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Introduction

The theory of Reliability and Life Testing has its roots in the research into the performance of engineered systems that was spawned by the applications arising in the second World War. An example of early studies of reliability issues is the work of Abraham Wald who, as a member of the heralded Statistical Research Group at Columbia University, treated the problem of estimating the vulnerability of aircraft used in WWII from data on "hits" taken by the planes that returned from various missions. Wald's work led to the addition of reinforcement of particularly vulnerable sections of the fuselage which ultimately led to a higher rate of returning aircraft. Wald's research on these problems was declassified in the late 1970s and is described in detail by Mangel and Samaniego [55].

Important advances in Reliability Theory were made in the early 1950s. Particularly notable is the paper by Epstein and Sobel [34], where optimal estimates are obtained for the mean lifetime of systems based on "type II" (or "order-statistic") censored data assumed to be drawn from an exponential distribution. The significance of that work was that it clearly demonstrated that characteristics of the population could be efficiently estimated from early failures, that is, from the first r systems to fail among the n systems placed on test. Grenander's [40] paper on nonparametric inference in reliability was highly influential. Zelen's [75] edited proceedings of a 1962 conference on the statistical theory of Reliability drew attention to the field and highlighted early research in the area.

A quantum leap in the development of a comprehensive theory of Reliability occurred concurrently with the formation of a statistical research team at Boeing Aircraft Company. That team, which achieved critical mass in the early 1960s, had as its core members James Esary, Albert Marshall, Frank Proschan and Sam Saunders. During their decade together at Boeing, they developed many of the key concepts, models and methods of modern Reliability Theory. This core group, in collaboration with Richard Barlow, Z. W.

Birnbaum, Ingram Olkin and others, published seminal work in each of the three primary subfields in the area: *structural reliability*, which concentrates on the way systems are designed and how these designs influence system performance, *stochastic reliability*, which concentrates on modeling the lifetime characteristics of systems and their components, and *statistical reliability*, which concentrates on the process of drawing inferences about general characteristics of systems from experimental data on their performance. Among the best known products of the Boeing group and its affiliates are papers by Birnbaum, Esary and Saunders [9] on the theory of coherent systems, by Proschan [59] on the occurrence of apparent improved performance over time (decreasing failure rate) in data sets consisting of combined failure data from several systems, the classic text on Mathematical Reliability by Barlow and Proschan [5], the important paper by Birnbaum, Esary and Marshall [8] on nonparametric modeling in reliability and the introduction of a multivariate exponential distribution as a shock model by Marshall and Olkin [56].

The basic theory and tools of structural reliability were pioneered by Birnbaum, Esary and Saunders in a seminal Technometrics paper published in 1961. In that work, the authors created a framework for studying the basic connection between the performance of a system and the performance of the components of which it is composed. Their study established the "structure function" as the predominant tool for distinguishing among systems and for determining whether one system will outperform another. In this sense, this class of functions can be used, though not with great ease, as an index on all systems of interest, and one might select one system over others on the basis of the characteristics of its structure function.

The aim of this monograph is to present a systematic examination of an alternative tool in structural reliability – system signatures. Both theory about, and applications of, system signatures are presented with a view toward demonstrating that this tool constitutes a powerful and versatile device for resolving a variety of problems in Reliability Theory, particularly those involving comparative analysis. I will begin, in Chapter 2, with a review of the traditional ideas and tools of structural reliability as found, for example, in Barlow and Proschan [6]. We formally define the notion of a coherent system and utilize structure functions and their properties as a vehicle for studying system behavior and for comparing one system with another. Central to this discussion is the important role of path sets and cut sets in studying the performance properties of coherent systems. The well-known representations of the structure function in terms of minimal path sets or minimal cut sets are developed. A constructive description of the class of all coherent systems of a given size is presented, and the intriguing open problem of counting the number of coherent systems of order n, for arbitrary fixed n, is discussed. The connection between the structure function and the reliability of a coherent system is presented, and the "reliability polynomial" is introduced for treatment of the i.i.d. case.

In Chapter 3, we introduce the notion of "signatures" of coherent systems in components with i.i.d. lifetimes and provide some guidance on computing and interpreting them. The problem of comparing two complex systems has typically been complicated by the fact that the traditional tools for characterizing system designs have proven to be rather awkward as indices in optimization problems. As we shall see, the existence of an easily interpretable summary of fixed dimension for the essential characteristics of the systems whose components have i.i.d. lifetimes has made the analytical investigation of many of these problems possible. Multiple examples of signature calculations are given, and mention is made of the elements of combinatorial mathematics that are relevant to such calculations. Under the assumption that the components of the systems to be considered have i.i.d. lifetimes, the distribution (and density and failure rate, if they exist) of the system's lifetime T will be represented explicitly as a function of the system's signature and the underlying distribution F of the component lifetimes. These representations are used with some regularity throughout the remainder of the monograph. The notion of signature is extended beyond the class of coherent systems to the family of all stochastic mixtures of coherent systems of a given size (to be referred to as "mixed systems"), and the motivation for doing so is discussed in detail.

In Chapters 4 and 5, the utility of signatures is demonstrated in various reliability contexts. Chapter 4 is dedicated to applications of system signatures to closure and preservation theorems in reliability and to the role that signatures can play in the comparison of coherent systems or mixtures of them. First, we present a description of the "IFR closure problem" and provide a characterization, in terms of system signatures, of systems whose lifetime distributions have an increasing failure rate whenever its components have i.i.d. lifetimes with an increasing failure rate. We then present a collection of preservation theorems showing that certain types of orderings of system signatures imply like orderings of the corresponding system lifetime distributions. Since the calculation of the lifetime distributions of complex systems is a challenging (and often unsolved) problem which makes the direct comparison of system lifetimes a tenuous matter, the utility of comparing some relatively simple summaries for two system designs and knowing immediately that one system has a longer lifetime (in some stochastic sense) is clearly useful. In Section 4.3, an example involving stochastic comparisons of different types of redundancy in coherent systems illustrates the utility of the preservation results developed in the preceding section.

Since the ordering conditions on signatures in the preservation theorems presented in Section 4.2 prove to be sufficient but not necessary for the ordering of system lifetimes, we turn, in the final section of Chapter 4, to the investigation of possible necessary and sufficient conditions (NASCs) for specific types of orderings to hold for the lifetimes T_1 and T_2 of two systems in i.i.d. components. For each of the contexts in which preservation theorems are established, NASCs are obtained for the ordering of system lifetimes. Interestingly, the precise crossing properties of the survival functions or failure rates of two systems of interest can be determined by the behavior of certain functions that depend on the systems' designs only through the respective system signatures. Results of this latter type lead to insights that extend beyond the partial ordering of systems via properties of their signatures. In situations in which systems are not comparable in the usual stochastic senses, it is possible to characterize the crossing behavior of pairs of survival functions or failure rates, as well as the alternating monotonicity of the likelihood ratio, through the precise behavior of the functions used in establishing NASCs for stochastic domination. This latter extension provides a vehicle for fully understanding the relative real-time behavior of the lifetime distributions, failure rates and density functions of competing systems.

In Chapter 5, several comparisons between pairs of special-purpose systems are pursued. In particular, direct and indirect majority systems are contrasted. Signatures are also employed in establishing monotonicity properties of consecutive k-out-of-n systems and in studying the limiting behavior of survival functions and failure rates of arbitrary mixed systems. In section 5.4, we present an important augmentation to the preceding theory on the comparison of two systems. The usual forms of stochastic comparisons (stochastic, hazard-rate and likelihood-ratio ordering) are powerful when they are applicable, but they have the limitation of inducing only a partial ordering on the class of coherent (or mixed) systems. Simply put, it is easy to find pairs of systems that are not comparable under any of these orderings. In Section 5.4, we consider the use of "stochastic precedence," introduced in Arcones, Kvam and Samaniego [3], which classifies system 2 as better than system 1 if $P(T_1 \leq T_2) \geq 1/2$. This criterion leads to definitive comparisons between any two systems of arbitrary size. An explicit formula for computing $P(T_1 \leq T_2)$ is displayed and a signature-based NASC is given for the inequality $P(T_1 \leq T_2) \geq 1/2$ to hold.

Chapter 6 is dedicated to the study of signatures in the context of network reliability. The chapter begins with a brief introduction to basic ideas and vocabulary of communication networks. The signature is a well-defined concept in each of several types of network problems, including two-terminal, k-terminal and all-terminal reliability (focusing, respectively, on whether two terminals, k terminals or all terminals in a network can communicate with each other). The treatment of network reliability includes a review the theory of "dominations" as introduced and developed by Satyanarayana and his co-workers. This is followed by a derivation of a closed-form functional relationship between dominations and the signature vector. The utility of this connection is illustrated by a comparison between two distinct networks with 9 terminals and 27 edges (i.e., 27 paths between pairs of terminals).

Problems in the field of "Reliability Economics" have been, until recently, largely resistant to analytical treatment. In Chapter 7, we will consider the problem of searching for the optimal system of order n relative to a specific criterion function which depends on both a system's performance and its cost. The solution to this problem assumes i.i.d. component lifetimes and makes essential use of the $(n-1)$-dimensional simplex of signatures of all stochastic mixtures of coherent systems of order n. The latter strategy turns what has heretofore been treated as a large discrete optimization problem (i.e., finding the best coherent system) into a continuous problem (i.e., finding the best mixed system) to which the methods of differential calculus can be applied. Given such a framework, optimal systems are identified through the signatures that maximize the chosen criterion function. Examples are given in which the optimal system is a non-degenerate mixture of coherent systems and every coherent system is inferior to it. Since the solutions obtained depend on a known underlying lifetime distribution F of the competing systems' components, a complete solution, usable in practice, would entail the estimation of this distribution or its relevant features. Chapter 7 closes with a treatment of the statistical problem that must be solved in order for the optimality results of Chapter 7 to be applicable in practice. The final chapter of this monograph is dedicated to a brief discussion of extensions of, and results related to, the theory and applications treated in Chapters 3 - 7, some further references to related work, and a description of several open problems of interest.

As indicated in the outline presented in this chapter, there's a good deal of work to be done. Let us now proceed with the program described above.