Lecture 6
Second-Order Linear Equations

Consider the homogeneous linear second-order DE with variable coefficients

$$p_0(x)y'' + p_1(x)y' + p_2(x)y \;=\; 0, \tag{6.1}$$

where $p_0(x)$ (> 0), $p_1(x)$ and $p_2(x)$ are continuous in J. There does not exist any method to solve it except in a few rather restrictive cases. However, the results below follow immediately from the general theory of first-order linear systems, which we shall present in later lectures.

Theorem 6.1. There exist exactly two solutions $y_1(x)$ and $y_2(x)$ of (6.1) which are linearly independent (essentially different) in J; i.e., there does not exist a constant c such that $y_1(x) = cy_2(x)$ for all $x \in J$.

Theorem 6.2. Two solutions $y_1(x)$ and $y_2(x)$ of (6.1) are linearly independent in J if and only if their *Wronskian* defined by

$$W(x) \;=\; W(y_1, y_2)(x) \;=\; \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \;=\; y_1(x)y_2'(x) - y_2(x)y_1'(x) \tag{6.2}$$

is different from zero for some $x = x_0$ in J.

Theorem 6.3. For the Wronskian defined in (6.2) the following Abel's identity (also known as the Ostrogradsky–Liouville formula) holds:

$$W(x) \;=\; W(x_0)\exp\left(-\int_{x_0}^{x}\frac{p_1(t)}{p_0(t)}dt\right), \quad x_0 \in J. \tag{6.3}$$

Thus, if the Wronskian is zero at some $x_0 \in J$, then it is zero for all $x \in J$.

Theorem 6.4. If $y_1(x)$ and $y_2(x)$ are solutions of (6.1) and c_1 and c_2 are arbitrary constants, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (6.1). Further, if $y_1(x)$ and $y_2(x)$ are linearly independent, then any solution $y(x)$ of (6.1) can be written as $y(x) = \overline{c}_1y_1(x) + \overline{c}_2y_2(x)$, where $\overline{c}_1$ and $\overline{c}_2$ are suitable constants.

Now we shall show that, if one solution $y_1(x)$ of (6.1) is known (by some clever method), then we can employ variation of parameters to find the second solution of (6.1). For this we let $y(x) = u(x)y_1(x)$ and substitute this in (6.1) to get

$$p_0(uy_1)'' + p_1(uy_1)' + p_2(uy_1) \;=\; 0,$$

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or

$$p_0u''y_1 + 2p_0u'y_1' + p_0uy_1'' + p_1u'y_1 + p_1uy_1' + p_2uy_1 \;=\; 0,$$

or

$$p_0u''y_1 + (2p_0y_1' + p_1y_1)u' + (p_0y_1'' + p_1y_1' + p_2y_1)u \;=\; 0.$$

However, since y_1 is a solution of (6.1), the above equation with $v = u'$ is the same as

$$p_0y_1v' + (2p_0y_1' + p_1y_1)v \;=\; 0, \tag{6.4}$$

which is a first-order equation, and it can be solved easily provided $y_1 \neq 0$ in J. Indeed, multiplying (6.4) by y_1/p_0, we get

$$(y_1^2v' + 2y_1'y_1v) + \frac{p_1}{p_0}y_1^2v \;=\; 0,$$

which is the same as

$$(y_1^2v)' + \frac{p_1}{p_0}(y_1^2v) \;=\; 0$$

and hence

$$y_1^2v \;=\; c\exp\left(-\int^x \frac{p_1(t)}{p_0(t)}dt\right),$$

or, on taking $c = 1$,

$$v(x) \;=\; \frac{1}{y_1^2(x)}\exp\left(-\int^x \frac{p_1(t)}{p_0(t)}dt\right).$$

Hence, the second solution of (6.1) is

$$y_2(x) \;=\; y_1(x)\int^x \frac{1}{y_1^2(t)}\exp\left(-\int^t \frac{p_1(s)}{p_0(s)}ds\right)dt. \tag{6.5}$$

Example 6.1. It is easy to verify that $y_1(x) = x^2$ is a solution of the DE

$$x^2y'' - 2xy' + 2y \;=\; 0, \quad x \neq 0.$$

For the second solution we use (6.5), to obtain

$$y_2(x) \;=\; x^2\int^x \frac{1}{t^4}\exp\left(-\int^t\left(-\frac{2s}{s^2}\right)ds\right)dt \;=\; x^2\int^x \frac{1}{t^4}t^2dt \;=\; -x.$$

Now we shall find a particular solution of the nonhomogeneous equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y \;=\; r(x). \tag{6.6}$$

For this also we shall apply the method of variation of parameters. Let $y_1(x)$ and $y_2(x)$ be two solutions of (6.1). We assume $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$ is a solution of (6.6). Note that $c_1(x)$ and $c_2(x)$ are two unknown functions, so we can have two sets of conditions which determine $c_1(x)$ and $c_2(x)$. Since

$$y' = c_1 y_1' + c_2 y_2' + c_1' y_1 + c_2' y_2$$

as a first condition, we assume that

$$c_1' y_1 + c_2' y_2 = 0. \tag{6.7}$$

Thus, we have

$$y' = c_1 y_1' + c_2 y_2'$$

and on differentiation

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'.$$

Substituting these in (6.6), we find

$$c_1(p_0 y_1'' + p_1 y_1' + p_2 y_1) + c_2(p_0 y_2'' + p_1 y_2' + p_2 y_2) + p_0(c_1' y_1' + c_2' y_2') = r(x).$$

Clearly, this equation in view of the fact that $y_1(x)$ and $y_2(x)$ are solutions of (6.1) is the same as

$$c_1' y_1' + c_2' y_2' = \frac{r(x)}{p_0(x)}. \tag{6.8}$$

Solving (6.7), (6.8) we find

$$c_1' = -\frac{y_2(x) r(x)/p_0(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}}, \qquad c_2' = \frac{y_1(x) r(x)/p_0(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}};$$

and hence a particular solution of (6.6) is

$$\begin{aligned} y_p(x) &= c_1(x) y_1(x) + c_2(x) y_2(x) \\ &= -y_1(x) \int^x \frac{y_2(t) r(t)/p_0(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} dt \; + \; y_2(x) \int^x \frac{y_1(t) r(t)/p_0(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} dt \\ &= \int^x H(x,t) \frac{r(t)}{p_0(t)} dt, \end{aligned} \tag{6.9}$$

where

$$H(x,t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} \Bigg/ \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}. \tag{6.10}$$

The general solution of (6.6) which is obtained by adding this particular solution with the general solution of (6.1) appears as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \tag{6.11}$$

The following properties of the function $H(x,t)$ are immediate:

(i) $H(x,t)$ is defined for all $(x,t) \in J \times J$;

(ii) $\partial^j H(x,t)/\partial x^j, \quad j = 0,1,2$ are continuous for all $(x,t) \in J \times J$;

(iii) for each fixed $t \in J$ the function $z(x) = H(x,t)$ is a solution of the homogeneous DE (6.1) satisfying $z(t) = 0, \; z'(t) = 1$; and

(iv) the function

$$v(x) \;=\; \int_{x_0}^{x} H(x,t)\frac{r(t)}{p_0(t)}dt$$

is a particular solution of the nonhomogeneous DE (6.6) satisfying $y(x_0) = y'(x_0) = 0$.

Example 6.2. Consider the DE

$$y'' + y \;=\; \cot x.$$

For the corresponding homogeneous DE $y'' + y = 0$, $\sin x$ and $\cos x$ are the solutions. Thus, its general solution can be written as

$$\begin{aligned}
y(x) &= c_1\cos x + c_2\sin x + \int^x \frac{\begin{vmatrix} \sin t & \cos t \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix}} \frac{\cos t}{\sin t}dt \\
&= c_1\cos x + c_2\sin x - \int^x (\sin t\cos x - \sin x\cos t)\frac{\cos t}{\sin t}dt \\
&= c_1\cos x + c_2\sin x - \cos x\int^x \cos t dt + \sin x\int^x \frac{\cos^2 t}{\sin t}dt \\
&= c_1\cos x + c_2\sin x - \cos x\sin x + \sin x\int^x \frac{1-\sin^2 t}{\sin t}dt \\
&= c_1\cos x + c_2\sin x - \cos x\sin x - \sin x\int^x \sin t dt + \sin x\int^x \frac{1}{\sin t}dt \\
&= c_1\cos x + c_2\sin x + \sin x\int^x \frac{\operatorname{cosec} t(\operatorname{cosec} t - \cot t)}{(\operatorname{cosec} t - \cot t)}dt \\
&= c_1\cos x + c_2\sin x + \sin x \; \ln[\operatorname{cosec} x - \cot x].
\end{aligned}$$

From the general theory of first-order linear systems, which we shall present in later lectures, it also follows that if the functions $p_0(x)$ (> 0), $p_1(x)$, $p_2(x)$, and $r(x)$ are continuous on J and $x_0 \in J$, then the initial value problem: (6.6) together with the *initial conditions*

$$y(x_0) \;=\; y_0, \quad y'(x_0) \;=\; y_1 \tag{6.12}$$

has a unique solution.

Now we shall show that second-order DEs with constant coefficients can be solved explicitly. In fact, to find the solution of the equation

$$y'' + ay' + by \ = \ 0, \tag{6.13}$$

where a and b are constants, as a first step we look back at the equation $y' + ay = 0$ (a is a constant) for which all solutions are multiples of $y = e^{-ax}$. Thus, for (6.13) also some form of exponential function would be a reasonable choice and would utilize the property that the differentiation of an exponential function e^{rx} always yields a constant multiplied by e^{rx}.

Thus, we try $y = e^{rx}$ and find the value(s) of r. For this, we have

$$r^2e^{rx} + are^{rx} + be^{rx} \ = \ (r^2 + ar + b)e^{rx} \ = \ 0,$$

which gives

$$r^2 + ar + b \ = \ 0. \tag{6.14}$$

Hence, e^{rx} is a solution of (6.13) if r is a solution of (6.14). Equation (6.14) is called the *characteristic equation.* For the roots of (6.14) we have the following three cases:

1. Distinct real roots. If r_1 and r_2 are real and distinct roots of (6.14), then e^{r_1x} and e^{r_2x} are two solutions of (6.13) and its general solution can be written as

$$y(x) \ = \ c_1e^{r_1x} + c_2e^{r_2x}.$$

In the particular case when $r_1 = r, \ r_2 = -r$ (then the DE (6.13) is $y'' - r^2y = 0$) we have

$$\begin{aligned} y(x) \ &= \ c_1e^{rx} + c_2e^{-rx} \ = \ \left(\frac{A+B}{2}\right)e^{rx} + \left(\frac{A-B}{2}\right)e^{-rx} \\ &= \ A\left(\frac{e^{rx}+e^{-rx}}{2}\right) + B\left(\frac{e^{rx}-e^{-rx}}{2}\right) \ = \ A\cosh rx + B\sinh rx. \end{aligned}$$

2. Repeated real roots. If $r_1 = r_2 = r$ is a repeated root of (6.14), then e^{rx} is a solution. To find the second solution, we let $y(x) = u(x)e^{rx}$ and substitute it in (6.13), to get

$$e^{rx}(u'' + 2ru' + r^2u) + ae^{ru}(u' + ru) + bue^{rx} \ = \ 0,$$

or

$$u'' + (2r + a)u' + (r^2 + ar + b)u \ = u'' + (2r + a)u' \ = \ 0.$$

Now since r is a repeated root of (6.14) it follows that $2r + a = 0$ and hence $u'' = 0$, i.e., $u(x) = c_1 + c_2x$. Thus,

$$y(x) \ = \ (c_1 + c_2x)e^{rx} \ = \ c_1e^{rx} + c_2xe^{rx}.$$

Hence, the second solution of (6.13) is xe^{rx}.

3. Complex conjugate roots. Let $r_1 = \mu + i\nu$ and $r_2 = \mu - i\nu$ where $i = \sqrt{-1}$, so that

$$e^{(\mu \pm i\nu)x} \; = \; e^{\mu x}(\cos \nu x \pm i \sin \nu x).$$

Since for the DE (6.13) real part, i.e., $e^{\mu x} \cos \nu x$ and the complex part, i.e., $e^{\mu x} \sin \nu x$ both are solutions, the general solution of (6.13) can be written as

$$y(x) \; = \; c_1 e^{\mu x} \cos \nu x + c_2 e^{\mu x} \sin \nu x.$$

In the particular case when $r_1 = i\nu$ and $r_2 = -i\nu$ (then the DE (6.13) is $y'' + \nu^2 y = 0$) we have $y(x) = c_1 \cos \nu x + c_2 \sin \nu x$.

Finally, in this lecture we shall find the solution of the *Cauchy–Euler equation*

$$x^2 y'' + axy' + by \; = \; 0, \quad x > 0. \tag{6.15}$$

We assume $y(x) = x^m$ to obtain

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m \; = \; 0,$$

or

$$m(m-1) + am + b \; = \; 0. \tag{6.16}$$

This is the characteristic equation for (6.15), and as earlier for (6.14) the nature of its roots determines the solution:

Real, distinct roots $m_1 \neq m_2$: $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$,

Real, repeated roots $m = m_1 = m_2$: $y(x) = c_1 x^m + c_2 (\ln x) x^m$,

Complex conjugate roots $m_1 = \mu + i\nu$, $m_2 = \mu - i\nu$: $y(x) = c_1 x^\mu \cos(\nu \ln x) + c_2 x^\mu \sin(\nu \ln x)$.

In the particular case

$$x^2 y'' + xy' - \lambda^2 y \; = \; 0, \quad x > 0, \quad \lambda > 0 \tag{6.17}$$

the characteristic equation is $m(m-1) + m - \lambda^2 = 0$, or $m^2 - \lambda^2 = 0$. The roots are $m = \pm\lambda$ and hence the solution of (6.17) appears as

$$y(x) \; = \; c_1 x^\lambda + c_2 x^{-\lambda}. \tag{6.18}$$

Problems

6.1. Let $y_1(x), y_2(x), y_3(x)$ and $\lambda(x)$ be differentiable functions in J. Show that for all $x \in J$,

(i) $W(y_1, y_2 + y_3)(x) = W(y_1, y_2)(x) + W(y_1, y_3)(x)$;
(ii) $W(\lambda y_1, \lambda y_2)(x) = \lambda^2(x)W(y_1, y_2)(x)$;
(iii) $W(y_1, \lambda y_1)(x) = \lambda'(x)y_1^2(x)$.

6.2. Show that the functions $y_1(x) = c\ (\neq 0)$ and $y_2(x) = 1/x^2$ satisfy the nonlinear DE $y'' + 3xyy' = 0$ in $(0, \infty)$, but $y_1(x) + y_2(x)$ does not satisfy the given DE. (This shows that Theorem 6.4 holds good only for the linear equations.)

6.3. Given the solution $y_1(x)$, find the second solution of the following DEs:

(i) $(x^2 - x)y'' + (3x - 1)y' + y = 0 \quad (x \neq 0, 1), \quad y_1(x) = (x - 1)^{-1}$.
(ii) $x(x - 2)y'' + 2(x - 1)y' - 2y = 0 \quad (x \neq 0, 2), \quad y_1(x) = (1 - x)$.
(iii) $xy'' - y' - 4x^3 y = 0 \quad (x \neq 0), \quad y_1(x) = \exp(x^2)$.
(iv) $(1 - x^2)y'' - 2xy' + 2y = 0 \quad (|x| < 1), \quad y_1(x) = x$.

6.4. The differential equation

$$xy'' - (x + n)y' + ny = 0$$

is interesting because it has an exponential solution and a polynomial solution.

(i) Verify that one solution is $y_1(x) = e^x$.
(ii) Show that the second solution has the form $y_2(x) = ce^x \int^x t^n e^{-t} dt$. Further, show that with $c = -1/n!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Note that $y_2(x)$ is the first $n+1$ terms of the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

6.5. For the differential equation

$$y'' + \delta(xy' + y) = 0,$$

verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution. Find its second solution.

6.6. Let $y_1(x) \neq 0$ and $y_2(x)$ be two linearly independent solutions of the DE (6.1). Show that $y(x) = y_2(x)/y_1(x)$ is a nonconstant solution of the DE

$$y_1(x)y'' + \left(2y_1'(x) + \frac{p_1(x)}{p_0(x)}y_1(x)\right)y' = 0.$$

6.7. Let $y_1(x)$ and $y_2(x)$ be solutions of the DE

$$y'' + p_1(x)y' + p_2(x)y = 0 \qquad (6.19)$$

in J. Show the following:

(i) If $y_1(x)$ and $y_2(x)$ vanish at the same point in J, then $y_1(x)$ is a constant multiple of $y_2(x)$.

(ii) If $y_1(x)$ and $y_2(x)$ have maxima or minima at the same point in the open interval J, then $y_1(x)$ and $y_2(x)$ are not the linearly independent solutions.

(iii) If $W(y_1, y_2)(x)$ is independent of x, then $p_1(x) = 0$ for all $x \in J$.

(iv) If $y_1(x)$ and $y_2(x)$ are linearly independent, then $y_1(x)$ and $y_2(x)$ cannot have a common point of inflexion in J unless $p_1(x)$ and $p_2(x)$ vanish simultaneously there.

(v) If $W(y_1, y_2)(x^*) = y_1(x^*) = 0$, then either $y_1(x) = 0$ for all $x \in J$, or $y_2(x) = (y_2'(x^*)/y_1'(x^*))y_1(x)$.

6.8. Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of (6.19), and $W(x)$ be their Wronskian. Show that

$$y'' + p_1(x)y' + p_2(x)y \;=\; \frac{W}{y_1}\frac{d}{dx}\left(\frac{y_1^2}{W}\frac{d}{dx}\left(\frac{y}{y_1}\right)\right).$$

6.9. Show that the DE (6.1) can be transformed into a first-order nonlinear DE by means of a change of dependent variable

$$y \;=\; \exp\left(\int^x f(t)w(t)dt\right),$$

where $f(x)$ is any nonvanishing differentiable function. In particular, if $f(x) = p_0(x)$, then show that (6.1) reduces to the Riccati equation,

$$w' + p_0(x)w^2 + \frac{p_0'(x) + p_1(x)}{p_0(x)}w + \frac{p_2(x)}{p_0^2(x)} \;=\; 0. \qquad (6.20)$$

6.10. If $w_1(x)$ and $w_2(x)$ are two different solutions of the DE (6.20) with $p_0(x) = 1$, i.e.,

$$w' + w^2 + p_1(x)w + p_2(x) \;=\; 0, \qquad (6.21)$$

then show that its general solution $w(x)$ is given by

$$\frac{w(x) - w_1(x)}{w(x) - w_2(x)}\exp\left(\int^x (w_1(t) - w_2(t))dt\right) \;=\; c_1.$$

Further, if $w_3(x)$ is another known solution of (6.21), then

$$\frac{w(x) - w_3(x)}{w(x) - w_2(x)} \;=\; c_2\frac{w_1(x) - w_3(x)}{w_1(x) - w_2(x)}.$$

6.11. Find the general solution of the following homogeneous DEs:

(i) $y'' + 7y' + 10y = 0$.

(ii) $y'' - 8y' + 16y = 0$.

(iii) $y'' + 2y' + 3y = 0$.

6.12. Find the general solution of the following nonhomogeneous DEs:

(i) $y'' + 4y = \sin 2x$.

(ii) $y'' + 4y' + 3y = e^{-3x}$.

(iii) $y'' + 5y' + 4y = e^{-4x}$.

6.13. Show that if the real parts of all solutions of (6.14) are negative, then $\lim_{x\to\infty} y(x) = 0$ for every solution of (6.13).

6.14. Show that the solution of the initial value problem

$$y'' - 2(r+\beta)y' + r^2 y \;=\; 0, \quad y(0) \;=\; 0, \quad y'(0) \;=\; 1$$

can be written as

$$y_\beta(x) \;=\; \frac{1}{2\sqrt{\beta(2r+\beta)}}\left[e^{[r+\beta+\sqrt{\beta(2r+\beta)}]x} - e^{[r+\beta-\sqrt{\beta(2r+\beta)}]x}\right].$$

Further, show that $\lim_{\beta\to 0} y_\beta(x) = xe^{rx}$.

6.15. Verify that $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of

$$x^3 y'' + x^2 y' - xy \;=\; 0.$$

Use this information and the variation of parameters method to find the general solution of

$$x^3 y'' + x^2 y' - xy \;=\; x/(1+x).$$

Answers or Hints

6.1. Use the definition of Wronskian.

6.2. Verify directly.

6.3. (i) $\ln x/(x-1)$. (ii) $(1/2)(1-x)\ln[(x-2)/x]-1$. (iii) e^{-x^2}. (iv) $(x/2)\times \ln[(1+x)/(1-x)] - 1$.

6.4. (i) Verify directly. (ii) Use (6.5).

6.5. $e^{-\delta x^2/2}\int^x e^{\delta t^2/2}dt$.

6.6. Use $y_2(x) = y_1(x)y(x)$ and the fact that $y_1(x)$ and $y_2(x)$ are solutions.

6.7. (i) Use Abel's identity. (ii) If both attain maxima or minima at x_0, then $\phi_1'(x_0) = \phi_2'(x_0) = 0$. (iii) Use Abel's identity. (iv) If x_0 is a common point of inflexion, then $\phi_1''(x_0) = \phi_2''(x_0) = 0$. (v) $W(x^*) = 0$ implies $\phi_2(x) = c\phi_1(x)$. If $\phi_1'(x^*) = 0$, then $\phi_1(x) \equiv 0$, and if $\phi_1'(x^*) \neq 0$ then $c = \phi_2'(x^*)/\phi_1'(x^*)$.

6.8. Directly show right-hand side is the same as left-hand side.

6.9. Verify directly.

6.10. Use the substitution $w = z + w_1$ to obtain $z' + (2w_1 + p_1(x))z + z^2 = 0$, which is a Bernoulli equation whose multiplier is $z^{-2}\exp(-\int^x (2u_1 + p_1)dt)$. Hence, if w_1 is a solution of (6.21), then its integrating factor is $(w - w_1)^{-2}\exp(-\int^x (2u_1 + p_1)dt)$. Now use Theorem 3.4.

6.11. (i) $c_1e^{-2x} + c_2e^{-5x}$. (ii) $(c_1 + c_2x)e^{4x}$. (iii) $c_1e^{-x}\cos\sqrt{2}x + c_2e^{-x} \times \sin\sqrt{2}x$.

6.12. (i) $c_1\cos 2x + c_2\sin 2x - \frac{1}{4}x\cos 2x$. (ii) $c_1e^{-x} + c_2e^{-3x} - \frac{1}{2}xe^{-3x}$ (iii) $c_1e^{-x} + c_2e^{-4x} - \frac{1}{3}xe^{-4x}$.

6.13. Use explicit forms of the solution.

6.14. Note that $\sqrt{\beta(\beta + 2r)} \to 0$ as $\beta \to 0$.

6.15. $c_1x + (c_2/x) + (1/2)[(x - (1/x))\ln(1 + x) - x\ln x - 1]$.