Lecture 7

Preliminaries to Existence and Uniqueness of Solutions

So far, mostly we have engaged ourselves in solving DEs, tacitly assuming that there always exists a solution. However, the theory of existence and uniqueness of solutions of the initial value problems is quite complex. We begin to develop this theory for the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{7.1}$$

where $f(x, y)$ will be assumed to be continuous in a domain D containing the point (x_0, y_0). By a solution of (7.1) in an interval J containing x_0, we mean a function $y(x)$ satisfying (i) $y(x_0) = y_0$, (ii) $y'(x)$ exists for all $x \in J$, (iii) for all $x \in J$ the points $(x, y(x)) \in D$, and (iv) $y'(x) = f(x, y(x))$ for all $x \in J$.

For the initial value problem (7.1) later we shall prove that the continuity of the function $f(x, y)$ alone is sufficient for the existence of at least one solution in a sufficiently small neighborhood of the point (x_0, y_0). However, if $f(x, y)$ is not continuous, then the nature of the solutions of (7.1) is quite arbitrary. For example, the initial value problem

$$y' = \frac{2}{x}(y - 1), \quad y(0) = 0$$

has no solution, while the problem

$$y' = \frac{2}{x}(y - 1), \quad y(0) = 1$$

has an infinite number of solutions $y(x) = 1 + cx^2$, where c is an arbitrary constant.

The use of integral equations to establish existence theorems is a standard device in the theory of DEs. It owes its efficiency to the smoothening properties of integration as contrasted with coarsening properties of differentiation. If two functions are close enough, their integrals must be close enough, whereas their derivatives may be far apart and may not even exist. We shall need the following result to prove the existence, uniqueness, and several other properties of the solutions of the initial value problem (7.1).

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Theorem 7.1. Let $f(x,y)$ be continuous in the domain D, then any solution of (7.1) is also a solution of the integral equation

$$y(x) \ = \ y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{7.2}$$

and conversely.

Proof. Any solution $y(x)$ of the DE $y' = f(x,y)$ converts it into an identity in x, i.e., $y'(x) = f(x, y(x))$. An integration of this equality yields

$$y(x) - y(x_0) \ = \ \int_{x_0}^{x} f(t, y(t))dt.$$

Conversely, if $y(x)$ is any solution of (7.2) then $y(x_0) = y_0$ and since $f(x,y)$ is continuous, differentiating (7.2) we find $y'(x) = f(x, y(x))$. ∎

While continuity of the function $f(x,y)$ is sufficient for the existence of a solution of (7.1), it does not imply uniqueness. For example, the function $f(x,y) = y^{2/3}$ is continuous in the entire xy-plane, but the problem $y' = y^{2/3}$, $y(0) = 0$ has at least two solutions $y(x) \equiv 0$ and $y(x) = x^3/27$. To ensure the uniqueness we shall begin with the assumption that the variation of the function $f(x,y)$ relative to y remains bounded, i.e.,

$$|f(x, y_1) - f(x, y_2)| \ \leq \ L|y_1 - y_2| \tag{7.3}$$

for all $(x, y_1),\ (x, y_2)$ in the domain D. The function $f(x,y)$ is said to satisfy a uniform *Lipschitz condition* in any domain D if the inequality (7.3) holds for all point-pairs $(x, y_1),\ (x, y_2)$ in D having the same x. The nonnegative constant L is called the *Lipschitz constant.*

The function $y^{2/3}$ violates the Lipschitz condition in any domain containing $y = 0$, whereas the function $f(x,y) = x - y$ satisfies the Lipschitz condition in $D = \mathbb{R}^2$ with $L = 1$. As an another example, the function $f(x,y) = e^y$ satisfies the Lipschitz condition in $D = \{(x,y) : \ x \in \mathbb{R},\ |y| \leq c\}$ with $L = e^c$, where c is some positive constant.

Obviously, if inequality (7.3) is satisfied in D, then the function $f(x,y)$ is continuous with respect to y in D; however, it is not necessarily differentiable with respect to y, e.g., the function $f(x,y) = |y|$ is not differentiable in $\mathbb{R}^2$ but satisfies (7.3) with $L = 1$.

If the function $f(x,y)$ is differentiable with respect to y, then it is easy to compute the Lipschitz constant. In fact, we shall prove the following theorem.

Theorem 7.2. Let the domain D be convex and the function $f(x,y)$ be differentiable with respect to y in D. Then for the Lipschitz condition

(7.3) to be satisfied, it is necessary and sufficient that

$$\sup_D \left| \frac{\partial f(x,y)}{\partial y} \right| \;\leq\; L. \tag{7.4}$$

Proof. Since $f(x,y)$ is differentiable with respect to y and the domain D is convex, for all $(x,y_1),\ (x,y_2) \in D$ the mean value theorem provides

$$f(x,y_1) - f(x,y_2) \;=\; \frac{\partial f(x,y^*)}{\partial y}(y_1 - y_2),$$

where y^* lies between y_1 and y_2. Thus, in view of (7.4) the inequality (7.3) is immediate.

Conversely, inequality (7.3) implies that

$$\left| \frac{\partial f(x,y_1)}{\partial y_1} \right| \;=\; \lim_{y_2 \to y_1} \left| \frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2} \right| \;\leq\; L. \qquad \blacksquare$$

To prove the existence, uniqueness, and several other properties of the solutions of (7.1), we shall also need a *Gronwall's-type integral inequality*, which is contained in the following result.

Theorem 7.3. Let $u(x)$, $p(x)$ and $q(x)$ be nonnegative continuous functions in the interval $|x - x_0| \leq a$ and

$$u(x) \;\leq\; p(x) + \left| \int_{x_0}^{x} q(t)u(t)dt \right| \quad \text{for} \quad |x - x_0| \leq a. \tag{7.5}$$

Then the following inequality holds:

$$u(x) \;\leq\; p(x) + \left| \int_{x_0}^{x} p(t)q(t) \exp\left(\left| \int_{t}^{x} q(s)ds \right| \right) dt \right| \quad \text{for} \quad |x - x_0| \leq a. \tag{7.6}$$

Proof. We shall prove (7.6) for $x_0 \leq x \leq x_0 + a$ whereas for $x_0 - a \leq x \leq x_0$ the proof is similar. We define

$$r(x) \;=\; \int_{x_0}^{x} q(t)u(t)dt$$

so that $r(x_0) = 0$, and

$$r'(x) \;=\; q(x)u(x).$$

Since from (7.5), $u(x) \leq p(x) + r(x)$, it follows that

$$r'(x) \;\leq\; p(x)q(x) + q(x)r(x),$$

which on multiplying by $\exp\left(-\int_{x_0}^x q(s)ds\right)$ is the same as

$$\left(\exp\left(-\int_{x_0}^x q(s)ds\right) r(x)\right)' \ \le\ p(x)q(x)\exp\left(-\int_{x_0}^x q(s)ds\right).$$

Integrating the above inequality, we obtain

$$r(x) \ \le\ \int_{x_0}^x p(t)q(t)\exp\left(\int_t^x q(s)ds\right)dt$$

and now (7.6) follows from $u(x) \le p(x) + r(x)$. ∎

Corollary 7.4. If in Theorem 7.3 the function $p(x) \equiv 0$, then $u(x) \equiv 0$.

Corollary 7.5. If in Theorem 7.3 the function $p(x)$ is nondecreasing in $[x_0, x_0 + a]$ and nonincreasing in $[x_0 - a, x_0]$, then

$$u(x) \ \le\ p(x)\exp\left(\left|\int_{x_0}^x q(t)dt\right|\right) \quad \text{for} \quad |x - x_0| \le a. \tag{7.7}$$

Proof. Once again we shall prove (7.7) for $x_0 \le x \le x_0 + a$ and for $x_0 - a \le x \le x_0$ the proof is similar. Since $p(x)$ is nondecreasing from (7.6) we find

$$\begin{aligned} u(x) \ &\le\ p(x)\left[1 + \int_{x_0}^x q(t)\exp\left(\int_t^x q(s)ds\right)dt\right] \\ &=\ p(x)\left[1 - \int_{x_0}^x \frac{d}{dt}\exp\left(\int_t^x q(s)ds\right)dt\right] \\ &=\ p(x)\exp\left(\int_{x_0}^x q(t)dt\right). \quad ∎ \end{aligned}$$

Corollary 7.6. If in Theorem 7.3 functions $p(x) = c_0 + c_1|x - x_0|$ and $q(x) = c_2$, where $c_0,\ c_1$ and c_2 are nonnegative constants, then

$$u(x) \ \le\ \left(c_0 + \frac{c_1}{c_2}\right)\exp(c_2|x - x_0|) - \frac{c_1}{c_2}. \tag{7.8}$$

Proof. For the given functions $p(x)$ and $q(x)$, in the interval $[x_0, x_0 + a]$ inequality (7.6) is the same as

$$\begin{aligned} u(x) \ &\le\ c_0 + c_1(x - x_0) + \int_{x_0}^x [c_0 + c_1(t - x_0)]c_2 e^{c_2(x-t)}dt \\ &=\ c_0 + c_1(x - x_0) + \left\{-[c_0 + c_1(t - x_0)e^{c_2(x-t)}\Big|_{x_0}^x - \frac{c_1}{c_2}e^{c_2(x-t)}\Big|_{x_0}^x\right\} \\ &=\ c_0 + c_1(x - x_0) - c_0 - c_1(x - x_0) + c_0 e^{c_2(x-x_0)} - \frac{c_1}{c_2} + \frac{c_1}{c_2}e^{c_2(x-x_0)} \\ &=\ \left(c_0 + \frac{c_1}{c_2}\right)\exp(c_2(x - x_0)) - \frac{c_1}{c_2}. \quad ∎ \end{aligned}$$

Finally, in this lecture we recall several definitions and theorems from real analysis which will be needed in Lectures 8 and 9.

Definition 7.1. The sequence of functions $\{y_m(x)\}$ is said to converge uniformly to a function $y(x)$ in the interval $[\alpha, \beta]$ if for every real number $\epsilon > 0$ there exists an integer N such that whenever $m \geq N$, $|y_m(x) - y(x)| \leq \epsilon$ for all x in $[\alpha, \beta]$.

Theorem 7.7. Let $\{y_m(x)\}$ be a sequence of continuous functions in $[\alpha, \beta]$ that converges uniformly to $y(x)$. Then $y(x)$ is continuous in $[\alpha, \beta]$.

Theorem 7.8. Let $\{y_m(x)\}$ be a sequence converging uniformly to $y(x)$ in $[\alpha, \beta]$, and let $f(x, y)$ be a continuous function in the domain D such that for all m and x in $[\alpha, \beta]$ the points $(x, y_m(x))$ are in D. Then

$$\lim_{m\to\infty} \int_\alpha^\beta f(t, y_m(t))dt \;=\; \int_\alpha^\beta \lim_{m\to\infty} f(t, y_m(t))dt \;=\; \int_\alpha^\beta f(t, y(t))dt.$$

Theorem 7.9 (Weierstrass' M-Test). Let $\{y_m(x)\}$ be a sequence of functions with $|y_m(x)| \leq M_m$ for all x in $[\alpha, \beta]$ with $\sum_{m=0}^\infty M_m < \infty$. Then $\sum_{m=0}^\infty y_m(x)$ converges uniformly in $[\alpha, \beta]$ to a unique function $y(x)$.

Definition 7.2. A set S of functions is said to be equicontinuous in an interval $[\alpha, \beta]$ if for every given $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_1,\ x_2 \in [\alpha, \beta]$, $|x_1 - x_2| \leq \delta$ then $|y(x_1) - y(x_2)| \leq \epsilon$ for all $y(x)$ in S.

Definition 7.3. A set S of functions is said to be uniformly bounded in an interval $[\alpha, \beta]$ if there exists a number M such that $|y(x)| \leq M$ for all $y(x)$ in S.

Theorem 7.10 (Ascoli–Arzela Theorem). An infinite set S of functions uniformly bounded and equicontinuous in $[\alpha, \beta]$ contains a sequence which converges uniformly in $[\alpha, \beta]$.

Theorem 7.11 (Implicit Function Theorem). Let $f(x, y)$ be defined in the strip $T = [\alpha, \beta] \times \mathbb{R}$, and continuous in x and differentiable in y, also $0 < m \leq f_y(x, y) \leq M < \infty$ for all $(x, y) \in T$. Then the equation $f(x, y) = 0$ has a unique continuous solution $y(x)$ in $[\alpha, \beta]$.

Problems

7.1. Show that the initial value problem

$$y'' \;=\; f(x, y), \quad y(x_0) \;=\; y_0, \quad y'(x_0) \;=\; y_1, \tag{7.9}$$

where $f(x,y)$ is continuous in a domain D containing the point (x_0, y_0), is equivalent to the integral equation

$$y(x) \; = \; y_0 + (x - x_0)y_1 + \int_{x_0}^{x} (x-t)f(t, y(t))dt.$$

7.2. Find the domains in which the following functions satisfy the Lipschitz condition (7.3), also find the Lipschitz constants:

(i) $\dfrac{y}{(1+x^2)}$. (ii) $\dfrac{x}{(1+y^2)}$. (iii) $x^2\cos^2 y + y\sin^2 x$.

(iv) $|xy|$. (v) $y + [x]$. (vi) $x^2y^2 + xy + 1$.

7.3. By computing appropriate Lipschitz constants, show that the following functions satisfy the Lipschitz condition in the given domains:

(i) $x\sin y + y\cos x, \quad |x| \le a, \quad |y| \le b.$

(ii) $x^3 e^{-xy^2}, \quad 0 \le x \le a, \quad |y| < \infty.$

(iii) $x^2 e^{x+y}, \quad |x| \le a, \quad |y| \le b.$

(iv) $p(x)y + q(x), \quad |x| \le 1, \quad |y| < \infty$ where $p(x)$ and $q(x)$ are continuous functions in the interval $|x| \le 1$.

7.4. Show that the following functions do not satisfy the Lipschitz condition (7.3) in the given domains:

(i) $$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}, \quad |x| \le 1, \quad |y| \le 2.$$

(ii) $$f(x,y) = \begin{cases} \dfrac{\sin y}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad |x| \le 1, \quad |y| < \infty.$$

7.5. Let $u(x)$ be a nonnegative continuous function in the interval $|x - x_0| \le a$, and $C \ge 0$ be a given constant, and

$$u(x) \; \le \; \left| \int_{x_0}^{x} C u^{\alpha}(t) dt \right|, \quad 0 < \alpha < 1.$$

Show that for all x in $|x - x_0| \le a$,

$$u(x) \; \le \; \left[C(1-\alpha)|x - x_0| \right]^{(1-\alpha)^{-1}}.$$

7.6. Let c_0 and c_1 be nonnegative constants, and $u(x)$ and $q(x)$ be nonnegative continuous functions for all $x \ge 0$ satisfying

$$u(x) \; \le \; c_0 + c_1 \int_0^x q(t)u^2(t)dt.$$

Show that for all $x \ge 0$ for which $c_0 c_1 \int_0^x q(t)dt < 1$,

$$u(x) \;\le\; c_0 \left[1 - c_0 c_1 \int_0^x q(t)dt\right]^{-1}.$$

7.7. Suppose that $y = y(x)$ is a solution of the initial value problem $y' = yg(x,y),\; y(0) = 1$ on the interval $[0,\beta]$, where $g(x,y)$ is a bounded and continuous function in the (x,y) plane. Show that there exists a constant C such that $|y(x)| \le e^{Cx}$ for all $x \in [0,\beta]$.

∗7.8. Suppose $\alpha > 0,\; \gamma > 0,\; c_0,\; c_1,\; c_2$ are nonnegative constants and $u(x)$ is a nonnegative bounded continuous solution of either the inequality

$$u(x) \;\le\; c_0 e^{-\alpha x} + c_1 \int_0^x e^{-\alpha(x-t)} u(t)dt + c_2 \int_0^\infty e^{-\gamma t} u(x+t)dt, \quad x \ge 0,$$

or the inequality

$$u(x) \;\le\; c_0 e^{\alpha x} + c_1 \int_x^0 e^{\alpha(x-t)} u(t)dt + c_2 \int_{-\infty}^0 e^{\gamma t} u(x+t)dt, \quad x \le 0.$$

If

$$\beta \;=\; \frac{c_1}{\alpha} + \frac{c_2}{\gamma} \;<\; 1,$$

then in either case, show that

$$u(x) \;\le\; (1-\beta)^{-1} c_0 e^{-[\alpha - (1-\beta)^{-1} c_1]|x|}.$$

∗7.9. Suppose $a,\; b,\; c$ are nonnegative continuous functions on $[0,\infty)$ and $u(x)$ is a nonnegative bounded continuous solution of the inequality

$$u(x) \;\le\; a(x) + \int_0^x b(x-t)u(t)dt + \int_0^\infty c(t)u(x+t)dt, \quad x \ge 0,$$

where $a(x) \to 0,\; b(x) \to 0$ as $x \to \infty$. If

$$\int_0^\infty [b(t) + c(t)]dt \;<\; 1,$$

then show that $u(x) \to 0$ as $x \to \infty$.

7.10. Show that the sequence $\{nx/(nx+1)\},\; 0 \le x \le 1$ converges pointwise to the function $f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \le 1. \end{cases}$

7.11. Show that the sequence $\left\{nx^2/(nx+1)\right\},\; 0 \le x \le 1$ converges uniformly to the function $f(x) = x$. Further, verify that

$$\lim_{n\to\infty} \int_0^1 \frac{nx^2}{nx+1}dx \;=\; \int_0^1 \lim_{n\to\infty} \frac{nx^2}{nx+1}dx \;=\; \frac{1}{2}.$$

***7.12.** Show the following:

(i) In the Ascoli–Arzela theorem (Theorem 7.10), the interval $[\alpha, \beta]$ can be replaced by any finite open interval (α, β).

(ii) The Ascoli–Arzela theorem remains true if instead of uniform boundedness on the whole interval (α, β), we have $|f(x_0)| < M$ for every $f \in S$ and some $x_0 \in (\alpha, \beta)$.

Answers or Hints

7.1.
$$\begin{aligned} y(x) &= y_0 + (x - x_0)y_1 + \int_{x_0}^{x} \left[\int_{x_0}^{t} f(s, y(s))ds\right] dt \\ &= y_0 + (x - x_0)y_1 + \left[t \int_{x_0}^{t} f(s, y(s))ds\Big|_{x_0}^{x} - \int_{x_0}^{x} tf(t, y(t))dt\right] \\ &= y_0 + (x - x_0)y_1 + \int_{x_0}^{x}(x - t)f(t, y(t))dt. \end{aligned}$$

7.2. (i) $\mathbb{R}^2$, 1. (ii) $|x| \le a,\ |y| < \infty,\ (3\sqrt{3}/8)a$. (iii) $|x| \le a,\ |y| < \infty,\ a^2 + 1$. (iv) $|x| \le a,\ |y| < \infty,\ a$. (v) $\mathbb{R}^2$, 1. (vi) $|x| \le a,\ |y| \le b,\ 2a^2 b + a$.

7.3. (i) $a + 1$. (ii) $\max\{2a^3, 2a^4\}$. (iii) $a^2 e^{a+b}$. (iv) $\max_{-1 \le x \le 1} |p(x)|$.

7.4. (i) $|f(x, y) - f(x, 0)| = |x^3 y/(x^4 + y^2)| \le L|y|$, i.e., $|x^3/(x^4 + y^2)| \le L$; however, along the curve $y = x^2$ this is impossible. (ii) $|f(x, y) - f(x, 0)| = |x^{-1} \sin y| \le L|y|$; but, this is impossible.

7.5. For $x \in [x_0, x_0 + a]$ let $r(x) = \int_{x_0}^{x} Cu^{\alpha}(t)dt$ so that $r'(x) < C(r(x) + \epsilon)^{\alpha}$, where $\epsilon > 0$ and $r(x_0) = 0$. Integrate this inequality and then let $\epsilon \to 0$.

7.6. Let $r(x) = c_0 + c_1 \int_0^x q(t)u^2(t)dt$ so that $r'(x) < c_1 q(x)(r(x) + \epsilon)^2$, where $\epsilon > 0$ and $r(0) = c_0$. Integrate this inequality and then let $\epsilon \to 0$.

7.7. Use Corollary 7.6.

7.10. Verify directly.

7.11. Verify directly.