Chapter 2

DISCOUNTED COST MODELS WITH BACKORDERS

2.1. Introduction

One of the most important developments in the inventory theory has been to show that (s, S) policies are optimal for a class of dynamic inventory models with random periodic demands and fixed ordering costs. Under an (s, S) policy, if the inventory level at the beginning of a period is less than the reorder point s, then a sufficient quantity must be ordered to achieve an inventory level S, the order-up-to level, upon replenishment. There are a number of papers in the literature devoted to proving the optimality of (s, S) policies under a variety of assumptions. However, in real-life inventory problems, some of these assumptions do not hold. It is our purpose to relax these assumptions toward realism and still demonstrate the optimality of (s, S)-type policies.

The nature of the demand process is an important assumption in stochastic inventory models. With possible exceptions of Karlin and Fabens (1960) and Iglehart and Karlin (1962), classical inventory models have assumed demand in each period to be a random variable independent of demands in other periods and of environmental factors other than time. However, as elaborated in Song and Zipkin (1993), many randomly changing environmental factors, such as fluctuating economic conditions and uncertain market conditions in different stages of a product life cycle, can have a major effect on demand. For such situations, the Markov chain approach provides a natural and flexible alternative for modeling the demand process. In such an approach, environmental factors are represented by the *demand state* or the *state-of-the-world* of a Markov process, and demand in a period is a random variable with its distribution function dependent on the demand state in that period.

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Furthermore, the demand state can also affect other parameters of the inventory system such as the cost functions.

Another feature that is not usually treated in the classical inventory models, but is often observed in real life is the presence of various constraints on ordering decisions and inventory levels. For example, there may be periods, such as weekends and holidays, during which deliveries cannot take place. Also, the maximum inventory that can be accommodated is often limited by a finite storage space. On the other hand, one may wish to keep the amount of inventory above a certain level to reduce the chance of a stockout, and ensure satisfactory service to customers.

While some of these features are dealt with in the literature in a piecemeal fashion, we will formulate a sufficiently general model that has models with one or more of these features as special cases and still retain the optimal policy to be of (s, S)-type. Thus, our model considers more general demands, costs, and constraints than most of the fixed cost inventory models in the literature.

The plan of this chapter is as follows. The next section contains a review of relevant models and how our model relates to them. In Section 2.3, we develop a general finite horizon inventory model with a Markovian demand process. In Section 2.4, we state the dynamic programming equations for the problem and the results on the uniqueness of the solution and the existence of an optimal feedback or Markov policy. In Section 2.5, we use some properties of K-convex functions, derived in Appendix C, to show that the optimal policy for the finite horizon model under consideration is still of (s, S)-type, with the policy parameters s and S dependent on the demand state and the time remaining. The nonstationary infinite horizon version of the model is examined in Section 2.6. The cyclic demand case is treated in Section 2.7. The analysis of models incorporating no-ordering periods and those with the shelf capacity and service level constraints is presented in Section 2.8. Section 2.9 concludes the chapter.

2.2. Review of the Related Literature

Classical papers on the optimality of (s, S) policies in dynamic inventory models with stochastic demands and fixed setup costs include those of Arrow *et al.*(1951), Dvoretzky *et al.* (1953), Karlin (1958a), Scarf (1960), Iglehart (1963b), and Veinott (1966). Scarf develops the concept of K-convexity and uses it to show that (s, S) policies are optimal for finite horizon inventory problems with fixed ordering costs. That a stationary (s, S) policy is optimal for the stationary infinite horizon problem is proved by Iglehart (1963b). Furthermore, Bensoussan *et al.* (1983) provide a rigorous formulation of the problem with nonstationary but stochastically independent demand. They also deal with the issue of the existence of optimal feedback policies, along with a proof of the optimality of an (s, S)-type policy in the nonstationary finite, as well as infinite horizon cases.

The effect of a randomly changing environment in inventory models with fixed costs received only limited attention in the early literature. Karlin and Fabens (1960) have introduced a Markovian demand model similar to ours. They indicate that given the Markovian demand structure in their model, it appears reasonable to postulate an inventory policy of (s, S)-type with a different set of critical numbers for each demand state. But they consider the analysis to be complex, and concentrate instead on optimizing only over the restricted class of ordering policies, each characterized by a single pair of critical numbers, s and S, irrespective of the demand state.

Song and Zipkin (1993) present a continuous-time, discrete-state formulation with a Markov-modulated Poisson demand and with linear costs of inventory and backlogging. They show the optimality of a state-dependent (s, S) policy when the ordering cost consists of both a fixed cost and a linear variable cost. An algorithm for computing the optimal policy is also developed using a modified value iteration approach.

The basic model presented in the next section extends the classical Karlin and Fabens model in two significant ways. It generalizes the cost functions that are involved and it optimizes over the natural class of all history-dependent ordering policies. In relation to Song and Zipkin (1993), we consider more general demands (see Remark 2.3) and state-dependent convex inventory/backlog costs without a standard assumption made in the literature on backlog and purchase costs; (see Remarks 2.2 and 2.1). The nonstationary infinite horizon model extends Bensoussan *et al.* (1983) to allow for Markovian demands and more general asymptotic behavior on the shortage cost as the shortage becomes large; (see Remark 2.1).

2.3. Formulation of the Model

Let us consider an inventory problem over a finite number of periods $\langle n, N \rangle = \{n, n+1, \ldots, N\}$ and an initial inventory of x units at the beginning of period n, where n and N are any given integers satisfying $0 \leq n \leq N < \infty$. The demand in each period is assumed to be a random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and not necessarily identically distributed. More specifically, the demand distributions in successive periods are defined as below.

Consider a finite collection of demand states $\mathbb{I} = \{1, 2, \ldots, L\}$, and let i_k denote the demand state in period k. We assume that i_k, $k \in [n, N]$, with known initial demand state i_n is a Markov chain over $\mathbb{I}$ with the transition matrix $P = \{p_{ij}\}$. Thus,

$$0 \leq p_{ij} \leq 1, \; i \in \mathbb{I}, j \in \mathbb{I}, \;\text{ and }\; \sum_{j=1}^{L} p_{ij} = 1, \; i \in \mathbb{I}.$$

Let a nonnegative random variable ξ_k denote the demand in a given period k, $k = 0, \ldots, N-1$. Demand ξ_k depends only on period k and the demand state in that period, by which we mean that it is independent of past demand states and past demands. We denote its cumulative probability distribution by $\Phi_{i,k}(x)$, when the demand state $i_k = i$. In the following period, if the state changes to state j, which happens with probability p_{ij}, then the demand distribution is $\Phi_{j,k+1}$ in that period. We further assume that for a positive constant D,

$$\mathsf{E}(\xi_k | i_k = i) = \int_0^{\infty} x d\Phi_{i,k}(x) \leq D < \infty, \; k = 0, \ldots, N-1, \; i \in \mathbb{I}. \quad (2.1)$$

This is not a very restrictive assumption from an applied perspective.

We denote by

$$\begin{aligned} &\mathcal{F}_l^k, \text{ the } \sigma\text{-algebra generated by } && \{i_l, \ldots, i_{k-1}, i_k; \xi_l, \ldots, \xi_{k-1}\}, \\ & && 0 \leq l \leq k \leq N, \\ &\mathcal{F}^k = \mathcal{F}_0^k. \end{aligned} \quad (2.2)$$

Since $i_k, \; k = 1, \ldots, N$, is a Markov chain and ξ_k depends only on i_k, we have

$$\begin{aligned} \mathsf{E}(\xi_k | \mathcal{F}^k) &= \mathsf{E}(\xi_k | i_0, i_1, \ldots, i_k; \xi_0, \xi_1, \ldots, \xi_{k-1}) \\ &= \mathsf{E}(\xi_k | i_k). \end{aligned} \quad (2.3)$$

An admissible decision (ordering quantities) for the problem on the interval $[n, N]$ with initial state $i_n = i$ can be denoted as

$$U = (u_n, \ldots, u_{N-1}), \quad (2.4)$$

where u_k is a nonnegative $\mathcal{F}_n^k$-measurable random variable. In simpler terms, this means that decision u_k depends only on the past information. Note that since i_n is known in period n, $\mathcal{F}_n^n = (\Omega, \emptyset)$, and hence

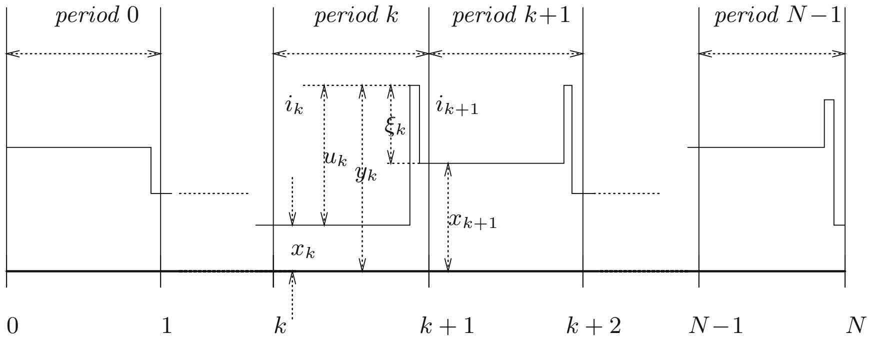

Figure 2.1. Temporal conventions used for the discrete-time inventory problem.

u_n is deterministic. Moreover, it should be emphasized that this class of admissible decisions is larger than the class of admissible feedback policies. Let us also denote a policy of not ordering anything at all as $\mathbf{0} = (0, \ldots, 0)$.

Ordering quantities are decided upon at the beginning of each period. Demand in each period is supposed to occur at the end of the period after the order has been delivered; (see Figure 2.1 for the temporal conventions used). Unsatisfied demand is carried forward as backlog. The inventory balance equations are defined by

$$\begin{cases} x_{k+1} = x_k + u_k - \xi_k, \ k = n, \ldots, N-1, \\ x_n = x, \ \text{ initial inventory level}, \\ i_k, k = n, \ldots, N, \ \text{Markov chain with transition matrix } P, \\ i_n = i, \ \text{ initial state}, \end{cases}$$

where x_k is the surplus level at the beginning of period k, u_k is the quantity ordered at the beginning of period k, i_k is the demand state in period k, and ξ_k is the demand in period k. Note that $x_k > 0$ represents an inventory of x_k and $x_k < 0$ represents a backlog (or shortage) of $-x_k$. Also, the initial state i and the initial inventory level x are assumed to be arbitrarily given.

Furthermore, we specify the relevant costs and the assumptions they satisfy.

(i) The purchase or production cost is expressed as

$$c_k(i, u) = K_k^i \mathbb{1}_{u>0} + c_k^i u, \quad k \in \langle 0, N-1 \rangle, \tag{2.5}$$

where the fixed ordering costs are $K_k^i \geq 0$, the variable costs are $c_k^i \geq 0$, and $\mathbb{1}_{u>0}$ equals 1 when $u > 0$ and equals 0 when $u \leq 0$.

(ii) The surplus (or inventory/backlog) cost functions $f_k(i, \cdot)$ are convex, and they are asymptotically linear, i.e.,

$$f_k(i, x) \leq C(1 + |x|) \quad \text{for some } C > 0, \quad k \in \langle 0, N \rangle. \tag{2.6}$$

The objective function to be minimized is the expected value of all the costs incurred during the interval $\langle n, N \rangle$ with $i_n = i$ and $x_n = x$, i.e.,

$$J_n(i, x; U) = \mathsf{E}\left\{ \sum_{k=n}^{N-1} [c_k(i_k, u_k) + f_k(i_k, x_k)] + f_N(i_N, x_N) \right\}, \tag{2.7}$$

where $U = (u_n, \ldots, u_{N-1})$ is a *history-dependent* or *nonanticipative* admissible decision (order quantities) for the problem and $u_N = 0$. The inventory balance equations are given by

$$x_{k+1} = x_k + u_k - \xi_k, \; k \in \langle n, N-1 \rangle. \tag{2.8}$$

Finally, we define the value function for the problem over $\langle n, N \rangle$ with $i_n = i$ and $x_n = x$ to be

$$v_n(i, x) = \inf_{U \in \mathcal{U}} J_n(i, x; U), \tag{2.9}$$

where $\mathcal{U}$ denotes the class of all admissible decisions. Note that the existence of an optimal policy is not required to define the value function. Of course, once the existence is established, the "inf" in (2.9) can be replaced by "min".

2.4. Dynamic Programming and Optimal Feedback Policy

In this section we develop the dynamic programming equations satisfied by the value function. We then provide a verification theorem (Theorem 2.2), which states that the cost associated with the feedback or Markov policy obtained from the solution of the dynamic programming equations, equals the value function of the problem on $\langle 0, N \rangle$.

Let $\mathbb{B}_0$ denote the class of all continuous functions from $\mathbb{I} \times \mathbb{R}$ into $\mathbb{R}^+$ and the pointwise limits of sequences of these functions; (see Feller (1971)). Note that it includes piecewise-continuous functions. Let $\mathbb{B}_1$ be the space of functions in $\mathbb{B}_0$ that are of linear growth, i.e., for any $b \in \mathbb{B}_1$, $0 \leq b(i, x) \leq C_b(1+|x|)$ for some $C_b > 0$. Let $\mathbb{C}_1$ be the subspace of functions in $\mathbb{B}_1$ that are uniformly continuous with respect to $x \in \mathbb{R}$. For any $b \in \mathbb{B}_1$, we define the norm

$$\| b \| = \max_i \sup_x \frac{b(i,x)}{1+|x|}$$

and the operator

$$\begin{aligned} F_{k+1}b(i,y) &= \mathsf{E}[b(i_{k+1}, y-\xi_k)|i_k = i] \\ &= \sum_{j=1}^{L} \{\mathsf{P}(i_{k+1} = j | i_k = i)\mathsf{E}[b(j, y-\xi_k)|i_k = i]\} \\ &= \sum_{j=1}^{L} p_{ij} \int_0^\infty b(j, y-\xi) d\Phi_{i,k}(\xi). \end{aligned} \tag{2.10}$$

In addition to Assumptions (i) and (ii) on costs, we also require that for $k = 0, 1, \ldots, N-1$,

$$c_k^i x + F_{k+1}(f_{k+1})(i,x) \to +\infty \text{ as } x \to \infty. \tag{2.11}$$

REMARK 2.1 Condition (2.11) means that either the unit ordering cost $c_k^i > 0$ or the expected holding cost $F_{k+1}(f_{k+1})(i,x) \to +\infty$ as $x \to \infty$, or both. Condition (2.11) is borne out of practical considerations and is not very restrictive. In addition, it rules out such unrealistic trivial cases as the one with $c_k^i = 0$ and $f_k(i,x) = 0, x \geq 0$, for each i and k, which implies ordering an infinite amount whenever an order is placed. The condition generalizes the usual assumptions made by Scarf (1960) and others that the unit inventory carrying cost $h > 0$. Furthermore, because of an essential asymmetry between the inventory side and the backlog side we need not impose a condition like (2.11) on the backlog side assumed in Bensoussan *et al.* (1983) and Bertsekas (1976). Whereas we can order any number of units to decrease backlog or build inventory, it is not possible to sell anything more than the demand in order to decrease inventory or increase backlog. If it were possible, then the condition like (2.11) as $x \to -\infty$ would be needed to make backlog more expensive than the revenue obtained by sale of units, asymptotically. In the special case of stationary linear backlog costs, this would imply $p > c$ (or $p > \alpha c$ if costs are discounted at the rate $\alpha, 0 < \alpha \leq 1$), where p is the unit backlog cost. But since sales in excess of demand are not allowed, we are able to dispense with the condition like (2.11) on the backlog side or the standard assumption $p > c$ (or $p > \alpha c$) as in Scarf (1960) and others, or the strong assumption $p > \alpha c^i$ for each i as in Song and Zipkin (1993).

Using the principle of optimality, we can write the following dynamic programming equations for the value function

$$\begin{cases} v_n(i,x) &= f_n(i,x) + \inf_{u\geq 0} \{c_n(i,u) \\ & \quad + \mathsf{E}[v_{n+1}(i_{n+1}, x+u-\xi_n)|i_n = i]\} \\ &= f_n(i,x) + \inf_{u\geq 0} \{c_n(i,u) + F_{n+1}(v_{n+1})(i,x+u)\}, \\ & \qquad\qquad\qquad\qquad n \in \langle 0, N-1\rangle, \\ v_N(i,x) &= f_N(i,x). \end{cases} \tag{2.12}$$

The next two theorems are fundamental. Together, they prove that (i) the dynamic programming equations have a solution in an appropriate space, (ii) the infima in these equations are attained, (iii) these infima provide an optimal feedback control within the class of admissible controls, and (iv) the solution of the equations is the value function. Theorem 2.1 proves (i) and (ii). Theorem 2.2 takes the solution of the dynamic programming equations and the infima obtained in Theorem 2.1 and goes on to prove (iii) and (iv). Its proof uses the property (vii) of conditional expectations given in Appendix B.2. Note, furthermore, that from (iv) it follows that the solution of the dynamic programming equations is unique in the defined space.

THEOREM 2.1 *The dynamic programming equations* (2.12) *define a sequence of functions in* $\mathbb{C}_1$. *Moreover, there exists a function* $\hat{u}_n(i,x)$ *in* $\mathbb{B}_0$, *which provides the infimum in* (2.12) *for any* x.

Proof. We proceed by induction. By Assumption (ii) on the function f_N and Theorems C.2.1 and A.1.3, v_N is in $\mathbb{C}_1$. Now, assume that $v_{n+1}(i,x)$ belongs to $\mathbb{C}_1$ for $n < N$. Consider points x such that $|x| \leq M$. It follows from (2.12) that $v_n(i,x) \geq f_n(i,x)$ for all $n, i,$ and x. Let

$$B^M_{n,i} = \max_{|x|\leq M}\{c_n(i,0) + F_{n+1}(v_{n+1})(i,x)\}.$$

We know that $B^M_{n,i} < \infty$ because $v_{n+1} \in \mathbb{C}_1$. Let $y = x+u$. Then, we have

$$\begin{aligned} c_n(i,u) + F_{n+1}(v_{n+1})(i,x+u) &\geq K^i_n + c^i_n(y-x) + F_{n+1}(f_{n+1})(i,y) \\ &\geq K^i_n - c^i_n M + c^i_n y + F_{n+1}(f_{n+1})(i,y). \end{aligned}$$

Because of (2.11), there is a constant $\bar{u}^M_{n,i}$ such that for all $y > \bar{u}^M_{n,i} - M$, we have

$$c^i_n y + F_{n+1}(f_{n+1})(i,y) > B^M_{n,i} + c^i_n M - K^i_n.$$

Since $y > \bar{u}^M_{n,i} - M$ is implied by $u > \bar{u}^M_{n,i}$, we have for all $u > \bar{u}^M_{n,i}$,

$$c_n(i,u) + F_{n+1}(v_{n+1})(i,x+u) > B^M_{n,i} \geq c_n(i,0) + F_{n+1}(v_{n+1})(i,x).$$

Consequently, any $u > \bar{u}^M_{n,i}$ cannot be the infimum in (2.12). Therefore in (2.12), we can restrict u by the constraint $0 \leq u \leq \bar{u}^M_{n,i}$ for all the points x satisfying $|x| \leq M$, without loss of optimality.

Since the function

$$\psi_n(i,x;u) = c_n(i,u) + F_{n+1}(v_{n+1})(i,x+u)$$

is l.s.c. and bounded from below, its minimum over a compact set is attained. Moreover, from the Selection Theorem A.1.7, we know that there exists a Borel function $\hat{u}^M_n(i,x)$ such that

$$\psi_n(i,x;\hat{u}^M_n(i,x)) = \inf_{0 \leq u \leq \bar{u}^M_{n,i}} \psi_n(i,x;u), \ \forall x.$$

With the definition

$$\hat{u}_n(i,x) = \hat{u}^M_n(i,x) \ \text{ for } M-1 < |x| \leq M,$$

we obtain a Borel function such that

$$\psi_n(i,x,\hat{u}_n(i,x)) = \inf_{u \geq 0} \psi_n(i,x,u), \ \forall x.$$

Now for $|x_1 - x_2| \leq \delta$, we have

$$\begin{aligned} &|\psi_n(i,x_1,u) - \psi_n(i,x_2,u)| \\ &\quad = |F_{n+1}(v_{n+1})(i,x_1+u) - F_{n+1}(v_{n+1})(i,x_2+u)| \\ &\quad \leq \sum_{j=1}^{L} p_{ij} \sup_{|x_1 - x_2| \leq \delta} |v_{n+1}(j,x_1) - v_{n+1}(j,x_2)|, \end{aligned}$$

from which together with (2.5), (2.6) and (2.12), it follows easily that $v_n(i,x)$ is uniformly continuous in x. Since

$$\inf_{u \geq 0} \psi_n(i,x,u) \leq c_n(i,0) + \| F_{n+1} \| \| v_{n+1} \| (1+|x|),$$

we can use (2.12) and (2.6) to conclude that $v_n(i,x) \in \mathbb{C}_1$. □

To solve the problem of minimizing $J_0(i,x;U)$, we use $\hat{u}_n(i,x)$ of Theorem 2.1 to define

$$\left\{ \begin{array}{lcl} \hat{u}_n & = & \hat{u}_n(i_n,\hat{x}_n), \ n \in \langle 0, N-1 \rangle \quad \text{with } i_0 = i, \\ \hat{x}_{n+1} & = & \hat{x}_n + \hat{u}_n - \xi_n, \ n \in \langle 0, N-1 \rangle \quad \text{with } \hat{x}_0 = x. \end{array} \right. \tag{2.13}$$

THEOREM 2.2 (Verification Theorem) *The decision* $\hat{U}$=$(\hat{u}_0,\hat{u}_1,\dots,\hat{u}_{N-1})$ *is optimal for the problem* $J_0(i,x;U)$. *Moreover,*

$$v_0(i,x) = \min_{U\in\mathcal{U}} J_0(i,x;U). \tag{2.14}$$

Proof. Let $U = (u_0,\dots,u_{N-1})$ be any admissible decision. Without loss of generality, we may assume that $\mathsf{E}c_n(i_n,u_n) < \infty$, $\mathsf{E}f_n(i_n,x_n) < \infty$, $n \in \langle 0, N-1\rangle$ and $\mathsf{E}f_N(i_N,x_N) < \infty$. Otherwise, $J_0(i,x;U) = \infty$ and U cannot be optimal since $J_0(i,x;\mathbf{0}) < \infty$ in view of (2.1) and Assumptions (i) and (ii).

Because $v_N(i_N,x_N) = f_N(i_N,x_N)$, it follows that $\mathsf{E}v_N(i_N,x_N) < \infty$. We proceed by induction. Assume that $\mathsf{E}v_{n+1}(i_{n+1},y_{n+1}) < \infty$. Next, using (2.8), the property (B.2-vii) of conditional expectations, the Markovian property (2.3), the independence assumption of ξ_n, and the notation (2.10), we obtain

$$\begin{aligned}
&\mathsf{E}\{v_{n+1}(i_{n+1},x_{n+1})|i_0,\dots,i_n,\xi_0,\dots,\xi_{n-1}\}\\
&\quad= \mathsf{E}\{v_{n+1}(i_{n+1},x_n+u_n-\xi_n)|i_0,\dots,i_n,\xi_0,\dots,\xi_{n-1}\}\\
&\quad= \mathsf{E}\{v_{n+1}(i_{n+1},y-\xi_n)|i_0,\dots,i_n,\xi_0,\dots,\xi_{n-1}\}_{y=x_n+u_n}\\
&\quad= \mathsf{E}\{v_{n+1}(i_{n+1},y-\xi_n)|i_n\}_{y=x_n+u_n}\\
&\quad= F_{n+1}(v_{n+1})(i_{n+1},y)_{y=x_n+u_n}\\
&\quad= F_{n+1}(v_{n+1})(i_{n+1},x_n+u_n) \text{ a.s.}
\end{aligned} \tag{2.15}$$

Now using (2.12), since U is admissible but not necessarily optimal, we can assert that

$$v_n(i_n,x_n) \le f_n(i_n,x_n) + c_n(i_n,u_n) + F_{n+1}(v_{n+1})(i_{n+1},x_n+u_n) \text{ a.s.},$$

and from the relation (2.15), we can derive

$$\begin{aligned}
v_n(i_n,x_n) \le\ & f_n(i_n,x_n) + c_n(i_n,u_n)\\
&+\mathsf{E}\{v_{n+1}(i_{n+1},x_{n+1})|i_0,\dots,i_n,\xi_0,\dots,\xi_{n-1}\} \text{ a.s.}
\end{aligned}$$

By taking the expectation of both sides of the above inequality, we obtain

$$\mathsf{E}v_n(i_n,x_n) \le \mathsf{E}(f_n(i_n,x_n) + c_n(i_n,u_n)) + \mathsf{E}(v_{n+1}(i_{n+1},x_{n+1})). \tag{2.16}$$

It follow from (2.16) that $\mathsf{E}v_n(i_n,x_n) < \infty$ and, therefore, (2.16) holds for all $n \in \langle 0, N\rangle$. Summing from 0 to $N-1$, we get

$$v_0(i,x) \le J_0(i,x;U). \tag{2.17}$$

Now consider the decision $\hat{U}$. From the definition of $\hat{u}_n(i_n, x)$ as the Borel function that attains the infimum in (2.12), and proceeding as above, we can also obtain

$$\mathsf{E}v_n(i_n, \hat{y}_n) = \mathsf{E}(f_n(i_n, \hat{y}_n) + c_n(i_n, \hat{u}_n)) + \mathsf{E}(v_{n+1}(i_{n+1}, y_{n+1})).$$

Note that $\hat{x}_0 = x$ is deterministic and $v_0(i,x) \in \mathbb{C}_1$. Thus, $\mathsf{E}v_0(i_0, \hat{x}_0) = v_0(i,x) < \infty$, and we can prove recursively that $\mathsf{E}c_n(i_n, \hat{u}_n) < \infty$, $n \in \langle 0, N-1 \rangle$ and $\mathsf{E}f_n(i_n, \hat{x}_n) < \infty$, $\mathsf{E}v_n(i_n, \hat{x}_n) < \infty$, $n \in \langle 0, N \rangle$. Adding up for n from 0 to $N-1$, it follows that

$$v_0(i,x) = J_0(i, x; \hat{U}).$$

This and the inequality (2.17) complete the proof. □

Taken together, Theorems 2.1 and 2.2 establish the existence of an optimal feedback policy. This means that there exists a policy in the class of admissible policies whose objective function value equals the value function defined by (2.9), as well as a Markov (or feedback) policy which gives the same objective function value. Furthermore, the solution $v_0(i,x)$ obtained in Theorem 2.1 is the value function.

2.5. Optimality of (s, S)-type Ordering Policies

We impose an additional condition on the costs under which the optimal feedback policy $\hat{u}_n(i,x)$ turns out to be an (s,S)-type policy. For $n \in \langle 0, N-1 \rangle$ and $i \in \mathbb{I}$, let

$$K_n^i \geq \bar{K}_{n+1}^i \equiv \sum_{j=1}^{L} p_{ij} K_{n+1}^j \geq 0. \tag{2.18}$$

Remark 2.2 Condition (2.18) means that the fixed cost of ordering in a given period with demand state i should be no less than the expected fixed cost of ordering in the next period. The condition is a generalization of the similar conditions used in the standard models. It includes the cases of the constant ordering costs ($K_n^i = K$, $\forall i, n$) and the nonincreasing ordering costs ($K_n^i \geq K_{n+1}^j$, $\forall i, j, n$). The latter case may arise on account of the learning curve effect associated with fixed ordering costs over time. Moreover, when all the future costs are calculated in terms of their present values, even if the undiscounted fixed cost may increase over time, Condition (2.18) still holds as long as the rate of increase of the fixed cost over time is less than or equal to the discount rate.

Theorems C.2.2 and C.2.3 are included in Appendix C to provide the required existing results on K-convex functions or their extensions. We can now derive the following result.

THEOREM 2.3 *Assume* (2.18) *in addition to the assumptions made in Section* 2.3. *Then there exists a sequence of numbers* $s_{n,i}, S_{n,i}$, $n \in \langle 0, N-1\rangle, i \in \mathbb{I}$, *with* $s_{n,i} \le S_{n,i}$, *such that the optimal feedback policy is*

$$\hat{u}_n(i,x) = (S_{n,i} - x)\mathbb{1}_{x<s_{n,i}}. \tag{2.19}$$

Proof. The dynamic programming equations (2.12) can be written as

$$\begin{cases} v_n(i,x) = f_n(i,x) - c_n^i x + h_n(i,x), & n \in \langle 0, N-1\rangle,\ i \in \mathbb{I}, \\ v_N(i,x) = f_N(i,x), & i \in \mathbb{I}, \end{cases} \tag{2.20}$$

where

$$h_n(i,x) = \inf_{y \ge x} [K_n^i \mathbb{1}_{y>x} + z_n(i,y)] \tag{2.21}$$

and

$$z_n(i,y) = c_n^i y + F_{n+1}(v_{n+1})(i,y). \tag{2.22}$$

From (2.5) and (2.12), we have $v_n(i,x) \ge f_n(i,x)$, $\forall n \in \langle 0, N-1\rangle$. From Theorem 2.1, we know that $v_n \in \mathbb{C}_1$. These, along with (2.11), ensure for $n \in \langle 0, N-1\rangle$ and $i \in \mathbb{I}$, that $z_n(i,y) \to +\infty$ as $y \to \infty$, and $z_n(i,y)$ is uniformly continuous.

In order to apply Theorem C.2.3 to obtain (2.19), we need only to prove that $z_n(i,x)$ is K_n^i-convex. According to Theorem C.2.2, it is sufficient to show that $v_{n+1}(i,x)$ is K_{n+1}^i-convex. This is done by induction. First, $v_N(i,x)$ is convex by definition and, therefore, K-convex for any $K \ge 0$. Let us now assume that for a given $n \le N-1$ and i, $v_{n+1}(i,x)$ is K_{n+1}^i-convex. By Theorem C.2.2 and Condition (2.18), it is easy to see that $z_n(i,x)$ is $\bar{K}_{n+1}^i$-convex, hence also K_n^i-convex. Then, Theorem C.2.3 implies that $h_n(i,x)$ is K_n^i-convex. Therefore, $v_n(i,x)$ is K_n^i-convex. This completes the induction argument.

Thus, it follows that $z_n(i,x)$ is K_n^i-convex for each n and i. Since $z_n(i,y) \to +\infty$ when $y \to \infty$, we apply Theorem C.2.3 to obtain the desired $s_{n,i}$ and $S_{n,i}$. According to Theorem 2.2, the (s,S)-type policy defined in (2.19) is optimal. □

REMARK 2.3 Theorem 2.3 can be easily extended to allow for a constant leadtime in the delivery of orders. The usual approach is to replace the surplus level by the so-called *surplus position.* It can also be generalized to Markovian demands with discrete components and countably many states.

REMARK 2.4 In the standard model with $L = 1$, Veinott (1966) gives an alternate proof to the one by Scarf (1960) based on K-convexity. For this, he does not need a condition like (2.11), but requires other assumptions instead.

2.6. Nonstationary Infinite Horizon Problem

We now consider an infinite horizon version of the problem formulated in Section 2.3. By letting $N = \infty$ and $U = (u_n, u_{n+1}, \ldots)$, the extended real-valued objective function of the problem becomes

$$J_n(i, x; U) = \sum_{k=n}^{\infty} \alpha^{k-n} \mathsf{E}[c_k(i_k, u_k) + f_k(i_k, x_k)], \tag{2.23}$$

where α is a given discount factor, $0 < \alpha \leq 1$. The dynamic programming equations are

$$v_n(i, x) = f_n(i, x) + \inf_{u \geq 0} \{c_n(i, u) + \alpha F_{n+1}(v_{n+1})(i, x+u)\},\ n = 0, 1, 2, \ldots. \tag{2.24}$$

In what follows, we will show that there exists a solution of (2.24) in class $\mathbb{C}_1$, which is the value function of the infinite horizon problem; (see also Remark 2.5). Moreover, the decision that attains the infimum in (2.24) is an optimal feedback policy. Our method is that of successive approximation of the infinite horizon problem by longer and longer finite horizon problems.

Therefore, we examine the finite horizon approximation $J_{n,m}(i, x; U)$, $m \geq 1$, of (2.23), which is obtained by the first m-period truncation of the infinite horizon problem of minimizing $J_n(i, x; U)$, i.e.,

$$J_{n,m}(i, x; U) = \sum_{k=n}^{n+m-1} \alpha^{k-n} \mathsf{E}[c_k(i_k, u_k) + f_k(i_k, x_k)]. \tag{2.25}$$

Let $v_{n,m}(i, x)$ be the value function of the truncated problem, i.e.,

$$v_{n,m}(i, x) = \inf_{U \in \mathcal{U}} J_{n,m}(i, x; U). \tag{2.26}$$

Since (2.26) is a finite horizon problem on the interval $\langle n, n+m \rangle$, we may apply Theorems 2.1 and 2.2 and obtain its value function by solving the dynamic programming equations

$$\begin{cases} v_{n,m+1}(i, x) &= f_n(i, x) + \inf_{u \geq 0} \{c_n(i, u) + \alpha F_{n+1}(v_{n+1,m})(i, x+u)\}, \\ v_{n+m,0}(i, x) &= 0. \end{cases} \tag{2.27}$$

Moreover, $v_{n,0}(i,x) = 0$, $v_{n,m} \in \mathbb{C}_1$, and the infimum in (2.26) is attained.

It is not difficult to see that the value function $v_{n,m}$ increases in m. In order to take its limit as $m \to \infty$, we need to establish an upper bound on $v_{n,m}$. One possible upper bound on $\inf_{U \in \mathcal{U}} J_n(i,x;u)$ can be obtained by computing the objective function value associated with a policy of never ordering anything. With the notation $\mathbf{0} = \{0, 0, \ldots\}$ for this policy, let us write

$$\begin{aligned} w_n(i,x) &= J_n(i,x;\mathbf{0}) \\ &= f_n(i,x) + \mathsf{E}\left\{ \sum_{k=n+1}^{\infty} \alpha^{k-n} f_k(i_k, x - \sum_{j=1}^{k-1} \xi_j) | i_n = i \right\}. \end{aligned} \quad (2.28)$$

In a way similar to Section I.5.1 of Chapter 4 in Bensoussan *et al.* (1983), it is easy to see that given (2.6), $w_n(i,x)$ is well-defined and is in $\mathbb{C}_1$. Furthermore, in class $\mathbb{C}_1$, w_n is the unique solution of

$$w_n(i,x) = f_n(i,x) + \alpha F_{n+1}(w_{n+1})(i,x). \quad (2.29)$$

We can state the following result for the infinite horizon problem.

THEOREM 2.4 *Assume* (2.5) *and* (2.6). *Then, we have*

$$0 = v_{n,0} \le v_{n,1} \le \ldots \le v_{n,m} \le w_n \quad (2.30)$$

and as $m \to \infty$

$$v_{n,m} \uparrow v_n, \text{ a solution of (2.24) in } \mathbb{B}_1. \quad (2.31)$$

Furthermore, $v_n \in \mathbb{C}_1$, *and we can obtain* $\hat{U} = \{\hat{u}_n, \hat{u}_{n+1}, \ldots\}$ *for which the infimum in* (2.24) *is attained. Moreover,* $\hat{U}$ *is an optimal feedback policy, i.e.,*

$$v_n(i,x) = \min_{U \in \mathcal{U}} J_n(i,x;U) = J_n(i,x;\hat{U}). \quad (2.32)$$

Proof. By definition, $v_{n,0} = 0$. Let $\tilde{U}_{n,m} = \{\tilde{u}_n, \tilde{u}_{n+1}, \ldots, \tilde{u}_{n+m-1}\}$ be a minimizer of (2.25). Thus,

$$\begin{aligned} v_{n,m}(i,x) &= J_{n,m}(i,x;\tilde{U}_{n,m}) \ge J_{n,m-1}(i,x;\tilde{U}_{n,m}) \\ &\ge \min_{U \in \mathcal{U}} J_{n,m-1}(i,x;U) = v_{n,m-1}(i,x). \end{aligned}$$

It is also obvious from (2.25) and (2.28) that $v_{n,m}(i,x) \le J_{n,m}(i,x;\mathbf{0}) \le w_n(i,x)$. This proves (2.30). Since $v_{n,m} \in \mathbb{C}_1$, we have

$$v_{n,m}(i,x) \uparrow v_n(i,x) \le w_n(i,x), \quad (2.33)$$

with $v_n(i,x)$ l.s.c., and hence in $\mathbb{B}_1$. Next, we show that v_n satisfies the dynamic programming equations (2.24). Observe from (2.27) and (2.30) that for each m, we have

$$v_{n,m}(i,x) \le f_n(i,x) + \inf_{u\ge 0}\{c_n(i,u) + \alpha F_{n+1}(v_{n+1,m})(i,x+u)\}.$$

Thus, in view of (2.33), we obtain

$$v_n(i,x) \le f_n(i,x) + \inf_{u\ge 0}\{c_n(i,u) + \alpha F_{n+1}(v_{n+1})(i,x+u)\}. \tag{2.34}$$

In order to obtain the reverse inequality, let $\hat{u}_{n,m}$ attain the infimum on the RHS of (2.27). From (2.5) and (2.6), we obtain that

$$c_n^i \hat{u}_{n,m}(i,x) \le \alpha F_{n+1}(v_{n+1,m})(i,x) \le \alpha(1+M)\ \| w_n \|\ (1+|x|),$$

where $\|\cdot\|$ is the norm defined on $\mathbb{B}_1$. This provides us with the bound

$$0 \le \hat{u}_{n,m}(i,x) \le M_n(1+|x|). \tag{2.35}$$

For $l > m$, we see from (2.27) that

$$\begin{aligned} v_{n,l+1}(i,x) &= f_n(i,x) + c_n(i,\hat{u}_{n,l}(x)) \\ &\quad +\alpha F_{n+1}(v_{n+1,l})(i,x+\hat{u}_{n,l}(x)) \\ &\ge f_n(i,x) + c_n(i,\hat{u}_{n,l}(x)) \\ &\quad +\alpha F_{n+1}(v_{n+1,m})(i,x+\hat{u}_{n,l}(x)). \end{aligned} \tag{2.36}$$

Fix m and let $l\to\infty$. In view of (2.35), we can, for any given n, i and x, extract a subsequence $\hat{u}_{n,l'}(i,x)$ such that $\hat{u}_{n,l'}(i,x)\to \bar{u}_n(i,x)$. Since $v_{n+1,m}$ is uniformly continuous in m and c_n is l.s.c., we can pass to the limit on the RHS of (2.36). Noting that the left-hand side converges as well, we obtain

$$\begin{aligned} v_n(i,x) &\ge f_n(i,x) + c_n(i,\bar{u}_n(x)) + \alpha F_{n+1}(v_{n+1,m})(i,x+\bar{u}_n(x)) \\ &\ge f_n(i,x) + \inf_{u\ge 0}\{c_n(i,u) + \alpha F_{n+1}(v_{n+1,m})(i,x+u)\}. \end{aligned}$$

This, along with (2.33), (2.34), and the fact that $v_n(i,x)\in\mathbb{B}_1$, proves (2.31).

Next we prove that $v_n\in\mathbb{C}_1$. Let us consider the problem (2.25) again. From (2.23),

$$J_n(i,x';U) - J_n(i,x;U)$$

$$= \sum_{k=n}^{\infty} \alpha^{k-n} \mathsf{E} \left[f_k \left(i_k, x' + \sum_{j=n}^{k-1} u_j - \sum_{j=n+1}^{k} \xi_i \right) - f_k \left(i_k, x + \sum_{j=n}^{k-1} u_j - \sum_{j=n+1}^{k} \xi_i \right) \right].$$

From (2.6), we have

$$|J_{n,m}(i,x';U) - J_{n,m}(i,x;U)| \le \sum_{l=n}^{\infty} \alpha^{l-n} C|x'-x| = C|x'-x|/(1-\alpha),$$

which implies $|v_{n,m}(i,x') - v_{n,m}(i,x)| \le C|x'-x|/(1-\alpha)$. By taking the limit as $m \to \infty$, we have $|v_n(i,x') - v_n(i,x)| \le C|x'-x|/(1-\alpha)$, from which it follows that $v_n \in \mathbb{C}_1$. Therefore, there exists a function $\hat{u}_n(i,x)$ in $\mathbb{B}_0$ such that

$$\begin{aligned} c_n(i, \hat{u}_n(i,x)) &+ \alpha F_{n+1}(v_{n+1})(i, x + \hat{u}_n(i,x)) \\ &= \inf_{u \ge 0} \{c_n(i,u) + \alpha F_{n+1}(v_{n+1})(i, x+u)\}. \end{aligned}$$

Hence, we have

$$v_n(i,x) = J_n(i,x;\hat{U}) \ge \inf_{U \in \mathcal{U}} J_n(i,x;U).$$

But for any arbitrary admissible control U, we also know that $v_n(i,x) \le J_n(i,x;U)$. Therefore, we conclude that

$$v_n(i,x) = J_n(i,x;\hat{U}) = \min_{U \in \mathcal{U}} J_n(i,x;U).$$

□

REMARK 2.5 We should indicate that Theorem 2.4 does not imply that there is a unique solution of the dynamic programming equations (2.24). There may well be other solutions. Moreover, one can show that the value function is the minimal positive solution of (2.24). It is also possible to obtain a uniqueness proof under additional assumptions.

With Theorem 2.4 in hand, we can now prove the optimality of an (s,S)-type policy for the nonstationary infinite horizon problem.

THEOREM 2.5 *Assume* (2.5), (2.6), *and* (2.11) *hold for the infinite horizon problem. Then, there exists a sequence of numbers* $s_{n,i}, S_{n,i}$, $n =$

$0, 1, \ldots,$ *with* $s_{n,i} \le S_{n,i}$ *for each* $i \in \mathbb{I}$, *such that the optimal feedback policy is* $\hat{u}_n(i, x) = (S_{n,i} - x)\mathbb{1}_{x < s_{n,i}}$.

Proof. Let v_n denote the value function. Define the functions z_n and h_n as in Section 2.5. We know that $z_n(i, x) \to \infty$ as $x \to +\infty$ and $z_n(i, x) \in \mathbb{C}_1$ for all n and $i \in \mathbb{I}$.

We now prove that v_n is K_n^i-convex. Using the same induction as in Section 2.5, we can show that $v_{n,k}(i, x)$, as defined in (2.26), is K_n^i-convex. This induction is possible since we know that $v_{n,k}(i, x)$ satisfies the dynamic programming equations (2.27). It is clear from the definition of K-convexity and from taking the limit as $k \to \infty$, that the value function $v_n(i, x)$ is also K_n^i-convex.

From Theorem 2.4, we know that $v_n \in \mathbb{C}_1$ and that v_n satisfies the dynamic programming equations (2.24). Therefore, we can obtain an optimal feedback policy $\hat{U} = \{\hat{u}_n, \hat{u}_{n+1}, \ldots\}$ that attains the infimum in (2.24). Because v_n is K_n^i-convex, $\hat{u}_n$ can be expressed as in Theorem 2.5. □

2.7. Cyclic Demand Model

Cyclic or seasonal demand often arises in practice. Such a demand represents a special case of the Markovian demand, where the number of demand states L is given by the cycle length, and

$$p_{ij} = \begin{cases} 1, & \text{if } j = i + 1, i = 1, \ldots, L - 1, \text{ or } i = L, j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we assume that the cost functions and density functions are all time invariant. The result is a considerably simplified optimal policy, i.e., only L pairs of (s_n, S_n) need to be computed.

We can state the following corollary to Theorem 2.5.

Corollary 2.1 *In the infinite horizon inventory problem with the demand cycle of L periods, let* n_1 *and* n_2 $(n_1 < n_2)$ *be any two periods such that* $n_2 = n_1 + m \cdot L$, $m = 1, 2, \ldots$. *Then, we have* $s_{n_1} = s_{n_2}$ *and* $S_{n_1} = S_{n_2}$.

2.8. Constrained Models

In this section, we incorporate some additional constraints that often arise in practice. We show that (s, S)-type policies continue to remain optimal for the extended models.

2.8.1 No-ordering Periods

Consider the special situation in which ordering is not possible in certain periods (for example, suppliers do not accept orders on weekends). We will show that the following theorem holds in such a situation.

THEOREM 2.6 *In the problem with some no-ordering periods, the optimal policy is still of (s, S)-type for any period, except when the ordering is not allowed.*

Proof. To stay with our earlier notation, it is no loss of generality to continue assuming the setup cost to be K_m^i in a no-ordering period m with the demand state i; clearly, setup costs are of no use in no-ordering periods. The definition (2.21) is revised as

$$h_n(i,x) = \begin{cases} z_n(i,x), & \text{in a no-ordering period } n, \\ \inf_{y \geq x} [K_n^i \mathbb{1}_{y>x} + z_n(i,y)], & \text{otherwise,} \end{cases} \tag{2.37}$$

and $z_n(i,y)$ is defined as before in (2.22). Using the same induction argument as in the proof of Theorem 2.3, we can show that $h_n(i,x)$ and $v_n(i,x)$ are K_n^i-convex if ordering is allowed in period n. If ordering is disallowed in period n, then $h_n(i,x) = z_n(i,x)$, which is $\bar{K}_{n+1}^i$-convex, and therefore also K_n^i-convex. Thus, in both cases $v_n(i,x)$ is K_n^i-convex. □

REMARK 2.6 Theorem 2.6 can be generalized to allow for supply uncertainty as in Parlar *et al.* (1995). One needs to replace u_k in (2.8) by $a_k u_k$, where $\mathsf{P}\{a_k = 1 | i_k = i\} = q_k^i$ and $\mathsf{P}\{a_k = 0 | i_k = i\} = 1 - q_k^i$, and modify (3.2) appropriately.

2.8.2 Storage and Service Level Constraints

Let $B < \infty$ denote an upper bound on the inventory level. Moreover, to guarantee a reasonable measure of service level, we introduce a chance constraint requiring that the probability of the ending inventory falling below zero in any given period does not exceed $1 - \alpha_p$ for a specified $\alpha_p \in (0, 1]$, known as Type 1 service level. Thus,

$$\mathsf{P}\{x_{k+1} < 0\} \leq 1 - \alpha_p, \ k \in \langle 0, N-1 \rangle.$$

As an example, if we set $\alpha_p = 0.95$, then we are requiring that we satisfy the demand in any given period with at least 95% probability.

Given the demand state i in period k and the inventory dynamics (2.8), we can write the above condition as $\Phi_{i,k}(x_k + u_k) \geq \alpha_p$, which can be converted into $x_k + u_k \geq A_k^i$, where $A_k^i = \inf(a | \Phi_{i,k}(a) \geq \alpha_p)$, referred to as the safety stock in period k that guarantees the Type 1 service level α_p in that period. If we define the *quantile function* $\Phi_{i,k}^{-1}(\eta) = \inf(a | \Phi_{i,k}(a) \geq \eta)$, then we can also write $A_k^i = \Phi_{i,k}^{-1}(\alpha_p)$.

The dynamic programming equations can be written as (2.22), where $z_n(i, y)$ is as in (2.22) and

$$h_n(i, x) = \inf_{y \geq x, A_n^i \leq y \leq B} [K_n^i \mathbb{1}_{y>x} + z_n(i, y)], \tag{2.38}$$

provided $A_n^i \leq B$, $n \in \langle 0, N-1 \rangle, i \in \mathbb{I}$; if not, then there is no feasible solution, and $h_n(i, x) = \inf \emptyset \equiv \infty$. This time, since y is bounded by $B < \infty$, Theorem 2.1 can be relaxed as follows.

THEOREM 2.7 *The dynamic programming equations* (2.20) *with* (2.38) *define a sequence of l.s.c. functions on* $(-\infty, B\,]$. *Moreover, there exists a function* $\hat{u}_n(i, x)$ *in* $\mathbb{B}_0$, *which attains the infimum in* (2.20) *for any* $x \in (-\infty, B\,]$.

With $\hat{u}_n(i, x)$ of Theorem 2.7, it is possible to prove Theorem 2.2, also known as the verification theorem, for the constrained case. We now show that the optimal policy is of (s, S)-type.

THEOREM 2.8 *There is a sequence of numbers* $s_{n,i}$, $S_{n,i}$, $n \in \langle 0, N-1 \rangle$, $i \in \mathbb{I}$, *with* $s_{n,i} \leq S_{n,i}$ *and* $A_n^i \leq S_{n,i} \leq B$ *such that optimal feedback policy* $\hat{u}_n(i, x) = (S_{n,i} - x)\mathbb{1}_{x<s_{n,i}}$ *is optimal for the model with capacity and service constraints defined above.*

Proof. First note that Theorem C.2.3 holds when g is l.s.c. and K-convex on $(-\infty, B\,]$, $B < \infty$. Also, by Theorem C.2.2 (iii) and (iv), one can see that $\mathsf{E}g(x-\xi)$ is K-convex on $(-\infty, B\,]$ since $\xi \geq 0$. Because g is l.s.c., it is easily seen that $\mathsf{E}g(x-\xi)$ is l.s.c. on $(-\infty, B\,]$. Furthermore, by Theorem 2.7, v_n is l.s.c. on $(-\infty, B\,]$. With these observations in mind, the proof of Theorem 2.3 can easily be modified to complete the proof. □

REMARK 2.7 A constant integer leadtime $\tau \geq 1$ can also be included in this model, with the surplus level replaced by the surplus position and with the lower bound A_k^i properly redefined in terms of the distribution of the total demand during the leadtime; (see, e.g., Porteus (1971) or Zipkin (2000)).

2.9. Concluding Remarks and Notes

This chapter, based on Sethi and Cheng (1997), develops various realistic extensions of the classical dynamic inventory model with stochastic demands. The models consider demands that are dependent on a finite state Markov chain including demands that are cyclic. Some constraints commonly encountered in practice, namely no-ordering periods, finite storage capacities, and service levels, are also treated. Both finite and infinite horizon cases are studied. It is shown that all of these models, not unlike the classical model, exhibit the optimality of (s, S)-type policies.