Chapter 3
Representable Group Functors

Throughout this chapter, by "ring" we mean a non-zero commutative ring with unity.

3.1 Introduction to Representable Group Functors

Let A be a commutative R-algebra, and let F be the covariant functor defined as $F = \mathrm{Hom}_{R\text{-alg}}(A, -)$. Let S be an object in $\mathrm{Ob}(\mathfrak{I}_{R\text{-alg}})$. What structure on A do we need to endow $F(S)$ with a binary operation?

Let $\Delta : A \to A \otimes_R A$ be an R-algebra homomorphism. For $a \in A$, we shall write the image of a as

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)},$$

where $a_{(1)}, a_{(2)} \in A$. This is the **Sweedler notation** for $\Delta(a)$. It is important to note that "(1)"and "(2)"are not subscripts in the usual sense: $a_{(1)}$ records the left components of the tensors in the expansion of $\Delta(a)$, while $a_{(2)}$ records the right components in $\Delta(a)$.

Since Δ is an R-algebra homomorphism, $\Delta(ab) = \Delta(a)\Delta(b)$ and one writes

$$\Delta(ab) = \left(\sum_{(a)} a_{(1)} \otimes a_{(2)}\right)\left(\sum_{(b)} b_{(1)} \otimes b_{(2)}\right) = \sum_{(a,b)} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}.$$

Note that $\Delta(a_{(i)})$, $i \geq 1$, is written

$$\Delta(a_{(i)}) = \sum_{(a_{(i)})} a_{(i)(1)} \otimes a_{(i)(2)}.$$

R.G. Underwood, *An Introduction to Hopf Algebras*, DOI 10.1007/978-0-387-72766-0_3,

By Proposition 2.4.3, the R-algebra homomorphism Δ determines a natural transformation of representable functors

$$^{a}\Delta : \mathrm{Hom}_{R\text{-alg}}(A \otimes_R A, -) \to \mathrm{Hom}_{R\text{-alg}}(A, -),$$

which by Proposition 2.4.2 can be written as

$$^{a}\Delta : F \times F \to F,$$

where

$$^{a}\Delta_S : F(S) \times F(S) \to F(S),$$

for an R-algebra S. The morphism $^{a}\Delta_S$ (binary operation on $F(S)$) is given by the rule

$$^{a}\Delta_S(f, g)(x) = (f, g)\Delta(x) = \sum_{(x)} f(x_{(1)})g(x_{(2)})$$

for $f, g \in F(S)$, $x \in A$. We set

$$f * g = {}^{a}\Delta_S(f, g).$$

Example 3.1.1. Let $A = RG$ with G a finite Abelian group, and let $F = \mathrm{Hom}_{R\text{-alg}}(RG, -)$ be the corresponding functor. Then the map $\Delta : RG \to RG \otimes_R RG$ defined by

$$\sum_{\tau \in G} a_\tau \tau \mapsto \sum_{\tau \in G} a_\tau (\tau \otimes \tau)$$

is an R-algebra homomorphism and determines a binary operation on $F(S)$ given by

$$(f * g)\left(\sum_{\tau \in G} a_\tau \tau\right) = \sum_{\tau \in G} a_\tau f(\tau) g(\tau).$$

Like all binary operations on sets, those on $F(S)$ can be associative or commutative, admit an identity or inverses, and so forth. Since we have defined a binary operation on $F(S)$ using an algebra map, we can also describe the properties of the binary operation by specifying conditions on this algebra map. We first look at conditions for commutativity and associativity.

The map $t : A \otimes_R A \to A \otimes_R A$ defined as $\mathrm{t}(a \otimes b) = b \otimes a$ is the **twist map**.

Proposition 3.1.1. *Let $\Delta : A \to A \otimes_R A$ be an R-algebra map that satisfies*

$$\Delta(a) = t(\Delta(a)) \tag{3.1}$$

for all $a \in A$. Then the corresponding binary operation on $F(S) = \mathrm{Hom}_{R\text{-alg}}(A, S)$ is commutative.

Proof. We show that $(f * g)(a) = (g * f)(a)$ for all $a \in A$. We have

$$\begin{aligned}(f * g)(a) &= (f, g)\sum_{(a)} a_{(1)} \otimes a_{(2)}\\ &= (f, g)\sum_{(a)} a_{(2)} \otimes a_{(1)} \quad \text{by (3.1)}\\ &= \sum_{(a)} f(a_{(2)})g(a_{(1)})\\ &= \sum_{(a)} g(a_{(1)})f(a_{(2)}) \quad \text{since } S \text{ is commutative}\\ &= (g, f)\sum_{(a)} a_{(1)} \otimes a_{(2)}\\ &= (g * f)(a).\end{aligned}$$

□

Let $I : A \to A$ denote the identity map. Let $f, g, h \in F(S)$. For $a \otimes b \otimes c \in A \otimes A \otimes A$, put $(f, g, h)(a \otimes b \otimes c) = f(a)g(b)h(c)$.

Proposition 3.1.2. *Let $\Delta : A \to A \otimes_R A$ be an R-algebra map that satisfies*

$$(I \otimes \Delta)\Delta(a) = (\Delta \otimes I)\Delta(a) \tag{3.2}$$

for all $a \in A$. Then the corresponding binary operation on $F(S)$ is associative.

Proof. We show that $(f * (g * h))(a) = ((f * g) * h)(a)$. Now,

$$\begin{aligned}(f * (g * h))(a) &= \sum_{(a)} f(a_{(1)})(g * h)(a_{(2)})\\ &= \sum_{(a)} f(a_{(1)}) \sum_{(a_{(2)})} g\left(a_{(2)_{(1)}}\right) h\left(a_{(2)_{(2)}}\right)\\ &= \sum_{(a,a_{(2)})} f(a_{(1)})g\left(a_{(2)_{(1)}}\right) h\left(a_{(2)_{(2)}}\right)\\ &= (f, g, h) \sum_{(a,a_{(2)})} a_{(1)} \otimes a_{(2)_{(1)}} \otimes a_{(2)_{(2)}}\\ &= (f, g, h) \sum_{(a,a_{(1)})} a_{(1)_{(1)}} \otimes a_{(1)_{(2)}} \otimes a_{(2)} \quad \text{by (3.2)}\\ &= \sum_{(a,a_{(1)})} f\left(a_{(1)_{(1)}}\right) g\left(a_{(1)_{(2)}}\right) h(a_{(2)})\end{aligned}$$

$$= \sum_{(a)} \sum_{(a_{(1)})} f\left(a_{(1)_{(1)}}\right) g\left(a_{(1)_{(2)}}\right) h(a_{(2)})$$

$$= \sum_{(a)} (f * g)(a_{(1)}) h(a_{(2)})$$

$$= ((f * g) * h)(a).$$

□

What condition on Δ guarantees the existence of a multiplicative identity element? Let $m : A \otimes_R A \to A$ denote the multiplication map of A defined as $m(\sum a \otimes b) = \sum ab$, and let $\lambda : R \to S$ denote the structure map of the commutative R-algebra S. We have $\lambda(1_R) = 1_S$.

Proposition 3.1.3. *Let $\epsilon : A \to R$ be an R-algebra homomorphism for which*

$$m(I \otimes \epsilon)\Delta(a) = a = m(\epsilon \otimes I)\Delta(a) \tag{3.3}$$

for $a \in A$. Then the R-algebra homomorphism $\lambda\epsilon : A \to S$ satisfies

$$(\lambda\epsilon) * f = f = f * (\lambda\epsilon)$$

for all $f \in F(S)$. Thus $\lambda\epsilon$ is a left and right identity for the binary operation on $F(S)$.

Proof.

$$((\lambda\epsilon) * f)(a) = \sum_{(a)} \lambda(\epsilon(a_{(1)})) f(a_{(2)})$$

$$= f\left(\sum_{(a)} \epsilon(a_{(1)}) a_{(2)}\right)$$

$$= f(a) \quad \text{by (3.3)}.$$

In a similar manner, one can show that $f = f * (\lambda\epsilon)$. □

What condition on Δ guarantees the existence of multiplicative inverse elements?

Proposition 3.1.4. *Let $f \in F(S)$. Let $\sigma : A \to A$ be an R-algebra homomorphism for which*

$$m(I \otimes \sigma)\Delta(a) = \epsilon(a)1_A = (\sigma \otimes I)\Delta(a) \tag{3.4}$$

for $a \in A$. Then the R-algebra homomorphism $f\sigma : A \to S$ satisfies

$$(f\sigma) * f = \lambda\epsilon = f * (f\sigma).$$

Thus $f\sigma$ is a left and right inverse for f with respect to the binary operation on $F(S)$.

Proof. We have

$$\begin{aligned}((f\sigma) * f)(a) &= \sum_{(a)} f(\sigma(a_{(1)}))f(a_{(2)}) \\ &= f\left(\sum_{(a)} \sigma(a_{(1)})a_{(2)}\right) \\ &= f(\epsilon(a)1_A) \quad \text{by (3.4)} \\ &= \epsilon(a)f(1_A) \\ &= \epsilon(a)1_S \\ &= \lambda(\epsilon(a)).\end{aligned}$$

Likewise, one has $\lambda\epsilon = f * (f\sigma)$. □

So we have arrived at the following.

Proposition 3.1.5. *Let* $F = \mathrm{Hom}_{R\text{-alg}}(A, -)$ *be a functor, together with additional* R*-algebra maps*

$$\Delta : A \to A \otimes_R A, \quad \epsilon : A \to R, \quad \sigma : A \to A,$$

that satisfy conditions (3.2), (3.3), *and* (3.4), *respectively. Then, for each* $S \in \mathrm{Ob}(\Im_{R\text{-alg}})$, *the set* $F(S)$ *is a group under the binary operation* $*$.

Proof. As one can easily verify, $F(S)$ together with $*$ satisfies the requirements for $F(S)$ to be a group. □

The functor F in Proposition 3.1.5 is a **representable group functor**, which is also called an **affine group scheme** or an ***R*-group scheme**. The R-algebra A is the **representing algebra of** F; we write $R[F] = A$. Note that F is a functor from the category of commutative R-algebras to the category of groups, where the morphisms are homomorphisms of groups. The map Δ is the **comultiplication map of** A, ϵ is the **counit map of** A, and σ is the **coinverse map of** A. When necessary to avoid confusion, we shall denote the comultiplication, counit, and coinverse maps of A by Δ_A, ϵ_A, and σ_A, respectively.

Here are some important examples of R-group schemes.

The ring R itself as an R-algebra represents the R-group scheme $F = \mathrm{Hom}_{R\text{-alg}}(R, -)$. The comultiplication on R is defined by $\Delta(1) = 1 \otimes 1$, the counit is defined by $\epsilon(1) = 1$, and the coinverse is given by $\sigma(1) = 1$. For an R-algebra S, $F(S)$ consists of a single element, the R-algebra structure map $\lambda : R \to S$, and thus F is the **trivial *R*-group scheme** denoted by **1**, or more simply 1 when the context is clear.

For a non-trivial example, let $R[X]$ denote the algebra of polynomials in the indeterminate X. Then $F = \mathrm{Hom}_{R\text{-alg}}(R[X], -)$ is an R-group scheme, with comultiplication defined by $\Delta(X) = X \otimes 1 + 1 \otimes X$, counit defined by $\epsilon(X) = 0$, and coinverse given by $\sigma(X) = -X$.

Let us examine this group scheme more closely. Let S be an R-algebra. The group $F(S)$ consists of all R-algebra maps $\phi : R[X] \to S$. These homomorphisms are precisely the evaluation homomorphisms and are determined by sending the indeterminate X to some element a in S. We see that $F(S)$ consists of the algebra maps $\phi_a : R[X] \to S$, where $X \mapsto a$, $a \in S$. We have $\phi_a(1_{R[X]}) = 1_S$.

Now, how does the group product $*$ in $F(S)$ work? Let ϕ_a, ϕ_b be elements of $F(S)$, and let $m : S \otimes_R S \to S$ denote the multiplication map of S. Then

$$\begin{aligned}(\phi_a * \phi_b)(X) &= m(\phi_a \otimes \phi_b)\Delta(X)\\ &= m(\phi_a \otimes \phi_b)(X \otimes 1_{R[X]} + 1_{R[X]} \otimes X)\\ &= m(a \otimes 1_S) + m(1_S \otimes b)\\ &= a + b,\end{aligned}$$

and so $\phi_a * \phi_b = \phi_{a+b}$. We identify $F(S)$ with the additive group $S, +$ of the ring S. For this reason, the group functor F is called the **additive *R*-group scheme**, denoted by $\mathbf{G}_a$.

For another example, let $R[X_1, X_2]$ be the R-algebra of polynomials in the indeterminates X_1, X_2. Let $I = (X_1X_2 - 1)$, and consider the quotient ring $R[X_1, X_2]/I$. There is an isomorphism of R-algebras,

$$f : R[X_1, X_2]/I \to R[X, X^{-1}],\ X \text{ indeterminate},$$

defined by $X_1 \mapsto T$, $X_2 \mapsto X^{-1}$. The functor $F = \mathrm{Hom}_{R\text{-alg}}(R[X, X^{-1}], -)$ is an R-group scheme with comultiplication Δ defined by $\Delta(X) = X \otimes X$, counit defined as $\epsilon(X) = 1$, and coinverse given as $\sigma(X) = X^{-1}$.

Let us see how this group scheme works. Let S be an R-algebra. The group $F(S)$ consists of the R-algebra maps $\phi : R[X, X^{-1}] \to S$. These maps are determined by sending the variable X to some element $\phi(X)$ in S. But in order for ϕ to be a ring homomorphism, we must have $\phi(X^{-1}) = (\phi(X))^{-1}$, and so this element must be a unit of S. We see that $F(S)$ consists of all algebra maps $\phi_u : R[X, X^{-1}] \to S$, where $X \mapsto u$, $u \in U(S)$.

Now, how is the group product in $F(S)$ defined? Let ϕ_u, ϕ_v be elements of $F(S)$. Then

$$\begin{aligned}(\phi_u * \phi_v)(X) &= m(\phi_u \otimes \phi_v)\Delta(X)\\ &= m(\phi_u \otimes \phi_v)(X \otimes X)\\ &= m(u \otimes v)\\ &= uv.\end{aligned}$$

Thus, $\phi_u * \phi_v = \phi_{uv}$, and we identify $F(S)$ with the multiplicative group of units in S. This is the **multiplicative *R*-group scheme**, which is denoted by $\mathbf{G}_m$.

Here are two more examples of R-group schemes. Let $A = R[X]/(X^n - 1)$. Then $F = \mathrm{Hom}_{R\text{-alg}}(A, -)$ is an R-group scheme with $\Delta(X) = X \otimes X$, $\epsilon(X) = 1$, and $\sigma(X) = X^{n-1} = X^{-1}$. An element $\phi \in F(S)$ is determined by sending X to an element s in S for which $s^n = 1$. For this reason, F is the **multiplicative group of the** n**th roots of unity**, denoted by μ_n.

Next, let $A = R[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}]$ be the polynomial algebra in the indeterminates $X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}$. Let J be the principal ideal of A generated by $X_{1,1}X_{2,1} - X_{1,2}X_{2,2} - 1$, and consider the quotient ring $B = A/J$. Then $F = \mathrm{Hom}_{R\text{-alg}}(B, -)$ is an R-group scheme where $F(S)$ is the (multiplicative) group of 2×2 matrices M with entries in S with $\det(M) = 1$. This is the **special linear group (scheme) of order** 2, denoted by $\mathbf{SL}_2$. We leave it as an exercise to formulate the comultiplication, counit, and coinverse maps on the representing algebra B.

3.2 Homomorphisms of R-Group Schemes

What are the maps between R-group schemes?

Definition 3.2.1. Let $\psi : F \to G$ be a natural transformation of R-group schemes. Then ψ is a **homomorphism of R-group schemes** if $\psi_S : F(S) \to G(S)$ is a homomorphism of groups for all $S \in \mathrm{Ob}(\Im_{R\text{-alg}})$.

By Yoneda's Lemma, the homomorphism $\psi : F \to G$ corresponds to an R-algebra homomorphism $\phi : B \to A$ with $R[F] = A$, $R[G] = B$. If ψ is a homomorphism of R-group schemes, what can we infer about the map ϕ?

By Yoneda's Lemma, the comultiplication map $\Delta_A : A \to A \otimes_R A$ corresponds to the associated map $\alpha : \mathrm{Hom}_{R\text{-alg}}(A \otimes_R A, -) \to \mathrm{Hom}_{R\text{-alg}}(A, -)$. There is a map $\alpha_{A\otimes_R A} : \mathrm{Hom}_{R\text{-alg}}(A \otimes_R A, A \otimes_R A) \to \mathrm{Hom}_{R\text{-alg}}(A, A \otimes_R A)$. Observe that $\alpha_{A\otimes_R A}$ is the group product in $\mathrm{Hom}_{R\text{-alg}}(A, A \otimes_R A)$.

Let $I \in \mathrm{Hom}_{R\text{-alg}}(A, A)$ be the identity map, and let $I \otimes I \in \mathrm{Hom}_{R\text{-alg}}(A \otimes_R A, A \otimes_R A)$ be defined as $(I \otimes I)(a \otimes b) = a \otimes b$. Now, for $x \in B$,

$$
\begin{aligned}
\psi_{A\otimes_R A}(\alpha_{A\otimes A}(I \otimes I))(x) &= \alpha_{A\otimes_R A}(I \otimes I)(\phi(x)) \\
&= (I \otimes I)(\Delta_A(\phi(x))) \\
&= \Delta_A(\phi(x)).
\end{aligned}
$$

Also by Yoneda's Lemma, the comultiplication map $\Delta_B : B \to B \otimes_R B$ corresponds to the associated map $\beta : \mathrm{Hom}_{R\text{-alg}}(B \otimes_R B, -) \to \mathrm{Hom}_{R\text{-alg}}(B, -)$. There is a map $\beta_{A\otimes_R A} : \mathrm{Hom}_{R\text{-alg}}(B \otimes_R B, A \otimes_R A) \to \mathrm{Hom}_{R\text{-alg}}(B, A \otimes_R A)$; $\beta_{A\otimes_R A}$ is the group operation in $\mathrm{Hom}_{R\text{-alg}}(B, A \otimes_R A)$.

Since ψ is a group homomorphism,

$$\begin{aligned}\psi_{A\otimes_R A}(\alpha_{A\otimes A}(I\otimes I))(x) &= \beta_{A\otimes_R A}(\psi_A(I)\otimes\psi_A(I))(x)\\ &= (\psi_A(I)\otimes\psi_A(I))(\Delta_B(x))\\ &= (I\otimes I)(\phi\otimes\phi)(\Delta_B(x))\\ &= (\phi\otimes\phi)(\Delta_B(x)),\end{aligned}$$

and so the R-algebra homomorphism ϕ must satisfy the property

$$\Delta_A(\phi(x)) = (\phi\otimes\phi)\Delta_B(x) \tag{3.5}$$

for all $x \in B$.

Moreover, since the identity element maps to the identity element under a group homomorphism,

$$\begin{aligned}\epsilon_A(\phi(x))1_R &= \lambda_R(\epsilon_A(\phi(x)))\\ &= (\lambda_R\epsilon_A)(\phi(x))\\ &= \psi_R(\lambda_R\epsilon_A)(x)\\ &= (\lambda_R\epsilon_B)(x)\\ &= \epsilon_B(x)1_R,\end{aligned}$$

and so ϕ satisfies

$$\epsilon_A(\phi(x)) = \epsilon_B(x) \tag{3.6}$$

for all $x \in B$. Finally,

$$\psi_A(I_A\sigma_A)(x) = (I_A\sigma_A)(\phi(x)) = \sigma_A(\phi(x)),$$

which since ψ_A is a group homomorphism equals

$$(\psi_A(I_A)\sigma_B)(x) = I_A\phi(\sigma_B(x)) = \phi(\sigma_B(x)),$$

and so ϕ satisfies

$$\sigma_A(\phi(x)) = \phi(\sigma_B(x)). \tag{3.7}$$

So we have arrived at the following characterization of a homomorphism of R-group schemes.

Definition 3.2.2. Let $\psi : F \to G$ be a natural transformation of R-group schemes with $R[F] = A$, $R[G] = B$. Then ψ is a **homomorphism of R-group schemes** if the corresponding map $\phi : B \to A$ satisfies conditions (3.5), (3.6), and (3.7).

Actually, it suffices to show that condition (3.5) holds in Definition 3.2.2.

Proposition 3.2.1. *Let $\psi : F \to G$ be a natural transformation of R-group schemes with $R[F] = A$, $R[G] = B$. Then ψ is a homomorphism of R-group schemes if the corresponding map $\phi : B \to A$ satisfies the condition*

$$(\phi \otimes \phi)\Delta_B(x) = \Delta_A(\phi(x))$$

for all $x \in B$.

Proof. Exercise. □

Let $\psi : F \to G$ be a homomorphism of R-group schemes with $R[F] = A$, $R[G] = B$. Let $\phi : B \to A$ denote the corresponding R-algebra homomorphism. Our goal is to arrive at a suitable definition of the kernel of ψ. One would expect $\ker(\psi)$ to be a group scheme over R. We know that $\psi_S : F(S) \to G(S)$ is a group homomorphism for each $S \in \mathrm{Ob}(\Im_{R\text{-alg}})$, and so to describe the kernel of ψ we consider $\ker(\psi_S)$. Since $\lambda\epsilon_B$ is the identity element of $G(S)$, we have

$$\begin{aligned}\ker(\psi_S) &= \{f \in F(S) :\ \psi_S(f)(x) = (\lambda\epsilon_B)(x), \forall x \in B\}\\ &= \{f \in F(S) :\ f(\phi(x)) = \lambda(\epsilon_B(x)), \forall x \in B\}.\end{aligned}$$

Now A can be viewed as a B-algebra with scalar multiplication defined as $b \cdot a = \phi(b)a$ for $a \in A$, $b \in B$. Likewise, R can be viewed as a B-algebra with scalar multiplication defined by $b \cdot r = \epsilon_B(b)r$, and S is a B-algebra with $b \cdot s = \lambda(\epsilon_B(b))s$ for $s \in S$.

One has the tensor product $A \otimes_B R$, which is also a B-algebra, and the representable functor

$$N = \mathrm{Hom}_{B\text{-alg}}(A \otimes_B R, -),$$

which is defined on the category of commutative B-algebras.

Proposition 3.2.2. *Let S be an R-algebra and a B-algebra with scalar multiplication defined as $b \cdot s = \lambda(\epsilon_B(b))s$. Then $N(S) = \ker(\psi_S)$.*

Proof. By Proposition 2.4.2,

$$\mathrm{Hom}_{B\text{-alg}}(A \otimes_B R, S) = \mathrm{Hom}_{B\text{-alg}}(A, S) \times \mathrm{Hom}_{B\text{-alg}}(R, S).$$

Since $\mathrm{Hom}_{B\text{-alg}}(R, S)$ consists only of the map λ, $N(S)$ is identified with the elements $f \in \mathrm{Hom}_{B\text{-alg}}(A, S)$. We have

$$
\begin{aligned}
f(\phi(x)) &= f(\phi(x)1_A) \\
&= f(x \cdot 1_A) \\
&= x \cdot f(1_A) \\
&= x \cdot 1_S \\
&= \lambda(\epsilon_B(x))1_S \\
&= (\lambda\epsilon_B)(x).
\end{aligned}
$$

Thus $N(S) = \ker(\psi_S)$. □

Thus we have the kernel of ψ described as a representable functor on the category of B-algebras. Our next step is to translate to R-algebras.

The **augmentation ideal of** B, denoted by B^+, is the kernel of the counit map $\epsilon_B : B \to R$. We have the short exact sequence of R-modules

$$0 \to B^+ \to B \to R \to 0,$$

and so as R-modules $R \cong B/B^+$. But B/B^+ is also a B-algebra through $b \cdot \overline{c} = \overline{bc}$, and so

$$A \otimes_B R \cong A \otimes_B (B/B^+) \cong A/\phi(B^+)A$$

as R-algebras. Thus there is a representable functor on the category of commutative R-algebras defined as

$$\mathrm{Hom}_{R\text{-alg}}(A/\phi(B^+)A, -).$$

We identify $\mathrm{Hom}_{R\text{-alg}}(A/\phi(B^+)A, S)$ with $N(S)$. (Note that S is viewed simultaneously as an R-algebra and a B-algebra.)

We claim that N is an R-group scheme. To prove this, we will need some lemmas.

Lemma 3.2.1. *Let B be an R-algebra, and let J be an ideal of B. Then there is an isomorphism of R-algebras*

$$B/J \otimes_R B/J \cong (B \otimes_R B)/(J \otimes_R B + B \otimes_R J).$$

Proof. First note that there is an R-algebra map

$$\alpha : B \otimes_R B \to B/J \otimes_R B/J,$$

defined as $\alpha(a \otimes b) = \overline{a} \otimes \overline{b}$. Now $J \otimes_R B + B \otimes_R J \subseteq \ker(\alpha)$, and so there exists an R-algebra map

$$\overline{\alpha} : (B \otimes_R B)/(J \otimes_R B + B \otimes_R J) \to B/J \otimes_R B/J$$

with $\overline{\alpha}(\overline{a \otimes b}) = \overline{a} \otimes \overline{b}$.

Next, let β denote the canonical surjection of R-algebras

$$\beta : B \otimes_R B \to (B \otimes_R B)/(J \otimes_R B + B \otimes_R J)$$

defined as $\beta(a \otimes b) = \overline{a \otimes b}$. Since $\beta(J \otimes 1) = \beta(1 \otimes J) = 0$, there exists an R-algebra map

$$\overline{\beta} : B/J \otimes_R B/J \to (B \otimes_R B)/(J \otimes_R B + B \otimes_R J)$$

defined as $\overline{\beta}(\overline{a} \otimes \overline{b}) = \overline{a \otimes b}$. Clearly, $(\overline{\alpha})^{-1} = \overline{\beta}$, and thus $\overline{\beta}$ is an isomorphism. □

We apply Lemma 3.2.1 to the ideal B^+ to show that $\Delta_B(B^+) \subseteq B \otimes_R B^+ + B^+ \otimes_R B$.

Lemma 3.2.2. *Let B^+ denote the augmentation ideal of B. Then $\Delta_B(B^+) \subseteq B \otimes_R B^+ + B^+ \otimes_R B$.*

Proof. Let $\Delta_R : R \to R \otimes_R R$ denote the comultiplication map of the trivial group scheme $\mathrm{Hom}_{R\text{-alg}}(R, -)$, and let $(\epsilon \otimes \epsilon) : B \otimes_R B \to R \otimes_R R$ be the R-algebra map defined by $a \otimes b \to \epsilon(a) \otimes \epsilon(b)$ for $a, b \in B$. For all $b \in B$,

$$\begin{aligned}
(\epsilon \otimes \epsilon)\Delta_B(b) &= \sum_{(b)} \epsilon(b_{(1)}) \otimes \epsilon(b_{(2)}) \\
&= (\epsilon \otimes 1) \sum_{(b)} b_{(1)} \otimes \epsilon(b_{(2)}) \\
&= (\epsilon \otimes 1) \sum_{(b)} b_{(1)}\epsilon(b_{(2)}) \otimes 1 \\
&= (\epsilon \otimes 1)(b \otimes 1) \quad \text{by (3.3)} \\
&= \epsilon(b) \otimes 1 \\
&= \Delta_R(\epsilon(b)).
\end{aligned}$$

Thus, for $b \in B^+$,

$$(\epsilon \otimes \epsilon)\Delta_B(b) = \Delta_R(\epsilon(b)) = 0,$$

and so

$$\Delta_B(B^+) \subseteq \ker(\epsilon \otimes \epsilon). \tag{3.8}$$

Now, by Lemma 3.2.1 there is an R-algebra isomorphism

$$\alpha : B/B^+ \otimes_R B/B^+ \to (B \otimes_R B)/(B \otimes_R B^+ + B^+ \otimes_R B),$$

which yields a surjective homomorphism of R-algebras

$$\beta : B \otimes_R B \to (B \otimes_R B)/(B \otimes_R B^+ + B^+ \otimes_R B)$$

since $R \otimes_R R \cong B/B^+ \otimes_R B/B^+$. Consequently,

$$\ker(\epsilon \otimes \epsilon) \subseteq \ker(\beta) = B \otimes_R B^+ + B^+ \otimes_R B,$$

and so, by (3.8), $\Delta_B(B^+) \subseteq B \otimes_R B^+ + B^+ \otimes_R B$. □

Lemma 3.2.3. *Let $\psi : F \to G$ be a homomorphism of R-group schemes with $\phi : B \to A$ the corresponding map of R-algebras. Then:*

(i) $\Delta_A(\phi(B^+)A) \subseteq \phi(B^+)A \otimes_R A + A \otimes_R \phi(B^+)A$.
(ii) $\epsilon_A(\phi(B^+)A) = 0$.
(iii) $\sigma_A(\phi(B^+)A) \subseteq \phi(B^+)A$.

Proof. We prove (i) and leave (ii) and (iii) as exercises. We have

$$\begin{aligned}\Delta_A(\phi(B^+)A) &\subseteq \Delta_A(\phi(B^+))(A \otimes_R A)\\ &\subseteq ((\phi \otimes \phi)(\Delta_B(B^+)))(A \otimes_R A)\\ &\subseteq ((\phi \otimes \phi)(B \otimes_R B^+ + B^+ \otimes_R B))(A \otimes_R A) \quad \text{by Lemma 3.2.2}\\ &\subseteq \phi(B^+)A \otimes_R A + A \otimes_R \phi(B^+)A.\end{aligned}$$

□

We are now in a position to show that N is an R-group scheme.

Proposition 3.2.3. *The representable functor $N = \mathrm{Hom}_{R\text{-alg}}(A/\phi(B^+)A, -)$ is an R-group scheme.*

Proof. Let $C = A/\phi(B^+)A$, and put $J = \phi(B^+)A \otimes_R A + A \otimes_R \phi(B^+)A$. We show that there exist R-algebra maps $\Delta : C \to C \otimes_R C$, $\epsilon : C \to R$, and $\sigma : C \to C$ that satisfy conditions (3.2), (3.3), and (3.4), respectively.

First, let $\Delta_A : A \to A \otimes_R A$ denote the comultiplication of A, and let $\beta : A \otimes_R A \to (A \otimes_R A)/J$ be the canonical surjection of R-algebras. By Lemma 3.2.3(i), $\Delta_A(\phi(B^+)A) \subseteq J$, and so there exists an R-algebra map

$$\overline{\beta\Delta_A} : C \to (A \otimes_R A)/J.$$

By Lemma 3.2.1, there is an isomorphism

$$\overline{\alpha} : (A \otimes_R A)/J \to C \otimes_R C,$$

and so the map defined as $\Delta = \overline{\alpha}\,\overline{\beta\Delta_A}$ is an R-algebra map $\Delta : C \to C \otimes_R C$, which satisfies the condition

$$(I \otimes \Delta)\Delta = (\Delta \otimes I)\Delta$$

since Δ_A satisfies condition (3.2).

Next, let $\epsilon_A : A \to R$ be the counit map. By Lemma 3.2.3(ii), there exists an R-algebra map $\epsilon : C \to R$ that satisfies, for all $c \in C$,

$$(I \otimes \epsilon)\Delta(c) = c = (\epsilon \otimes I)\Delta(c),$$

since ϵ_A satisfies (3.3).

Finally, by Lemma 3.2.3(iii), there is a map $\sigma : C \to C$, induced from σ_A, that evidently satisfies the condition

$$(I \otimes \sigma)\Delta(c) = \epsilon(c)1 = (\sigma \otimes I)\Delta(c)$$

for all $c \in C$. Thus $N = \mathrm{Hom}_{R\text{-alg}}(C, -)$ is an R-group scheme. □

Now, we can make the following definition.

Definition 3.2.3. Let $\psi : F \to G$ be a homomorphism of R-group schemes, and let $\phi : B \to A$ denote the corresponding homomorphism of R-algebras. Then the **kernel of** ψ is the R-group scheme defined as

$$N = \mathrm{Hom}_{R\text{-alg}}(A/\phi(B^+)A, -).$$

Let $\psi : F \to G$ be a homomorphism of R-group schemes with $\ker(\psi) = N$. We define an **exact sequence of *R*-group schemes** to be the sequence

$$1 \to N \to F \stackrel{\psi}{\to} G$$

with $R[N] = A/\phi(B^+)A$, $R[F] = A$, and $R[G] = B$. For each $S \in \mathrm{Ob}(\Im_{R\text{-alg}})$, the sequence

$$1 \to N(S) \to F(S) \to G(S)$$

is an exact sequence of groups with $N(S) = \ker(F(S) \to G(S))$.

3.3 Short Exact Sequences

In the preceding section, we showed that a homomorphism of R-group schemes gives rise to an exact sequence of R-group schemes. This is analogous to the situation for ordinary abstract groups. The analogy breaks down, however, when we consider short exact sequences. For abstract groups, an exact sequence always extends to a short exact sequence, but this is not the case for R-group schemes. The problem is that the map $S \mapsto F(S)/N(S)$ is not always an R-group scheme; that is, this map does not necessarily take the form of $\mathrm{Hom}_{R\text{-alg}}(Q, -)$ for some R-algebra Q.

Let M, L be modules over a ring S, let T be a ring, and let $\varrho : S \to T$ be a ring homomorphism. We will consider T an S-module with $s \cdot t = \varrho(s)t$.

Definition 3.3.1. A ring homomorphism $\varrho : S \to T$ is **flat** if, whenever $\alpha : M \to L$ is an injection of S-modules, the map $\varphi : M \otimes_S T \to L \otimes_S T$ defined as $\varphi(m \otimes 1) = \alpha(m) \otimes 1$ is also an injection.

As an example, we prove the following.

Proposition 3.3.1. *Let f be a non-nilpotent element of the ring S. Then the localization map $S \to S_f$, $s \mapsto s/1$, is flat.*

Proof. Let $\alpha : M \to L$ be an injection of S-modules, and let $\varphi : M \otimes_S S_f \to L \otimes_S S_f$ be the map defined as $\varphi(m \otimes 1) = \alpha(m) \otimes 1$. Suppose $\varphi(m \otimes 1) = \varphi(n \otimes 1)$, so that $\alpha(m) \otimes 1 = \alpha(n) \otimes 1$. It follows that $\alpha(m - n) \otimes 1 = 0$, and so there exists an element $f^i \in \{1, f, f^2, \dots\}$ such that $f^i \alpha(m - n) = 0$. Thus $\alpha(f^i(m-n)) = \alpha(f^i m - f^i n) = 0$, and hence $f^i m = f^i n$ since α is an injection. Consequently, $m \otimes f^i = n \otimes f^i$, and so $m \otimes 1 = n \otimes 1$ since f^i is a unit of S_f. It follows that φ is an injection. □

Lemma 3.3.1. *Let $\varrho : S \to T$ be flat, and suppose that $P \cdot T \neq T$ for every maximal ideal P of S. If M is a non-zero S-module, then $M \otimes_S T$ is non-zero.*

Proof. Let $m \neq 0$ be an element of M, and let I be the annihilator ideal of m. Then $Sm \cong S/I$. Since $S/I \cong Sm \subseteq M$, the flatness of ϱ implies the existence of an injection $(S/I) \otimes_S T \to M \otimes_S T$. Note that $(S/I) \otimes_S T \cong T/(I \cdot T)$. There exists a maximal ideal P with $I \subseteq P$; hence $T/(I \cdot T) \neq 0$ since $T/(P \cdot T) \neq 0$. It follows that $M \otimes_S T \neq 0$. □

Lemma 3.3.2. *Let $\varrho : S \to T$ be flat, and suppose that $P \cdot T \neq T$ for every maximal ideal P of S. Let $\alpha : M \to N$ be a map of S-modules, and let $\alpha' : M \otimes_S T \to N \otimes_S T$ be the induced map defined by $m \otimes t \mapsto \alpha(m) \otimes t$. Then, if α' is an injection, so is α.*

Proof. Suppose that $\alpha : M \to N$ has non-zero kernel L. Then, by Lemma 3.3.1, $L \otimes_S T \neq 0$. By the flatness of ϱ, $L \otimes_S T \to M \otimes_S T$ is an injection. Since $L \otimes_S T$ is in the kernel of α', α' is not an injection, which proves the lemma. □

Definition 3.3.2. A flat map $S \to T$ is **faithfully flat** if the map $\varphi : M \to M \otimes_S T$ defined as $\varphi(m) = m \otimes 1$ is an injection for all S-modules M.

The localization map $S \to S_f$ may not be faithfully flat, though it can be used to build a faithfully flat map. Let $\{f_1, f_2, \dots, f_n\}$ be a finite set of non-nilpotent elements of S, and suppose that the ideal generated by $\{f_1, f_2, \dots, f_n\}$ is S. Then the map $\varrho : S \to \prod_{i=1}^{n} S_{f_i}$ defined as $s \mapsto ((s/1)_{f_i})$ is faithfully flat. (Prove this as an exercise. Hint: Use Lemma 3.3.1.)

Proposition 3.3.2. *Let $\varrho : S \to T$ be faithfully flat. Then ϱ is an injection.*

Proof. Since $S \to T$ is faithfully flat, the map $\varphi : S \to S \otimes_S T = T$ is an injection. □

Proposition 3.3.3. *Let $\alpha : S \to Q$ be a ring homomorphism, and consider Q an S-module with $s \cdot q = \alpha(s)q$. Suppose $S \to T$ is faithfully flat. Then $Q \to Q \otimes_S T$, $q \mapsto q \otimes 1$, is faithfully flat.*

Proof. Let M be a Q-module (also an S-module), and let $\varrho : M \to M \otimes_Q (Q \otimes_S T)$ be the Q-module map defined by $m \mapsto m \otimes (1 \otimes 1)$. There is an isomorphism $\phi : M \otimes_Q (Q \otimes_S T) \to M \otimes_S T$ defined by $m \otimes (q \otimes t) \mapsto q \cdot m \otimes t$. Now, $\phi\varrho : M \to M \otimes_S T$ is an injection by the faithful flatness of $S \to T$. Consequently, ϱ is an injection, and so $Q \to Q \otimes_S T$ is faithfully flat. □

Proposition 3.3.4. *Let $S \to T$ be faithfully flat, and let $x \in \operatorname{Spec} S$. Then the induced map $S_x \to T \otimes_S S_x$ is faithfully flat.*

Proof. Let M be an S_x-module, and let $\varphi : M \to M \otimes_{S_x} (T \otimes_S S_x)$ be the map defined as $m \mapsto m \otimes (1 \otimes 1)$. Note that M is also an S-module with

$$M \otimes_{S_x} (T \otimes_S S_x) \cong (M \otimes_S T) \otimes_S S_x.$$

Since $S \to T$ is faithfully flat, there is an injection $M \to M \otimes_S T$ given as $m \mapsto m \otimes 1$. Consequently, φ is also an injection. □

Faithful flatness is a critical condition in view of the following.

Proposition 3.3.5. *Let $\varrho : S \to T$ be a flat ring homomorphism. Then ϱ is faithfully flat if and only if the associated map ${}^a\varrho : \operatorname{Spec} T \to \operatorname{Spec} S$ is surjective.*

Proof. Assume that $\varrho : S \to T$ is faithfully flat, and let $x \in \operatorname{Spec} S$. By Proposition 3.3.4, $\varrho_x : S_x \to T \otimes_S S_x$ is faithfully flat, and therefore S_x/xS_x injects into

$$(S_x/xS_x) \otimes_{S_x} (T \otimes_S S_x) \cong (T \otimes_S S_x)/(T \otimes_S xS_x).$$

It follows that $xS_x = S_x \cap (T \otimes_S xS_x)$. Thus, $T \otimes_S xS_x$ is a proper ideal of $T \otimes_S S_x$. By Proposition 1.1.1, $T \otimes_S xS_x$ is contained in a prime ideal J' of $T \otimes_S S_x$. The preimage of J' under the structure map $T \to T \otimes_S S_x$ is the prime ideal J in $\operatorname{Spec} T$. One then has ${}^a\varrho(J) = x$, and consequently ${}^a\varrho$ is surjective.

For the converse, suppose that ${}^a\varrho : \operatorname{Spec} T \to \operatorname{Spec} S$ is surjective. Let P be a maximal ideal of S. Then there exists a prime ideal $Q \in \operatorname{Spec} T$ with ${}^a\varrho(Q) = P$, and so $\varrho(P) = Q$ with $P \cdot T = \varrho(P)T = QT = Q \neq T$.

Let M be an S-module. Define a map $\varphi : M \otimes_S T \to (M \otimes_S T) \otimes_S T$ by the rule $\varphi(m \otimes t) = m \otimes 1 \otimes t$, and define a map $\omega : (M \otimes_S T) \otimes_S T \to M \otimes_S T$ by the rule $\omega(m \otimes t \otimes v) = m \otimes tv$. Then $\omega\varphi$ is the identity on $M \otimes_S T$, and so φ is an injection. An application of Lemma 3.3.2 then implies that $M \to M \otimes_S T$, $m \mapsto m \otimes 1$, is an injection. Therefore, $\varrho : S \to T$ is faithfully flat. □

If Spec S and Spec T are endowed with the Zariski topology, then faithful flatness is equivalent to the notion that Spec T is an open covering of Spec S.

We are now in a position to define surjectivity for homomorphisms of group schemes.

Definition 3.3.3. The homomorphism of R-group schemes $\psi : F \to G$ is an **epimorphism** if for each $g \in G(S)$ there is a faithfully flat R-algebra map $\varrho : S \to T$ for which $g' \in \psi_T(F(T))$, where $g' \in G(\varrho)(g)$.

Suppose $\psi : F \to G$ is an epimorphism. Since $\psi_T(F(T)) \cong F(T)/N(T)$, we say that G is the **quotient sheaf of** F **by** N and write $G = F/N$. We define a **short exact sequence of** R**-group schemes** to be the sequence

$$1 \to N \to F \to F/N = G \to 1.$$

If the corresponding map of a homomorphism $\psi : F \to G$ is faithfully flat, then the homomorphism is an epimorphism.

Proposition 3.3.6. *Let $\psi : F \to G$ be a homomorphism of R-group schemes with $R[F] = A$, $R[G] = B$. Suppose that the corresponding algebra map $\phi : B \to A$ is faithfully flat; that is, suppose that* Spec $A \to$ Spec B *is surjective. Then ψ is an epimorphism of group schemes.*

Proof. Let $g \in G(S)$. We consider A a B-algebra with $b \cdot a = \phi(b)a$, for $a \in A$, $b \in B$, and consider S a B-module with $b \cdot s = g(b)s$, $b \in B$, $s \in S$. Let $S' = S \otimes_B A$ denote the tensor product over B. By Proposition 3.3.3, the map $S \to S'$ defined by $s \mapsto s \otimes 1$ is faithfully flat. Let $f \in \mathrm{Hom}_{R\text{-alg}}(A, S')$ be defined by $a \mapsto 1 \otimes a$. Then

$$\begin{aligned} f(\phi(b)) &= 1 \otimes \phi(b) \\ &= 1 \otimes \phi(b)1 \\ &= 1 \otimes b \cdot 1 \\ &= b \cdot 1 \otimes 1 \\ &= g(b) \otimes 1. \end{aligned}$$

Define a map $g' : B \to S'$ by $b \mapsto g(b) \otimes 1$. Then $\psi_{S'}(f)(b) = g'(b)$, and ψ is an epimorphism. □

Remark 3.3.1. Let $\psi : F \to G$ be an epimorphism of R-group schemes. Let $g \in G(S)$ be the trivial element of the group $G(S)$; that is, suppose $g = \lambda\epsilon_B$. Then S is a B-module with $b \cdot s = (\lambda\epsilon_B)(b)s$. For $f \in \mathrm{Hom}_{B\text{-alg}}(A, S)$,

$$\begin{aligned} f(\phi(b)) &= f(\phi(b)1) \\ &= f(b \cdot 1) \\ &= b \cdot f(1) \\ &= b \cdot 1 \\ &= \lambda(\epsilon_B(b)). \end{aligned}$$

In this case, there exists an element $h \in F(S) = \mathrm{Hom}_{R\text{-alg}}(A, S)$ with

$$\psi_S(h)(b) = f(\phi(b)) = \lambda(\epsilon_B(b)),$$

and so S' can be taken to be S. Indeed, we can take any

$$h \in \mathrm{Hom}_{R\text{-alg}}(A/\phi(B^+)A, S) \subseteq \mathrm{Hom}_{R\text{-alg}}(A, S).$$

Thus, the collection of all preimages of $\lambda\epsilon_B \in G(S)$ coincides with the kernel of $\psi : F \to G$ given in Definition 3.2.3.

3.4 An Example

In this section, we present an important example of a short exact sequence of group schemes.

Let K be a field, let $\mathbf{G}_m$ denote the multiplicative group scheme represented by $K[X, X^{-1}]$, and let $\mathbf{G}'_m$ denote a copy that is represented by $K[Y, Y^{-1}]$. Let m denote multiplication in the K-algebra $K[X, X^{-1}]$, and let I denote the identity map on $K[X, X^{-1}]$. For an integer $l \geq 2$, define

$$m^{(l-1)} = m(I \otimes m)(I \otimes I \otimes m) \cdots (\underbrace{I \otimes I \otimes \cdots \otimes I}_{l-2} \otimes m);$$

$$\Delta^{(l-1)} = (\underbrace{I \otimes I \otimes \cdots \otimes I}_{l-2} \otimes \Delta) \cdots (I \otimes I \otimes \Delta)(I \otimes \Delta)\Delta.$$

Then there exists a natural transformation of group schemes

$$p : \mathbf{G}_m \to \mathbf{G}'_m,$$

where $p_S : \mathbf{G}_m(S) \to \mathbf{G}'_m(S)$ is defined by

$$p_S(f)(Y) = m^{(p-1)}(\underbrace{f \otimes f \otimes \cdots \otimes f}_{p})\Delta^{(p-1)}(X).$$

Since $\Delta(X) = X \otimes X$, we have $p_S(f)(Y) = f(X)^p = f(X^p)$. The K-algebra map corresponding to p is $\phi : K[Y, Y^{-1}] \to K[X, X^{-1}]$, defined by $\phi(Y) = X^p$.

Put $\Delta' = \Delta_{K[Y,Y^{-1}]}$, $\epsilon' = \epsilon_{K[Y,Y^{-1}]}$, and $\sigma' = \sigma_{K[Y,Y^{-1}]}$. Since

$$(\phi \otimes \phi)\Delta'(Y) = \phi(Y) \otimes \phi(Y) = X^p \otimes X^p = \Delta(X^p) = \Delta(\phi(Y)),$$

$$\phi(\epsilon'(Y)) = 1 = \epsilon(\phi(Y)),$$

and

$$\phi(\sigma'(Y)) = \phi(Y^{-1}) = X^{-p} = \sigma(\phi(Y)),$$

p is a homomorphism of group schemes called the p**th power map.**

The augmentation ideal $K[Y, Y^{-1}]^+$ is $(Y-1)$. Thus the kernel of p is the group scheme N represented by the K-algebra

$$K[X, X^{-1}]/\phi((Y-1))K[X, X^{-1}] \cong K[X, X^{-1}]/(X^p - 1).$$

Thus there is an exact sequence of K-group schemes

$$1 \to \mu_p \to \mathbf{G}_m \xrightarrow{p} \mathbf{G}'_m.$$

In fact, we have the following.

Proposition 3.4.1. *The map $p : \mathbf{G}_m \to \mathbf{G}'_m$ is an epimorphism of group schemes.*

Proof. Note that $K[Y, Y^{-1}]$ is the localization $M^{-1}K[Y]$ at the multiplicative set $M = \{1, Y, Y^2, \ldots\}$. Thus, by Proposition 1.1.4, Spec $K[Y, Y^{-1}]$ consists of ω together with the collection

$$\{q(Y)K[Y, Y^{-1}] : \ q(Y) \text{ is irreducible over } K \text{ and } Y \notin q(Y)K[Y]\}.$$

Let $(q(Y)) \in$ Spec $K[Y, Y^{-1}]$, $q(Y) \neq 0$. We have $\phi((q(Y))) = (q(X^p))$, which is contained in some maximal ideal $(r(X))$ of Spec $K[X, X^{-1}]$. Now $\phi^{-1}((r(X)))$ is a prime ideal of $K[Y, Y^{-1}]$ containing $(q(Y))$, and hence $\phi^{-1}((r(X))) = (q(Y))$. Thus $p((r(X))) = (q(Y))$. Moreover, $p(\omega) = \omega$. Consequently, the map of spectra

$$p : \text{Spec } K[X, X^{-1}] \to \text{Spec } K[Y, Y^{-1}]$$

(which we also denote by p) is surjective, and so, by Proposition 3.3.6, p is an epimorphism. □

Let S be an R-algebra and let $g \in \mathbf{G}'_m(S)$. Since $p : \mathbf{G}_m \to \mathbf{G}'_m$ is an epimorphism, there is an R-algebra S' and a faithfully flat map $S \to S'$ (a Zariski covering Spec $S' \to$ Spec S) for which g has a preimage in $\mathbf{G}_m(S')$.

We compute the structure of S'. The algebra map $g : K[Y, Y^{-1}] \to S$ is determined by sending Y to a unit $a \in S$. There exists a faithfully flat map $\varrho : S \to S'$ with $S' = S \otimes_{K[Y,Y^{-1}]} K[X, X^{-1}]$. In S',

$$\begin{aligned}(1 \otimes X)^p &= 1 \otimes X^p \\ &= 1 \otimes \phi(Y) \\ &= g(Y) \otimes 1 \\ &= a \otimes 1.\end{aligned}$$

And so, identifying a with $a \otimes 1$, one has $S' \cong S[T]/(T^p - a)$ for T indeterminate.

We have the short exact sequence of group schemes

$$1 \to \mu_p \to \mathbf{G}_m \xrightarrow{p} \mathbf{G}'_m \to 1 \tag{3.9}$$

with $\mathbf{G}_m/\mu_p = \mathbf{G}'_m$. Note that there are elements $S \in \mathrm{Ob}(\Im_{R\text{-alg}})$ for which

$$1 \to \mu_p(S) \to \mathbf{G}_m(S) \xrightarrow{p_S} \mathbf{G}'_m(S) \to 1$$

is not a short exact sequence of abstract groups; that is, there are R-algebras S for which $\mathbf{G}_m(S)/\mu_p(S) \neq (\mathbf{G}_m/\mu_p)(S)$.

In Chapter 8, we shall employ short exact sequence (3.9) in the case where K is a field containing $\mathbb{Q}_p$.

3.5 Chapter Exercises

Exercises for §3.1

1. Referring to Example 3.1.1, prove that the map $\Delta : \mathbb{Z}G \to \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$ defined by $\Delta(\sum_{\tau \in G} a_\tau) = \sum_{\tau \in G} a_\tau(\tau \otimes \tau)$ is a $\mathbb{Z}$-algebra homomorphism.
2. Let F be an R-group scheme, and let S be a commutative R-algebra. Show that the left/right identity element for $F(S)$ is unique.
3. Let R be a commutative ring with unity of characteristic 2. Let F be an R-group scheme with $R[F] = A$, and let S be a commutative R-algebra. Suppose that $f \in F(S)$ has order 2 in $F(S)$, and assume that $a \in A$ satisfies $\Delta(a) = a \otimes 1 + 1 \otimes a$. Prove that $\lambda\epsilon(a) = 0$.
4. Let F be an R-group scheme represented by the R-algebra A. Suppose that for all $a \in A$ and $\phi, \alpha, \beta \in F(S)$,

$$\sum_{(a)} \phi(a_{(1)})\alpha(a_{(2)}) = \sum_{(a)} \phi(a_{(1)})\beta(a_{(2)}).$$

   Show that $\alpha = \beta$.

Exercises for §3.2

5. Prove Proposition 3.2.1.
6. Prove Lemma 3.2.3, parts (ii) and (iii).
7. Compute the augmentation ideal of $R[\mathbf{G}_a]$.
8. Compute the augmentation ideal of $R[\mathbf{G}_m]$.
9. Let $\mathbf{G}_{m,\mathbb{Z}}$ and $\mathbf{G}_{a,\mathbb{Z}}$ denote the multiplicative and additive $\mathbb{Z}$-group schemes, respectively. Prove that $\psi : \mathbf{G}_{m,\mathbb{Z}} \to \mathbf{G}_{a,\mathbb{Z}}$ defined as $\psi_S(x) = 0$ for all $x \in \mathbf{G}_{m,\mathbb{Z}}(S)$ is the only homomorphism of $\mathbf{G}_{m,\mathbb{Z}}$ into $\mathbf{G}_{a,\mathbb{Z}}$.

Exercises for §3.3

10. Suppose $A \to B$ and $B \to C$ are flat maps of commutative rings. Prove that $A \to C$ is flat.
11. Let A and B be commutative rings. Show that $B \to A$ is faithfully flat if and only if $M \otimes_B A = 0$ implies that $M = 0$ for all B-modules M.
12. Let A and B be commutative rings, and suppose that $B \to A$ is faithfully flat. Show that $m \cdot A \neq A$ for every maximal ideal m of B.
13. Show that the inclusion $\mathbb{Q} \to \mathbb{Q}(\sqrt{2})$ is faithfully flat.

Exercises for §3.4

14. Consider the short exact sequence of $\mathbb{Q}$-group schemes

$$1 \to \mu_p \to \mathbf{G}_m \xrightarrow{p} \mathbf{G}'_m \to 1.$$

(a) Find a $\mathbb{Q}$-algebra S for which the sequence

$$1 \to \mu_p(S) \to \mathbf{G}_m(S) \xrightarrow{p} \mathbf{G}'_m(S) \to 1$$

is a short exact sequence of abstract groups.

(b) Find a $\mathbb{Q}$-algebra S for which the sequence

$$1 \to \mu_p(S) \to \mathbf{G}_m(S) \xrightarrow{p} \mathbf{G}'_m(S) \to 1$$

fails to be short exact.