Chapter 7

WHAT IS MATHEMATICS? PHILOSOPHICAL COMMENTS

Abstract: This chapter addresses the issues that have been raised so far from a philosophical point of view. An extended metaphor of Middle Earth is used to describe a more relativistic view of mathematics.

Keywords: philosophy of mathematics, relativism, applications of mathematics

1. MIDDLE EARTH

What about the Plato's ideal world inhabited by mathematical objects? Does it exist? Yes. Is a circle real? For sure. Is there such a thing as a prime number? Of course there is.

That world and the mathematical objects in it exist just like Tolkien's Middle Earth. Mathematics is a created world, a world of the human imagination, and, like Middle Earth, we can write about it, film it, become part of it in our minds and emotions. Also like Middle Earth mathematics has been expanded upon by others apart from Tolkien (despite his family's best attempts to preserve copyright), notably as Peter Jackson's film crew and actors gave more substance to the appearances and actions of the creatures and environment that make up that world. Mathematics has, it is true, a longer history, and many more screenwriters, but it can be thought of as an academic Middle Earth.

Can mathematics be compared to such a flight of imagination? Isn't there something much more contingent, much more true about the mathematical world than there is about Middle Earth? Once we have the number 1 and the number 2, then no mathematical Tolkien could have written anything other than $1 + 1 = 2$. Once we construct a circle and its diameter, and then draw a triangle on the diameter to a point on the circumference, it is not just geometric poetic licence that says that the angle at the circumference will be a right angle. And it is not just a muse's whisper that requires that right-angled triangle to have sides that obey the Pythagorean relationship. These things must be so.

The mistake is to think that this situation does not exist for Middle Earth. If you are a hobbit of Middle Earth, and you get yourself into deep trouble with the Forces of Evil, then, in your moment of dire need, lo, the Elves will come to your aid. It cannot be otherwise. For if it was otherwise it would not be Middle Earth!! It would be some other fantasy. The elements of Middle Earth were created in just such a relationship. Elves help humans: that characteristic is part of what Elves are. It was determined by their history (as written in the book) 2000 years before the time of Frodo and Sam and Bilbo Baggins. (Notice how we talk of these people as if they were real, with real histories, real names, real lives to be led—just as we talk of numbers and circles as if they were real objects that can be held, turned over, or combined with each other).

In the same way, if 1 + 1 does not equal 2, then we are not talking about the world of mathematics, we are in some other world. The number objects 1 and 2 were created into just the relationship embodied by 1 + 1 = 2. That is what mathematics is. Circles and triangles and angles were also created into their relationships.

But when Tolkien wrote *Lord of the Rings*, he had all the relationships and consequences worked out in advance. As the mathematicians write mathematics, the consequences of some of their supposed imaginative constructions are still being discovered, many are suspected but not yet proven, and still more are not yet known—or so the hundreds of budding mathematicians hope.

In order to properly understand the nature of mathematics, it is necessary to think of Lord of the Rings the computer game. Version 1 will closely resemble the book, and the relationships will be preserved intact, and the game will involve consequences of alliances, with some randomised luck thrown in: outcomes of individual battles; weather; perhaps the timing of the crumbling of the bridges in the Mines of Mordor. But then Version 2 will come out. A new, improved version. A few more subtleties. Perhaps some group of Elves will remember that they also had a 3000 year old pact with a group of Orcs who helped them in a time of need, and if, in the grand battle, this group of Elves come up against the Orcs then they might walk away. The writer of Version 2 will inject some of his or her own imagination, and may create some additional history for the characters. This will not be anything that contradicts Tolkien's original vision, but Tolkien did not have time nor space to write the complete history of Middle Earth, to describe each and every possible combination of relationship and circumstance. Plenty more can be written that is consistent with the copyright original.

The Version 2 Elves are not the same as Tolkien's Elves then? Yes and no. The computer game will be recognisable as Middle Earth, no contradictions will be involved, but the Elves will have evolved under new requirements. This has also happened in mathematics: Euclidean geometry with its circles and triangles and embedded relationships is now viewed as one of many possible geometries depending on the axioms. Non-Euclidean geometries, such as projective geometry, do not contradict Euclidean geometry, but evolved from it with new writers and new consequences as the literature of mathematics continued to be written.

Eventually, there will be Middle Earth Version 3. In this multi-dimensioned, multi-media extravaganza, Middle Earth is seen to be part of a greater world. Bilbo Baggins returns from his travels bringing new technologies and new perspectives. Middle Earth is a very special case of a universal fight between good and evil, tyranny and justice, truth and falsehood. The new technologies allow the Hobbits to understand more about what is right and to use this knowledge in their lives. The horizons for creating new beings and new relationships extends indefinitely, although Middle Earth remains intact as the literary historical origin of the edifice built upon it.

Now *that* is like mathematics. The words 'dimension', 'technology', 'special case' and 'new relationships' were used in the previous paragraph with intent. The parallels I leave for the reader.

The idea embodied in calling mathematics Middle Earth is not original. Wittgenstein seems to be saying the same thing when he says that a mathematical statement is a prescription or a rule (1956, I–30, 33). That is, every mathematical statement is saying "This is how it should be if you want to be in a mathematical world"—just as Tolkein's descriptions of Orcs can be seen as statements of what certain things are like in the Middle Earth.

Middle Earth is also a braid with many origins. If we regard Tolkein's novel as an allegory on good and evil, then it is one of many such allegories arising in many different cultures. They exist alongside each other, borrow from each other, can be discussed in relation to each other. This is parallel to mathematics being seen as a QRS-system alongside other QRS-systems.

Even within Tolkein's creation there could be many fibres, as copycat writers pick up his themes, or abridged versions for young children or radio plays emerge, or a modern writer picks up an incident in the story and creates a new work.

Notice also that Middle Earth bears more than a passing resemblance to our experiential reality. We recognise the practical level (the characters ride horses, live in houses, and suffer the weather), the contextual level (things fall under gravity, landscapes are earth-like), and the human level (the morals, emotions, and physical constraints like wounds and illness are all familiar). Tolkein's world is consistent and speaks to us about our experience. Just like mathematics.

2. MATHEMATICAL WORLDS

The evidence from language, and other reflections, have led us to the idea of a braid of many strands. The strand that is NUC-mathematics has been discussed above. What about the others? These different strands have sometimes been referred to as different mathematical worlds.

The idea that there can be several mathematical worlds is far from new. In Western literature, it was described by Oswald Spengler (1926, 1956). His grand conception was that a mathematic (singular) was a feature of each cultural era (like art or architecture), and that all such features grow, flourish and decline contemporaneously in every culture. Spengler focussed on the conception of number. Number, he claimed, is a representation of thought, of a conception of the world. The difference between the Classical idea and the modern Western idea of number, for example, is that number is regarded as measure-ment in the former, and as a relation in the latter. The important point made by Spengler is that this is not a development, but in each era there is a destruction of the concept of number of the previous era, and the generation of a new one. Eighty years after Spengler, the question of whether mathematics develops gradually, or whether old concepts die and are replaced by new concepts, is still being debated (Gillies, 1992).

Mathematical worlds are also discussed by Sal Restivo (Restivo, 1983, 1992; Restivo, Van Bendegem, & Fischer, 1993). He believes that "all talk about mathematics is social talk" based on the Marxist view that all human activities are social activities and social products. Restivo notes two interpretations of the relationship between mathematics and culture. The weak view is that mathematics is a social and cultural phenomenon, so that mathematical ideas and activities vary from culture to culture, and that the results of the various cultural mathematics together make up world mathematics. The weak view is adequately demonstrated by examining mathematics from different historical periods and cultures. The strong view challenges the idea that all cultural traditions in mathematics contribute to the same mathematics. Rather it assumes different mathematics' and incommensurability between them.

This leads Restivo to describe 'math worlds' (Restivo, Van Bendegem, & Fischer, 1993, p. 249–50). He notes that pure mathematical concepts appear objective when they are communicated, hence mathematics is a social world of people communicating about their ideas—agreeing, disagreeing, arguing. Mathematics is not a world of triangles, symbols, rules of argument; it is a world of networks of people talking about ideas. The social practice generates the objects and the results of mathematics through naming and arguing.

Both Spengler's and Restivo's views are culturally based and refer as much to the nature of the surrounding culture as they do to the nature of the mathematics in the worlds being described. Can we get closer in to the mathematics?

Do different mathematical worlds mean, as asked in the Introduction, that mathematics as an academic discipline is somehow different in different parts of the world? A bridge designed using mathematical theory surely stands (or falls) in the same way independently of the country it is built in, or of the language of the person who solved the equations of its design? Surely 1 + 1 = 2 in Alaska, Nigeria, Tahiti and Singapore?

Think about the bridge for a moment. The technical part of building a bridge involves resolving the forces that might make it fall down. This is largely an empirical matter—does it fall down or not? In 2003 the remains of a Bronze Age bridge was found in Wessex, England. There were no mathematicians around in England during the Bronze Age, according to histories of the subject, but clearly there were effective bridgebuilders. There were mathematicians around in 25BC to help build the Pont Saint-Martin, one of the oldest surviving bridges, but their mathematics would not have included the techniques that would be used today in the same circumstances. One wonders what mathematics was being used in 1756 by William Edwards, the builder of the Pontypridd Bridge. He had to reconstruct the bridge several times before he got the rise-to-span ratio correct: an expensive trial and error procedure.

The point here is that the mathematics is how we make sense of the technology we need. The experiences of bridge-building, the talking about these experiences and what ideas explain them, the use of mathematical techniques that have been developed in other situations, lead mathematicians to develop effective ways of describing bridges before they are built and communicating about whether they are likely to stay built (that is, effective ways of designing them). A bridge does not stay up because of the mathematics. It stays up because it is built effectively. Mathematics is one way of discussing what "built effectively" means.

So, yes, a bridge designed using a mathematical theory stands (or falls) in the same way independently of the country it is built in, or of the language of the person who solved the equations of its design. But many mathematical theories may adequately describe why the bridge stands or falls. The techniques of engineering mathematics are wonderfully detailed and can cope with a vast range of potential bridges—bigger and bigger as the mathematics and the materials develop. The mathematical theory used to design the bridge stands (or falls) on the success of the bridge—this statement is not the same as saying there is only one possible mathematical theory.

Notice that the "correctness" of the mathematical theory is something of an empirical matter. If the bridge falls down then the builder needs to think again (like William Edwards). More accurately, the correctness or appropriateness of the application of the theory is an empirical matter. What about the pure mathematical theory itself? Surely it is right or wrong.

Hence we come to $1 + 1 = 2$. Surely the equation is correct in Alaska, Nigeria, Tahiti and Singapore? I need some help here, and I am calling on Wittgenstein.

3. WITTGENSTEINIAN MATHEMATICAL WORLDS

Many different writers have made commentaries on Wittgenstein's writing. I prefer Shanker's (1987) interpretation. He notes that Wittgenstein was concerned that mathematical philosophy should look at how mathematical expressions are used, and at the logic of such expressions, not at whether mathematical expressions refer to anything "real" or not. When this is done, it becomes clear that mathematical expressions are rules, not descriptions. Mathematics is neither a description of the world nor a useful science-like theory: it is a system, the statements of which are the rules which must be used to make meaning within that system.

Grammatical analysis reveals that sometimes we use mathematical expressions as if they were part of familiar syntactical domains, and Wittgenstein believes that this is the source of traditional philosophical argument. For example, treating '15' as a thing, and its divisors as discoveries to be made is a Platonist/realist domain; or treating a mathematical 'group' as an arbitrary construction which could have been otherwise is a constructivist domain. At different times either of these grammatical similarities seem more appropriate. However, we cannot thereby argue that one or the other is correct. Mathematical syntax has is its own domain to be analysed for its logical grammar irrespective of how, or when, it is similar to a Platonist or to a constructivist domain.

Let me note again that English, the language we have come to use for mathematics, tends to make mathematical ideas into objects. We talk of mathematical objects because that is what the English language makes available for talking, but it is just a way of talking. Bishop (1988, 1990) identifies the objectifying tendency of mathematics to be one of the values inherent in the subject. For NUC-mathematics, this is because of its Indo-European linguistic roots. A non-objectifying mathematics is possible.

Wittgenstein claims that mathematical statements are normative descriptions of how the world is seen, of what is meant by being intelligible. We cannot have 'intelligible' communities who divide by zero, or who calculate 24×30 as 712, or who measure differently, because such communities would not see numbers and counting as the same sort of thing or activity as we do, thus they would not be intelligible. 1 + 1 = 2 is always the case because this is the standard of the correct use of numbers in the discourse of the mathematical world we inhabit. That is how we agree to count. It does not make sense to say that $1 + 1 \neq 2$ (Shanker, 1987, p. 303).

People in a different mathematical world will not be talking about the same idea if they use the symbols '1', '+', '=', '2' and do not accept that 1 + 1 = 2. The clash of different mathematical worlds is obvious when the same word is used to describe different ideas. We have discussed what happens when mathematicians have different views on continuity or probability, or different cultures have different views on navigation or shape. Any community or culture is free to make its own sense of the world. Mathematics is the name we give to how it chooses to express the sense of quantity, relationships, or space.

Rotman (1987, p. 2) makes the same point with respect to symbols, and similarly rejects the idea of mathematical things being prior to mathematical signs. It is not the case that mathematics was "there", was then "discovered and named", and then remained unchanged.

He compares mathematics with art and finance. We have "the natural but mistaken notion that a painting is simply a depiction and money a representation of some economic reality". That is to say, we often treat pictures as if they showed us reality: "That picture captures the colour of the ocean on a stormy day", we might say, but actually it just invokes in us the sense we have when viewing a stormy ocean, and would not do that if we had never seen a stormy ocean. The lie to pictures representing reality is most clearly found in those pictures of impossible images, Escher's etching of the never-ending steps being a good example.

Similarly, we treat money as if it represented some actual commodity, when what it actually represents is "value", and what that means changes with our actions. This is clear when we say things like: "Bill Gates lost half a billion dollars on the stock market this week". This does not represent any actual change in things that he owns.

It is the same with mathematical signs. Rotman focuses on the role of zero, because it demonstrates that numbers do not represent any thing. If we regarded numbers as representing a reality, even the reality of our action of counting, then zero is a problem, since it represents the absence of that reality. As soon as we allow zero to be a number, then we must give up the idea that some thing is being represented—by zero, or by any other number. Numbers are seen to be signs that we use. They are not things, nor do they represent any thing. "Numbers signify the activity of one who counts" (Rotman, 1987, pp. 8–9).

Both Rotman and Wittgenstein make the point that mathematical symbols and expressions are made and remade repeatedly. These are not individually created, but are public, culturally dependent forms of communication.

If mathematics is the way mathematicians talk, then the cultural influences on that talk (the language of discourse, the meanings of words and symbols at the time of the talk) create different mathematics. If mathematics is a set of normative rules, then they could have been different. We accept different rules of grammar in different languages, and the other ways of talking about the world that those languages generate.

Another mind-game. I once spent some time in Guiyang in Guizhou province in southern China where my wife was teaching English. There I met an American linguist who was studying the indigenous Miao language, a member of the Hmong-Mien family of languages. I asked him whether his American-learned linguistic theory was adequate to describe everything he found in the Miao language. He replied that yes it was, although he sometimes had to bend it a bit, or create new categories within that theory.

Now the mind-game. I then asked whether he thought that a hypothetical Miao linguist, who had studied linguistics built up around Hmong-Mien languages, or Sino-Tibetan languages in general, would be able to use his linguistics to describe American English. The reply was predictable: yes, but probably that linguistics would have to be bent a little. What I now asked was this: after the American had twisted his linguistics to fit Miao, and the Miao had twisted his linguistics to fit American English, would the resulting two linguistic systems be the same? My intuition (and my friend agreed) is that the answer is no. Linguistics can be different. Mathematical worlds can similarly be different.

As Shanker points out (1987, p. 319), the possibility of different mathematical worlds does not mean that mathematics is arbitrary, and thereby opens the way for mathematical anarchy. We are free to construct the grammatical rules of mathematics, but the grammar comes before truth, it determines what makes sense. The rules therefore cannot be true or false. Neither predetermined meaning, nor reality, can be used to justify such rules.

An example that shows that predetermined meaning does not justify the rules of mathematical discourse is when anomalies or contradictions emerge in mathematical investigations. The most famous of these are the paradoxes of self-reference that Bertrand Russell attempted to resolve, but more commonly known as the barber paradox: if a barber shaves all those who do not shave themselves, then who shaves the barber himself? Mathematicians' attempts to satisfactorily define concepts such as sets to resolve the difficulty get tangled up in contradictions and impossibilities within their own frame of reference. Meanings of mathematical ideas evolve. That is to say, the grammar of mathematics, what is accepted as making sense, evolves, as we communicate more and more about mathematical ideas.

A final point about Wittgenstein's mathematical worlds. What happens when different mathematical systems meet? Wittgenstein's answer is that there are no 'gaps' in mathematics. Each system is complete at any moment. It is not waiting to be added to with new mathematics. Thus (Shanker, 1987, p. 329), any connection between two worlds is not in the same space as either of the worlds. The interconnections are not waiting to be discovered. We choose whether or not to make connections between systems, and if we do then the connections create a new system.

4. MATHEMATICS AND EXPERIENCE

We have looked at bridges (applied mathematics) and 1 + 1 = 2 (pure mathematics). Let us say a little more about the relationship between the two. When do numbers apply to the real world?

What we forget most of the time is that numbers are mathematical ways of talking, they are not aspects of the world. In some situations numbers (as they have been constructed in mathematical talk) are useful models of the real world, and sometimes the ways we use numbers mathematically do not fit at all to the quantitative aspects of the world we wish to talk about.

We saw this happening in the story about fractions. The "rules" for being sensible with numbers, including fractions, do not apply in every situation in which we wish to represent quantity by one number divided by another. It is possible to make the rules apply by putting alternative interpretations on the word "add" and then using the rules as normal. This example highlights the relationship between the mathematical world and reality. I can make the mathematical world apply by interpreting the situation—mathematics does not just apply automatically. It is not real. Nor is the situation that fits the rules (in the fractions case adding pieces of pies) more mathematically correct or privileged than the other situations.

The next example shows how we move between contexts within mathematics without acknowledging their differences. First, consider counting a large stock of books. Let us say that there are 25 cartons of 50 books. How many books altogether? We know that the mathematical way of talking called multiplication can apply to this situation: multiply the two numbers together: 25 × 50 = 1250 books.

Now consider the measurement of rectangular areas. We also regard this as a matter of multiplying two numbers representing the length and breadth of the rectangle. Thus an area 25 metres by 50 metres is calculated : 25 × 50 = 1250 square metres. But these are not numbers in the same way that 25 is the number of cartons of books. For area, the 25 and 50 represent measurements which have errors. (Mathematically we take account of this by modelling them as Real numbers. The number of books and cartons are Whole numbers, or possibly fractions in the case of the cartons).

We only know measurements within a certain accuracy. It is extremely difficult to measure anything to four significant digits, let alone five or six. In this example, giving normal rounding off, 25 and 50 could represent values as high as 25.4 and 54 or as low as 24.5 and 45, respectively. Multiplying these maximum or minimum values gives areas of 1371.6 sq. metres and 1102.5 sq. metres, so the range of possible actual areas varies by over 260 sq. metres, about 20%.

We explain away the discrepancy by a theory of errors—or in a mathematics textbook we just say "an area of exactly 25 metres by 50 metres" which is nonsense. We act as if we can interpret both situations by the same meaning of number, and disguise the fact that they are fundamentally different conceptions.

Their similarity is sufficient for a large range of practical situations, of course, but it can lead us into trouble. If we come to think of multiplication as the same as these applications, then we will have problems when we multiply negative numbers.

5. RECURRENT HISTORY: BACHELARD

We have examined mathematical objects, and seen that they are possible creations within a mathematical world generated by a social community. But what about objectivity? Is there no way of judging between different mathematical worlds on some objective basis? Is there no way to dismiss as nonsense some mathematical worlds that purport to make sense of our experience of quantity, relationships and space?

First, how do we explain that, historically, our idea of what constitutes rationality has changed? The French philosopher Gaston Bachelard, writing in the 1930s, makes an attempt. He describes a historically relative notion of objectivity which allows for changing conceptions of mathematical objects and of rationality, (see Smith, 1982; Tiles, 1984).

For Bachelard, mathematics allows us to create new realities using new structures of knowledge. Bachelard's key idea is that objectivity is an ideal rather than a reality. At any one time we may think that we see clearly how things are, or that we know how to discover the truth, or that we understand what makes a proof. However these ideas change over time, that is, the sense of objectivity is illusory. Objectivity is not, however, nothing. Conceptions of mathematics at different times depend on changing notions of rationality, each successive change being regarded as being more objective than the last. There is a progression towards a better, and then a still better, understanding of the things that must be taken into account to get an objective view.

A consequence of this analysis is that there are many different historical standpoints from which to view mathematics, each of which is correct *at that time* and each of which explains previous views. Each such view gains its apparent objectivity because of the wide agreement amongst mathematicians about the view, and because it is seen to arise from previous views and encompass them. This historical explanation allows for the development of mathematics over time, and for the changing, creative nature of mathematical ideas, while retaining the objectivity required of the discipline.

Bachelard's idea is called recurrent history because history keeps being re-evaluated in the light of present knowledge: mathematicians look at their own practices and conceptions in the light of other practices and conceptions; modify, reinterpret, discard, or adopt particular practices; and retain the knowledge of how and why this was done as part of their mathematical understanding. These changes are themselves the subject of critical reflection when further advances are made.

I suggest that a similar situation can exist between contemporaneous mathematical worlds. Mathematicians from different worlds can look at their own practices and conceptions in the light of the practices and conceptions of other worlds; modify, reinterpret, discard, or adopt particular practices; and retain the knowledge of how and why this was done as part of their mathematical understanding. In the same way that we do not reject as wrong historical practices and conceptions (only see them as consistent within their historical context and use that knowledge to inform the present), so too could mathematicians from each world acknowledge the other mathematics within their context and use the knowledge to reflect on their own.

If this is true, how is it that not all mathematicians acknowledge that this process has gone on? Mathematicians have a consciousness of change, of what motivated particular thoughts, new ideas and so on, but they are not necessarily conscious that this is a culturally relative process. Most mathematicians regard their subject as universal and from their point of view it is. If opposing ideas arise, whether internally or from a different mathematical world, then there is eventually a cognitive shift to accommodate the clash of domains. When this is achieved the sense of universality returns. It is only when this process is reflected upon that we see the relativity of the past situation. Universality, like Bachelard's concept of objectivity, is an (unattainable) ideal that guides mathematical development. It is illusory in that any claim to universality may be challenged by an awareness of a different culturally-based view; but it is real because, at any given time for any particular person, there is a complete explanation for the domain of mathematical concepts.

Bachelard's description of recurrent history is helpful when we want to describe different mathematical worlds. Particular conceptions of mathematics begin and end, but also live on, in the critical role played by the historical definition of present conceptions. The end occurs when a new conception encompasses the past ones and resolves any conflict that has arisen. Culturo-mathematical worlds are also temporary in the sense that they end when a new world arises out of two meeting ones. However each world lives on in the critical role played by the conflicts of the meeting.

Mathematical practices are quickly accommodated, and can usually be transported across cultural boundaries without much difficulty because they are very generalised, have broad areas of applicability, and can therefore adapt to a wide range of activities. However the interplay between mathematical worlds is not so visible because the resolution of conflicting conceptions gets played out through many practices. This explains why mathematical conceptions of minor cultures become colonised: the mathematical conception with the wider range of applicability will accommodate different practices more readily.

6. UNIVERSAL OR RELATIVE

I will try to sum up where we have got to philosophically by dealing directly with the question about universality and relativity. Where does the evidence from language lead me? Is mathematics universal, or is it relative? My answer, predictably, is both. It depends what you mean. I can see two senses in which mathematics is universal, and two senses in which it is relative.

The first universal sense arises from the fact that, if you are in a particular mathematical world, then it is possible to look at another mathematical world and see it in your terms. For example, Bishop (1988) identifies six pre-mathematical practices which are present in every culture: counting, measuring, locating, designing, playing, and explaining. Bishop is not saying that these activities are equivalently defined in every culture; he is saying that he can identify in any culture activities which come under each of these headings as far as he is concerned. This leaves open the question as to whether numbers exist in some real sense because everyone counts, or triangles exist because everyone designs; or the continuum exists because everyone measures. These 'objects' could be conceptual tools with no existence beyond the conceiver. This sense of universality does not imply a Platonist reality.

The second sense in which mathematics is universal results from the fact that, if you acknowledge mathematics at all, then you must acknowledge conventional NUC-mathematics. For, if you don't, then it is difficult to justify your use of the label 'mathematics'. Mathematics exists as a knowledge category, recognised by a very large proportion of humans in every culture. To call something else mathematics, is not making sense of the use of that word. NUC-mathematics is universal because it is part of the meaning of ‘mathematics’.

These senses of universal mathematics do not mean that the subject is static. A person may hold a differing view of mathematics from the conventional one to the extent that a debate may take place through which mathematics may change its conventional meaning. Development is possible. For this to occur, however, there must be one of two situations. Either the unconventional viewer acknowledges that the conventional view has legitimacy and the onus is on them to convince others that a change is justified (for example, Joseph's writings on non-European aspects of mathematics (1991)); or there may be more than one community of convention, mutually acknowledged by the other as having a right to the debate (such as the communities of standard and non-standard analysis, or Bayesian and Frequentist statisticians).

Now relativity. The first sense in which mathematics is relative is that it can change. This change is more than just an evolutionary building on what has gone before, it involves revolutionary change in the sense that fundamental ways of thinking can change (see Gillies, 1992, Section 2.3). Completely different mathematical concepts, which are subsumable neither by existing ones, nor by some new, overarching generalisation, are possible. In other words, a new mathematical concept may arise which radically changes existing mathematics because it cannot be integrated into mathematics as presently understood in any other way.

The second sense in which mathematics is relative is that mathematics is not the only way to see the world, nor is it the only way to see those aspects of the world having to do with number, relationships, or space. Other people may see things that I might call mathematical in entirely different terms.

To summarise: if we are to ask whether there is, in fact, another mathematics equal in power to NUC-mathematics, then the answer is no. On the other hand, if we are to ask whether mathematics could have been different, then the answer is yes.

Historically, the line of progress of mathematics could have been otherwise. We cannot know what theory of mathematics we might now have, nor whether this hypothetical theory would be more comprehensive, more sophisticated, more applicable, or 'better' by any other criterion. It is not possible to completely rewrite history (Lakatos, 1978).

The sociology of mathematics will help us identify how divergent ideas may have changed the path of mathematical development: to identify the turning points and decision points; to specify the socio-cultural conditions which determined particular paths; and to trace paths as far as possible. The anthropology of mathematics will help us explore the existence of other paths and other mathematical worlds (even in embryonic form). Both of those have a historical orientation. An ethnomathematician's task is to explore—in the present—the consequences of different worlds for mathematics: first to understand where they were/are leading, and then to reflect on them mathematically.

The lack of more than one contemporary, sophisticated mathematics does not imply the universality of the one we know—it only contributes to our feeling of its truth. There is potential for divergent mathematical development, which I call contemporary relativity.

7. EVIDENCE, REFLECTIONS, & CONSEQUENCES

We have used the evidence from the language of everyday mathematical talk to reflect upon mathematics, and have come to some far-reaching suggestions about the nature of mathematics. One of these conclusions is that mathematics and language evolved together. Does this mean that we can suggest things about language and linguistics from this evidence?

Questions of whether languages evolved from a common proto-language, and whether there are linguistically universal concepts, are intricately tied up with the arguments of this book. For example, if it is argued that mathematics develops differently in different languages, then it might still be possible to have a single, universal mathematics if there are some things that are linguistically the same, no matter what language you speak. Mathematics could be exactly those things that are universal.

I believe that the weight of evidence presented in this book opposes such an idea. To the extent that we regard language as the cultural expression of a world view, there appear to be quantitative, spatial, and relational aspects of some world views that are not essentially the same. We do, indeed, still talk of 'quantitative' aspects of each world view—but this is just our way of talking and making sense of the differences we see.

What about whether different languages evolved from a single common language? If there are such different conceptions of mathematics embedded in languages, then this is evidence that all languages did not evolve from one language—or if they did then it was before some elementary quantitative, spatial, and relationship conceptions were formed. The latter possibility seems unlikely, therefore the mathematical evidence suggests that some languages must have evolved independently.

However, I am not a linguist, and the debate about such things contains much more evidence than that from mathematics (Chomsky, 1998; Pinker, 1994).

Part I of this book presents some evidence to illustrate that mathematical ideas are represented in fundamentally different ways in the everyday talk of different languages. It also explored how some of these could evolve in different directions or into different structures in mathematics. The idea being put forward is that there could be different mathematical worlds, or that mathematics could have evolved in another way from the one that we know.

Part II examines the consequences for mathematics. It describes the origins and evolution of mathematics from a stance which accepts the possibility of other mathematical worlds. Further illustrations are given, and evidence and supporting views of others from both the history, anthropology, and sociology of mathematics is presented. What emerges is a picture of mathematics as a plaited braid of many strands, that merge and split, fold back and tangle—but a braid in which there is no 'one way' unless you are looking from inside one of the strands.

This picture leads to some philosophical reflections about mathematics, particularly to the writings of Wittgenstein as interpreted by Shanker, and to a way of conceiving both the universality and the relativity of mathematics as meaningful.

The final section of the book looks at the consequences of this point of view for mathematics education. The evidence from different languages makes us think again about how we might approach teaching, particularly to students whose language is not Indo-European, or not the same as our own. We must also think about what it is that we are teaching, and the underlying experiences and dispositions that will lead to high levels of creativity and application in the mathematical and information sciences and their applications.

To finish this section, let us remind ourselves that what has been said has been said before. The idea that language and mathematical thought are inextricably linked is not new, nor is the recognition of the potential for new mathematics embedded in other languages. Benjamin Whorf has already been quoted. He also said (1956, p. 245):

> …an important field for working out new order systems, akin to, yet not identical with, present mathematics, lies in more penetrating investigation than has yet been made of languages remote in type from our own.