8

Summary and Discussion

8.1 Introduction

There are many problems of probability and statistics in which characterizing a large and awkward space of objects by a simpler index of the space facilitates analysis and makes the identification of optimal or at least rational solutions possible. The notion of a sufficient statistic, one that can reduce the data to a simple summary measure without loss of information about the unknown features of the model involved, is perhaps the quintessential example of this phenomenon. In linear model theory, results on dimension reduction have the same aim, though the possibility of such reduction without some (at least minor) loss of information is rarely possible. In this latter case, the compromise is generally deemed to be worth making. The theory and applications of system signatures can be thought of in the same way. The signature of a system is a characteristic of the system's design which captures an essential feature of that design. Specifically, it provides a measure of how component failures influence system failures when the components are independent and have the same lifetime distributions. As mentioned earlier, this leveling of the playing field among the components' theoretical performance allows one to focus exclusively on system design. Signatures are deterministic measures that are properly classified as tools within the field of Structural Reliability, providing information solely about the design of the corresponding system.

Regarding the information lost in using a signature vector as a proxy for a particular system design, there are two sources of lost information that require mention. The first is that there is not a one-to-one correspondence between systems and signatures. There are, for example, only 17 distinct signatures among the 20 different coherent systems of order 4. There are three pairs of systems of order 4 that have the same lifetime distributions when the component lifetimes are i.i.d. with a common distribution F. Of course, all 20 system lifetime distributions would differ if one relaxes the i.i.d. assumption. Secondly, the lifetime distributions of the components of most real systems cannot reasonably be considered identical. The independence of the component lifetimes is a less restrictive assumption, but dependencies can certainly occur due to stress, wear-out or early failures, and a theory for comparing two systems which relaxes both the independence and identically distributed assumptions could well be viewed as the ultimate goal of the type of analyses developed in this monograph.

This final chapter has several purposes. I will present, in Section 8.2, an overview of the theory and applications of system signatures, summarizing what I consider to be the highlights of the present monograph. This will include some brief commentary on the definition and interpretation of system signatures, related representations of system lifetime distributions, preservation and characterization results based on traditional stochastic orders, alternative signature-based metrics for comparing systems, the relationship between dominations and signatures in the context of communication networks and the search for optimal systems in a Reliability Economics setting. Possible extensions of the developments mentioned above are discussed in Section 8.3, where I attempt to provide some indication of the extent of generalization that appears to be feasible. In Section 8.4, I will review some signature-related literature that has not been mentioned in this monograph but gives further evidence of the broad applicability of the concept. Finally, in Section 8.5, I will mention a number of open problems for which solutions would be most welcome. In the spirit of the great mathematician Paul Erdös, I will offer financial rewards for published solutions to these problems. Being of comparatively modest means, however, I cannot match the tantalizing offers that Erdös enjoyed sprinkling throughout his lectures. I will pay 50 cents for solutions to easy problems and 1 dollar for solutions to hard ones. While these miserly offers won't serve as much of an incentive for anyone to work on these problems, I will count on old-fashioned self satisfaction, plus the right to add something like "Winner of the Samaniego Prize for Contributions to Signature Theory - 2043" to one's resume, as sufficient inducement for readers to spend at least a few minutes considering the problems I will mention. (Don't worry about the year of the prize; a generous endowment has been added to my will which will sustain this prize indefinitely. Indeed, because of this endowment, I am able to extend the range of the prize to any contribution to signature theory that I or my descendents deem to be "not bad.")

8.2 A Retrospective Overview

The signature of a system of order n whose components have i.i.d. lifetimes with common distribution F has been defined as an n-dimensional probability vector $\mathbf{s}$ whose ith element is $s_i = P(T = X_{i:n})$, where T is the system's lifetime and $X_{i:n}$ is the ith ordered component lifetime. As the order statistics of a random sample are stochastically ordered, it is clear from the definition of signatures that probability vectors $\mathbf{s}$ which place most of their weight on the larger integers in the set $\{1, 2, \ldots, n\}$ will correspond to the better performing systems simply because these systems will tend to fail later, that is, they will fail upon the failure of one of the larger order statistics. It has been shown that these particular proxies for system designs give rise to representations of systems' survival functions, and also of the systems' density and failure rate functions when the underlying component distribution F is absolutely continuous. These representations are used as essential tools in studying the performance of individual systems in i.i.d. components and in comparing such systems with each other. In the latter context, it is shown in Chapter 4 that the existence of certain ordering relationships between the signatures of two systems ensures that a similar relationship holds between the systems' lifetimes. Such preservation results are established for stochastic, hazard-rate and likelihood-ratio ordering between signature vectors of the same size. The sufficient conditions of these preservation theorems are extended in Section 4.4 to necessary and sufficient conditions on two signature vectors for the aforementioned relationships between system lifetimes to hold. Theorem 3.2, a result that establishes a recursive relationship between a given system's signature and that of a system of an arbitrary larger size having the same lifetime distribution, renders comparisons between systems of different sizes feasible.

Both of the developments mentioned above – the signature-based representations of system behavior and the relationships between signatures that imply or characterize similar relationships between system lifetimes – hold for all coherent systems and hold as well for all stochastic mixtures of coherent systems. It has been argued that the notion of mixed systems is more than a mathematical artifact which extends the reach of some theoretical results of interest and serves as a useful tool in certain optimization problems. Indeed, in applications in which a given system is to be used repeatedly, a mixed system represents a potential selection among systems. Its implementation can be physically realized through a simple process of randomization. In the i.i.d. setting studied here, employing a mixed system involves the selection, in each particular instance of its use, of a coherent system chosen according to a fixed probability distribution. Further, since for any (coherent or mixed) system in i.i.d. components, there exists a mixture of k-out-of-n systems with the same lifetime distribution, one can restrict attention to the class of k-out-of-n systems in carrying out the randomized selection of a coherent system at each stage of the application. In Chapter 7, examples of problems are given in which the optimal system relative to a chosen criterion function is not a coherent system but rather a nondegenerate mixture of k-out-of n systems. Thus, in selected circumstances, a particular mixed system can exhibit better expected behavior than any competing coherent system and can thus be reasonably recommended for practical use. The natural domain of applicability of mixed systems is in settings in which the opportunity to select a system for a particular purpose occurs repeatedly.

The various forms of stochastic ordering considered in Chapter 4 have the common characteristic of generating only a partial order among signature vectors or system lifetimes. Even the most liberal of these orderings, the "st" order, will not apply to all possible pairs of coherent systems, and there are uncountably many pairs of mixed systems that are not comparable via stochastic ordering. In Section 5.4, this limitation is addressed through the consideration of an alternative metric between system lifetimes. If T_1 and T_2 are the lifetimes of two mixed systems in i.i.d. components and the orders of these systems are potentially different, then T_1 is said to stochastically precede T_2 if and only if $P(T_1 \leq T_2) \geq 1/2$. Three characteristics of the "sp" metric that make it especially useful in comparing systems are that (i) any two mixed systems of arbitrary size are necessarily comparable, that is, the first is better than, equivalent to or worse than the second, and (ii) the relevant probability $P(T_1 \leq T_2)$ is independent of the underlying common component lifetime distribution F, that is, it is distribution-free (provided only that F is continuous) and (iii) a closed form expression for computing $P(T_1 \leq T_2)$ is available (and is given in Lemma 5.2). This type of comparison offers a potential refinement of comparisons via the traditional stochastic orderings when the latter yield inconclusive results.

The comparative reliability of communication networks is a research area that abounds with problems having both theoretical interest and practical importance. As is typical in the field, a given network is pictured as an undirected graph with a certain number of vertices and with a set of edges connecting different pairs of vertices. The primary problem of interest is the determination of the probability that a given set of vertices can communicate with another set. Two scenarios of special interest are the "two-terminal" problem, where interest is restricted to the question of whether two particular vertices are connected, and the "all terminal" problem, where the probability that each vertex can communicate with every other vertex is of primary interest. In these and other communication network problems, much attention has been given to the problem of computing the reliability of the network of interest. In Chapter 6, our focus is directed at one particularly efficient mode of computation of network reliability – Satyanarayana's theory of dominations. The main goal of that chapter is to identify an explicit relationship between the domination vector $\mathbf{d}$ of a given network and its signature vector $\mathbf{s}$ in the form $\mathbf{s} = g(\mathbf{d})$. Such a formula allows one to exploit simultaneously the computational efficiency of dominations and the utility and interpretive power of signatures in the comparative analysis of networks. In Theorem 6.1, the relationship between dominations and signatures is clarified through an explicit expression of the form $\mathbf{s} = \mathbf{P}^{-1}\mathbf{M}^{-1}\mathbf{d}$, where the matrices $\mathbf{P}$ and $\mathbf{M}$ are specifically identified. The comparison of the networks displayed in Figure 7.3 provides a striking illustration of the potential benefits of the joint use of these two tools.

The generality of the main result in Chapter 6 should be noted. In that chapter, we first call attention to the two different forms generally used to represent the reliability polynomial of a mixed system based on components with i.i.d. lifetimes in Chapter 2. The standard form and pq-form of these polynomials were displayed in equations (2.23) and (2.24). Then the relationship was derived between the domination vector $\mathbf{d}$ and the signature vector $\mathbf{s}$ which, respectively, define the coefficients of the reliability polynomials of a given communication network in standard and pq forms. This relationship, in the form $\mathbf{s} = g(\mathbf{d})$, is displayed explicitly in Theorem 6.1. The fact that this theorem applies equally to the respective coefficients of the reliability polynomials of mixed systems was not stated explicitly in Chapter 6, but is readily apparent from the algebraic developments in that chapter. The general problem solved in Chapter 6 is that of obtaining the exact relationship between the vectors $\mathbf{d}$ and $\mathbf{s}$ in the two polynomials

$$\sum_{j=1}^{n} d_j p^j \quad \text{and} \quad \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j}.$$

This is, of course, precisely the same problem one faces when transforming the reliability polynomial of a mixed system from standard to pq form. It is thus apparent that the relationship between $\mathbf{d}$ and $\mathbf{s}$ in Theorem 6.1 provides the required link in the "systems" setting as well. This connection makes it possible to exploit in tandem the computational advantages of dominations and the broad utility of signatures in the comparative analysis of mixed systems.

Chapter 7 is dedicated to a particular problem in the area of Reliability Economics. Specifically, we are interested in the problem of finding optimal system designs relative to a class of criterion functions depending on both a system's performance and its cost. The criterion functions employed (see (7.5)) depend on a system's design solely through its signature, and they have the desirable property that they are increasing functions of system performance and decreasing functions of system cost. In a case of special interest, the criterion function can be viewed as a system's "performance per unit cost" (PPUC), but the class of criteria considered includes functions that admit to a variety of other interpretations. The optimization problem considered in Chapter 7 is divided into two mutually exclusive cases, and the precise nature of the optimal design is obtained for each. In the first case (corresponding to $r = 1$, i.e., the PPUC case alluded to above), it is shown that the criterion function can be maximized by a given coherent system (indeed, by a particular k-out-of-n system), while in the complementary case, where $r \neq 1$, an optimal system may be represented as a stochastic mixture of at most two k-out-of-n systems. Several examples are given in which the class of coherent systems are dominated by a particular mixed system and are thus suboptimal. Finally, the problem of estimating the features of the underlying component lifetime distribution F upon which the criterion function depends is treated in Section 7.5. The availability of an auxiliary sample of N i.i.d. component lifetimes is assumed. For the particular case in which the measure of performance used in the criterion function is the expected lifetime of the system (and thus depends on F only through the expected order statistics $\{\mu_{i:n}, i = 1, \ldots, n\}$ of component lifetimes), consistent, asymptotically normal estimates of the parameters $\mu_{i:n}$ are obtained. This leads to the important practical conclusion that the system that maximizes the estimated criterion function will be, for arbitrary $\varepsilon > 0$, an ε-optimal system relative to the true criterion function if N is sufficiently large.

8.3 Desiderata

The theory and applications of system signatures treated in this monograph have been developed under the assumption that the systems on which we have focused are based on components with i.i.d. lifetimes. Since this is an overarching assumption and since signature vectors are well defined with or without this assumption, it is natural to explore possible generalizations of signatures in which the i.i.d. assumption is relaxed. We will begin such an exploration in this section with a view toward making the case that certain generalizations are in fact both feasible and useful. Before tackling this issue, however, it seems worth expanding upon the defense of signatures as defined herein. In Chapter 3, we argued that signatures based on an i.i.d. assumption on component lifetimes have the conceptual benefit of "evening the playing field" among system designs we might wish to compare. Further, because the signature provides information about a system that is a function of the system design alone, it is a valuable measure of system characteristics that can be useful quite apart from the consideration of the behavior of the components one might deal with in practice. For example, if one system has a signature vector that stochastically dominates that of a second system, then the fact that the second system performs better in a particular application constitutes an indication that the lifetimes of the components are either exhibiting some form of dependence or are quite differently distributed or both. Since information on the behavior of a system's components is not always available or easy to obtain, the insight about components gained from observed system behavior can be helpful. Finally, when the i.i.d. assumption is a reasonable approximation to the true behavior of a system's components, signature-based calculations of the system's theoretical behavior can be expected to provide useful approximations.

One generalization that can be developed involves replacing the i.i.d. assumption by the assumption that component lifetimes are exchangeable. As noted by Kochar, Mukerjee and Samaniego [51], the representation in (3.5) of the system survival function holds under this less stringent assumption, that is, the identity

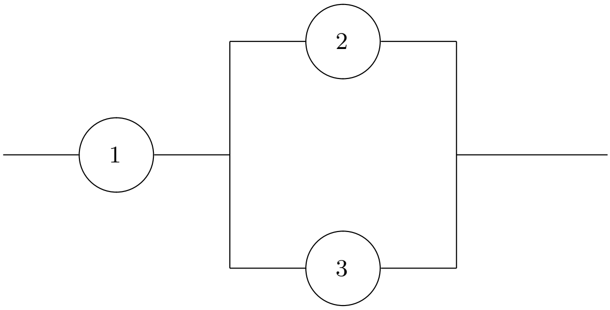
$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t) \tag{8.1}$$

holds when component lifetimes are exchangeable. Thus, one possible direction of further research is to seek to establish the results presented in this monograph under the weaker assumption of exchangeability. However, since exchangeability is but another way to quantify the notion that components behave in a similar manner, such generalizations are not likely to make an appreciable difference in practical applications. Let us, therefore, consider for a moment a more useful generalization, that is, the relaxation of the i.i.d. assumption to the case in which components have independent lifetimes with possibly different lifetime distributions. The signature of an n-component system whose components have independent lifetimes is defined as before, that is, the signature vector $\mathbf{s}$ is an n-dimensional probability vector whose ith element is given by $s_i = P(T = X_{i:n})$, where T is the failure time of the system and $X_{i:n}$ is the lifetime of the ith component to fail. However, representation results such as those in Chapter 3, while not entirely lost, will emerge in a somewhat more cumbersome form. For example, the representation in (8.1) becomes

$$P(T > t) = \sum_{i=1}^{n} s_i P(T > t \mid T = X_{i:n}) \ , \tag{8.2}$$

and the conditional probability in (8.2) can be computationally complex. The signature vector itself may be computed as the sum of the probabilities of all permutations of the component failure times that correspond to system failure upon the ith component failure. Since the signature vector provides an indication of how long a system will tend to last, it is useful to have it in hand. Although there are no existing results under the sole assumption of independent component lifetimes that state that the domination of one signature over another in some stochastic sense implies some form of domination of the respective system lifetimes, it still makes some intuitive sense to utilize the system with the dominating signature. The potential for theoretical results in this setting will not be pursued further here. Instead, we turn to an example of this latter setting in which the signature vector and the survival function are obtained for the system displayed in the figure below.

Fig. 8.1. A 3-component series-parallel system



Example 8.1. Consider the 3-component coherent system in Figure 8.1 above. Let us assume that the system's components have independent exponential lifetimes X_1, X_2, X_3 with

$$X_i \sim \text{Exp}(\lambda_i) \quad \text{for } i = 1, 2, 3 \,.$$

We first compute the signature vector of the system. Note that the system will fail upon the first component failure if and only if $T = X_1$. Thus, the permutations of the component failure times that result in system failure upon the first component failure are $\{X_1 < X_2 < X_3\}$ and $\{X_1 < X_3 < X_2\}$. We thus obtain s_1 as

$$\begin{aligned} s_1 &= \int_0^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1\lambda_2\lambda_3 \exp\{-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)\} \; \mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \\ &\quad + \int_0^\infty \int_{x_1}^\infty \int_{x_3}^\infty \lambda_1\lambda_2\lambda_3 \exp\{-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)\} \, \mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_1 \\ &= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)} + \frac{\lambda_1\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)} \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \,. \end{aligned}$$

It follows that

$$s_2 = \frac{\lambda_2+\lambda_3}{\lambda_1+\lambda_2+\lambda_3} \,.$$

The survival function of the system may be computed directly as

$$\begin{aligned} P(T > t) &= P(X_1 > t, X_1 < X_2 < X_3) + P(X_1 > t, X_1 < X_3 < X_2) \\ &\quad + P(X_1 > t, X_2 < X_1 < X_3) + P(X_3 > t, X_2 < X_3 < X_1) \\ &\quad + P(X_1 > t, X_3 < X_1 < X_2) + P(X_2 > t, X_3 < X_2 < X_1). \end{aligned} \tag{8.3}$$

The first two probabilities in (8.3) correspond to system failure upon the first component failure (since $T = X_{1:3}$ only if $X_{1:3} = X_1$). A typical calculation of such probabilities would proceed as follows:

$$\begin{aligned}
&P(X_1 > t, X_1 < X_2 < X_3) \\
&\quad = \int_t^\infty \int_{x_1}^\infty \int_{x_2}^\infty \lambda_1\lambda_2\lambda_3 \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3\}\ \mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \\
&\quad = \int_t^\infty \int_{x_1}^\infty \lambda_1\lambda_2 \exp\{-\lambda_1 x_1 - (\lambda_2+\lambda_3)x_2\}\ \mathrm{d}x_2\mathrm{d}x_1 \\
&\quad = \frac{\lambda_2}{\lambda_2+\lambda_3}\int_t^\infty \lambda_1 \exp\{-(\lambda_1+\lambda_2+\lambda_3)x_1\}\ \mathrm{d}x_1 \\
&\quad = \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_2+\lambda_3)} \exp\{-(\lambda_1+\lambda_2+\lambda_3)t\}\ .
\end{aligned}$$

From this we may infer that the first two terms on the RHS of (8.3) add to

$$P\left(T > t, T = X_{1:3}\right) = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\exp\{-(\lambda_1+\lambda_2+\lambda_3)t\}\ . \tag{8.4}$$

The expression in (8.4) is, of course, equal to

$$s_1 \times P\left(T > t \mid T = X_{1:3}\right)\ .$$

Proceeding similarly, one may obtain an expression for $P\left(T > t, T = X_{2:3}\right)$ which is equivalent to the sum of the last four terms in (8.3). Evaluating the integrals associated with these four terms yields

$$\begin{aligned}
P\left(T > t, T = X_{2:3}\right) &= \exp\{-(\lambda_1+\lambda_3)t\}\left(1 - \exp\{-\lambda_2 t\}\right) \\
&\quad + \exp\{-(\lambda_1+\lambda_2)t\}\left(1 - \exp\{-\lambda_3 t\}\right) \\
&\quad + \frac{\lambda_2+\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\exp\{-(\lambda_1+\lambda_2+\lambda_3)t\}\ . \qquad (8.5)
\end{aligned}$$

Adding (8.4) and (8.5), we obtain the final expression

$$\begin{aligned}
P(T > t) &= \exp\{-(\lambda_1+\lambda_3)t\}\left(1 - \exp\{-\lambda_2 t\}\right) \\
&\quad + \exp\{-(\lambda_1+\lambda_2)t\}\left(1 - \exp\{-\lambda_3 t\}\right) \\
&\quad + \exp\{-(\lambda_1+\lambda_2+\lambda_3)t\}\ . \qquad (8.6)
\end{aligned}$$

The reader will notice that the expression in (8.6) can also be obtained utilizing three independent Bernoulli variables associated with the events $\{X_i > t\}$ for $i = 1, 2$ and 3.

Problems at the next level of generalization, where the i.i.d. assumption is relaxed in its entirety, are likely to resist solution for some time. There are a variety of reasons for this. First, the modeling of dependence in lifetime distributions is itself a challenging problem, with only a few models available that are both tractable and easily interpreted. Among such models, the Marshall-Olkin [56] multivariate exponential (MVE) model is the best known. But even this model, which is well-motivated as a shock model and highly tractable, has found rather limited applicability in practice. One of the reasons for this is the complexity of the model in higher dimensions, where, in the most general case, $2^n - 1$ unknown parameters are required to describe the distribution of an n-dimensional vector of component lifetimes. In computing signature vectors under an MVE assumption, there is a further difficulty. Since the MVE is not absolutely continuous with respect to Lebesgue measure of the appropriate dimension, and indeed gives positive probability to the events that two or more component lifetimes are equal (via the action of a shock which causes several components to fail simultaneously), the term $P(T = X_{i:n})$ has some ambiguity. One could impose some convention on the interpretation of the term (such as that the event occurs if $i - 1$ failures preceded the failure of the system, and one or more components then fail simultaneously, causing the system to fail). But whatever convention is adopted, it is clear that the calculation of signature vectors will be substantially more complex than when component lifetimes are i.i.d. according to some continuous distribution F. Although some tractability is lost in using continuous multivariate lifetime models, the fact that all the component lifetimes are different with probability 1 at least removes the ambiguity discussed above. Time will tell whether the extension of the notion of signatures to the general multivariate domain proves feasible and useful.

8.4 Some Additional Related Literature

There are a variety of further applications of system signatures which merit mention. We present a brief summary.

Boland [16] derives the signatures of indirect majority systems and executes a comparison of such systems with direct majority systems of the same size. He proves that the signature vector of an indirect majority system of odd order n is symmetric about $(n + 1)/2$, and uses this fact to show that, for $n = R \times S$, the expected lifetime of an n-component indirect majority system exceeds the expected lifetime of a direct majority system of size n when the components have i.i.d. lifetimes with a common exponential distribution. Some of this work is presented without proof in Section 5.1 and is applied to the problem studied there.

Shaked and Suarez-Llorens [66] compare the information content of reliability experiments when components are assumed to have i.i.d. lifetimes distributed according to a two-parameter exponential distribution. They introduce the “convolution ordering” and provide sufficient conditions in terms of this ordering for one experiment to be more informative than another. They also use conditions on the signature of a system to obtain certain information inequalities. Specifically, when a system has a signature vector of the form $(0, \ldots, 0, s_k, \ldots, s_n)$ and its components have i.i.d. exponential lifetimes, its lifetime is dominated in the information ordering by a k-out-of-n system with similarly distributed components. An analogous result is shown to hold for the dispersive ordering.

Belzunce and Shaked [7] define the new and useful concept of "failure profiles" for studying the behavior of systems with independent but not necessarily identically distributed component lifetimes. In describing their main results, I will define terms slightly differently and more simply than is done in the referenced paper (eliminating, for example, their use of the term "admissible"), but the results to be described are isomorphic to theirs. A *failure profile* of a coherent system is a pair (I, i), where I is a set of components and $i \notin I$, such that I is a path set of τ and $I \cup \{i\}$ is a cut set of τ. Belzunce and Shaked demonstrate the relevance of failure profiles in two standard formulations of component importance. In their Theorem 2.5, they obtain a useful representation of the density of the lifetime of a system based on components with independent lifetimes in terms of its collection of failure profiles and the individual densities and distribution functions of the system's components. This result generalizes the representation (3.8) of the density of the system lifetime in the i.i.d. case. In their Theorem 3.5, they prove the likelihood ratio ordering between two competing systems (assuming only independent component lifetimes from distributions that are allowed to vary) under specific conditions on the underlying component distributions and on the failure profiles of the two systems. They utilize this latter result to establish a likelihood ratio ordering result for two systems with i.i.d. component lifetimes whose respective signature vectors have a particular form.

Khaledi and Shaked [48]) study the behavior of the conditional residual system lifetime given that a certain number of components are known to be working. The motivation for this study is the fact that, for some systems, it is possible to design a warning mechanism which alerts the user, before the system fails, that at least a certain number of components are still functioning. The authors' main interest is the comparison of two systems conditional on such information. They provide conditions on the signature vectors of the two n-component systems, and on the component distributions, which ensure that the conditional system lifetimes of two competing systems, given that at least $n - i + 1$ components are functioning, are stochastically ordered. For example, they prove the following result.

Theorem 8.1. *Let F_1 and F_2 be two continuous distributions on $(0, \infty)$. Let τ_1 and τ_2 be coherent systems of order n based on components with i.i.d.*

lifetimes

$$X_1, X_2, \ldots, X_n \sim F_1 \quad \textit{and} \quad Y_1, Y_2, \ldots, Y_n \sim F_2 \ ,$$

and let

$$X_{1:n}, X_{2:n}, \ldots, X_{n:n} \quad \textit{and} \quad Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$$

be the corresponding order statistics. Denote the signatures of τ_1 *and* τ_2 *by* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *and their lifetimes* T_1 *and* T_2*. Suppose* $\mathbf{s}_1$ *is of the form* $(0, \ldots, 0, s_{1j}, \ldots, s_{1n})$ *and* $\mathbf{s}_2$ *is of the form* $(0, \ldots, 0, s_{2j}, \ldots, s_{2n})$*. If* $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$ *and* $F_1 \leq_{hr} F_2$*, then for* $i \leq j$*,*

$$F_{T_1 - y \mid X_{i:n} > y} \leq_{st} F_{T_2 - y \mid Y_{i:n} > y} \ .$$

They also obtain the following complementary result for component distributions F_1 and F_2 that are "reverse hazard rate ordered" (denoted by $\leq_{\mathrm{rh}}$), that is, for which $F_2(t) \ / \ F_1(t)$ is increasing in t.

Theorem 8.2. *Let* F_1 *and* F_2 *be two continuous distributions on* $(0, \infty)$*. Let* τ_1 *and* τ_2 *be coherent systems of order* n *based on components with i.i.d. lifetimes*

$$X_1, X_2, \ldots, X_n \sim F_1 \quad \textit{and} \quad Y_1, Y_2, \ldots, Y_n \sim F_2 \ ,$$

and let

$$X_{1:n}, X_{2:n}, \ldots, X_{n:n} \quad \textit{and} \quad Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$$

be the corresponding order statistics. Denote the signatures of τ_1 *and* τ_2 *by* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *and their lifetimes* T_1 *and* T_2*. Suppose* $\mathbf{s}_1$ *is of the form* $(s_{11}, \ldots, s_{1i}, 0, \ldots, 0)$ *and* $\mathbf{s}_2$ *is of the form* $(s_{21}, \ldots, s_{2i}, 0, \ldots, 0)$*. If* $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$ *and* $F_1 \leq_{rh} F_2$*, then for* $i \leq j$*,*

$$F_{T_1 - y \mid X_{j:n} > y} \leq_{st} F_{T_2 - y \mid Y_{j:n} > y} \ .$$

Further, under specific conditions on the signature vector of the system, Khaledi and Shaked [48] obtain upper and lower bounds for

$$E[T - y \mid X_{i:n} > y] \ ,$$

where T and $X_{i:n}$ are the lifetime and ith ordered component failure time of a given system.

Navarro and Shaked [58] utilize system signatures in showing hazard-rate ordering among independently drawn minima $\{X_{1:1}, X_{1:2}, \ldots, X_{1:n}, \ldots\}$ and in studying the limiting behavior of failure rates of selected systems. They develop a representation of the system survival function as a linear combination of the survival functions of minima such as those above, calling the vector of coefficients in the expression the "minimal signature" of the system. For signatures as defined in this monograph, conditions are given under which the ratio of the failure rates of an n-component system with a given signature and that of a k-out-of-n system is asymptotically equal to 1.

8.5 Some Open Problems of Interest

8.5.1 The Ordering of Expected System Lifetimes

In section 5.4, we treated the comparison of systems via stochastic precedence. This metric has the appealing characteristic of rendering all pairs of mixed systems comparable, making it always possible to judge one system as better or worse than (or equivalent to) the other. It would, of course, be useful to have definitive results using the even simpler and most commonly used metric for system performance, the expected system lifetime. It should be noted that comparisons of the expected lifetimes of two systems may be "too rough" a comparison in some problems, as it completely ignores the variability in system lifetime. Further, it is possible that a system whose expected lifetime exceeds that of a second system will be less reliable than the second system at the systems' planned mission time. It is nonetheless of interest to know when one could expect that, on average, one system will last longer than another. While the condition $ET_1 \leq ET_2$, where T_1 and T_2 are the lifetimes of the two systems involved, is a fairly weak stochastic relationship (implied, for example, by $T_1 \leq_{\mathrm{st}} T_2$), questions about this ordering are likely to arise more often in applications than questions about the more stringent relationships discussed in Chapters 4 and 5. Thus, the goal of finding conditions which guarantee that the above ordering of expectations holds seems worthy of attention. Boland and Samaniego [20] discuss this problem and note that for two given systems having components with i.i.d. lifetimes $\sim F$, it is possible for $ET_1 \leq ET_2$ when $F = F_1$ and for $ET_2 \leq ET_1$ when $F = F_2$. However, they prove the following result for a particular group of small systems.

Theorem 8.3. *Consider two mixed systems of order $n = 3$ based on coherent systems in i.i.d. components with lifetime distribution F. Denote their respective signature vectors as $\mathbf{s}_1$ and $\mathbf{s}_2$. Then $ET_1 \leq ET_2$ for all distribution functions F if and only if $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$.*

The extension of this result to systems of arbitrary order n has not been shown, nor have counterexamples been identified which demonstrate that the result fails to hold for other values of n. The former possibility depends on the special properties of the spacings between order statistics and appears to be quite challenging. Counterexamples are inherently quirky, so it is difficult to assess the level of difficulty in showing that the theorem above does not hold for general n if, in fact, that is the case.

8.5.2 Other Preservation Results

In Section 4.2, it was shown that three specific versions of the stochastic ordering of system signatures carry over to the lifetimes of these systems. There are, of course, numerous other orderings for which preservation results may hold (or fail to hold). Shaked and Shantikumar [65] discuss a host of alternative orderings of univariate distributions. Among these are mean residual life ordering, reverse hazard rate ordering, dispersive ordering, the Laplace transform and the moment generating function orderings, convex ordering and star-shaped ordering. The latter two orderings are also discussed in Chapter 4 of Barlow and Proschan [6]. These and a number of other formulations of univariate ordering have been found useful in reliability. If ORD represents any given ordering among them, then it would be of interest to know whether or not the implication $\mathbf{s}_1 \leq_{\mathrm{ORD}} \mathbf{s}_2 \Rightarrow T_1 \leq_{\mathrm{ORD}} T_2$ holds for systems in i.i.d components. I am inclined to classify problems consisting of proofs or counterexamples for these implications as dollar-valued. For all those disposed to think about such problems, it seems appropriate to say at this time: ready, set, go!

8.5.3 The limiting monotonicity of $r_T(t)$

In Section 5.3, the asymptotic behavior of the failure rate of an arbitrary mixed system was examined, and its limiting value was explicitly identified. This result was established by Block, Dugas and Samaniego [11] using the failure rate representation in (3.11). A natural question that arises in this same context concerns the potential relationship between the monotonicity of the component failure rate $r(t)$ and that of the system's failure rate $r_T(t)$. More specifically, it would be of interest to identify conditions that imply that, for sufficiently large t, the system failure rate $r_T(t)$ is strictly increasing (decreasing) if and only if the common component failure rate $r(t)$ is eventually strictly increasing (decreasing).

8.5.4 Further Results on Stochastic Precedence

In section 5.4, it was shown that, using the metric of stochastic precedence, any pair of mixed systems based on components with i.i.d. lifetimes $\sim F$ are comparable, with either one being superior to the other or the two systems being sp-equivalent. Stochastic precedence is well defined when both systems are based on components with i.i.d. lifetimes with differing component distributions F_1 and F_2. Indeed, the representation in (3.28) was shown to hold under these more general conditions. However, the computation of $P(T_1 \leq T_2)$ is considerably more challenging in this latter scenario. Results facilitating the comparison of two system lifetimes when the systems are based on components with independent lifetimes but different distributions would be a worthwhile extension of the results in section 5.4. Hollander and Samaniego [43] demonstrate the feasibility of such generalizations, providing a formula for the exact calculation of the probability $P(T_1 \leq T_2)$ when F_2 is in the class of Lehmann alternatives to F_1, that is, when $F_2(t) \equiv [F_1(t)]^k$ for some $k > 0$. The comparative analysis of systems in the general setting in which $F_1 \neq F_2$ will require further investigation.

8.5.5 Uniformly Optimal Networks

Let us, for concreteness, limit our discussion to the "all-terminal" problem for communication networks, that is, to the problem of determining whether or not all the vertices of a given network can communicate with each other. Among all networks of a given size, that is, within the class of $G(v, n)$ networks with v vertices and n edges, a network is said to be *uniformly optimal* if the probability that all vertices can communicate is maximal. Even under the simplifying assumption that that edges are independent and have a common reliability p, the problem of identifying uniformly optimal networks (that is, networks that are optimal for all $p \in (0, 1)$) in the class $G(v, n)$ is an open problem and appears to be a quite challenging one. In certain special cases, the problem has been solved, but results to date are quite limited. Boesch et al. [14], for example, identified a uniformly optimal network (UON) in the class $G(v, v+1)$. The general problem is complicated by the fact that, for some values of v and n, no uniformly optimal network exists, as demonstrated by Myrvold et al. [57]. Thus, the open problems that remain include the problem of characterizing those classes of networks for which a uniformly optimal network exists and, given such a class, identifying the UON explicitly. Because of the challenging nature of these problems, certain intermediate problems are also of interest. For example, direct comparisons among two or more networks of special interest or between two subclasses of networks of the same size, can be of use in particular applications.

In Chapter 6, it was shown that the signature vectors of competing networks can be a useful tool in comparing their reliability. To my knowledge, the tool has not yet been applied in the search for UONs. To illustrate the utility of signatures in this context, we provide a brief illustration in a problem alluded to above. Let us consider the class of networks in the class $G(5, 6)$. Three particular networks in this class are displayed below.

Fig. 8.2. Network $G_1(5,6)$

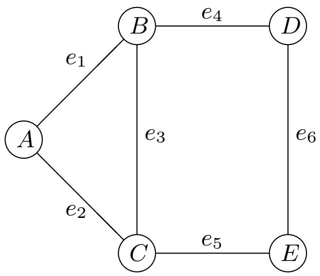


Fig. 8.3. Network $G_2(5,6)$

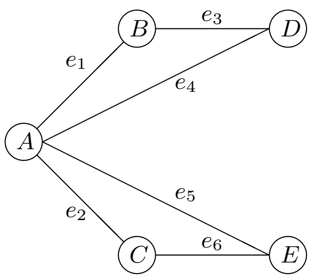
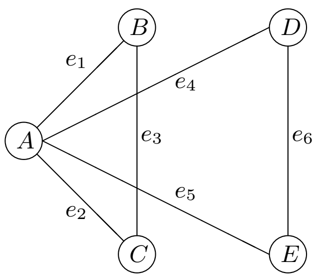


Fig. 8.4. Network $G_3(5,6)$



The three networks above can be compared using the associated reliability polynomials computed via domination theory. Alternatively, one can identify the signatures of the three systems as $\mathbf{s}_1 = (0, 4/15, 11/15, 0, 0, 0)$, $\mathbf{s}_2 = (0, 2/5, 3/5, 0, 0, 0)$ and $\mathbf{s}_3 = (0, 2/5, 3/5, 0, 0, 0)$, from which we see that G_2 and G_3 are equivalent and that both are inferior to G_1 (with $\mathbf{s}_i \geq_{\mathrm{lr}} \mathbf{s}_1$ for $i = 2$ and 3). In larger problems, where existence and uniqueness questions remain unresolved and available methods of finding UONs, if they exist, amount to numerical searches, the ability to establish the superiority of one network over another by comparing their signatures, and the possibility of optimizing network reliability as a function of the signature vector, should offer some hope of successfully attacking these challenging problems.

8.5.6 Other Problems in Reliability Economics

I'll begin by making brief mention of two problems in Reliability Economics that are related to the problem treated in Chapter 7, both involving the search for an optimal system design relative to a criterion such as that in (7.5) under specified constraints. One obvious class of open problems involves maximizing the criterion function (7.5) among systems that are mixtures of a fixed sub-collection of coherent systems. Another class of open problems would involve maximizing the criterion function under a budgetary constraint such as $\sum_{i=1}^{n} c_i \leq K$. Both of these problems are of practical interest, as the selection of a system will often be restricted to the choice among certain available systems and mixtures thereof, and there are often budgetary limits that restrict the selection of the system one might purchase. In either of these constrained scenarios, the optimal system is likely to differ from the optimal systems identified in Chapter 7. Problems involving the characterization of optimal solutions in constrained Reliability Economics contexts constitute a set of interesting open problems of some practical importance.

Example 8.2. As an illustration of the first of these problems, consider the problem of selecting among stochastic mixtures of the two coherent systems of order $n = 3$ having signature vectors $\mathbf{s}_1 = (1/3, 2/3, 0)$ and $\mathbf{s}_2 = (0, 2/3, 1/3)$ respectively. Let us take $r = 1$ in the criterion function in (7.5). Without loss of generality, we set $c_1 = 1$ and allow c_2 and c_3 to be arbitrary values satisfying $1 < c_2 < c_3$. We will take the vector $\mathbf{a}$ to be equal to $(1/4, 1/2, 3/4)$, the expected order statistics of the Uniform distribution $U[0, 1]$. Then the criterion function of the two systems above will be

$$m_1 = \frac{5/12}{1/3 + c_2(2/3)} \quad \text{and} \quad m_2 = \frac{7/12}{c_2(2/3) + c_3(1/3)} \, .$$

A mixed system giving weights p and $(1 - p)$ to these two systems will have criterion function equal to

$$m_3 = \frac{p(5/12) + (1 - p)(7/12)}{p(1/3) + c_2(2/3) + c_3(1/3)(1 - p)} \, .$$

It's easy to verify that

$$m_1 < m_3 \quad \text{if and only if} \quad c_3 < \frac{7}{5} + c_2 \, \frac{4}{5} \, ,$$

while

$$m_2 < m_3 \quad \text{if and only if} \quad c_3 > \frac{7}{5} + c_2\,\frac{4}{5}\,.$$

It follows that if $1 < c_2 < c_3$ and $c_3 = 7/5 + c_2(4/5)$, both coherent systems above, as well as all stochastic mixtures of them, yield the same value of the criterion function, while for any other choice of $c_3 > c_2 > 1$, the criterion function is uniquely maximized by one of the coherent systems or the other. In all cases, there exists a coherent system (among the two available) that is optimal relative to the chosen criterion function.

The example above suggests the conjecture that, when $r = 1$ in the criterion function in (7.5), there exists a system in any collection of coherent systems which will be optimal within the class of all mixtures of these systems. This conjecture agrees with the result in Theorem 7.1 in the case that one seeks an optimal system among the stochastic mixtures of all coherent systems. While it is not readily apparent that Theorem 7.2 generalizes in the same way, it seems reasonable to conjecture that, when $r \neq 1$ in (7.5), an optimal system within any collection of coherent systems and their mixtures can be found within the class of mixtures of two systems in the collection. The method of proof used in establishing Theorem 7.2 (showing, essentially, that any mixture of three systems in the collection can be improved upon by an appropriate mixture of two) may well be successful in showing this.

We will also comment briefly on the problem of searching for an optimal system under cost constraints. In the criterion function in (7.5), the cost of a given n-component system design is quantified in terms of the positive constants $c_1 < c_2 < \cdots < c_n$. Note that if the constraint $\sum_{i=1}^{n} c_i \leq K$ will place no restriction on the choice of system if in fact $c_n \leq K$. This is the case because the optimal system in the unconstrained problem is either a k-out-of-n system costing $c_k \leq K$ or a mixture of two k-out-of-n systems costing $pc_i + (1-p)c_j \leq K$. Thus, for any value of r in (7.5), the system that optimizes the criterion function overall will satisfy the constraint and is thus obviously optimal in the constrained problem. At the other extreme, if $K < c_1$, then there is no system that will satisfy the constraint. Intermediate problems in which $c_1 < K < c_n$ will require individual optimal solutions that may well differ from the optimal solution in the unconstrained problem. We conjecture that, in such constrained problems, the optimal solutions will be of a form similar to those given in Theorems 7.1 and 7.2, and that the methods of proof utilized in those theorems can be adapted to obtain the new results. Without repeating the commentary above, we mention that a similar set of considerations arise when the budgetary constraint of interest is on the total cost of a mixed system, that is, has the form $\sum_{i=1}^{n} c_i s_i \leq K$.

The problem of finding an optimal design while accounting for both performance and cost is but one of many optimization problems of interest in the general area of Reliability Economics. Among topics in reliability in which performance and cost enter in a central way are the areas of maintenance and repair. The varied policies employed in these areas (including, for example, block replacement policies and maintenance through the use of spare parts) have both reliability and economic implications. While there is a literature on optimization on these topics, it remains to be seen whether the problems of interest can benefit from formulations based on system signatures.

8.5.7 Wholly New Stuff

Hey, don't be greedy. I've got to leave something for me to do!