Chapter 8

LEARNING MATHEMATICS

Abstract: The conclusions of the book are examined for their implications for mathematics education, and an argument is made for attention to be paid to the communicative aspects of mathematics during its development in schools. This includes more exploration at all levels of education, and the importance of informing students of the nature of mathematics. Some notes on assessment are made.

Keywords: communication mathematics teaching, meta-mathematics, explorations in mathematics

1. CONCLUSIONS THROUGH EDUCATIONAL EYES

Learning mathematics has been an enigma for many thousands of years. On the one hand it seems so straight forward. Counting things feels natural, and young children often find numbers playful, reciting them as songs, for example. We all have adequate spatial intuitions. We find our way around our familiar environment without apparent effort. Those living in a city easily make complex routing decisions, and those living in the country develop a directional sense that is reliable and automatic. In many activities we perform mathematical tasks with ease: we intuitively estimate the trajectory of balls in many sports; weavers and sewers and designers manipulate patterns and shapes in sophisticated ways; people build model or real houses and boats that are robust and balanced; and the ever present money transactions in modern life are routinely handled with efficiency.

On the other hand, mathematics classrooms have been places of fear and puzzlement for many, probably since they first appeared in China around 1000BC (Swetz, 1974). They have received bad press throughout literature, and Math Phobia has now become a buzz-word

(Burns, 1998; Clawson, 1991; Tobias, 1995). Many people experience the mathematics classroom as a place of pleasure and wonder, but even this positive aspect is often turned by society into a negative one. Those for whom formal mathematics education is easy and a pleasure are routinely transformed into oddities or nerds.

The enigma of learning mathematics, and the best teaching methods for it, have been discussed explicitly since at least Greek times when Socrates put forward his method. What light can we shed on this enigma by reflecting on language and mathematics? What are the implications for education of the conclusions of Parts I and II? After looking through educational eyes at the conclusions already reached about mathematics, this chapter discusses what our reflections tell us about the nature of mathematics learning in general. I suggest a conclusion about the role of abstraction that is at odds with general thinking, and make some comments about the role of mathematical play and creativity. We finish by examining implications that can be drawn for formal classroom teaching.

The second chapter in this section discusses two particular language contexts. I argue that multilingual environments are a rich source of learning rather than ones filled with problems, and then I discuss the particular situation of indigenous education. Indigenous groups are faced with an interesting dilemma. They learn mathematics in a distinct cultural-linguistic context—how can they study an international subject while retaining the integrity of a minority world view?

In order to keep focussed on the conclusions that have been generated from the evidence from language, I summarise the five main conclusions.

The most important conclusion is that mathematics and language develop together. Historically this has been so, with each of these two areas of human activity affecting the other. It continues to be so, as new language and mathematics is generated in new areas of human interest: computer environments; space exploration; biological modelling; the mathematics of finance. The co-development of mathematics and language happens at both a macro- and a micro-level. At a macro-level they both respond to social and political demands. At the micro-level, the vocabulary and syntax of mathematical discourse responds to that of the language being spoken (and the world view represented therein), as well as to the mathematical needs. The discourse may then affect the direction of mathematical development.

The educational perspective on this conclusion concerns the development of mathematical ideas in an individual. To what extent is the historical link with language also present in personal mathematical development? At first there does not seem to be a necessary connection between the two, but two threads of argument suggest otherwise. The evidence strongly suggests that mathematics as it has evolved does have strong ties with particular language characteristics, and that these need to be established for an individual in some way or other if this mathematics is to be easily understood. Also, the evidence about the difference between mathematical discourse and everyday language means that, even if your language is consonant with NUC-mathematics, there are still changes in your language that need to be made to correctly understand, communicate, and use mathematical ideas.

A second conclusion, related to the first, is the idea that mathematics arises after, not before, human activity, in response to human thinking and communicating about quantity, relationships, and space within particular socio-cultural environments. An educational perspective asks whether (or in what way) socio-cultural context (including language) might be important for understanding a mathematical concept. For example, does the gambling origins of probability theory mean that an understanding of gambling is necessary (or helpful) for statistical education? Will a child who has only experienced probability in the more Bayesian environment of predicting the outcome of a sporting event, have difficulty conceptualising long-run Frequentist ideas? My view is that these are likely to be important considerations in mathematical learning.

A third conclusion of Parts I and II is that mathematics could be different. A corollary of this is that there are still many undeveloped mathematical ideas. This statement does not only refer to advanced level research mathematics. There are still undeveloped ideas in pre-formalised mathematics, elementary mathematics, and at every subsequent level.

The educational perspective on this conclusion is that mathematics is far from a complete and established set of concepts and relationships that can be presented to anyone learning the subject. Nor is mathematics a body of ideas that all children will come to discover in a natural way, even if they are given appropriate activities. At every level there are alternatives to be acknowledged and the possibility of exploring them. The depth of this conclusion cannot be overemphasised. It is not just a question of different techniques of multiplication—it is a question of what operations are possible and sensible at all, or, at an even more basic level, the possible different conceptions of quantity that can exist in a formal way. The conclusion means not only that a mathematics classroom should be open to unconventional mathematics, but also that it must exhibit unconventional ideas, particularly if we wish students to understand what the human activity we call mathematics is all about. A further impli cation is that conventional mathematics must be explained as just that—conventional.

A fourth conclusion is that mathematics is created by communicating, that is, mathematics arises within the communication. I am not just saying that mathematics arises because of the need to communicate, nor just that mathematics is recorded by communication (writing it down as a journal article, for example). Mathematics is created in the act of communication—even the mathematics that is reportedly created in intuitive flashes of an individual when they are alone. The ideas of such flashes do not become mathematics until they are formalised and related to other ideas—until they become part of a system.

An educational perspective asks whether this implies that mathematics is learned through communication. This perspective also focuses on the nature of the communication, and the role played by different people in it. More critical, however, is the idea that mathematical knowledge is therefore never finished, never completed. Whatever understanding a learner reaches is always an understanding of the communication that has just happened—further communication will generate further mathematical understanding.

The most fundamental conclusion of this book is that each language contains its own mathematical world. The worlds may be implicit, of small scope, and/or undeveloped, but these worlds exist—they are not just rudimentary versions of conventional mathematics, nor are they simple, unformalised mathematics. These worlds represent systems of meaning concerned with quantity, relationships, or space, and are, in some sense, incommensurable with NUC-mathematics.

An educational theorist, faced with this conclusion, is likely to ask for justification that one world is the subject of curricular attention while another is not. They are unlikely to accept as a sole answer that one world is more extensive, or more developed. Rather they will want to know about the relationship between this world and the particular learners for whom the curriculum applies. They will evaluate the justification on educational criteria (for example the overall aims of education) as much as on mathematical grounds. And they will ask whether one world needs to be exclusive of others. This issue is especially important for indigenous mathematics education.

A more direct educational issue relating to mathematical worlds is the psychological question of the extent to which an individual is wedded to one world view, and whether (or how) this will affect their understanding of another world view. This question has long been asked by mathematics educators as they search for answers to the differential performance of particular groups in various educational environments. It will be obvious to the reader that my view is that learners are more affected by their world view than is commonly acknowledged.

There are three further issues related to mathematical language that can be viewed from an educational perspective. Mathematical language change is in the direction of more similarity. In other words, different languages are evolving to express QRS ideas in ways that are more and more the same. Is this good for education because it means that there is more uniformity and less need to accommodate differences, or is it detrimental for education because it means that variety and versatility are being lost?

Mathematical language (not just mathematics) evolves from the physical and social environment. To what extent does the everyday meaning and environmental origin of mathematical vocabulary and discourse interfere with or enhance mathematical meaning? Teachers need to take into account the conditions under which the everyday meaning of a mathematical word can contribute to the development of mathematical understanding.

Finally, mathematical language is more consonant with some languages, and less consonant with others. In what ways is this a problem (for example, speakers of less consonant languages might find mathematical constructions difficult), and in what ways is this an advantage (for example, a wide difference between natural language and mathematical discourse may emphasise the particular nature of mathematical discourse and reduce the interferences mentioned in the previous paragraph).

This summary of the main conclusions and what they mean from an educational point of view sets the scene for a look at mathematical learning, mathematical teaching, and mathematics education in the particular contexts of multilingual and indigenous peoples' education.

2. BECOMING A BETTER GOSSIP

I take the hand of my three-year-old granddaughter as we jump down the cobbled steps in the narrow street of the old town. Jump-"one". Jump-"two". She knows this game, and we count for a while. Then I start again: jump-"two". Silence. Jump-"four". "You missed three, Pa-Bill." "I don't like three," I say. The inevitable "why" and I make it clear that it is part of the game: "let's pretend", I say, and that is enough, she knows how to pretend. Soon we have a rhythm: jump-"two", "you missed one"; jump-"four", "you missed three"; jump-"six", "you missed five"; and so on. She did not, as it happened, demand to take the lead with her own sequence, but I would not have been surprised. Young children can play games better than most, and can generate complex games at the drop of a hat, remembering and changing rules as they go along.

For the two of us, what had been the counting numbers became just a sequence of words that were part of a game. We were not counting any more, since 2, 4, 6, … is not how we record single jumps, we were game-playing. We were at the very beginnings of talking about relations between numbers as abstract objects, as opposed to their practical application as recording the act of counting.

Young children also understand relationships between people, and can articulate them, often embarrassingly. It is said that a two-year-old is the best guru you can have. Watch one go around a room full of adults and systematically elicit reactions from every one. Sibling rivalries and playground positioning are more evidence. This is not to say that their awareness is conscious or their actions deliberate—but at some level young children understand complex human relationships. Why not mathematical ones?

It is noted in Part II that Keith Devlin describes mathematics as the same sort of activity as gossip. That is, mathematics is talk about relationships, but at a higher level of abstraction: it is about relationships between mathematical ideas, not between people. The important thing he notes is that it is the same kind of talk (Devlin, 2001, p. 244):

> … [A] mathematician is someone for whom mathematics is a soap opera. … I am not referring to the mathematical community but to mathematics itself. The 'characters' in the mathematical soap opera are not people but mathematical objects … . The facts and relationships .. are not births and deaths, marriages and love affairs, but mathematical facts and relationships. … The secret of all those people who seem to be "good at maths" [is] not that they have different brains. It's just that they have found a way to use a standard issue brain in a slightly different way.

Given that mathematics is created in communication, that mathematics happens in the act of gossiping, then the trick to doing mathematics is to do what everyone has no difficulty doing, but do it with abstract ideas. There is good evidence that young children do know about relationships and act on that knowledge. There is also good evidence that they can play with relationships in an abstract way: they play games with rules all the time, and they both articulate and manipulate rules explicitly. Furthermore they can play games with rules about mathematical ideas also. Children do not need to have 3 follow 2, they do not need to have the 'correct' number of objects to refer to. They can suspend their dependence on reality if that is part of the game. All young children can do mathematics in this very real sense. All older people can too.

A relevant question to be asked is how this ability can be nurtured. How can I go about increasing my ability to think and act mathematically? A likely answer is to practice 'gossiping' with abstractions as often as possible, or, if I am responsible for young children, to play such abstract games whenever the opportunity arises. We need to establish a wide base of real experiences from which to abstract, and we should develop a large background of gossiping about abstractions. Advanced mathematical development is unlikely to happen until there have been a lot of abstraction experiences.

In the light of this conclusion it is interesting to note that a common educational response for children who are having difficulty with school mathematics is to give them more concrete problems, to reduce the abstraction by giving problems for which they can refer to real world situations. This strategy does increase the base of real experience, but it does nothing about increasing the base of abstract activity that is also needed to appreciate formal mathematics. In many cases sufficient real experience is already present, and so a better strategy would be to undertake abstract activity in an appropriate way—that is, at a level of game-playing rather than within formal mathematics.

A reflection on young children's development of the ability to gossip about abstract things is suggested by Oliver Sacks' (1991) book about sign language. Sacks presents the evidence that the groundwork for the ability to understand language as a concept is laid down before age eight. In other words, if a child has not experienced language by this age, if, say, they have been isolated from speaking human contact, then they will never really 'get it'. Even if they subsequently join a language community, they may learn to communicate, but will never properly develop linguistic skills. To the extent that mathematics is like language as a cognitive function, we can infer that the same is true: if there is no experiential base of abstract gossip before some early age, perhaps it will never fully develop. Could this be the key to Math Phobia (Burns, 1998; Clawson, 1991; Tobias, 1995) or the widespread phenomenon of people who say they never understood mathematics beyond routine and real world based arithmetic and geometric activities?

Another feature of children's mathematical gossip (that is, children's abstract play) is that they are explicitly aware that this is a game, that there are rules, and that the relationships are under their control. This feature sometimes disappears in a formal mathematics classroom. Mathematics is not an inevitable body of knowledge. Understanding it and doing it requires a consciousness of the 'rules' and the awareness that they are rules or conventions. Such awareness is particularly needed at the early stages where we often act as if there is nothing to be surprised about. The examples of fractions and multiplication are cases in point. In the real world multiplication is never commutative—it is only the abstraction of multiplication that is commutative. Often the numbers do not even represent the same kind of thing: 5 packets of biscuits at $3.80 each cost $19.00. Two of these numbers represent money, the other represents a counting number. We expect children to multiply in this situation, and to understand that multiplication is commutative. They need to know that this is the game.

Formal mathematics language is subtly different from everyday gossip. Think about the codes that develop amongst small communities of gossipers: phrases that take on special meanings so that an outsider might not get the full meaning of a statement. Mathematics has its own codes. The unexplained introduction of the codes of mathematics (that is, mathematics that is already formalised) may cause confusion. For example, at the elementary level, if a child is familiar with numbers describing "how many" and then, without apparent change of discourse, they hear numbers talked about as objects that can be manipulated independently of things being counted, it is no wonder that they become confused at what is going on.

Having a language that is in congruence with mathematics may be a two-edged sword educationally. On the one hand, it seems that there will be no cognitive disruption for students approaching mathematics. The way they have used numbers in everyday conversation will slowly evolve into the mathematical use of numbers, and no troubles will result. On the other hand, perhaps the way that concepts change without being noted or explained causes some of the problems experienced by young children? Is this the cause of widespread claims that people do not have a mathematical mind?

Reasons for concluding that such difficulties exist can be found in the history of mathematics. Rotman (1987, p. 8) records the difficulty mathematicians had with transforming the idea of nothing into a number. How can nothing be something? Nothing is the absence of something, even the absence of number—it cannot be a number itself. Are such difficulties replicated in some mathematics learners today, or does their common experience of zero appearing on a calculator overcome this particular language shift?

Mathematics as abstract gossip—the idea has led us to think that children should have more abstraction, not less, and that being aware of the rules of the game is an essential feature. What other implications can we draw for mathematics learning?

3. FROM 1 TO 100: PLAYING & EXPLORING

It is a curious feature of mathematics education that we expect and encourage exploratory and playful mathematical activity in very young children, and in advanced research mathematicians, but in between we sit students down to do exercises and listen to teachers or lecturers explain how it is. There are now secondary school classrooms where exploratory activity is encouraged, but as soon as the spectre of national examinations or international testing surveys looms close, mathematical activity reverts to closed exercises and transmission teaching.

I acknowledge some reasons for this: bureaucratic pressures to cover a defined syllabus; policy pressures to report in a particular way; pragmatic pressures to help students respond to the types of tests they will face; and teachers simply doing what they know best. However, from the point of view of learning to do mathematics well and effectively, and in order to experience the joy and beauty of mathematics, the removal of exploratory and playful opportunities from learning activity at secondary and tertiary levels is a very strange thing to do.

First of all, mathematical exploration and play is always possible, at any level. Within the environment of existing mathematics there are (have always been) educational resources full of wonderful open questions. However there is something more. We concluded above that mathematics could have been different. This conclusion does not just apply to research mathematics—it applies from the very first experiences with numbers and shapes, to beginning algebra, to practical and theoretical statistics, and to any branch of advanced mathematics. We can always do mathematical exploration outside the confines of NUC-mathematics. It is nearly always possible to change some basic assumption of mathematics, and to genuinely explore or play in a new environment. The Double Origin and Active Geometries discussed in Part I are examples.

Exploration and play are always possible: is it always a good idea to do it? One reason for playing with mathematics is because exploration is an interesting and efficient way to exhibit the nature of mathematics. Mathematics could have been (still can be) different. There are many untapped potential ideas that can be explored, and may even turn out to be useful or applicable. Having mathematical ability includes an attitude towards mathematics that assumptions can, and should, be questioned, and that changing (or creating new) assumptions leads to new ideas. Experiencing mathematics outside the normal conventions is the most direct and the most powerful way of developing these attitudes.

Another reason for mathematical play and exploration is that the ability to change mathematical contexts deliberately is part of the skill of doing mathematics. Not only is it necessary to be able to question mathematical assumptions, it is also necessary to step outside conceptual conventions. Many of these conventions are unrecognised and language-based, thus developing mathematical ability includes taking every opportunity to practice “thinking outside the square”. Both changing assumptions within mathematics and conceptualising in original ways are useful habits at any level. We need to keep challenging established ideas at every stage in our learning.

Questioning and challenging assumptions are not just useful habits, they are vital skills for a mathematician. We should therefore be particularly concerned that exploratory and playful activity is largely absent in undergraduate mathematics—this is exactly where it should be most in evidence. In these classes we have collected the best young mathematical minds a society has. Why are they deprived of a mathematical activity that is both one of the most pleasurable and also one of the most important for their future work? The freedom of university study where new ideas and novel learning experiences abound is exactly the right environment for exploratory mathematics, but in mathematics at this level the approach is usually more closed and structured than ever before.

Not only is mathematical play and exploration necessary to understand the nature of mathematics, and necessary to be able to do mathematics, it is also necessary for the process of learning mathematics. Mathematics is created in the act of communicating human activity directed towards making sense of quantitative, relational, and spatial aspects of the world. Many argue that learning mathematics must reproduce the historical development of mathematics, the ontogenetic argument (Fauvel & van Maanen, 2001). If this is accepted then reproducing the exploratory experience is a vital component. However even if this argument is not accepted, there is still a need for these activities during learning. Mathematics is the abstract systematisation of experiences; it is a process as much as it is the result of a process. Learning mathematics cannot therefore just be learning about the completed system, it must also be learning the process—and there is no way to do that without undertaking the process. You cannot learn to drive a car from a book about driving.

Communication is a key element of the process. In order for communication to happen, not only do we need relevant experiences to communicate about, but we also need to have a reason to communicate, and, just as important in this case, a need to communicate formally. If the communication is about pre-formalised mathematics, then students will not learn the process of formalising for themselves.

In other words, they need original mathematical experiences to communicate. They need to be excited enough about them to both want to communicate them, and challenged enough to communicate about them precisely. Play and exploration are the first stage of this process.

4. CREATING MATHEMATICS THROUGH TALKING

What does the conclusion that mathematics is created in the act of communicating mean for a learner? What are the special requirements of this form of communication?

I now need to abandon the metaphor of gossip. Mathematics is not gossip. Devlin only says that doing mathematics is in some ways like gossiping. Mainly, they are both about relationships. There are other ways in which mathematics is not at all like gossip. We can easily associate the adjective 'idle' to gossip, but formalised mathematics is far from idle. It is purposeful and directed. Gossip is rarely reproduced exactly: it is usually elaborated and embellished. Mathematics is deliberately created in such a way that it can be exactly repeated. Gossip thrives on ambiguity, suggestion, and nuance. Formalised mathematics, on the other hand, needs to be as precise and unambiguous as possible.

Learning mathematics is learning to communicate in particular ways about relationships. Part of learning the ability to formalise includes understanding the reason for formalising. Only through communicating back and forth can the need for precision of meaning become evident; and it is only by passing ideas through chains of communication that the need for reproducibility is experienced. Important ideas need to be communicated, and the more important they are, the more accurately and consistently they need to be communicated. Mathematical ideas must be systematised to be communicated, thus mathematics is created. This is why mathematics and language develop together.

In this process a mathematical world is created. Mathematics and language evolve together to create a world that is not the same as the real experiences from which it originated.

Take the example of 'membership'. The concept is a familiar one: we are members of a family, we are members of clubs, we hold membership cards to prove the relationship. Membership means that we are included, that we are part of a group

Here is a very simple problem involving family membership and its internal relations. Maria and Pedro Oliveras (who have since died) were married and their children were four sons: Carlos, Salvador, Garcia, and Juan. These sons are now the only members of the Oliveras family. One day, in a cafe, two members of the Oliveras family are standing at the bar. Are they brothers? Yes, they must be.

Now here is a mathematical problem that appears parallel, but is not because 'member' has a subtly different meaning. Let S be the set containing the four sons of the Oliveras family: Carlos, Salvador, Garcia and Juan. Mathematically this is written: S = {Carlos, Salvador, Garcia, Juan}. Thus each son is a member of the set S, we write Carlos $\in$ S. A mathematical question is: will two members of the set S always be brothers? The answer is "No". The reason is that, mathematically, the same member may be selected twice. That is "two members of the set S" includes the possibility of 'Juan' and 'Juan' being chosen. Juan is not his own brother.

This is confusing because we do not usually apply mathematical membership to people. The problem clarifies a little if I change set S to be the collection of *names* {Carlos, Salvador, Garcia, Juan}. Now if I ask two students to each choose a name, and ask whether the people corresponding to those names are brothers, it is more obvious that the two students could choose the same name.

It clarifies even further if I ask a parallel problem about numbers. Let B be the set {2, 4, 6, 8}. Let x and y be members of B and add $x + y$. Will the result always be a number between 5 and 15? No, because x and y can have the same value—they can both be the same member of the set B. For example they could both be 2. Now 2 + 2 = 4 (which is less than 5). Or both could be 8, and 8 + 8 = 16 (which is more than 15).

However this confusion of the meaning of membership and choosing x and y from a set is a common one. Ferrari (1999) did some research with a similar example using undergraduate mathematics students and found that even in a clearly mathematical context at an advanced level, the everyday meaning of membership interfered with their understanding of the mathematical question.

The mathematical world is not the same as the experiential world. The language changes, as do the concepts. Learning to be part of that world involves learning how it is created, and students need to experience the process of evolution. Being presented with the mathematical world, its concepts and language completely formed, will not help anyone to learn to be part of the evolution.

The formalisation of mathematical communication is not just a record of abstraction, it is also a way to enable abstraction to happen. Mathematics is not just gossip about abstraction, it is the formation of abstraction through communication. Once an abstract idea has been formalised it is available for further abstraction, an idea described by Piaget (1953), and developed for advanced mathematics as APOS theory by Dubinsky (1991). Once an idea has been formalised it is available again, layer upon layer of abstraction. For example, the joining of two collections is formalised as the arithmetical operation of addition; addition and subtraction and other operations are formalised as algebraic binary operations; binary operations and the objects they operate on are formalised as group theory; groups and their fundamental properties are formalised as topology; and so on and so on. This layering of abstraction is the real depth of mathematics, and is a clear example of the way mathematics and language must develop together. It also makes clear that learning mathematics must involve communicating about it.

The mathematical learner has one further task. A mathematician must also be able to talk about the process of abstraction in which they are engaged. There needs to be a meta-level language so that the mathematician can discuss the possibilities available for abstraction in any particular situation. The evidence from language shows that there are usually several directions we can take when making an abstraction. In order that a fuller range of choices is available, and that mathematical (rather than linguistic or experiential) decisions are made between them, the process needs to be articulated.

5. SOME THOUGHTS ABOUT TEACHING MATHEMATICS

Before we consider the act of teaching, a few words on why we want to teach mathematics. If mathematics is not the highest expression of human thought (as Plato claimed), or even the science of what is clear by itself (as Jacobi suggested), then why should it be such a pervasive subject in our learning institutions? If it is, as I have claimed, a language dependent, context dependent, historically dependent view of the world, why is it endowed with such importance? If NUC-mathematics is not the only one possible, why does it have pre-eminence in curricula world-wide?

Mathematics is important for all the usual reasons: NUC-mathematics is the foundation of science and technology. It provides a suite of techniques and tools for business, engineering, medicine, architecture and design, navigation, astronomy, social science, and many other fields. It continues to enthral many great minds. Mathematics does turn out to be beautiful as well as unexpectedly effective (Wigner, 1960; Hamming, 1980). (Both its beauty and effectiveness are sourced in its connections with language and the evolution of abstract ways of thinking based on human experience). These reasons would suffice for mathematics' place in education. But another reason for teaching mathematics emerges from this book: mathematics helps us make personal sense of the world.

Now let us turn to teaching. What are the implications of the conclusions from language for those who facilitate, design, or control mathematics learning?

Note that all the ideas about learning detailed above have their parallels in teaching. If abstraction activities are needed at an early age, then teachers have a responsibility to provide them. Those responsible for young children can (and do) play many pre-mathematical games. They play with numbers in ways that do not involve counting; they draw plans of buildings and playgrounds, they draw maps of neighbourhoods, and they ask questions about the numbers, plans and maps. They tell stories that involve classification systems, and relations such as inclusion and size comparisons. These are all abstract experiences in quantity, space and relations.

Games can also be played with argumentation and logic. I once watched my brother at the zoo with my daughter (who was about four or five at the time). "There's a big animal," he said, looking at a rhinoceros, "it must be an elephant". "No," came the reply, "it's a rhinoceros". "But it's grey, and elephants are grey," he responded. "But it has a horn," she replied. "So have elephants—they have two ivory tusks and this has got two horns". "But elephants have trunks". "This elephant hasn't grown it yet". And so it went on, he forcing her to justify her statements, and countering them, and continuing the argument with other elephant characteristics (ears that flick, tail with a tuft of hair, mud on its legs, the noise it makes—of course the rhinoceros did not make any noise so she could not deny that he had got it wrong). It was not long before she turned the tables on him at another cage. The game of false or incomplete logic with ridiculous conclusions has been a family staple ever since.

I argue above that playing and exploring are vital parts of learning mathematics at all levels. Teachers, therefore, have an important role to provide opportunities, to model such activity, and to value it within their courses. This applies at university level, as much as it does in schools.

A teaching implication of the way mathematics is created through communication is the need to be explicit about the difference between everyday and formal mathematical talk. For a teacher not only does this mean that they should talk about this difference, for example, when discussing set membership, but they should also point out places where our everyday language is not quite adequate for mathematical discourse. An example arises from the unique feature of the Dhivehi language referred to in Part I.

In Dhivehi, we can refer to 'the book' by using the root word for book, *fot*. We can also refer to 'a book' meaning a particular but non-specified book, as in the sentence 'John was carrying a book when he fell into the water'. This sense of book is indicated by the suffix *–aku*, thus *fotaku*. There is a different word if we wish to refer to any book at all, as in the sentence 'John asked for a book to put on his papers so they would not blow away'. Here the book is a general book from the class of books. In Dhivehi this sense is indicated with a different suffix: -*ek*, thus *fotek*. The distinction is sometimes important in mathematics, but can be overlooked. An example occurs when drawing graphs of functions.

In the graph in Fig. 8-1 the variable x and the function $f(x)$ are each used in two ways, and these ways are different in the same way as the two different uses of 'book' described above. The 'x' in the expression $f(x) = 2x^2 + 1$ is any value of x at all—x is a variable. But the meaning of 'x' in the label P(x, $f(x)$) and the label on the horizontal axis is a particular, but unspecified, value of x. In this situation it is more correct to label the particular value as x_1, but often teachers do not do this, and slip between particular and general uses of a variable without thinking—to the confusion of their students.

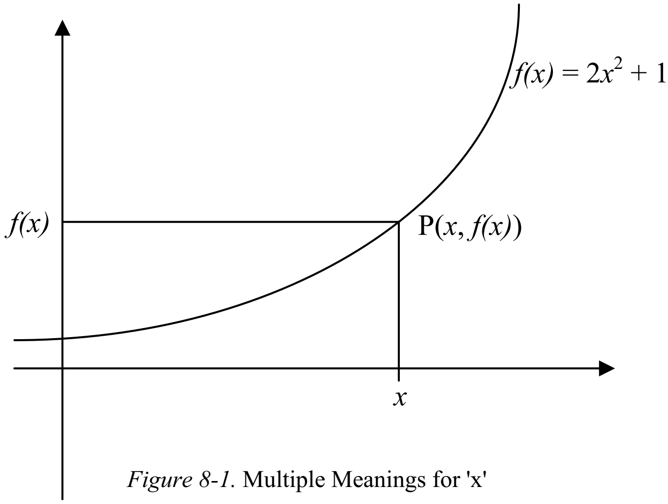

Figure 8-1. Multiple Meanings for 'x'

One more implication. The need to communicate, the need to play, the need to explore, and the need to learn about mathematics means that those charged with teaching the subject must themselves be more mathematically literate than ever before. If a teacher is to recognise, follow, and utilise the diverse mathematical thinking of children, then the more links, experiences, and applications on which to draw the better. They must know other ways of approaching the same idea; they must sense different directions in which the idea can be taken; they must be able to make use of cognitive conflicts that arise and new situations the children imagine.

In a world where the mathematical background of teachers is a cause for concern in many countries, an increasing mathematical demand on teachers may not be welcome—but it cannot be ignored.

6. NOTES ON ASSESSMENT

First, let us remember that all the learning activities described above are linked to assessment. They are linked both because assessment is part of the pedagogical process, and they are linked because most formal institutionalised learning has a summative assessment requirement.

Assessment can be a way in which aspects of the pedagogical process are valued. Mathematics is a gatekeeper well beyond its real status, thus those parts of mathematics that are measured become part of the process for deciding on vocational and educational opportunities. Hence exactly those parts of mathematics receive focus and teacher input. So much is commonly understood.

The problem is, that if the conclusions from language are true, then what is needed for successful mathematical activity is exceedingly difficult—if not impossible—to measure. How do you evaluate creating a base of abstract experiences? By their definition, experiences are many and varied, and you cannot know in advance which ones will be used in later, formal, mathematical, abstraction activities. How are playing and creativity to be measured? The very act of attempting to measure them will kill them as play or as creativity. How is communicative mathematising to be measured? This latter might be partially possible with one-on-one interviews and recording group activities, but is hardly practicable as a routine for all students. The Numeracy Programme developed in New Zealand in 2002 onwards does just this: teachers are helped to evaluate each child's position on a framework of mathematical development through interviews. Irrespective of possible benefits, the practicality of such interviews as a regular part of the mathematics classroom leaves little time for other types of teacher/student interaction.

We are left on the horns of a dilemma. Either these vital features of mathematics education are not assessed and will not be valued (probably leading to being neglected by teachers and students alike), or they are assessed badly at a high cost in terms of time, teacher resources, and impact on the activity itself.

I am convinced that we need to wrestle with the first of these horns, not because of the resource cost of the second (if it is important enough, resources are usually found), but because assessment will ultimately kill these vital activities.

This means that having a variety of abstraction experiences, indulging in mathematical creativity and play, and communicating mathematically all need to be given high value in some way other than by assessment. This can be done by individual teachers—but such a solution is unlikely to be universally adopted. Another strategy is to highlight this activity amongst mathematical practitioners (not just mathematicians, but also system analysts, designers, engineers, information scientists, and so on).

The conclusions of this book lead us to downplay assessment for further reasons. Two of the conclusions about mathematics are, first, that mathematics is in continual formation, and second that mathematics is open in the sense that it could have been otherwise. The first means that it is never finished, the second means that there is always another way of perceiving, conceiving, or receiving mathematical ideas.

Given these parameters, assessment of learning mathematics as a whole is impossible. Assessment needs to be against something, a framework, a standard, another performance. But if mathematical learning is forever unfinished, and if it proceeds along any of a myriad of pathways, then there is no way of creating the basis for judgement. Any assessment that takes place compromises the nature of mathematics.

A final note on assessment concerns the conclusion that mathematics and language develop together: it is not possible to have mathematics without language.

A research project aimed at investigating the situation of senior secondary students with Mandarin as their first language learning mathematics in English involved giving them the same test in English and Mandarin (Neville-Barton & Barton, 2004). These students had done all their education in China except for the last few months. Their English proficiency was not high. Not surprisingly, the performance was better in Mandarin, but a large variation emerged between questions. Students performed the questions with technical vocabulary, complex syntax, or an unfamiliar context much better in the Mandarin version. One question, however, was done better in the English version. This was a question involving the concept of *gradient*.

The teachers reported that this concept was the only one in the test which had been taught for the first time in English, and that the term does not translate easily into Mandarin. This may explain the result, but it begs the question: what is the true mathematical understanding of these students? If some parts of mathematics are understood in English, and others in Mandarin, then what sort of test can evaluate mathematics? A bilingual test is not the answer, because even within one language there are many ways of expressing an idea, and many different associations for what might appear to be the same mathematical process—take, for example, the concepts of anti-differentiation and integration.

The chimera of mathematical ability, let alone the measurement of this ability, disappears into the mists of language, no matter how precise we think we are.