Chapter 8

MODELS WITH DEMAND INFLUENCED BY PROMOTION

8.1. Introduction

This chapter deals with a stochastic inventory model in which the probability distribution of the product demand in any given period depends on some environmental factors, as well as on whether or not the product is promoted in the period. The problem is to obtain optimal inventory ordering and product promotion decisions jointly so as to maximize the total profit.

Such problems arise in many business environments, where marketing tools such as promotions are often used to stimulate consumer demand. The coordination between marketing and inventory management becomes critical in their decision making. Traditionally, marketing is mainly concerned with satisfying customers while manufacturing is primarily interested in production efficiency. Conflicts may arise between the two business functions because of their different primary focus areas. Furthermore, it is necessary to evaluate the trade-off between the benefit brought by higher sales, and the increased costs caused by promotion and holding more inventory. On the other hand, a promotion plan must be supported by a coordinated procurement plan to ensure that sufficient stock is available to meet the stimulated demand.

It is obviously desirable to adopt an integrated decision making system by considering the two types of decisions jointly. However, most works in inventory literature assume exogenous demand and do not deal with promotion decisions explicitly. Early efforts in integrating promotional decisions have been mostly focused on pricing.

A typical approach for modeling the price demand relationship is to assume that the price-dependent portion of the demand is either additive

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or multiplicative to the base demand; (see Young (1978)). Karlin and Carr (1962), Mills (1962), and Zabel (1970) studied the price demand models with this assumption, and obtained structural results for the additive and/or multiplicative cases. Furthermore, Thowsen (1975) showed that the optimal pricing/inventory policy is a base-stock list price policy under some additional assumptions. That is, if the initial inventory level is below the base-stock level, then that stock level is replenished and the list price is charged; if the initial inventory level is above the base-stock level, then nothing is ordered, and a price discount is offered.

A dynamic inventory/advertising model with partially controlled additive demand is analyzed by Balcer (1983), who obtains a joint optimal inventory and advertising strategy under certain restrictions. However, in the Balcer model, the demand/advertising relationship is deterministic because only the deterministic component of demand is affected by the advertising decision.

Sogomonian and Tang (1993) formulate a mixed integer program for an integrated promotion and production decision problem, and devise a “longest path” algorithm for finding an optimal joint promotion and production plan. Their model assumes that demand at each period depends on the time elapsed since the last promotion, the level of the last promotion, and the retail price of the current period.

In this chapter, we analyze the joint promotion/inventory management problem for a single item in the context of Markov decision processes (MDP). Specifically, we assume that consumer demand is affected by a Markov process that depends on promotion decisions. The state variable of the Markov process represents the demand state brought about by changing environmental factors as well as promotion decisions. The demand state in a period, in turn, determines the distribution of the random demand in that period.

At the beginning of each period, a decision maker observes the demand state and the inventory level, and then decides on (a) whether or not to promote and (b) how much to order. The MDP is modeled so that demand, and hence the revenue, will increase in the following period after the product is promoted. However, there is a fixed cost for promoting the product.

We solve the finite horizon problem via a dynamic programming approach. We show that there is a threshold inventory level P for each demand state such that if the threshold is exceeded, then it is desirable to promote the product. For the linear ordering cost case, the optimal inventory replenishment policy is a base-stock type policy.

Our model differs from the existing models in the literature in several ways. First, we do not stipulate a deterministic functional form of the relationship between promotion and demand. This allows the maximum flexibility for modeling the demand uncertainty in the presence of promotions. Second, we model the demand as an MDP, which provides a better modeling approach for demand under the influence of marketing activities and other uncertain environmental factors.

We should note that Sethi and Zhang (1994,1995) have developed a general stochastic production/advertising model with unreliable machines and random demand influenced by the level of advertising via a Markov process. They developed the dynamic programming equations and show how various hierarchical solutions of the problem can be constructed. They do not study the nature of optimal solutions. Their focus is on proving that the constructed hierarchical solutions are approximately optimal.

The Markov demand modeling technique adopted in this chapter extends the existing Markov-modulated demand models by introducing promotion decisions. It provides a new, flexible approach to model demand that depends not only on uncertain environmental factors, but is also influenced by promotion decisions. While we limit the scope of our analysis by including only the promotion decision, other factors that potentially affect customer demand could also be incorporated in the model in a similar way.

The remainder of the chapter is organized as follows. In Section 8.2, we present the mathematical formulation of the promotion/inventory problem, and provide a preliminary analysis of the problem in a general setting. The assumptions required for obtaining additional structural results of the model are discussed in Section 8.3. In Section 8.4, we provide the optimality proof for the optimal promotion/inventory policies. Solutions with simplified parameters are also discussed. Further extensions of the model, including those with price discounts as a promotion device, are briefly discussed in Section 8.5. A numerical study is presented in Section 8.6. The chapter concludes with some concluding remarks and end notes in Section 8.7.

8.2. Formulation of the Model

In most classical inventory models, it is usually assumed that both the price and the demand of the product are parameters and not decision variables. Hence, the revenue from sales is independent of any decision variables and can be neglected from the model except for the situation in which demand not satisfied directly from inventory cannot be backlogged. Furthermore, even in this lost sales case, the situation can be handled by including the loss in revenue as a penalty cost of the unsatisfied demand. Therefore, an optimal inventory policy can be determined by minimizing the total cost. However, in our model, promotion decisions will affect the future demand, and hence the future revenue. Thus, it is important to include the revenue explicitly in the objective function of the model.

When both revenues and costs are included in the objective function, one usually formulates a profit maximization problem. Nevertheless, in order to preserve the similarity with the approach commonly used in classical inventory literature, we will formulate the problem as a cost minimization problem by treating revenues as negative costs. This is equivalent to formulating the problem as a profit maximization problem.

8.2.1 Notation

We will formulate a discrete-time N-period problem consisting of period $0, 1, 2, \ldots, N-1$. For the temporal conventions used in our discrete-time setting, (see Figure 2.1 of Chapter 2), we introduce the following notation.

m_k = the promotion decision in period k such that
$$m_k = \begin{cases} 1, & \text{if the product is promoted in period } k \\ 0, & \text{otherwise;} \end{cases}$$

$p_{ij}(m)$ = the transition probability that the demand state changes from i to j in one period if the promotion value is m;

$\{i_k\}$ = a discrete MDP with the transition matrix $P = \{p_{ij}(m_k)\}$;

ξ_k = the demand in period k; it depends on i_k but not on k, and is independent of past demand states and past demands;

$\varphi_i(\cdot)$ = the conditional density function of ξ_k when $i_k = i$;

$\Phi_i(\cdot)$ = the distribution function corresponding to φ_i;

μ_i = $\mathsf{E}\{\xi_k | i_k = i\}$;

u_k = the nonnegative order quantity in period k;

x_k = the inventory level at the *beginning* of period k;

c_k = the unit purchase cost in period k, $k \in \langle 0, N-1 \rangle$;

c_N = the shortage cost per unit in period N;

r_k = the unit revenue in period k, $k \in \langle 0, N-1 \rangle$;

r_N = the salvage cost per unit in period N;

A_k = the promotion cost in period k when the product is promoted in the period;

h_k = the unit inventory holding cost in period k assessed on the ending inventory in the period;

p_k = the unit backlogging cost in period k assessed on the backlog at the end of the period;

$l_k(z)$ = $h_k z^+ + p_k z^-$, the inventory/backlog cost function in period k when z is the ending inventory in the period, $z^+ = \max(0, z)$, and $z^- = -\min(0, z)$.

REMARK 8.1 All of the cost/revenue parameters are assumed to be independent of the demand states for the sake of simplicity in exposition. Given that the unit revenue is independent of the promotion decision, this may imply that the model does not allow price discount. However, as it will be shown in Section 8.5, the price discount can easily be incorporated in the model by treating it as a part of the promotion cost.

8.2.2 An MDP Formulation

We suppose that an order is placed at the beginning of a period and delivered instantaneously. Also the promotion decision is made at the beginning of the period. Subsequently, the actual demand materializes during the period. The unsatisfied portion of the demand, if any, is carried forward as backlog.

The expected one-period inventory cost for period $k \in \langle 0, N{-}1\rangle$, given the demand state i, is

$$\begin{aligned} L_k(i,y) &= \mathsf{E}[l_k(y-\xi_k)|i_k=i] \\ &= \int_0^y h_k(y-\xi)\varphi_i(\xi)d\xi + \int_y^\infty p_k(\xi-y)\varphi_i(\xi)d\xi, \quad (8.1) \end{aligned}$$

where y is the amount of stock available at the beginning of a period after the order, if any, is delivered.

We assume that the revenue is received when the demand occurs. Hence, the total expected net cost in period k, given the initial inventory level x, the order quantity $u \geq 0$, and the promotion decision m, can be expressed as

$$\begin{aligned} G_k(i,x;m,u) &= \mathsf{E}[l_k(x+u-\xi_k) + A_k \mathbb{1}_{m=1} + c_k u - r_k \xi_k | i_k = i] \\ &= L_k(i,x+u) + A_k \mathbb{1}_{m=1} + c_k u - r_k \mu_i. \end{aligned}$$

In the last period $N{-}1$, if the ending inventory is positive, it is salvaged at r_N per unit; if the ending inventory is negative, the shortage is met at c_N per unit. Thus, the terminal cost function can be defined as

$$g_N(x) = c_N x^- - r_N x^+. \tag{8.2}$$

Since the unit salvage value is normally less than the purchase price, we have

$$c_N \geq r_N \geq 0. \tag{8.3}$$

Note that c_N and r_N have different meanings than c_k and r_k, $k \in \langle 0, N{-}1\rangle$.

Let $J_n(i, x; M, U)$, $n \in \langle 0, N-1\rangle$, denote the total expected cost, including the terminal cost, when the system is operated under the promotion policy $M = \{m_n, m_{n+1}, \ldots, m_{N-1}\}$ and the ordering policy $U = \{u_n, u_{n+1}, \ldots, u_{N-1}\}$ from period n through period $N-1$, given the initial demand state i and the inventory level x at the beginning of period n. That is,

$$J_n(i, x; M, U) = \mathsf{E}\left[\sum_{k=n}^{N-1} G_k(i_k, x_k; m_k, u_k) + g_N(x_N)\right]. \tag{8.4}$$

The inventory balance equations are given by

$$\begin{cases} x_{k+1} = x_k + u_k - \xi_k, \ k \in \langle n, N{-}1\rangle, \text{ with} \\ x_n = x, \ \text{ the beginning inventory level in period } n. \end{cases} \tag{8.5}$$

The objective is to determine $M = \{m_n, m_{n+1}, \ldots, m_{N-1}\}$ and $U = \{u_n, u_{n+1}, \ldots, u_{N-1}\}$ to minimize the expected total cumulative cost. Denote $v_n(i, x)$ to be the infimum of $J_n(i, x; M, U)$, i.e.,

$$v_n(i, x) = \inf_{M,U} J_n(i, x; M, U). \tag{8.6}$$

Then, $v_n(i, x)$ satisfies the dynamic programming equations

$$\begin{cases} v_n(i, x) &= \inf_{m; u \geq 0} \{G_n(i, x; m, u) \\ & \quad + \mathsf{E}[v_{n+1}(i_{n+1}, x + u - \xi_n) \mid i_n = i]\}, \ n \in \langle 0, N{-}1\rangle, \\ v_N(i, x) &= g_N(x). \end{cases} \tag{8.7}$$

Note that in this MDP formulation, the additional decision variable m takes values in a finite set. Thus, the results on the existence of optimal policies and the verification theorem can be obtained in a fashion similar to those in Chapters 2 and 4. We refer the readers to Bertsekas and Shreve (1976) for further discussion on the related issues.

Now, we rewrite (8.7) as

$$v_n(i, x) = \min_{m; y \geq x} \{G_n(i, x; m, y - x) + \mathsf{E}[v_{n+1}(i_{n+1}, y - \xi_n) \mid i_n = i]\}$$

$$\begin{aligned} &= -c_n x + \min_{m;y\geq x}\{A_n \mathbb{1}_{m=1} + c_n y + L_n(i,y) - r_n\mu_i \\ &\quad + \int_0^\infty \sum_{j=1}^{L} p_{ij}(m) v_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi\}, \\ &\qquad\qquad n = 0,\ldots,N-1, \end{aligned} \tag{8.8}$$

$$v_N(i,x) = g_N(x). \tag{8.9}$$

To simplify our analysis, we convert (8.8) and (8.9) to an alternative set of dynamic programming equations in terms of functions w_n, using the relation

$$v_n(i,x) = w_n(i,x) - c_n x, \quad n = 0,\ldots,N. \tag{8.10}$$

The resulting dynamic programming equations are

$$\begin{aligned} w_n(i,x) &= \min_{m;y\geq x}\{A_n \mathbb{1}_{m=1} + (c_{n+1} - r_n)\mu_i + (c_n - c_{n+1})y \\ &\quad + L_n(i,y) + \int_0^\infty \sum_{j=1}^{L} p_{ij}(m) w_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi\}, \\ &\qquad\qquad n = 0,\ldots,N-1, \end{aligned} \tag{8.11}$$

$$w_N(i,x) = g_N(x) + c_N x. \tag{8.12}$$

Let us define for $n = 0,\ldots,N-1$,

$$\begin{aligned} g_n(i,y,m) &= (c_{n+1} - r_n)\mu_i + (c_n - c_{n+1})y + L_n(i,y) \\ &\quad + \int_0^\infty \sum_{j=1}^{L} p_{ij}(m) w_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi, \end{aligned} \tag{8.13}$$

$$q_n(i,x,m) = A_n \mathbb{1}_{m=1} + \min_{y\geq x} g_n(i,y,m). \tag{8.14}$$

Then, we have for $n = 0,\ldots,N-1$,

$$w_n(i,x) = \min_m\{q_n(i,x,m)\}. \tag{8.15}$$

For $n = N$, we use (8.2) and (8.12) and write

$$w_N(i,x) = g_N(x) + c_N x = \begin{cases} 0, & x < 0, \\ (c_N - r_N)x, & x \geq 0. \end{cases} \tag{8.16}$$

8.2.3 The Newsvendor Problem – a Myopic Solution

The value function $w_n(i,x)$ given by (8.11) and (8.12), consists of two components – the cost incurred in the current period and the cost to go. A myopic solution can be obtained by ignoring the cost to go, i.e., by ignoring the last term in (8.11). Since a promotion affects only the demand in the future, it will always be true that $m = 0$ in a myopic solution. The remaining problem becomes a newsvendor-type problem, in which the only decision variable is the inventory position y (after the order is delivered). Let us denote the cost function of the newsvendor problem by

$$\begin{aligned}\min_y g_n^b(i,y) &= (c_{n+1} - r_n)\mu_i + (c_n - c_{n+1})y \\ &\quad + L_n(i,y), \quad n \in \langle 0, N-2 \rangle. \end{aligned} \tag{8.17}$$

We can obtain the minimum of $g_n^b(i,y),\ n \in \langle 0, N-2\rangle$, at

$$\bar{S}_{n,i} = \Phi_i^{-1}\left(\frac{c_{n+1} - c_n + p_n}{h_n + p_n}\right), \tag{8.18}$$

provided the various unit costs satisfy the condition

$$-p_n \le c_{n+1} - c_n \le h_n. \tag{8.19}$$

When $c_{n+1} - c_n + p_n < 0$, we have $\bar{S}_{n,i} = 0$. In this case, the "newsvendor" is always better off by postponing the purchase since the saving of $c_n - c_{n+1}$ from the decreased purchase cost outweighs the backlogging cost p_n. On the other hand, when $c_{n+1} - c_n > h_n$, we have $\bar{S}_{n,i} = \infty$. This means that the "newsvendor" makes money on each unsold unit by salvaging it at c_{n+1} in period $n+1$, which is greater than the unit purchase plus holding cost. In our dynamic inventory problem, Condition (8.19) is nothing but the usual condition which rules out any speculative motive. Henceforth, we will assume that the costs satisfy (8.19).

It is clear from Assumption 8.2 (i) that

$$\bar{S}_{n,i'} \ge \bar{S}_{n,i}, \quad \forall n \in \langle 0, N-1\rangle \text{ and } i' > i. \tag{8.20}$$

We further assume that

$$\bar{S}_{n+1,i} \ge \bar{S}_{n,i}, \quad n \in \langle 0, N-2\rangle, \text{ and } i \in \mathbb{I}. \tag{8.21}$$

Assumption (8.21) is equivalent to assuming that $(c_{n+1} - c_n + p_n)/(h_n + p_n)$ is nondecreasing in n. When $c_{n+1} = c_n,\ n \in \langle 0, N-1\rangle$, (8.21) would imply that h_n/p_n is nonincreasing in n.

8.2.4 Joint Optimal Promotion and Inventory Policies

In this section, we will characterize the optimal policies derived from the dynamic programming equations by identifying the special structures that an optimal policy exhibits under certain conditions.

We will first define the optimal policy for the inventory/promotion problem as follows.

THEOREM 8.1 *The optimal inventory policy for the problem defined by* (8.11) *can be characterized by two base-stock levels, and the optimal promotion policy is of multithreshold type.*

The proof of this theorem is given in Section 8.4, after additional assumptions have been made and the relevant preliminary results are provided in Section 8.3.

The first structural property of the optimal policy is like that of the standard inventory models with linear ordering costs; a base-stock policy is still optimal. Moreover, the optimal base-stock level in each period is contingent on the demand state and the promotion decision to be made. The two base-stock levels correspond to two different decisions on promotions, respectively.

While the optimality of a base-stock policy with the base-stock level given by $S^m_{n,i}$, immediately follows from the quasi-convexity of $g_n(i,x,m)$ (the proof will be provided in Section 8.4), the promotion policy is determined by the relative position of the two functions $q_n(i,x,0)$ and $q_n(i,x,1)$. We define the difference between the two functions as

$$\Delta q_n(i,x) = q_n(i,x,0) - q_n(i,x,1), \quad n = 0,\ldots,N-1.$$

Given i and x, if $\Delta q_n(i,x) \geq 0$, then it is optimal to promote; otherwise, it is optimal not to promote.

Let us consider the equation

$$\Delta q_n(i,x) = 0. \tag{8.22}$$

Suppose there exists a finite number of real roots of this equation. Let us label these roots as $P^k_{n,i}$, $k = 1,2,\ldots,N_r$, such that $P^1_{n,i} < P^2_{n,i} < \ldots < P^{N_r}_{n,i}$. If there exists no real root of (8.22), we let $N_r = 1$ and $P^1_{n,i} = -\infty$ when $\Delta q_n(i,x) > 0$, $\forall x$, and $P^1_{n,i} = \infty$ when $\Delta q_n(i,x) < 0$, $\forall x$.

It follows that the optimal promotion policy is a function of x and is implemented in the following fashion. When $x \leq P^1_{n,i}$, do not promote; when $P^1_{n,i} < x \leq P^2_{n,i}$, promote; when $P^2_{n,i} < x \leq P^3_{n,i}$, do not promote; and so on. We may call this type of policy *a multithreshold policy.* □

8.3. Assumptions and Preliminaries

In order to provide a simple and meaningful characterization to the optimal policy for the somewhat general MDP problem formulated in Section 8.3, we introduce some additional assumptions.

8.3.1 Quasi-convexity

To obtain the optimality of simple form policies for our model, we would like to have the function $g_n(i, y, m)$ to be quasi-convex in y according to Definition C.1.2. For this purpose, we make the following important, albeit restrictive, assumption on the demand density functions.

ASSUMPTION 8.1 *The demand density functions* $\varphi_i(\cdot)$, $i = 1, \ldots, L$, *are assumed to be of Pólya frequency of order* 2 *(PF*$_2$*) according to Definition* C.1.1.

Assumption 8.1 is, in fact, a popular assumption made in the inventory literature; (see Porteus (1971)). As Porteus and others have argued, such a condition is not very restrictive because the class of PF_2 densities is not lacking significant members. It contains all exponential densities and all finite convolutions of such densities. Furthermore, any mean μ and any variance in the range $[\mu^2/n, \mu^2]$ can be produced with a convolution of n exponential densities.

We will further note that such a condition on demand densities is not necessary for our results. However, we use this condition for the convenience of mathematical derivation for our results.

8.3.2 Stochastic Dominance

In the inventory literature (see the review in Section 8.1), it is usually assumed that demand consists of a deterministic component and a stochastic component, with promotion activities affecting only the deterministic component. Such an assumption provides a simplified mathematical modeling approach for describing the promotion/demand relationship. However, it does not accurately reflect the real-world situation, since the effect of promotional activities on demand cannot be predetermined with certainty. To capture the stochastic nature of the promotion/demand relationship, we will use a more realistic modeling approach based on the concept of *stochastic dominance* for our joint inventory/promotion decision problem.

Modeling the demand uncertainty induced by environmental factors and promotional activities in a stochastic fashion has not been attempted in the inventory literature. Our models represent a new approach to demand modeling. Based on the stochastic ordering relations introduced above, we are able to describe the relationship between demand and promotion activities in mathematical terms. See Section B.5 for additional information on stochastic dominance and some of the commonly used definitions of stochastic ordering.

For the purpose of the analysis carried out in this chapter, we adopt Definition B.5.1 of stochastic ordering.

ASSUMPTION 8.2

(i) *There exists a stochastic ordering relation between demands in different demand states such that*

$$\zeta_1 \underset{\text{st}}{\leq} \zeta_2 \underset{\text{st}}{\leq} \cdots \underset{\text{st}}{\leq} \zeta_L.$$

(ii) *For any* $1 \leq l \leq L$,

$$\sum_{j=l}^{L} p_{ij}(1) \geq \sum_{j=l}^{L} p_{ij}(0), \text{ and} \tag{8.23}$$

(iii)

$$\sum_{j=l}^{L} p_{ij}(m) \text{ is nondecreasing in } i. \tag{8.24}$$

REMARK 8.2 In simpler terms, Assumption 8.2 (i) means that the demand in a higher demand state is more likely to be larger than that in a lower demand state. Assumption 8.2 (ii) reflects the fact that a promotion would more likely lead to a demand state with a stochastically larger demand. Assumption 8.2 (iii) means that if the current demand state is higher, the next period is more likely to be in a demand state with a stochastically larger demand. Together, these assumptions will ensure that a promotion not only always generates a stochastically greater demand in the next period, but also has a positive impact on future demand.

Assumption 8.2 may not apply to the cases in which customers build inventories during an on-sale period and purchase less afterwards. However, our current model is more suitable to the situation where promotions are made in the form of advertising.

ASSUMPTION 8.3 $p_{ij}(1) = 0, \forall j < i$ *and* $p_{ij}(0) = 0, \forall i < j.$

REMARK 8.3 Assumption 8.3 means that the demand state in the next period after the product is promoted cannot be stochastically smaller than that in the current period. Likewise, if the product is not promoted in the current period, the demand state in the next period cannot be stochastically greater than that in the current period. Assumption 8.3 is stronger than Assumption 8.2, in the sense that some of the transition probabilities are fixed at zero. However, the nonzero elements of the transition probability matrix are not binding by Assumption 8.2.

The purpose of making these assumptions is to overcome some mathematical difficulties that are involved in proving the optimality of the simple form policies introduced in Section 8.4.2.

8.4. Structural Results

We have formulated a profit-maximizing inventory/promotion decision problem as a cost minimization problem, and have developed the dynamic programming equations for it.

In Section 8.2, we have characterized the general structure of the optimal policies without detailed proofs. We have shown that the optimal policies for (8.7) can be expressed in simple forms. Specifically, the inventory policy still retains the simplicity of base-stock policies, while the promotion policy is of threshold type.

In this section, we will provide the proof for the optimality of the policies defined in Theorem 8.1. Furthermore, we will show that the optimal policies can be further simplified under certain conditions.

8.4.1 Quasi-convexity of g_n

Now we present some useful properties of the dynamic programming equations and functions involved in (8.11-8.15).

LEMMA 8.1 *Denote the minimizer of* $g_n(i, y, m)$ *by* $S^m_{n,i}$. *For* $i = 1, \ldots, L$ *and* $m = 0, 1$,

(a) $g_n(i, y, m)$ *is quasi-convex in* y *with* $S^m_{n,i} \geq 0$ *for* $n = 0, \ldots, N-1$;

(b) $w_n(i, x)$ *is nondecreasing in* x *and constant when* $x \leq 0$ *for* $n = 0, \ldots, N$.

Proof. In view of $c_N \geq r_N$, part(b) is obvious from (8.16) when $n = N$. It is easy to see from (8.13) and (8.15) that (b) follows directly from (a) for $n < N-1$.

We will now prove part (a) for $n = 0, \ldots, N-1$ by induction. Let us define

$$\bar{g}_n(i, y, m) = (c_n - c_{n+1})y + l_n(y) + \sum_{j=1}^{L} p_{ij}(m) w_{n+1}(j, y),$$

where $l_n(y)$ is the surplus cost function as defined in Section 8.2.1. Then, (8.13) can be written as

$$g_n(i, y, m) = (c_n - r_n)\mu_i + \int_0^\infty \bar{g}_n(i, y - \xi, m)\varphi_i(\xi)d\xi.$$

When $n = N-1$, $w_{n+1}(j, y) = w_N(j, y)$ is nondecreasing in y and is a constant when $y \leq 0$ by definition. From the definition of $l_n(y)$, we have

$$(c_n - c_{n+1})y + l_n(y) = (c_n - c_{n+1} + h_n)y^+ + (p_n - c_{n+1} + c_n)y^-,$$

which is convex in y with the minimum at 0. Therefore, $\bar{g}_n(i, y, m)$ is quasi-convex in y. Furthermore, the demand density φ_i is assumed to be PF$_2$. By Theorem C.1.1, $\int_0^\infty \bar{g}_n(i, y - \xi, m)\varphi_i(\xi)d\xi$, and hence, $g_n(i, y, m)$ is also quasi-convex in y with its minimizer $S^m_{n,i} \geq 0$. Now, we have completed the induction for the base case $n = N-1$.

Let us assume that (a) holds for $n = k < N-1$ and prove that (a) is true for $n = k - 1$. Since (a) is assumed to be true for $k = n$, it follows that (b) is true for $k = n$, i.e., $w_{n+1}(j, y)$ is nondecreasing in y and constant when $y \leq 0$. Using the same line of reasoning used above for the base case, we can show that for $n = k-1$, $g_n(i, y, m)$ is quasi-convex in y with its minimizer $S^m_{n,i} \geq 0$. Therefore, (a) is proved for $n = k - 1$, and the induction is completed. □

8.4.2 Simple Form Solutions

In this subsection we will demonstrate that under certain conditions, the optimal solution exhibits some special structures that will simplify its computation greatly. Furthermore, the simplicity of the special structures allows for easier interpretations and implementation of optimal policies. The simple form solutions will be obtained with additional assumptions made about demand and transition probabilities of the Markov chain.

8.4.2.1 (S^0, S^1, P) Policies

We will show in this subsection that if a stochastic dominance relationship exists between the demands with and without promotions, the optimal promotion and ordering policy for the finite horizon problem can be characterized as an (S^0, S^1, P) *policy*, which is defined below.

DEFINITION 8.1 An (S^0, S^1, P) policy is specified by three control parameters S^0, S^1, and P with $S^0 \leq S^1$. Under an (S^0, S^1, P) policy, the product is promoted when the initial inventory x is above or equal to P. Furthermore, an order is placed to increase the inventory position to S^1

when $x < S^1$ and the product is promoted, and to S^0 when $x < S^0$ and the product is not promoted.

This is a special case of the optimal policies described in Section 8.4. Here, the multithreshold policy for promotion decisions is simplified to a single threshold policy.

First, we show some monotonicity properties that are needed for deriving the new results.

LEMMA 8.2 *Let $c_N = r_N$ and let Assumption 8.2 hold. Then for* $1 \le i < i' \le L$,
(a) $w_n(i, x) \ge w_n(i', x)$, $n \in \langle 0, N\rangle$;
(b) $S^1_{n,i'} \ge S^1_{n,i}$, $n \in \langle 0, N-1\rangle$; *and*
(c) $S^1_{n+1,i} \ge S^1_{n,i}$ *and* $S^1_{n,i} \ge S^0_{n,i}$, $n \in \langle 0, N-1\rangle$.

Proof. We prove (a) by induction. When $n = N$, $w_N(i, x) = 0$ by definition. Suppose that $w_{n+1}(i, x)$ is nonincreasing in i. First, let us prove that $g_n(i, y, m) \ge g_n(i', y, m), \forall m, y$. Since $w_{n+1}(j, y)$ is nonincreasing in j, according to Assumption 8.2 (iii), $\sum_{j=1}^{L} p_{ij}(m)w_{n+1}(j, y)$ is also nonincreasing in i. Furthermore, by (B.3) we have

$$\begin{aligned}\int_0^\infty \sum_{j=1}^{L} p_{ij}(m)w_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi \\ \ge \int_0^\infty \sum_{j=1}^{L} p_{ij}(m)w_{n+1}(j, y-\xi)\varphi_{i'}(\xi)d\xi.\end{aligned}$$

Using Assumption 8.2 (ii), we obtain that

$$\begin{aligned}\int_0^\infty \sum_{j=1}^{L} p_{ij}(m)w_{n+1}(j, y-\xi)\varphi_{i'}(\xi)d\xi \\ \ge \int_0^\infty \sum_{j=1}^{L} p_{i'j}(m)w_{n+1}(j, y-\xi)\varphi_{i'}(\xi)d\xi.\end{aligned}$$

On account of

$$g_n(i, y, m) = g_n^b(i, y) + \int_0^\infty \sum_{j=1}^{L} p_{ij}(m)w_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi \qquad (8.25)$$

and (8.13), it is clear that

$$g_n(i, y, m) \ge g_n(i', y, m).$$

In view of (8.14) and the quasi-convexity of g_n, we have

$$q_n(i,x,m) \geq q_n(i',x,m), \forall i' > i, x, m.$$

Finally, from (8.15), we conclude that

$$w_n(i,x) \geq w_n(i',x),$$

which proves (a).

Let us examine (8.25) again. For $n = N-1$, it is clear that $S^1_{N-1,i} = \bar{S}_{N-1,i}$ since $w_N(i,x) = 0$. Hence, from (8.20) we know that $S^1_{N-1,i'} \geq S^1_{N-1,i}, \forall i' > i$. When $n = N-2$, from Assumption 8.3 we have

$$\begin{aligned} & \int_0^\infty \sum_{j=1}^{L} p_{ij}(1) w'_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi \\ = & \int_0^\infty \sum_{j=i}^{L} p_{ij}(1) w'_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi, \end{aligned}$$

which is equal to 0 when $y \leq \bar{S}_{N-1,i}$ and is nonnegative otherwise, according to Lemma 8.1 (b). By Assumption (8.21), $\bar{S}_{N-2,i} \leq \bar{S}_{N-1,i}$. Thus, $S^1_{N-2,i} = \bar{S}_{N-2,i}$. Therefore, by (8.20), $S^1_{N-2,i'} \geq S^1_{N-2,i}, \forall i' > i$. Repeating this step for $n = N-3, \ldots, 0$, we complete the proof for (b).

Since $S^1_{n,i} = \bar{S}_{n,i}$, then according to (8.21), $S^1_{n+1,i} \geq S^1_{n,i}$. Now let us examine (8.25) with $m = 0$. Since

$$\int_0^\infty \sum_{j=1}^{L} p_{ij}(0) w'_{n+1}(j, y-\xi)\varphi_i(\xi)d\xi \geq 0,$$

we have $S^0_{n,i} \leq \bar{S}_{n,i}$, i.e., $S^0_{n,i} \leq S^1_{n,i}$. Thus, (c) is proved. □

REMARK 8.4 The assumption $c_N = r_N$ is used in this proof for convenience in the exposition. However, it can be relaxed by recalculating $\bar{S}_{N-1,i}$ with the last item included in (8.13).

Because of Assumption 8.3, we are able to determine the optimal base-stock level in a myopic fashion, which is reflected in part (b) of Lemma 8.2. We are now ready to show that the optimal policies for the N-period problem are of (S^0, S^1, P) type.

THEOREM 8.2 *When $x \leq S^1_{n,i}$, an $(S^0_{n,i}, S^1_{n,i}, P_{n,i})$ policy is optimal for the problem defined by* (8.10-8.13) *under Assumption 8.2, with $P_{n,i}$ given by the smallest root of equation* (8.22).

Proof. From the quasi-convexity of $g_n(i,y,m)$, $g'_n(i,y,m) \geq 0$ when $y \leq S^m_{n,i}$. By (8.14), we have $q'_n(i,x,m) = 0$ when $x \leq S^m_{n,i}$, and $q'_n(i,x,m) \geq 0$ when $x > S^m_{n,i}$. Also, we know that $S^0_{n,i} \leq S^1_{n,i}$ by Lemma 8.2 (c). Hence, for $x \leq S^1_{n,i}$,

$$\frac{\partial}{\partial x}(q_n(i,x,0) - q_n(i,x,1)) \geq 0,$$

which means that there exists at most one real root for (8.22) when $x \leq S^1_{n,i}$. Hence, a simple threshold policy is optimal for the promotion decision when $x \leq S^1_{n,i}$. □

REMARK 8.5 First, $x \leq S^1_{n,i}$ is not a necessary condition for Theorem 8.2. It is used for obtaining a simple proof. Second, the condition is not very restrictive and is satisfied for all $n \geq k$ if $x_k \leq S^1_{k,i}$ for any period $k \geq 0$ with demand state i. This can be seen from the following argument. By Assumption 8.3, if the product is promoted in period k, the demand state j in period $k+1$ will not be lower than i. Since $S^1_{k,i} \leq S^1_{k+1,j}$, and $\forall j \geq i$ and $S^0_{k,i} \leq S^1_{k,i}$, $\forall i$ as shown in Lemma 8.2 (c), then as soon as $x_k \leq S^1_{k,i}$ for any $k \geq 0$, we have $x_n \leq S^1_{n,i}$ for all subsequent periods $n = k+1, k+2, \ldots, N-1$ and all possible i. In view of these arguments, the condition will hold when $x_0 = 0$, which is often the case.

REMARK 8.6 Without further specifying cost and density functions, we may have three typical cases in terms of the relative position of $P_{n,i}$ as given below.

- Case 1. $P_{n,i} = -\infty$. In this case, a promotion is always desired. The corresponding optimal inventory policy is a base-stock policy with the base-stock level given by $S^1_{n,i}$. In fact, this is the only case in which $P_{n,i} < S^0_{n,i}$, since $q'_n(i,x,m) = 0$ when $x \leq S^m_{n,i}$, $m = 1, 2$.

- Case 2. $S^0_{n,i} < P_{n,i} \leq S^1_{n,i}$. In this case, a promotion is desirable only when the initial inventory level exceeds the critical level $P_{n,i}$. Two base-stock levels $S^0_{n,i}$ and $S^1_{n,i}$ are applied, respectively, depending on whether or not the product is promoted.

- Case 3. $S^0_{n,i} \leq S^1_{n,i} < P_{n,i}$. This case is similar to Case 2 except that no order will be placed once the product is promoted.

8.4.2.2 (S,P) Policies

In this subsection, we replace Assumptions 8.2 (i) and 8.2 (iii) by the following.

ASSUMPTION 8.4 *We assume that* $\xi_1 \le \xi_2 \le \dots \xi_L$, *in the sense of the first moment ordering.*

REMARK 8.7 Although Assumption 8.4 is more restrictive than Assumption 8.2 (i) used in the previous subsection, it still represents a generalization of the commonly used assumption of the additive random demands in the literature. On the positive side, we can now drop Assumption 8.3 for the results to be obtained in this subsection. We further note that Assumption 8.4 can be replaced by some weaker assumptions under which the results in this subsection would still hold.

THEOREM 8.3 *Under Assumptions* 8.4, 8.2 (ii), *and* 8.2 (iii), *a special case of the* (S^0, S^1, P) *policy, i.e., the* (S, P) *policy with* $S_{n,i} = S^0_{n,i} = S^1_{n,i}$, *is optimal for the model.*

Proof. Assumption 8.4 is equivalent to

$$\varphi_1(x-\mu_1) = \varphi_2(x-\mu_2) = \cdots = \varphi_L(x-\mu_L)$$

in view of Definition B.5.3. Since ξ_1 is nonnegative, i.e., $\varphi_1(x)=0, \forall x \le 0$, it follows that

$$\varphi_i(x) = 0, \;\; \forall x \le \mu_i - \mu_1, \; i = 1, \dots, L. \tag{8.26}$$

Also from (8.18),

$$\bar{S}_{n,1} - \mu_1 = \bar{S}_{n,2} - \mu_2 = \cdots = \bar{S}_{n,L} - \mu_L. \tag{8.27}$$

The proof is done by induction. For $k = N-1$, $S_{N-1,i} = S^0_{N-1,i} = S^1_{N-1,i} = \bar{S}_{N-1,i}$ for all i. Assume for $k = n+1$ that $S_{n+1,i} = S^0_{n+1,i} = S^1_{n+1,i} = \bar{S}_{n+1,i}$ for all i.

It is known from Lemma 8.1 (b) that $w'_{n+1}(i,y) = 0, \; \forall y \le S^0_{n+1,i}$, and $w'_{n+1}(i,y) \ge 0, \; \forall y > S^0_{n+1,i}$. Thus,

$$\sum_{j=1}^{L} p_{ij}(m) w'_{n+1}(j,y) \begin{cases} = 0, & y \le S^0_{n+1,1} = \bar{S}_{n+1,1}, \\ \ge 0, & \text{otherwise.} \end{cases}$$

Furthermore, by (8.26) and (8.27),

$$\int_0^\infty \sum_{j=1}^{L} p_{ij}(m) w'_{n+1}(j, y-\xi)\varphi_i(\xi) d\xi \begin{cases} = 0, & y \le \bar{S}_{n+1,1} + \mu_i - \mu_1 \\ & \qquad = \bar{S}_{n+1,i}, \\ \ge 0, & \text{otherwise.} \end{cases}$$

Since $\bar{S}_{n,i} \le \bar{S}_{n+1,i}$, we conclude that $S_{n,i} = S^0_{n,i} = S^1_{n,i} = \bar{S}_{n,i}$. □

8.5. Extensions

In this section, we briefly discuss some extensions of our model.

Infinite Horizon Problems. When the planning horizon is infinite, the formulation of the model can be presented as follows. The total cost of the infinite horizon problem to be minimized is given by

$$J_n(i, x; M, U) = \mathsf{E} \sum_{k=n}^{\infty} \alpha^{k-n} G_k(i_k, x_k; m_k, u_k), \tag{8.28}$$

where $0 < \alpha < 1$ is a discount factor. The inventory balance equations are given by

$$\begin{cases} x_{k+1} = x_k + u_k - \xi_k, \ k = n, n+1, \ldots, \text{ with} \\ x_n = x, \ \text{ the initial inventory level.} \end{cases} \tag{8.29}$$

Denote the minimum of $J_n(i, x; M, U)$ by $v_n(i, x)$, i.e.,

$$v_n(i, x) = \inf_{M,U} J_n(i, x; M, U). \tag{8.30}$$

Then, $v_n(i, x)$ satisfies the dynamic programming equations

$$\begin{aligned} v_n(i, x) \quad = \quad & \min_{m; u \geq 0} \{G_n(i, x; m, u) \\ & + \alpha \ \mathsf{E}[v_{n+1}(i_{n+1}, x + u - \xi_n) | i_n = i]\}, \\ & n = 0, 1, \ldots. \end{aligned} \tag{8.31}$$

By assuming stationary data, we can suppress the time index in the formulation and expect that there exists a stationary optimal policy which does not depend on time. (Issues regarding the existence of an optimal feedback-type policy in infinite horizon models have been addressed in Bertsekas and Shreve (1976).) Therefore, the DP equation can be written as

$$\begin{aligned} v(i, x) \quad = \quad & \min_{m; u \geq 0} \{G(i, x; m, u) \\ & + \alpha \sum_{j=1}^{L} p_{ij}(m) \int_0^{\infty} v(j, x + u - \xi) \varphi_i(\xi) d\xi\}. \end{aligned} \tag{8.32}$$

Since the infinite horizon problem can be considered as a limiting case of the finite horizon problem when $N \to \infty$, its optimal policy will have the same structure as the optimal policy in the finite horizon case, i.e., base-stock inventory policies and the threshold promotion policy, or (S^0, S^1, P) policy, under the corresponding assumptions made in the finite horizon problem.

Lost Sales. The dynamic programming equations can be written as

$$\left\{\begin{array}{rcl} v_n(i,x) & = & \min\limits_{m;u\geq 0}\{G_n(i,x;m,u) \\ & & +\int_0^\infty [v_{n+1}(i_{n+1},(x+u-\xi)^+)\varphi_i(\xi)d\xi\}, \\ & & \qquad\qquad\qquad 0\leq n\leq N-1, \\ v_N(i,x) & = & -c_N x^+. \end{array}\right. \tag{8.33}$$

For this problem, we only need to verify that

$$g_n(i,y,m)=c_n y+L_n(i,y)+\int_0^\infty \sum_{j=1}^L p_{ij}(m)v_{n+1}(j,(y-\xi)^+)\varphi_i(\xi)d\xi$$

is also quasi-convex in y, where $L_n(i,y)$ is defined as

$$L_n(i,y)=\int_0^\infty [l_n(i,y-\xi)-r_n(y-\mathsf{E}(y-\xi)^+)]\varphi_i(\xi)d\xi.$$

Furthermore, we have

$$\begin{array}{rcl} g_n(i,y,m) & = & (c_n-r_n)y+\displaystyle\int_0^\infty [l_n(i,y-\xi)+r_n\mathsf{E}(y-\xi)^+ \\ & & +\displaystyle\sum_{j=1}^L p_{ij}(m)w_{n+1}(j,(y-\xi)^+) \\ & & -c_{n+1}\mathsf{E}(y-\xi)^+]\varphi_i(\xi)d\xi \\ & = & \displaystyle\int_0^\infty [l_n(i,y-\xi)+(r_n-c_n)(y-\xi)^- \\ & & +(c_n-c_{n+1})(y-\xi)^+ \\ & & +\displaystyle\sum_{j=1}^L p_{ij}(m)w_{n+1}(j,(y-\xi)^+)]\varphi_i(\xi)d\xi. \end{array}$$

It is easy to verify that

$$l_n(i,y)+(r_n-c_n)y^-+(c_n-c_{n+1})y^++\sum_{j=1}^L p_{ij}(m)w_{n+1}(j,y^+)$$

is quasi-convex in y. Therefore, $g_n(i,y,m)$ is also quasi-convex in y since the density of ξ_i is assumed to be PF_2.

Price Discounts. Price discounts can be considered a part of the promotion cost. Let b_k be a proportional price discount offered if the product is promoted in period k. Then, the actual unit revenue will be $(1-b_k)r_k$. By considering the revenue loss as part of the promotion cost, the problem has the same form as the original problem.

Multiple Promotion Levels. We assume that the promotion effort can take on one of M discrete levels. These discrete levels could represent promotions communicated through different advertising media. At different promotion levels, the promotion expenditures incurred are also different.

Carryover Effect of Promotions. First of all, the carryover effect of a promotion can be partially captured by the Markovian transition law associated with the demand process. However, we may also incorporate this factor explicitly in the following way. If we assume that the effect of a promotion lasts $J-1$ periods, then we can redefine the demand state as

$$\tilde{i} = (i, j), \ i = 1, \ldots, L, \ j = 0, \ldots, J-1,$$

where j is the number of periods elapsed from the last promotion. In such a way, we can transform the new problem to the standard form and obtain a similar solution.

Nonlinear Inventory Cost Function. By assuming the demand density function to be PF_2, we may relax the requirements on the inventory cost function. We only require $l_k(x)$ to be quasi-convex in x and attain its minimum at $x = 0$. It is clear that such a generalized form of $l_k(x)$ will not change the results obtained in this chapter.

Case	c	α	p	q	A	S_1^0, S_2^0, S_3^0	S_1^1, S_2^1, S_3^1
3.1	0.50	0.90	1.00	5.00	2.00	12,12,16	16,17,17
3.2	1.00	0.90	1.00	2.00	2.00	8,8,12	12,15,15
3.3	2.00	0.90	1.00	2.00	2.00	0,0,0	0,0,0
3.4	2.50	0.90	1.00	2.00	2.00	0,0,0	0,0,0
3.5	1.00	0.90	2.00	4.00	2.00	9,9,13	13,16,16
3.6	1.00	0.90	1.00	4.00	2.00	11,11,15	15,16,16
3.7	1.00	0.90	1.00	5.00	2.00	11,11,15	15,17,17
3.8	0.50	0.90	1.00	2.00	2.00	9,9,13	13,16,16
3.9	0.50	0.90	1.00	1.00	2.00	7,7,10	10,14,14
3.10	0.50	0.95	1.00	2.00	2.00	9,9,13	13,16,16
3.11	0.50	0.85	1.00	2.00	2.00	9,9,13	13,16,16

Table 8.1. Numerical results for the MDP model with uniform demand distribution.

Case	c	α	p	q	A	S_1^0, S_2^0, S_3^0	S_1^1, S_2^1, S_3^1
4.1	0.50	0.90	1.00	5.00	2.00	8,8,14	14,19,19
4.2	1.00	0.90	1.00	2.00	2.00	4,4,10	10,15,15
4.3	2.00	0.90	1.00	2.00	2.00	0,0,0	0,0,0
4.4	2.50	0.90	1.00	2.00	2.00	0,0,0	0,0,0
4.5	1.00	0.90	2.00	4.00	2.00	6,6,12	12,17,17
4.6	1.00	0.90	1.00	4.00	2.00	7,7,13	13,18,18
4.7	1.00	0.90	1.00	5.00	2.00	7,7,13	13,18,18
4.8	0.50	0.90	1.00	2.00	2.00	6,6,12	12,17,17
4.9	0.50	0.90	1.00	1.00	2.00	3,3,9	9,14,14
4.10	0.50	0.95	1.00	2.00	2.00	6,6,12	12,17,17
4.11	0.50	0.85	1.00	2.00	2.00	6,6,12	12,17,17

Table 8.2. Numerical results for the MDP model with truncated normal demand distribution.

8.6. Numerical Results

In this section, we design a comparison study to demonstrate the advantage of the joint promotion and inventory decision making. For the same sample data, two types of policies are computed. One is the (S^0, S^1, P) policy, which we have shown to be optimal under certain conditions. The other is the policy that one would use in a decentralized system, i.e., promotion decisions and inventory decisions are made separately. More specifically, we assume that in a decentralized system, the information about the current inventory level is not considered for the promotion decision making purpose. A promotion will be conducted only if the expected revenue increase exceeds the promotion cost.

Our numerical results confirm that the joint decision making leads to better performance than the decentralized decision making. Tables 8.1 and 8.2 are the results for the cases with uniform demand distributions and truncated normal distributions, respectively. The parameters c, α, h, p and A are the unit purchase cost, the discount factor, the unit holding cost, the unit shortage cost, and the promotion cost, respectively. Figures 8.1-8.4 plot some of the cost functions obtained by two types of decision making systems corresponding to the Cases 3.1-3.4 in Table 8.1 . The solid lines (OP) are the results using our MDP models, while the dotted lines (DC) represent the results obtained based on the decentralized system.

The base-stock values computed using the two different approaches happen to be the same in these particular cases. However, the values of the cost functions are different. In each case, our model performs no worse than the decentralized model.

8.7. Concluding Remarks and Notes

In this chapter based on Cheng and Sethi (1999a) and Cheng (1996), we have developed an MDP model for a joint inventory/promotion decision problem, where the state variable of the MDP represents the demand state brought about by changing environmental factors as well as promotion decisions. Optimal inventory and promotion decision policies in a finite horizon setting are obtained via dynamic programming. Under certain conditions, we show that there is a threshold inventory level P for each demand state such that if the threshold is exceeded, then it is desirable to promote the product. For the proportional ordering cost case, the optimal inventory replenishment policy is a base-stock type policy with the optimal base-stock level dependent on the promotion decision. Several extensions of the model are also discussed. We also provided numerical results in Section 8.6 to demonstrate the benefit achieved by the joint inventory/promotion decision making.

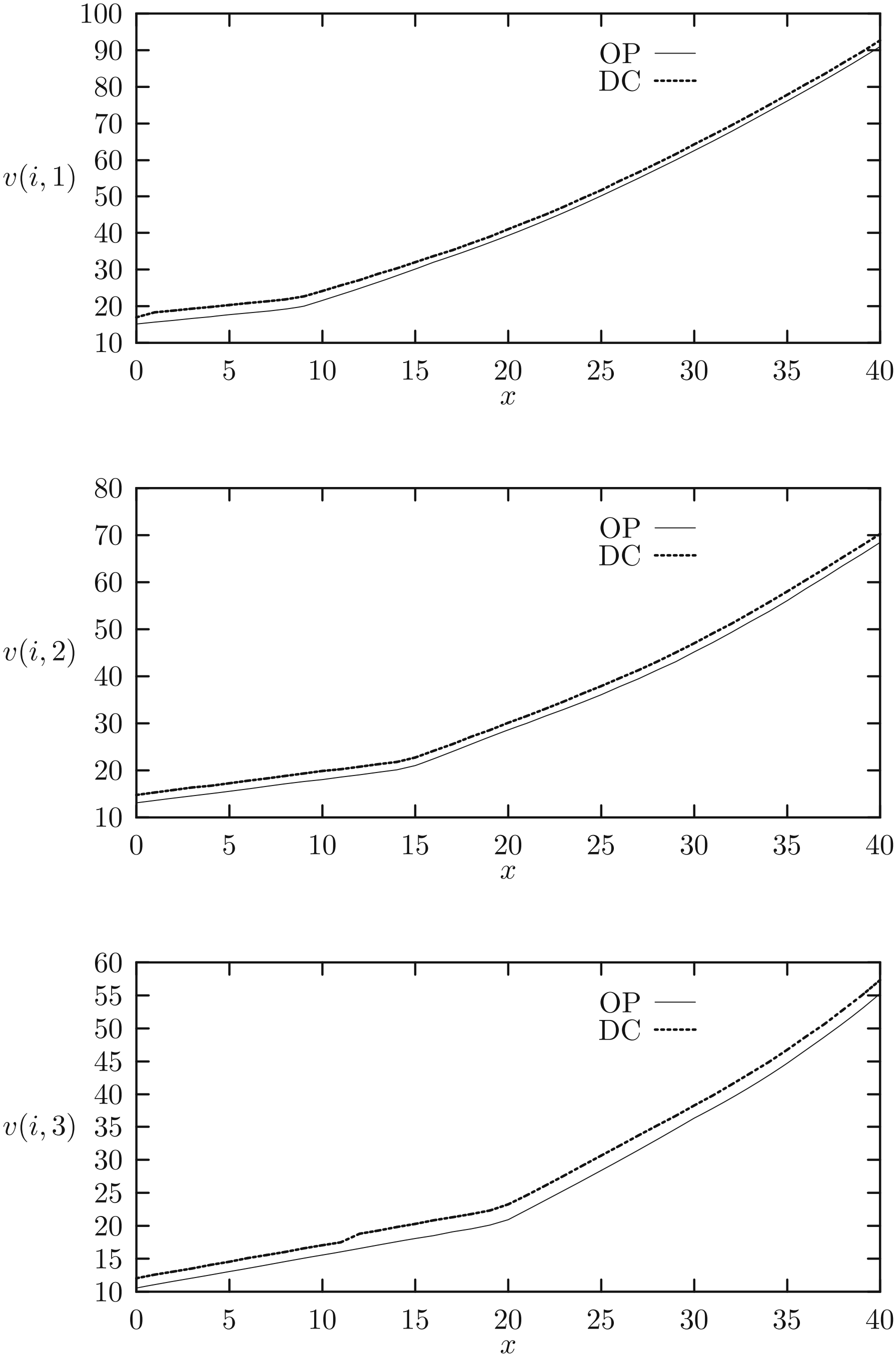

Figure 8.1. Numerical results for Case 3.1.

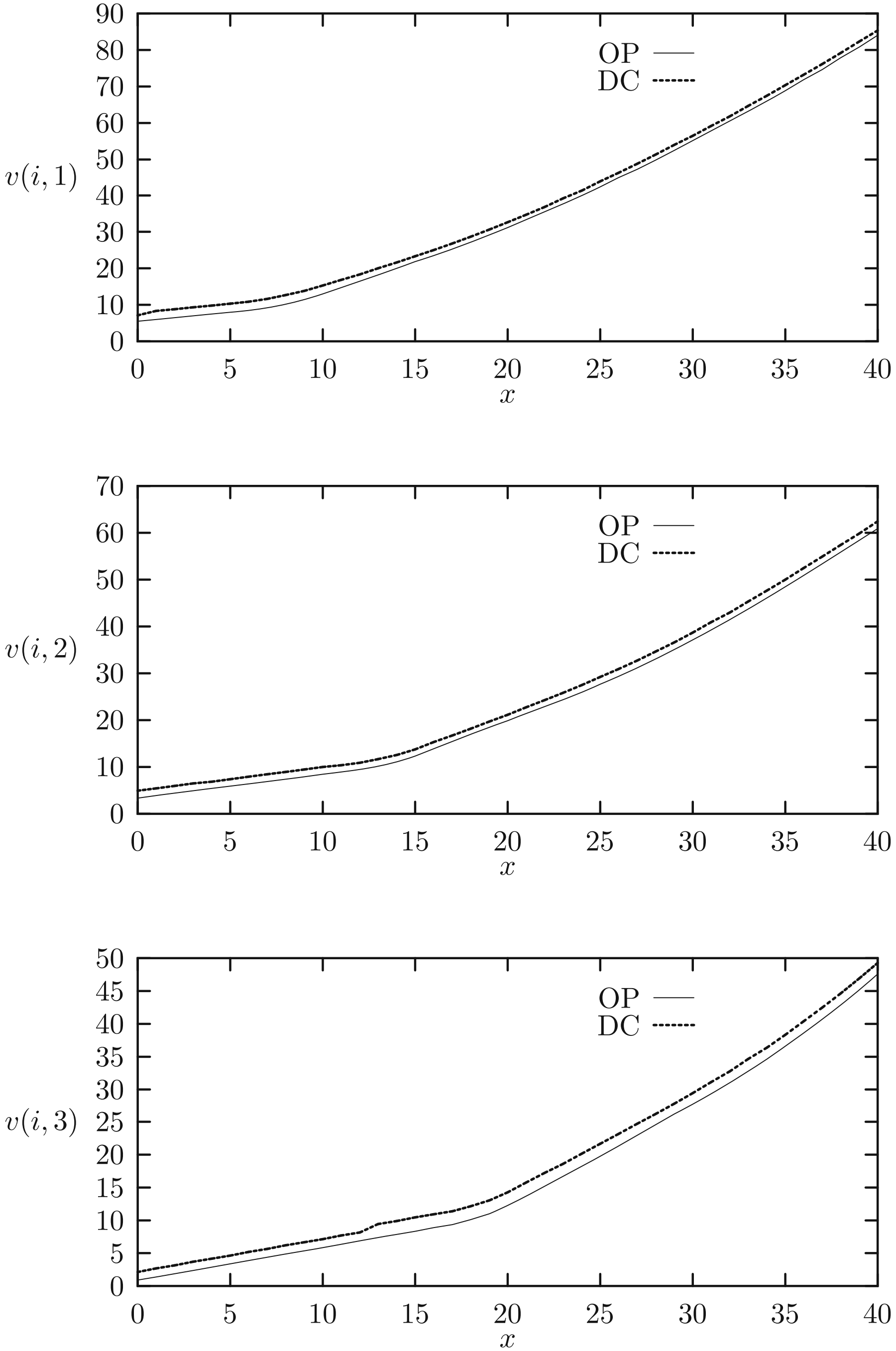

Figure 8.2. Numerical results for Case 3.2.

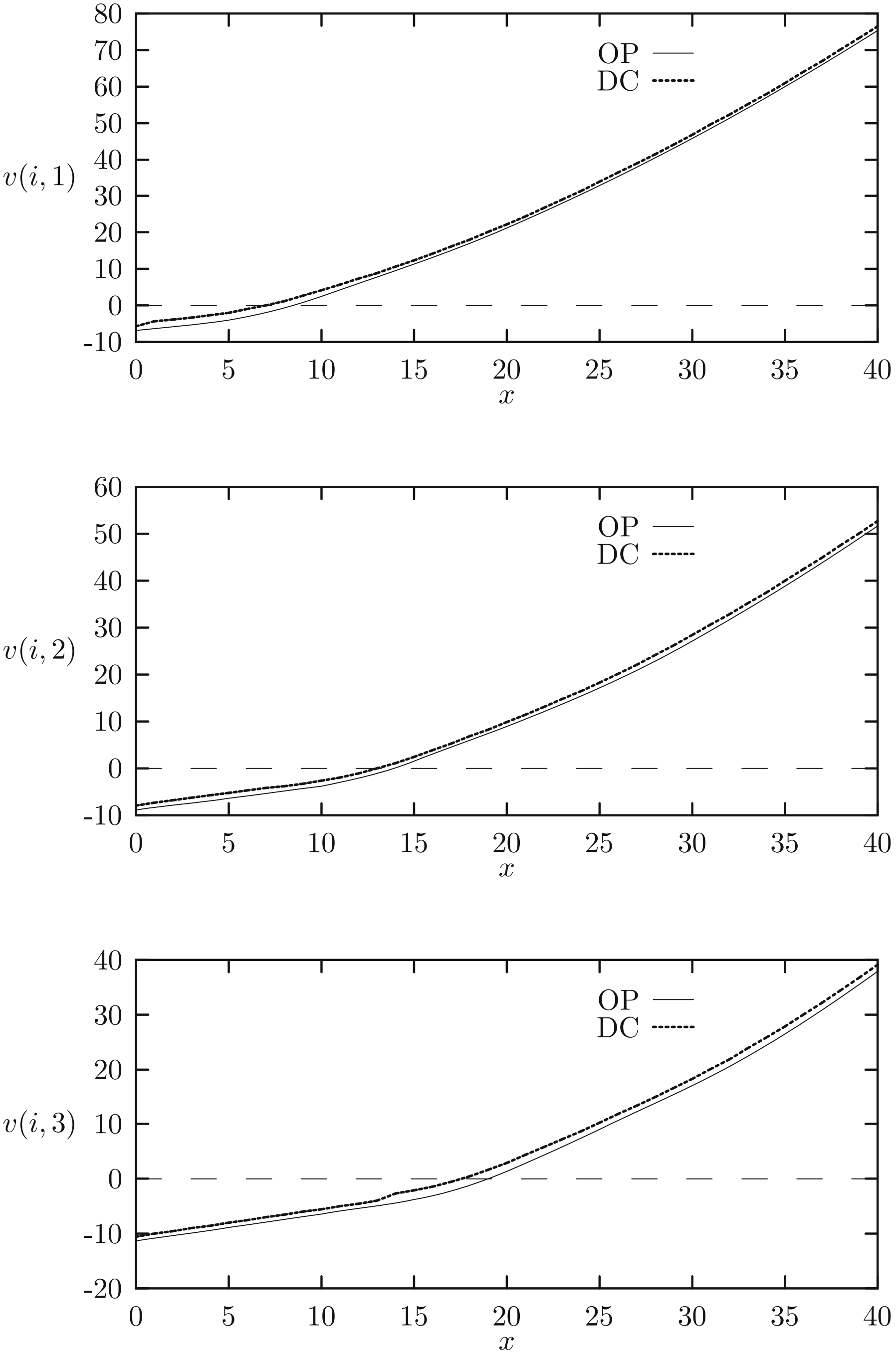

Figure 8.3. Numerical results for Case 3.3.

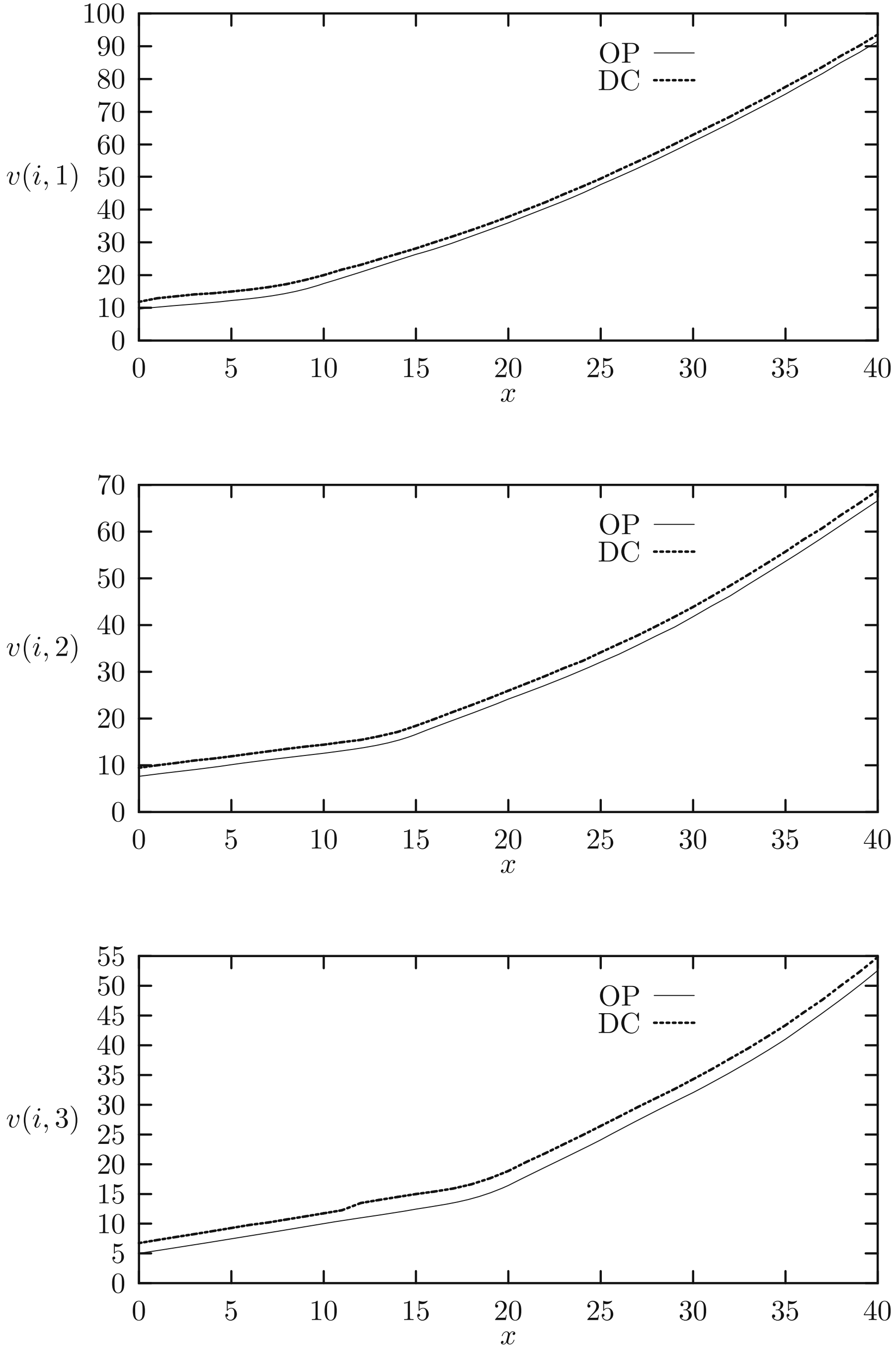

Figure 8.4. Numerical results for Case 3.4.