where $\mathbf{W}$ is the matrix of external forces and $\mathbf{F}$ is the matrix of internal string forces (that is, the *ith* column of $\mathbf{W}$ summed with the *ith* column of $\mathbf{F}$ is the total force vector acting on node $\mathbf{n}_i$). Hence, from string connectivity, $\mathbf{F} = -\mathbf{N}\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S$, and for an infinitesimal matrix displacement $\mathbf{\Delta_N}$

$$\mathrm{trace}((\mathbf{W}+\mathbf{F})^T\mathbf{\Delta_N}) = \mathrm{trace}(\mathbf{\Delta_N})^T(\mathbf{W}+\mathbf{F}) \;\; = \mathrm{trace}\,\mathbf{\Delta_N}^T(\mathbf{W}-\mathbf{N}\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S).$$

Recalling that $\mathbf{N} = \mathbf{Q}\,\mathbf{\Psi}^T + \mathbf{Y}$ so that $\mathbf{\Delta_N} = \mathbf{\Delta_Q}\mathbf{\Psi}^T$, then

$$\mathrm{trace}\,\mathbf{\Delta}_{\mathbf{N}}^T(\mathbf{W}+\mathbf{F}) = \mathrm{trace}\,\mathbf{\Psi}\mathbf{\Delta}_{\mathbf{Q}}^T(\mathbf{W}-\mathbf{N}\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S) \qquad = \mathrm{trace}\,\mathbf{\Delta}_{\mathbf{Q}}^T\mathbf{F}_{\mathbf{Q}},$$

where $\mathbf{F_Q} = (\mathbf{W} - \mathbf{N}\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S)\mathbf{\Psi} = (\mathbf{W} - (\mathbf{Q}\,\mathbf{\Psi}^T + \mathbf{Y})\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S)\mathbf{\Psi}$. In the absence of external forces, expression (5.43) follows from vectorization:

$$\mathbf{f_q} = \left(\mathbf{\Psi}^T \otimes \mathbf{I}_3\right)\mathbf{w} - \left(\mathbf{\Psi}^T\mathbf{C}_S^T\,\mathbf{\Gamma}\,\mathbf{C}_S \otimes \mathbf{I}_3\right)\left((\mathbf{\Psi}\otimes\mathbf{I}_3)\,\mathbf{q} + \mathbf{y}\right),$$

and using $\mathbf{\Phi} = \mathbf{\Psi} \otimes \mathbf{I}_3$.

Equations of motion

Newtonian approach The two equations that describe the constrained system are

$$J\,\tilde{\mathbf{b}}\,\ddot{\mathbf{b}} = \tau, \qquad\qquad \phi = \mathbf{b}^T\mathbf{b} - 1 = 0. \tag{5.52}$$

Here we will completely ignore the original constraint $\phi = 0$ and its first derivative, $\dot{\phi} = 0$, but we will honor the second derivative of the constraint $\ddot{\phi} = 0$. Assembling these into a single equation yields

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{b}^T \end{bmatrix} \ddot{\mathbf{b}} = \begin{bmatrix} J^{-1}\tau \\ -\|\dot{\mathbf{b}}\|^2 \end{bmatrix}. \tag{5.53}$$

The task of solving (5.53) for $\ddot{\mathbf{b}}$ is simply a linear algebra problem. Uniqueness of the solution is guaranteed by linear independent columns of the matrix coefficient of $\ddot{\mathbf{b}}$. We prove this linear independence by noting that

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{b}^T \end{bmatrix}^T \begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{b}^T \end{bmatrix} = \|\mathbf{b}\|^2\mathbf{I}.$$

The above identity can be expanded to provide an expression for the square of a skew-symmetric matrix

$$\tilde{\mathbf{b}}^2 = \mathbf{b}\mathbf{b}^T - \mathbf{b}^T\mathbf{b}\mathbf{I} = -\mathbf{P}(\mathbf{b})\|\mathbf{b}\|^2, \tag{5.54}$$

in terms of the projection matrix (5.27).

Next we use the properties of the unique Moore–Penrose inverse [SIG98] to conclude that the unique Moore–Penrose inverse of the coefficient matrix of the left-hand side of (5.53) is

$$\begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{b}^T \end{bmatrix}^+ = \begin{bmatrix} -\tilde{\mathbf{b}} & \mathbf{b} \end{bmatrix} / \|\mathbf{b}\|^2.$$

Using this fact and $\tau = \tilde{\mathbf{b}}\sum_i^n \eta_i \mathbf{f}_i = \tilde{\mathbf{b}}\,\mathbf{f}_b$, the unique solution of (5.53) is

$$\begin{aligned} \ddot{\mathbf{b}} &= -J^{-1}\tilde{\mathbf{b}}\,\tau/\|\mathbf{b}\|^2 - \mathbf{b}(\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2 \\ &= J^{-1}\mathbf{P}(\mathbf{b})\,\mathbf{f}_b - \mathbf{b}(\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2, \end{aligned}$$

where we have used (5.54).

Lagrangian approach From the Lagrangian function (5.29) and with the assumption that $f = 0$ the equations of motion of the rod are given by

$$m\,\ddot{\mathbf{r}} = \frac{d}{dt}\partial_{\dot{\mathbf{r}}}L = \partial_{\mathbf{r}}L = \mathbf{f}_{\mathbf{r}},$$
$$J\,\ddot{\mathbf{b}} = \frac{d}{dt}\partial_{\dot{\mathbf{b}}}L = \partial_{\mathbf{b}}L = \mathbf{f}_{\mathbf{b}} - J\xi\mathbf{b}, \tag{5.55}$$
$$\mathbf{0} = 2\,J^{-1}\partial_{\xi}L = \mathbf{b}^T\mathbf{b} - 1. \tag{5.56}$$

Now multiply (5.55) by $\mathbf{b}^T$ on the left and differentiate (5.56) twice with respect to time to obtain

$$J\,\mathbf{b}^T\ddot{\mathbf{b}} = \mathbf{b}^T\mathbf{f}_{\mathbf{b}} - J\xi\|\mathbf{b}\|^2, \qquad \mathbf{b}^T\ddot{\mathbf{b}} + \|\dot{\mathbf{b}}\|^2 = 0,$$

from where

$$\xi = (\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2 + J^{-1}\mathbf{b}^T\mathbf{f}_{\mathbf{b}}/\|\mathbf{b}\|^2.$$

Equation (5.32) follows after substituting ξ into (5.55) and using the definition of the projection matrix (5.27). Hence, the final form of the equations of motion is

$$m\,\ddot{\mathbf{r}} = \mathbf{f}_{\mathbf{r}},$$
$$J\,\ddot{\mathbf{b}} = \mathbf{P}(\mathbf{b})\mathbf{f}_{\mathbf{b}} - J(\|\dot{\mathbf{b}}\|/\|\mathbf{b}\|)^2\mathbf{b}. \tag{5.57}$$

Note that the two terms on the right-hand side of (5.57) are orthogonal to each other. This feature is expected to lend some efficiency to the numerical simulations, although we have not tried to quantify this.

5.6.2 Constrained Class 1 Tensegrity Structures

Note that when $\mathbf{\Phi} \neq \mathbf{\Psi} \otimes \mathbf{I}_3$ one should compute the vector of generalized forces $\mathbf{f}_{\mathbf{q}}$ using the vectorial version of the principle of virtual work

$$\delta_{\mathbf{n}}^T\,\mathbf{f} = \delta_{\mathbf{q}}^T\,\mathbf{t}_{\mathbf{q}},$$

after recalling that $\mathbf{n} = \mathbf{\Phi}\mathbf{q} + \mathbf{y}$ so that $\delta_{\mathbf{n}} = \mathbf{\Phi}\delta_{\mathbf{q}}$ and consequently

$$\delta_{\mathbf{n}}^T\,\mathbf{f} = \delta_{\mathbf{q}}^T\,\mathbf{\Phi}^T\mathbf{f} = \delta_{\mathbf{q}}^T\,\mathbf{t}_{\mathbf{q}},$$

where

$$\mathbf{t}_{\mathbf{q}} = \mathbf{\Phi}^T\mathbf{f} = -\mathbf{\Phi}^T(\mathbf{C}_S^T\mathbf{\Gamma}\mathbf{C}_S \otimes \mathbf{I}_3)(\mathbf{\Phi}\,\mathbf{q} + \mathbf{y}),$$

which is (5.43).

Single constrained rigid rod

Here is a proof that $\mathbf{M}(\mathbf{q})$ is nonsingular when $\mathbf{J} \succ 0$. Matrix $\mathbf{M}(\mathbf{q})$ is block lower-triangular; therefore, it is nonsingular whenever its diagonal blocks are nonsingular. In this case this means that $\mathbf{M}(\mathbf{q})$ is nonsingular if and only if its first diagonal block

$$\Sigma := m\,\mathbf{I} - f^2 J^{-1}\mathbf{E}^T\mathbf{P}(\mathbf{b})\mathbf{E}$$

is nonsingular. Recall that $\mathbf{q} \in \mathcal{Q}$ implies $\mathbf{b} \neq 0$ so that $\mathbf{P}(\mathbf{b})$ is well defined and that $\mathbf{E}$ is an orthonormal constant matrix, that is, it is full column rank and $\mathbf{E}^T\mathbf{E} = \mathbf{I}$. Therefore,

$$\Sigma = \mathbf{E}^T\Theta\mathbf{E}, \qquad \Theta := (m - f^2 J^{-1})\mathbf{I} + f^2 J^{-1}\mathbf{b}\mathbf{b}^T/\|\mathbf{b}\|^2.$$

Because $\mathbf{J} \succ 0$ we have that $J > 0$ and $m > f^2 J^{-1} \geq 0$. Therefore,

$$\Theta \succeq (m - f^2 J^{-1})\mathbf{I} \succ 0 \qquad \Longrightarrow \qquad \Sigma = \mathbf{E}^T\Theta\mathbf{E} \succ 0.$$

Hence $\mathbf{M}(\mathbf{q})$ is nonsingular.