3

System Signatures

The basic notion of a coherent system has been defined and illustrated in the preceding chapter, and various properties and tools have been established to assist in the study of such systems. Structure functions are admittedly quite revealing. They are in one-to-one correspondence with the coherent systems themselves and provide a way of indexing systems and also of comparing them. Also, they are unambiguous summaries of a system's design, and are more useful summaries than schematic diagrams or flow charts, which often look different but may correspond to one and the same system. We must, however, acknowledge that the artillery we have discussed thus far has some limitations. Since the number of coherent systems of order n grows exponentially with n, the indexing of systems through their structure functions tends to be of limited use in problems involving comparisons or optimization among systems. Structure functions are complex algebraic expressions that, in general, admit to multiple equivalent representations. For example, the expressions in (2.5) and (2.11) look quite different but are, in fact, equivalent forms of the structure function of the bridge system in Figure 2.2. In this chapter, we introduce an alternative index which, although less general than a structure function, has the virtues of being both quite manageable and easily interpreted. Most importantly, for systems of order n, this index is of fixed dimension; in fact, it resides in a bounded simplex in n-dimensional Euclidean space. We call this index the system's signature. Its precise meaning is specified below.

Definition 3.1. *Let τ represent a coherent system of order n. Assume that the lifetimes of the system's n components are independent and identically distributed (i.i.d.) according to the (continuous) distribution F. The* signature *of the system τ, denoted by $\mathbf{s}_\tau$, or simply by $\mathbf{s}$ when the corresponding system is clear from the context, is an n-dimensional probability vector whose ith element s_i is equal to the probability that the ith component failure causes the system to fail. In brief, $s_i = P(T = X_{i:n})$, where T is the failure time of the* *system and* $X_{i:n}$ *is the* i*th order statistic of the* n *component failure times, that is, the time of the* i*th component failure.*

Before illustrating the concept of system signatures, it seems advisable to scrutinize the definition above. In particular, it seems reasonable to question the wisdom of the i.i.d. assumption on component lifetimes. The notion of signature, as a certain probability vector, is well defined without this assumption, but the assumption is nonetheless made, and is made for good reason. Signatures will be used primarily in the comparison of system designs. It should be noted that a comparison between two systems with quite different component characteristics may well be either misleading or inconclusive. It is clear, for example, that a series system with four highly reliable components will outperform a four-component parallel system with relatively poor components. If the probability that the components of the series system last beyond a fixed mission time is 0.9, its reliability at that mission time is 0.6561, while that of a parallel system having four components with reliability 0.1 is 0.3439. It is clear, however, that parallel systems are preferable, in a general sense, to series systems. Indeed, the former's structure function uniformly dominates the latter's. Once the i.i.d. assumption is made, any remaining differences in system performance must be attributable to the system's design. In that sense, the assumption levels the playing field so that one has a basis for comparing the designs themselves. From an analytical point of view, signatures, as defined above in the i.i.d. setting, provide three major advantages. They allow one to utilize the tools of combinatorial mathematics for the calculation of system characteristics. Also, the well-known distribution theory for the order statistics of an i.i.d. sample from a continuous distribution F is available for studying the performance of a system with a given signature. Finally, signatures depend only on the permutation distribution of the n observed failure times and do not depend on the underlying distribution F. The signature vector can therefore be viewed as a pure measure of a system's design.

More can be said about the comparisons we will indulge in as we proceed. We will, for example, be primarily interested in comparing systems of the same order. While one could, in some instances, be interested in comparing systems of different sizes, it is far more common to compare systems of the same size, essentially investigating questions such as "Which of several possible configurations of components would be preferable for certain specific purposes?" In the words of the great Eastern philosopher Confucius, comparing apples to oranges is a rather fruitless endeavor. A second issue that should be addressed before proceeding is the possibility that results depending on signature vectors as defined above might in fact be irrelevant in studying and comparing the performance of real systems whose components are neither independent nor identically distributed. In addressing this concern, it should be said that, in any application of signature-related results in which the foundations of their definition are in doubt, one should proceed with considerable caution. It is probably worth adding that signature-based results may be inexact in such applications but are not necessarily irrelevant. Any mathematical result can only give guidance in real applications, as the assumptions under which the result is developed can't be checked with certainty in a given practical situation. What signatures do is tell us something about the design of the associated system. Knowing that one design is better than another (everything else being equal) is useful information as one diverges from the basic assumption of i.i.d. component lifetimes. If, for example, the component lifetimes could be considered independent and, while not identically distributed, nonetheless roughly comparable, selecting the system with a better signature should lead to better performance. Although an exact analysis would of course be desirable, characterizing system performance in non i.i.d. settings is a formidable analytical task, quite unlike the i.i.d. setting to be studied here.

The computation of system signatures is, in essence, a combinatorial exercise. That doesn't mean that it's simple. It only means that there is a well-organized body of knowledge and tools that can be applied to such problems. To describe the counting problem of interest, let's suppose that the random variables $X_1, X_2, \ldots, X_n$ represent the failure times of the components of the n-component system under study. Since the Xs are assumed to be i.i.d. from some continuous distribution on $(0, \infty)$, the $n!$ permutations of these n distinct failure times are equally likely. As noted above, the ith element of $\mathbf{s}$ can be obtained as the probability $s_i = P(T = X_{i:n})$, where T is the failure time of the system and $X_{i:n}$ is the ith order statistic (that is, the ith smallest value) among the i.i.d. failure times $X_1, X_2, \ldots, X_n$. Equivalently, we may obtain s_i as the ratio of n_i, the number of orderings for which the ith component failure causes system failure, to $n!$, the total number of possible orderings of the n failure times. The essential feature of the calculation of signatures is the counting of the number of permutations of the n potential component failure times that correspond with system failure upon the ith failure among the n components. Since T resides in the set $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ with probability one, it follows that the signature $\mathbf{s}$ is a probability vector, that is, $s_i \geq 0$ for all i and $\sum_{i=1}^{n} s_i = 1$.

We now turn to the computation of the signature vector for some simple coherent systems. As an example of this computation, consider the three-component system pictured in Figure 3.1 below. The failure times X_1, X_2 and X_3 of the three components of this system can be ordered in $3! = 6$ ways, and these six possible permutations are equally likely due to the i.i.d. assumption. The "order-statistic equivalent" for the system failure time T is shown below for each permutation of the component failure times.

Fig. 3.1. A 3-component system with structure function $\varphi^*(\mathbf{x}) = x_1(x_2+x_3-x_2x_3)$

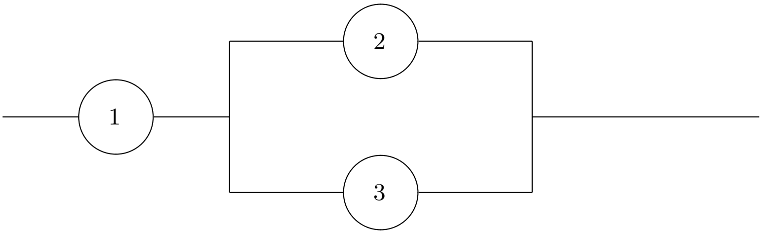

Table 3.1. The ordered component failure time which causes failure of system φ^* above

Ordered Component Failure Times	Order Statistic Equal to System Failure Time T
$X_1 < X_2 < X_3$	$X_{1:3}$
$X_1 < X_3 < X_2$	$X_{1:3}$
$X_2 < X_1 < X_3$	$X_{2:3}$
$X_2 < X_3 < X_1$	$X_{2:3}$
$X_3 < X_1 < X_2$	$X_{2:3}$
$X_3 < X_2 < X_1$	$X_{2:3}$

It follows that the system above has signature vector $\mathbf{s} = (1/3, 2/3, 0)$. It is easy to show that the five distinct coherent systems of order 3 have the signatures $(1,0,0), (0,1,0), (0,0,1), (1/3,2/3,0)$ and $(0,2/3,1/3)$. The first three of these signatures correspond to the i-out-of-3 systems for $i = 1,2,3$, and the fifth corresponds to the system in which one component is in parallel with a series system in the other two components. While the combinatorics involved in calculating the signatures of systems of a given order can be fairly complex, it is worth noting that, via the notion of "duality," the amount of calculation can be cut roughly in half, as the signature of a system's "dual" can be obtained from the system's signature via symmetry arguments. Of the five systems mentioned above, the first system is the dual of the third and the fourth is the dual of the fifth. Table 3.2 provides the signatures of the 20 distinct coherent systems of order 4.

Table 3.2. Coherent Systems of Order 4

System	Minimal cut sets	Signature
1	{1}, {2}, {3}, {4}	(1, 0, 0, 0)
2	{1}, {2}, {3, 4}	(1/2, 1/2, 0, 0)
3	{1}, {2, 3}, {2, 4}	(1/4, 7/12, 1/6, 0)
4	{1}, {2, 3}, {2, 4}, {3, 4}	(1/4, 3/4, 0, 0)
5	{1}, {2, 3, 4}	(1/4, 1/4, 1/2, 0)
6	{1, 2}, {1, 3}, {1, 4}	(0, 1/2, 1/4, 1/4)
7	{1, 2}, {1, 3}, {1,4}, {2, 3}	(0, 2/3, 1/3, 0)
8	{1, 2}, {1, 3}, {1,4}, {2, 3}, {2, 4}	(0, 5/6, 1/6, 0)
9	{1, 2}, {1, 3}, {1,4}, {2, 3}, {2, 4}, {3, 4}	(0, 1, 0, 0)
10	{1, 2}, {1, 3}, {2, 4}, {3, 4}	(0, 2/3, 1/3, 0)
11	{1, 2}, {2, 4}, {3, 4}	(0, 1/2, 1/2, 0)
12	{1, 2}, {3, 4}	(0, 1/3, 2/3, 0)
13	{1, 2}, {1, 3}, {1, 4}, {2, 3, 4}	(0, 1/2, 1/2, 0)
14	{1, 2}, {1, 3}, {2, 3, 4}	(0, 1/3, 2/3, 0)
15	{1, 2}, {1, 3, 4}, {2, 3, 4}	(0, 1/6, 5/6, 0)
16	{1, 2}, {1, 3, 4}	(0, 1/6, 7/12, 1/4)
17	{1, 2, 3}, {1, 2, 4}	(0, 0, 1/2, 1/2)
18	{1, 2, 3}, {1, 2, 4}, {1, 3, 4}	(0, 0, 3/4, 1/4)
19	{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}	(0, 0, 1, 0)
20	{1, 2, 3, 4}	(0, 0, 0, 1)

We now establish a fundamental property of a system's signature $\mathbf{s}$, namely, that the distribution of the system lifetime T, given i.i.d. components lifetimes with c.d.f. F, can be expressed as a function of $\mathbf{s}$ and F alone. The following representation is drawn from Samaniego [61].

Theorem 3.1. *Let $X_1, \ldots, X_n$ be the i.i.d. component lifetimes of an n-component coherent system with signature $\mathbf{s}$, and let T be the system's lifetime. Then*

$$\overline{F}_T(t) \equiv P(T > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j \left(\overline{F}(t)\right)^{n-j} . \tag{3.1}$$

Proof. We first note that the system fails concurrently with the failure of one of its components, so that T will necessarily take on the values of one of the order statistics $X_{i:n}$ of the sample $X_1, \ldots, X_n$, that is, $T \in \{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ with probability 1. Then, utilizing the Law of Total Probability and the i.i.d. assumption on component lifetimes, we may write

$$\begin{aligned}P(T>t) &= \sum_{i=1}^{n} P\left(T>t, T=X_{i:n}\right)\\ &= \sum_{i=1}^{n} P\left(T>t \mid T=X_{i:n}\right) P\left(T=X_{i:n}\right)\\ &= \sum_{i=1}^{n} s_i P\left(X_{i:n}>t\right)\\ &= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j \left(\overline{F}(t)\right)^{n-j} . \qquad \blacksquare\end{aligned}$$

By interchanging the order of the summations in (3.1), the representation of the system survival function in (3.1) may be written in the alternative form

$$\overline{F}_T(t) = \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{j} (F(t))^j \left(\overline{F}(t)\right)^{n-j} . \tag{3.2}$$

If one considers the chances that a system based on n i.i.d components is working at a fixed point in time t_0, then setting $p = \overline{F}(t_0)$ and $q = F(t_0)$, we derive the reliability polynomial $h(p)$ in pq form from (3.2). Two equivalent versions of h are

$$h(p) = \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{j} q^j p^{n-j} \text{ and } h(p) = \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j} . \tag{3.3}$$

The representation in (3.2) can also be written as a function involving the odds $G(t) = F(t)/\overline{F}(t)$ of failure vs. survival. This form of the representation of $\overline{F}_T(t)$ in terms of signatures will prove to be quite useful in the sequel, and is recorded below:

$$\overline{F}_T(t) = \left(\overline{F}(t)\right)^n \sum_{j=0}^{n-1} \left(\sum_{i=j+1}^{n} s_i \right) \binom{n}{j} (G(t))^j . \tag{3.4}$$

A more detailed proof of Theorem 3.1 invoking Lemma 8.3.11 of Randles and Wolfe [60] on the independence of the order statistics $\{X_{i:n}\}$ and the ranks of the original observations $X_1, X_2, \ldots, X_n$ is given in Kochar, Mukerjee and Samaniego [51].

The proof of Theorem 3.1 contains an elementary fact that is of independent interest and will also prove quite useful in the sequel. Note that the survival function of the system lifetime T may be written in terms of the survival functions of the order statistics of the component failure times, that is,

$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t) \; . \tag{3.5}$$

Utilizing the well-known identity for positive random variables Y, namely,

$$EY = \int_0^\infty \overline{F}(y) \, \mathrm{d}y \; , \tag{3.6}$$

another useful connection between the system lifetime and the ordered failure times follows:

$$ET = \sum_{i=1}^{n} s_i E X_{i:n} \; . \tag{3.7}$$

The representation in (3.1) can be applied to obtain useful representations of a system's density function and failure rate when F is absolutely continuous. For example, the density function $f(x)$ may be obtained from (3.1) as follows.

Corollary 3.1. *Let $X_1, \ldots, X_n \sim F$ be the i.i.d. component lifetimes of an n-component coherent system with signature* $\mathbf{s}$*, and let T be the system's lifetime. If F is absolutely continuous, then*

$$f_T(t) = -(\partial/\partial t) P(T > t) = \sum_{i=1}^{n} i \; s_i \binom{n}{i} \left(F(t)\right)^{i-1} \left(\overline{F}(t)\right)^{n-i} f(t) \; . \tag{3.8}$$

Proof. Differentiating $\overline{F}_T(t)$ in formula (3.1) yields an interior summation with alternating signs in which all elements but the one shown in (3.8) cancel out. ■

It follows that the system failure rate $r_T(t)$, defined as the ratio

$$\frac{f_T(t)}{\overline{F}_T(t)} \; ,$$

can be written in terms of the signature vector $\mathbf{s}$ and the underlying component distribution F. The ratio of the density in (3.8) to the survival function in (3.1) may be algebraically simplified to obtain a useful representation of the system's failure rate.

Corollary 3.2. *Consider an n-component coherent system with signature* $\mathbf{s}$*, and assume that the component lifetimes $X_1, \ldots, X_n$ are i.i.d. with distribution F and density f. Let T be the system lifetime. Then*

$$r_T(t) = \frac{\sum_{i=1}^{n} i \; s_i \binom{n}{i} \left(F(t)\right)^{i-1} \left(\overline{F}(t)\right)^{n-i+1}}{\sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} \left(F(t)\right)^{j} \left(\overline{F}(t)\right)^{n-j}} \, r(t) \; , \tag{3.9}$$

where $r(t) = \left(f(t)/\overline{F}(t)\right)$, the common failure rate of the components.

An equivalent and occasionally more useful version of (3.9) is the following:

$$r_T(t) = \frac{\sum_{i=0}^{n-1}(n-i)s_{i+1}\binom{n}{i}(F(t))^i\left(\overline{F}(t)\right)^{n-i}}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n}s_j\right)\binom{n}{i}(F(t))^i\left(\overline{F}(t)\right)^{n-i}}\, r(t)\ , \tag{3.10}$$

or, in terms of the odds function $G(t) = F(t)/\overline{F}(t)$,

$$r_T(t) = \frac{\sum_{i=0}^{n-1}(n-i)s_{i+1}\binom{n}{i}(G(t))^i}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n}s_j\right)\binom{n}{i}(G(t))^i}\, r(t)\ . \tag{3.11}$$

We will be interested in the comparison of two systems with i.i.d. components. As is clear from equation (3.1), the lifetime of a coherent system with i.i.d. components depends on the structure of the system only through the signature **s**. Indeed, if two systems in i.i.d. components have the same signature, the stochastic behavior of their lifetimes is identical. It is natural to ask if two different coherent systems can have the same signature. The answer is yes; one can see from Table 3.2 that the four-component systems labeled as systems 11 and 13 have the same signature. The twenty coherent systems of order four give rise to precisely 17 distinct signatures.

While the class of all coherent systems of a given size (or even the overall collection of coherent systems of arbitrary order) is arguably the collection of systems on which one would wish to concentrate in a particular application, the class does have some limitations which will lead us to broaden our perspective. I mentioned earlier that the number of coherent systems of order n is not precisely known for general n and is quite large, even for moderate size n. For any fixed n, the space of coherent systems of order n is, obviously, discrete. As we shall see in the sequel, this has some negative consequences, both mathematically and practically. The mathematical difficulty is that in problems in which an optimal coherent system is sought, one typically must focus on finding approximately optimal systems via some appropriate discrete search algorithm. In other words, problems aimed at finding optimal coherent systems tend to be analytically intractable. The practical problem with coherent systems is less apparent at this point, but will become quite clear in certain specific problems taken up in Chapter 7. Briefly, the fact is that it is possible to expand the class of coherent systems to a larger collection and that the solution to certain optimality problems lies outside of the subset of coherent systems. We will see that, in certain problems, one can actually do better, in a sense that will be made specific, by using a "system" that is not coherent. The expansion pursued below is based on the familiar notion of "randomization." Indeed, what will be advocated here, in selected circumstances, is the process of selecting a coherent system at random. This process leads to the concept of a mixed system, to which I now turn. Mixed systems were first treated in Boland and Samaniego [20].

Let us suppose that we have an (essentially) unlimited supply of components whose lifetimes are i.i.d. with common distribution F. Consider the collection of all coherent systems of order n. We could, in principle, have a warehouse in which all such systems are in stock, and we could conceive of the possibility of making available, upon demand, any particular coherent system with n i.i.d. components. While this collection may be quite large, it is finite. Limiting ourselves to this collection may have negative consequences, as alluded to above. Let us consider, instead, the process of selecting a coherent system at random according to a fixed and known probability distribution $\mathbf{p}$. Let's suppose that the probability vector $\mathbf{p}$ is m-dimensional and gives positive weight to m distinct coherent systems of order n with signature vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m$ (each assumed to have components whose lifetimes are i.i.d with distribution F). Then it is clear that

$$\begin{aligned} &P(\text{system fails upon the } i\text{th component failure}) \\ &= \sum_{k=1}^{m} P(k\text{th syst. chosen}) P(i\text{th comp. failure kills syst.} \mid k\text{th syst. chosen}) \\ &= \sum_{k=1}^{m} p_k s_{ki} \ . \end{aligned} \tag{3.12}$$

It follows that the signature $\mathbf{s}^*$ associated with the process of selecting among these m systems according to the probability distribution $\mathbf{p}$ is the vector equal to the mixture of the signature vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m$, that is, $\mathbf{s}^* = \sum_{k=1}^{m} p_k \mathbf{s}_k$.

To what extent does the consideration of stochastic mixtures of coherent systems (hereafter referred to as "mixed systems") broaden the class of coherent systems? The broadening is, in fact, quite substantial. The signature of a k-out-of-n system is the n-dimensional unit vector $\mathbf{s}_{k:n} = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 as the vector's kth element. It is thus clear that any probability vector $\mathbf{p}$ in the simplex

$$\left\{ \mathbf{p} \in [0,1]^n : \sum_{i=1}^{n} p_i = 1 \right\} \tag{3.13}$$

is the signature of a mixed system, namely, the system which mixes the k-out-of-n systems with mixing distribution $\mathbf{p}$. This simple observation follows from (3.12) since the n-dimensional probability vector $\mathbf{p}$ can be written as

$$\mathbf{p} = \sum_{k=1}^{n} p_k \mathbf{s}_{k:n} \ . \tag{3.14}$$

Expanding the space of coherent systems of order n to the space of all mixed systems of order n has a number of mathematical benefits. It makes the index of the systems under consideration continuous, taking values in the simplex in (3.13). Thus, instead of being restricted to discrete search methods and approximate solutions to optimization problems of interest, such problems will be amenable to analytical treatment using the tools of differential calculus. As mentioned earlier, the solutions to certain optimization problems turn out to be mixed rather than coherent systems; in such problems, the potential exists for improving over the "score" of any coherent system by using a particular mixed system. One further comment about the utility of mixed systems should be made: the representation results in (3.1) - (3.11), as well as all signature-related results to be presented in the sequel, apply equally to coherent or mixed systems. Indeed, since mixed systems include coherent systems as special cases, that is, as degenerate mixtures placing all their mass on a single coherent system, it suffices to say that all signature-related results in this monograph apply to the entire class of mixed systems.

But a critical question remains to be answered. Do mixed systems correspond to a physical reality that can be utilized in practice? If not, the potential mathematical advantages alluded to above are merely window-dressing. I will therefore briefly indulge in some apologetics for the concept of mixtures of coherent systems. To a decision theorist, or to a statistician working in the field of survey sampling or experimental design, the notion of randomization is both natural and essential. In the first instance, it is clear from the theory of games that the best strategies often involve randomization, and the decision theorist's goal is to select the "best" decision rule available, a goal that, in certain situations, leads to some form of randomization. In the latter fields, randomization is seen as a tool that protects the statistician from both known and unknown biases. Sir Ronald Fisher, the brilliant pioneer in the field of experimental design (and in mathematical statistics generally) was an early and strong advocate of randomization in designed experiments. What we have advocated above is nothing more than randomization within the class of coherent systems. The fact that a mixed system can be physically realized by a simple randomization process is the key observation in recognizing the concept's practical utility.

To utilize a mixed system, one simply selects a coherent system at random according to a particular mixing distribution, and one then uses the coherent system so chosen. The signature of the mixed system is interpreted as an expectation or long-run average over many applications of the mixed system. The signature's first element, for example, will be the limiting probability that the first component failure causes the system to fail. This interpretation suggests that the natural domain of application of mixed systems is a scenario in which such a system will be used many times. Thus, its failure history will, in the long run, be well represented by the mixed system's signature. Recalling (3.14), we note that, in the i.i.d. setting, any system, coherent or mixed, will have the same long run performance as a mixture of k-out-of-n systems. The mixing distribution would simply be the signature vector of the desired system. An important and useful implication of these considerations is that any coherent or mixed system of order n will have the same expected performance as a specific mixture of k-out-of-n systems. The "warehouse" mentioned on page 29 could therefore provide a client with a mixture of k-out-of-n systems whose performance matched the client's requirements, that is, had the desired signature.

I mentioned above that a non-degenerate mixture of coherent systems is not itself coherent. This may not be self evident, especially in view of the fact that each application of a mixed system amounts to picking a particular coherent system at random. Of course, successive applications of the mixed system will typically result in the use of different coherent systems from one application to the next. The latter fact notwithstanding, every application of a mixed system selects a coherent system with probability one. Since every coherent system is monotone, every mixed system also enjoys the monotonicity property. Fixing a non-working component cannot adversely affect the performance of a mixed system. So what is incoherent about a mixed system? The incoherence comes from the following technicality: only the components of the coherent system selected for use are relevant. The state of the components of all other (unused) coherent systems to which the mixed system gives weight makes no difference to the success or failure of the system actually chosen. One could argue that in the course of repeated applications, all these other coherent systems will come into play. But relevance is defined in terms of a single application of the system and, according to that definition, a mixed system will have irrelevant components. This technicality proves to be of minor importance, since when one focuses on relevance over repeated trials, the issue vanishes entirely.

One final comment about the utility of mixtures should be made. The intended domain for use of mixed systems is in applications in which a large number of systems will be purchased and used. As will be seen in Chapter 7, the best available system, when both performance and cost are taken into consideration, may well be a non-degenerate mixture of coherent systems. In some settings, the exclusive use of a particular coherent system might be justified by convenience considerations or by its being "almost optimal" relative to the criterion used to rank systems. The fact remains that one may be able to achieve better results by randomizing among two or more coherent systems. In using the optimal "mixed system," one must recognize that the system used in a given instance will be suboptimal, but that mixing these systems according to the prescribed recipe produces an optimal result when averaged over the many times in which the mixed system is applied.

In the next two chapters, we will consider a variety of problems in which our primary interest will be in comparing one coherent (or mixed) system with another. The most natural comparison one can make is between two systems of the same size. The preservation theorems of Section 4.2 and the characterization results in section 4.4 deal precisely with that type of comparison. But there are also occasions in which one is interested in comparing two systems in i.i.d. components whose orders n and m do not happen to match (say, with $n < m$). Our approach to such a problem is to attempt to "convert" the smaller system into an equivalent system of order m, (that is, a system in m i.i.d. components with exactly the same lifetime distribution as the smaller system). If this were possible, we would be in a position to use the results of Sections 4.2 and 4.4 on comparing two systems of the same size. It is not obvious, at first view, that the equivalence mentioned can in fact be realized. The fact that it can is a consequence of the following theorem. Assuming i.i.d. component lifetimes, this result ensures that for any system of size n (a positive integer assumed to be smaller than the integer m), there exists a system of size m with the same lifetime distribution. To get from the smaller system to the larger one requires $m - n$ consecutive applications of the result below.

Theorem 3.2. *Let* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *be the signature of a coherent or mixed system based on* n *i.i.d. components with common lifetime distribution* F. *Then the coherent or mixed system with* $(n+1)$*-components with i.i.d. lifetimes* $\sim F$ *and corresponding to the signature vector*

$$\mathbf{s}^* = \left(\frac{n}{n+1}s_1, \frac{1}{n+1}s_1 + \frac{n-1}{n+1}s_2, \frac{2}{n+1}s_2 + \frac{n-2}{n+1}s_3, \ldots, \right.$$
$$\left. \frac{n-1}{n+1}s_{n-1} + \frac{1}{n+1}s_n, \frac{n}{n+1}s_n \right) \tag{3.15}$$

has the same lifetime distribution as the n*-component system with signature* $\mathbf{s}$.

Proof. Assume that all components in the following discussion have i.i.d. lifetimes with common distribution F. We wish to show that, for a given n-component mixed system with signature $\mathbf{s}$, there is an $(n+1)$-component system (namely, the system with signature $\mathbf{s}^*$) with the same lifetime distribution, thereby being stochastically equivalent to the original system. It suffices to prove (3.15) when the original system is an arbitrary k-out-of-n system, that is, when $\mathbf{s}$ is the unit vector

$$\mathbf{s}_{k:n} = (0, \ldots, 0, 1, 0, \ldots, 0) \;, \tag{3.16}$$

where only the kth element is different from zero. We will show that this n-dimensional signature is equivalent to the $(n+1)$-dimensional signature

$$\mathbf{s}^* = \frac{n-k+1}{n+1}\mathbf{s}_{k:n+1} + \frac{k}{n+1}\mathbf{s}_{k+1:n+1}$$
$$= \left(0, \ldots, 0, \frac{n-k+1}{n+1}, \frac{k}{n+1}, 0, \ldots, 0\right) \;, \tag{3.17}$$

where the two non-zero elements of $\mathbf{s}^*$ are in the kth and $(k+1)$st positions. To see that this implies the claimed equivalence in the theorem, assume that, for any k, the signature in (3.16) is equivalent to the signature in (3.17). Then, since an arbitrary n-dimensional signature $\mathbf{s}$ may be written as a mixture of the signatures of k-out-of-n systems, that is, as

$$\mathbf{s} = \sum_{i=1}^{n} s_i \mathbf{s}_{i:n} \ , \tag{3.18}$$

one may use the equivalence of (3.16) and (3.17) to convert $\mathbf{s}$ into the vector in (3.15).

A direct proof of the equivalence of the signatures in (3.16) and (3.17) could proceed by determining the validity of equating the corresponding survival functions in (3.1). We will examine such an equation at a fixed time point t and denote by p and q, respectively, the probability of the success or failure of any given component at time t; thus, we will let $q = F(t)$ and $p = \overline{F}(t)$. As we will see, the equation of interest holds for all values of p and thus for all choices of t. Writing the reliability polynomial as on the left-hand side of (3.3) for the signature vectors in (3.16) and (3.17), and setting these expressions equal to each other, yields

$$\sum_{j=0}^{k-1} \binom{n}{j} q^j p^{n-j} = \frac{n-k+1}{n+1} \sum_{j=0}^{k-1} \binom{n+1}{j} q^j p^{n+1-j} + \frac{k}{n+1} \sum_{j=0}^{k} \binom{n+1}{j} q^j p^{n+1-j} \ . \tag{3.19}$$

One could then replace each "q" in equation (3.19) by "$1-p$," write both sides of the equation as polynomials in p, and then verify that the coefficients of p^k on either side of (3.19) are equal for $k = 0, 1, \ldots, n+1$. This approach to the desired identity is clearly a cumbersome algebraic exercise. We shall, instead, demonstrate that the equality in (3.19) is valid by working with an alternative representation of the equation.

Suppose that X is a binomial random variable and that Y is a Bernoulli variable, with $X \sim B(n, q)$ and $Y \sim B(1, q)$, and assume that X and Y are independent. (Note that $q = F(t)$ is being used as the probability of "success" in the Bernoulli trials associated with these variables.) We can then identify equation (3.19) as the following equation involving X and Y:

$$P(X < k) = \frac{n-k+1}{n+1} P(X+Y < k) + \frac{k}{n+1} P(X+Y < k+1) \ . \tag{3.20}$$

It is evident that equation (3.20) can be rewritten as

$$P(X < k) = P(X + Y < k) + \frac{k}{n+1} P(X + Y = k) \; . \tag{3.21}$$

Upon dividing both sides of (3.21) by $P(X < k)$, we note that

$$\frac{P(X + Y < k)}{P(X < k)}$$

is equal to the conditional probability $P(X + Y < k \mid X < k)$. We thus see that (3.21) holds if and only if

$$P(X + Y \geq k \mid X < k) = \frac{k}{n+1} \, \frac{P(X + Y = k)}{P(X < k)} \; . \tag{3.22}$$

We will now demonstrate that (3.22) holds. We do so by displaying a sequence of equivalent statements, the last of which is transparently true. We begin by exploiting the fact that Y is a Bernoulli variable and is independent of X; this allows us to rewrite (3.22) as

$$q \, \frac{P(X = k-1)}{P(X < k)} = \frac{k}{n+1} \, \frac{qP(X = k-1) + pP(X = k)}{P(X < k)} \tag{3.23}$$

$$\begin{aligned} \iff & q \binom{n}{k-1} q^{k-1} p^{n-k+1} \\ & = \frac{k}{n+1} \left(q \binom{n}{k-1} q^{k-1} p^{n-k+1} + p \binom{n}{k} q^k p^{n-k} \right) \end{aligned} \tag{3.24}$$

$$\iff \binom{n}{k-1} q^k p^{n-k+1} = \frac{k}{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) q^k p^{n-k+1} \tag{3.25}$$

$$\iff \binom{n}{k-1} = \frac{k}{n+1} \binom{n+1}{k} \; . \tag{3.26}$$

We thus see that equation (3.19) is equivalent to the trivial relationship between binomial coefficients displayed in (3.26). Since (3.19) is equivalent to the fact that the systems with the signatures $\mathbf{s}$ and $\mathbf{s}^*$ in the statement of the theorem have identical distributions, the proof is complete. ■

The form of the signature $\mathbf{s}^*$ in (3.15) makes it evident that, when $\mathbf{s}$ is symmetric, the equivalent signature $\mathbf{s}^*$ will inherit this property. This result is recorded as

Corollary 3.3. *Let* $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ *be the signature of a mixed system based on* n *i.i.d. components with common lifetime distribution* F*, and let* $\mathbf{s}^*$ *be the signature in* (3.15) *of the equivalent system with* $(n+1)$*-components based on i.i.d. lifetimes* $\sim F$*. If* $\mathbf{s}$ *is symmetric, that is, if* $s_i = s_{n-i+1}$ *for all* i*, then* $\mathbf{s}^*$ *is symmetric as well. Furthermore, if* (3.15) *is applied repeatedly to a signature vector* $\mathbf{s}_1$ *of length* n *to obtain an equivalent signature* $\mathbf{s}_2$ *of length* $m > n$*, then the symmetry of* $\mathbf{s}_1$ *implies the symmetry of* $\mathbf{s}_2$*.*

The concept of the duality of two systems was defined formally in Chapter 2. Since taking advantage of duality relationships reduces the calculation of the signatures of all coherent systems of a given size approximately in half, it is useful to record the relationship between the signature of a system and its dual. This is done in the following theorem. For a proof, see Kochar, Mukerjee and Samaniego [51].

Theorem 3.3. *Let* $\mathbf{s}$ *be the signature of a coherent system* φ *whose* n *components have i.i.d. lifetimes, and let* $\mathbf{s}^D$ *be the signature of its dual system* φ^D*. Then*

$$s_i = s^D_{n-i+1} \quad \text{for } i = 1, 2, \ldots, n\,. \tag{3.27}$$

We note that Theorem 3.3 holds as well for the broader class of mixed systems of order n.

We close this chapter with a useful notion and a related expression to which we will return in Chapter 5. In (3.5), we displayed an explicit representation of the survival function of a system in terms of its signature vector and the survival functions of the order statistics corresponding to the n i.i.d. component failure times. A similar representation may be developed for the probability $P(T_1 \leq T_2)$, where T_1 and T_2 are the lifetimes of mixed systems of orders n and m based on two independent i.i.d. samples of sizes n and m from underlying distributions F_1 and F_2, respectively. The representation of interest is drawn from Hollander and Samaniego [43], whose proof differs from the one below in that it utilizes integral representations of probabilities of the form $P(X \leq Y)$. Here, we provide a simpler, direct proof.

Theorem 3.4. *Let* T_1 *and* T_2 *represent the lifetimes of mixed systems of orders* n *and* m *with respective signatures* $\mathbf{s}_1$ *and* $\mathbf{s}_2$*. Assume that the* n *components of system 1 have i.i.d. lifetimes governed by the continuous distribution* F_1*, and let* $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ *be the corresponding ordered component lifetimes. Similarly, assume that the* m *components of system 2 have i.i.d. lifetimes governed by the continuous distribution* F_2*, and let* $\{Y_{1:m}, Y_{2:m}, \ldots, Y_{m:m}\}$ *be the corresponding ordered component lifetimes. Finally, assume the two samples are independent. Then*

$$P(T_1 \leq T_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_{1i} s_{2j} P\left(X_{i:n} \leq Y_{j:m}\right) \tag{3.28}$$

Proof. Using the Law of Total Probability and the i.i.d. assumption on component lifetimes, we may obtain the desired result by writing

$$
\begin{aligned}
&P(T_1 \leq T_2) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} P(T_1 = X_{i:n})P(T_2 = Y_{j:m})P(T_1 \leq T_2 \mid T_1 = X_{i:n}, T_2 = Y_{j:m}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} s_{1i}s_{2j}P(X_{i:n} \leq Y_{j:m}). \qquad \blacksquare
\end{aligned}
$$