Lecture 5
First-Order Linear Equations

Let in the DE (3.1) the functions M and N be $p_1(x)y - r(x)$ and $p_0(x)$, respectively, then it becomes

$$p_0(x)y' + p_1(x)y = r(x), \tag{5.1}$$

which is a first-order linear DE. In (5.1) we shall assume that the functions $p_0(x),\ p_1(x),\ r(x)$ are continuous and $p_0(x) \neq 0$ in J. With these assumptions the DE (5.1) can be written as

$$y' + p(x)y = q(x), \tag{5.2}$$

where $p(x) = p_1(x)/p_0(x)$ and $q(x) = r(x)/p_0(x)$ are continuous functions in J.

The corresponding homogeneous equation

$$y' + p(x)y = 0 \tag{5.3}$$

obtained by taking $q(x) \equiv 0$ in (5.2) can be solved by separating the variables, i.e., $(1/y)y' + p(x) = 0$, and now integrating it to obtain

$$y(x) = c\exp\left(-\int^x p(t)dt\right). \tag{5.4}$$

In dividing (5.3) by y we have lost the solution $y(x) \equiv 0$, which is called the *trivial solution* (for a linear homogeneous DE $y(x) \equiv 0$ is always a solution). However, it is included in (5.4) with $c = 0$.

If $x_0 \in J$, then the function

$$y(x) = y_0 \exp\left(-\int_{x_0}^x p(t)dt\right) \tag{5.5}$$

clearly satisfies the DE (5.3) in J and passes through the point (x_0, y_0). Thus, it is the solution of the initial value problem (5.3), (1.10).

To find the solution of the DE (5.2) we shall use *the method of variation of parameters* due to Lagrange. In (5.4) we assume that c is a function of x, i.e.,

$$y(x) = c(x)\exp\left(-\int^x p(t)dt\right) \tag{5.6}$$

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and search for $c(x)$ so that (5.6) becomes a solution of the DE (5.2). For this, substituting (5.6) into (5.2), we find

$$\begin{aligned} c'(x)\exp\left(-\int^x p(t)dt\right) & -c(x)p(x)\exp\left(-\int^x p(t)dt\right) \\ & +c(x)p(x)\exp\left(-\int^x p(t)dt\right) \;=\; q(x), \end{aligned}$$

which is the same as

$$c'(x) \;=\; q(x)\exp\left(\int^x p(t)dt\right). \tag{5.7}$$

Integrating (5.7), we obtain the required function

$$c(x) \;=\; c_1 + \int^x q(t)\exp\left(\int^t p(s)ds\right)dt.$$

Now substituting this $c(x)$ in (5.6), we find the solution of (5.2) as

$$y(x) \;=\; c_1\exp\left(-\int^x p(t)dt\right) + \int^x q(t)\exp\left(-\int_t^x p(s)ds\right)dt. \tag{5.8}$$

This solution $y(x)$ is of the form $c_1u(x) + v(x)$. It is to be noted that $c_1u(x)$ is the general solution of (5.3) and $v(x)$ is a particular solution of (5.2). Hence, the general solution of (5.2) is obtained by adding any particular solution of (5.2) to the general solution of (5.3).

From (5.8) the solution of the initial value problem (5.2), (1.10) where $x_0 \in J$ is easily obtained as

$$y(x) \;=\; y_0\exp\left(-\int_{x_0}^x p(t)dt\right) + \int_{x_0}^x q(t)\exp\left(-\int_t^x p(s)ds\right)dt. \tag{5.9}$$

This solution in the particular case when $p(x) \equiv p$ and $q(x) \equiv q$ simply reduces to

$$y(x) \;=\; \left(y_0 - \frac{q}{p}\right)e^{-p(x-x_0)} + \frac{q}{p}.$$

Example 5.1. Consider the initial value problem

$$xy' - 4y + 2x^2 + 4 \;=\; 0, \quad x \neq 0, \quad y(1) = 1. \tag{5.10}$$

Since $x_0 = 1,\ y_0 = 1,\ p(x) = -4/x$ and $q(x) = -2x - 4/x$, from (5.9) the solution of (5.10) can be written as

$$\begin{aligned} y(x) &= \exp\left(\int_1^x \frac{4}{t}dt\right) + \int_1^x \left(-2t - \frac{4}{t}\right) \exp\left(\int_t^x \frac{4}{s}ds\right) dt \\ &= x^4 + \int_1^x \left(-2t - \frac{4}{t}\right) \frac{x^4}{t^4} dt \\ &= x^4 + x^4 \left(\frac{1}{x^2} + \frac{1}{x^4} - 2\right) \;=\; -x^4 + x^2 + 1. \end{aligned}$$

Alternatively, instead of using (5.9), we can find the solution of (5.10) as follows. For the corresponding homogeneous DE $y' - (4/x)y = 0$ the general solution is cx^4, and a particular solution of the DE (5.10) is

$$\int^x \left(-2t - \frac{4}{t}\right) \exp\left(\int_t^x \frac{4}{s}ds\right) dt \;=\; x^2 + 1,$$

and hence the general solution of the DE (5.10) is $y(x) = cx^4 + x^2 + 1$. Now in order to satisfy the initial condition $y(1) = 1$ it is necessary that $1 = c + 1 + 1$, or $c = -1$. The solution of (5.10) is therefore $y(x) = -x^4 + x^2 + 1$.

Suppose $y_1(x)$ and $y_2(x)$ are two particular solutions of (5.2), then

$$\begin{aligned} y_1'(x) - y_2'(x) &= -p(x)y_1(x) + q(x) + p(x)y_2(x) - q(x) \\ &= -p(x)(y_1(x) - y_2(x)), \end{aligned}$$

which implies that $y(x) = y_1(x) - y_2(x)$ is a solution of (5.3). Thus, if two particular solutions of (5.2) are known, then $y(x) = c(y_1(x) - y_2(x)) + y_1(x)$ as well as $y(x) = c(y_1(x) - y_2(x)) + y_2(x)$ represents the general solution of (5.2). For example, $x + 1/x$ and x are two solutions of the DE $xy' + y = 2x$ and $y(x) = c/x + x$ is its general solution.

The DE $(xf(y) + g(y))y' = h(y)$ may not be integrable as it is, but if the roles of x and y are interchanged, then it can be written as

$$h(y)\frac{dx}{dy} - f(y)x \;=\; g(y),$$

which is a linear DE in x and can be solved by the preceding procedure. In fact, the solutions of (1.9) and $dx/dy = 1/f(x, y)$ determine the same curve in a region in $\mathbb{R}^2$ provided the function f is defined, continuous, and nonzero. For this, if $y = y(x)$ is a solution of (1.9) in J and $y'(x) = f(x, y(x)) \neq 0$, then $y(x)$ is monotonic function in J and hence has an inverse $x = x(y)$. This function x is such that

$$\frac{dx}{dy} \;=\; \frac{1}{y'(x)} \;=\; \frac{1}{f(x, y(x))} \quad \text{in} \quad J.$$

Example 5.2. The DE

$$y' = \frac{1}{(e^{-y} - x)}$$

can be written as $dx/dy + x = e^{-y}$ which can be solved to obtain $x = e^{-y}(y + c)$.

Certain nonlinear first-order DEs can be reduced to linear equations by an appropriate change of variables. For example, it is always possible for the *Bernoulli equation*

$$p_0(x)y' + p_1(x)y = r(x)y^n, \quad n \neq 0, 1. \tag{5.11}$$

In (5.11), $n = 0$ and 1 are excluded because in these cases this equation is obviously linear.

The equation (5.11) is equivalent to the DE

$$p_0(x)y^{-n}y' + p_1(x)y^{1-n} = r(x) \tag{5.12}$$

and now the substitution $v = y^{1-n}$ leads to the first-order linear DE

$$\frac{1}{1-n}p_0(x)v' + p_1(x)v = r(x). \tag{5.13}$$

Example 5.3. The DE $xy' + y = x^2y^2$, $x \neq 0$ can be written as $xy^{-2}y' + y^{-1} = x^2$. The substitution $v = y^{-1}$ converts this DE into $-xv' + v = x^2$, which can be solved to get $v = (c - x)x$, and hence the general solution of the given DE is $y(x) = (cx - x^2)^{-1}$.

As we have remarked in Lecture 2, we shall show that if one solution $y_1(x)$ of the Riccati equation (2.14) is known, then the substitution $y = y_1 + z^{-1}$ converts it into a first-order linear DE in z. Indeed, we have

$$\begin{aligned} y_1' - \frac{1}{z^2}z' &= p(x)\left(y_1 + \frac{1}{z}\right)^2 + q(x)\left(y_1 + \frac{1}{z}\right) + r(x) \\ &= (p(x)y_1^2 + q(x)y_1 + r(x)) + p(x)\left(\frac{2y_1}{z} + \frac{1}{z^2}\right) + q(x)\frac{1}{z} \end{aligned}$$

and hence

$$-\frac{1}{z^2}z' = (2p(x)y_1 + q(x))\frac{1}{z} + p(x)\frac{1}{z^2},$$

which is the first-order linear DE

$$z' + (2p(x)y_1 + q(x))z + p(x) = 0. \tag{5.14}$$

Example 5.4. It is easy to verify that $y_1 = x$ is a particular solution of the Riccati equation $y' = 1 + x^2 - 2xy + y^2$. The substitution $y = x + z^{-1}$

converts this DE to the first-order linear DE $z' + 1 = 0$, whose general solution is $z = (c - x),\ x \neq c$. Thus, the general solution of the given Riccati equation is $y(x) = x + 1/(c - x),\ x \neq c$.

In many physical problems the nonhomogeneous term $q(x)$ in (5.2) is specified by different formulas in different intervals. This is often the case when (5.2) is considered as an *input–output* relation, i.e., the function $q(x)$ is an *input* and the solution $y(x)$ is an *output* corresponding to the input $q(x)$. Usually, in such situations the solution $y(x)$ is not defined at certain points, so that it is not continuous throughout the interval of interest. To understand such a case, for simplicity, we consider the initial value problem (5.2), (1.10) in the interval $[x_0, x_2]$, where the function $p(x)$ is continuous, and

$$q(x) \;=\; \begin{cases} q_1(x), & x_0 \le x < x_1 \\ q_2(x), & x_1 < x \le x_2. \end{cases}$$

We assume that the functions $q_1(x)$ and $q_2(x)$ are continuous in the intervals $[x_0, x_1)$ and $(x_1, x_2]$, respectively. With these assumptions the "solution" $y(x)$ of (5.2), (1.10) in view of (5.9) can be written as

$$y(x) = \begin{cases} y_1(x) = y_0 \exp\left(-\int_{x_0}^{x} p(t)dt\right) + \int_{x_0}^{x} q_1(t) \exp\left(-\int_{t}^{x} p(s)ds\right) dt, \\ \hfill x_0 \le x < x_1 \\ y_2(x) = c \exp\left(-\int_{x_1}^{x} p(t)dt\right) + \int_{x_1}^{x} q_2(t) \exp\left(-\int_{t}^{x} p(s)ds\right) dt, \\ \hfill x_1 < x \le x_2. \end{cases}$$

Clearly, at the point x_1 we cannot say much about the solution $y(x)$, it may not even be defined. However, if the limits $\lim_{x\to x_1^-} y_1(x)$ and $\lim_{x\to x_1^+} y_2(x)$ exist (which are guaranteed if both the functions $q_1(x)$ and $q_2(x)$ are bounded at $x = x_1$), then the relation

$$\lim_{x\to x_1^-} y_1(x) \;=\; \lim_{x\to x_1^+} y_2(x) \tag{5.15}$$

determines the constant c, so that the solution $y(x)$ is continuous on $[x_0, x_2]$.

Example 5.5. Consider the initial value problem

$$\begin{aligned} y' - \frac{4}{x}y \;&=\; \begin{cases} -2x - \dfrac{4}{x}, & x \in [1, 2) \\ x^2, & x \in (2, 4] \end{cases} \\ y(1) \;&=\; 1. \end{aligned} \tag{5.16}$$

In view of Example 5.1 the solution of (5.16) can be written as

$$y(x) \;=\; \begin{cases} -x^4 + x^2 + 1, & x \in [1, 2) \\ c\dfrac{x^4}{16} + \dfrac{x^4}{2} - x^3, & x \in (2, 4]. \end{cases}$$

Now the relation (5.15) gives $c = -11$. Thus, the continuous solution of (5.16) is

$$y(x) = \begin{cases} -x^4 + x^2 + 1, & x \in [1, 2) \\ -\dfrac{3}{16}x^4 - x^3, & x \in (2, 4]. \end{cases}$$

Clearly, this solution is not differentiable at $x = 2$.

Problems

5.1. Show that the DE (5.2) admits an integrating factor which is a function of x alone. Use this to obtain its general solution.

5.2. (Principle of Superposition). If $y_1(x)$ and $y_2(x)$ are solutions of $y' + p(x)y = q_i(x),\ i = 1, 2$, respectively, then show that $c_1y_1(x) + c_2y_2(x)$ is a solution of the DE $y' + p(x)y = c_1q_1(x) + c_2q_2(x)$, where c_1 and c_2 are constants.

5.3. Find the general solution of the following DEs:

(i) $y' - (\cot x)y = 2x \sin x$.

(ii) $y' + y + x + x^2 + x^3 = 0$.

(iii) $(y^2 - 1) + 2(x - y(1 + y)^2)y' = 0$.

(iv) $(1 + y^2) = (\tan^{-1} y - x)y'$.

5.4. Solve the following initial value problems:

(i) $y' + 2y = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}, \quad y(0) = 0$.

(ii) $y' + p(x)y = 0, \quad y(0) = 1$, where $p(x) = \begin{cases} 2, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}$

5.5. Let $q(x)$ be continuous in $[0, \infty)$ and $\lim_{x\to\infty} q(x) = L$. For the DE $y' + ay = q(x)$, show the following:

(i) If $a > 0$, every solution approaches L/a as $x \to \infty$.

(ii) If $a < 0$, there is one and only one solution which approaches L/a as $x \to \infty$.

5.6. Let $y(x)$ be the solution of the initial value problem (5.2), (1.10) in $[x_0, \infty)$, and let $z(x)$ be a continuously differentiable function in $[x_0, \infty)$ such that $z' + p(x)z \le q(x),\ z(x_0) \le y_0$. Show that $z(x) \le y(x)$ for all x in $[x_0, \infty)$. In particular, for the problem $y' + y = \cos x,\ y(0) = 1$ verify that $2e^{-x} - 1 \le y(x) \le 1,\ x \in [0, \infty)$.

5.7. Find the general solution of the following nonlinear DEs:

(i) $2(1+y^3)+3xy^2y'=0$.
(ii) $y+x(1+xy^4)y'=0$.
(iii) $(1-x^2)y'+y^2-1=0$.
(iv) $y'-e^{-x}y^2-y-e^x=0$.

***5.8.** Let the functions $p_0,\ p_1$, and r be continuous in $J=[\alpha,\beta]$ such that $p_0(\alpha)=p_0(\beta)=0,\ p_0(x)>0,\ x\in(\alpha,\beta),\ p_1(x)>0,\ x\in J$, and

$$\int_\alpha^{\alpha+\epsilon}\frac{dx}{p_0(x)} = \int_{\beta-\epsilon}^{\beta}\frac{dx}{p_0(x)} = \infty, \quad 0<\epsilon<\beta-\alpha.$$

Show that all solutions of the DE (5.1) which exist in (α,β) converge to $r(\beta)/p_1(\beta)$ as $x\to\beta$. Further, show that one of these solutions converges to $r(\alpha)/p_1(\alpha)$ as $x\to\alpha$, while all other solutions converge to ∞, or $-\infty$.

Answers or Hints

5.1. Since $M=p(x)y-q(x),\ N=1,\ [(M_y-N_x)/N]=p(x)$, and hence the integrating factor is $\exp(\int^x p(t)dt)$.

5.2. Use the definition of a solution.

5.3. (i) $c\sin x+x^2\sin x$. (ii) $ce^{-x}-x^3+2x^2-5x+5$. (iii) $x(y-1)/(y+1)=y^2+c$. (iv) $x=\tan^{-1}y-1+ce^{-\tan^{-1}y}$.

5.4. (i) $y(x)=\begin{cases}\frac{1}{2}(1-e^{-2x}), & 0\le x\le 1\\ \frac{1}{2}(e^2-1)e^{-2x}, & x>1\end{cases}$ (ii) $y(x)=\begin{cases}e^{-2x}, & 0\le x\le 1\\ e^{-(x+1)}, & x>1.\end{cases}$

5.5. (i) In $y(x)=y(x_0)e^{-a(x-x_0)}+[\int_{x_0}^x e^{at}q(t)dt]/e^{ax}$ take the limit $x\to\infty$. (ii) In $y(x)=e^{-ax}\left[y(x_0)e^{ax_0}+\int_{x_0}^\infty e^{at}q(t)dt-\int_x^\infty e^{at}q(t)dt\right]$ choose $y(x_0)$ so that $y(x_0)e^{ax_0}+\int_{x_0}^\infty e^{at}q(t)dt=0$ ($\lim_{x\to\infty}q(x)=L$). Now in $y(x)=-[\int_x^\infty e^{at}q(t)dt]/e^{ax}$ take the limit $x\to\infty$.

5.6. There exists a continuous function $r(x)\ge 0$ such that $z'+p(x)z=q(x)-r(x),\ z(x_0)\le y_0$. Thus, for the function $\phi(x)=y(x)-z(x),\ \phi'+p(x)\phi=r(x)\ge 0,\ \phi(x_0)=y_0-z(x_0)\ge 0$.

5.7. (i) $x^2(1+y^3)=c$. (ii) $xy^4=3(1+cxy),\ y=0$. (iii) $(y-1)(1+x)=c(1-x)(1+y)$. (iv) $e^x\tan(x+c)$.