Chapter 9

VANISHING DISCOUNT APPROACH VERSUS STATIONARY DISTRIBUTION APPROACH

9.1. Introduction

In Part III, we derived the structure of the optimal policy to minimize the long-run average cost by using the vanishing discount method. In the classical inventory literature, a stationary distribution approach is often used to minimize the long-run average cost. In this approach, the stationary distribution of the inventory levels is obtained for a specific class of policies, the best policy in this class is found, and then it is proven that this policy is average optimal. In this chapter, we review the stationary distribution approach in solving the simpler problem of an inventory model with i.i.d demands, and then show how the results of this analysis relate to those obtained by the vanishing discount approach.

In the context of inventory problems, Iglehart (1963b) and Veinott and Wagner (1965) were the first to study the issue of the existence of an optimal (s, S)-type policy for average cost problems with independent demands, linear holding and backlog costs, and ordering costs consisting of a fixed cost and a proportional variable cost. Iglehart obtained the stationary distribution of the inventory/backlog (or surplus) level given an (s, S) policy using renewal theory arguments (see also Karlin (1958a) and Karlin (1958b)), and developed an explicit formula for the stationary average cost $\mathcal{L}(s, S)$, $s \leq S$, associated with the policy.

Iglehart *assumed* that the function $\mathcal{L}(s, S)$ is continuously differentiable and that there exists a pair (s^*, S^*), $-\infty < s^* < S^* < \infty$, which minimizes $\mathcal{L}(s, S)$ and satisfies the first-order conditions for an interior local minimum. While he does not specify these assumptions explicitly, and certainly does not verify them, he *uses* them in showing the key result that the minimum average cost of a sequence of problems

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with increasing finite horizons approaches $\mathcal{L}(s^*, S^*)$ asymptotically as the horizon becomes larger. Veinott and Wagner (1965), with a short additional argument suggested by Derman (1965), were able to advance the Iglehart result into the optimality of the (s^*, S^*) policy in the special case of discrete demands; (see also Veinott (1966)). While Veinott and Wagner deal with the case of discrete demands, they assume, without proving, that Iglehart's results derived for the continuous demand case also hold for their case. It should also be mentioned that Derman's short additional argument also applies to the continuous demand case treated in Iglehart and proves that an $\mathcal{L}(s, S)$-minimizing pair (s^*, S^*), if there exists one, provides an optimal inventory policy; (see Section 9.7).

A good deal of research has been carried out in connection with the average cost (s, S) models since then. With the exception of Zheng (1991) and Huh *et al.* (2008), however, most of this research devoted to establishing the optimality of (s, S) strategies, uses bounds on the inventory position after ordering. Examples are Tijms (1972), Wijngaard (1975), and Küenle and Küenle (1977). On the other hand, quite a few papers are concerned with the computation of the (s^*, S^*) pair that minimizes $\mathcal{L}(s, S)$, and not with the issue of establishing the optimality of an (s, S) policy. Some examples are Stidham (1977), Zheng and Federgruen (1991), Federgruen and Zipkin (1984), Hu *et al.* (1993), and Fu (1994). For other references, the reader is directed to Zheng and Federgruen (1991), Porteus (1985), Sahin (1990), and Presman and Sethi (2006). Furthermore, this literature has generally assumed that together, the papers of Iglehart (1963b) and Veinott and Wagner (1965), have established the optimality of an (s, S)-type policy for the problem.[1] But, this is not quite the case, however, since the assumptions on $\mathcal{L}(s, S)$ *implicit* in Iglehart, to our knowledge, have not been satisfactorily verified.

We will compare the stationary cost analyses of Iglehart (1963b) and Veinott and Wagner (1965) and the vanishing discount approach used in Part II of this book. Both approaches prove the optimality of (s, S) strategies, but for somewhat different notions of the long-run average

[1] Another possible approach views continuous demand distributions as the limit of a sequence of discrete demand distributions. Such a limiting procedure could lead to a proof of optimality of (s, S) policies in the continuous demand case. However, this is by no means a trivial exercise.

cost to be minimized. It turns out that the optimal (s, S) strategies are optimal for these different notions of long-run average cost.

We will reproduce the results of Iglehart (1963b) and Veinott and Wagner (1965) in some detail. Some of the proofs of implicit assumptions are missing in Iglehart (1963b) with respect to the model under his consideration. Without these results, the Iglehart analysis cannot be considered complete. Moreover, these results are by no means trivial.[2]

For this purpose, we need to precisely specify Iglehart's model. Then for this model, we must show that there exists a pair (s^*, S^*), $-\infty < s^* \leq S^* < \infty$ that minimizes $\mathcal{L}(s, S)$. In order to accomplish this, we establish in Section 9.7, *a priori* bounds on the minimizing values of s and S. We should caution that any verification of these assumptions on $\mathcal{L}(s, S)$ should not use arguments that rely on the optimality of an (s, S) policy. With the bounds established in Section 9.7, the continuity of $\mathcal{L}(s, S)$ provides us with the existence of a minimum. Continuous differentiability of $\mathcal{L}(s, S)$ follows from the definition of the surplus cost function $L(y)$ and the assumption of a continuous density for the demand. We then show that if (s^*, S^*) with $s^* = S^*$ is a minimum, then there is another minimum (s, S) with $s < S$. It is then possible to assume that an interior solution always exists, and to obtain it by the first-order conditions of an interior minimum.

Tijms (1972) uses the theory of Markov decision processes (MDP) to prove the optimality of (s, S) strategies for a modified inventory problem with discrete demand. In particular, he imposes upper and lower bounds on the inventory position after ordering. These bounds provide a compact (finite) action space as well as bounded costs. Under these conditions, standard MDP results yield the optimality of an (s, S) strategy.

Zheng (1991) has provided a rigorous proof of the optimality of an (s, S) policy in the case of discrete demands for the model in Veinott and Wagner (1965). He was able to use the theory of countable state Markov decision processes in the case when the solution of the average cost optimality equation for the given problem is bounded, which is clearly not the case here since the inventory cost is unbounded. Note that this theory does not deal with the continuous demand case as it would involve an uncountable state MDP. Zheng relaxed the problem by allowing inventory disposals, and since the inventory costs are charged

[2] In fact, the motivation to write our paper arose from Example 9.4 in Section 9.4. The example shows that even for a well-behaved demand density function satisfying Iglehart's assumptions, the derivation of equation (9.12) crucial for the subsequent analysis requires additional arguments not given in Iglehart's paper. The example also shows that in some cases a base-stock policy can be optimal even in the presence of a positive fixed ordering cost.

on ending inventories in his problem, he obtained a bounded solution for the average cost optimality equation of the relaxed problem that involves a dispose-down-to-S component. But the dispose-down-to-S component of the optimal policy would be invoked in the relaxed problem only in the first period (and only when the initial inventory is larger than S), which has no influence on the long-run average cost of the policy. It follows, therefore, that the (s, S) policy, without the dispose-down-to-S component, will also be optimal for the original problem.

The plan of the chapter is as follows. In Section 9.2, we state the problem under consideration. Section 9.3 summarizes the results of Iglehart relevant to the average cost minimization problem. Furthermore, we point out exactly which implicit assumptions have been used by Iglehart without verification. We develop an example in Section 9.4 to show that even under the quite restrictive assumption of the existence of a continuous demand density, the assumptions implicit in the Iglehart analysis are not necessarily satisfied. In Section 9.5, we derive asymptotic bounds on the minimum cost function. In Section 9.6, we review the analysis contained in Veinott and Wagner (1965) that is devoted to the solution of the average cost problem in the case of discrete demands. To the extent that they use Iglehart's analysis for their solution, we show how their paper is not quite complete and how it can be completed. Section 9.7 contains the proofs needed for the completion of Iglehart's analysis and for establishing the optimality of an (s, S) policy in the continuous demand case. Section 9.8 lists results that connect the stationary distribution approach and the vanishing discount approach; both are undertaken to prove the existence of an optimal (s, S) strategy for the average cost inventory problem. Section 9.9 concludes the chapter.

9.2. Statement of the Problem

In this section we formulate a stationary one-product periodic review inventory model with the following notation and assumptions.

(i) The surplus (inventory/backlog) level at the beginning of period k prior to ordering is denoted by x_k. Unsatisfied demand is fully backlogged.

(ii) The surplus level after ordering, but before demand realizes, in period k is denoted by y_k. Orders arrive immediately.

(iii) The one period demands ξ_k, $k = 1, 2, \ldots$, are i.i.d. and the demand distribution has a density $\varphi(\cdot)$. Let μ denote the mean demand. Assume $0 < \mu < \infty$.

(iv) The surplus cost function $L(y)$, where y is the surplus level immediately after ordering, is given by

$$L(y) = \mathsf{E}(h([y - \xi]^+)) + \mathsf{E}(p([y - \xi]^-)),$$

where $h(\cdot)$ and $p(\cdot)$ represent holding and shortage cost functions, respectively. Furthermore, $L(y)$ is assumed to be convex and finite for all y.

(v) The ordering cost when an amount u is ordered is given by

$$\hat{c}(u) = K\mathbb{1}_{u>0} + cu,\ c \geq 0,\ K > 0.$$

Given an ordering up to policy $Y = (y_1, y_2, \ldots)$, the inventory balance equation is

$$x_{k+1} = y_k - \xi_k.$$

Let $f_n(x|Y)$ denote the expected total cost for an n-period problem with the initial inventory $x_1 = x$ when the order policy Y is used, i.e.,

$$f_n(x|Y) = \mathsf{E}\left\{\sum_{k=1}^{n}[\hat{c}(y_k - x_k) + L(y_k)]\right\}.$$

The objective is to minimize the expected long-run average cost

$$a(x|Y) = \liminf_{n\to\infty} \frac{1}{n} f_n(x|Y) \tag{9.1}$$

over the class of all nonanticipative or history-dependent policies $\mathcal{Y}$. In what follows we use $f_n(x|s,S)$ and $a(x|s,S)$ instead of $f_n(x|Y)$ and $a(x|Y)$, with a slight abuse of notation, if Y is a stationary (s,S) strategy.

The model described above is investigated in a more general setting of Markovian demand and polynomially growing surplus cost in Chapter 6 for the average cost objective function

$$J(x|Y) = \limsup_{n\to\infty} \frac{1}{n} f_n(x|Y). \tag{9.2}$$

There we establish the average cost optimality equation and prove the optimality of an (s,S)-type policy by using the vanishing discount approach.

A policy minimizing either (9.1) or (9.2) does not necessarily minimize the other. However, if an optimal policy Y^* with respect to (9.1) is such that $\lim_{n\to\infty}(1/n)f_n(x|Y^*)$ exists, then this limit is less than or equal to both objective functions associated with any policy $Y \in \mathcal{Y}$. On the other hand, if a policy Y^* is optimal with respect to (9.2) and if $\lim_{n\to\infty}(1/n)f_n(x|Y^*)$ exists, then Y^* may still not minimize (9.1). For this reason, Veinott and Wagner (1965) consider the objective function (9.1) to be the *stronger* of the two; often the term *more conservative* is used instead. In Sections 9.7 and 9.8, we complete Iglehart's stationary state analysis and use Derman's short additional argument to obtain an (s, S) policy, which is optimal with respect to (9.1) and has the limit $(1/n)f_n(x|s, S))$ as $n \to \infty$. Thus, this (s, S) policy also minimizes both objective functions (9.1) and (9.2). In addition, by combining the stationary approach with the dynamic programming approach, we show in Section 9.8 that *any* (s, S) policy that is optimal with respect to (9.2) is also optimal with respect to (9.1).

9.3. Review of Iglehart (1963b)

In this section we will summarize the results of a paper by Iglehart (1963b) relevant to the problem of minimizing the long-run average cost, and point out the implicit assumptions which have been used in his paper without verification.

Let $f_n(x)$ denote the minimal total cost for the n-period problem when the initial surplus level is x, i.e.,

$$f_n(x) = \min_{\mathcal{Y}} f_n(x|Y).$$

The sequence of functions $(f_m(x))_{m=1}^n$ satisfies the dynamic programming equation

$$f_n(x) = \min_{y\geq x}[\hat{c}(y-x) + L(y) + \int_0^\infty f_{n-1}(y-\xi)\varphi(\xi)d\xi]. \tag{9.3}$$

Furthermore, it is known that $f_m(x)$ is K-convex, and an optimal strategy minimizing the total cost is determined by a sequence $(s_m, S_m)_{m=1}^n$ of real numbers with $s_m \leq S_m$, such that the optimal order quantity in period m is

$$u_m = \begin{cases} S_m - x_m & \text{if} \quad x_m \leq s_m, \\ 0 & \text{if} \quad x_m > s_m, \end{cases}$$

where x_m denotes the surplus level at the beginning of the m^{th} period.[3]

It is obvious that $\lim_{n\to\infty} f_n(x) = \infty$ for all initial surplus levels x. Iglehart investigates the asymptotic behavior of the function $f_n(x)$ for large n. Heuristic arguments suggest that for a stationary infinite hori-

[3]Well-known papers dealing with this finite horizon problem are those of Scarf (1960), Schäl (1976), and Veinott (1966).

zon inventory problem, a stationary strategy should be optimal. Furthermore, it is reasonable to expect that this stationary strategy is of (s, S)-type.

Iglehart obtains the stationary distribution of the surplus level and the expected one-period cost under any given (s, S) strategy satisfying $-\infty < s \le S < \infty$. He uses the result of Karlin (1958a,b) that the surplus level x_n at the beginning of period n converges in distribution to a random variable whose distribution has the density

$$f(x) = \begin{cases} \dfrac{m(S-x)}{1+M(\Delta)}, & s < x \le S, \\ \dfrac{h(\Delta, s-x)}{1+M(\Delta)}, & x \le s, \end{cases} \tag{9.4}$$

where $\Delta := S - s$, $M(\cdot)$ and $m(\cdot)$ are the renewal function and the renewal density associated with $\varphi(\cdot)$, respectively, and $h(\Delta, \cdot)$ is the density of the order quantity in excess of Δ. Note that by the Elementary Renewal Theorem B.3.1, we have $M(t)/t \to 1/\mu$ as $t \to \infty$. Also $M(t) \to \infty$ as $t \to \infty$.

Furthermore, the one-period cost $C(x)$, given the initial surplus x, is

$$C(x) = \begin{cases} K + L(S) + c(S-x), & x \le s, \\ L(x), & s < x \le S. \end{cases} \tag{9.5}$$

Averaging $C(x)$ with respect to $f(x)$ yields the following formula for the stationary cost $\mathcal{L}(s, S)$ per period corresponding to the strategy parameters s and S:

$$\begin{aligned} \mathcal{L}(s,S) &= \int_{-\infty}^{s} \Big[K + L(S) + c(S-x)\Big] f(x)dx + \int_{s}^{S} L(x)f(x)dx \\ &= \frac{K + L(S) + \int_s^S L(x)m(S-x)dx}{1 + M(S-s)} + c\mu. \end{aligned} \tag{9.6}$$

Remark 9.1 It follows from the convergence in distribution of the surplus level in period n that the expected cost $\mathsf{E}(C(x_n))$ in period n converges to $\mathcal{L}(s, S)$. Therefore, we have

$$a(x|s,S) = \lim_{n\to\infty} \frac{1}{n} f_n(x|s,S) = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \mathsf{E}(C(x_n)) \right) = \mathcal{L}(s,S).$$

In many cases, it is more convenient to define the stationary cost in terms of Δ and S:

$$\tilde{\mathcal{L}}(\Delta, S) := \mathcal{L}(S-\Delta, S) = \frac{K + L(S) + \int_0^\Delta L(S-x)m(x)dx}{1+M(\Delta)} + c\mu, \quad (9.7)$$

or in terms of s and Δ:

$$\begin{aligned}\hat{\mathcal{L}}(s,\Delta) &:= \mathcal{L}(s, s+\Delta) \\ &= \frac{K + L(s+\Delta) + \int_0^\Delta L(s+\Delta-x)m(x)dx}{1+M(\Delta)} + c\mu. \quad (9.8)\end{aligned}$$

Iglehart then attempts to minimize $\mathcal{L}(s, S)$ with respect to s and S. Implicitly, he assumes $\mathcal{L}(s, S)$ to be continuously differentiable and that the minimum is attained for some s^* and S^* satisfying $-\infty < s^* < S^* < \infty$, i.e., the minimum is attained at an interior point and not at the boundary $s = S$ of the feasible parameter set. Therefore, he can take derivatives of $\hat{\mathcal{L}}(s, \Delta)$ with respect to s and Δ and obtain necessary conditions for the minimum by setting them equal to zero:

$$\frac{\partial \hat{\mathcal{L}}(s,\Delta)}{\partial s} = L'(s+\Delta) + \int_0^\Delta L'(s+\Delta-x)m(x)dx = 0 \quad (9.9)$$

and

$$\begin{aligned}&\frac{\partial \hat{\mathcal{L}}(s,\Delta)}{\partial \Delta} \\ &= \frac{L'(s+\Delta) + \int_0^\Delta L'(s+\Delta-x)m(x)dx + L(s)m(\Delta)}{1+M(\Delta)} \\ &\quad - \frac{K + L(s+\Delta) + \int_0^\Delta L(s+\Delta-x)m(x)dx}{(1+M(\Delta))^2} m(\Delta) = 0. \quad (9.10)\end{aligned}$$

Note that if $s^* = S^*$, Condition (9.9) must be relaxed to $\partial\hat{\mathcal{L}}/\partial s \leq 0$. Combining the necessary Conditions (9.9) and (9.10), one obtains

$$(1+M(\Delta))L(s) - K - L(s+\Delta) - \int_0^\Delta L(s+\Delta-x)m(x)dx)m(\Delta) = 0. \quad (9.11)$$

Assuming further that $m(\Delta^*) = m(S^* - s^*) > 0$ for the minimizing values s^* and S^*, Iglehart obtains the following formula, crucial for his subsequent analysis, by dividing by $m(\Delta)$:

$$L(s^*) = \frac{K + L(s^* + \Delta^*) + \int_0^{\Delta^*} L(s^* + \Delta^* - x)m(x)dx}{1 + M(\Delta^*)} = \mathcal{L}(s^*, S^*) - c\mu. \tag{9.12}$$

Let us recapitulate Iglehart's implicit assumptions here. For his subsequent analysis, he requires that

(I1) there is a pair (s^*, S^*) with $-\infty < s^* < S^* < \infty$ (i.e., an interior solution) that minimizes $\mathcal{L}(s, S)$,

(I2) $\mathcal{L}(s, S)$ is continuously differentiable, and

(I3) the minimizing pair (s^*, S^*) satisfies $m(S^* - s^*) > 0$.

We will see that there is always a pair (s^*, S^*) satisfying Assumption (I1) since $K > 0$. On the other hand, in general, not every minimizer of $\mathcal{L}(s, S)$ satisfies Assumption (I1) even though $K > 0$; (see Example 9.4). Assumption (I2) is satisfied because of the continuous differentiability of $L(\cdot)$ and the existence of a continuous density, which implies continuous differentiability of $M(\cdot)$. Assumption (I3) is more difficult to deal with. In Example 9.4, Assumption (I3), i.e., $m(S^* - s^*) > 0$, is violated for all minimizing (s^*, S^*) pairs. The fact that Assumption (I3) does not hold in general is a problem that can be fixed. It turns out that it is not Assumption (I3) itself but rather equation (9.12) derived with the help of this assumption that is crucial for the subsequent analysis. In Section 9.7, we prove that there is always a minimizing pair (s^*, S^*) that satisfies (9.12), even if $m(S^* - s^*) = 0$ for all minimizing pairs.

9.4. An Example

We develop an example with $K > 0$, in which $s = S$ is optimal and $m(S^* - s^*) = 0$ for all minimizing pairs of parameters (s^*, S^*). To do so, we prove two preliminary results.

LEMMA 9.1 *Let $L(\cdot)$ be a convex function with* $\lim_{x\to\pm\infty} L(x) = \infty$. *Then for $0 \le x_1 \le x_2$,*

$$\min_S \{L(S) + L(S - x_1)\} \le \min_S \{L(S) + L(S - x_2)\}.$$

Proof. Because of the limit property and the convexity of $L(\cdot)$, $L(S) + L(S - x_2)$ attains its minimum. Let S_2 be a minimum point of $L(S) +$

$L(S-x_2)$. Let us first assume that $L(S_2) \geq L(S_2-x_2)$. Then, it follows from the convexity of $L(\cdot)$ that $L(S_2-x_2+x_1) \leq L(S_2)$, and we obtain

$$\begin{aligned}\min_S\{L(S)+L(S-x_1)\} &\leq L(S_2-x_2+x_1)+L(S_2-x_2)\} \\ &\leq L(S_2)-L(S_2-x_2) \\ &= \min_S\{L(S)+L(S-x_2)\}.\end{aligned}$$

If $L(S_2) < L(S_2-x_2)$, we conclude from the convexity of $L(\cdot)$ that $L(S_2-x_1) \leq L(S_2-x_2)$. Thus,

$$\begin{aligned}\min_S\{L(S)+L(S-x_1)\} &\leq L(S_2)+L(S_2-x_1)\} \\ &\leq L(S_2)+L(S_2-x_2) \\ &= \min_S\{L(S)+L(S-x_2)\},\end{aligned}$$

which completes the proof. □

LEMMA 9.2 *Let* $L(\cdot)$ *be a convex function with* $\lim_{x\to\pm\infty} L(x) = \infty$. *Then*

$$\lim_{D\to\infty}\min_S\{L(S)+L(S-D)\} = \infty.$$

Proof. Fix $D > 0$ and let S^* denote a fixed minimum point of $L(\cdot)$. It is easy to see that

$$L(S) \geq \min\{L(S^*+D/2), L(S^*-D/2)\} \text{ for } |S-S^*| \geq D/2.$$

Also, substituting $S-D$ for S,

$$L(S-D) \geq \min\{L(S^*+D/2), L(S^*-D/2)\} \text{ for } |S-D-S^*| \geq D/2.$$

Since at least one of the above conditions is satisfied and $L(x) \geq L(S^*)$ for all x, we obtain

$$L(S)+L(S-D) \geq \min\{L(S^*+D/2), L(S^*-D/2)\}+L(S^*).$$

Therefore, it follows that

$$\begin{aligned}&\lim_{D\to\infty}\min_S\{L(S)+L(S-D)\} \\ &\quad\geq \lim_{D\to\infty}\min\{L(S^*+D/2), L(S^*-D/2)\}+L(S^*) = \infty,\end{aligned}$$

and the proof is completed. □

For the purpose of this example, we assume the unit purchase cost $c = 0$. It can easily be extended to the case $c > 0$. Let the one-period demand consist of a deterministic component $D \geq 0$ and a random component d. Let $\varphi(\cdot)$ be the density of d, where $\varphi(\cdot)$ is continuous on $(-\infty, \infty)$, $\varphi(t) = 0$ for $t \leq 0$, and $\varphi(t) > 0$ for $t \in (0, \varepsilon)$ for some $\varepsilon > 0$. The density of the one-period demand is then given by $\varphi^D(t) = \varphi(t - D)$. It is continuous on the entire real line. We denote the renewal density and the renewal function with respect to φ^D by m^D and M^D, respectively. Let

$$L^D(x) = \int_{-\infty}^{\infty} l(x - \xi)\varphi^D(\xi)d\xi$$

be the expected one-period surplus cost function. Let $L(\cdot) = L^0(\cdot)$. Then, we have

$$L^D(x) = L(x - D).$$

Now the stationary cost function, given s and S, is

$$\mathcal{L}^D(s, S) = \frac{K + L^D(S) + \int_s^S L^D(x)m^D(S - x)dx}{1 + M^D(S - s)},$$

and in view of $\Delta = S - s$, we define

$$\tilde{\mathcal{L}}^D(\Delta, S) = \mathcal{L}^D(S - \Delta, S) = \frac{K + L^D(S) + \int_0^\Delta L^D(S - x)m^D(x)dx}{1 + M^D(\Delta)}.$$

We will show that there is a constant D_0 such that for all $D > D_0$, the pair that minimizes the stationary average cost is of the following form

> (s^*, S^*) is a minimum point of $\mathcal{L}^D(s, S)$ if, and only if, S^* minimizes $L^D(S)$ and $0 \leq S^* - s^* \leq D$.

It is obvious that because of $\varphi^D(t) = 0$ for all $t \leq D$, $m^D(t) = 0$ and $M^D(t) = 0$ for $t \leq D$. Therefore, for any given $\Delta \leq D$,

$$\min_S \tilde{\mathcal{L}}^D(\Delta, S) = K + \min_S L^D(S) = K + \min_S L(S).$$

We will show that for a sufficiently large D,

$$\min_S \tilde{\mathcal{L}}^D(\Delta, S) > K + \min_S L(S) \text{ for all } \Delta > D.$$

To do so, we consider three cases that arise when $\Delta > D$ for any given D.

- Case 1. Let $\Delta \in \{\Delta : 0 < M^D(\Delta) \leq 1\}$. Then,

$$
\begin{aligned}
&\min_S \tilde{\mathcal{L}}^D(\Delta, S) \\
&= \frac{K + \min_S \left\{ L^D(S) + \int_0^\Delta L^D(S-x) m^D(x) dx \right\}}{1 + M^D(\Delta)} \\
&\geq \frac{K + \min_S \left\{ L^D(S) + \min_{x \in [D,\Delta]} \{L^D(S-x)\} M^D(\Delta) \right\}}{1 + M^D(\Delta)} \\
&= K + \frac{1}{1 + M^D(\Delta)} \Bigg(\min_S \Big\{ (1 - M^D(\Delta)) L^D(S) \\
&\quad + \min_{x \in [D,\Delta]} \{L^D(S-x) + L^D(S) - K\} M^D(\Delta) \} \Big\} \Bigg) \\
&\geq K + \frac{1}{1 + M^D(\Delta)} \Bigg((1 - M^D(\Delta)) \min_S L^D(S) \\
&\quad + M^D(\Delta) \min_S \{ \min_{x \in [D,\Delta]} \{L^D(S-x) + L^D(S) - K\}\} \Bigg).
\end{aligned} \tag{9.13}
$$

Using Lemma 9.1, we have

$$
\begin{aligned}
&\min_S \{ \min_{x \in [D,\Delta]} \{L^D(S-x) + L^D(S) - K\}\} \\
&\quad = \min_S \{ \min_{x \in [D,\Delta]} \{L(S-D-x) + L(S-D) - K\}\} \\
&\quad = \min_S \{ \min_{x \in [D,\Delta]} \{L(S-x) + L(S) - K\}\} \\
&\quad = \min_S \{L(S-D) + L(S) - K\}.
\end{aligned} \tag{9.14}
$$

On account of Lemma 9.2, this expression tends to infinity as $D \to \infty$. Therefore, there is a $D_1 > 0$ such that

$$
\min_S \{L(S-D) + L(S) - K\} > 2 \min_S L(S) \text{ for all } D \geq D_1. \tag{9.15}
$$

Thus from (9.13)–(9.15), we have

$$
\min_S \tilde{\mathcal{L}}^D(\Delta, S) > K + \min_S L(S) \text{ for all } D \geq D_1,
$$

implying that Δ in this case cannot be a minimizer.

- Case 2. Let $\Delta \in \{\Delta : 1 < M^D(\Delta) \le 2\}$. Then,

$$\begin{aligned}
\tilde{\mathcal{L}}^D(\Delta, S) &= \frac{K + L^D(S) + \int_0^\Delta L^D(S-x)m^D(x)dx}{1 + M^D(\Delta)} \\
&\ge \frac{K + \frac{1}{2}\int_0^\Delta (L^D(S-x) + L^D(S))m^D(x)dx}{3} \\
&\ge \frac{1}{6}\min_{x\in[D,\Delta]}\{L^D(S-x) + L^D(S)\}M^D(\Delta) \\
&\ge \frac{1}{6}\min_{x\in[D,\Delta]}\{L^D(S-x) + L^D(S)\}.
\end{aligned}$$

In view of Lemma 9.1, taking the minimum with respect to S yields

$$\begin{aligned}
\min_S \tilde{\mathcal{L}}^D(\Delta, S) &\ge \frac{1}{6}\min_S\{\min_{x\in[D,\Delta]}\{L^D(S-x) + L^D(S)\}\} \\
&= \frac{1}{6}\min_S\{L^D(S-D) + L^D(S)\} \\
&= \frac{1}{6}\min_S\{L(S-D) + L(S)\}. \qquad (9.16)
\end{aligned}$$

From Lemma 9.2 we know that this expression tends to infinity as $D \to \infty$. Therefore, there is a $D_2 > 0$ such that

$$\min_S\{L(S-D) + L(S)\} > 6(K + \min_S L(S)) \text{ for all } D \ge D_2. \quad (9.17)$$

Then, from (9.16) and (9.17),

$$\min_S \tilde{\mathcal{L}}^D(\Delta, S) > K + \min_S L(S) \text{ for all } D \ge D_2,$$

and thus Δ in Case 2 cannot be optimal.

- Case 3. Let $\Delta \in \{\Delta : 2 < M^D(\Delta)\}$. Now we define $C := 3(K + \min_S L(S)) = 3(K + \min_S L^D(S))$, $S_l := \min\{S : L^D(S) \le C\}$, and $S_u := \max\{S : L^D(S) \le C\}$. It follows from the convexity of $L^D(\cdot)$ and the fact that $\lim_{x\to\pm\infty} L^D(x) = \infty$ that S_l and S_u are finite. Let $D_0 := \max\{D_1, D_2, S_u - S_l\}$ and choose $D \ge D_0$. Then we obtain

$$\begin{aligned}
\tilde{\mathcal{L}}^D(\Delta, S) &= \frac{K + L^D(S) + \int_0^\Delta L^D(S-x)m^D(x)dx}{1 + M^D(\Delta)} \\
&\ge \frac{\int_0^\Delta L^D(S-x)m^D(x)dx}{1 + M^D(\Delta)}.
\end{aligned}$$

It is easy to conclude from the definitions of S_l and S_u that $L^D(S - x) \geq C$ for $x \in [0, \Delta] \setminus [S - S_u, S - S_l]$. Because

$$[0, \Delta] \setminus [S - S_u, S - S_l] = [0, \Delta] \setminus [(S - S_u)^+, (S - S_l)^+],$$

we obtain

$$\tilde{\mathcal{L}}^D(\Delta, S) \geq \frac{CM(\Delta) - C(M^D((S - S_l)^+) - M^D((S - S_u)^+))}{1 + M^D(\Delta)}. \tag{9.18}$$

Since $0 \leq (S - S_l)^+ - (S - S_u)^+ \leq S_u - S_l \leq D$, it is clear that there is almost surely at most one renewal between $(S - S_u)^+$ and $(S - S_l)^+$, and therefore $M^D((S - S_l)^+) - M^D((S - S_u)^+) \leq 1$. Thus,

$$\tilde{\mathcal{L}}^D(\Delta, S) \geq \frac{C(M^D(\Delta) - 1)}{1 + M^D(\delta)} > \frac{C}{3} = K + \min_S L(S).$$

Therefore, for $D > D_0$ and all $\Delta = S - s > D$, we have

$$\min_S \mathcal{L}^D(s, S) = \min_S \tilde{\mathcal{L}}^D(\Delta, S) \geq \min_S \mathcal{L}^D(S, S).$$

Because $\mathcal{L}^D(s, S)$ is constant in s for $S - D \leq s \leq S$, it is clear that the minimizing point (s^*, S^*) satisfies the desired conditions. Furthermore, because $m^D(t) = 0$ for $0 \leq t \leq D$, it holds that

$$m^D(S^* - s^*) = 0 \text{ for all minimum points of } \mathcal{L}^D(s, S).$$

9.5. Asymptotic Bounds on the Optimal Cost Function

As the horizon n of the inventory problem becomes large, it is reasonable to expect that the optimal strategy parameters (s_m, S_m) and (s_{m+1}, S_{m+1}) for small m do not differ significantly and that the optimal strategy tends to a stationary one. On the other hand, if a stationary strategy is applied, the inventory level tends towards a steady state. The minimum cost per period that one can achieve in the steady state is $k := \mathcal{L}(s^*, S^*)$. If the system approaches the steady state fast enough, one could expect the difference $f_n(x) - nk$ to be uniformly bounded with respect to n for any x. In Section 4 of his paper, Iglehart obtains bounds on $f_n(x)$ in terms of an explicitly given solution $\psi(\cdot)$ of the equation

$$\psi(x) = \min_{y \geq x} [\hat{c}(y - x) + L(y) - k + \int_0^\infty \psi(y - \xi)\varphi(\xi) d\xi]. \tag{9.19}$$

He proves that the function $\psi(\cdot)$ defined as

$$\psi(s^* + y) = \begin{cases} -cy, & y \le 0, \\ L(y+s^*) - k + \int\limits_0^\infty \psi(y+s^*-\xi)\varphi(\xi)d\xi, & y > 0, \end{cases} \tag{9.20}$$

satisfies (9.19). The pair (s^*, S^*) is a minimizer of $\mathcal{L}(s, S)$, which satisfies (9.12).

Briefly, the proof goes as follows. First, Iglehart verifies that the function $\psi(\cdot)$ defined in (9.20) is K-convex.

Remark 9.2 It should be mentioned that $\psi(\cdot)$ is also K-convex if we replace k by any value larger than $\mathcal{L}(s^*, S^*) = L(s^*) + c\mu$.

In the next step, he shows that the function

$$G(y) = cy + L(y) + \int\limits_0^\infty \psi(y-\xi)\varphi(\xi)d\xi,$$

which represents the function to be minimized in (9.19), attains its minimum at $y = S^*$, and $G(s^*) = K + G(S^*)$ for $\psi(\cdot)$ defined in (9.20). To show this, it is essential that (9.12) holds.

The K-convexity of $G(\cdot)$ follows from the K-convexity of $\psi(\cdot)$. Now we return to equation (9.19) written in terms of $G(\cdot)$, i.e.,

$$\psi(x) = -k - cx + \min_{y \ge x}[K\mathbb{1}_{y>x} + G(y)],$$

and transform its RHS. For $x \le s^*$, the minimum is attained for $y = S^*$ and we get

$$\begin{aligned} & -k - cx + \min_{y \ge x}[K\mathbb{1}_{y>x} + G(y)] \\ &= -k - cx + K + G(S^*) = -k - cx + G(s^*) \\ &= -k - cx + cs^* + L(s^*) + \int\limits_0^\infty \psi(s^*-\xi)\varphi(\xi)d\xi \\ &= -c(x - s^*). \end{aligned}$$

For $x > s^*$, the minimum is attained for $y = x$ and we obtain

$$\begin{aligned} & -k - cx + \min_{y \ge x}[K\mathbb{1}_{y>x} + G(y)] \\ &= -k - cx + G(x) \\ &= -k + L(x) + \int\limits_0^\infty \psi(x-\xi)\varphi(\xi)d\xi. \end{aligned}$$

Therefore, the function $\psi(\cdot)$ defined in (9.20) actually satisfies (9.19). Observe that we can write explicitly,

$$G(y+s^*) = \begin{cases} c(s^*+\mu) + L(y+s^*) & \text{for} \quad y < 0, \\ c(s^*+\mu) + L(s^*) + L(y+s^*) & \\ \qquad + \int_0^y L(y+s^*-\xi)m(\xi)d\xi & \\ \qquad -L(s^*)[1+M(y)] & \text{for} \quad y \geq 0. \end{cases}$$

The main result of this section is that for given $W \in \mathbb{R}$, there are constants r and R depending on W such that the inequalities

$$nk + \psi(x) - r \leq f_n(x) \leq nk + \psi(x) + R, \text{ for } x \leq W, \tag{9.21}$$

hold. The assertion is proved by induction. First, Iglehart shows that the optimal order levels S_k for the n-period problem are uniformly bounded from above, i.e., the bound does not depend on n. Then he chooses a constant W larger than this bound and proves that the inequality holds for $n = 1$. Since $f_1(x)$ and $\psi(x)$ are both linear with slope $-c$ for $x \leq \min\{s_1, s\}$, we can set

$$r = \min_{\min\{s_1,s\} \leq x \leq W} \{f_1(x) - \psi(x) - k\}$$

and

$$R = \max_{\min\{s_1,s\} \leq x \leq W} \{f_1(x) - \psi(x) - k\}.$$

The induction step for $n = N+1$ uses (9.3) and (9.19). Note that because $x \leq W$ and $S_n \leq W$, the minimum in (9.3) is attained for some $y \leq W$, and the inequality (9.21) can be used for $n = N$.

Because $\psi(\cdot)$ is continuous and therefore $\psi(x) < \infty$ for any x, it follows from (9.21) that

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = k. \tag{9.22}$$

It should be mentioned that the proof of (9.21) requires only that $\psi(\cdot)$ solves (9.19) and that $\psi(x)$ is linear with slope $-c$ for all x smaller than some finite constant. Moreover, it follows from (9.22) that there is no such solution for $k \neq L(s^*) + c\mu$.

In Sections 6 and 7, Iglehart investigates the limiting behavior of the function $g_n(x) = f_n(x) - nk$ as $n \to \infty$. He proves that for $K = 0$,

$$\lim_{n\to\infty} g_n(x) = \psi(x) + A,$$

where $\psi(\cdot)$ is given by (9.20) and A is a constant. For $K > 0$, he is only able to obtain

$$\limsup_{n\to\infty} g_n(x) \leq \psi(x) + B$$

for a constant B. For $K > 0$, he also conjectures that $\liminf_{n\to\infty} g_n(x) \geq \psi(x) + B$, which would lead to $\lim_{n\to\infty} g_n(x) = \psi(x) + B$.

9.6. Review of the Veinott and Wagner Paper

Veinott and Wagner (1965) deal with the inventory problem introduced in Section 9.2 with one essential difference. They consider the demand ξ_i in period i to be a *discrete* random variable taking nonnegative integer values. They assume that one-period demands $\xi_1, \xi_2, \ldots$ are i.i.d. random variables with the probability

$$\mathsf{P}(\xi_i = k) = \varphi(k), \qquad k = 0, 1, \ldots, \quad i = 1, 2, \ldots.$$

We only recapitulate here the results of Veinott and Wagner that are important in the context of the existence of an optimal (s, S) strategy. We restrict our attention to the case of the zero leadtime for convenience in exposition.

The discrete renewal density and the renewal function are defined as

$$\begin{aligned} m(k) &= \sum_{i=1}^{\infty} \varphi^i(k), \\ M(k) &= \sum_{i=1}^{\infty} \Phi^i(k) = \sum_{j=0}^{k} m(j), \qquad k = 0, 1, \ldots, \end{aligned}$$

where φ^i and Φ^i denote the probabilities and the cumulative distribution function of the i-fold convolution of the demand distribution, respectively.

Employing a renewal approach or a stationary probability approach, Veinott and Wagner derive a formula for the stationary average cost $a(x|s, S)$, given a particular stationary (s, S) strategy. Since the unit purchase cost does not influence the optimal strategy, the formulas are derived for the case $c = 0$. An extension to $c > 0$ is straightforward. Veinott and Wagner (VW) obtain

$$a(x|s, S) = \mathcal{L}_{\mathrm{VW}}(s, S) = \frac{K + L(S) + \sum_{i=0}^{S-s} L(S - j)m(j)}{1 + M(S - s)}$$

for integer values of s and S. It should be mentioned that this function does not depend on the initial surplus x.

For their discrete demand case, Veinott and Wagner claim that, just as in Iglehart's continuous demand density case, a minimizing pair (s^*, S^*)

of $\mathcal{L}_{\mathrm{VW}}(s,S)$ would satisfy (9.22), i.e.,

$$\lim_{n\to\infty}\frac{1}{n}f_n(x)=\mathcal{L}_{\mathrm{VW}}(s^*,S^*). \tag{9.23}$$

Furthermore, in the appendix of their paper, Veinott and Wagner establish the bounds for the parameters minimizing $\mathcal{L}_{\mathrm{VW}}(s,S)$. The proofs of these bounds, derived in their paper for computational purposes, do not require the existence of a stationary optimal (s,S) strategy in the average cost case, but critically depend on the discrete nature of the demand.

Having established bounds on the minimizers of $\mathcal{L}_{\mathrm{VW}}(s,S)$, it is clear that the discrete function $\mathcal{L}_{\mathrm{VW}}(s,S)$ attains its minimum for an integer pair (s^*,S^*).

However, the claim by Veinott and Wagner that a discrete analog of Iglehart's analysis yields (9.23) for the discrete demand case, requires some additional arguments not included in their paper. Observe that a completion of Iglehart's analysis not only requires the existence of a finite minimizer of the stationary average cost function, but it also needs the minimizer to satisfy equation (9.12), which with $c=0$ reduces to

$$\mathcal{L}(s^*,S^*)=L(s^*). \tag{9.24}$$

In general, the integer minimizer of $\mathcal{L}_{\mathrm{VW}}(s,S)$ will not satisfy this condition.

Subsequently, in order to establish the dynamic programming equation in the MDP context, Tijms (1972) has shown that there are integer minimizers (s^*,S^*) of $\mathcal{L}(s,S)$ such that

$$L(s^*-1)\geq\mathcal{L}(s^*,S^*)\geq L(s^*). \tag{9.25}$$

For our purpose, it immediately follows from (9.25) and the continuity of $L(x)$ that there is an $s^\# < S^*$ such that

(a) $(\lfloor s^\#\rfloor,S^*)$ is an integer minimizer of $\mathcal{L}_{\mathrm{VW}}(s,S)$, implying that $\mathcal{L}_{\mathrm{VW}}(\lfloor s^\#\rfloor,S^*)=\mathcal{L}_{\mathrm{VW}}(s^*,S^*)$, and

(b) $\mathcal{L}_{\mathrm{VW}}(\lfloor s^\#\rfloor,S^*)=L(s^\#)$,

where $\lfloor x\rfloor$ denotes the largest integer smaller or equal to x. Using the pair $(s^\#,S^*)$ and the function $\mathcal{L}(s,S):=\mathcal{L}_{\mathrm{VW}}(\lfloor s\rfloor,S)$ in Iglehart's analysis, we can get the desired formula (9.24).

Once (9.24) is established, the short additional argument suggested by Derman and used by Veinott and Wagner would provide the optimality of a stationary (s,S) strategy for the average cost inventory problem.

Specifically, it follows from the definition of f_n that $f_n(x)\leq f_n(x|Y)$ for all initial values x and all history-dependent strategies $Y\in\mathcal{Y}$. Thus,

$$a(x|s^*,S^*) \;=\; \mathcal{L}(s^*,S^*)=\lim_{n\to\infty}\frac{1}{n}f_n(x)$$

$$= \liminf_{n\to\infty} \frac{1}{n} f_n(x) \le \liminf_{n\to\infty} \frac{1}{n} f_n(x|Y) = a(x|Y),$$

for any history-dependent strategy $Y \in \mathcal{Y}$. Therefore, (s^*, S^*) is average optimal.

We now return to the continuous demand case of Iglehart.

9.7. Existence of Minimizing Values of s and S

In this section, we first establish *a priori* bounds on the values of s and S that minimize $\mathcal{L}(s, S)$. Once that is done, the continuity of $\mathcal{L}(s, S)$ ensures the existence of a solution (s^*, S^*) that minimizes $\mathcal{L}(s, S)$. While Veinott and Wagner have proved bounds on the minimizing s and S, their proofs use the fact that the demands are discrete, and there is no obvious way to transfer their proofs to the continuous demand case. Additionally, they employ a vanishing discount argument. Since the discounted problem is not within the scope of the original Iglehart paper, we will provide bounds on the minimizing parameters (s, S) using only the properties of the stationary cost function.

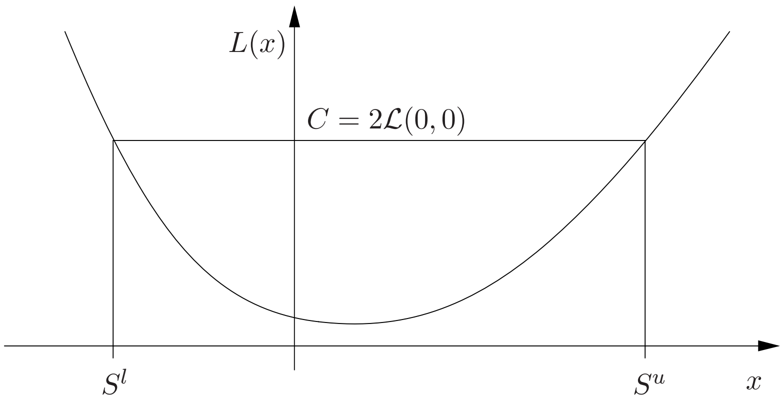

Figure 9.1. Definitions of S^l and S^u

LEMMA 9.3 *There is a constant $\bar{S}$ such that for all $S \ge \bar{S}$ and $s \le S$, the stationary cost $\mathcal{L}(s, S) > \mathcal{L}(0, 0)$.*

Proof. Let $C = 2\mathcal{L}(0, 0) > 0$. Choose S^l and S^u such that $S^l \le S^u$ and $L(S^l) = L(S^u) = C$; (see Figure 9.1). Let $S \ge S^u$. Our proof requires three cases to be considered: $s \le S^l$, $S^l < s < S^u$, and $S^u \le s \le S$.

- Case $s \le S^l$. In this case, $M(S - S^l) \le M(S - s)$, which we will use later. From the cost formula (9.5) and the stationary probability density (9.4), we have

$$\begin{aligned}
\mathcal{L}(s,S) &\geq \int_s^S L(x)f(x)dx \geq C\int_s^{S^l} f(x)dx + C\int_{S^u}^S f(x)dx \\
&= \frac{C}{1+M(S-s)}\left[\int_s^S m(S-x)dx - \int_{S^l}^{S^u} m(S-x)dx\right] \\
&= \frac{C}{1+M(S-s)}\left[M(S-s) - M(S-S^l) + M(S-S^u)\right] \\
&= C - C\left[\frac{1+M(S-S^l)-M(S-S^u)}{1+M(S-s)}\right] \\
&\geq C - C\left[\frac{1+M(S-S^l)-M(S-S^u)}{1+M(S-S^l)}\right] \\
&= C\left[\frac{M(S-S^u)}{1+M(S-S^l)}\right]. \qquad (9.26)
\end{aligned}$$

- Case $S^l < s < S^u$. In this second case, we use (9.5) and (9.4) to immediately obtain

$$\begin{aligned}
\mathcal{L}(s,S) \geq \int_{S^u}^S L(x)f(x)dx &= \frac{C}{1+M(S-s)}\int_{S^u}^S m(S-x)dx \\
&= C\left[\frac{M(S-S^u)}{1+M(S-s)}\right].
\end{aligned}$$

But in this case, $S-s \leq S-S^l$. Therefore,

$$\mathcal{L}(s,S) \geq C\left[\frac{M(S-S^u)}{1+M(S-S^l)}\right]. \qquad (9.27)$$

Since $M(t)/t \to 1/\mu$ as $t \to \infty$, the expression in the square brackets in (9.26) and (9.27), which does not depend on s, goes to one as $S \to \infty$, i.e.,

$$\begin{aligned}
\lim_{S\to\infty} \frac{M(S-S^u)}{1+M(S-S^l)} &= \lim_{S\to\infty} \frac{M(S-S^u)}{M(S-S^l)} \\
&= \lim_{S\to\infty} \frac{M(S-S^u)}{S-S^u}\,\frac{S-S^l}{M(S-S^l)}\,\frac{S-S^u}{S-S^l} \\
&= \frac{1}{\mu}\cdot\mu\cdot 1 = 1.
\end{aligned}$$

Therefore, there is an $\bar{S} \geq S^u$, independent of s, such that $M(S-S^u)$ $/(1+M(S-S^l)) > 1/2$ for all $S \geq \bar{S}$. This means that

$$\mathcal{L}(s,S) > \mathcal{L}(0,0) \text{ for all } S \geq \bar{S} \text{ and } s < S^u,$$

which proves the lemma in the first two cases.

- Case $S^u \leq s \leq S$. Finally, for this third case, we use (9.5) to obtain

$$\mathcal{L}(s,S) = \int_{-\infty}^{s} [K + L(S)]f(x)dx + \int_{s}^{S} L(x)f(x)dx \geq C > \mathcal{L}(0,0).$$

This completes the proof. □

LEMMA 9.4 *For S^l defined in Lemma* 9.3*, $\mathcal{L}(s,S) > \mathcal{L}(0,0)$ for all $S \leq S^l$ and $s \leq S$.*

Proof. For $S \leq S^l$, it is clear from (9.5) that

$$\mathcal{L}(s,S) = \int_{-\infty}^{s} [K + L(S)]f(x)dx + \int_{s}^{S} L(x)f(x)dx \geq C > \mathcal{L}(0,0).$$

This completes the proof. □

It is easy to see that together, Lemmas 9.3 and 9.4, prove that the minimizing value of S lies in the set $[S^l, \bar{S}]$. In the next lemma, we show that the minimizing value of s is bounded as well.

LEMMA 9.5 *There is a constant $\bar{s}$ such that for all $s \leq \bar{s}$ and $S \geq s$, $\mathcal{L}(s,S) > \mathcal{L}(0,0)$.*

Proof. Let S^l and $\bar{S}$ be defined as in Lemma 9.3. Let $s \leq S^l$. For any S satisfying $s \leq S \leq S^l$ or $s \leq S^l \leq \bar{S} \leq S$, it follows from Lemmas 9.3 and 9.4 that $\mathcal{L}(s,S) > \mathcal{L}(0,0)$. Therefore, we can restrict our attention to the values of S which satisfy $S^l \leq S \leq \bar{S}$. Then, from (9.5) and (9.4) we have

$$\begin{aligned} \mathcal{L}(s,S) &\geq \int_{s}^{S^l} L(x)f(x)dx \\ &\geq C\int_{s}^{S^l} f(x)dx = \frac{C}{1+M(S-s)}\left[\int_{s}^{S^l} m(S-x)dx\right] \end{aligned}$$

$$= \frac{C}{1+M(S-s)}\left[M(S-s)-M(S-S^l)\right]$$
$$= C - C\left[\frac{1+M(S-S^l)}{1+M(S-s)}\right]$$
$$\geq C - C\left[\frac{1+M(\bar{S}-S^l)}{1+M(S^l-s)}\right]. \tag{9.28}$$

Since $M(t) \to \infty$ as $t \to \infty$, the expression in the square brackets in (9.28), which does not depend on S, goes to zero as $s \to -\infty$. Therefore, there is an $\bar{s} \leq S^l$, independent of S, such that $(1+M(\bar{S}-S^l))/(1+M(S^l-s)) < 1/2$ for all $s \leq \bar{s}$. This means that

$$\mathcal{L}(s,S) > C/2 = \mathcal{L}(0,0) \text{ for all } s \leq \bar{s} \text{ and } S > s,$$

and the proof is completed. □

REMARK 9.3 The proofs of Lemmas 9.3 and 9.4 can be easily extended to nondifferentiable renewal functions M by using Lebesgue-Stieltjes integrals. Therefore, the assertions of these two lemmas also hold for demand distributions which do not have densities.

THEOREM 9.1 *If the one-period demand has a density, then the function $\mathcal{L}(s,S)$, defined on $-\infty < S < \infty$, $s \leq S$, attains its minimum. Furthermore, if (s^*,S^*) is a minimum point of $\mathcal{L}(s,S)$, then $S^l \leq S^* \leq \bar{S}$ and $\bar{s} \leq s^* \leq S^*$.*

Proof. Because of Lemmas 9.3–9.5, the search for a minimum point can be restricted to the compact set $\{(s,S) : S^l \leq S \leq \bar{S}, \bar{s} \leq s \leq S\}$. It immediately follows from the existence of a density of the one-period demand that $\mathcal{L}(s,S)$ is continuous, and therefore it attains its infimum over the compact set. □

It remains to show that the minimum is attained at an interior point and that there is a minimum point that satisfies (9.12).

LEMMA 9.6 *If $K > 0$, then there is a pair (s^*,S^*), with $s^* < S^*$, that minimizes $\mathcal{L}(s,S)$.*

Proof. We distinguish three cases.

- Case 1. If $m(x)$ is identically zero on $[0,\varepsilon]$ for some $\varepsilon > 0$, it immediately follows that $\tilde{\mathcal{L}}(\Delta,S)$ is constant for $\Delta \in [0,\varepsilon]$ and any fixed S. Therefore, if $(0,S^*)$ minimizes $\tilde{\mathcal{L}}(\Delta,S)$, so does (ε,S^*), which is an interior point.

- Case 2. Now let $m(0) > 0$. Any minimal point of $\hat{\mathcal{L}}(s,\Delta)$ satisfies (9.9). Let s^* be a solution of (9.9) for $\Delta = 0$. Then it follows from (9.10) that

$$\left.\frac{\partial \hat{\mathcal{L}}(s^*,\Delta)}{\partial \Delta}\right|_{\Delta=0} = -Km(0) < 0$$

and therefore, $\Delta = 0$ cannot be a minimizer.

- Case 3. If $m(\cdot)$ does not satisfy either of the two cases above, it follows from the continuity of m as a sum of convolutions of continuous densities that there is an $\varepsilon > 0$ such that $m(x) > 0$ for all $x \in (0,\varepsilon]$. This property implies that $M(x) > 0$ for $x > 0$. For a fixed S, it holds that

$$\begin{aligned} &\tilde{\mathcal{L}}(S,\Delta) - \tilde{\mathcal{L}}(S,0) \\ &= \frac{K + L(S) + \int_0^\Delta L(S-x)m(x)dx}{1+M(\Delta)} - (K + L(S)) \\ &= \frac{1}{1+M(\Delta)}\left(-M(\Delta)(K+L(S)) + L(S-\eta)M(\Delta)\right) \quad (9.29)\end{aligned}$$

for some $\eta \in [0,\Delta]$. Since $M(\Delta) > 0$ for $\Delta \in (0,\varepsilon]$ and L is continuous, it follows that for sufficiently small Δ

$$\tilde{\mathcal{L}}(S,\Delta) - \tilde{\mathcal{L}}(S,0) < 0,$$

and $\Delta = 0$ cannot be a minimizer. □

Lemma 9.7 *There are s^* and S^*, with $s^* < S^*$, which minimize $\mathcal{L}(s,S)$ and, at the same time, satisfy* (9.12).

Proof. It follows from Lemma 9.6 that there is an interior minimizer $(s^\#, S^*)$ of $\mathcal{L}(s,S)$. If $m(S^* - s^\#) > 0$, it immediately follows from Iglehart's analysis that (9.12) is satisfied.

If $m(S^* - s^\#) = 0$, we define

$$\Delta_0 = \inf\{\Delta \geq 0 : m(x) \equiv 0 \text{ on } [\Delta, S^* - s^\#]\}$$

and

$$\Delta_1 = \sup\{\Delta \geq 0 : m(x) \equiv 0 \text{ on } [S^* - s^\#, \Delta]\}.$$

Obviously for $\Delta \in [\Delta_0,\Delta_1]$, the pair $(S^*-\Delta, S^*)$ minimizes $\mathcal{L}$. Let $\varepsilon > 0$. Then we have

$$\tilde{\mathcal{L}}(\Delta_1, S^*) - \tilde{\mathcal{L}}(\Delta_1 + \varepsilon, S^*) \leq 0.$$

Therefore, it follows that

$$\left(K + L(S^*) + \int_0^{\Delta_1} L(S^* - \xi)m(\xi)d\xi\right)(1 + M(\Delta_1 + \varepsilon)) - \left(K + L(S^*) + \int_0^{\Delta_1+\varepsilon} L(S^* - \xi)m(\xi)d\xi\right)(1 + M(\Delta_1)) \leq 0$$

or

$$\left(K + L(S^*) + \int_0^{\Delta_1} L(S^* - \xi)m(\xi)d\xi\right)(M(\Delta_1 + \varepsilon) - M(\Delta_1)) - (1 + M(\Delta_1)) \int_{\Delta_1}^{\Delta_1+\varepsilon} L(S^* - \xi)m(\xi)d\xi \leq 0.$$

Applying the Mean Value Theorem A.1.9 and dividing by $(M(\Delta_1+\varepsilon) - M(\Delta_1))(1 + M(\Delta_1))$, which is strictly positive by the definition of Δ_1 and the monotonicity of $M(\cdot)$, we obtain

$$\frac{K + L(S^*) + \int_0^{\Delta_1} L(S^* - \xi)m(\xi)d\xi}{1 + M(\Delta_1)} \leq L(S^* - \eta)$$

for some $\eta \in [\Delta_1, \Delta_1 + \varepsilon]$. Since $L(y)$ is continuous, we find for $\varepsilon \to 0$,

$$\mathcal{L}(\Delta_1, S^*) - c\mu = \frac{K + L(S^*) + \int_0^{\Delta_1} L(S^* - \xi)m(\xi)d\xi}{1 + M(\Delta_1)} \leq L(S^* - \Delta_1). \tag{9.30}$$

If $\Delta_0 > 0$, we analogously find

$$\tilde{\mathcal{L}}(\Delta_0, S^*) \geq L(S^* - \Delta_0). \tag{9.31}$$

If $\Delta_0 = 0$, it is easy to see from (9.7) that

$$\tilde{\mathcal{L}}(0, S^*) - c\mu = K + L(S^*) > L(S^*). \tag{9.32}$$

Since $\tilde{\mathcal{L}}(\cdot, S^*)$ and $L(\cdot)$ are both continuous, it follows from (9.30)-(9.32) that there is a $\Delta^* > 0$ such that

$$\tilde{\mathcal{L}}(\Delta^*, S^*) - c\mu = L(S^* - \Delta^*),$$

i.e., the pair $(s^*, S^*) := (S^* - \Delta^*, S^*)$ is an interior minimizer that satisfies (9.12). □

Lemma 9.7 finally establishes the existence of an interior minimizer of the stationary cost function that satisfies equation (9.12) as required for Iglehart's analysis. With that analysis completed, the following result is easily established with the help of the same short additional argument that Derman suggested to Veinott and Wagner.

THEOREM 9.2 *The parameters s^* and S^*, obtained in Lemma 9.7, determine a stationary (s, S) strategy which is average optimal.*

REMARK 9.4 It follows from (9.9) that for any minimizer (s^*, S^*) of $\mathcal{L}(s, S)$, we have $s^* \leq \operatorname{argmin} L(y)$. Therefore, for any two minimizers (s_1^*, S_1^*) and (s_2^*, S_2^*) of $\mathcal{L}(s, S)$ that satisfy (9.12), it holds that

$$L(s_1^*) = \mathcal{L}(s_1^*, S_1^*) - c\mu = k - c\mu = \mathcal{L}(s_2^*, S_2^*) - c\mu = L(s_2^*).$$

Since $k - c\mu > \min_S L(S)$, $L(x)$ is convex, and $\lim_{x \to \pm\infty} L(x) = \infty$, it follows that $s_1^* = s_2^*$.

9.8. Stationary Distribution Approach versus Dynamic Programming and Vanishing Discount Approach

In Chapters 5 and 6, we have established the average optimality of an (s, S) strategy in the more general setting of Markovian demand. We use dynamic programming and a vanishing discount approach to obtain the average cost optimality equation and show that it has a K-convex solution, which provides an (s, S) strategy that minimizes (9.2). Furthermore, we prove a verification theorem stating that any stable policy (defined later in the section; see (9.35)) satisfying the average cost optimality equation is average optimal.

More specifically, we prove that there is a policy Y^* that minimizes the average cost defined by

$$J(x|Y) = \limsup_{n \to \infty} \frac{1}{n} f_n(x|Y)$$

over all history-dependent policies $Y \in \mathcal{Y}$. Furthermore, this policy Y^* can be represented as an (s, S) policy. In addition, we show that this policy also minimizes the criterion

$$a(x|Y) = \liminf_{n \to \infty} \frac{1}{n} f_n(x|Y)$$

over the class of all stable policies.[4]

Moreover, the completion of Iglehart's analysis in the previous section allows us to drop the stability restriction on the class of admissible strategies and to obtain the stronger result that an optimal (s,S) strategy also minimizes $a(x|Y)$ over *all* history-dependent policies $Y \in \mathcal{Y}$. Theorem 9.3 that follows connects the two approaches.

For the average cost problem under consideration, the average cost optimality equation derived in Chapter 6 is given by

$$\psi(x) = \min_{y \geq x}[\hat{c}(y-x) + L(y) - \lambda + \int_0^\infty \psi(y-\xi)\varphi(\xi)d\xi]. \tag{9.33}$$

A pair (λ^*, ψ^*) such that

$$\psi^*(x) = \min_{y \geq x}[\hat{c}(y-x) + L(y) - \lambda^* + \int_0^\infty \psi^*(y-\xi)\varphi(\xi)d\xi]$$

is called a *solution* of (9.33). Note that for $\lambda = k$, (9.33) reduces to equation (9.19), specified by Iglehart.

THEOREM 9.3 *Let* (λ^*, ψ^*) *be a solution of the average cost optimality equation* (9.33)*. Let* ψ^* *be continuous and let the minimizer on the RHS of* (9.33) *be given by*

$$y(x) = \begin{cases} S^* & \text{if } x \leq s^*, \\ x & \text{if } x > s^*, \end{cases}$$

for $-\infty < s^* \leq S^* < \infty$*. Then,*

(a) *the pair* (s^*, S^*) *minimizes* $\mathcal{L}(s,S)$*,*

(b) (s^*, S^*) *satisfies* (9.12)*, and*

(c) *for all history-dependent policies* $Y \in \mathcal{Y}$*, it holds that*

$$\lambda^* = k = \lim_{n\to\infty} \frac{1}{n} f_n(x|s^*, S^*) \leq \liminf_{n\to\infty} \frac{1}{n} f_n(x|Y).$$

[4]Bounds on the action space imposed by Tijms (1972) imply that the admissible policies he considers are stable. In this case, either of the criteria — lim inf or lim sup — can be used in the MDP context.

Proof. To prove (a) we assume to the contrary that the pair (s^*, S^*) does not minimize $\mathcal{L}(s,S)$. Then, there is another strategy (s,S) with $\mathcal{L}(s^*,S^*) > \mathcal{L}(s,S)$. Therefore, in view of Remark 9.1, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} f_n(x|s^*,S^*) = \mathcal{L}(s^*,S^*) > \mathcal{L}(s,S) = \limsup_{n\to\infty} \frac{1}{n} f_n(x|s,S). \tag{9.34}$$

Equation (9.34) contradicts the optimality of (s^*,S^*) proved in Chapter 6 for the average cost objective function (9.2). Therefore, (s^*,S^*) minimizes $\mathcal{L}(s,S)$.

It is shown in Chapter 6 that the λ^* from the solution of the average cost optimality equation is equal to the minimum of the average cost defined in (9.2), and is therefore equal to $k = \mathcal{L}(s^*,S^*)$. Knowing this, part (c) of Theorem 9.3 immediately follows from (a) and Theorem 9.2.

For the proof of (b), we note that ψ^* can by expressed as in (9.20). As shown in Iglehart, ψ^* is continuous if and only if (9.12) holds. This proves part (b) of Theorem 9.3. □

THEOREM 9.4 *If* $\lim_{x\to-\infty}[cx+L(x)] = \infty$, *then there is a unique (up to a constant) continuous bounded-from-below* K*-convex function* ψ^* *such that* (λ^*,ψ^*) *is a solution of the average cost optimality equation* (9.33). *Furthermore,* λ^* *is equal to the minimal average cost with respect to either* (9.1) *or* (9.2).

Proof. Because of the K-convexity of the solution, the minimizer on the RHS of (9.19) is given by

$$y(x) = \begin{cases} S^* & \text{if} \quad x \le s^*, \\ x & \text{if} \quad x > s^*, \end{cases}$$

for some not necessarily finite $s^* \le S^*$. Since $\lim_{x\to-\infty}(cx + L(x)) = \infty$ and ψ^* is bounded from below, it follows that

$$\lim_{x\to-\infty} \left[cx + L(x) - k + \int_0^\infty \psi^*(x-\xi)\varphi(\xi)d\xi\right] = \infty,$$

and therefore, $s^* > -\infty$. Since $\lim_{x\to\infty} L(x) = \infty$ and ψ^* is bounded from below, it is obvious that

$$\lim_{y\to\infty} \left[cy + L(y) - k + \int_0^\infty \psi^*(y-\xi)\varphi(\xi)d\xi\right] = \infty,$$

and therefore, $S^* < \infty$.

It is easy to show that the (s, S) strategy with finite parameters s^* and S^* is *stable with respect to* ψ^*, i.e.,

$$\lim_{n\to\infty} \frac{\psi^*(x_n)}{n} = 0. \tag{9.35}$$

Given this fact, it is proved in Chapter 6 that (s^*, S^*) is an optimal strategy with respect to the average cost defined in (9.2) with minimal average cost λ^*. It follows from Theorem 9.3 that $\lambda^* = k$.

By Theorem 9.3, (s^*, S^*) minimizes $\mathcal{L}(s, S)$ and satisfies (9.12). It follows from Remark 9.4 that s^* is unique. The K-convexity and the parameter s^* uniquely determine the solution (9.20) of (9.19) (up to a constant). Therefore, (λ^*, ψ^*) is the unique solution of (9.33) with the desired properties. □

REMARK 9.5 In the case of a constant unit shortage cost p, the condition $\lim_{x\to-\infty}(cx + L(x)) = \infty$ is equivalent to the requirement $p > c$. This condition is only introduced to simplify the proof and can be dropped altogether.

9.9. Concluding Remarks and Notes

In this chapter, based on Beyer and Sethi (1999), we have reviewed the classical papers of Iglehart (1963b) and Veinott and Wagner (1965), treating single-product average cost inventory problems. We have pointed out some conditions that are assumed implicitly but not proved in these papers, and we have proved them rigorously. In particular, we have provided bounds for any pair (s^*, S^*) minimizing the stationary one-period cost $\mathcal{L}(s, S)$, using only the properties of the stationary cost function.

The main purpose of our analysis in this chapter has been to complete the stationary distribution analyses of Iglehart and Veinott and Wagner and relate it to the vanishing discount approach. Therefore, we have stayed with the relatively restrictive assumptions on the demand distribution made by Iglehart. However, these assumptions are not necessary for the results obtained in this chapter. Indeed, it can be shown that even for general demand distributions consisting of discrete and continuous components, the corresponding stationary cost function attains its minimum provided the expected values of all the quantities required in the analysis exist. Furthermore, there is a pair minimizing $\mathcal{L}(s, S)$ that satisfies (9.12) and that can in turn be proved to be average optimal.

We have also introduced the connection between the stationary distribution approach and the dynamic programming approach to the problem. While dynamic programming is applicable even in problems defying a stationary analysis but with objective function (9.2), the stationary analysis, when possible, can prove optimality with respect to the more conservative objective function (9.1). Finally, by combining both approaches, we have shown that an (s, S) policy – optimal with respect to either of the objective functions (9.1) or (9.2) – is also optimal with respect to the other.

Finally, we mention that Presman and Sethi (2006) have developed a unified stationary distribution approach that deals with both the discounted and long-run average cost problems in continuous time. Bensoussan *et al.* (2005b), on the other hand, use the dynamic programming approach in the continuous-time case, which use the theory of quasivariational inequalities. They only deal with the discounted problem, however.