CHAPTER XI

Brownian Motion: The LIL and Some Fine-Scale Properties

In this chapter we analyze the *growth* of the Brownian paths $t \mapsto B_t$ as $t \to \infty$. We will see by a property of "time inversion" of Brownian motion that this leads to small-scale properties as well. First, however, let us record some basic properties of the Brownian motion that follow somewhat directly from its definition.

Theorem 11.1. Let $B = \{B_t : t \geq 0\}$ be a standard one-dimensional Brownian motion starting at 0. Then

1. *(Symmetry)* $W_t := -B_t$, $t \geq 0$, is a standard Brownian motion starting at 0.
2. *(Homogeneity and Independent Increments)* $\{B_{t+s} - B_s : t \geq 0\}$ is a standard Brownian motion independent of $\{B_u : 0 \leq u \leq s\}$, for every $s \geq 0$.
3. *(Scale-Change Invariance).* For every $\lambda > 0$, $\{B_t^{(\lambda)} := \lambda^{-\frac{1}{2}} B_{\lambda t} : t \geq 0\}$ is a standard Brownian motion starting at 0.
4. *(Time-Inversion Invariance)* $W_t := tB_{1/t}$, $t > 0$, $W_0 = 0$, is a standard Brownian motion starting at 0.

Proof. Each of these is obtained by showing that the conditions defining a Brownian motion are satisfied. In the case of the time-inversion property one may apply the strong law of large numbers to obtain continuity at $t = 0$. That is, if $0 < t_n \to 0$ then write $s_n = 1/t_n \to \infty$ and $N_n := [s_n]$, where $[\cdot]$ denotes the greatest integer function, so that by the strong law of large numbers, with probability one

$$W_{t_n} = \frac{1}{s_n} B_{s_n} = \frac{N_n}{s_n} \frac{1}{N_n} \sum_{j=1}^{N_n} (B_i - B_{i-1}) + \frac{1}{s_n}(B_{s_n} - B_{N_n}) \to 0,$$

since $B_i - B_{i-1}$, $i \geq 1$, is an i.i.d. mean-zero sequence, $N_n/s_n \to 1$, and $(B_{s_n} - B_{N_n})/s_n \to 0$ a.s. as $n \to \infty$ (see Exercise 1). ■

In order to prove our main result of this section, we will make use of the following important inequality due to Paul Lévy.

Proposition 11.2 ***(Lévy's Inequality).*** Let $X_j, j = 1, \ldots, N$, be independent and symmetrically distributed (about zero) random variables. Write $S_j = \sum_{i=1}^{j} X_i, 1 \leq j \leq N$. Then, for every $y > 0$,

$$P\left(\max_{1\leq j\leq N} S_j \geq y\right) \leq 2P(S_N \geq y) - P(S_N = y) \leq 2P(S_N \geq y).$$

Proof. Write $A_j = [S_1 < y, \ldots, S_{j-1} < y, S_j \geq y]$, for $1 \leq j \leq N$. The events $[S_N - S_j < 0]$ and $[S_N - S_j > 0]$ have the same probability and are independent of A_j. Therefore

$$\begin{aligned}
P\left(\max_{1\leq j\leq N} S_j \geq y\right) &= P(S_N \geq y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N < y]) \\
&\leq P(S_N \geq y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N - S_j < 0]) \\
&= P(S_N \geq y) + \sum_{j=1}^{N-1} P(A_j)P([S_N - S_j < 0]) \\
&= P(S_N \geq y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N - S_j > 0]) \\
&\leq P(S_N \geq y) + \sum_{j=1}^{N-1} P(A_j \cap [S_N > y]) \\
&\leq P(S_N \geq y) + P(S_N > y) \\
&= 2P(S_N \geq y) - P(S_N = y). \qquad (11.1)
\end{aligned}$$

This establishes the basic inequality. ■

Corollary 11.3. For every $y > 0$ one has for any $t > 0$,

$$P\left(\max_{0\leq s\leq t} B_s \geq y\right) \leq 2P(B_t \geq y).$$

Proof. Partition $[0,t]$ by equidistant points $0 < u_1 < u_2 < \cdots < u_N = t$, and let $X_1 = B_{u_1}, X_{j+1} = B_{u_{j+1}} - B_{u_j}$, $1 \le j \le N-1$, in the proposition. Now let $N \to \infty$, and use the continuity of Brownian motion. ■

Remark 11.1. It is shown in the text on stochastic processes that $P(\max_{0\le s\le t} B_s \ge y) = 2P(B_t \ge y)$. Thus Lévy's inequality is sharp in its stated generality. The following proposition concerns the **simple symmetric random walk** defined by $S_0 = 0, S_j = X_1 + \cdots + X_j$, $j \ge 1$, with $X_1, X_2, \ldots$ i.i.d. ± 1-valued with equal probabilities. It demonstrates the remarkable strength of the **reflection method** used in the proof of the lemma, allowing one in particular to compute the distribution of the maximum of a random walk over a finite time.

Proposition 11.4. For the simple symmetric random walk one has for every positive integer y,

$$P\left(\max_{0\le j\le N} S_j \ge y\right) = 2P(S_N \ge y) - P(S_N = y).$$

Proof. In the notation of Lévy's inequality given in Proposition 11.2 one has, for the present case of the random walk moving by ± 1 units at a time, that $A_j = [S_1 < y, \ldots, S_{j-1} = y]$, $1 \le j \le N$. Then in (11.1) the probability inequalities are all equalities for this special case. ■

Theorem 11.5 ***(Law of the Iterated Logarithm (LIL) for Brownian Motion).*** Each of the following holds with probability one:

$$\overline{\lim}_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1, \qquad \underline{\lim}_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = -1.$$

Proof. Let $\varphi(t) := \sqrt{2t\log\log t}, t > 0$. Let us first show that for any $0 < \delta < 1$, one has with probability one that

$$\overline{\lim}_{t\to\infty} \frac{B_t}{\varphi(t)} \le 1 + \delta. \tag{11.2}$$

For arbitrary $\alpha > 1$, partition the time interval $[0,\infty)$ into subintervals of exponentially growing lengths $t_{n+1} - t_n$, where $t_n = \alpha^n$, and consider the event

$$E_n := \left[\max_{t_n \le t \le t_{n+1}} \frac{B_t}{(1+\delta)\varphi(t)} > 1\right].$$

Since $\varphi(t)$ is a nondecreasing function, one has, using Corollary 11.3, a scaling property, and Lemma 5 from Chapter X, that

$$\begin{aligned} P(E_n) &\le P\left(\max_{0\le t\le t_{n+1}} B_t > (1+\delta)\varphi(t_n)\right) \\ &\le 2P\left(B_1 > \frac{(1+\delta)\varphi(t_n)}{\sqrt{t_{n+1}}}\right) \\ &\le \sqrt{\frac{2}{\pi}}\frac{\sqrt{t_{n+1}}}{(1+\delta)\varphi(t_n)}e^{-\frac{(1+\delta)^2\varphi^2(t_n)}{2t_{n+1}}} \le c\frac{1}{n^{(1+\delta)^2/\alpha}} \end{aligned} \tag{11.3}$$

for a constant $c > 0$ and all $n \ge (\log\alpha)^{-1}$. For a given $\delta > 0$ one may select $1 < \alpha < (1+\delta)^2$ to obtain $P(E_n\ i.o.) = 0$ from the Borel–Cantelli lemma (Part I). Thus we have (11.2). Since $\delta > 0$ is arbitrary we have with probability one that

$$\overline{\lim}_{t\to\infty}\frac{B_t}{\varphi(t)} \le 1. \tag{11.4}$$

Next let us show that with probability one,

$$\overline{\lim}_{t\to\infty}\frac{B_t}{\varphi(t)} \ge 1. \tag{11.5}$$

For this consider the independent increments $B_{t_{n+1}} - B_{t_n}$, $n \ge 1$. For $\theta = \frac{t_{n+1}-t_n}{t_{n+1}} = \frac{\alpha-1}{\alpha} < 1$, using Feller's tail probability estimate (Lemma 5, Chapter X) and Brownian scale change,

$$\begin{aligned} P\left(B_{t_{n+1}} - B_{t_n} > \theta\varphi(t_{n+1})\right) &= P\left(B_1 > \sqrt{\frac{\theta}{t_{n+1}}}\varphi(t_{n+1})\right) \\ &\ge c'e^{-\theta\log\log t_{n+1}} \\ &\ge cn^{-\theta} \end{aligned} \tag{11.6}$$

for suitable constants c, c' depending on α and for all sufficiently large n. It follows from the Borel–Cantelli Lemma (Part II) that with probability one,

$$B_{t_{n+1}} - B_{t_n} > \theta\varphi(t_{n+1})\ i.o. \tag{11.7}$$

Also, by (11.4) and replacing $\{B_t : t \ge 0\}$ by the standard Brownian motion $\{-B_t : t \ge 0\}$,

$$\underline{\lim}_{t\to\infty}\frac{B_t}{\varphi(t)} \ge -1,\ a.s. \tag{11.8}$$

Since $t_{n+1} = \alpha t_n > t_n$, we have

$$\frac{B_{t_{n+1}}}{\sqrt{2t_{n+1}\log\log t_{n+1}}} = \frac{B_{t_{n+1}} - B_{t_n}}{\sqrt{2t_{n+1}\log\log t_{n+1}}} + \frac{1}{\sqrt{\alpha}}\frac{B_{t_n}}{\sqrt{2t_n(\log\log t_n + \log\log\alpha)}}. \tag{11.9}$$

Now, using (11.7) and (11.8), it follows that with probability one,

$$\overline{\lim}_{n\to\infty}\frac{B_{t_{n+1}}}{\varphi(t_{n+1})} \geq \theta - \frac{1}{\sqrt{\alpha}} = \frac{\alpha-1}{\alpha} - \frac{1}{\sqrt{\alpha}}. \tag{11.10}$$

Since $\alpha > 1$ may be selected arbitrarily large, one has with probability one that

$$\overline{\lim}_{t\to\infty}\frac{B_t}{\varphi(t)} \geq \overline{\lim}_{n\to\infty}\frac{B_{t_{n+1}}}{\varphi(t_{n+1})} \geq 1. \tag{11.11}$$

This completes the computation of the limit superior. To get the limit inferior simply replace $\{B_t : t \geq 0\}$ by $\{-B_t : t \geq 0\}$. ■

The time inversion property for Brownian motion turns the law of the iterated logarithm (LIL) into a statement concerning the degree (or lack) of *local smoothness.* (Also see Exercise 5).

Corollary 11.6. Each of the following holds with probability one:

$$\overline{\lim}_{t\to 0}\frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1, \qquad \underline{\lim}_{t\to 0}\frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = -1.$$

EXERCISES

Exercise Set XI

1. Use Feller's tail estimate (Lemma 5, Chapter X). to prove that $\max\{|B_i - B_{i-1}| : i = 1, 2, \ldots, N+1\}/N \to 0$ a.s. as $N \to \infty$.
2. Show that with probability one, standard Brownian motion has arbitrarily large zeros. [*Hint*: Apply the LIL.]
3. Fix $t \geq 0$ and use the law of the iterated logarithm to show that $\lim_{h\to 0}\frac{B_{t+h}-B_t}{h}$ exists only with probability zero. [*Hint*: Check that $Y_h := B_{t+h} - B_t, h \geq 0$, is distributed as standard Brownian motion starting at 0. Consider $\frac{1}{h}Y_h = \frac{Y_h}{\sqrt{2h\log\log(1/h)}}\frac{\sqrt{2h\log\log(1/h)}}{h}$.]
4. For the simple symmetric random walk, find the distributions of the extremes: (a) $M_N = \max\{S_j : j = 0, \ldots, N\}$, and (b) $m_N = \min\{S_j : 0 \leq j \leq N\}$.

5. (*Lévy Modulus of Continuity*[1]) Use the wavelet construction $B_t := \sum_{n,k} Z_{n,k} S_{n,k}(t)$, $0 \le t \le 1$, of standard Brownian motion to establish the following fine-scale properties.
   (i) Let $0 < \delta < \frac{1}{2}$. With probability one there is a random constant K such that if $|t-s| \le \delta$ then $|B_t - B_s| \le K\sqrt{\delta \log \frac{1}{\delta}}$. [*Hint*: Fix N and write the increment as a sum of three terms: $B_t - B_s = Z_{00}(t-s) + \sum_{n=0}^{N} \sum_{k=2^n}^{2^{n+1}-1} Z_{n,k} \int_s^t H_{n,k}(u)du + \sum_{n=N+1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} Z_{n,k} \int_s^t H_{n,k}(u)du = a + b + c$. Check that for a suitable (random) constant K' one has $|b| \le |t-s|K' \sum_{n=0}^{N} n^{\frac{1}{2}} 2^{\frac{n}{2}} \le |t-s|K' \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{N} 2^{\frac{N}{2}}$, and $|c| \le K' \sum_{n=N+1}^{\infty} n^{\frac{1}{2}} 2^{-\frac{n}{2}} \le K' \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{N} 2^{-\frac{N}{2}}$. Use these estimates, taking $N = [-\log_2(\delta)]$ such that $\delta 2^N \sim 1$, to obtain the bound $|B_t - B_s| \le |Z_{00}|\delta + 2K'\sqrt{-\delta \log_2(\delta)}$. This is sufficient since $\delta < \sqrt{\delta}$.]
   (ii) The modulus of continuity is sharp in the sense that with probability one, there is a sequence of intervals $(s_n, t_n), n \ge 1$, of respective lengths $t_n - s_n \to 0$ as $n \to \infty$ such that the ratio $\frac{B_{t_n} - B_{s_n}}{\sqrt{-(t_n - s_n)\log(t_n - s_n)}}$ is bounded below by a positive constant. [*Hint*: Use Borel–Cantelli I together with Feller's tail probability estimate for the Gaussian distribution to show that $P(A_n\ i.o.) = 0$, where $A_n := [|B_{k2^{-n}} - B_{(k-1)2^{-n}}| \le c\sqrt{n2^{-n}}, k = 1, \ldots, 2^n]$ and c is fixed in $(0, \sqrt{2\log 2})$. Interpret this in terms of the certain occurrence of the complimentary event $[A_n\ i.o.]^c$.]
   (iii) The paths of Brownian motion are a.s. nowhere differentiable.

[1] The calculation of the modulus of continuity for Brownian motion is due to Lévy, P. (1937), *Théorie de l'addition des variables aléatores*, Gauthier-Villars, Paris. However this exercise follows Pinsky, M. (1999): Brownian continuity modulus via series expansions, *J. Theor. Probab.* **14** (1), 261–266.