CHAPTER 9

Parametric Lorenz Curves: Models and Applications

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Abstract

The Lorenz curve (LC) is an important instrument for analyzing the size of distribution of income or wealth and inequality. Finding an appropriate functional form is an important practical and theoretical problem. In this chapter we study parametric models for the LC and some important applications.

The basic properties that a function should satisfy in order to be a genuine LC are discussed. Next, we study the different ways for generating parametric families of LCs, as well as some of their basic properties, including their relationship with the underlying income distribution function. The basic parametric models proposed in the literature are studied, including the Pareto, lognormal and other important families of LCs.

Some general strategies to obtain extensions and generalizations of the basic parametric models are presented. One of the main applications of LCs is the study of inequality. We begin studying different measures of inequality together with their expressions in terms of the LC. These measures include the Gini index and some of their generalizations proposed by Kakwani (1980) and Yitzhaki (1983). Their corresponding expressions for the proposed parametric families of LCs will be obtained. The Lorenz ordering is also studied. The Lorenz ordering is a partial order that allows the comparison of two distributions when its corresponding LCs do not intersect. Some basic properties of this order are studied, including the effect of transformations, its relations with other partial orderings and their application to important parametric income distributions. The recent proposal of multivariate versions of the LC are studied. Finally, some applications of the Lorenz curve are presented.

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1 Introduction

The merits of parametric methods as opposed to non-parametric methods for the construction of indices and inequality measures for income distributions have been pointed out by Slottje (1990) and Ryu and Slottje (1996, 1999). These authors conclude, among other things, that the indices should be constructed using the parametric method and then the results checked using a non-parametric method. In this regard, the Lorenz curve (LC) is an essential instrument for analyzing the size distribution of income, wealth and inequality and the problem of finding an appropriate functional form is both an important practical and theoretical problem.

Some recent advances have contributed to the current development of this research instrument. New ways to specify the Lorenz curve have been developed and studied (see Section 4). On the other hand, the Lorenz ordering has been characterized in important families of income distributions (see Kleiber and Kotz (2003) and Section 6). The interest in and development of multivariate inequality measures as well as the multivariate versions of the Lorenz curve (see Section 8) have led to an increase in the amount of research devoted to this area. In this chapter we study parametric models for the LC and some important applications.

The contents of this chapter are as follows. In Section 2 we study basic properties of the LC, including their relationship with the underlying income distribution function. Section 3 reviews the LC of some important income models, including the following distributions: classical Pareto, lognormal, Singh-Maddala and Dagum type I. There exists a variety of approaches for the construction of parametric families of LC's. In Section 4 we study the different ways of generating parametric families of LC and some general strategies to obtain extensions and generalizations of the basic parametric models. One of the main applications of the LC's is the study of inequality. Inequality indices derived from the Lorenz curve and other classical inequality measures are studied in Section 5. The Lorenz ordering is a partial order that allows us to compare two distributions when their corresponding LC's do not intersect. Properties of this order, and their application to important parametric income distributions are studied in Section 6. Section 7 presents some variations of the LC. The recent proposal of multivariate versions of the LC are studied in Section 8. Finally, some applications of the Lorenz curve are presented in Section 9.

2 The Lorenz Curve. Basic Properties

The Lorenz curve is defined by points $(p, L(p))$, where p represents the cumulative proportion of income-receiving units, and $L(p)$ the cumulative proportion of incomes, when the incomes are arranged in ascending order of magnitude.

In the empirical case, if we denote the ordered individual incomes in the population by $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$, then for $i = 1, 2, \ldots, n$

$$L(\frac{i}{n}) = \frac{\sum_{j=1}^{i} x_{j:n}}{\sum_{j=1}^{n} x_{j:n}}. \tag{9.1}$$

The points $(\frac{i}{n}, L(\frac{i}{n}))$ are then linearly interpolated to complete the corresponding Lorenz curve.

Now, our next step is to extend (9.1) to the continuous case. If n is large the distribution of incomes within the population can be approximated by a continuous distribution function $F(x)$, with density $f(x)$ related by $F(x) = \int_0^x f(y)dy$. The interpretation here is similar to the previous one: for each positive x, $F(x)$ represents or approximates the proportion of individuals in the population whose income is less than or equal to x. Now, we consider the k-moment distribution of the population $F_{(k)}(x)$ defined by

$$F_{(k)}(x) = \frac{\int_0^x y^k dF(y)}{\int_0^\infty y dF(y)}, \ \ k = 1, 2, \ldots \tag{9.2}$$

where the denominator is assumed to be finite. If we set $k = 1$ in (9.2), then for each x, $F_{(1)}(x)$ represents the proportion of the total incomes which accrues to individuals with incomes less than or equal to x. The Lorenz curve corresponding to the distribution F can be described as the set of points,

$$(F(x), F_{(1)}(x)) \tag{9.3}$$

defined in the unit square, where x ranges from 0 to ∞ completed if necessary by linear interpolation.

An expression for the Lorenz curve can be constructed using the parametric representation (9.3). We may write

$$L(p) = F_{(1)}(F^{-1}(p)). \tag{9.4}$$

To use formula (9.4) we obviously need closed form expressions for $F_{(1)}$ and F^{-1}.

Let $\mathscr{L}$ be the class of all non-negative random variables with positive finite expectations. For a random variable X in $\mathscr{L}$ with cumulative distribution function F_X we define its inverse distribution function by

$$F_X^{-1}(y) = \inf\{x : F_X(x) \geq y\} \tag{9.5}$$

Note that the mathematical expectation of X is $\mu_X = \int_0^1 F_X^{-1}(y)dy$. According to Gastwirth (1971) we have the following definition.

Definition 9.1. Let $X \in \mathscr{L}$ with cumulative distribution function F_X and inverse distribution function F_X^{-1}. The Lorenz curve L_X corresponding to X is defined by

$$L_X(p) = \frac{1}{\mu_X} \int_0^p F_X^{-1}(y)dy, \ \ 0 \leq p \leq 1. \tag{9.6}$$

This definition contains the definition provided by (9.1) in the case of a finite population and (9.2) in the continuous case. In formula (9.6) F_X^{-1} is piecewise continuous and the integrals can be assumed to be ordinary Riemann integrals.

From definition (9.6) we can show that a Lorenz curve will be a continuous, non-decreasing convex function that is differentiable almost everywhere in $[0,1]$ and $L(0)=0$ and $L(1)=1$. These are properties that we expect to characterize an LC. A formal characterization of a Lorenz curve attributed to Gaffney and Anstin by Pakes (1981) is the following.

Theorem 9.1. *Suppose $L(p)$ is defined and continuous on $[0,1]$ with second derivative $L''(p)$. The function $L(p)$ is a Lorenz curve if and only if*

$$L(0)=0,\ \ L(1)=1,\ \ L'(0^+)\geq 0,\ \ L''(p)\geq 0\ \ in\ \ (0,1). \tag{9.7}$$

The Lorenz curve determines the distribution of X up to a scale factor transformation. This is true since $F_X^{-1}(x)=\mu_X L'(x)$ almost everywhere and F_X^{-1} will determine F_X. Concerning the probability density function $f_X(x)$ associated with a Lorenz curve $L(p)$, we have the following result (Arnold, 1987).

Theorem 9.2. *If $L''(p)$ exists and is positive everywhere in an interval (x_1,x_2), then F_X has a finite positive density in the interval $(\mu L'(x_1^+),\mu L'(x_2^-))$ which is given by*

$$f_X(x)=\frac{1}{\mu L''(F_X(x))}. \tag{9.8}$$

As an illustration of these results, we consider Chotikapanich's LC defined in (9.28). The cumulative distribution function corresponding to this LC model is

$$F(x;k,\mu)=\frac{1}{k}\log\left(\frac{x}{c_k\mu}\right),\ \ c_k\mu\leq x\leq c_k\mu e^k,$$

where $c_k=k/(e^k-1)$ and $F(x;k,\mu)=0$ if $x\leq c_k\mu$ and $F(x;k,\mu)=1$ if $x\geq c_k\mu e^k$. Note that the cdf depends on k and a new scale parameter μ which represent the population mean.

From a geometric viewpoint it is natural to enquire whether an LC exhibits symmetry. A Lorenz curve is symmetric if

$$L[1-L(p)]=1-p,\ \ 0\leq p\leq 1. \tag{9.9}$$

If a random variable X has mean μ and density $f_X(x)$, its LC is symmetric if and only if

$$\frac{f_X(\mu^2/x)}{f_X(x)}=\left(\frac{x}{\mu}\right)^3,$$

for every x with $f_X(x)>0$.

3 Lorenz Curves of Some Classical Income Distributions

In this section we review the LC corresponding to some important models of income distributions. We begin with models corresponding to income distributions with closed expressions for the inverse cdf which can be integrated so that and then Gastwirth's formula can be used. The first example corresponds to the classical Pareto distribution (see Arnold (1983)) with cumulative distribution function

$$F_X(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\alpha}, \ x \geq \sigma \tag{9.10}$$

where $\sigma > 0$ is a scale parameter and $\alpha > 0$ a shape parameter. For the Pareto distribution the quantile function is,

$$F_X^{-1}(y) = \sigma(1-y)^{-1/\alpha}, \ 0 < y < 1$$

and the mean $\mu_X = \alpha\sigma/(\alpha-1)$ if $\alpha > 1$. Using (9.6) we obtain

$$L_X(p) = \frac{\alpha-1}{\alpha\sigma}\int_0^p \sigma(1-y)^{-1/\alpha}dy = 1-(1-p)^{1-1/\alpha}, \ 0 < p < 1 \tag{9.11}$$

provided $\alpha > 1$.

The Singh-Maddala distribution is one of the most popular distributions used in practice to fit income and wealth data (Kleiber and Kotz, 2003). This distribution was obtained by Singh and Maddala (1976) by considering the hazard rate of income. Let X be a random variable with Singh-Maddala distribution with cdf,

$$F_X(x) = 1 - \frac{1}{[1+(x/\sigma)^a]^q}, \ x > 0 \tag{9.12}$$

where $a, q, \sigma > 0$. Definition (9.12) corresponds to the Pareto IV distribution, in the Arnold (1983) Pareto hierarchy. If $q > 1/a$ then using expression (9.6) the Lorenz curve of (9.12) is,

$$\begin{aligned} L_X(p) &= \frac{1}{\mu_X}\int_0^p \sigma[(1-y)^{-1/q}-1]^{1/a}dy \\ &= \frac{\sigma q}{\mu_X}\int_0^z t^{1/a}(1-t)^{q-1/a-1}dt \\ &= I_z(1+1/a, q-1/a), \ 0 \leq p \leq 1 \end{aligned}$$

where $z = 1-(1-p)^{1/q}$ and $I_x(a,b)$ denotes the incomplete beta function ratio defined as $(0 < x < 1)$

$$I_x(a,b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt}. \tag{9.13}$$

Another important income distribution is the Dagum type I distribution (Dagum, 1977) with cdf

$$F(x) = [1 + (x/\sigma)^{-a}]^{-q}, \;\; x > 0 \tag{9.14}$$

where $a, q, \sigma > 0$. This distribution is related with the Singh-Maddala distribution by the inverse transformation $1/X$. Since the quantile function is available in closed form, the LC can be written as (Dagum, 1977),

$$L(p) = I_z(q + 1/a, 1 - 1/a), \;\; 0 \leq p \leq 1 \tag{9.15}$$

where $z = p^{1/q}$, $a > 1$ and $I_x(a,b)$ is defined in (9.13). The Gini index is given by,

$$G = \frac{\Gamma(q)\Gamma(2q + 1/a)}{\Gamma(2q)\Gamma(q + 1/a)} - 1.$$

Another group of income distributions corresponds to families where all of the kth moment distributions and the original distribution belong to the same family so that formulas (9.3) or (9.4) can be applied. Consider a lognormal distribution, for which the cumulative distribution function is given by

$$F(x) = \Phi(\frac{\log x - \mu}{\sigma}), \;\; x > 0 \tag{9.16}$$

where Φ denotes the cdf of the standard normal distribution. This distribution will be denoted by $X \sim \mathscr{LN}(\mu, \sigma^2)$. The inverse of the cdf is $F^{-1}(x) = \exp[\mu + \sigma\Phi^{-1}(x)]$ and the cdf of the kth moment distributions is again lognormal and is given by Aitchison and Brown (1957),

$$X_{(k)} \sim \mathscr{LN}(\mu + k\sigma^2, \sigma^2), \;\; k = 1, 2, \ldots \tag{9.17}$$

In particular $X_{(1)} \sim \mathscr{LN}(\mu + \sigma^2, \sigma^2)$. Now, by introducing F^{-1} and $F_{(1)}(x)$ in formula (9.4) we obtain the LC corresponding to the lognormal distribution which is given by

$$L(p) = \Phi(\Phi^{-1}(p) - \sigma), \;\; 0 < p < 1. \tag{9.18}$$

The gamma distribution is another popular distribution used in analysis of income and wealth data. The pdf of a gamma distribution is

$$f(x) = \frac{x^{\alpha-1}e^{-x/\sigma}}{\sigma^\alpha \Gamma(\alpha)}, \;\; x > 0 \tag{9.19}$$

where $\sigma > 0$ is a scale and $\alpha > 0$ a shape parameter. A random variable with pdf (9.19) will be denoted as $X \sim \mathscr{G}(\alpha, \sigma)$. The gamma distribution includes as particular cases the exponential ($\alpha = 1$) and the chi-square distribution ($\alpha = n/2$, $n = 1, 2, \ldots$). The cdf of the gamma distribution can be written as

$$F(x) = \frac{\gamma(\alpha, x/\sigma)}{\Gamma(\alpha)}, \;\; x > 0 \tag{9.20}$$

where $\gamma(a,x)$ denotes the incomplete gamma function defined as,

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt \tag{9.21}$$

with $a,x > 0$. The kth moment distribution is distributed again as a gamma distribution, that is, $X_{(k)} \sim \mathscr{G}(\alpha + k, \sigma)$ and thus the Lorenz curve can be expressed in a parametric fashion using (9.3) and (9.20). We thus have that

$$(p, L(p)) = \left(\frac{\gamma(\alpha, x/\sigma)}{\Gamma(\alpha)}, \frac{\gamma(\alpha+1, x/\sigma)}{\Gamma(\alpha+1)}\right), \ \ x > 0. \tag{9.22}$$

Sarabia and Castillo (2005) have obtained expressions for the LC and the Gini index for a general class of max-stable income distributions.

In order to complete this section we include the LC corresponding to a discrete random variable. Let X be a geometric distribution with probability mass function $\Pr(X = k) = pq^{k-1}$, $k = 1,2,\ldots$ with $0 < p < 1$ and $q = 1 - p$. Using formula (9.6), the LC is (Gastwirth, 1971)

$$L(u) = 1 - kq^{k-1} + (k-1)q^k + kp[u - (1 - q^{k-1})],$$

if $1 - q^{k-1} \le u \le 1 - q^k$, $k = 1,2,\ldots$ The Gini index is given by (Dorfman, 1979): $G = (1-p)/(2-p)$.

Table 9.1 summarized the Lorenz curves and the Gini index of some important income distributions.

Table 9.1: Lorenz curves and Gini indices of Classical Income Distributions.

Distribution	Lorenz Curve	Gini Index
Uniform $\mathscr{U}[a,b]$	$L(p) = \frac{2ap+(b-a)p^2}{a+b}$	$G = \frac{b-a}{3(a+b)}$
Exponential[1]	$L(p) = p + (1 + \frac{\mu}{\sigma})^{-1}(1-p)\log(1-p)$	$G = \frac{\sigma}{2(\mu+\sigma)}$
Classical Pareto	$L(p) = 1 - (1-p)^{1-1/\alpha}$	$G = \frac{1}{2\alpha-1}$
Singh-Maddala	$L(p) = I_z(1+1/a, q-1/a)$ where $z = 1-(1-p)^{1/q}$	$G = 1 - \frac{\Gamma(q)\Gamma(2q-1/a)}{\Gamma(q-1/a)\Gamma(2q)}$
Dagum	$L(p) = I_z(q+1/a, 1-1/a)$ where $z = p^{1/q}$	$G = \frac{\Gamma(q)\Gamma(2q+1/a)}{\Gamma(2q)\Gamma(q+1/a)} - 1$
Lognormal	$L(p) = \Phi(\Phi^{-1}(p) - \sigma)$	$G = 2\Phi(\frac{\sigma}{\sqrt{2}}) - 1$
Classical Gamma	$(p, L(p)) = (\frac{\gamma(\alpha,x/\sigma)}{\Gamma(\alpha)}, \frac{\gamma(\alpha+1,x/\sigma)}{\Gamma(\alpha+1)})$	$G = \frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha+1)}$

[1]Exponential distribution with cdf $F(x) = 1 - e^{-(x-\mu)/\sigma}$ if $x > \mu$, with $\mu, \sigma > 0$.

4 Models of Parametric Lorenz Curves

There exists a variety of approaches for the construction of parametric families of LC's. The first obvious approach consists of starting from an appropriate parametric family of income distribution functions and obtaining the corresponding LC by analytically using representations (9.4) or (9.6), as we have seen in the previous Section. A second approach consists of selecting parametric families of simple curves satisfying the required conditions for Lorenz curves given in Theorem 9.1. This method usually leads to complicated distribution functions, but may be flexible enough for fitting empirical Lorenz curves.

Several parametric models have been proposed in using the second approach. The pioneer model was established by Kakwani and Podder (1973), who proposed the functional form,

$$L(p) = p^{\alpha} e^{-\beta(1-p)},\ \ 0 \le p \le 1, \tag{9.23}$$

with $\beta > 0$ and $\alpha \ge 1$ (see also Rao and Tam (1987)). An alternative parameterization of this model was provided by Gupta (1984). Kakwani and Podder (1976) also proposed a new parametric model based on a geometric motivation. This model expresses a point of the LC as (x, y), where y is the length of the ordinate from LC on the egalitarian line and x is the distance of the ordinate from the origin along the egalitarian line. This model was completed by Rasche *et al.* (1980) who proposed the family of curves

$$L(p) = [1-(1-p)^{\alpha}]^{\beta},\ \ 0 \le p \le 1 \tag{9.24}$$

where $0 < \alpha \le 1$ and $\beta \ge 1$. If $\beta = 1$ we obtain the LC (9.11) corresponding to the classical Pareto distribution, and if $\alpha = 1/\beta$ a symmetric LC is obtained according to definition (9.9).

Using several well-known sets of data Villaseñor and Arnold (1989) observed that segments of ellipses frequently fit data surprisingly well. The class of elliptical LC is given by

$$L(p; \alpha, \beta, \delta) = \frac{1}{2}\left[(a - \beta p) - \sqrt{a^2 + bp + cp^2}\right] \tag{9.25}$$

where $a = \alpha + \beta + \delta + 1 > 0$, $b = -2a\beta - 4\delta$, $c = \beta^2 - 4\alpha$, $\alpha + \delta \le 1$, and $\delta \ge 0$. Equation (9.25) implies that any point (p_i, q_i) must satisfy $y_i = \alpha x_i + \beta z_i + \delta w_i$, $i = 1, 2, \ldots, n$, where $y_i = q_i(1 - q_i)$, $x_i = p_i^2 - q_i$, $z_i = q_i(p_i - 1)$, and $w_i = p_i - q_i$. This is a linear function of α, β and δ and the least square estimation method can be applied. Using this fact, robust estimation methods have been proposed by Castillo *et al.* (1998). This functional form provides excellent fit and the associated distribution and density functions are available in closed form. In a similar geometric context and from a proposal by Aggarwal (1984) and Aggarwal and Singh (1984), Arnold (1986) considered a hyperbolic functional form for the LC given by

$$L(p;\alpha,\beta) = \frac{p[1+(\alpha-1)p]}{1+(\alpha-1)+\beta(1-p)},\ \ 0 \le p \le 1 \tag{9.26}$$

where $\alpha, \beta > 0$ and $\alpha - \beta < 1$. Models (9.25) and (9.26) can be considered to be within the class of general quadratic Lorenz curves (Villaseñor and Arnold, 1989). The circular LC was considered by Ogwang and Rao (1996).

Arnold *et al.* (1987) proposed a class of LC of the form,

$$L(p;\sigma) = F(F^{-1}(p) - \sigma),\ \ \sigma \ge 0, \tag{9.27}$$

where $F(\cdot)$ is any strongly unimodal cdf. For instance, if $F = \Phi$, we obtain the LC (9.18), corresponding to a classical lognormal distribution.

Chotikapanich (1993) proposed the uniparametric model,

$$L(p;k) = \frac{e^{kp}-1}{e^k-1},\ \ 0 \le p \le 1 \tag{9.28}$$

where $k > 0$ and where the limit case $k \to 0$ corresponds to the egalitarian line. With several data sets the model outperforms those of Kakwani and Podder (1976) and Rasche *et al.* (1980) in terms of the Gini coefficient estimation but is not as good for predicting expenditures shares.

Sarabia (1997) considered an alternative method for the construction of LC specifying an appropriate quantile function, and using it to generate the LC. Using the generalized Tukey's Lambda distribution, this author obtained a family of nested models, which, in the most general case, is

$$L(p) = \pi_1 p + \pi_2 p^{\alpha_1} + (1-\pi_1-\pi_2)[1-(1-p)^{\alpha_2}],\ \ 0 \le p \le 1,$$

where $0 \le \pi_1, \pi_2 \le 1$, $\alpha_1 \ge 1$ and $0 < \alpha_2 \le 1$. This model is a mixture of the egalitarian line, the power LC and the classical Pareto LC.

Another important model was considered by Basmann *et al.* (1990), which extend Kakwani and Podder's model (9.23). Ryu and Slottje (1996) introduced two flexible functional form approaches to approximate Lorenz curves, an exponential polynomial and a Bernstein polynomial expansion. Holm (1993) has based his model on the principle of maximum entropy and Sarabia and Pascual (2002) on linear exponential loss functions.

Recent research on the Lorenz curve (Basmann *et al.*, 1990; Ryu and Slottje, 1996; Ogwang and Rao, 2000) has shown that some families of LCs approximate some segments of the income distributions well but not others segments. In the next subsection we propose some general strategies to obtain extensions and generalizations of the basic parametric models.

4.1 A Hierarchical Family

Recently, Sarabia *et al.* (1999) have suggested a general method for obtaining a hierarchical family of LC that unifies and synthesizes some of the previous proposals, as well as providing good fit in the whole the range of the data. If we begin with any Lorenz curve L_0 the following curves are also Lorenz curves that generalize the initial model L_0:

$$L_1(p;\alpha) = p^{\alpha}L_0(p),\ \ \alpha \geq 1 \ \text{ or } \ 0 \leq \alpha < 1 \ \text{ and } \ L_0'''(p) \geq 0, \tag{9.29}$$
$$L_2(p;\gamma) = [L_0(p)]^{\gamma},\ \ \gamma \geq 1, \tag{9.30}$$
$$L_3(p;\alpha,\gamma) = p^{\alpha}[L_0(p)]^{\gamma},\ \ \alpha \geq 1 \ \text{ or } \ 0 \leq \alpha < 1 \ \text{ and } \ L_0'''(p) \geq 0, \tag{9.31}$$

An advantage of this method is that Lorenz ordering results are obtained. Equations (9.29) and (9.30) are ordered with respect to their parameters α and γ and a combination of these cases yield ordering results for (9.31).

This method allows for the generation of a hierarchy of Lorenz curves starting from an initial curve L_0. A relevant family is generated from

$$L_0(p) = L_0(p;k) = 1-(1-p)^k,\ \ 0 < k \leq 1,$$

which is the LC (9.11) associated to the classical Pareto distribution. Since $L_0'''(p;k) > 0$ we can apply results in a general way. We can consider the parametric family of Lorenz curves,

$$L_1(p;k,\alpha) = p^{\alpha}[1-(1-p)^k],\ \ \alpha \geq 0 \tag{9.32}$$
$$L_2(p;k,\gamma) = [1-(1-p)^k]^{\gamma},\ \ \gamma \geq 1, \tag{9.33}$$
$$L_3(p;k,\alpha,\gamma) = p^{\alpha}[1-(1-p)^k]^{\gamma},\ \ \alpha \geq 0,\ \gamma \geq 1, \tag{9.34}$$

which is called the Pareto hierarchy of Lorenz curves, since they originate from the Pareto distribution. Family (9.32) coincides with the family proposed by Ortega *et al.* (1991) and (9.33) with the family proposed by Rasche *et al.* (1980). A detailed study of the family (9.34) can be found in Sarabia *et al.* (1999). The method has been used to generate other families of Lorenz curves beginning with different choices for L_0. If we begin with the Chotikapanich LC given in (9.28), we obtain a new family of LC, called the exponential family of LC by Sarabia *et al.* (2001). This approach was also used by Sarabia and Pascual (2002). Table 9.2 summarized the Pareto LC family.

4.2 Mixture Lorenz Curve

A possible solution for obtaining better fit consists in building more complex models combining some of the classical models using convex linear combinations of LCs. The proposals of Sarabia (1997) and Ogwang and Rao (2000) respond to this idea.

Table 9.2: The Pareto Lorenz curve family.

Lorenz Curve	Gini Index
$L_0(p;k) = 1-(1-p)^k$	$G = \frac{1-k}{1+k}$
$L_1(p;k,\alpha) = p^{\alpha}[1-(1-p)^k]$	$G = 1-2[B(\alpha+1,1)-B(\alpha+1,k+1))]$
$L_2(p;k,\gamma) = [1-(1-p)^k]^{\gamma}$	$G = 1-\frac{2}{k}[B(1/k,\gamma+1)]$
$L_3(p;k,\alpha,\gamma) = p^{\alpha}[1-(1-p)^k]^{\gamma}$	$G = 1-2\sum_{i=0}^{\infty}\frac{\Gamma(i-\gamma)}{\Gamma(i+1)\Gamma(-\gamma)}B(\alpha+1,ki+1)$

In this sense, one of the reasons that can explain the lack of fit in some LC's is the existence of some factor of heterogeneity in the population (for example, age, gender or education), so the LC varies from some individuals to others. If we compose the initial LC with the heterogeneity (described in terms of a known pdf) we obtain a new LC called a mixture LC (Sarabia *et al.*, 2005). If $L(p;\theta)$ denotes a LC, and we assume that θ varies according to an absolutely continuous density function $\pi(\theta)$ with support on a set $\Theta \subset R$, the expression

$$\tilde{L}(p) = \int_{\Theta} L(p;\theta)\pi(\theta)d\theta$$

defines a genuine LC. Several mixture LC models have been proposed by Sarabia *et al.* (2005). For example, if a power LC is composed with a gamma distribution, we obtain the LC,

$$L(p;\alpha,\sigma) = \frac{p}{(1-\sigma\log p)^{\alpha}}.$$

5 Inequality Measures Derived from the Lorenz Curve

The two best known measures of inequality which are directly related to the Lorenz curve are the Gini and Pietra indices. Both indices can be viewed as alternative forms of measuring the distance between the Lorenz curve and the egalitarian line. The Gini index is defined as twice the area between the egalitarian line and the Lorenz curve

$$G_X = 2\int_0^1 [p - L_X(p)]dp = 1-2\int_0^1 L_X(p)dp. \tag{9.35}$$

There are several alternative expressions of the Gini index equivalent to (9.35). One of the most important is

$$G_X = 1 - \frac{E(X_{1:2})}{\mu} = 1 - \frac{1}{\mu}\int_0^\infty [1 - F_X(x)]^2 dx \tag{9.36}$$

where $X_{1:2}$ is the smaller of a sample of size 2 coming from the cdf F_X. This expression is useful when we have a closed form for the cumulative distribution function (see, for example, Cronin (1979)).

A second important inequality measure is the Pietra index, which is defined as the maximal vertical deviation between the Lorenz curve and the egalitarian line

$$P_X = \max_{0 \le p \le 1} \{p - L_X(p)\}. \tag{9.37}$$

If we assume that F is strictly increasing on its support, the function $p - L_X(p)$ will be differentiable everywhere on $(0,1)$ and its maximum will be reached when $1 - F^{-1}(x)/\mu$ is zero, that is, when $x = F(\mu)$. The value of $p - L_X(p)$ in this point is given by

$$P_X = F(\mu) - \frac{1}{\mu}\int_0^{F(\mu)} [\mu - F^{-1}(y)]dy = \frac{1}{2\mu}\int_0^\infty |z - \mu| dF(z),$$

in consequence

$$P_X = \frac{E|X - \mu|}{2\mu},$$

which is an alternative formula for the Pietra index. For the classical Pareto distribution (9.10) the mean is $\mu = \alpha/(\alpha - 1)$ if $\alpha > 1$ and $F(\mu) = 1 - (\alpha/(\alpha - 1))^{-\alpha}$ and thus the Pietra index is

$$P_X = F(\mu) - L(F(\mu)) = \frac{(\alpha - 1)^{\alpha - 1}}{\alpha^\alpha}.$$

There are several generalizations of the Gini index proposed in the literature. Mehran (1976)considered the general class of linear measures of the form

$$I(w) = \int_0^1 [p - L_X(p)]dw(p), \tag{9.38}$$

where $w(p)$ is some increasing function which allows value judgments about inequality to be incorporated. Note that $I(w)$ is always compatible with the Lorenz order. If we take $w(p) = 2p$, $0 \le p \le 1$, we obtain the Gini index.

Another important generalization of the Gini index was proposed by Yitzhaki (1983). This author proposed the generalized Gini index defined as

$$G_\nu = 1 - \nu(\nu - 1)\int_0^1 (1 - p)^{\nu - 2} L_X(p) dp, \tag{9.39}$$

where $\nu > 1$. If $\nu = 2$ we obtain the Gini index. When ν increases, higher weights are attached to small incomes. The limit case when ν goes to infinity depends on the lowest income, expressing the judgement introduced by Rawls, that social welfare depends only on the poorest society member. On the other hand, it can be proved that (Muliere and Scarsini, 1989)

$$G_\nu = 1 - \frac{E(X_{1:\nu})}{\mu_X},$$

which can also be seen as a generalization of (9.36). For the classical Pareto LC (9.11) Yitzhaki's index (9.39) is,

$$G_\nu = \frac{\nu - 1}{\alpha\nu - 1}, \;\; \alpha, \nu > 1.$$

Arnold (1983, p. 109) has proposed next generalization of the Gini index,

$$\tilde{G}_n = 1 - \frac{E(X_{1:n+1})}{E(X_{1:n})}.$$

The Gini index corresponds to the case $n = 1$. The set of all such indices $\{\tilde{G}_n : n = 1, 2, \dots\}$ determines the parent distribution up to a scale factor.

Another two important inequality measures deserve our attention: the Atkinson (1970) inequality measures and the generalized entropy indices. The Atkinson inequality indices are defined as

$$A(\varepsilon) = 1 - \left[\int_0^\infty (x/\mu)^{1-\varepsilon} dF(x)\right]^{1/(1-\varepsilon)}, \;\; \varepsilon > 0, \tag{9.40}$$

where ε is a parameter that controls the inequality aversion. The limit cases $\varepsilon \to 1$ and $\varepsilon \to \infty$ are

$$\begin{aligned} A(1) &= 1 - \frac{1}{\mu}\exp\left\{\int_0^\infty \log(x) dF(x)\right\}, \\ A(\infty) &= 1 - \frac{F^{-1}(0)}{\mu}. \end{aligned}$$

The generalized entropy indices are

$$G(\theta) = \frac{1}{\theta(\theta - 1)}\int_0^\infty \left[(x/\mu)^\theta - 1\right] dF(x), \;\; \theta \neq 0, 1 \tag{9.41}$$

and

$$\begin{aligned} G(0) &= \int_0^\infty \log(\mu/x) dF(x), \\ G(1) &= \int_0^\infty (x/\mu)\log(x/\mu) dF(x). \end{aligned}$$

These two latter indices are known as the Theil coefficients. Indices (9.40) and (9.41) can be written in terms of the LC using the formulas,

$$A(\varepsilon) = 1 - \left\{ \int_0^1 [L_X'(p)]^{1-\varepsilon} dp \right\}^{1/(1-\varepsilon)}, \ \varepsilon > 0 \tag{9.42}$$

$$G(\theta) = \frac{1}{\theta(\theta-1)} \int_0^1 \left\{ [L_X'(p)]^{\theta} - 1 \right\} dp, \ \theta \neq 0,1 \tag{9.43}$$

These formulas allow these indices to be obtained directly from the Lorenz curve without the necessity of knowing the underlying cumulative distribution function. For the classical Pareto distribution with LC (9.11), using (9.43) the generalized entropy index is given by ($\theta \neq 0,1$),

$$G(\theta) = \frac{1}{\theta(\theta-1)} \left[\left(1 - \frac{1}{\alpha}\right)^{\theta} \frac{\alpha}{\alpha-\theta} - 1 \right]$$

where $\alpha > \max\{1, \theta\}$.

6 Lorenz Order

In this section we study the Lorenz ordering and its applications to the most important income distributions, including the members of the family proposed by McDonald (1984). Lorenz curves can be used to define an ordering in the space of the $\mathscr{L}$ distributions. If two distribution functions have associated Lorenz curves which do not intersect, they can be ordered without ambiguity in terms of welfare functions which are symmetric, increasing and quasiconcave (Atkinson, 1970); (Dasgupta *et al.*, 1973; Shorrocks, 1983).

Definition 9.2. Let X and Y be random variables belonging to $\mathscr{L}$ class. The Lorenz order $\leq_L$ on $\mathscr{L}$ is defined by,

$$X \leq_L Y \iff L_X(p) \geq L_Y(p), \ \forall p \in [0,1]. \tag{9.44}$$

If $X \leq_L Y$, then X exhibits less inequality than Y in the Lorenz sense. Note that the Lorenz order is a partial order and is invariant with respect to scale transformation. We present two relevant examples of the Lorenz order:

- Let $X_i \sim \mathscr{P}a(\alpha_i, \sigma_i)$, $i = 1,2$ be Pareto distributions with cdf (9.10). Then:

$$X_1 \leq_L X_2 \iff \alpha_1 \geq \alpha_2.$$

- Let $X_i \sim \mathscr{LN}(\mu_i, \sigma_i)$, $i = 1,2$ be lognormal distributions with cdf (9.16). Then:

$$X_1 \leq_L X_2 \iff \sigma_1 \leq \sigma_2.$$

The proof of these results is direct by checking the Lorenz curve. Other stronger definitions of stochastic orderings are useful in this context. Let X and Y be random variables in $\mathscr{L}$ with distribution functions F_X and F_Y. Star-shaped ordering is defined as follows (Arnold, 1987).

Definition 9.3. We say that X is star-shaped with respect to Y, and we write $X \leq_* Y$ if $F_X^{-1}(x)/F_Y^{-1}(x)$ is a non-increasing function of x.

This definition is specially useful when the quantile function is available in a closed form. The star-shaped ordering implies the Lorenz ordering.

Theorem 9.3. *Suppose that $X, Y \in \mathscr{L}$. If $X \leq_* Y$, then $X \leq_L Y$.*

The proof of this result is as follows. Without loss of generality we may assume that $E(X) = E(Y) = 1$, since both orders are scale invariant. Then,

$$L_X(p) - L_Y(p) = \int_0^p [F_X^{-1}(y) - F_Y^{-1}(y)]dy.$$

Now, since $F_X^{-1}(y)/F_Y^{-1}(y)$ is a non-increasing function, the integrand is first positive and then negative as y ranges from 0 to 1. In consequence the integral assumes its smallest value when $p = 1$. Thus, $L_X(p) - L_Y(p) \geq L_X(1) - L_Y(1) = 1 - 1 = 0$, and $X \leq_L Y$.

This result was used by Wilfling (1996) for proving the Lorenz ordering in the Singh-Maddala family (see below).

The next theorem established by Fellman (1976), examines the Lorenz order between a random variable X and a transformation $g(X)$.

Theorem 9.4. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying*

1. *$g(x) > 0$ for all $x > 0$,*
2. *$g(x)$ is non-decreasing on $[0, \infty)$ and $g(x)/x$ is non-decreasing on $(0, \infty)$.*

If $g(X) \in \mathscr{L}$ then $g(X) \leq_L X$.

Let us now focus our attention on three important income distributions proposed in the literature. The generalized gamma (GG) and generalized beta of the first and second kind (GB1 and GB2) (see McDonald (1984)) are defined in terms of their probability density functions ($a, p, q, \sigma > 0$):

$$f_{GG}(x; a, p, \sigma) = \frac{ax^{ap-1} e^{-(x/\sigma)^a}}{\sigma^{ap}\Gamma(p)}, \;\; x \geq 0, \tag{9.45}$$

$$f_{GB1}(x; a, p, q, \sigma) = \frac{ax^{ap-1}[1 - (x/\sigma)^a]^{q-1}}{\sigma^{ap}B(p,q)}, \;\; 0 \leq x \leq b \tag{9.46}$$

$$f_{GB2}(x; a, p, q, \sigma) = \frac{ax^{ap-1}}{\sigma^{ap}B(p,q)[1 + (x/\sigma)^a]^{p+q}}, \;\; x \geq 0 \tag{9.47}$$

and 0 otherwise. The parameter σ in (9.45), (9.46) and (9.47) is a scale parameter and, due to the fact that the Lorenz ordering is invariant with respect to scale changes, it can be assumed without loss of generality that it is equal to 1. Thus we will represent them as $X \sim GG(a, p)$, $X \sim GB1(a, p, q)$ and $X \sim GB2(a, p, q)$.

These models include an important number of income distributions proposed in the literature. The generalized gamma includes the usual gamma distribution ($GG(1,p) \equiv G(p)$), the Weibull distribution ($GG(a,1) \equiv W(a)$) and the exponential distribution ($GG(1,1) \equiv E(1)$). The GB2 includes the usual beta distribution of the second kind ($GB2(1,p,q) \equiv B2(p,q)$), the Singh-Maddala distribution ($GB2(a,1,q) \equiv SM(a,q)$), the Dagum (1977) distribution ($GB2(a,p,1) \equiv D(a,p)$), the Lomax distribution ($GB2(1,1,q) \equiv L(q)$) and the Fisk distribution ($GB2(a,1,1) \equiv F(a)$). Both of the generalized beta distributions include the generalized gamma as a limiting case.

The next result provides the Lorenz order within the family of generalized gamma distributions defined in (9.45) (Taillie, 1981; Wilfling, 1996).

Theorem 9.5. *Let $X_i \sim GG(a_i, p_i)$, $i = 1,2$ be generalized gamma distributions. Then,*

$$X_1 \geq_L X_2 \iff a_1 \leq a_2 \text{ and } a_1 p_1 \leq a_2 p_2.$$

For the GB2 family, the Lorenz ordering can be verified for certain parametric configurations (Kleiber, 1999).

Theorem 9.6. *Let $X_i \sim GB2(a_i, p_i, q_i)$, $i = 1,2$ be GB2 distributions with finite means. Then*

1. *If $a_1 \leq a_2$, $a_1 p_1 \leq a_2 p_2$ and $a_1 q_1 \leq a_2 q_2$ then $X_1 \geq_L X_2$.*
2. *If $X_1 \geq_L X_2$ then $a_1 p_1 \leq a_2 p_2$ and $a_1 q_1 \leq a_2 q_2$.*

This theorem leaves open some parameter configurations of the kind $a_1 \leq a_2$, $p_1 \geq p_2$ and $q_1 \geq q_2$, with $a_1 p_1 \geq a_2 p_2$ and $a_1 q_1 \geq a_2 q_2$. In spite of these holes, this result allows a complete characterization of many subfamilies coming from GB2 distribution. Some important cases are the following:

- Let $X_i \sim SM(a_i, q_i)$, $i = 1,2$ be Singh-Maddala distributions with cdf given in (9.12). Then (Wilfling and Krämer, 1993; Wilfling, 1996):

$$X_1 \geq_L X_2 \iff a_1 q_1 \leq a_2 q_2, \text{ and } a_1 \leq a_2.$$

- Let $X_i \sim B2(p_i, q_i, \sigma_i)$, $i = 1,2$ be beta distributions of the second kind. Then:

$$X_1 \geq_L X_2 \iff p_1 \leq p_2, \text{ and } q_1 \leq q_2.$$

- Let $X_i \sim D(a_i, q_i)$, $i = 1,2$ be Dagum distributions with cdf (9.14). Then (Kleiber, 1996, 1999)

$$X_1 \geq_L X_2 \iff a_1 q_1 \leq a_2 q_2, \text{ and } a_1 \leq a_2.$$

The following results (Sarabia *et al.*, 2002) establish some additional Lorenz orderings involving the three families of distributions (9.45)-(9.47).

Theorem 9.7. *Assume that one of the following conditions holds:*

1. *Let $X \sim GG(\tilde{a}, \tilde{p})$ and $Y \sim GB2(a, p, q)$, with $aq > 1$, $\tilde{a} \geq a$ and $\tilde{a}\tilde{p} \geq a$.*

2. *Let* $X \sim GB1(a,p,q)$ *and* $Y \sim GB2(\tilde{a},\tilde{p},\tilde{q})$*, with* $\tilde{a}\tilde{q} > 1$*,* $a \geq \tilde{a}$*,* $ap \geq \tilde{a}\tilde{p}$ *and* $aq \geq \tilde{a}\tilde{q}$*.*
3. *Let* $X \sim GB1(a,p,q)$ *and* $Y \sim GG(\tilde{a},\tilde{p})$*, with* $a \geq \tilde{a}$*,* $ap \geq \tilde{a}\tilde{p}$*.*

Then: $X \leq_L Y$

A whole range of literature is available for studying sampling theory of Lorenz curves (Beach and Davidson (1983) and Bishop *et al.* (1989) among others). The problem of making inequality comparison when Lorenz curves intersect has been studied by Shorrocks and Foster (1987) and Davies and Hoy (1995).

7 Variations of the Lorenz Curve

The generalized Lorenz curve (GLC) introduced by Shorrocks (1983) is the most important variation of the LC. The LC is scale invariant and is thus only an indicator of relative inequality. However, it does not provide a complete basis for making social welfare comparisons. The Shorrocks proposal is the generalized Lorenz curve defined as

$$GL_X(p) = \mu_X \cdot L_X(p) = \int_0^p F_X^{-1}(y)dy, \ \ 0 \leq p \leq 1. \tag{9.48}$$

Note that $GL_X(0) = 0$ and $GL_X(1) = \mu_X$. A distribution with a dominating GLC provides greater welfare according to all concave increasing social welfare functions defined on individual incomes (Kakwani (1984) and Davies *et al.* (1998)). On the other hand, the GLC is no longer scale-free and in consequence it determines any distribution with finite mean. The order induced by (9.48) is the second-order stochastic dominance

$$X_1 \leq_{GL} X_2 \iff \int_0^x F_1(y)dy \leq \int_0^x F_2(y)dy, \ \ x \geq 0,$$

which has been studied by Thistle (1989). This order is a new partial ordering, and sometimes it allows a bigger percentage of curves to be ordered than in the Lorenz ordering case. The normative interpretations for the restrictions required on the class of social welfare function to satisfy a GLC dominance have been studied by Shorrocks and Foster (1987) and Davies and Hoy (1994) among others.

Other variations of the LC have been proposed. The absolute Lorenz curve introduced by Moyes (1987) is defined by,

$$AL_X(p) = \mu_X \cdot [L_X(p) - p] = \int_0^p [F_X^{-1}(u) - \mu_X]du, \ \ 0 < p < 1.$$

Note that the new definition changes scale invariance with location invariance. Zenga (1984) defined next concentration curve,

$$ZC(p) = 1 - \frac{F^{-1}(p)}{F_{(1)}^{-1}(p)}, \ \ 0 < p < 1,$$

which is scale free and belongs to the unit square.

8 Multivariate Lorenz Curves

We finish this chapter about Lorenz curves with their extensions to higher dimensions. Although the use of multivariate income data is becoming increasingly more habitual, the proposals of multivariate Lorenz curves are very recent. The pioneer work in this field is due to Taguchi (1972a,b) and Arnold (1987). A recent multivariate version of the LC is based on the concept of Lorenz zonoid of the population introduced by Koshevoy (1995) and Koshevoy and Mosler (1996). Their idea is based on a vision of the usual LC as a convex region bordered by $L(p)$ and $\tilde{L}(p)$, where $\tilde{L}(p) = 1 - L(1-p)$ is the dual Lorenz curve. With this idea, the area between these two curves is the classical Gini index.

The multivariate Lorenz curve is a generalization of this concept to $d+1$ space. Consider the set $\mathscr{L}^d$ of probability distribution functions on R_+^d that have finite and strictly positive expectations $\mu_j = \int_{\mathbb{R}_+} x_j dF(x)$, $j = 1,2,\ldots,d$ and set
$\tilde{\mathbf{x}} = (\tilde{x_1},\ldots,\tilde{x_j})^\top$, $\tilde{x_j} = \frac{x_j}{\mu_j}$, $j = 1,2,\ldots,d$.

Then, $\tilde{\mathbf{X}}$ is the normalization of $\mathbf{X}$ with expectation $\mathbf{1}_d = (1,\ldots,1)^\top$. For $F \in \mathscr{L}^d$, the set

$$LZ(F) = \{\mathbf{z} \in R^{d+1} : \mathbf{z} = (z_0, z_1, \ldots, z_d) = \zeta(h)\}$$

where

$$\zeta(h) = \left(\int_{R_+^d} h(\mathbf{x}) dF(\mathbf{x}), \int_{R_+^d} h(\mathbf{x}) \tilde{\mathbf{x}} dF(\mathbf{x}) \right)$$

for every measurable function $h : R_+^d \to [0,1]$, is called the Lorenz zonoid. The Lorenz zonoid is a convex compact subset of the unit hypercube in R_+^{d+1} containing the origin and the point $\mathbf{1}_{d+1}$ in R^{d+1}. Now, we define a generalization of the LC. For $F \in \mathscr{L}^d$, let us consider the set

$$Z(F) = \{\mathbf{y} \in R_+^d : \mathbf{y} = \int_{R_+^d} h(\mathbf{x}) \tilde{\mathbf{x}} dF(\mathbf{x}), h : R_+^d \to [0,1], \text{measurable}\},$$

which is called the F zonoid.

Note that if $(z_0, z_1, \ldots, z_d) \in LZ(F)$, then $(z_1, \ldots, z_d) \in Z(F)$. The F zonoid is contained in the unit cube on R_+^d and consists of all total portion vectors held by subpopulations. If $d = 1$, $Z(F)$ is the unit interval. For a given $(z_1, \ldots, z_d) \in Z(F)$, we have $(z_0, z_1, \ldots, z_d) \in LZ(F)$ if and only if z_0 is in the closed interval between the smallest and the largest percentage of the population by which the portion vector $(z_1, \ldots, z_d)$ is held. This leads us to the definition of an inverse Lorenz function. The function $l_F : Z(F) \to R_+$ defined as

$$l_F(\mathbf{y}) = \max\{t \in R_+ : (t, \mathbf{y}) \in LZ(F)\},$$

is called the inverse Lorenz function of F. Its graph is the Lorenz surface of F. In terms of a distribution of commodities, the function $l_F(\mathbf{y})$ is equal to the maximum percentage of the population whose total portion amounts to $\mathbf{y}$. The multivariate order is defined as the set inclusion ordering of Lorenz zonoids

$$F \geq_{LZ} G \iff LZ(F) \supseteq LZ(G),$$

and implies the usual Lorenz ordering of all marginal distributions. Finally, the multivariate Gini index is defined as the volume of their Lorenz zonoid $LZ(F)$

$$\mathbf{G} = \text{vol}[LZ(F)] = \frac{E(|\text{det}\mathbf{Q}_F|)}{(d+1)!\prod_{j=1}^{d} E(X_j)},$$

where $\mathbf{Q}_F$ is the $(d+1)\times(d+1)$ matrix with rows $(1,\mathbf{X}_i)$, $i = 1,2,\ldots,d+1$, and $\mathbf{X}_1,\ldots,\mathbf{X}_{d+1}$ are i.i.d. with cdf F.

The Lorenz zonoid order and the multivariate Gini index appear to be good choices as suitable d-dimensional analogs of the Lorenz order and the Gini index. However, there are some problems. Sometimes, the zonoid can have zero volume for some non-degenerate distributions. In response to this, Mosler (2002) has provided a modified definition to rectify this problem. Several alternative definitions for a Lorenz order among d-dimensional non-negative random vectors have been proposed by Arnold (2007).

9 Applications of the Lorenz Curves

Application of Lorenz curves and associated concentration measures is encountered in a broad spectrum of modern scientific fields. Many authors in very different areas of investigation have realized the usefulness of these instruments. Atkinson (1970), in his seminal and influential paper showed that the rules for ordering risky prospects can be written in terms of Lorenz curves (Hadar and Russell, 1969; Hanoch and Levy, 1969; Rothschild and Stiglitz, 1970). Perhaps the greatest number of applications can be found in the usual field of income distributions and poverty (Sen, 1976) but also in the field of finance. In this last field, rules for ordering risky prospects using the Gini index and for the evaluation of risky assets have been studied and developed (Yitzhaki, 1982; Shalit and Yitzhaki, 1984).

Other applications include the use of the Lorenz/Leimkuhler concentration curves in informetric contexts (Burrell, 2005), Lorenz curves of cumulative electricity consumption (Jacobson and Kammen, 2005), LC and Gini index to assess yield inequality within paddocks (Sadras and Bongiovanni, 2004) or characterization of the early growth inequality of ninety crosses of Chinese fir (Ma *et al.*, 2006), to mention but a few examples.

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